

Yosra Barkaoui

**Products of
nonnegative
selfadjoint
operators, linear
relations, and
their local spectral
theory**



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TIIVISTELMÄ

Yksi tämän väitöskirjan keskeisistä kohteista on suljettujen, yleisesti rajoittamattomien Hilbert-avaruuden operaattoreiden T luokka, joiden jäsenet voidaan faktoroida kahden ei-negatiivisen itseadjungoidun operaattorin A ja B tuloksi $T = AB$. Tapaus, jossa jompikumpi operaattoreista A tai B on rajoitettu tai vaihtoehtoisesti, kun jommallakummalla niistä on rajoitettu käänteisoperaattori, mahdollistaa lokaalin spektraaliteorian kehittämisen tällaisille operaattoreille. Tapaus, jossa A on rajoitettu operaattori, karakterisoidaan ensin adjungaatin T^* kvasiaffinisudella operaattoriin $S = S^* \geq 0$. Toinen karakterisointi johdetaan yleistämällä ensin Sebestyén'in teoreema rajoitettujen operaattoreiden luokasta rajoittamattomien operaattoreiden luokkaan. Lisäksi todistetaan Sebestyén'in epäyhtälön käänteinen versio, jonka osoitetaan olevan yhteydessä T :n kvasi-affinisuuteen johonkin ei-negatiiviseen operaattoriin $S = S^* \geq 0$.

Tästä saadaan operaattoreiden T toinen alaluokka, jossa A :n käänteisoperaattori on rajoitettu. Tätä tarkoitusta varten Sebestyénin epäyhtälö laajennetaan myös lineaaristen relaatioiden tapaukseen. Lisäksi väitöskirjassa tutkitaan näiden kahden alaluokan yhteyttä heikkoon similaarisuuteen sekä T :n kvasi-affinisuuteen operaattoriin $S = S^* \geq 0$. Lopuksi väitöskirjassa todistetaan lokaaliin spektraaliteoriaan liittyviä tuloksia tapauksessa, jossa A on rajoitettu operaattori; mm. yksiarvoinen laajennusominaisuus (SVEP), Dunfordin ominaisuus (c) ja ennen kaikkea se, että tällaiseen luokkaan kuuluvat operaattorit ovat \mathbb{C} -yleistettyjä skalaarioperaattoreita. Tämä ominaisuus on keskeisessä asemassa, koska sen ansiosta myös kaikkien operaattoreiden T , jotka ovat kvasi-affiineja jonkun ei-negatiivisen itseadjungoidun operaattorin S kanssa, spektri toteuttaa yhtälön $\sigma(T) = \sigma(S)$.

Avainsanat: Ei-negatiivisten itseadjungoitujen operaattoreiden tulot. Käänteinen epäyhtälö. Ei-negatiiviset lineaariset relaatiot. Kvasi-affinisuus. Lokaali spektraaliteoria.

ABSTRACT

One of the key objects in this thesis is a class of closed, in general, unbounded Hilbert space operators T which admit a factorization into a product $T = AB$ of two nonnegative selfadjoint operators A and B in a Hilbert space. The case where one of the operators A or B is bounded or, alternatively, when one of them admits a bounded inverse, appears to be most prominent for further analysis and leads to a development of local spectral theory for such operators. The subclass, where A is bounded, is characterized first by means of quasi-affinity of T^* to an operator $S = S^* \geq 0$. Another characterization is established by first generalizing Sebestyén theorem from case of bounded operators to the present setting of unbounded operators.

Moreover, a reversed version of Sebestyén inequality is proved and shown to be connected to the quasi-affinity of T to $S = S^* \geq 0$. This gives rise to the second subclass of operators T , where the inverse of A is bounded. For this purpose Sebestyén inequality is extended even to the case of linear relations. Furthermore, the connection between these two subclasses and weak-similarity as well as quasi-affinity to some $S = S^* \geq 0$ is investigated. Finally, this thesis establishes different local spectral properties of the first class including the single valued extension property (SVEP), the Dunford's property (c) and, more significantly the fact that operators in such a class are \mathbb{C} -generalized scalar. This property plays a central role as it also shows that any operator T that is quasi-affine to a nonnegative selfadjoint operator S satisfies $\sigma(T) = \sigma(S)$.

Keywords: Products of nonnegative selfadjoint operators. Reversed inequality. Nonnegative linear relations. Quasi-affinity. Local spectral theory.

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LIST OF PUBLICATIONS

The dissertation is based on the following three articles:

- (I) Barkaoui, Y., and Hassi,S (2025) Product of nonnegative selfadjoint operators in unbounded settings. <https://doi.org/10.48550/arXiv.2507.14404>
- (II) Barkaoui, Y., and Mnif,M (2025) Three equivalent conditions for spectral decomposable linear relations <https://doi.org/10.1007/s40065-025-00503-5>
- (III) Barkaoui, Y., and Hassi,S (2025) Local spectral theory for unbounded product of nonnegative selfadjoint operators <https://doi.org/10.1007/s11785-025-01807-2>

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AUTHOR'S CONTRIBUTION

Publication I: “Product of nonnegative selfadjoint operators in unbounded settings ”

The original idea for this article was proposed by Seppo Hassi. The work was carried out in collaboration, and the results were developed with continuous guidance and discussions. The author contributed to the proofs and writing under the supervision of Seppo Hassi.

Publication II: “Three equivalent conditions for spectral decomposable linear relations ”

This article is based on two previous papers co-authored with the author's previous supervisor, Maher Mnif, dealing with necessary and sufficient conditions for extended spectral decomposability of a closed linear relation T . Motivated by this work, the author proposed a further study of the spectral properties of T . This led to a synthesis of the two above-mentioned papers and resulted in a unified characterization of spectral decomposability. The results presented in this article were established and written by the author under the supervision of Maher Mnif.

Publication III: “Local spectral theory for unbounded product of non-negative selfadjoint operators ”

This article is a natural continuation of Publication I. The work was developed in collaboration, benefiting from regular discussions and feedback from the supervisor Seppo Hassi.

1 INTRODUCTION

Nonnegative selfadjoint operators play a central role in quantum mechanics and functional analysis. In physics, they model measurable quantities such as energy and momentum, where the non-negativity constraint implies natural restrictions so that no negative kinetic energy can be obtained. In mathematics, these operators are of interest in spectral theory, semigroups and the theory of partial differential equations. This thesis, in fact, studies the product of such operators in the unbounded framework of complex Hilbert spaces involving local spectral theory.

From a historical point of view, the first insights into this problem were obtained in finite-dimensional contexts: in 1988, Wu (1988) proved that a necessary and sufficient condition for an operator $T = AB$ to be factorized as the product of two positive matrices A and B for it is to be similar to a positive matrix. When dealing with infinite-dimensional Hilbert spaces, the issue becomes richer and more complex and, in fact, Contino, Dritschel, Maestripieri, and Marcantognini (2021) proved that similarity to a nonnegative operator is no longer sufficient to characterize the product of two nonnegative operators, since they established the following characterization:

$$\begin{aligned} T \text{ is similar to a nonnegative operator} \\ \Updownarrow \\ T = AB \text{ with } A, B \in \mathbf{B}^+(\mathfrak{H}) \text{ and, in addition, } A \text{ or } B \text{ is invertible,} \end{aligned} \tag{1.1}$$

where $\mathbf{B}(\mathfrak{H})$ and $\mathbf{B}^+(\mathfrak{H})$ stand for the set of all bounded and bounded nonnegative operators on \mathfrak{H} , respectively. Even weaker conditions than similarity, such as quasi-similarity and quasi-affinity have also proven to be insufficient to fully characterize such a product. Indeed, for $T = AB$, $A, B \in \mathbf{B}^+(\mathfrak{H})$, they proved that

$$T \text{ is quasi-affine to } S \in \mathbf{B}^+(\mathfrak{H}) \Leftrightarrow T = A_1 B_1, A_1, B_1 \in \mathbf{B}^+(\mathfrak{H}) \text{ and } A_1 \text{ is injective.}$$

So they rather characterized such a product by means of Sebestyén inequality proved in Sebestyén (1983) as follows:

$$T = AB, A, B \in \mathbf{B}^+(\mathfrak{H}) \Leftrightarrow TT^* \leq XT^* \text{ for some } X \in \mathbf{B}^+(\mathfrak{H}). \tag{1.2}$$

Hence, a natural approach to improve the above results is either to pursue weaker concepts than quasi-affinity or to relax certain conditions on T .

One of the main purposes in this dissertation is to investigate these questions and to extend the above results to the setting of unbounded operators T . More precisely, a complete study is first carried out when a closed operator T belongs to the following class of operators:

$$\mathcal{L}_l^{+2}(\mathfrak{H}) := \{T = AB, A \in \mathbf{B}^+(\mathfrak{H}) \text{ and } B = B^* \geq 0\},$$

where B is in general unbounded. It will be seen in Barkaoui and Hassi (2025b) that (1.1) remains true for unbounded operators T , where the product representation AB belongs to $\mathcal{L}_l^{+2}(\mathfrak{H})$. Furthermore, every element of this class satisfies an equality analogous to the one appearing in (1.2). More generally, for closed operators T and B such that T^*B is selfadjoint, Sebestyén theorem is generalized to the unbounded context as follows:

$$X\overline{B_0} \subseteq T \text{ for some } X \in \mathbf{B}^+(\mathfrak{H}) \iff T^*T \leq \lambda T^*B, \quad (1.3)$$

for the restriction $B_0 := B \upharpoonright \text{dom } T^*B$ of B . Obviously, in the particular case where $\text{dom } T^*B$ is a core for B , i.e., $\overline{B_0} = B$, (1.3) is instead stated for B . This covers the bounded setting in which the equivalence (1.3) holds with equality $T = XB$. However, for the unbounded setting where $B \neq \overline{B_0}$, one only gets the following implication when T^*B is selfadjoint:

$$XB \subseteq T \implies T^*T \leq \lambda T^*B \leq \lambda^2 B^*B. \quad (1.4)$$

Note that the reversed implication is not true unless $B^*T = T^*B$, as proved in Barkaoui and Hassi (2025b).

The inclusion in (1.4) represents a good motivation for describing the connection between the class $\mathcal{L}_l^{+2}(\mathfrak{H})$ and the notion of quasi-affinity to a nonnegative selfadjoint operator as shown in the following result:

$$T^* \text{ is quasi-affine to } S = S^* \geq 0 \iff \mathcal{L}_l^{+2}(\mathfrak{H}) \ni AB \subseteq T \text{ with } \overline{\text{ran}} A = \mathfrak{H} \quad (1.5)$$

Motivated by (1.3), this induces the following new characterization of Sebestyén inequality by means of quasi-affinity to some $S = S^* \geq 0$:

$$\begin{aligned} T^*T \leq \lambda T^*B \text{ with } \text{dom } T \subseteq \text{dom } B \text{ for some } \lambda \geq 0, B = B^* = \overline{B_0} \geq 0 \\ \iff \\ T = A\overline{B_0} \in \mathcal{L}_l^{+2}(\mathfrak{H}) \text{ with } \overline{\text{ran}} A = \mathfrak{H} \\ \iff \\ T^* \text{ is } G\text{-quasi-affine to } S = S^* \geq 0 \text{ with } \text{dom } T \subseteq \text{dom } \overline{B_F \upharpoonright \text{dom}(T^*B_F)} \\ \text{and } B_F = G^{-1}S^{\frac{1}{2}}S^{\frac{1}{2}}(G^{-1})^*. \end{aligned} \quad (1.6)$$

It is quite important to note that these equivalences have been extended to fully cover the entire class $\mathcal{L}_l^{+2}(\mathfrak{H})$, thanks to the implication in (1.4).

An additional major objective pursued in this work is to investigate the reversed inequality

$$T^*T \geq \eta AT, \quad \eta > 0, \quad (1.7)$$

and prove analogues for the characterizations in (1.3) and (1.5). The idea to get further characterizations here is to make a connection to the initial Sebestyén inequality (1.3) by taking inverses in the operator inequality (1.7). This has motivated a further generalization of the above results to the case of nondensely defined op-

erators as well as multivalued linear operators (linear relations); see Baskakov and Zagorskii (2007); Cross (1998); Derkach, Hassi, Malamud, and De Snoo (2006); Von Neumann (1950).

For the reversed inequality (1.7), quasi-affinity of T , rather than T^* , to S arises and leads to a new class defined by

$$\mathcal{L}_l^{+2}(\mathfrak{H}) := \{T = BA, B^{-1} \in \mathbf{B}^+(\mathfrak{H}) \text{ and } A = A^* \geq 0\}.$$

It is proved that whenever T is G -quasi-affine to $S = S^* \geq 0$ with $\rho(\overline{G^* S^{\frac{1}{2}} S^{\frac{1}{2}} G T}) \neq \emptyset$, then the reverse inequality holds for some $A = A^* \geq 0$:

$$T^* T \geq \frac{1}{\lambda} AT.$$

In fact, behind this proof appears the notion of Friedrichs extension of a nonnegative (symmetric operator). More importantly, the following analogue of (1.5) is established:

$$T \text{ is quasi-affine to some } S = S^* \geq 0 \iff T \subseteq BA \in \mathcal{L}_l^{+2}(\mathfrak{H}). \quad (1.8)$$

It is clear from (1.8) and (1.5) that there is no direct relation between $\mathcal{L}_l^{+2}(\mathfrak{H})$ and $\mathcal{L}_l^{+2}(\mathfrak{H})$. However, if T is quasi-similar to $S = S^* \geq 0$ then one can construct $T_1 \in \mathcal{L}_l^{+2}(\mathfrak{H})$ and $T_2 \in \mathcal{L}_l^{+2}(\mathfrak{H})$ such that

$$T_1 \subseteq T \subseteq T_2.$$

Finally we turn our attention to the spectral as well as local spectral theory of the class $\mathcal{L}_l^{+2}(\mathfrak{H})$. Just as in the finite-dimensional case, we prove that any operator $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ has a positive spectrum whenever $\rho(T) \neq \emptyset$. This latter condition plays a crucial role in questions related to W -similarity and similarity to nonnegative selfadjoint operators, ultimately leading to the following equivalence for a closed operator T with $\rho(T) \neq \emptyset$

$$T \text{ is } \mathcal{W}\text{-similar to } S = S^* \geq 0 \iff T \in \mathcal{L}_l^{+2}(\mathfrak{H}) \iff T \in \mathcal{L}_l^{+2}(\mathfrak{H}).$$

More specifically, it is proved that any $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ with $\rho(T) \neq \emptyset$ admits the single valued extension property (SVEP) and, more generally, is a \mathbb{C} -generalized scalar operator. The latter property is a useful key for the study of further local spectral properties of T such as the Dunford's property. It also implies that if $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ with $\rho(T) \neq \emptyset$ is quasi-affine to $S = S^* \geq 0$ then the spectra of T and S coincide. In this case, \mathfrak{H} can be described via the local spectral subspaces of both S and T^* as follows:

$$\mathfrak{H} = \mathfrak{H}_S(\sigma(T)) = \mathfrak{H}_{T^*}(\sigma(S^*)). \quad (1.9)$$

However, in general, when T is quasi-affine to a closed operator T_2 , only the first equality in (1.9) holds, i.e., $\mathfrak{H} = \mathfrak{H}_{T_2}(\sigma(T))$. All these properties are the objective of Barkaoui and Hassi (2025a) where the optimal pairs in $\mathcal{L}_l^{+2}(\mathfrak{H})$ are treated. It is

worth noting that all the above results cover, in particular, the classical framework of bounded operators.

2 LINEAR RELATIONS

Multivalued linear operators (linear relations) were first introduced by Von Neumann (1950), and then have been studied by various mathematicians (e.g Bennewitz (2006); Coddington (1973); Cross (1998); Derkach et al. (2006); Favini and Yagi (1993); Langer and Textorius (1977)) for their extensive applications in digital imaging, differential inclusions, game theory, mathematical economics, and other fields of applied mathematics (see, for instance Kaczynski (2008)). What is particularly interesting about the theory of linear relations is that we can define and analyze both the inverse and the adjoint without requiring either injectivity or domain density. This is in fact the main reason for their use in our work, in particular for taking inverses and adjoints in Sebestyén inequality and extending it to the general case of unbounded linear relations (including linear operators) in \mathfrak{H} .

Here $(\mathfrak{H}, (\cdot, \cdot)_{\mathfrak{H}})$ is a complex Hilbert space, and a *linear relation* is a mapping T from a linear subspace $\text{dom } T$ of \mathfrak{H} into the collection of non-empty subsets of a complex Hilbert space $(\mathfrak{K}, (\cdot, \cdot)_{\mathfrak{K}})$ such that

$$\alpha Tx + \beta Ty = T(\alpha x + \beta y) \quad \text{for all } x, y \in \text{dom } T \text{ and } \alpha, \beta \in \mathbb{C} \setminus \{0\}.$$

The class of all linear relations from $T : \mathfrak{H} \rightarrow \mathfrak{K}$ is denoted by $R(\mathfrak{H}, \mathfrak{K})$. If $\mathfrak{H} = \mathfrak{K}$, one simply says that T is a linear relation on \mathfrak{H} or writes $T \in R(\mathfrak{H})$. Note that a relation $T \in R(\mathfrak{H}, \mathfrak{K})$ is entirely determined by its graph given by

$$T := G(T) = \{(x, y) \in \mathfrak{H} \times \mathfrak{K} ; x \in \text{dom } T \text{ and } y \in Tx\}.$$

In particular, the *inverse* of $T \in R(\mathfrak{H}, \mathfrak{K})$, the *sum*, the *componentwise sum* and the *product* of $T_1 \in R(\mathfrak{H}, \mathfrak{K})$ and $T_2 \in R(\mathfrak{K}, \mathfrak{V})$ are respectively defined by

$$\begin{aligned} T^{-1} &:= \{(y, x) \in \mathfrak{K} \times \mathfrak{H} ; (x, y) \in T\}, \\ T_1 + T_2 &:= \{(x, y_1 + y_2) \mid (x, y_1) \in T_1, (x, y_2) \in T_2\}, \\ T_1 \hat{+} T_2 &:= \{(x + u, y + v) \mid (x, y) \in T_1, (u, v) \in T_2\}, \\ T_1 T_2 &:= \{(x, z) \in \mathfrak{H} \times \mathfrak{V} ; (x, y) \in T_1 \text{ and } (y, z) \in T_2 \text{ for some } y \in \mathfrak{K}\}. \end{aligned}$$

The *range*, *kernel* and *multivalued part* of a relation $T \in R(\mathfrak{H}, \mathfrak{K})$ are defined by

$$\begin{aligned} \text{ran } T &= \{y \in \mathfrak{K} : (x, y) \in T \text{ for some } x \in \mathfrak{H}\}, \\ \text{ker } T &= \{x \in \mathfrak{H} : (x, 0) \in T\}, \\ \text{mul } T &= \{y \in \mathfrak{K} : (0, y) \in T\}, \end{aligned}$$

respectively. The closure of $\text{ran } T$ (respectively, $\text{dom } T$) is denoted by $\overline{\text{ran } T}$ (respectively, $\overline{\text{dom } T}$).

