Mathematics Interdisciplinary Research **9** (3) (2024) 315 – 331

Original Scientific Paper

Timeline and Wavelets Method for Pricing Cash-or-Nothing Options

*Saeed Vahdati** *and Foad Shokrollahi*

Abstract

This study investigates the application of the Haar wavelet method as an innovative and effective approach for valuing financial derivatives, particularly cash-or-nothing options. Valuing derivatives is a complex task in finance, requiring advanced numerical methods that can adapt to various models and scenarios. Cash-or-nothing options are popular for their simplicity and cost-effectiveness in market speculation and risk hedging, but their pricing is challenging due to several influencing factors. The study provides a comprehensive overview of the Haar wavelet method, demonstrating through numerical examples its precision and stability in option pricing. Additionally, it examines critical risk parameters, such as delta and gamma, essential for managing and hedging risks associated with these options.

Keywords: Option, Cash-or-nothing option, Method of timeline, Haar wavelets, Black-Scholes model.

2020 Mathematics Subject Classification: 65T60, 91G60, 91G20.

How to cite this article
S. Vahdati and F. Shokrollahi, Timeline and wavelets method for pricing cash -or-nothing options, *Math. Interdisc. Res.* **9** (3) (2024) 315-331.

1. Introduction

The valuation of options is an essential component of contemporary finance, serving as a determining factor in investment decision-making, risk management, and

*Corresponding author (E-mail: s.vahdati@khc.ui.ac.ir & sdvahdati@gmail.com)
Academic Editor: Mehrdad Lakestani
Received 6 December 2023, Accepted 2 February 2024
DOI: 10.22052/MIR.2024.253952.1448

trading. To estimate the fair value of options, numerous numerical methods have been devised, ranging from the conventional Black-Scholes model to more sophisticated approaches. The present literature review delves into the numerical methods and their practical implementations in the domain of option pricing, emphasizing the use of Haar wavelets in particular. The Black-Scholes model, proposed by Black and Scholes in 1973 [1], is a notable advancement in the domain of option pricing as it provides a mathematical equation for determining the value of European options. However, it is crucial to recognize that this method has numerous limitations, one of which is its assumption of uniform volatility. Researchers have extended this framework to include intricacies observed in real-life scenarios, leading to the development of computational techniques like Monte Carlo simulation [2] and finite difference methods [3, 4].

Binomial tree models, such as the Cox-Ross-Rubinstein model (CRR) [5], are widely used for pricing American options. These models employ time discretization and utilize a tree structure to compute option pricing. Scientists have made additional improvements to these models in order to address unusual alternatives and intricate payoffs [6].

The Haar wavelet transform, initially developed for signal processing, has recently gained recognition in the finance industry. The efficient representation of discontinuities makes it a compelling choice for option pricing. The application of Haar wavelets in pricing European options and stochastic differential equations has been studied in [7, 8].

Numerical approaches are essential for options that include intricate features such as early exercise rights or dividends. The utilization of the Haar wavelet method has demonstrated potential in effectively accommodating these intricacies. The use of wavelet-based numerical approaches in finance is growing due to their ability to capture irregular patterns in financial time series data. Atilla Cifter [9] utilized wavelet analysis to enhance the precision of implied volatility estimation, showcasing the promise of wavelets in the fields of risk management and option pricing. Luis and Oosterlee [10] introduced a method for pricing European options based on the wavelet approximation method and the characteristic function. They approximate the density function associated to the underlying asset price process by a finite combination of jth order B-splines, and recover the coefficients of the approximation from the characteristic function.

Liu et. al [11] analyzed a nonlinear fractional Black and Scholes model, and they found the solution by using a numerical method, based on a mixture of efficient techniques. In particular, they combine (1) the Haar wavelet integration method which transforms the PDEs into a system of algebraic equations, (2) the homotopy perturbation method in order to linearize the problem, and (3) the variational iteration method for fractional option pricing problems. Devendra and Komal [12] presented a Haar wavelet-based approximation for pricing American options under linear complementarity formulations. They also proposed [13] a two-dimensional Haar wavelet-based approximation technique to study the sensitivities of the price of an option. Other types of wavelets are employed to address diverse linear or nonlinear problems. Chebyshev cardinal wavelets [14] have been used to solve nonlinear Volterra integral equations of the second kind, whereas Legendre wavelets [15] have been utilized for the fractional order reverse osmosis desalination model.

2. Options

Options are financial instruments that grant their holders the privilege, without imposing a duty, to purchase (call option) or sell (put option) a designated underlying asset, such as stocks, commodities, or currencies, at a predefined price (strike price) during a specified timeframe (expiration date). Options are extensively utilized in financial markets and serve multiple crucial purposes.

Hedging is a key application of options, mostly used for risk management purposes. Options can be utilized by investors and corporations as a means of safeguarding their portfolios or assets against unfavorable price fluctuations.

Numerous traders employ options for speculative endeavors. If individuals anticipate an increase in the price of the underlying asset, they may elect to purchase call options. Conversely, if they anticipate a decrease in price, they may choose to purchase put options. Speculation in this context enables traders to generate profits from price fluctuations without possessing the underlying asset, hence offering the advantages of leverage and the potential for increased returns.

Utilizing options can enhance the diversification of investment portfolios. By incorporating more choices into a collection of conventional assets, investors can augment risk-adjusted returns and diminish the overall risk of their portfolio.