The adjoint of $T \in R(\mathfrak{H})$ is denoted by T^* and it is defined as

$$T^* := \{(x, x') \in \mathfrak{K} \times \mathfrak{H}; (x', y)_{\mathfrak{H}} = (x, y')_{\mathfrak{K}} \text{ for all } (y, y') \in T\}.$$

A linear relation T is said to be *closed* if $G(T)$ is closed. The *closure* of T is denoted by \overline{T} and often also denoted by T^{**} . If $\text{mul } T^{**} = \text{mul } T$ then T is called *closable*. Hence, it is clear that T is a closable operator if and only if $\text{mul } T^{**} = \{0\}$. Clearly, $H^* = (\overline{H})^*$ and it follows directly from the definition that

$$(H^*)^{-1} = (H^{-1})^*.$$

Some further elementary properties of adjoint of relations are collected from Cross (1998) in the next proposition.

Proposition 2.1. *For a linear relation $H \in R(\mathfrak{H}, \mathfrak{K})$ the following statements hold:*

(i)(i)

1. $(\text{dom } H)^\perp = \text{mul } H^*$ and $(\text{dom } H^*)^\perp = \text{mul } \overline{H}$;
2. $\ker H^* = (\text{ran } H)^\perp$ and $(\ker H^*)^\perp = \overline{\text{ran } H}$;
3. H^* is an operator if and only if $\text{dom } H$ is dense in \mathfrak{H} ;
4. H is a closable operator if and only if $\text{dom } H^*$ is dense in \mathfrak{K} .

The operator part of a closed relation $T \in CR(\mathfrak{H}, \mathfrak{K})$ is given by $T_s = P_s T$, where P_s stands for the orthogonal projection onto $(\text{mul } T)^\perp = \overline{\text{dom } T^*}$. If T is closed then T_s is closed and one has $T = T_s \hat{\oplus} T_{\text{mul}}$, where $T_{\text{mul}} := \{0\} \times \text{mul } T$. Here $CR(\mathfrak{H}, \mathfrak{K})$ denotes the set of all closed linear relations from \mathfrak{H} to \mathfrak{K} .

In what follows one often needs to determine the adjoint and the closure of the product of linear relations T and S . Unlike the case of bounded operators one has only the inclusion

$$(ST)^* \supseteq T^* S^*.$$

The following lemma contains a useful condition for the preceding inclusion to be an equality.

Lemma 2.2. *(Derkach, Hassi, Malamud, & de Snoo, 2009, Lemma 2.9) Let $T \in CR(\mathfrak{H}, \mathfrak{K})$ and $S \in CR(\mathfrak{K}, \mathfrak{V})$ be closed linear relations. If $\text{dom } S$ is closed and $\text{ran } T \subseteq \text{dom } S$ then*

$$(ST)^* = T^* S^*. \tag{2.1}$$

In particular, (2.1) holds when S is a bounded everywhere defined linear operator.

3 NONNEGATIVE SELFADJOINT SINGLE-VALUED AND MULTIVALUED OPERATORS

A linear relation $T \in R(\mathfrak{H})$ is said to be *symmetric* if $\langle x', x \rangle_{\mathfrak{H}} \in \mathbb{R}$ for all $(x, x') \in T$ or, equivalently, if $T \subseteq T^*$. In particular, if T is closed then $T_s \subseteq T$ and $T^* \subseteq (T_s)^*$, so that

$$T^*T \subseteq (T_s)^*T = T^*P_sT = T^*T_s \subseteq (T_s)^*T_s. \quad (3.1)$$

A linear relation $T \in R(\mathfrak{H})$ is said to be *nonnegative* if $\langle x', x \rangle_{\mathfrak{H}} \in \mathbb{R}^+$ for all $(x, x') \in T$ and writes $T \geq 0$. If $T = T^*$ then T is called *selfadjoint*. In particular, for $T \in CR(\mathfrak{H})$, the product T^*T is a nonnegative selfadjoint relation; see (Behrndt, Hassi, & De Snoo, 2020, Lemma 1.5.8). In this case, equality holds in (3.1) and one has

$$T^*T = (T_s)^*T = T^*P_sT = T^*T_s = (T_s)^*T_s.$$

Hence, $\text{mul } T^*T = \text{mul } T^* = (\text{dom } T)^\perp$.

Note that, if T is a nonnegative selfadjoint relation then there exists a unique nonnegative selfadjoint relation K in \mathfrak{H} , denoted by $K = T^{\frac{1}{2}}$, such that $K^2 = T$. Moreover, one has

$$T^{\frac{1}{2}} = (T_s)^{\frac{1}{2}} \oplus T_{\text{mul}}. \quad (3.2)$$

Definition 3.1. Let T and S be nonnegative selfadjoint relations in \mathfrak{H} . Then $T \leq S$, if

$$\text{dom } S^{\frac{1}{2}} \subseteq \text{dom } T^{\frac{1}{2}} \text{ and } \|(T_s)^{\frac{1}{2}}x\| \leq \|S_s^{\frac{1}{2}}x\|, \quad x \in \text{dom } S^{\frac{1}{2}}. \quad (3.3)$$

Lemma 3.2. (Barkaoui & Hassi, 2025b, Lemma 2.1) Let $X \in \mathbf{B}^+(\mathfrak{K})$ and $R \in R(\mathfrak{H}, \mathfrak{K})$, and let $\alpha \in [0, 1]$. If XR^{**} is closed (closable, respectively), then $X^\alpha R^{**}$ is closed (closable, respectively) and

$$(R^*X^\alpha)^* = X^\alpha R^{**}.$$

Analogously, if $\ker X = \{0\}$ and $R^{**}X^{-1}$ is closed, then $R^{**}X^{-\alpha}$ is closed and

$$(X^{-\alpha}R^*)^* = R^{**}X^{-\alpha}.$$

3.1 Nonnegative selfadjoint forms

A *sesquilinear form* $\mathfrak{t}[\cdot, \cdot]$ in \mathfrak{H} is a mapping from $\mathfrak{D} \times \mathfrak{D}$ to \mathbb{C} , where \mathfrak{D} is a linear subspace of \mathfrak{H} , such that $\mathfrak{t}[\cdot, \cdot]$ is linear with respect to the first entry and anti-linear with respect to the second one. The domain $\text{dom } \mathfrak{t}$ is defined by $\text{dom } \mathfrak{t} = \mathfrak{D}$. The form \mathfrak{t} is said to be *symmetric* if $\mathfrak{t}[\varphi, \psi] = \overline{\mathfrak{t}[\psi, \varphi]}$ for all $\varphi, \psi \in \text{dom } \mathfrak{t}$. The corresponding *quadratic form* $\mathfrak{t}[\cdot]$ is defined by $\mathfrak{t}[\varphi] = \mathfrak{t}[\varphi, \varphi]$, $\varphi \in \text{dom } \mathfrak{t}$, and induces

the polarization formula

$$\mathfrak{t}[\varphi, \psi] = \frac{1}{4}(\mathfrak{t}[\varphi + \psi] - \mathfrak{t}[\varphi - \psi]) + \frac{i}{4}(\mathfrak{t}[\varphi + i\psi] - \mathfrak{t}[\varphi - i\psi])$$

for $\varphi, \psi \in \text{dom } \mathfrak{t}$ which is easily checked. For instance, the inner product $(\cdot, \cdot)_{\mathfrak{H}}$ of \mathfrak{H} is a form defined on all of \mathfrak{H} .

A symmetric form \mathfrak{t} is nonnegative if $\mathfrak{t}[\varphi] \geq 0$ for all $\varphi \in \text{dom } \mathfrak{t}$ and one writes $\mathfrak{t} \geq 0$. For $\mathfrak{t}_1, \mathfrak{t}_2 \geq 0$, one writes $\mathfrak{t}_1 \leq \mathfrak{t}_2$ if

(3.4)

A nonnegative form \mathfrak{t} is *closable* if for any $(\varphi_n) \subseteq \text{dom } \mathfrak{t}$ such that $\varphi_n \xrightarrow{n \rightarrow \infty} 0$ and $\mathfrak{t}[\varphi_n - \varphi_m] \xrightarrow{n, m \rightarrow \infty} 0$, one has $\mathfrak{t}[\varphi_n] \xrightarrow{n \rightarrow \infty} 0$. It is said to be *closed* if for any $(\varphi_n) \subseteq \text{dom } \mathfrak{t}$ such that $\varphi_n \xrightarrow{n \rightarrow \infty} \varphi \in \mathfrak{H}$ and $\mathfrak{t}[\varphi_n - \varphi_m] \xrightarrow{n, m \rightarrow \infty} 0$ one has $\varphi \in \text{dom } \mathfrak{t}$ and $\mathfrak{t}[\varphi_n - \varphi] \xrightarrow{n \rightarrow \infty} 0$.

Note that every nonnegative linear relation $S \in R(\mathfrak{H})$ generates a closable nonnegative form \mathfrak{t}_S via

$$\mathfrak{t}_S[\varphi, \psi] = (\varphi', \psi)_{\mathfrak{H}}, \quad (\varphi, \varphi'), (\psi, \psi') \in S, \quad \text{dom } \mathfrak{t}_S = \text{dom } S; \quad (3.5)$$

see (Behrndt et al., 2020, Lemma 5.1.17). In this case, $S^{**} \geq 0$ and the closures of \mathfrak{t}_S and $\mathfrak{t}_{S^{**}}$ coincide. Furthermore, the closure $\tilde{\mathfrak{t}}_S$ of \mathfrak{t}_S is a nonnegative closed form which satisfies $\text{dom } \tilde{\mathfrak{t}}_S \subseteq \overline{\text{dom } S}$ and gives rise to a nonnegative selfadjoint relation which is called the Friedrichs extension S_F of S , by the next theorem.

A particular case of (3.5) where S is also selfadjoint is considered in the first representation theorem stated below.

Theorem 3.3. (First representation theorem) *If \mathfrak{t} is a closed nonnegative form in \mathfrak{H} then there exists a nonnegative selfadjoint relation H in \mathfrak{H} such that $\text{dom } H \subset \text{dom } \mathfrak{t}$ and*

$$\mathfrak{t}[\varphi, \psi] = (\varphi', \psi) \text{ for every } (\varphi, \varphi') \in H \text{ and } \psi \in \text{dom } \mathfrak{t}. \quad (3.6)$$

The nonnegative selfadjoint relation H is uniquely determined by (3.6).

Theorem 3.4. (Second representation theorem) *Let \mathfrak{t} be the closed nonnegative form mentioned in Theorem 3.3. Then, $\text{dom } \mathfrak{t} = \text{dom } H_s^{\frac{1}{2}}$, and*

$$\mathfrak{t}[\varphi, \psi] = (H_s^{1/2}\varphi, H_s^{1/2}\psi), \quad \varphi, \psi \in \text{dom } \mathfrak{t}.$$

4 LOCAL SPECTRAL THEORY IN $\mathcal{L}_l^{+2}(\mathfrak{H})$

The following definitions are collected from Aiena, Trapani, and Triolo (2014); Erdelyi and Wang (1985); Laursen and Neumann (2000) and they are used in Barkaoui and Hassi (2025a).

Definition 4.1. An operator $T \in CO(\mathfrak{H})$ is said to have the *single valued extension property* (abbreviated SVEP) at $\alpha_0 \in \mathbb{C}$ if, for every neighborhood U_{α_0} of α_0 , the only analytic function $f_{\alpha_0} : U_{\alpha_0} \rightarrow \text{dom } T$, which satisfies

$$0 = (T - \mu I)f_{\alpha_0}(\mu) \quad \text{for all } \mu \in U_{\alpha_0}$$

is the function $f_{\alpha_0} \equiv 0$. The operator T is said to have the SVEP if it has the SVEP at every $\alpha_0 \in \mathbb{C}$.

Definition 4.2. Let $T \in CO(\mathfrak{H})$. (i)(i)

1. The *local resolvent set* of T at a point $x \in \mathfrak{H}$ is the set $\rho_T(x)$ of all $\lambda_0 \in \mathbb{C}$ for which there exist an open neighborhood U_{λ_0} of λ_0 in \mathbb{C} and an analytic function $f_{x,\lambda_0} : U_{\lambda_0} \rightarrow \text{dom } T$ which satisfies

$$x = (T - \mu)f_{x,\lambda_0}(\mu) \quad \text{for all } \mu \in U_{\lambda_0}.$$

The set $\mathbb{C} \setminus \rho_T(x)$ is called the *local spectrum* of T at the point x and denoted by $\sigma_T(x)$.

2. The *analytic spectral subspace* of T at a subset F of \mathbb{C} is given by

$$\mathfrak{H}_T(F) = \{x \in \mathfrak{H} ; \sigma_T(x) \subseteq F\}.$$

If $\mathfrak{H}_T(F)$ is closed for all closed $F \subseteq \mathbb{C}$, then T has the *Dunford's property*, for short property (C).

3. $\mathfrak{H}_T(F)$ stands for the *glocal spectral subspace* of T at a closed set $F \subseteq \mathbb{C}$ and it is defined as the set of all $x \in \mathfrak{H}$ for which there exists an analytic function $\ell : \mathbb{C} \setminus F \rightarrow \text{dom } T$ such that

$$x = (T - \mu I)\ell(\mu) \quad \text{for all } \mu \in \mathbb{C} \setminus F.$$

The operator T is said to have property (k) if T has the SVEP and (C); cf.(Shengwang & Erdelyi, 1986, Definition 2.2). It is shown in Barkaoui and Hassi (2025a) that every operator T in $\mathcal{L}_l^{+2}(\mathfrak{H})$ with a nonempty resolvent has the property (k). This in fact follows from the fact that T is a generalized scalar operator; cf. Sussmann (1973). Note that in the bounded framework, operators of this type are known to be decomposable in the sense Laursen and Neumann (2000). More generally, the

decomposability for the class of closed and continuous linear relations in complex Banach spaces is treated in this thesis and it is characterized by means of local spectral properties; see Barkaoui and Mnif (2025). Let us recall from Barkaoui and Mnif (2022) a couple of notations and the notion of extended spectral decomposability for linear relations.

In the sequel, X denotes a complex Banach space and $CR_c(X)$ is the class of all closed and continuous linear relations T from X to X . Here a relation $T : X \rightarrow X$ is said to be *continuous* if $\|T\| := \|Q_T T\| < \infty$, where $Q_T : X \rightarrow X/\overline{\text{mul } T}$ denotes the *natural quotient map* with domain X and kernel $\ker Q_T := \overline{\text{mul } T}$. cf. Cross (1998). In the particular case where $T \in CR(X)$, i.e. T is a closed, if $\text{dom } T$ is closed such that $X = \text{dom } T \oplus \overline{\text{mul } T}$, then $T \in CR_c(X)$ and one writes $T \in CR_c^\oplus(X)$.

The *resolvent set* of a relation $T \in CR(X)$ is denoted by $\rho(T)$ and defined as the set of all $\eta \in \mathbb{C}$ for which $\ker(T - \eta I) = \{0\}$ and $\text{ran}(T - \eta I) = X$. The *spectrum* of T is the set $\sigma(T) := \mathbb{C} \setminus \rho(T)$. The *extended spectrum* of T is a subset of \mathbb{C}_∞ defined as

$$\tilde{\sigma}(T) := \begin{cases} \sigma(T) & \text{if } T \in B(X) \\ \sigma(T) \cup \{\infty\} & \text{otherwise,} \end{cases}$$

where $B(X)$ is the class of all bounded operators on X ; see Baskakov and Zagorskii (2007).

For $\eta \in \rho(T)$, the *resolvent function* of T is the function $R(\cdot, T) : \rho(T) \rightarrow B(X)$ such that $R(\eta, T) = (T - \eta I)^{-1}$. A closed linear subspace $Y \subseteq X$ is called *T -strongly invariant* if $R(\eta, T)Y \subseteq Y$ for all $\eta \in \rho(T)$. The *strong restriction* of T to Y is the relation $T|_s Y \in CR(Y)$ whose resolvent is $R(\cdot, T|_s Y) : \rho(T) \rightarrow B(Y)$ which satisfies $R(\cdot, T|_s Y)(\eta) := R(\eta, T)|_Y$ for all $\eta \in \rho(T)$.

Definition 4.3. A relation $T \in CR_c(X)$ with $\rho(T) \neq \emptyset$ is called *extended spectral decomposable* if for every open cover $\{U, V\}$ of the extended complex plane $\mathbb{C}_\infty : \mathbb{C} \cup \{\infty\}$ by an open bounded subset U of \mathbb{C} and an open subset V of \mathbb{C}_∞ , there exist two T -strongly invariant closed linear subspaces M and N of X with the following properties: (i)(i)

1. $\tilde{\sigma}(T|_s M) = \sigma(T|_s M) \subseteq U$, $\tilde{\sigma}(T|_s N) \subseteq V$;
2. $T(N) \subseteq N$ and $\mathcal{P}(N) \subseteq N$ for some bounded linear projection $\mathcal{P} : X \rightarrow X$ verifying $\text{ran } \mathcal{P} = \text{dom } T$;
3. $X = M + N$.

In Theorem 4.5, the extended spectral decomposability is characterized by means of local spectral properties presented in the definition below; these definitions can be found in Barkaoui and Mnif (2022, 2023a, 2023b).