Options are frequently employed in the commodities market to mitigate the risk associated with price volatility. Producers and customers can utilize options to establish fixed prices for future deliveries, guaranteeing consistency in their financial planning and business activities.

In the subsequent subsections, we assume that the stock price adheres to a geometric Brownian motion with a consistent drift (μ) and volatility (σ).

$$dS(t) = \mu S(t)d\tau + \sigma dW(t), \tag{1}$$

which $W(t)$ is a Brownian motion. In each of the shapes presented in Figure 1, five sample trajectories of stock prices are illustrated. Various values for drift and volatility have been taken into account to emphasize the distinctions in the range of stock price fluctuations across these different scenarios.

2.1 Cash-or-nothing options

A cash-or-nothing option is a binary option that provides a predetermined payout or no payoff at all when the option expires, depending on the fulfillment of a certain condition. For a cash-or-nothing option, the condition usually pertains to the value of the underlying asset.

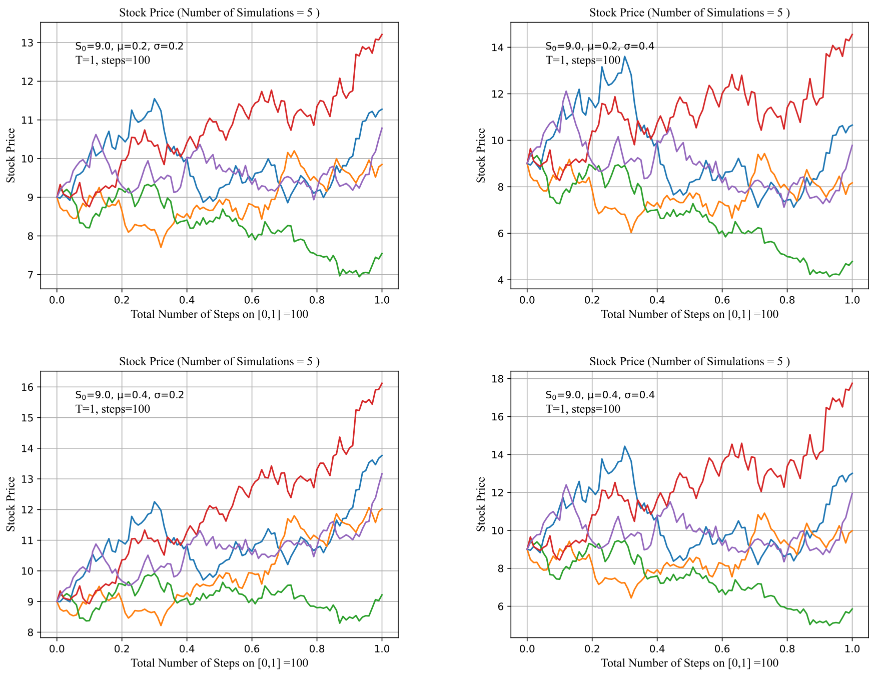

Figure 1: Sample pathes of stock price with different values of μ and σ.

Cash-or-nothing options exhibit a binary or "all-or-nothing" payout structure. At expiration, the option holder will receive a predefined "cash" amount if the specified condition is met, or nothing if the condition is not met.

The available choices can be derived from a range of underlying assets, such as equities, foreign exchange (forex), commodities, or indices. The condition for the payout of the option is frequently linked to the price or performance of the underlying asset.

There are two main categories of cash-or-nothing options: cash-or-nothing call options and cash-or-nothing put options.

1. **Cash-or-nothing call option:** A fixed cash amount is paid out by this type if, upon expiration, the price of the underlying asset exceeds the strike price. If the condition is not satisfied (specifically, if the price is equal to or lower than the strike price), the option holder does not get any compensation.

2. **Cash-or-nothing put option:** This particular type of option provides a predetermined cash payout if, upon reaching the expiration date, the price of the underlying asset is lower than the striking price. If the condition is not satisfied (specifically, if the price is equal to or higher than the strike

price), the option holder does not get any compensation.

Cash-or-nothing options possess a pre-established strike price, which acts as the benchmark for assessing the fulfillment or non-fulfillment of the condition. The predetermined cash amount, referred to as the "cash" component, is fixed and independent of the extent of the price disparity between the strike price and the asset's price at the end of the contract. Cash-or-nothing options, similar to other alternatives, possess an expiration date, which signifies the point in time when the payout for the option holder is determined.

Cash-or-nothing options are characterized by their simplicity and comprehensibility, which contributes to their popularity among traders seeking binary risk-reward outcomes. An inherent benefit of cash-or-nothing options is that the maximum potential loss for a trader is limited to the original price paid for the option. Unlike typical options, which may lead to endless losses for the buyer, this option offers a different outcome.

2.1.1 Black-Scholes model for cash-or-nothing options

The Black-Scholes model, alternatively referred to as the Black-Scholes-Merton (BSM) model, holds significant prominence in contemporary financial theory. This mathematical equation calculates the theoretical value of derivatives by considering the influence of time and various risk factors, while also considering other investment instruments. Created in 1973, it continues to be widely recognized as one of the most effective methods for determining the price of an options contract.

The payoff of a cash-or-nothing call option at the expiry date is

$$C^{\text{Cash}}(S(T),T) = \begin{cases} A, & S(T) > E, \\ 0, & S(T) < E. \end{cases} \tag{2}$$

For a cash-or-nothing put option, the following applies:

$$P^{\text{Cash}}(S(T),T) = \begin{cases} A, & S(T) < E, \\ 0, & S(T) > E. \end{cases} \tag{3}$$

In both of the equations mentioned above, the value of A is held constant, and E represents the striking price.

Define $C^{\text{cash}}(S,t)$ as the value of the cash-or-nothing call option for a given asset price S and time t. Similarly, let $P^{\text{cash}}(S,t)$ represent the value of the cash-or-nothing put option for the same asset price S and time t. By employing the hedging argument [16], we may derive the subsequent equations

$$\frac{\partial C^{\text{Cash}}}{\partial t} + \frac{1}{2}\sigma^2 S \frac{\partial^2 C^{\text{Cash}}}{\partial S^2} + rS\frac{\partial C^{\text{Cash}}}{\partial S} - rC^{\text{cash}} = 0. \tag{4}$$

The cash-or-nothing call payoff function provides the final time conditions

$$\lim_{t\to T^-} C^{\text{Cash}}(S,T) = \begin{cases} A, & S > E, \\ \frac{A}{2}, & S = E, \\ 0, & S < E. \end{cases} \tag{5}$$

When the value of S is equal to zero, the asset remains at zero indefinitely, resulting in a payout of zero. This establishes the boundary condition.

$$C^{\text{Cash}}(0,t) = 0, \qquad \text{for all} \;\; 0 \leq t \leq T. \tag{6}$$

For a sufficiently big value of S, the option is highly likely to yield the amount A. Therefore, after discounting for interest, we determine that

$$C^{\text{Cash}}(S,t) \approx Ae^{-r(T-t)}, \qquad \text{for large} \;\; S. \tag{7}$$

A cash-or-nothing put and a cash-or-nothing call with identical strike prices and expiry dates can be used in a portfolio to establish a relationship known as cash-or-nothing put-call parity

$$C^{\text{Cash}}(S,t) + P^{\text{Cash}}(S,t) = Ae^{-r(T-t)}. \tag{8}$$

Figure 2 illustrates the graphical representation of the payoff functions associated with both call and put options. This depiction offers a comparative analysis of the potential financial outcomes for investors engaging in these derivative instruments.

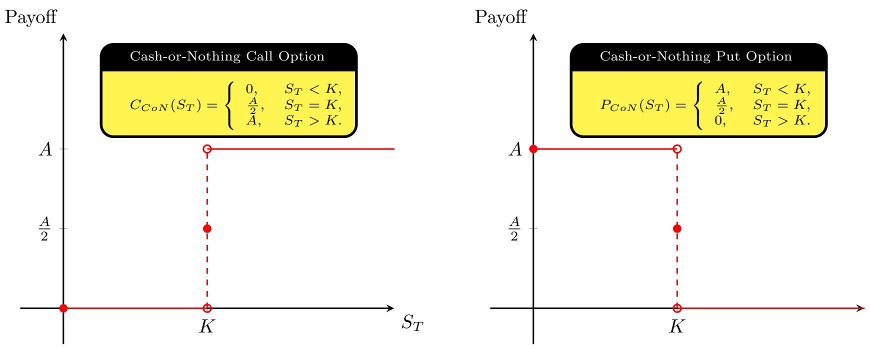

Figure 2: **Left:** Payoff function for cash-or-nothing call option. **Right:** Payoff function for cash-or-nothing put option.

3. Haar Wavelets and function approximation

Haar wavelets are one of the earliest and simplest types of wavelets used in signal processing and numerical analysis. They were developed by the Hungarian mathematician Alfréd Haar in the early 20th century[17]. Haar's work laid the foundation for wavelet analysis, a mathematical framework that became increasingly important in various scientific and engineering applications.

In this work, he presented a compact and orthogonal set of functions that later became known as Haar wavelets. These wavelets were the first known example of a wavelet basis. Haar wavelets are piecewise constant and piecewise linear functions. They consist of a mother scaling function (ϕ) and a mother wavelet function (ψ) with compact support. The scaling and wavelet functions exhibit specific localization and orthogonality properties, which make them valuable for signal processing and solving partial differential equations. One of the key features of Haar wavelets is their orthogonality. This means that the wavelet functions are mutually uncorrelated. The orthogonality property simplifies the decomposition and reconstruction of signals and data, contributing to efficient analysis. The definition of the Haar scaling function is as follows:

$$\varphi(x) = \begin{cases} 1, & 0 \leq x < 1, \\ 0, & \text{elsewhere.} \end{cases} \tag{9}$$

This function will precisely define the Haar wavelet as follows:

$$\psi(x) = \varphi(2x) - \varphi(2x-1) = \begin{cases} 1, & 0 \leq x < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq x < 1. \end{cases} \tag{10}$$

Let $j = 0, 1, 2, \ldots,$ $m = 2^j$, $k = 0, 1, 2, \ldots, m-1$ and $i = m + k + 1$. Put $h_1(x) = \varphi(x)$ and

$$h_i(x) = \begin{cases} 1, & \alpha \leq x < \beta, \\ -1, & \beta \leq x < \gamma, \\ 0, & \text{elsewhere,} \end{cases} \tag{11}$$

where

$$\alpha = \frac{k}{m}, \quad \beta = \frac{2k+1}{2m}, \quad \gamma = \frac{k+1}{m}. \tag{12}$$

The Haar wavelet family is generated by implementing the previously stated definitions. If f is a square integrable function on the interval $[0, 1)$, it can be expressed as a linear combination of the Haar wavelet family in the following manner.