Definition 4.4. Let $T \in CR(X)$.
(i)(i)

1. The *local resolvent set* of T at $x_0 \in X$ is the set of all $\eta_0 \in \mathbb{C}$ for which there exist an open neighborhood U_{η_0} of η_0 in \mathbb{C} and an analytic function $f_{x_0, \eta_0} : U_{\eta_0} \rightarrow \text{dom } T$ such that

$$x_0 \in (T - \mu)f_{x_0, \eta_0}(\mu) \quad \text{for all } \mu \in U_{\eta_0}.$$

The set *local spectrum* of T at x_0 is the set $\mathbb{C} \setminus \rho_T(x_0)$ and it is denoted by $\sigma_T(x_0)$. For $F \subseteq \mathbb{C}$, the set

$$X_T(F) := \{x \in X ; \sigma_T(x) \subseteq F\}$$

is called the *local spectral subspace* of T at a subset F . The *glocal spectral subspace* of T at a closed set $M \subseteq \mathbb{C}$ is the set of all $x \in X$ for which there exists an analytic function $h_x : H \rightarrow \text{dom } T$ such that

$$x \in (T - \mu I)h_x(\mu) \quad \text{for all } \mu \in \mathbb{C} \setminus M.$$

2. The *extended local resolvent set* of T at a point x_0 , denoted by $\widetilde{\rho}_T(x_0)$, as the set of all $\lambda_0 \in \mathbb{C}_\infty$ for which there are an open neighborhood U_{λ_0} in \mathbb{C}_∞ of λ_0 and a function $\widetilde{f}_{x_0, \lambda_0} \in P(U_{\lambda_0}, \text{dom } T)$ such that

$$x_0 \in (\mu I - T)\widetilde{f}_{x_0, \lambda_0}(\mu) \quad \text{for all } \mu \in U_{\lambda_0} \cap \mathbb{C}.$$

Here, $P(U, X) = \{f \in H^\sharp(U, X) \mid f(\infty) = 0 \text{ if } \infty \in U\}$, where $H^\sharp(U, X)$ stands for the space of all analytic functions from an open set $U \subseteq \mathbb{C}_\infty$ to X . The *extended local spectrum* of T at $w_0 \in \mathbb{C}_\infty$ is given by $\widetilde{\sigma}_T(w_0) := \mathbb{C}_\infty \setminus \widetilde{\rho}_T(w_0)$.

3. For a set $F \subseteq \mathbb{C}_\infty$, the *extended local spectral subspace* of T at F is defined by

$$\widetilde{X}_T(F) := \{w \in X ; \widetilde{\sigma}_T(w) \subseteq F\}.$$

If, in addition, F is closed then the *extended glocal spectral subspace* of T at F as the set $\widetilde{\mathcal{X}}_T(F)$ of all $x \in X$ for which there exists a function $\widetilde{\Psi} \in P(\mathbb{C}_\infty \setminus F, \text{dom } T)$ such that

$$x \in (T - \mu I)\widetilde{\Psi}(\mu) \quad \text{for all } \mu \in (\mathbb{C}_\infty \setminus F) \cap \mathbb{C}.$$

4. T is said to have the *extended decomposition property* (\mathcal{E}_δ) if for every open cover $\{U, V\}$ of \mathbb{C}_∞ , where U is an open bounded subset of \mathbb{C} and V is an open subset of \mathbb{C}_∞ , the following identity holds:

$$X = \mathcal{X}_T(\overline{U}) + \widetilde{\mathcal{X}}_T(\overline{V}).$$

5. If $T \in CR_c(X)$ and $\widetilde{X}_T(F)$ is closed for every closed subset $F \subseteq \mathbb{C}_\infty$, then

T is said to have *extended Dunford's property* (\mathcal{E}_C), shortly property (\mathcal{E}_C).

6. If $T \in CR_c(X)$, then T has the *extended Bishop's property* (\mathcal{E}_β), shortly the property (\mathcal{E}_β), if for every open set $U \subseteq \mathbb{C}_\infty$ and every sequences analytic functions $(f_n)_{n \in \mathbb{N}} \subseteq P(U, \text{dom } T)$, $(g_n)_{n \in \mathbb{N}} \subseteq H^\sharp(U, X)$ such that $g_n(\lambda) \in (T - \lambda I)f_n(\lambda)$, for all $\lambda \in U \cap \mathbb{C}$, the following implication holds

$$g_n \xrightarrow[n \rightarrow +\infty]{} 0 \text{ in } H^\sharp(U, X) \implies f_n \xrightarrow[n \rightarrow +\infty]{} 0 \text{ in } P(U, \text{dom } T).$$

7. Let Y be a T -invariant subspace, i.e, $T(Y) \subseteq Y$. The *coinduced* linear operator $T/M : X/M \rightarrow X/M$ induced by T on the quotient space X/M is defined by:

$$T/M\tilde{x} := \widetilde{T\alpha} = Q_M T(\alpha) \quad \text{for all } \tilde{x} \in \text{dom}(T/M) := \frac{\text{dom } T + M}{M},$$

where $x = \alpha + m$, $(\alpha, m) \in \text{dom } T \times M$ and $Q_M : X \rightarrow X/M$ is the natural quotient map.

The following theorem follows from a combination of (Barkaoui & Mnif, 2025, Theorem 3.10) and (Barkaoui & Mnif, 2025, Theorem 3.13).

Theorem 4.5. *Let $T \in CR_c^\oplus(X)$. Then, the following assertions are equivalent:*

- (i) T is an extended decomposable linear relation;
- (ii) T has the property (\mathcal{E}_C) and $\widetilde{\sigma}(T/\widetilde{\mathfrak{H}}_T(F)) \subseteq \overline{\widetilde{\sigma}(T) \setminus F}$ for all closed subsets F of \mathbb{C}_∞ ;
- (iii) T has the properties (\mathcal{E}_β) and (\mathcal{E}_δ);
- (iv) T has the properties (\mathcal{E}_C) and (\mathcal{E}_δ).

5 CONCLUSION

5.1 Summary of articles

I. Product of nonnegative selfadjoint operators in unbounded settings

In this paper, the factorization of a closed, in general, unbounded operator T into a product $T = AB$ of two nonnegative selfadjoint operators A and B is studied. Already the special case, where A or B is bounded, leads to new results and is of wider interest, since the problem is connected to the notion of similarity of the operator T to a selfadjoint one, but, in fact, goes beyond this case. It is proved that this subclass of operators can be characterized not only by means of quasi-affinity of T^* to an operator $S = S^* \geq 0$, but also via Sebestyén inequality, a result known in the setting of bounded operators T . Another subclass of operators T , where A or B has a bounded inverse, leads to a similar analysis. This gives rise to a reversed version of Sebestyén inequality which is introduced in the paper. It is also shown that this second subclass can be characterized by means of quasi-affinity of T , rather than T^* , to an operator $S = S^* \geq 0$. Furthermore, the connection between these two classes and weak-similarity as well as quasi-affinity to some $S = S^* \geq 0$ is investigated. Finally, the special case where S is bounded is considered.

II. Three equivalent conditions for spectral decomposable linear relation

This paper extends the following characterizations of the spectral decomposability from the case of bounded linear operators to that of multivalued linear ones:

$$\begin{aligned} T_0 \text{ is decomposable} &\Leftrightarrow T_0 \text{ has } (C) \text{ and } \sigma(T_0/X_{T_0}(F)) \subseteq \overline{\sigma(T_0) \setminus F} \text{ for every} \\ &\text{closed subset } F \text{ of } \mathbb{C} \\ &\Leftrightarrow T_0 \text{ has } (\delta) \text{ and } (C) \\ &\Leftrightarrow T_0 \text{ has } (\beta) \text{ and } (\delta), \end{aligned}$$

where (δ) , (C) , and (β) are the properties of decomposition, Dunford and Bishop, respectively. Moreover, $X_{T_0}(F)$ and $T_0/X_{T_0}(F)$ denote the local spectral subspace of T_0 with respect to F and the induced operator by T_0 on the quotient space $X/X_{T_0}(F)$, respectively.

III. Local spectral theory for unbounded products of non-negative selfadjoint operators

An unbounded product $T = AB$ of two nonnegative selfadjoint operators A and B , where $\rho(T) \neq \emptyset$ and A is bounded, is proved to have the single valued extension property (SVEP) and, more significantly, to be a \mathbb{C} -generalized scalar operator. This property plays a central role as, it not only implies the Dunford's property for the product AB , but it also allows any operator T that is quasi-affine to a selfadjoint operator S to satisfy $\sigma(T) = \sigma(S)$. These results have required a deeper study of the spectral connection between operators S having SVEP and Dunford properties and operators quasi-affine to S .

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Publications

Article I

**PRODUCT OF NONNEGATIVE SELFADJOINT OPERATORS
IN UNBOUNDED SETTINGS**

YOSRA BARKAOUI AND SEPPPO HASSI

Abstract In this paper, necessary and sufficient conditions are established for the factorization of a closed, in general, unbounded operator $T = AB$ into a product of two nonnegative selfadjoint operators A and B . Already the special case, where A or B is bounded, leads to new results and is of wider interest, since the problem is connected to the notion of similarity of the operator T to a selfadjoint one, but, in fact, goes beyond this case. It is proved that this subclass of operators can be characterized not only by means of quasi-affinity of T^* to an operator $S = S^* \geq 0$, but also via Sebestyén inequality, a result known in the setting of bounded operators T . Another subclass of operators T , where A or B has a bounded inverse, leads to a similar analysis. This gives rise to a reversed version of Sebestyén inequality which is introduced in the present paper. It is shown that this second subclass, where A^{-1} or B^{-1} is bounded, can be characterized in a similar way by means of quasi-affinity of T , rather than T^* , to an operator $S = S^* \geq 0$. Furthermore, the connection between these two classes and weak-similarity as well as quasi-similarity to some $S = S^* \geq 0$ is investigated. Finally, the special case where S is bounded is considered.

1. INTRODUCTION

In 2021 M. Contino, M. A. Dritschel, A. Maestripieri, and S. Marcantognini [7] (see also [2]) showed that similarity to a bounded positive operator is no longer sufficient to characterize the product of two positive bounded operators in the settings of infinite-dimensional complex Hilbert space, contrary to that of finite-dimension; see [24]. More precisely, for a bounded operator $T \in B(\mathfrak{H})$ they established the following characterization for similarity:

$$(1.1) \quad \begin{array}{c} T \text{ is similar to a positive operator} \\ \Downarrow \\ T = AB \text{ with } A, B \in B^+(\mathfrak{H}) \text{ and, in addition, } A \text{ or } B \text{ is invertible,} \end{array}$$

where $B^+(\mathfrak{H})$ stands for the set of all bounded nonnegative operators on \mathfrak{H} ; see [7, Theorem 3.1]. This result remains true for unbounded operators T ; cf. Proposition 4.1. Even weaker conditions than similarity, such as quasi-similarity and quasi-affinity have also proven to be insufficient to fully characterize such a product.

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Instead, the product representation $T = AB$, $A, B \in B^+(\mathfrak{H})$ was characterized by means of Sebestyén inequality [21] as follows:

$$(1.2) \quad T = AB \quad \Leftrightarrow \quad TT^* \leq XT^* \text{ for some } X \in B^+(\mathfrak{H});$$

see [7, Theorem 4.5]. Hence, a natural approach to improve the above results is either to pursue weaker concepts than quasi-affinity or to relax certain conditions on T .

One of the main purposes in the present paper is to investigate these questions and to extend the above results to the setting of unbounded operators T . More precisely, a complete study is first carried out when a closed operator T belongs to the following class of operators:

$$\mathcal{L}_t^{+2}(\mathfrak{H}) = \left\{ T = AB; A \in B^+(\mathfrak{H}) \text{ and } B = B^* \geq 0 \right\},$$

where B is in general unbounded. It will be seen in Section 2 that every element of $\mathcal{L}_t^{+2}(\mathfrak{H})$ satisfies an equality analogous to the one appearing in (1.2). More generally, for closed operators T and B such that T^*B is selfadjoint, Sebestyén theorem [21] is generalized to the unbounded context as follows:

$$(1.3) \quad X\overline{B_0} \subseteq T \text{ for some } X \in B^+(\mathfrak{H}) \Leftrightarrow T^*T \leq \lambda T^*B,$$

for the restriction $B_0 := B \upharpoonright \text{dom } T^*B$ of B ; cf. Theorem 2.7. In the unbounded setting the restriction B_0 appears naturally, and, in fact, due to the equality

$$T^*B_0 = T^*\overline{B_0} = T^*B$$

the equivalence in (1.3) can restated just with B_0 . Obviously, in the particular case where $\text{dom } T^*B$ is a core for B , i.e., $\overline{B_0} = B$, (1.3) is instead stated for B . This covers the bounded setting in which (1.2) is true for $B \in B^+(\mathfrak{H})$ and the equivalence (1.3) holds with equality $T = XB$. However, for the unbounded setting where $B \neq \overline{B_0}$, it is necessary to consider further conditions including $B^*T = T^*B$ in order to state (1.3) for B ; see Proposition 2.10.

The inclusion in (1.3) represents a good motivation for describing the connection between the class $\mathcal{L}_t^{+2}(\mathfrak{H})$ and the notion of quasi-affinity to a nonnegative selfadjoint operator. Recall from [16, Definition 2.2] that T is said to be *quasi-affine* to some operator S if there exists an injective $G \in B(\mathfrak{H})$ such that $\overline{\text{ran } G} = \mathfrak{H}$ and the following inclusion holds:

$$(1.4) \quad GT \subseteq SG.$$

In the bounded case, treated in [7, Proposition 3.8], one can observe that the inclusion in (1.4) is equivalent to

$$(1.5) \quad S = \overline{GTG^{-1}} = (G^{-1})^*T^*G^*.$$

However, (1.5) need not hold anymore in the unbounded setting and this motivates the investigation of a possible connection between quasi-affinity to $S = S^* \geq 0$ and the existence of nonnegative selfadjoint extensions of GTG^{-1} , which in turn leads to the following characterization given in Proposition 2.14

$$(1.6) \quad \begin{array}{c} T \supseteq AB \in \mathcal{L}_t^{+2}(\mathfrak{H}) \text{ with } \overline{\text{ran } A} = \mathfrak{H} \\ \Updownarrow \\ T^* \text{ is quasi-affine to } S = S^* \geq 0. \end{array}$$

Motivated by (1.3), this induces the following new characterization of Sebestyén inequality by means of quasi-affinity to some $S = S^* \geq 0$:

$$\begin{aligned}
 (1.7) \quad & T^*T \leq \lambda T^*B \text{ with } \text{dom } T \subseteq \text{dom } B \text{ for some } \lambda \geq 0, B = B^* = \overline{B_0} \geq 0 \\
 & \quad \quad \quad \updownarrow \\
 & T = A \overline{B_0} \in \mathcal{L}_l^{+2}(\mathfrak{H}) \quad \text{with } \overline{\text{ran}} A = \mathfrak{H} \\
 & \quad \quad \quad \updownarrow \\
 & T^* \text{ is } G\text{-quasi-affine to } S = S^* \geq 0 \text{ with } \text{dom } T \subseteq \overline{\text{dom } B_F \upharpoonright \text{dom } (T^*B_F)} \\
 & \text{and } B_F = G^{-1}S^{\frac{1}{2}}S^{\frac{1}{2}}(G^{-1})^*;
 \end{aligned}$$

see Theorem 2.18.

The present setting of unbounded operators leads to further generalisations of the equivalences in (1.3) and (1.6). In particular, the next goal in this paper is to investigate the reversed inequality

$$(1.8) \quad T^*T \geq \eta AT, \quad \eta > 0,$$

and prove analogs for the characterizations in (1.3) and (1.6); see Theorem 3.3 and Corollary 3.6. The idea to get further characterizations here is to make connection to the initial Sebestyén inequality (1.3) by taking inverses in the operator inequality (1.8). This has motivated a further generalisation of the above results to the case of nondensely defined operators as well as multivalued linear operators (linear relations) in Theorem 3.1.

For the reversed inequality (1.8), quasi-affinity of T , rather than T^* , to S arises and leads to a new class different from $\mathcal{L}_l^{+2}(\mathfrak{H})$ defined by

$$\mathcal{L}_l^{+2}(\mathfrak{H}) = \{T = BA, B^{-1} \in B^+(\mathfrak{H}) \text{ and } A = A^* \geq 0\}.$$

In fact, Theorem 4.3 shows that:

$$(1.9) \quad T \subseteq BA \in \mathcal{L}_l^{+2}(\mathfrak{H}) \Leftrightarrow T \text{ is quasi-affine to some } S = S^* \geq 0.$$

In particular, if T is G -quasi-affine to S such that $\rho(\overline{G^*S^{\frac{1}{2}}S^{\frac{1}{2}}GT}) \neq \emptyset$, then

$$T^*T \geq \frac{1}{\lambda}AT$$

for some $A = A^* \geq 0$, which emphasizes the strong connection between the class $\mathcal{L}_l^{+2}(\mathfrak{H})$ and the reversed inequality.

It is clear from (1.9) and (1.6) that there is no direct relation between $\mathcal{L}_l^{+2}(\mathfrak{H})$ and $\mathcal{L}_l^{+2}(\mathfrak{H})$. However, if T is quasi-similar to $S = S^* \geq 0$ or, equivalently T and T^* are quasi-affine to S then one obtains

$$\mathcal{L}_l^{+2}(\mathfrak{H}) \ni T_1 \subseteq T \subseteq T_2 \in \mathcal{L}_l^{+2}(\mathfrak{H}).$$

In fact, behind this proof appears the notion of Friedrichs extension of a nonnegative (symmetric operator). More importantly, when $\rho(T) \neq \emptyset$ the operators T and T^* play a symmetric role with respect to stronger notions than quasi-similarity, namely W -similarity and similarity. This can be seen in Proposition 4.1 where the equivalence (1.1) remains valid even in the unbounded setting. In this case one obtains the following equivalences:

$$T \text{ is } W\text{-similar to } S = S^* \geq 0 \Leftrightarrow T \in \mathcal{L}_l^{+2}(\mathfrak{H}) \Leftrightarrow T \in \mathcal{L}_l^{+2}(\mathfrak{H}).$$

The assumption $\rho(T) \neq \emptyset$ is quite important also for the spectral properties of T (see [5]), in particular, if $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ such that $\rho(T) \neq \emptyset$ then

$$\sigma(T) \subseteq \mathbb{R}^+.$$

The last part of this paper deals with a particular case, where T is compared to a bounded nonnegative $S \in B^+(\mathfrak{H})$. Since both \mathcal{W} -similarity and similarity to such operators imply the boundedness of T , it is enough to restrict attention to quasi-affinity and quasi-similarity notions.

2. THE CLASS $\mathcal{L}_l^{+2}(\mathfrak{H})$ AND SEBESTYÉN INEQUALITY

In this section the emphasis will be on the following subclass of the closed operators in $CO(\mathfrak{H})$:

$$(2.1) \quad \mathcal{L}_l^{+2}(\mathfrak{H}) = \left\{ T = AB \in CO(\mathfrak{H}); A \in B^+(\mathfrak{H}) \text{ and } B = B^* \geq 0 \right\},$$

where B is in general a closed unbounded operator on \mathfrak{H} . Analogous to the bounded case, this class is characterized through Sebestyén inequality now involving unbounded operators. Further extensions are treated in Section 3.