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x), \tag{13}$$

where a_i are constants for $i = 1, 2, \cdots$. Assuming that J represents the highest possible value for j and setting M equal to 2^J, we may obtain an estimate for the square integrable function f on the interval $[0, 1)$ using the following method.

$$f(x) \simeq \sum_{i=1}^{2M} a_i h_i(x). \tag{14}$$

In the subsequent sections, we employ the following notations to streamline the calculations.

$$p_{i,1}(x) = \int_0^x h_i(t)dt, \tag{15}$$

$$p_{i,n+1} = \int_0^x p_{i,n}(t)dt, \quad n = 1, 2, \dots, \tag{16}$$

$$C_{i,n} = \int_0^1 p_{i,n}(x)dx, \quad n = 1, 2, \dots. \tag{17}$$

The relationships mentioned can be derived using the definition of the Haar wavelet and conducting initial calculations, as stated in [18].

$$p_{i,n}(x) = \begin{cases} 0, & 0 \le x < \alpha, \\ \frac{1}{n!}(x-\alpha)^n, & \alpha \le x < \beta, \\ \frac{1}{n!}\left[(x-\alpha)^n - 2(x-\beta)^n\right], & \beta \le x < \gamma, \\ \frac{1}{n!}\left[(x-\alpha)^n - 2(x-\beta)^n + (x-\gamma)^n\right], & \gamma \le x < 1, \end{cases} \tag{18}$$

and

$$C_{1,n} = \frac{1}{(n+1)!}\left[(x-\alpha)^{n+1} - 2(x-\beta)^{n+1} + (x-\gamma)^{n+1}\right], \tag{19}$$

where $n = 1, 2, \cdots$ and $i = 2, 3, \dots$. For the given scenario when i is equal to 1, the following statement holds:

$$p_{1,n} = \frac{x^n}{n!}, \ n = 1, 2, \dots, \tag{20}$$

and

$$C_{1,n} = \frac{1}{(n+1)!}, \ n = 1, 2, \dots. \tag{21}$$

4. Implementation of the method

The domain of the stock price (spatial domain) in Equation (4) is the interval $[0, \infty)$. To solve this problem using Haar wavelets, we begin by replacing the semi-finite interval $[0, \infty)$ with a finite interval $[0, S_{\max}]$. The value of $S_{\max}$ must be chosen carefully to ensure that a satisfactory level of approximation is achieved for the interval $[0, S_{\max}]$. In certain research papers, $S_{\max} = 4E$ is commonly used. Furthermore, we employ the newly introduced variable $x = \frac{S}{S_{\max}}$ to transform the spatial domain from $[0, S_{\max}]$ to $[0, 1]$, enabling us to effectively utilize Haar wavelets in addressing the problem. In order to convert the problem from a backward to a forward time domain, we utilize the variable $\tau = T - t$. By implementing the aforementioned modification of variables to Equation (4), we will derive the subsequent equations for the cash-or-nothing call option:

$$-\frac{\partial C^{\text{Cash}}}{\partial \tau} + \frac{1}{2}\sigma^2 x \frac{\partial^2 C^{\text{Cash}}}{\partial x^2} + rx\frac{\partial C^{\text{Cash}}}{\partial x} - rC^{\text{cash}} = 0, \ \tau \in [0, T], \ x \in [0, 1]. \tag{22}$$

4.1 Pricing cash-or-nothing options

By using a θ-weighted $(0 \leq \theta \leq 1)$ approach for the spatial component and utilizing the forward difference method for the temporal component of Equation (22), we obtain the following result:

$$\begin{aligned} & C_{k+1}^{Cash}(x) - \theta d\tau \left[\frac{\sigma^2}{2} x^2 C_{k+1,xx}^{Cash}(x) + rx C_{k+1,x}^{Cash}(x) - r C_{k+1}^{Cash}(x) \right] \\ = \; & C_k^{Cash}(x) + (1-\theta) d\tau \left[\frac{\sigma^2}{2} x^2 C_{k,xx}^{Cash}(x) + rx C_{k,x}^{Cash}(x) - r C_k^{Cash}(x) \right]. \end{aligned} \tag{23}$$

The function $C_k^{Cash}(x)$ represents the cash value at time τ_k. The value of τ_{k+1} is obtained by adding the time step $d\tau$ to the current value of τ_k. Now, calculate an approximation of the mixed-order derivative using Haar wavelets in the following manner:

$$C_{k+1,xx}^{Cash}(x) \approx \sum_{i=1}^{2M} \alpha_i h_i(x). \tag{24}$$

By performing calculations using integration and substitution [7], we obtain the following equations.

$$C_{k+1,x}^{Cash}(x) \approx \sum_{i=1}^{2M} \alpha_i (p_{i,1}(x) - p_{i,2}(1)) + C_{k+1}^{Cash}(1) - C_{k+1}^{Cash}(0), \tag{25}$$

$$C_{k+1}^{Cash}(x) \approx \sum_{i=1}^{2M} \alpha_i \left(p_{i,2}(x) - x p_{i,2}(1)\right) + x\left(C_{k+1}^{Cash}(1) - C_{k+1}^{Cash}(0)\right) + C_{k+1}^{Cash}(0). \tag{26}$$

For the purpose of simplification, we employ the following notations.