In the sequel $T \in LO(\mathfrak{H}, \mathfrak{K})$ stands for a linear operator from \mathfrak{H} to a complex Hilbert space \mathfrak{K} with domain $\text{dom } T$ and range $\text{ran } T$. In addition, one writes $T \in CO(\mathfrak{H}, \mathfrak{K})$ if T is closed. If $\mathfrak{K} = \mathfrak{H}$ then $CO(\mathfrak{H}) := CO(\mathfrak{H}, \mathfrak{K})$ and $LO(\mathfrak{H}, \mathfrak{K}) = LO(\mathfrak{H})$. In this case, T is said to be *symmetric* if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in \text{dom } T$. If $\langle Tx, x \rangle \geq 0$ for all $x \in \text{dom } T$, then T is *nonnegative*. It is *selfadjoint* when $\overline{\text{dom } T} = \mathfrak{H}$ and $T^* = T$. Note that if T is nonnegative and selfadjoint, then it admits a unique nonnegative selfadjoint square root which will be denoted by $T^{\frac{1}{2}}$; cf. [22, 23]. One writes $T \leq S$ for two nonnegative selfadjoint operators S and T if

$$\text{dom } S^{\frac{1}{2}} \subseteq \text{dom } T^{\frac{1}{2}} \quad \text{and} \quad \|T^{\frac{1}{2}}x\| \leq \|S^{\frac{1}{2}}x\| \text{ for all } x \in \text{dom } S^{\frac{1}{2}}.$$

The class of bounded operators from \mathfrak{H} to \mathfrak{K} is denoted by $B(\mathfrak{H}, \mathfrak{K})$ and in case $\mathfrak{K} = \mathfrak{H}$ this is appropriated to $B(\mathfrak{H})$. If $0 \leq T = T^* \in B(\mathfrak{H})$ then one writes $T \in B^+(\mathfrak{H})$.

If T is closed, then its Moore-Penrose inverse is denoted by $T^{(-1)}$. It satisfies the following equalities:

$$TT^{(-1)} = P_{\ker T^* \perp} I \upharpoonright \text{ran } T \quad T^{(-1)}T = P_{\ker T \perp} \upharpoonright \text{dom } T.$$

The *resolvent set* of $T \in CO(\mathfrak{H})$ is the set $\rho(T)$ of all $\mu \in \mathbb{C}$ for which $(T - \mu I)^{-1} \in B(\mathfrak{H})$. The *spectrum* of T is defined by $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

The next lemma provides a key ingredient for what follows. It treats both densely defined and nondensely defined operators, as well as linear relations; cf. Section 3. Note that its proof is based on [9, Lemma 2.9], where the equality

$$(2.2) \quad (ST)^* = T^*S^*$$

is established in the general case of linear relations. Recall that (2.2) is satisfied if $S \in B(\mathfrak{H})$ or T is invertible.

Lemma 2.1. *Let $X \in B^+(\mathfrak{K})$ and R be a linear relation from \mathfrak{H} to \mathfrak{K} , and let $\alpha \in [0, 1]$. If XR^{**} is closed (closable), then $X^\alpha R^{**}$ is closed (closable, respectively) and, moreover,*

$$(2.3) \quad (R^*X^\alpha)^* = X^\alpha R^{**}.$$

Analogously, if $\ker X = \{0\}$ and $R^{**}X^{-1}$ is closed (closable), then $R^{**}X^{-\alpha}$ is closed (closable, respectively) and

$$(2.4) \quad (X^{-\alpha}R^{**})^* = R^{**}X^{-\alpha}.$$

Proof. Let $(x_n, y_n) \in X^{\alpha}R^{**}$ be such that $(x_n, y_n) \xrightarrow{n \rightarrow +\infty} (x, y) \in \mathfrak{H} \times \mathfrak{K}$. Then, $y_n \in X^{\alpha-1}XR^{**}x_n$, and therefore

$$X^{1-\alpha}y_n \in X^{1-\alpha}X^{\alpha-1}XR^{**}x_n \subseteq XR^{**}x_n;$$

here $X^{\alpha-1}$ denotes a linear relation inverse of $X^{1-\alpha}$. Since $X^{1-\alpha} \in B^+(\mathfrak{K})$, one has $X^{1-\alpha}y_n \xrightarrow{n \rightarrow +\infty} X^{1-\alpha}y$ and $(x_n, X^{1-\alpha}y_n) \xrightarrow{n \rightarrow +\infty} (x, X^{1-\alpha}y)$. As $(x_n, X^{1-\alpha}y_n) \in G(XR^{**})$ and XR^{**} is closed, one concludes that $(x, X^{1-\alpha}y) \in G(XR^{**})$. On the other hand, $y \in X^{\alpha-1}X^{1-\alpha}y$, which implies that

$$y \in X^{\alpha-1}(X^{1-\alpha}y) = X^{\alpha-1}(XR^{**}x) = X^{\alpha}R^{**}x.$$

Consequently, $X^{\alpha}R^{**}$ is closed. To prove (2.3), it suffices to observe that

$$X^{\alpha}R^{**} = [(X^{\alpha}R^{**})^*]^* = [R^*(X^{\alpha})^*]^*.$$

If $\ker X = \{0\}$ and $R^{**}X^{-1}$ is closed, then $(R^{**}X^{-1})^{-1} = XR^{-1**}$ is closed. Thus, (2.4) follows immediately by applying (2.3) to R^{-1} and by taking the inverse. For the closability, it suffices to consider the case where $(x_n, y_n) \xrightarrow{n \rightarrow +\infty} (0, y)$. \square

Corollary 2.2. *If $T = AB \in \mathcal{L}_l^{+2}(\mathfrak{H})$ and T^{2^n} is closed for every $n \in \mathbb{N}$, then*

$$(2.5) \quad T^{2^n} = AS_n \in \mathcal{L}_l^{+2}(\mathfrak{H}) \quad \text{for all } n \in \mathbb{N},$$

where $(S_n)_{n \in \mathbb{N}}$ is a sequence of nonnegative selfadjoint unbounded operators such that $S_0 = B$ and $S_n = S_{n-1}AS_{n-1}$ for all $n \in \mathbb{N}^*$.

Proof. The case $n = 0$ is easily seen. For $n = 1$, one has $T^2 = A(BAB) = AS_1$ and

$$(2.6) \quad S_1 := BAB = S_0AS_0 = (A^{\frac{1}{2}}B)^*A^{\frac{1}{2}}B.$$

On the other hand $A \in B^+(\mathfrak{H})$ and $AB = T$ is closed, so by Lemma 2.1 $A^{\frac{1}{2}}B$ is closed. This proves, by (2.6) that $S_1 = S_1^* \geq 0$.

For $n = 2$, one has

$$T^{2^2} = A[(BAB)A(BAB)] = A(S_1AS_1) = AS_2,$$

where

$$(2.7) \quad S_2 = S_1AS_1 = (A^{\frac{1}{2}}S_1)^*A^{\frac{1}{2}}S_1.$$

But $AS_1 = ABAB = T^2$ is closed, by hypothesis, $A \in B^+(\mathfrak{H})$ and S_1 is closed, so $A^{\frac{1}{2}}S_1$ is closed by Lemma 2.1. Hence, (2.7) yields that $S_2 = S_2^* \geq 0$. Using again Lemma 2.1 and the fact that T^{2^n} is closed, one can conclude by induction that, for all $n \in \mathbb{N}$, S_n is a nonnegative selfadjoint unbounded operator such that $S_n = S_{n-1}AS_{n-1}$ and $T^{2^n} = AS_n \in \mathcal{L}_l^{+2}(\mathfrak{H})$. \square

It is worth mentioning that, in the bounded case, any element $T = AB \in \mathcal{L}_l^{+2}(\mathfrak{H})$ satisfies the following formula:

$$(2.8) \quad \sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\},$$

which easily implies the positivity of the spectrum of T . However, this is a bit more delicate when it comes to the unbounded case. In fact, (2.8) is not guaranteed

anymore unless some further spectral properties are added like $\rho(AB) \neq \emptyset$ and $\rho(BA) \neq \emptyset$; see Hardt et al. [11]. In particular, for any unbounded $T \in \mathcal{L}_1^{+2}(\mathfrak{H})$ with $\rho(T) \neq \emptyset$, it will be shown that $\sigma(T) \subseteq \mathbb{R}^+$. This motivates the next results.

Lemma 2.3. *Let $X \in B^+(\mathfrak{H})$ and $T \in CO(\mathfrak{H})$ be a densely defined operator such that XT is closed. Then,*

$$(2.9) \quad (X^{\frac{1}{2}}T^*X^{\frac{1}{2}})^* = X^{\frac{1}{2}}TX^{\frac{1}{2}}.$$

Moreover, if $T = T^*$ and $\rho(XT) \neq \emptyset$, then

$$(2.10) \quad \sigma(XT) = \sigma(X^{\frac{1}{2}}TX^{\frac{1}{2}}) \subseteq \mathbb{R},$$

in particular, $0 \in \rho(XT) \Leftrightarrow 0 \in \rho(X^{\frac{1}{2}}TX^{\frac{1}{2}})$.

Proof. Observe that

$$(2.11) \quad (X^{\frac{1}{2}}T^*X^{\frac{1}{2}})^* = (X^{\frac{1}{2}}(X^{\frac{1}{2}}T)^*)^* = (X^{\frac{1}{2}}T)^{**}(X^{\frac{1}{2}})^* = \overline{X^{\frac{1}{2}}TX^{\frac{1}{2}}}.$$

Since XT is closed, it follows from Lemma 2.1 that $X^{\frac{1}{2}}T$ is closed. This yields by (2.11) that $(X^{\frac{1}{2}}T^*X^{\frac{1}{2}})^* = X^{\frac{1}{2}}TX^{\frac{1}{2}}$.

Assume now that $\rho(XT) = \rho(X^{\frac{1}{2}}(X^{\frac{1}{2}}T)) \neq \emptyset$ and $T^* = T$. Then, (2.11) shows that $X^{\frac{1}{2}}TX^{\frac{1}{2}}$ is selfadjoint, and hence $\rho(X^{\frac{1}{2}}TX^{\frac{1}{2}}) = \rho(X^{\frac{1}{2}}(TX^{\frac{1}{2}})) \neq \emptyset$. Using [11, Lemma 2.2] and [11, Lemma 2.4], one then concludes that

$$(2.12) \quad \sigma(XT) \cup \{0\} = \sigma(X^{\frac{1}{2}}(X^{\frac{1}{2}}T)) \cup \{0\} = \sigma((X^{\frac{1}{2}}T)X^{\frac{1}{2}}) \cup \{0\} \subseteq \mathbb{R}.$$

Now assume that $0 \in \rho(XT)$. Then $\text{ran } XT = \mathfrak{H} = \text{ran } X^{\frac{1}{2}}$, and hence $X^{\frac{1}{2}}$ is invertible. This implies the invertibility of T , so $0 \in \rho(X^{\frac{1}{2}}TX^{\frac{1}{2}})$. Similarly, the invertibility of $X^{\frac{1}{2}}TX^{\frac{1}{2}}$ ensures that of T , which proves the remaining implication. Together with (2.12), this shows (2.10). \square

Thanks to the previous lemma, it will be seen in Proposition 2.5 how any element of $\mathcal{L}_1^{+2}(\mathfrak{H})$ is connected to a nonnegative selfadjoint operator. This connection is introduced in the following definition and it will be further developed in Section 4.

Definition 2.4. Let $T, S \in LO(\mathfrak{H})$. If there exists $G \in B(\mathfrak{H})$ such that $TG = GS$ then T is said to be *pre-similar* to S with interwining operator G .

Proposition 2.5. *If $T = AB \in \mathcal{L}_1^{+2}(\mathfrak{H})$ then $(A^{\frac{1}{2}}BA^{\frac{1}{2}})^* = A^{\frac{1}{2}}BA^{\frac{1}{2}}$ and T is pre-similar to $A^{\frac{1}{2}}BA^{\frac{1}{2}}$ with interwining operator $A^{\frac{1}{2}}$. Moreover, if $\rho(T) \neq \emptyset$, then*

$$\sigma(T) = \sigma(A^{\frac{1}{2}}BA^{\frac{1}{2}}) \subseteq \mathbb{R}^+.$$

Proof. Since by definition $A \in B^+(\mathfrak{H})$ and AB is closed, it follows from Lemma 2.3 that $S := A^{\frac{1}{2}}BA^{\frac{1}{2}}$ is a nonnegative selfadjoint operator such that $TA^{\frac{1}{2}} = A^{\frac{1}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = A^{\frac{1}{2}}S$. Hence, T is pre-similar to S . The remaining result follows immediately again from Lemma 2.3. \square

2.1. Sebestyén inequality. In this section, Sebestyén's theorem is generalized to the case of unbounded operators. The case of bounded operators was originally proved in [21], for a recent treatment see also [2, 7], where the following equivalence is stated for $T, B \in B(\mathfrak{H})$:

$$(2.13) \quad T^*T \leq \lambda T^*B, \lambda \geq 0 \quad \Leftrightarrow \quad T = XB \quad \text{for some } X \in B^+(\mathfrak{H}).$$

The following lemma serves as a first step towards the generalization of (2.13) and is a useful tool for some further results. The equivalence of (i) and (ii) holds even in the case of linear relations; cf. [12, Lemma 4.2], and for related results see also [18, Theorem 2.2], where $T \subseteq BY \Leftrightarrow \text{ran } T \subseteq \text{ran } B$ and [19, Lemma 3.1], where $YB \subseteq T \Leftrightarrow \ker B \subseteq \ker T$, respectively are established for linear relations T, B and Y .

Lemma 2.6. *Let $T, B : \mathfrak{H} \rightarrow \mathfrak{K}$ be closed densely defined linear operators. Then the following statements are equivalent:*

- (i) $YB \subseteq T$ has a solution $Y \in B(\mathfrak{K})$;
- (ii) $T^*T \leq c^2 B^*B$ for some $0 \leq c (= \|Y\|)$.

In this case, Y can be selected such that $\text{ran } Y \subseteq \overline{\text{ran } T}$ and $\ker B^ \subseteq \ker Y$. Furthermore, if T^*B is selfadjoint then the following implication holds*

$$(2.14) \quad YB \subseteq T \text{ for some } Y \in B^+(\mathfrak{K}) \Rightarrow T^*T \leq c_1 T^*B \leq c_2 B^*B,$$

*where $c_1, c_2 \geq 0$. In this case $\text{dom } B \subseteq \text{dom } (T^*B)^{\frac{1}{2}} \subseteq \text{dom } T$ and*

$$(2.15) \quad T^*B = B^*YB = B^*T.$$

Proof. The implication (i) \Rightarrow (ii) is clear since $\|YBx\| \leq \|Y\|\|Bx\|$ for all $x \in \text{dom } B$. To see the reverse implication notice that $GBx = Tx$, $x \in \text{dom } B$ is a well-defined operator with $\|G\| \leq c$. Then, $Y \in B(\mathfrak{K})$ is obtained by continuation of G to $\overline{\text{ran } B}$ and using the zero extension to $(\text{ran } B)^\perp = \ker B^*$, so that $\ker B^* \subseteq \ker Y$.

Now, assume that T^*B is selfadjoint and $YB \subseteq T$ for some $Y \in B^+(\mathfrak{K})$. Then, $Y^{\frac{1}{2}}Y^{\frac{1}{2}}B \subseteq Y^{\frac{1}{2}}Y^{\frac{1}{2}}B = \overline{YB} \subseteq T$ and the first part of the lemma shows that there exists $0 \leq c_1 \leq \|Y^{\frac{1}{2}}\|$ such that

$$(2.16) \quad T^*T \leq c_1^2 (Y^{\frac{1}{2}}B)^* \overline{Y^{\frac{1}{2}}B}.$$

On the other hand, one has

$$T^*B \subseteq (YB)^*B = (Y^{\frac{1}{2}}Y^{\frac{1}{2}}B)^*B = (Y^{\frac{1}{2}}B)^*Y^{\frac{1}{2}}B \subseteq (Y^{\frac{1}{2}}B)^* \overline{Y^{\frac{1}{2}}B}.$$

Since T^*B is selfadjoint, it follows that

$$(2.17) \quad T^*B = B^*YB = (Y^{\frac{1}{2}}B)^* \overline{Y^{\frac{1}{2}}B},$$

which shows the first identity in (2.15). Moreover, one has

$$B^*T \subseteq T^*B = (Y^{\frac{1}{2}}B)^* \overline{Y^{\frac{1}{2}}B} = B^*(YB) \subseteq B^*T,$$

which means that

$$B^*T = T^*B = B^*YB = (Y^{\frac{1}{2}}B)^* \overline{Y^{\frac{1}{2}}B} \leq \|Y\|B^*B.$$

Combining this with (2.16) leads to

$$T^*T \leq c_1^2 T^*B \leq c_1^2 \|Y\|B^*B,$$

which completes the proof of (2.14), (2.15) and $\text{dom } B \subseteq \text{dom } (T^*B)^{\frac{1}{2}} \subseteq \text{dom } T$. \square

Motivated by Lemma 2.6, the next step towards the extension of the equivalence (2.13) is to address the implication in the following equivalence:

$$(2.18) \quad T^*T \leq \lambda T^*B, \lambda \geq 0 \Leftrightarrow XB \subseteq T \text{ for some } X \in B^+(\mathfrak{H}).$$

For this, begin by observing that in the general case of closed densely defined operators, and for $B_0 := B \upharpoonright \text{dom } T^*B$, one has $T^*B = T^*B_0 \subseteq T^*\overline{B_0} \subseteq T^*B$. This means that

$$(2.19) \quad T^*B = T^*B_0 = T^*\overline{B_0} = T^*B,$$

so the following equivalence holds for $\lambda \geq 0$

$$(2.20) \quad T^*T \leq \lambda T^*B \Leftrightarrow T^*T \leq \lambda T^*\overline{B_0}.$$

However, contrary to the bounded case where automatically $B_0 = \overline{B_0} = B$, one cannot expect the factorization $T = XB$ as in (2.13) since one only has

$$\overline{B_0} \subseteq B.$$

Thus, it becomes natural to restrict B to $\overline{B_0}$ in the following extension of Sebestyén theorem.