$$\begin{aligned} C_{k+1,xx}^{Cash}(\overrightarrow{\boldsymbol{x}}) &\approx \boldsymbol{\alpha}^{k+1} \boldsymbol{H}(\overrightarrow{\boldsymbol{x}}), & (27) \\ C_{k+1,x}^{Cash}(\overrightarrow{\boldsymbol{x}}) &\approx \boldsymbol{\alpha}^{k+1} \boldsymbol{P}(\overrightarrow{\boldsymbol{x}}) + \boldsymbol{c}^{k+1}, & (28) \\ C_{k+1}^{Cash}(\overrightarrow{\boldsymbol{x}}) &\approx \boldsymbol{\alpha}^{k+1} \boldsymbol{Q}(\overrightarrow{\boldsymbol{x}}) + \overrightarrow{\boldsymbol{x}}\, \boldsymbol{c}^{k+1} + \boldsymbol{d}^{k+1}, & (29) \end{aligned}$$

where

$$\begin{aligned} \overrightarrow{\boldsymbol{x}} &= [x_1 \;\; x_2 \;\; \cdots \;\; x_{2M}], \qquad (30) \\ \boldsymbol{\alpha}^{k+1} &= \left[\alpha_1^{k+1} \;\; \alpha_2^{k+1} \;\; \cdots \;\; \alpha_{2M}^{k+1}\right], \\ \boldsymbol{H} &= \left[h_1(\overrightarrow{\boldsymbol{x}}) \; h_2(\overrightarrow{\boldsymbol{x}}) \; \cdots \; h_{2M}(\overrightarrow{\boldsymbol{x}})\right]^T, \\ \boldsymbol{P} &= \left[p_{1,1}(\overrightarrow{\boldsymbol{x}}) - p_{1,2}(1) \;\; p_{2,1}(\overrightarrow{\boldsymbol{x}}) - p_{2,2}(1) \;\; \cdots \;\; p_{2M,1}(\overrightarrow{\boldsymbol{x}}) - p_{2M,2}(1)\right]^T, \\ \boldsymbol{Q} &= \left[p_{1,2}(\overrightarrow{\boldsymbol{x}}) - \overrightarrow{\boldsymbol{x}} p_{1,2}(1) \;\; p_{2,2}(\overrightarrow{\boldsymbol{x}}) - \overrightarrow{\boldsymbol{x}} p_{2,2}(1) \;\; \cdots \;\; p_{2M,2}(\overrightarrow{\boldsymbol{x}}) - \overrightarrow{\boldsymbol{x}} p_{2M,2}(1)\right]^T, \\ \boldsymbol{c}^{k+1} &= C_{k+1}^{Cash}(1) - C_{k+1}^{Cash}(0), \\ \boldsymbol{d}^{k+1} &= C_{k+1}^{Cash}(0). \end{aligned}$$

By substituting Equations (24), (25), and (26) into Equation (23), using the collocation points $x_j = \frac{j+0.5}{2M}$ and the notations introduced in (27)-(29), we can construct the following system of algebraic equations:

$$\boldsymbol{\alpha}^{k+1}\boldsymbol{G} = \boldsymbol{\alpha}^{k}\boldsymbol{N} + \boldsymbol{R_1} - \boldsymbol{R_2}, \tag{31}$$

where

$$\boldsymbol{G} = (1+\theta r d\tau)\boldsymbol{Q} - \theta r d\tau \overrightarrow{\boldsymbol{x}}\boldsymbol{P} - \frac{1}{2}\theta\sigma^2\overrightarrow{\boldsymbol{x}}^2\boldsymbol{H}, \tag{32}$$

$$\boldsymbol{N} = [1-(1-\theta)rd\tau]\,\boldsymbol{Q} + (1-\theta)rd\tau\overrightarrow{\boldsymbol{x}}\boldsymbol{P} + \frac{1}{2}(1-\theta)\sigma^2\overrightarrow{\boldsymbol{x}}^2\boldsymbol{H}, \tag{33}$$

$$\boldsymbol{R_1} = \boldsymbol{c}^{k+1}\overrightarrow{\boldsymbol{x}} + r\theta d\tau\boldsymbol{d}^{k+1} + \boldsymbol{d}^{k+1}, \tag{34}$$

$$\boldsymbol{R_2} = \boldsymbol{c}^{k}\overrightarrow{\boldsymbol{x}} - (1-\theta)rd\tau\boldsymbol{d}^{k} + \boldsymbol{d}^{k}. \tag{35}$$

The value of $\boldsymbol{\alpha}^1$ can be readily derived from the following equation.

$$C^{Cash}(0,\overrightarrow{\boldsymbol{x}}) = \boldsymbol{\alpha}^1\boldsymbol{Q} + \overrightarrow{\boldsymbol{x}}(C^{Cash}(0,1) - C^{Cash}(0,0)) + C^{Cash}(0,0). \tag{36}$$

5. Numerical examples

To validate the efficacy of the proposed approach, a series of test issues are solved. We seek to obtain a numerical solution for both the cash-or-nothing call option and the cash-or-nothing put option, which will serve as two test cases. For comparison, each numerical result has been juxtaposed with the precise answer for each scenario, which is displayed below. The findings demonstrate that the approach outlined in the research is a remarkably precise and effective one. Given the significance of option prices when initiating a trade, we have placed particular emphasis on error-checking during this stage. We performed all computations and simulations in this paper using Python 3.10.

5.1 Pricing cash-or-nothing call option

The cash-or-nothing call option pricing problem is described by a partial differential equation, which is accompanied by its corresponding boundary conditions. The subsequent equations are derived from the variable transformations covered in the preceding sections.

$$-C_{\tau}^{\text{Cash}} + \frac{1}{2}\sigma^2 x C_{xx}^{\text{Cash}} + rxC_{x}^{\text{Cash}} - rC^{\text{cash}} = 0,\ \tau\in[0,T],\ x\in[0,1], \tag{37}$$

$$C^{\text{Cash}}(x,T) = \begin{cases} A, & x > \frac{E}{S_{\max}}, \\ \frac{A}{2}, & x = \frac{E}{S_{\max}}, \\ 0, & x < \frac{E}{S_{\max}}, \end{cases} \tag{38}$$

$$
\begin{aligned}
C^{\text{Cash}}(0,t) &= 0, &(39)\\
C^{\text{Cash}}(1,t) &= Ae^{-r(T-t)}. &(40)
\end{aligned}
$$

Given Equations (37)-(40) and the values $r = 0.03$, $T = 1$, $K = A = 100$, and $d\tau = 0.01$, the numerical solutions are depicted in the figures. Figure 3 shows the numerical solution of this problem based on the Haar wavelets method. As previously mentioned, the option price at time $t = 0$ holds significant importance. Therefore, we will examine the error in the option price at this specific time. In Figure 4, on the left side, it is observed that when the value of J increases, the error value decreases rapidly. The errors curves for three distinct values of J are depicted in this figure. The right side of Figure 4 illustrates the option pricing at time $t = 0$ for different values of σ.

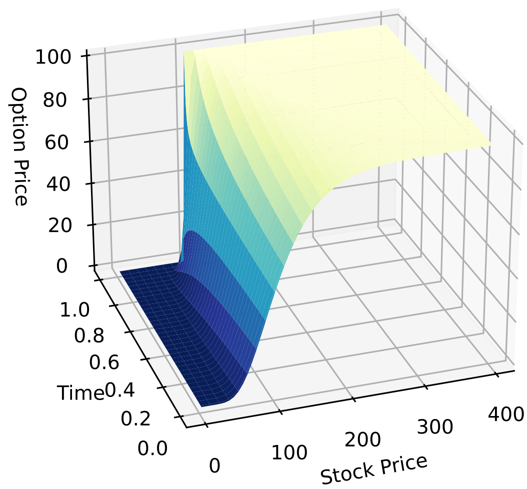

Figure 3: Plot of the approximate solution for the cash-or-nothing call option problem.

5.2 Pricing cash-or-nothing put option

Similarly, in the case of a cash-or-nothing put option, we have

$$
-P_{\tau}^{\text{Cash}} + \frac{1}{2}\sigma^2 x P_{xx}^{\text{Cash}} + rxP_{x}^{\text{Cash}} - rP^{\text{cash}} = 0,\ \ \tau \in [0,T],\ \ x \in [0,1], \qquad (41)
$$

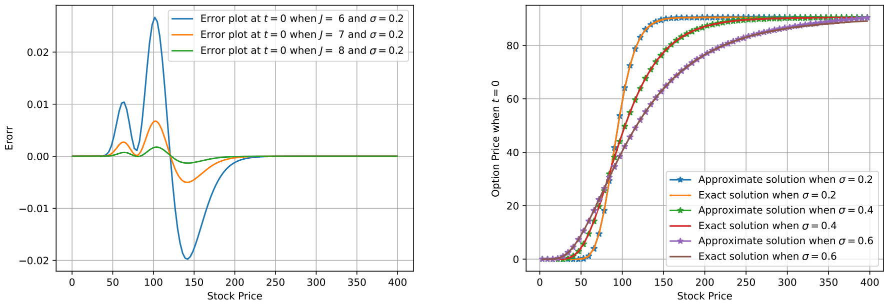

Figure 4: **Left:** Error plot at time $t = 0$ with varying values of J and $\sigma = 0.2$. **Right:** Exact and approximated pricing of a cash-or-nothing call option at time $t = 0$ with varying values of σ.

$$P^{\text{Cash}}(x,T) = \begin{cases} 0, & x > \frac{E}{S_{\max}}, \\ \frac{A}{2}, & x = \frac{E}{S_{\max}}, \\ A, & x < \frac{E}{S_{\max}}, \end{cases} \tag{42}$$

$$P^{\text{Cash}}(0,t) = Ae^{-r(T-t)}, \tag{43}$$

$$P^{\text{Cash}}(1,t) = 0. \tag{44}$$

Figure 5 shows the numerical solution of this problem based on the Haar wavelets method. In Figure 6, on the left side, it is observed that when the value of J increases, the error value decreases rapidly. The errors curves for three distinct values of J are depicted in this diagram. The right side of Figure 6 illustrates the option pricing at time $t = 0$ for different values of r.

5.3 The Greeks: Delta and Gamma

Attempting to forecast the price movement of a single option or a complex option position in response to market fluctuations can be a challenging endeavor. To fully comprehend the movement in the price of an option and its correlation with the underlying asset, it is crucial to grasp the contributing elements and their impact.