Theorem 2.7. *Let $T, B : \mathfrak{H} \rightarrow \mathfrak{K}$ be closed densely defined linear operators such that T^*B is a selfadjoint operator and let $B_0 = B \upharpoonright \text{dom } T^*B$. Then the following assertions are equivalent for some $0 \leq \lambda (= \|X\|)$:*

- (i) $T^*T \leq \lambda T^*B$;
- (ii) $X\overline{B_0} \subseteq T$ has a solution $X \in B^+(\mathfrak{K})$.

In this case

$$(2.21) \quad (B_0)^*X\overline{B_0} = T^*\overline{B_0} = (B_0)^*T$$

and, moreover, X can be chosen such that $\ker T^* \subseteq \ker X$ with $\|X\| \leq \lambda$. In particular,

$$(2.22) \quad T^*T \leq \lambda T^*\overline{B_0} \text{ and } \text{dom } T \subseteq \text{dom } \overline{B_0} \Leftrightarrow T = X\overline{B_0} \text{ for some } X \in B^+(\mathfrak{K}).$$

In this case $\ker X = \ker T^*$.

Proof. Assume (i). Then a direct application of Lemma 2.6 to T and $(T^*B)^{\frac{1}{2}}$ leads to the existence of $G_0 \in B(\mathfrak{H}, \mathfrak{K})$ such that $\overline{\text{ran}} G_0 \subseteq \overline{\text{ran}} T$, $\ker T^*B \subseteq \ker G_0$ and

$$(2.23) \quad G_0(\lambda T^*B)^{\frac{1}{2}} \subseteq T.$$

Hence

$$(2.24) \quad T^* \subseteq (\lambda T^*B)^{\frac{1}{2}}(G_0)^*$$

and

$$(2.25) \quad \lambda T^*B \subseteq (\lambda T^*B)^{\frac{1}{2}}\lambda(G_0)^*B.$$

Multiplying (2.25) from the left by $(\lambda T^*B)^{(-\frac{1}{2})}$, one obtains

$$(2.26) \quad P_{\ker(T^*B)^\perp} I_{\text{dom}(T^*B)^{\frac{1}{2}}} (\lambda T^*B)^{\frac{1}{2}} \subseteq P_{\ker(T^*B)^\perp} I_{\text{dom}(T^*B)^{\frac{1}{2}}} \lambda(G_0)^*B \subseteq \lambda(G_0)^*B.$$

This implies that

$$(P_{\ker(T^*B)^\perp} I_{\text{dom}(T^*B)^{\frac{1}{2}}} (\lambda T^*B)^{\frac{1}{2}}) \upharpoonright \text{dom } T^*B = \lambda(G_0)^*B \upharpoonright \text{dom } T^*B = \lambda(G_0)^*B_0,$$

and hence

$$(2.27) \quad (\lambda T^*B)^{\frac{1}{2}} \upharpoonright \text{dom } T^*B = \lambda(G_0)^*B_0.$$

Since $\text{dom } T^*B$ is a core for $(T^*B)^{\frac{1}{2}}$, i.e. $(T^*B)^{\frac{1}{2}} = \overline{(T^*B)^{\frac{1}{2}} \upharpoonright \text{dom } T^*B}$, one concludes that

$$(2.28) \quad (\lambda T^*B)^{\frac{1}{2}} = \overline{(\lambda T^*B)^{\frac{1}{2}} \upharpoonright \text{dom } T^*B} = \overline{\lambda(G_0)^*B_0} \supseteq \lambda(G_0)^*\overline{B_0}.$$

Together with (2.23) this implies that $\lambda G_0(G_0)^*\overline{B_0} \subseteq T$ and therefore

$$X\overline{B_0} \subseteq T$$

with $X = \lambda G_0(G_0)^* \in B^+(\mathfrak{K})$ so that $\|X\| \leq \lambda$.

The reverse implication as well as the equalities in (2.21) follow immediately from Lemma 2.6.

The inclusion $\ker T^* \subseteq \ker X$ follows easily from the construction of G_0 , the identity (2.24) and from the fact that $\ker(G_0)^* = \ker X$.

Now assume that $\text{dom } T \subseteq \text{dom } \overline{B_0}$ and $T^*T \leq \lambda T^*\overline{B_0}$. Then, the implication "(i) \Rightarrow (ii)" immediately yields that $X\overline{B_0} = T$ for some $X \in B^+(\mathfrak{K})$. For the converse, observe that $X\overline{B_0} = T$ is closed, so $X^{\frac{1}{2}}\overline{B_0}$ is closed by Lemma 2.1. Consequently,

$$T^*\overline{B_0} = (B_0)^*X^{\frac{1}{2}}X^{\frac{1}{2}}\overline{B_0} = (X^{\frac{1}{2}}\overline{B_0})^*X^{\frac{1}{2}}\overline{B_0}$$

is a nonnegative selfadjoint operator with $\text{dom } (T^*\overline{B_0})^{\frac{1}{2}} = \text{dom } X^{\frac{1}{2}}\overline{B_0} = \text{dom } \overline{B_0} = \text{dom } T$. Moreover, $T^*T = T^*X\overline{B_0} \leq \|X\|T^*\overline{B_0}$, which completes the argument. On the other hand, one has $T^* = B_0^*X$, so $\ker X \subseteq \ker T^*$. Consequently, $\ker T^* = \ker X$ by the first part of the proof. \square

Remark 2.8. (i) The inequality in item (i) of Theorem 2.7 induces the following new inequality

$$(2.29) \quad T^*\overline{B_0} \leq \mu(B_0)^*\overline{B_0},$$

where $\mu = \|X\|$. This follows from Lemma 2.6, (2.14). Notice that the inclusion $\overline{B_0} \subseteq B$ implies that $B^*B \leq B_0^*\overline{B_0}$, and hence (2.29) does not necessarily imply the inequality $T^*B \leq \gamma B^*B$, $\gamma \geq 0$.

(ii) The inequality (2.29) is not sufficient to prove item (i) of Theorem 2.7. However, one can always obtain the following equivalence

$$(2.30) \quad \begin{aligned} T^*T \leq \lambda T^*\overline{B_0} &\leq \lambda \mu(B_0)^*\overline{B_0} \\ &\Updownarrow \\ X\overline{B_0} \subseteq T &\text{ has a solution } X \in B^+(\mathfrak{K}). \end{aligned}$$

(iii) By construction, $\lambda = 0$ if and only if the solution $X = 0$, in which case $T = 0$.

Although Theorem 2.7 establishes the equivalence (2.18) only for B_0 , its proof reveals that an additional condition would allow the desired equivalence to hold for B , more generally. This can be seen in the following remark.

Remark 2.9. Following Remark 2.8, a particular case of Theorem 2.7 where $\text{dom } T^*B$ is a core for B leads to the following statements for $\lambda \geq 0$:

- (1) $XB \subseteq T$ for some $X \in B^+(\mathfrak{K}) \Leftrightarrow T^*T \leq \lambda T^*B$.
- (2) $T = XB$ for some $X \in B^+(\mathfrak{K}) \Leftrightarrow T^*T \leq \lambda T^*B$ and $\text{dom } T \subseteq \text{dom } B$.

In the absence of the additional core conditions stated in Remark 2.9, the question arises about the most appropriate generalization of (2.13) to the unbounded

setting. Motivated by (2.30), this question naturally leads to consider whether the converse of (2.14) in Lemma 2.6 is true. Since the latter implies that

$$\text{dom } B \subseteq \text{dom } (T^*B)^{\frac{1}{2}} \text{ and } B^*T = T^*B,$$

it becomes natural also to impose these conditions in the following result, which in fact constitutes the final step towards the objective of this subsection.

Proposition 2.10. *Let $T, B : \mathfrak{H} \rightarrow \mathfrak{K}$ be closed densely defined linear operators such that $T^*B = B^*T$ is selfadjoint and $\text{dom } (T^*B)^{\frac{1}{2}} = \text{dom } T$. Then the following assertions are equivalent for some $0 \leq \lambda$ ($= \|X\|$):*

- (i) $T^*T \leq \lambda T^*B$ and $\text{dom } B \subseteq \text{dom } (T^*B)^{\frac{1}{2}}$;
- (ii) $XB \subseteq T$ has a solution $X \in B^+(\mathfrak{K})$;
- (iii) $\overline{XB} = T$ has a solution $X \in B^+(\mathfrak{K})$.

In this case

$$(2.31) \quad B^*XB = T^*B = B^*T$$

and, moreover, X can be chosen such that $\ker T^* = \ker X$ with $\|X\| \leq \lambda$.

Proof. Assume (i). Then, following the same reasoning as in the proof of Theorem 2.7, (2.23) together with the fact that $\text{dom } T = \text{dom } (T^*B)^{\frac{1}{2}}$ gives

$$(2.32) \quad T = G_0(\lambda T^*B)^{\frac{1}{2}},$$

and hence

$$(2.33) \quad B^*T = B^*G_0(\lambda T^*B)^{\frac{1}{2}}.$$

As $B^*T = T^*B$ is nonnegative and selfadjoint, multiplying (2.33) from the right by $(B^*T)^{(-\frac{1}{2})}$ implies that

$$(2.34) \quad (B^*T)^{\frac{1}{2}} \left(P_{\ker B^*T^\perp} \upharpoonright (B^*T)^{\frac{1}{2}}(\text{dom } B^*T) \right) \subseteq B^*G_0.$$

Since $\text{dom } T^*B$ is a core for $(B^*T)^{\frac{1}{2}}$, the set $P_{\ker B^*T^\perp} \upharpoonright (B^*T)^{\frac{1}{2}}(\text{dom } B^*T)$ is dense in \mathfrak{H} and, therefore, $\overline{\text{dom } B^*G_0} = \mathfrak{H}$ by (2.34). Hence, G_0^*B is a closable operator which satisfies

$$(\lambda T^*B)^{\frac{1}{2}} = \overline{\lambda G_0^*B_0} \subseteq \overline{\lambda G_0^*B};$$

see (2.28). Therefore, $\text{dom } B \subseteq \text{dom } (T^*B)^{\frac{1}{2}}$ implies that

$$\lambda G_0^*B = \overline{\lambda G_0^*B} \upharpoonright \text{dom } B \subseteq \overline{\lambda G_0^*B} \upharpoonright \text{dom } (T^*B)^{\frac{1}{2}} = (\lambda T^*B)^{\frac{1}{2}} \subseteq \overline{\lambda G_0^*B}.$$

Consequently $\overline{\lambda G_0^*B} = (\lambda T^*B)^{\frac{1}{2}}$, which by (2.32) gives

$$T = \lambda G_0 \overline{G_0^*B} = \overline{\lambda G_0 G_0^*B} = \lambda G_0 G_0^*B \supseteq G_0 G_0^*B.$$

This completes the proof of the implication (i) \Rightarrow (iii) \Rightarrow (ii) for $X = G_0(G_0)^* \in B^+(\mathfrak{K})$. The implication (ii) \Rightarrow (i) together with the identity (2.31) is immediate from Lemma 2.6.

To see that $\ker T^* = \ker(G_0)^*$, observe from (2.32) that $T^* = (\lambda T^*B)^{\frac{1}{2}}(G_0)^*$ and hence $\ker X = \ker(G_0)^* \subseteq \ker T^*$. On the other hand, the inclusion $XB \subseteq T$ together with Lemma 2.6 shows that $\overline{\text{ran } X} \subseteq \overline{\text{ran } T}$ or, equivalently, $\ker T^* \subseteq \ker X$. Consequently, $\ker X = \ker T^*$. \square

Observe that, under the assumptions of Proposition 2.10, items (i) – (iii) are equivalent to the following statement for some $\lambda \geq 0$:

$$(2.35) \quad T^*T \leq \lambda T^*B \leq \lambda^2 B^*B \quad \text{and} \quad \text{dom}(T^*B)^{\frac{1}{2}} = \text{dom} T.$$

Moreover, some further necessary and sufficient conditions for (2.35) may be derived through the study of forms, as investigated in [4].

Corollary 2.11. *Let $T, B : \mathfrak{H} \rightarrow \mathfrak{K}$ be closed densely defined linear operators. Then the following assertions are equivalent for some $0 \leq \lambda (= \|X\|)$:*

- (i) T^*B is a selfadjoint operator such that $\text{dom} T \subseteq \text{dom} B \subseteq \text{dom}(T^*B)^{\frac{1}{2}}$ and $T^*T \leq \lambda T^*B = \lambda B^*T$;
- (ii) $XB = T$ has a solution $X \in B^+(\mathfrak{K})$.

Proof. If (i) holds then $\text{dom}(T^*B)^{\frac{1}{2}} \subseteq \text{dom} T$, and hence item (ii) easily follows from the implication (i) \Rightarrow (ii) of Proposition 2.10. Conversely, if $T = XB$ has a solution $X \in B^+(\mathfrak{H})$ then $X^{\frac{1}{2}}B$ is closed, by Lemma 2.1 and therefore $T^*B = B^*XB = B^*T = (X^{\frac{1}{2}}B)^*X^{\frac{1}{2}}B$ is a nonnegative selfadjoint operator. Hence, one concludes the result from Lemma 2.6, (2.14) and from the fact that $\text{dom} T = \text{dom} B$. \square

A consequence of Corollary 2.11 leads to the characterization of the class $\mathcal{L}_l^{+2}(\mathfrak{H})$ by means of Sebestyén inequality described in the following theorem, thereby generalizing [7, Theorem 4.5].

Theorem 2.12. *Let $T \in CO(\mathfrak{H})$ be a densely defined operator. Then, $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ if and only if $T^*T \leq T^*Y = YT$ admits a solution $Y = Y^* \geq 0$ such that T^*Y is selfadjoint and $\text{dom} T \subseteq \text{dom} Y \subseteq \text{dom}(T^*Y)^{\frac{1}{2}}$.*

Proof. The proof follows immediately by applying Corollary 2.11 to $B = Y$. \square

2.2. $\mathcal{L}_l^{+2}(\mathfrak{H})$ and quasi-affinity to $S = S^* \geq 0$. In this subsection, for the convenience of the reader, G -quasi-affinity refers to quasi-affinity already mentioned in the introduction. The following lemma provides a link between the G -quasi-affinity and the $|G|$ -quasi-affinity to a nonnegative selfadjoint operator, which will be useful in Subsection 4.3.

Lemma 2.13. *Let $T \in LO(\mathfrak{H})$ be a densely defined operator. Then the following assertions are equivalent:*

- (i) $\overline{GTG^{-1}} = G^{-1}T^*G^* \geq 0$ for a quasi-affinity $G \in B(\mathfrak{H})$;
- (ii) $X^{\frac{1}{2}}TX^{-\frac{1}{2}} = X^{-\frac{1}{2}}T^*X^{\frac{1}{2}} \geq 0$ is selfadjoint for a quasi-affinity $X \in B^+(\mathfrak{H})$.

Proof. Assume (i) and let $G = U|G|$ be the polar decomposition of G . Since G is a quasi-affinity, U is unitary. Setting $X := G^*G$, one sees that $X \in B^+(\mathfrak{H})$ is a quasi-affinity and

$$(2.36) \quad G^{-1}T^*G^* = U|G|^{-1}T^*|G|U^* = U(X^{-\frac{1}{2}}T^*X^{\frac{1}{2}})U^{-1}.$$

As U is unitary, one concludes from (2.36) and (i) that $X^{-\frac{1}{2}}T^*X^{\frac{1}{2}} \geq 0$ is selfadjoint and hence $X^{-\frac{1}{2}}T^*X^{\frac{1}{2}} = (X^{-\frac{1}{2}}T^*X^{\frac{1}{2}})^* = X^{-\frac{1}{2}}T^*X^{\frac{1}{2}} \geq 0$. The reverse implication is immediate. \square

The following theorem establishes a connection between the class $\mathcal{L}_l^{+2}(\mathfrak{H})$ and the quasi-affinity to nonnegative selfadjoint operators.

Proposition 2.14. *Let $T \in CO(\mathfrak{H})$ be densely defined. Then the following assertions are equivalent:*

- (i) T^* is quasi-affine to some $S = S^* \geq 0$;
- (ii) $T \supseteq AB \in \mathcal{L}_l^{+2}(\mathfrak{H})$ with $\overline{\text{ran}} A = \mathfrak{H}$;
- (iii) there exists a quasi-affinity $X \in B^+(\mathfrak{H})$ such that

$$0 \leq T^*X^{-1} \subseteq X^{-1}T \Leftrightarrow 0 \leq XT^* \subseteq TX.$$

Proof. (i) \Rightarrow (ii) Assume that T^* is G -quasi-affine to $S = S^* \geq 0$. Then the inclusion $GT^* \subseteq SG$ implies that

$$(2.37) \quad T^*(G^*G)^{-1} \subseteq G^{-1}S(G^{-1})^* := B_0 \geq 0$$

and hence $\overline{\text{dom}} B_0 = \overline{\text{ran}} G^*G = \mathfrak{H}$. Now, let B_F be the Friedrichs extension of B_0 (cf. [14]) and let $A = G^*G \in B^+(\mathfrak{H})$. Then (2.37) shows that $T^*A^{-1} \subseteq B_0 \subseteq B_F$, and therefore $AB_F \subseteq T$.

(ii) \Rightarrow (iii) Since $AB \subseteq T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ and $\overline{\text{ran}} A = \mathfrak{H}$, it follows that $\ker A = \{0\}$ and one has $0 \leq B \subseteq A^{-1}T$. Hence,

$$(2.38) \quad 0 \leq T^*A^{-1} \subseteq (A^{-1}T)^* \subseteq B \subseteq A^{-1}T = (T^*A^{-1})^*.$$

By taking $X = A$, one concludes that $0 \leq T^*X^{-1} \subseteq X^{-1}T$ or, equivalently, $XT^* \subseteq TX$. Moreover, it follows from (2.38) that $XT^* \subseteq XBX \geq 0$, which completes the proof of (iii).