Option traders frequently use the terms **Delta:=** $\frac{\partial C}{\partial S}$ (measures the change in an option's price resulting from a change in the underlying security), **Gamma:=** $\frac{\partial^2 C}{\partial S^2}$ (measures the rate of change in the delta for each one-point increase in the underlying asset), **Vega:=** $\frac{\partial C}{\partial \sigma}$ (measures the sensitivity of the price of an option

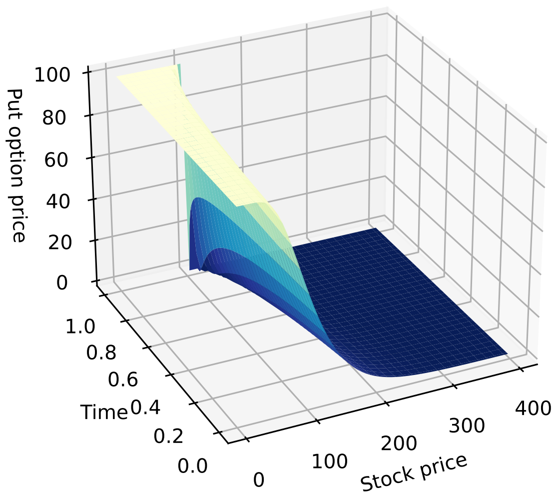

Figure 5: Plot of the approximate solution for the cash-or-nothing put option problem.

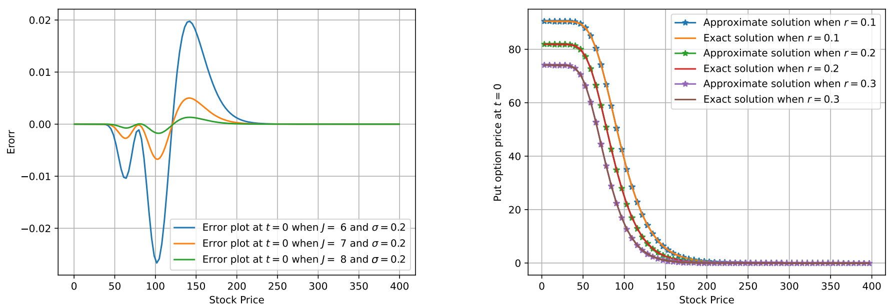

Figure 6: **Left:** Plotting the error at time $t = 0$ for various values of J. **Right:** Exact and approximated pricing of a cash-or-nothing put option at time $t = 0$ with varying interest rates.

to changes in volatility), and **Theta:** $= \dfrac{\partial C}{\partial t}$ (it explains the effect of time on the premium of the options purchased or sold) to describe the characteristics of their option holdings. The Greeks collectively refer to these phrases and provide a means to quantify the impact of various factors on the price of an option. The terminology may appear perplexing and daunting to novice option traders, but, when dissected, the Greeks denote uncomplicated ideas that can enhance one's comprehension of the risk and potential gain associated with an option position.

In this section, by employing the methodology utilized in the preceding section, it is possible to directly acquire two Greek individuals. Once the values of vectors $\boldsymbol{\alpha}^k$ is obtained in Equations (36) and (31), it is simply a matter of substituting these values into Equations (28) and (27) to determine the Greeks **Delta** and **Gamma** respectively. Figure 7 displays the Delta and Gamma surfaces for the cash-or-nothing call option. In this figure, we have utilized the following constants: $K = A = 100, r = 0.03, \sigma = 0.2, T = 1, J = 6$ and $d\tau = 0.001$. Delta is a risk metric utilized to approximate the price volatility of a derivative, such as an options contract, in response to a one-percent change in the underlying security. Additionally, the delta provides options traders with the hedging ratio required to achieve delta neutrality. The probability of finishing in the money is a third interpretation of the Delta of an option. Positive or negative delta values are possible, contingent upon the nature of the option. Gamma, which represents the rate of change in an option's Delta for each one-point movement in the price of the underlying asset, is an option risk metric. As previously stated, Delta indicates the magnitude of the premium (price) variation in an option in response to a one-point change in the price of the underlying asset. Consequently, Gamma represents the rate at which the price of an option reacts to changes in the underlying security price. As Gamma increases, the volatility in the option's price also increases. Gamma is a significant indicator of the degree to which the value of a derivative is convex in relation to the underlying asset. It is greatest when an option is in the money and decreases as the option moves further out of the money. Gamma is also greatest for options approaching their expiration date compared to those with a later date, assuming all other factors remain constant. It is utilized to determine how changes in the fundamental asset will impact the value of an option. By employing Delta-Gamma hedging, an options position is protected from fluctuations in the underlying asset. At this moment, singularities are possible due to the inheritance of the mentioned risk parameters and the discontinuity of the solution at the expiration date.

6. Conclusions

In this article, we have explored the application of the Haar wavelet method for pricing cash-or-nothing options, which are a type of binary or digital option that pays a fixed amount if the underlying asset is above or below a certain level at maturity. The Haar wavelet method can approximate the functions involved in the Black-Scholes partial differential equation, which is the standard model for

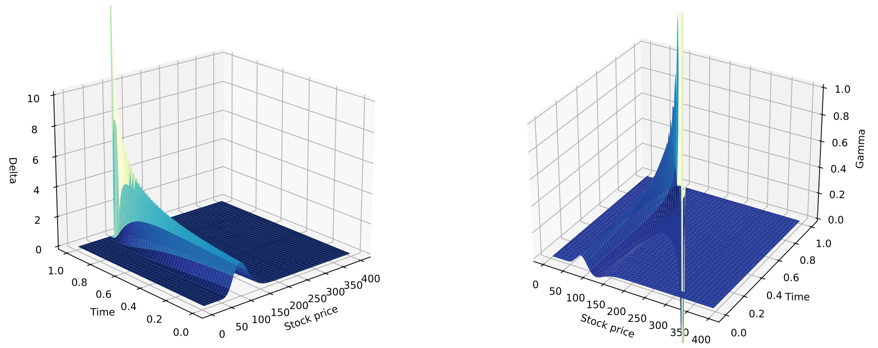

Figure 7: **Left:** Delta surface for the cash-or-nothing call option. **Right:** Gamma surface for the cash-or-nothing call option.