(iii) \Rightarrow (i) Since $\overline{\text{ran}} X = \mathfrak{H} = \overline{\text{dom}} T^*$ it follows that $T^*X^{-1} \geq 0$ is a densely defined operator whose Friedrichs extension is again denoted by B_F . Then $T^*X^{-1} \subseteq B_F$ and one has

$$X^{\frac{1}{2}}T^* \subseteq (X^{\frac{1}{2}}B_FX^{\frac{1}{2}})X^{\frac{1}{2}} \subseteq \left(\overline{X^{\frac{1}{2}}B_F^{\frac{1}{2}}(X^{\frac{1}{2}}B_F^{\frac{1}{2}})^*} \right) X^{\frac{1}{2}}.$$

This proves that T^* is $X^{\frac{1}{2}}$ -quasi-affine to $S := \overline{X^{\frac{1}{2}}B_F^{\frac{1}{2}}(X^{\frac{1}{2}}B_F^{\frac{1}{2}})^*} \geq 0$. \square

Remark 2.15. In the proof of Proposition 2.14 the operator B_0 in (2.37) is non-negative and densely defined. Hence the form generated by B_0 is closable and its closure has B_F , the Friedrichs extension, as the unique representing operator given by

$$(2.39) \quad B_F = (G^{-1}S^{\frac{1}{2}})\overline{S^{\frac{1}{2}}(G^{-1})^*},$$

cf. [14]. The proof also shows that if B is any nonnegative selfadjoint extension of B_0 then (ii) holds and (iii) follows by taking $X = A$.

The rest of this section is devoted to describe close relations between Sebestyén inequality and quasi-affinity to a nonnegative selfadjoint operator.

Corollary 2.16. *Let $T \in CO(\mathfrak{H})$ be a densely defined operator and let $S = S^* \geq 0$. If T^* is G -quasi-affine to S such that $\rho(T^*B_F) \neq \emptyset$, then there exists $\lambda > 0$ such that*

$$T^*T \leq \lambda T^*B_F,$$

where B_F is defined in (2.39).

Proof. Since T^* is G -quasi-affine to some $S = S^* \geq 0$, it follows from Proposition 2.14 and Remark 2.15 that $AB_F \subseteq T$ with $B_F = (G^{-1}S^{\frac{1}{2}})\overline{S^{\frac{1}{2}}(G^{-1})^*}$. Hence

$$T^*B_F \subseteq B_FAB_F = (A^{\frac{1}{2}}B_F)^*A^{\frac{1}{2}}B_F \subseteq (A^{\frac{1}{2}}B_F)^*\overline{A^{\frac{1}{2}}B_F} =: M \geq 0.$$

Since M is selfadjoint, T^*B_F is symmetric. On the other hand, $\rho(T^*B_F) \neq \emptyset$ by assumption and therefore T^*B_F is selfadjoint, too. Together with the fact that $\overline{AB_F} \upharpoonright \text{dom } T^*B_F \subseteq AB_F \subseteq T$ this yields $T^*T \leq \lambda T^*B$ for some $\lambda \geq 0$ by Theorem 2.7. \square

Note that a small adjustment to item (i) of Proposition 2.14 allows T to be written as the product of two nonnegative, in general, unbounded linear operators motivating the following result.

Proposition 2.17. *Let $T \in CO(\mathfrak{H})$ be densely defined. Then the following assertions are equivalent:*

- (i) T^* is G -quasi-affine to $S = S^* \geq 0$ with $\text{dom } T \subseteq \text{dom } G^{-1}S^{\frac{1}{2}}\overline{S^{\frac{1}{2}}(G^{-1})^*}$;
- (ii) $T = AB \in \mathcal{L}_l^{+2}(\mathfrak{H})$ with $\overline{\text{ran}} A = \mathfrak{H}$;
- (iii) there exists a quasi-affinity $X \in B^+(\mathfrak{H})$ such that

$$X^{-1}T = \overline{T^*X^{-1}} \geq 0,$$

where $\text{dom } X^{-1}T = \text{dom } T$.

Proof. (i) \Rightarrow (ii) Using the same arguments as in the proof of Proposition 2.14, one obtains $AB_F \subseteq T$. On the other hand, $B_F = G^{-1}S^{\frac{1}{2}}\overline{S^{\frac{1}{2}}(G^{-1})^*}$ by Remark 2.15 and hence the assumption $\text{dom } T \subseteq \text{dom } B_F$ yields $T = AB_F \in \mathcal{L}_l^{+2}(\mathfrak{H})$.

(ii) \Rightarrow (iii) For $X = A$ one has $X^{-1}T = B = B^* = \overline{(X^{-1}T)^*} = \overline{T^*X^{-1}} \geq 0$. Moreover, $\text{dom } T = \text{dom } B = \text{dom } X^{-1}T$.

(iii) \Rightarrow (i) Set $B = X^{-1}T$. Then $XB = XX^{-1}T \subseteq T$ and, since $\text{dom } T = \text{dom } B$ it follows that $T = XB \in CO(\mathfrak{H})$. Thus $X^{\frac{1}{2}}B$ is a closed densely defined operator by Lemma 2.1 and hence

$$(2.40) \quad S := X^{-\frac{1}{2}}TX^{\frac{1}{2}} = X^{\frac{1}{2}}BX^{\frac{1}{2}} = X^{\frac{1}{2}}B^{\frac{1}{2}}(X^{\frac{1}{2}}B^{\frac{1}{2}})^* \geq 0,$$

is a selfadjoint operator. Moreover, it follows from (2.40) that

$$S = (X^{-\frac{1}{2}}TX^{\frac{1}{2}})^* \supseteq X^{\frac{1}{2}}T^*X^{-\frac{1}{2}},$$

and therefore $X^{\frac{1}{2}}T^* \subseteq SX^{\frac{1}{2}}$. This proves that T^* is $X^{\frac{1}{2}}$ -quasi-affine to S . On the other hand, multiplying (2.40) from the left by $X^{-\frac{1}{2}}$ and from the right by $X^{\frac{1}{2}}$ shows that $X^{-\frac{1}{2}}SX^{\frac{1}{2}} = BX = T^*$. Thus $T^*X^{-1} \subseteq X^{-\frac{1}{2}}SX^{-\frac{1}{2}} \geq 0$, which implies that $B_0 := X^{-\frac{1}{2}}SX^{-\frac{1}{2}}$ is a densely defined operator such that

$$\overline{T^*X^{-1}} \subseteq \overline{B_0} \subseteq B_0^* \subseteq \overline{T^*X^{-1}}.$$

Consequently, $\overline{T^*X^{-1}} = \overline{B_0} = B_F$, where $B_F = X^{-\frac{1}{2}}S^{\frac{1}{2}}\overline{S^{\frac{1}{2}}X^{-\frac{1}{2}}}$ is the Friedrichs extension of B_0 and $\text{dom } B_F = \text{dom } \overline{T^*X^{-1}} = \text{dom } X^{-1}T = \text{dom } T$. \square

The reversed implication for Corollary 2.16 is established in the next result where a subclass of $\mathcal{L}_l^{+2}(\mathfrak{H})$ is characterized not only by Sebestyén inequality but also by quasi-affinity to a nonnegative selfadjoint operator.

Theorem 2.18. *Let $T \in CO(\mathfrak{H})$ be a densely defined operator with $\overline{\text{ran}} T = \mathfrak{H}$ and let $S = S^* \geq 0$. Then the following statements are equivalent:*

- (i) $T^*T \leq \lambda T^*B$ with $\text{dom } B \subseteq \text{dom } T \subseteq \text{dom } \overline{B} \upharpoonright \overline{T^*B}$ for some $\lambda \geq 0$ and $B = B^* \geq 0$;
- (ii) $T = A\overline{B} \upharpoonright \overline{T^*B} \in \mathcal{L}_l^{+2}(\mathfrak{H})$ with $\overline{\text{ran}} A = \mathfrak{H}$;

- (iii) T^* is G -quasi-affine to $S = S^* \geq 0$ with $\text{dom } T \subseteq \overline{\text{dom } B_F \upharpoonright \text{dom } (T^*B_F)}$, where $B_F = G^{-1}S^{\frac{1}{2}}S^{\frac{1}{2}}(G^{-1})^*$.

Proof. (i) \Leftrightarrow (ii) Observe from (2.22) in Theorem 2.7 that $T = \overline{AB_0} \subseteq AB$, where $A \in B^+(\mathfrak{H})$ and $B_0 = B \upharpoonright \text{dom } T^*B$. Since $\text{dom } B \subseteq \text{dom } T$ it follows that $T = \overline{AB_0} = AB$, and hence $\text{dom } B = \text{dom } \overline{B_0}$. This implies that $\overline{B_0} = B = B^* \geq 0$, and hence $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ with $\overline{\text{ran}} T = \overline{\text{ran}} A = \mathfrak{H}$. The reverse implication follows immediately from Theorem 2.7 by choosing $B = \overline{B} \upharpoonright \text{dom } T^*B$.

(ii) \Leftrightarrow (iii) Assume (ii). Then, it is clear from Proposition 2.17 that T^* is quasi-affine to some $S = S^* \geq 0$ with $\text{dom } T \subseteq \text{dom } B_F$. Moreover, the proof of Proposition 2.17 shows that $B_F = \overline{B} \upharpoonright \text{dom } T^*B$, which completes the argument of the direct implication. To see the reverse implication, observe from the Proposition 2.17 that $T = AB_F \in \mathcal{L}_l^{+2}(\mathfrak{H})$ with $\overline{\text{ran}} A = \mathfrak{H}$. Hence $\text{dom } T = \text{dom } B_F \subseteq \text{dom } \overline{B_F} \upharpoonright \text{dom } (T^*B_F)$, and thus $B_F = \overline{B_F} \upharpoonright \text{dom } (T^*B_F)$ satisfies (ii). \square

The next remark contains a variant of Theorem 2.18 and gives necessary and sufficient conditions for an operator T with $\overline{\text{ran}} T = \mathfrak{H}$ to be in $\mathcal{L}_l^{+2}(\mathfrak{H})$.

Remark 2.19. Let $T \in CO(\mathfrak{H})$ be a densely defined operator with $\overline{\text{ran}} T = \mathfrak{H}$ and let $S = S^* \geq 0$. Then the following statements are equivalent:

- (i) $T^*T \leq \lambda_0 T^*B \leq \lambda_1 B^*B$ for some $\lambda_0, \lambda_1 \geq 0$ with $\text{dom } T \subseteq \text{dom } B$, $T^*B = B^*T$ and $B = B^* \geq 0$;
- (ii) $T = AB \in \mathcal{L}_l^{+2}(\mathfrak{H})$;
- (iii) T^* is G -quasi-affine to $S = S^* \geq 0$ with $\text{dom } T \subseteq \text{dom } B_F$, where $B_F = G^{-1}S^{\frac{1}{2}}S^{\frac{1}{2}}(G^{-1})^*$.

Note that once Corollary 2.18 or Corollary 2.16 is applied to T , one would expect that the quasi-affinity of T to selfadjoint operators is connected to the Sebestyén inequality involving TT^* . However, it will be seen in Section 4 that the reversed inequality is more appropriate for such an approach and this will be achieved through a further study of linear relations, which will be discussed in the next section.

3. GENERALIZATION TO LINEAR RELATIONS

In this section an analog of Theorem 2.7 is established for the case where the operator inequality therein is reversed. For this purpose it is helpful to first prove Theorem 2.7 in a bit more general context where T and B are not assumed to be densely defined and, in fact, they will also be allowed to be multivalued linear relations. This needs some basic facts concerning ordering of semibounded selfadjoint relations; see [6, Section 5.2] and e.g. [20, 12].

Before stating the result, some key notions on linear relations in Hilbert spaces are recalled; for further details, the reader is referred to [8, 6, 1]. A linear relation (relation) T from \mathfrak{H} to \mathfrak{K} is a linear subspace of the Cartesian product $\mathfrak{H} \times \mathfrak{K}$. It is uniquely determined by its graph $G(T) = \{(x, y) \in \mathfrak{H} \times \mathfrak{K} : x \in \text{dom } T, y \in Tx\}$. Unless otherwise specified, the same notations, familiar for linear operators, will be used for linear relations. The *inverse* and the *adjoint* of T are respectively given by $G(T^{-1}) = \{(y, x) : (x, y) \in G(T)\}$ and $G(T^*) = \{(x, x') \in \mathfrak{K} \times \mathfrak{H} : \langle x', y \rangle = \langle x, y' \rangle \text{ for all } (y, y') \in G(T)\}$. For a closed operator T , the operator part is given by $T_s = P_s T$, where P_s stands for the orthogonal projection onto $(\text{mul } T)^\perp = \overline{\text{dom } T^*}$. Moreover, T_s is closed and T decomposes as $T = T_s \hat{\oplus} T_{\text{mul}}$, where $T_{\text{mul}} = (\{0\} \times$

$\text{mul } T$).

If $\mathfrak{K} = \mathfrak{H}$ and $\langle x', x \rangle \in \mathbb{R}$ for all $(x, x') \in G(T)$ then T is said to be *symmetric* or, equivalently, $T \subseteq T^*$. If $\langle x', x \rangle \in \mathbb{R}^+$ then T is *nonnegative* and one writes $T \geq 0$. Moreover, T is *selfadjoint* if $T = T^*$. Note that, if $T = T^* \geq 0$ then $T_s^{\frac{1}{2}} := (T_s)^{\frac{1}{2}} = (T^{\frac{1}{2}})_s$. For a closed linear relation T the product T^*T is a nonnegative selfadjoint relation; see [6, Lemma 1.5.8]. In particular, $T_s \subseteq T$ and $T^* \subseteq (T_s)^*$, so that

$$(3.1) \quad T^*T \subseteq (T_s)^*T = T^*P_sT = T^*T_s \subseteq (T_s)^*T_s$$

and here all inclusions prevail as equalities, since T^*T and $(T_s)^*T_s$ both are selfadjoint. Notice that if T is a closed operator, which is not densely defined, then T^*T is a selfadjoint relation with $\text{mul } T^*T = (\text{dom } T)^\perp$.

3.1. Sebestyén inequality for linear relations. The next result allows T and B to be closed linear relations such that the case of densely defined operators in Theorem 2.7 is explicitly included in it. It should be pointed out that, exactly as in the case of linear operators; cf. (2.19), one has

$$(3.2) \quad T^*B = T^*B \upharpoonright \text{dom } T^*B = \overline{T^*B \upharpoonright \text{dom } T^*B},$$

where $B \upharpoonright D := B \cap (D \times \mathfrak{K})$ denotes the restriction of the relation $B : \mathfrak{H} \rightarrow \mathfrak{K}$ to a linear subspace $D \subseteq \mathfrak{H}$.

Theorem 3.1. *Let $T, B : \mathfrak{H} \rightarrow \mathfrak{K}$ be closed linear relations such that $\text{mul } B \subseteq \ker(T_s)^*$ and T^*B is selfadjoint. Then, the following statements are equivalent for $B_0 := B \upharpoonright \text{dom } T^*B$ and for some $\lambda \geq 0$:*

- (i) $T^*T \leq \lambda T^*B$;
- (ii) $X\overline{B_0} \subseteq T$ has a solution $X \in B^+(\mathfrak{K})$.

In this case, X can be chosen such that $X\overline{B_0} \subseteq T_s$ and $\ker(T_s)^ \subseteq \ker X$ with $\|X\| \leq \lambda$. Moreover, in this case*

$$(3.3) \quad T^*\overline{B_0} = B_0^*X\overline{B_0} = B_0^*T.$$

In particular, the following assertions are equivalent for some $\lambda \geq 0$:

- (iii) $T^*T \leq \lambda T^*\overline{B_0}$ and $\text{dom } T \subseteq \text{dom } \overline{B_0}$;
- (iv) $T = X\overline{B_0} \dot{+} T_{\text{mul}}$ has a solution $X \in B^+(\mathfrak{K})$.

In this case, X can be chosen such that $T_s = X\overline{B_s}$ and $\ker X = \ker(T_s)^$.*

Proof. Observe that item (i) is equivalent to $(T_s)^*T_s \leq \lambda(T^*B)_s^{\frac{1}{2}}(T^*B)_s^{\frac{1}{2}}$, and hence the formula

$$\begin{aligned} G : \text{ran } (T^*B)_s^{\frac{1}{2}} &\longrightarrow \text{ran } T_s \\ (\lambda T^*B)_s^{\frac{1}{2}} f &\longmapsto T_s f, \quad f \in \text{dom } (T^*B)_s^{\frac{1}{2}}, \end{aligned}$$

defines a contractive operator from $\text{ran } (T^*B)_s^{\frac{1}{2}}$ into $\text{ran } T_s$, since

$$\|G(\lambda T^*B)_s^{\frac{1}{2}} f\|^2 = \|T_s f\|^2 \leq \|(\lambda T^*B)_s^{\frac{1}{2}} f\|^2$$

for all $f \in \text{dom } (T^*B)_s^{\frac{1}{2}}$. Moreover, G can be extended to an operator $G_0 \in B(\mathfrak{H}, \mathfrak{K})$ such that

$$(3.4) \quad \begin{aligned} \ker(G_0) &\supseteq (\text{ran } (T^*B)_s^{\frac{1}{2}})^\perp = \ker(T^*B)_s^{\frac{1}{2}} \oplus \text{mul } T^*B \\ &= \ker T^*B \oplus \text{mul } T^*B \end{aligned}$$

and $\overline{\text{ran}} G_0 \subseteq \overline{\text{ran}} T_s$, which is equivalent to $\text{ran} (G_0)^* \subseteq \overline{\text{ran}} (T^*B)_s$ and

$$(3.5) \quad \ker G_0^* \supseteq \ker (T_s)^* = \ker T^* \oplus \text{mul } T.$$

Thus,

$$(3.6) \quad G_0(\lambda T^*B)_s^{\frac{1}{2}} \subseteq T_s.$$

As T is closed, T_s is also closed and $T_s \subseteq T$. Hence,

$$T^* \subseteq (T_s)^* \subseteq (\lambda T^*B)_s^{\frac{1}{2}} G_0^*$$

which implies that

$$(3.7) \quad \lambda T^*B \subseteq (\lambda T^*B)_s^{\frac{1}{2}} \lambda G_0^* B.$$

By assumption $\text{mul } B \subseteq \ker(T_s)^*$, and hence $\text{mul } T^*B = \text{mul } T^*T = \text{mul } T^*$. On the other hand, $\text{mul } B \subseteq \ker(T_s)^* \subseteq \ker G_0^*$ (see (3.5)), so

$$(3.8) \quad G_0^*B = G_0^*(B_s + B_{\text{mul}}) = G_0^*B_s.$$

This yields by (3.7) that

$$(3.9) \quad (\lambda T^*B)_s \subseteq (\lambda T^*B)_s^{\frac{1}{2}} G_0^* B_s.$$

Multiplying (3.9) from the left by the Moore-Penrose inverse $(T^*B)_s^{(-\frac{1}{2})}$ gives

$$\begin{aligned} Q_s(I \upharpoonright \text{dom} (T^*B)_s^{\frac{1}{2}})(\lambda T^*B)_s^{\frac{1}{2}} &\subseteq Q_s I \upharpoonright \text{dom} (T^*B)_s^{\frac{1}{2}} \lambda G_0^* B_s \\ &\subseteq \lambda Q_s G_0^* B_s = \lambda G_0^* B_s, \end{aligned}$$

where Q_s is the orthogonal projection onto $\overline{\text{ran}} (T^*B)_s$. Consequently,

$$(\lambda T^*B)_s^{\frac{1}{2}} \upharpoonright \text{dom} T^*B \subseteq \lambda G_0^* B_s,$$

and hence $(\lambda T^*B)_s^{\frac{1}{2}} \upharpoonright \text{dom} T^*B = \lambda G_0^* B_s \upharpoonright \text{dom} T^*B$. Since $\text{dom} T^*B$ is a core for $(T^*B)_s^{\frac{1}{2}}$, one gets

$$(\lambda T^*B)_s^{\frac{1}{2}} = \overline{\lambda G_0^* B_s \upharpoonright \text{dom} T^*B} \supseteq \overline{\lambda G_0^* B_s \upharpoonright \text{dom} T^*B}.$$

Together with (3.6) and (3.8) this implies that

$$(3.10) \quad \lambda G_0 G_0^* \overline{B_0} = \lambda G_0 G_0^* \overline{B_s \upharpoonright \text{dom} T^*B} \subseteq G_0 (\lambda T^*B)_s^{\frac{1}{2}} \subseteq T_s$$

and, in particular,

$$\lambda G_0 G_0^* \overline{B_0} \subseteq T.$$

This proves (ii) for $X = \lambda G_0 G_0^* \in B^+(\mathfrak{R})$.

The inclusion $\ker T^* \subseteq \ker(T_s)^* \subseteq \ker X$ follows from (3.5) by fact that $\ker(G_0)^* = \ker X$.

For the reverse implication (ii) \Rightarrow (i), observe that

$$T^* \overline{B_0} \subseteq B_0^* X \overline{B_0} \subseteq (X^{\frac{1}{2}} \overline{B_0})^* X^{\frac{1}{2}} \overline{B_0},$$

and since $T^* \overline{B_0} = T^*B$ is selfadjoint also $B_0^* X \overline{B_0}$ is selfadjoint. Thus,

$$(3.11) \quad T^* \overline{B_0} = B_0^* X \overline{B_0}.$$

Now, $X^{\frac{1}{2}} X^{\frac{1}{2}} \overline{B_0} \subseteq \overline{X \overline{B_0}} \subseteq T$ and the same argument that was used in the proof of Lemma 2.6 shows that for $\lambda = \|X\|$ one has

$$T^*T \leq \lambda (X^{\frac{1}{2}} B)^* \overline{X^{\frac{1}{2}} \overline{B_0}} = \lambda B_0^* X \overline{B_0} = \lambda T^* \overline{B_0},$$

and (i) is proved.

To complete the proof of (3.3) observe that

$$B_0^*T \subseteq (T^*\overline{B_0})^* = T^*\overline{B_0} = B_0^*X\overline{B_0} \subseteq B_0^*T,$$

and hence $B_0^*T = B_0^*X\overline{B_0} = T^*\overline{B_0}$.

For the proof of the equivalence (iii) \Leftrightarrow (iv), it suffices to observe that item (iv) is equivalent to $X\overline{B_0} \subseteq T_s$ and $\text{dom } T = \text{dom } \overline{B_0}$, and conclude the result from the first part of the proof. In this particular case, one easily sees that $T_s = X\overline{B_0}$ and hence $(T_s)^* = B_0^*X$, which leads to $\ker X \subseteq \ker(T_s)^* \subseteq \ker X$. \square

As seen in Remark 2.8, one obtains from (3.3) the following inequality

$$T^*B \leq \mu(B_0)^*\overline{B_0}, \quad \mu = \|X\|,$$

which implies that $\text{dom } \overline{B_0} \subseteq \text{dom } (T^*B)^{\frac{1}{2}} \subseteq \text{dom } T$. Some further properties of T and B are collected in the next remark.

Remark 3.2. Under the assumptions of Theorem 3.1 the following further statements hold:

- (1) $(T_s)^*B = (T_s)^*B_s$.
- (2) $\text{mul } T^*B = \text{mul } T^*$ (equivalently, $\overline{\text{dom } T^*B} = \overline{\text{dom } T}$);
- (3) if $\overline{B_0} = B$ then $\text{mul } T \cap \text{dom } B^* \subseteq \ker B^* \subseteq \ker(B_s)^*$;
- (4) As noted above $T^*T = (T_s)^*T_s$; cf. (3.1). Likewise, if $\overline{B_0} = B$ then

$$(3.12) \quad (T_s)^*B_s = (B_s)^*T_s,$$

which implies that

$$(3.13) \quad T^*B = (T_s)^*B = (T_s)^*B_s = (B_s)^*XB_s,$$

where $X \in B^+(\mathfrak{H})$.

- (5) If $\overline{B_0} = B$ then the first item of Theorem 3.1 can be written with the operator part of T :

$$(T_s)^*T_s \leq \lambda(T_s)^*\overline{B_0}.$$

Indeed, the identity (1) follows easily from the inclusion $\text{mul } B \subseteq \ker(T_s)^* = \ker T^* \oplus \text{mul } T$ which implies that

$$(T_s)^*B = T^*P_s(B_s \widehat{\oplus} B_{\text{mul}}) = T^*P_sB_s = (T_s)^*B_s.$$

To see (2), apply (1) to get

$$\text{mul } T^* \subseteq \text{mul } T^*B \subseteq \text{mul } (T_s)^*B = \text{mul } (T_s)^*B_s = \text{mul } T^*.$$

Hence $\text{mul } T^* = \text{mul } T^*B$ or, equivalently, $\overline{\text{dom } T} = \overline{\text{dom } T^*B}$.

For the proof of (3), observe that $\text{mul } B^* \subseteq \text{mul } B^*T \subseteq \text{mul } (T^*B)^* = \text{mul } T^*B^*$. On the other hand, $\text{mul } T^*B \subseteq \text{mul } B^*$, by Remark 2.8 (i). Hence, $\text{mul } B^* = \text{mul } B^*T$, which means that

$$\text{mul } T \cap \text{dom } B^* \subseteq \ker B^* \subseteq \ker(B_s)^*.$$

For the proof of (4), observe that $XB = XB_s \subseteq T_s$ together with (3.3) and item (1) yields

$$(3.14) \quad (T_s)^*B_s = (T_s)^*B \subseteq (XB)^*B = B^*XB = T^*B \subseteq (T_s)^*B_s.$$

This means that $(T_s)^*B_s$ is selfadjoint and, moreover,

$$(3.15) \quad (T_s)^*B_s \subseteq (B_s)^*XB_s \subseteq (B_s)^*T_s \subseteq ((T_s)^*B_s)^* = (T_s)^*B_s.$$

A combination of (3.14) and (3.15) shows (3.12) and (3.13).

To see (5), observe from (3.13) and (3.2) that

$$(T_s)^*B = T^*B = T^*\overline{B_0} \subseteq (T_s)^*\overline{B_0} \subseteq (T_s)^*B,$$

which implies that $T^*B = (T_s)^*\overline{B_0}$. Together with (3.1), this implies that

$$T^*T \leq \lambda T^*B \Leftrightarrow (T_s)^*T_s \leq \lambda (T_s)^*\overline{B_0}.$$

3.2. Characterization of the reversed inequality. The following result shows that reversing Sebestyén inequality yields a new nonnegative, in general, unbounded solution X with $X^{-1} \in B^+(\mathfrak{H})$ rather than a bounded one as seen in Theorem 2.7 and Theorem 3.1. This motivates the study of a new unbounded product of nonnegative operators; see Section 4.

Theorem 3.3. *Let \mathfrak{K} be a complex Hilbert space and $T, B : \mathfrak{H} \rightarrow \mathfrak{K}$ be closed linear relations such that B^*T is selfadjoint and let $B_0 := B^* \cap (\mathfrak{K} \times \text{ran } B^*T)$. If $\ker B^* \subseteq \ker T^* \oplus \text{mul } T$ then the following assertions are equivalent for some $\eta > 0$:*

- (i) $T^*T \geq \eta \overline{B_0}T \geq 0$;
- (ii) $T \subseteq YB_0^*$ has a solution $Y^{-1} \in B^+(\mathfrak{K})$.

In this case, Y can be chosen such that $\ker T^* \oplus \text{mul } T \subseteq \text{mul } Y$ and

$$(3.16) \quad B^*T = \overline{B_0}T = \overline{B_0}YB_0^* = T^*B_0^*$$

In particular, the following statements are equivalent for some $\eta > 0$:

- (iii) $T^*T \geq \eta \overline{B_0}T$ with $\text{ran } T^* \subseteq \text{ran } \overline{B_0}$;
- (iv) $T^* = \overline{B_0}Y \dot{+} (\ker T^* \times \{0\})$ has a solution $Y^{-1} \in B^+(\mathfrak{K})$.

In this case, $\text{mul } T \oplus \ker T^* = \text{mul } Y$.

Proof. First observe that

$$(3.17) \quad B^*T = B_0T = \overline{B_0}T$$

is selfadjoint. Now, let $S := T^{*-1}$ and $A := (B^*)^{-1}$. Then, S and A are two closed linear relations such that $(S^*A)^* = (T^{-1}(B^*)^{-1})^* = ((B^*T)^*)^{-1} = (B^*T)^{-1} = S^*A$ and the assumption $\ker B^* \subseteq \ker T^* \oplus \text{mul } T$ is equivalent to $\text{mul } A \subseteq \text{mul } S \oplus \ker S^* = \ker(S_s)^*$. Now, using Remark 2.8 (iii), one can apply Theorem 3.1 to S and A which yields the following equivalences for $\lambda > 0$ and $A_0 := A \upharpoonright \text{dom } S^*A$:

- (1) $S^*S \leq \lambda S^*\overline{A_0}$;
- (2) $X\overline{A_0} \subseteq S$ has a solution $X \in B^+(\mathfrak{K})$, $X \neq 0$,

where X can be chosen such that $\ker(S_s)^* \subseteq \ker X$ and

$$(3.18) \quad S^*\overline{A_0} = A_0^*X\overline{A_0} = A_0^*S.$$

Equivalently, $\text{mul } T \oplus \ker T^* \subseteq \text{mul } Y$ for $Y = X^{-1}$. By taking inverses the equalities (3.18) can be rewritten as

$$\overline{B_0}T = \overline{B_0}YB_0^* = T^*B_0^*$$

using the fact that $B_0 = (A_0)^{-1}$. Combining this with (3.17) proves (3.16). Next, using [13, Lemma 3.3], or [6, Corollary 5.2.8] one has the following equivalence for some $\eta = \frac{1}{\lambda} > 0$:

- (1) $(S^*S)^{-1} \geq \eta (S^*\overline{A_0})^{-1}$;
- (2) $(X\overline{A_0})^{-1} = \overline{A_0}^{-1}X^{-1} \subseteq S^{-1}$ has a solution $X \in B^+(\mathfrak{K})$, $X \neq 0$.

By taking adjoints in (2) this equivalence can be rewritten as

- (1) $T^*T \geq \eta \overline{B_0}T$;
- (2) $T \subseteq YB_0^*$ has a solution $Y^{-1} \in B^+(\mathfrak{K})$.

Next, to see the equivalence (iii) \Leftrightarrow (iv), observe that (iii) is equivalent to $S^*S \leq \lambda S^*\overline{A_0}$ and $\text{dom } S \subseteq \text{dom } \overline{A_0}$, which is equivalent to $S = X\overline{A_0} + S_{\text{mul}}$, by Theorem 3.1. This last identity can be rewritten in the form

$$(T^*)^{-1} = Y^{-1}\overline{B_0}^{-1} \dot{+} (\{0\} \times \ker T^*) \Leftrightarrow T^* = \overline{B_0}Y \dot{+} (\ker T^* \times \{0\}).$$

Furthermore, it follows from Theorem 3.1 that $\ker S^* \oplus \text{mul } S = \ker X$, which means that $\text{mul } T \oplus \ker T^* = \text{mul } Y$. □

The following result is analogous to the first items of Remark 2.8 and Remark 2.9.

Remark 3.4. Under the assumptions of Theorem 3.3, one obtains from (3.16) the following implication

$$(3.19) \quad \begin{array}{c} T \subseteq YB_0^* \text{ has a solution } Y^{-1} \in B^+(\mathfrak{K}) \\ \Downarrow \\ B^*T \geq \mu \overline{B_0}B_0^*, \quad \mu = \frac{1}{\|Y^{-1}\|}. \end{array}$$

In particular, if (the graph of) B_0 is a core of B^* , i.e. $\overline{B_0} = B^*$, then the converse implication in (3.19) holds, i.e.,

$$(3.20) \quad \begin{array}{c} T \subseteq YB \text{ has a solution } Y^{-1} \in B^+(\mathfrak{K}) \\ \Updownarrow \\ B^*T \geq \mu B^*B, \quad \mu = \frac{1}{\|Y^{-1}\|}. \end{array}$$

Remark 3.5. The equivalence stated in (3.20) can also be established under conditions different from those given in Theorem 3.3, in particular, when $B^*T \geq 0$ is selfadjoint, $\text{mul } T \subseteq \ker(B_s)^*$ and $\text{dom } B^*T$ is a core for the operator part B_s . To see this, it suffices reverse the roles of B and T in Remark 2.9 and observe that $T \subseteq YB \Leftrightarrow Y^{-1}T \subseteq B$ for any $Y^{-1} \in B^+(\mathfrak{K})$.

The following corollary treats a particular case of Theorem 3.3, the case of densely defined operators with dense ranges.

Corollary 3.6. *Let \mathfrak{K} be a complex Hilbert space and $T, B : \mathfrak{H} \rightarrow \mathfrak{K}$ be closed densely defined linear operators such that B^*T is selfadjoint and $\overline{B_0} = B^*$. If $\overline{\text{ran}} T = \overline{\text{ran}} B = \mathfrak{H}$ then the following assertions are equivalent for some $\eta > 0$:*

- (i) $T^*T \geq \eta B^*T \geq 0$ with $\text{ran } T^* \subseteq \text{ran } B^*$;
- (ii) $T^* = B^*Y$ has a solution $Y^{-1} \in B^+(\mathfrak{K})$.

4. THE CLASS $\mathcal{L}_l^{+2}(\mathfrak{H})$ AND THE REVERSED INEQUALITY

In this section, the emphasis will be on the following class

$$(4.1) \quad \mathcal{L}_l^{+2}(\mathfrak{H}) := \{T = AB; A^{-1} \in B^+(\mathfrak{H}), B = B^* \geq 0\}$$

as a modification of the class $\mathcal{L}_l^{+2}(\mathfrak{H})$. In (4.1), A is invertible, i.e., belongs to the class $Gl(\mathfrak{H})$ of closed densely defined injective and onto operators on \mathfrak{H} . Denote by $GL(\mathfrak{H})$ the set of all bounded everywhere defined invertible operators and, moreover, one has $GL^+(\mathfrak{H}) := GL(\mathfrak{H}) \cap B^+(\mathfrak{H})$ and $Gl^+(\mathfrak{H}) := \{S \in Gl(\mathfrak{H}); S = S^* \geq 0\}$.

Note that $S \in GL^+(\mathfrak{H})$ if and only if S is a nonnegative selfadjoint operator with $\text{ran } S = \mathfrak{H}$.

The simpler case where T belongs to $\mathcal{L}_l^{+2}(\mathfrak{H}) \cap \mathcal{L}_l^{+2}(\mathfrak{H})$ will be treated in Section 4.1 and involves weak similarity as well as similarity to nonnegative selfadjoint operators, while the general case, treated in Section 4.2, is rather connected to quasi-affinity and quasi-similarity to nonnegative selfadjoint operators. These notions will appear to be significantly related to the reversed inequality treated in Section 3.2.

4.1. Similarity and \mathcal{W} -similarity to $S = S^* \geq 0$. An operator $T \in LO(\mathfrak{H})$ is said to be \mathcal{W} -similar to $S \in LO(\mathfrak{H})$ if there exists $G \in GL(\mathfrak{H})$ such that

$$GT \subseteq SG.$$

If $TG = GS$ then T is similar to S . In particular, if T is similar to a normal operator then it is said to be scalar; see [3, 10, 15] for general background on scalar operators. The next proposition characterizes \mathcal{W} -similarity to nonnegative selfadjoint operators with non-empty resolvent set.

Proposition 4.1. *Let $T \in CO(\mathfrak{H})$ be a densely defined linear operator. Then, the following assertions are equivalent:*

- (i) T is \mathcal{W} -similar to a nonnegative selfadjoint operator S with $\rho(T) \neq \emptyset$;
- (ii) $XT = T^*X$, where $X \in GL^+(\mathfrak{H})$ and $\sigma(T) \subseteq \mathbb{R}^+$;
- (iii) $T = X_1B_1$ with $X_1 \in GL^+(\mathfrak{H})$ and $B_1 = B_1^* \geq 0$ (respectively, $T^* = X_2B_2$ with $X_2 \in GL^+(\mathfrak{H})$ and $B_2 = B_2^* \geq 0$);
- (iv) $T = BX$, where $B = B^* \geq 0$ and $X \in GL^+(\mathfrak{H})$ (respectively, $T^* = B'Y$ with $B' = (B')^* \geq 0$ and $Y \in GL^+(\mathfrak{H})$);
- (v) There exist $W, Z \in GL^+(\mathfrak{H})$ such that $TW = (TW)^* \geq 0$ (respectively, $ZT = (ZT)^* \geq 0$);
- (vi) T is a scalar operator and $\sigma(T) \subseteq \mathbb{R}^+$.

If one of the above conditions holds, then

$$(4.2) \quad \overline{\text{ran } T} \dot{+} \ker T = \mathfrak{H}.$$

Proof. (i) \Rightarrow (ii) Since T is \mathcal{W} -similar to a nonnegative operator S , there exists $G \in GL(\mathfrak{H})$ such that $GT \subseteq SG$. Hence,

$$GTG^{-1} \subseteq S = S^* \subseteq (GTG^{-1})^*,$$

which shows that GTG^{-1} is symmetric. As $G \in GL(\mathfrak{H})$, one then has $\rho(GTG^{-1}) = \rho(T) \neq \emptyset$, and therefore

$$(4.3) \quad GTG^{-1} = S = (GTG^{-1})^* = G^{-1*}T^*G^*.$$

This yields that

$$G^*GT = T^*G^*G,$$

and the statement follows by taking $X = G^*G \in B^+(\mathfrak{H})$. Furthermore, (4.3) shows that $\sigma(T) = \sigma(GTG^{-1}) = \sigma(S) \subseteq \mathbb{R}^+$.