option pricing. We have also provided several examples of cash-or-nothing options and solved them by using the Haar wavelet method. The results have demonstrated the accuracy and efficiency of the Haar wavelet method in valuing cash-or-nothing options, as well as its adaptability to different pricing models, such as the Black-Scholes model and its extensions. The Haar wavelet method is a promising technique for solving option pricing problems, as it offers several advantages over other numerical methods, such as simplicity, stability, and scalability. Other types of options and financial derivatives, such as asset-or-nothing options, barrier options, and exotic options, can also benefit from the application of the Haar wavelet method. We hope that this article will inspire further research and development of the Haar wavelet method and its applications in finance.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

References

[1] F. Black and M. Scholes, The pricing of options and corporate liabilities, *J. Political Econ.* **81** (1973) 637 – 654.

[2] P. Glasserman, *Monte Carlo Methods in Financial Engineering*, Springer Science & Business Media, 2004.

[3] P. Wilmott, *Paul Wilmott Introduces Quantitative Finance*, John Wiley & Sons, 2007.

[4] D. J. Duffy, *Finite Difference Methods in Financial Engineering: A Partial Differential Equation Approach*, John Wiley & Sons, 2013.

[5] J. C. Cox, S. A. Ross and M. Rubinstein, Option pricing: a simplified approach, *J. Financ. Econ.* **7** (1979) 229 − 263, https://doi.org/10.1016/0304-405X(79)90015-1.

[6] R. Geske and H. E. Johnson, The american put option valued analytically, *J. Finance.* **39** (1984) 1511 − 1524, https://doi.org/10.1111/j.1540-6261.1984.tb04921.x.

[7] S. Vahdati, M. R. Ahmadi Darani and M. R. Ghanei, Haar wavelet-based valuation method for pricing European options, *Comput. Methods Differ. Equ.* **11** (2023) 281 − 290, https://doi.org/10.22034/CMDE.2022.52027.2177.

[8] S. Vahdati, A wavelet method for stochastic Volterra integral equations and its application to general stock model, *Comput. Methods Differ. Equ.* **5** (2017) 170 − 188.

[9] A. Cifter, Value-at-risk estimation with wavelet-based extreme value theory: evidence from emerging markets, *Physica A Stat. Mech. Appl.* **390** (2011) 2356 − 2367, https://doi.org/10.1016/j.physa.2011.02.033.

[10] L. Ortiz-Gracia and C. W. Oosterlee, Robust pricing of European options with wavelets and the characteristic function, *SIAM J. Sci. Comput.* **35** (2013) $B1055 - B1084$, https://doi.org/10.1137/130907288.

[11] L. Meng, M. Kexin, X. Ruyi, S. Mei and C. Cattani, Haar wavelet transform and variational iteration method for fractional option pricing models, *Math. Methods Appl. Sci.* **46** (2023) 8408−8417, https://doi.org/10.1002/mma.8343.

[12] D. Kumar and K. Deswal, Haar-wavelet based approximation for pricing American options under linear complementarity formulations, *Numer. Methods Partial Differential Equations* **37** (2021) 1091 − 1111, https://doi.org/10.1002/num.22568.

[13] D. Kumar and K. Deswal, Two-dimensional Haar wavelet based approximation technique to study the sensitivities of the price of an option, *Numer. Methods Partial Differential Equations* **38** (2022) 1195 − 1214, https://doi.org/10.1002/num.22729.

[14] B. Salehi, L. Torkzadeh and K. Nouri, Chebyshev cardinal wavelets for nonlinear volterra integral equations of the second kind, *Math. Interdisc. Res.* **7** (2022) 281 − 299, https://doi.org/ 10.22052/MIR.2022.243395.1325.

[15] O. Belhamiti and B. Absar, A numerical study of fractional order reverse osmosis desalination model using legendre wavelet approximation, *Iranian J. Math. Chem.* **8** (2017) 345 − 364, https://doi.org/10.22052/IJMC.2017.86494.1289.

[16] D. J. Higham, *An Introduction to Financial Option Valuation: Mathematics, Stochastics and Computation*, Cambridge University Press, 2004.

[17] A. Haar, Zur Theorie der orthogonalen Funktionensysteme, *Math. Ann.* **69** (1910) 331 − 371, https://doi.org/10.1007/BF01456326.

[18] S. −u. Islam, I. Aziz and A. S. Al-Fhaid, An improved method based on Haar wavelets for numerical solution of nonlinear integral and integro-differential equations of first and higher orders, *J. Comput. Appl. Math.* **260** (2014) 449−469, https://doi.org/10.1016/j.cam.2013.10.024.

Saeed Vahdati
Department of Mathematics,
Khansar Campus, University of Isfahan,
Isfahan, I. R. Iran.
e-mail: s.vahdati@khc.ui.ac.ir & sdvahdati@gmail.com

Foad Shokrollahi
Department of Mathematics and Statistics,
University of Vaasa,
P.O. Box 700, Fin-65101 Vaasa, Finland.
e-mail: foad.shokrollahi@uwasa.fi