(ii) \Rightarrow (iii) Let $T = X^{-1}T^*X$, $X \in GL^+(\mathfrak{H})$, and assume that $\sigma(T) \subseteq \mathbb{R}^+$. Then, $X^{\frac{1}{2}}TX^{-\frac{1}{2}} = X^{-\frac{1}{2}}T^*X^{\frac{1}{2}}$, and hence

$$(X^{\frac{1}{2}}TX^{-\frac{1}{2}})^* = X^{-\frac{1}{2}}T^*X^{\frac{1}{2}} = X^{\frac{1}{2}}TX^{-\frac{1}{2}}.$$

Since $X \in GL(\mathfrak{H})$ and $\sigma(T) \subseteq \mathbb{R}^+$, it follows that $\sigma(X^{\frac{1}{2}}TX^{-\frac{1}{2}}) = \sigma(T) \subseteq \mathbb{R}^+$, and therefore

$$(4.4) \quad S := X^{\frac{1}{2}}TX^{-\frac{1}{2}} = X^{-\frac{1}{2}}T^*X^{\frac{1}{2}} = S^* \geq 0.$$

Thus,

$$B_1 := X^{\frac{1}{2}}SX^{\frac{1}{2}} = X^{\frac{1}{2}}(X^{\frac{1}{2}}TX^{-\frac{1}{2}})X^{\frac{1}{2}} = XT = B_1^* \geq 0$$

and $T = X^{-1}B_1 = X_1B_1$, where $X_1 = X^{-1}$ is invertible.

To prove the remaining statement, observe from (4.4) that

$$B_2 := X^{-\frac{1}{2}}SX^{-\frac{1}{2}} = X^{-\frac{1}{2}}(X^{-\frac{1}{2}}T^*X^{\frac{1}{2}})X^{-\frac{1}{2}} = X^{-1}T^* = B_2^* \geq 0$$

and $T^* = XB_2$ with X invertible.

The equivalence (iii) \Leftrightarrow (iv) is direct.

(iii) \Rightarrow (v) Assume that $T = X_1B_1$ with $X_1 \in GL^+(\mathfrak{H})$. Then, for $Z := X_1^{-1} \in GL^+(\mathfrak{H})$ one has $ZT = B_1 = B_1^* = (ZT)^* \geq 0$.

Similarly, $T^* = X_2B_2 \in \mathcal{L}_l^{+2}(\mathfrak{H})$ with $X_2 \in GL^+(\mathfrak{H})$ and $W := X_2^{-1} \in GL^+(\mathfrak{H})$ yield that $T = B_2X_2$ and $TW = B_2 = B_2^* = (TW)^* \geq 0$.

(v) \Rightarrow (vi) Assume that there exists $W \in GL^+(\mathfrak{H})$ such that $S_0 = TW = S_0^* \geq 0$. Then, $W^{-\frac{1}{2}}S_0W^{-\frac{1}{2}} \geq 0$, $W^{\frac{1}{2}} \in GL(\mathfrak{H})$, and one has

$$W^{\frac{1}{2}}(W^{-\frac{1}{2}}S_0W^{-\frac{1}{2}}) = TW^{\frac{1}{2}}.$$

Similarly if $Z \in GL^+(\mathfrak{H})$ such that $S_1 = ZT = S_1^* \geq 0$, then $Z^{-\frac{1}{2}} \in GL(\mathfrak{H})$, $Z^{-\frac{1}{2}}S_1Z^{-\frac{1}{2}} = (Z^{-\frac{1}{2}}S_1Z^{-\frac{1}{2}})^* \geq 0$ and

$$TZ^{-\frac{1}{2}} = Z^{-\frac{1}{2}}(Z^{-\frac{1}{2}}S_1Z^{-\frac{1}{2}}).$$

In both cases, one concludes that T is similar to a nonnegative selfadjoint operator and $\sigma(T) = \sigma(S_0) = \sigma(S_1) \subseteq \mathbb{R}^+$. By definition T is a scalar operator.

(vi) \Rightarrow (i) If T is a scalar operator with $\sigma(T) \subseteq \mathbb{R}^+$ then it is easily seen that it is similar, and hence \mathcal{W} -similar to a nonnegative selfadjoint operator.

If one of the above conditions holds, then T is similar to $S = S^*$ and (4.2) follows directly from $\overline{\text{ran}} S \dot{+} \ker S = \mathfrak{H}$. \square

Remark 4.2. Note that in Proposition 4.1, the similarity and the \mathcal{W} -similarity to a nonnegative selfadjoint operator are the same, cf. (4.3).

4.2. $\mathcal{L}_l^{+2}(\mathfrak{H})$ and quasi-affinity to $S = S^* \geq 0$. Recall that in Section 2.2, the quasi-affinity of T^* to a nonnegative selfadjoint operator S is described through elements T in $\mathcal{L}_l^{+2}(\mathfrak{H})$. Unlike in the case of bounded operators, the quasi-affinity of T^* to S does not imply the one of T . The latter will rather be described by elements of $\mathcal{L}_l^{+2}(\mathfrak{H})$ in the following theorem.

Theorem 4.3. *Let $T \in CO(\mathfrak{H})$ be densely defined. Then the following statements are equivalent:*

- (i) T is quasi-affine to some $S = S^* \geq 0$;
- (ii) $T \subseteq BA \in \mathcal{L}_l^{+2}(\mathfrak{H})$;
- (iii) there exists a quasi-affinity $X \in B^+(\mathfrak{H})$ such that

$$0 \leq XT \subseteq T^*X.$$

Proof. (i) \Rightarrow (ii) Assume that T is G -quasi-affine to $S = S^* \geq 0$ and fix $A_0 := G^*SG$ and $B := (G^*G)^{-1}$. Then $B^{-1} \in B^+(\mathfrak{H})$ and the inclusion $GT \subseteq SG$ implies that $B^{-1}T = G^*GT \subseteq G^*SG = A_0 \geq 0$ with $\text{dom } A_0 = \mathfrak{H}$. Let now $A_F = A_F^* \geq 0$ be the Friedrichs extension of A_0 . Then

$$(4.5) \quad B^{-1}T \subseteq A_0 \subseteq A_F,$$

and, therefore, $T \subseteq BA_F \in \mathcal{L}_l^{+2}(\mathfrak{H})$.

(ii) \Rightarrow (iii) Since $T \subseteq BA \in \mathcal{L}_l^{+2}(\mathfrak{H})$ it follows that $B^{-1}T \subseteq A \subseteq (B^{-1}T)^* = T^*B^{-1}$. Hence, for $X = B^{-1} \in B^+(\mathfrak{H})$ one has $0 \leq XT \subseteq T^*X$.

(iii) \Rightarrow (i) Let $A_0 := XT \geq 0$. Then A_0 is densely defined. Let $A = A^* \geq 0$ be a selfadjoint extension of A_0 . Then clearly

$$(4.6) \quad XT \subseteq A \subseteq (XT)^* = T^*X.$$

Now let $S_0 := X^{-\frac{1}{2}}AX^{-\frac{1}{2}} \geq 0$. Then by multiplying (4.6) from the left and right by $X^{-\frac{1}{2}}$ and one obtains

$$(4.7) \quad 0 \leq X^{\frac{1}{2}}TX^{-\frac{1}{2}} \subseteq S_0$$

Since $\overline{\text{dom } TX^{-\frac{1}{2}}} = \overline{\text{ran } X} = \mathfrak{H}$ one concludes that S_0 is densely defined operator with the Friedrichs extension $S_F = (X^{-\frac{1}{2}}A^{\frac{1}{2}})A^{\frac{1}{2}}X^{-\frac{1}{2}}$. Multiplying (4.7) from the right by $X^{\frac{1}{2}}$ one gets

$$(4.8) \quad X^{\frac{1}{2}}T \subseteq S_0X^{\frac{1}{2}} \subseteq S_FX^{\frac{1}{2}},$$

which proves the quasi-affinity of T to S_F . \square

Remark 4.4. In the proof of Theorem 4.3, (i), $A_0 = G^*SG = (G^*S^{\frac{1}{2}})S^{\frac{1}{2}}G$ with $\overline{\text{dom } A_0} = \mathfrak{H}$. Hence $\overline{\text{dom } S^{\frac{1}{2}}G} = \mathfrak{H}$ and one has

$$(4.9) \quad 0 \leq A_0 = G^*SG \subseteq (S^{\frac{1}{2}}G)^*S^{\frac{1}{2}}G = \overline{G^*S^{\frac{1}{2}}S^{\frac{1}{2}}G} = A_F,$$

where A_F is the Friedrichs extension of A_0 . The proof also works for any nonnegative selfadjoint extension of A_0 , respectively, S_0 (see (4.8)).

The following result is the analog of Corollary 2.16. It shows a connection between the reversed inequality and quasi-affinity to a nonnegative selfadjoint operator.

Corollary 4.5. *Let $T \in CO(\mathfrak{H})$ be a densely defined operator let $S = S^* \geq 0$. If T is G -quasi-affine to S such that $\rho(A_FT) \neq \emptyset$, then*

$$T^*T \geq \frac{1}{\lambda}A_FT \geq 0$$

for some $\lambda > 0$, where A_F is given in (4.9).

Proof. Since T is G -quasi-affine to S one obtains from Theorem 4.3 that $T \subseteq BA_F$ where $A_F = A_F^* \geq 0$ and $B^{-1} = G^*G \in B^+(\mathfrak{H})$. Hence $B^{-1}T \subseteq A_F$ and $A_F \subseteq (B^{-1}T)^* = T^*B^{-1}$. Consequently,

$$A_FT \subseteq T^*B^{-1}T = T^*B^{-\frac{1}{2}}B^{-\frac{1}{2}}T \subseteq T^*B^{-\frac{1}{2}}(T^*B^{-\frac{1}{2}})^* =: F \geq 0.$$

Since $F = F^*$, it follows that A_FT is symmetric. On the other hand, $\rho(A_FT) \neq \emptyset$ by assumption, and therefore

$$A_FT = T^*B^{-1}T \leq \|B^{-1}\|T^*T \Rightarrow T^*T \geq \frac{1}{\lambda}A_FT \quad \text{with } \lambda = \|B^{-1}\|. \quad \square$$

Note that a small adjustment to item (i) of Theorem 4.3 allows T to be written as the product of two nonnegative, in general, unbounded linear operators motivating the following result.

Proposition 4.6. *Let $T \in CO(\mathfrak{H})$ be densely defined. Then, the following are equivalent:*

- (i) T is G -quasi-affine to some $S = S^* \geq 0$ such that $\overline{\text{dom } G^* S^{\frac{1}{2}} S^{\frac{1}{2}} G} \subseteq \text{dom } T$;
- (ii) $T = BA \in \mathcal{L}_l^{+2}(\mathfrak{H})$ and $\text{dom } T = \text{dom } A$;
- (iii) there exists a quasi-affinity $X \in B^+(\mathfrak{H})$ such that

$$(4.10) \quad XT = T^*X \geq 0.$$

Proof. (i) \Rightarrow (ii) Using the same argument as in the proof of Theorem 4.3 combined with Remark 4.4 one obtains that $B^{-1}T \subseteq A_F = \overline{G^* S^{\frac{1}{2}} S^{\frac{1}{2}} G}$, cf. (4.9). Now the assumption $\text{dom } A_F \subseteq \text{dom } T$ shows that

$$B^{-1}T = A_F.$$

Hence $T = BA_F \in \mathcal{L}_l^{+2}(\mathfrak{H})$ and $\text{dom } T = \text{dom } A_F$.

(ii) \Rightarrow (iii) Observe that $B^{-1}T \subseteq A$ and since $\text{dom } T = \text{dom } A$ one obtains

$$A = B^{-1}T = A^* = T^*B^{-1}.$$

Now take $X = B^{-1} \in B^+(\mathfrak{H})$ to get (4.10).

(iii) \Rightarrow (i) By assumption $A = XT \geq 0$ is selfadjoint. Now proceed as in the proof of Theorem 4.3. Then the operator

$$(4.11) \quad S_0 = X^{-\frac{1}{2}}AX^{-\frac{1}{2}} = X^{\frac{1}{2}}TX^{-\frac{1}{2}} \geq 0$$

is densely defined where its Friedrichs extension S_F satisfies (4.8) and T is quasi-affine to S_F . Multiplying (4.8) from the left by $X^{\frac{1}{2}}$ gives

$$XT = X^{\frac{1}{2}}S_0X^{\frac{1}{2}} \subseteq X^{\frac{1}{2}}S_FX^{\frac{1}{2}} \subseteq E_F,$$

where $E_F = \overline{X^{\frac{1}{2}}S_F^{\frac{1}{2}}S_F^{\frac{1}{2}}X^{\frac{1}{2}}}$ denotes the Friedrichs extension of $X^{\frac{1}{2}}S_FX^{\frac{1}{2}}$. Consequently, $XT = E_F$ and $\text{dom } E_F = \text{dom } T$, as required. \square

It is worth noticing that the quasi-affinity of T together with that of T^* gives raise to a new notion defined below, which will be characterized in Lemma 4.8.

Definition 4.7. [17, Definition 2.1] $T \in LO(\mathfrak{H})$ is said to be *quasi-similar* to $S \in LO(\mathfrak{H})$ if there exist two quasi-affinities $G_1, G_2 \in B(\mathfrak{H})$ such that

$$G_1T \subseteq SG_1 \quad \text{and} \quad G_2S \subseteq TG_2.$$

The next lemma contains a duality property of the quasi-affinity and characterizes the quasi-similarity to a nonnegative selfadjoint operator.

Lemma 4.8. *Let $T, S \in CO(\mathfrak{H})$ be a densely defined operators. Then the following statements are equivalent:*

- (i) T is G -quasi-affine to $S \Leftrightarrow S^*$ is G^* -quasi-affine to T^* ;
- (ii) T is quasi-similar to $S = S^* \Leftrightarrow T$ and T^* are quasi-affine to $S = S^*$.

Proof. (i) Let $S \in CO(\mathfrak{H})$. Then T is G -quasi-affine to $S \Leftrightarrow GT \subseteq SG \Leftrightarrow G^*S^* \subseteq T^*G^*$, i.e. S^* is G^* -quasi-affine to T^* .

- (ii) If T is quasi-similar to S , then there are two quasi-affinities $G_1, G_2 \in B(\mathfrak{H})$ such that $G_1T \subseteq SG_1$ and $G_2S \subseteq TG_2$. This shows that T is G_1 -quasi affine to S and, by (i), T^* is G_2^* -quasi-affine to S . Conversely, if T and T^* are quasi-affine to S , then it follows from (i) that there are two quasi-affinities $G_1, G_2 \in B(\mathfrak{H})$ with the property that $G_1T \subseteq SG_1$ and $G_2^*S \subseteq SG_2^*$. As G_2^* is a quasi-affinity, one concludes that T is quasi-similar to S . \square

The next result is now a consequence of Lemma 4.8, Theorem 4.3 and Proposition 4.6.

Corollary 4.9. *Let $T \in CO(\mathfrak{H})$ be a densely defined operator and let $S = S^* \geq 0$. If T is quasi-similar to S then there exist $T_1 \in \mathcal{L}_l^{+2}(\mathfrak{H})$ and $T_2 \in \mathcal{L}_l^{+2}(\mathfrak{H})$ such that*

$$T_1 \subseteq T \subseteq T_2.$$

In particular, if T and T^ are respectively G_1 and G_2 -quasi-affine to S such that $\text{dom}(G_1^*S^{\frac{1}{2}}S^{\frac{1}{2}}G_1) \subseteq \text{dom} T \subseteq \text{dom}(G_2^{*-1}S^{\frac{1}{2}}S^{\frac{1}{2}}G_2^{*-1})$, then*

$$(4.12) \quad T \in \mathcal{L}_l^{+2}(\mathfrak{H}) \cap \mathcal{L}_l^{+2}(\mathfrak{H}).$$

Proof. Assume that T is quasi-similar to $S = S^* \geq 0$. Then, by Lemma 4.8, there exist two quasi-affinities $G_1, G_2 \in B(\mathfrak{H})$ such that T and T^* are respectively G_1 and G_2 quasi-affine to S . A direct application of Proposition 2.14 and Theorem 4.3 implies the existence of $A_1 \in B^+(\mathfrak{H})$, $B_1 = B_1^* \geq 0$, $B_2^{-1} \in B^+(\mathfrak{H})$ and $A_2 = A_2^* \geq 0$ such that

$$(4.13) \quad \mathcal{L}_l^{+2}(\mathfrak{H}) \ni A_1B_1 \subseteq T \subseteq B_2A_2 \in \mathcal{L}_l^{+2}(\mathfrak{H}).$$

Now, assume that $\text{dom}(\overline{G_1^*S^{\frac{1}{2}}S^{\frac{1}{2}}G_1}) \subseteq \text{dom} T \subseteq \text{dom}(\overline{G_2^{*-1}S^{\frac{1}{2}}S^{\frac{1}{2}}G_2^{*-1}})$. Then equalities hold in (4.13) by Propositions 2.17 and 4.6, which proves (4.12). \square

4.3. Quasi-affinity and quasi-similarity to $S \in B^+(\mathfrak{H})$. It is worth noticing that if T is \mathcal{W} -similar or similar to a bounded nonnegative operator $S \in B^+(\mathfrak{H})$, then also T itself is bounded. In this case, its similarity to S is already dealt with in [7, Theorem 3.1]. The focus is therefore on quasi-affinity and quasi-similarity.

Proposition 4.10. *Let $T \in LO(\mathfrak{H})$ be a densely defined operator. Then the following statements are equivalent:*

- (i) T is G -quasi-affine to $S \in B^+(\mathfrak{H})$;
- (ii) $\overline{GTG^{-1}} = G^{-1*}T^*G^* \in B^+(\mathfrak{H})$ for a quasi-affinity $G \in B(\mathfrak{H})$;
- (iii) $\overline{X^{\frac{1}{2}}TX^{-\frac{1}{2}}} = X^{-\frac{1}{2}}T^*X^{\frac{1}{2}} \in B^+(\mathfrak{H})$ for a quasi-affinity $X \in B^+(\mathfrak{H})$.

In this case, $T \subseteq BA$, where $A, B^{-1} \in B^+(\mathfrak{H})$. Moreover, there exists a quasi-affinity $X \in B^+(\mathfrak{H})$ such that

$$(4.14) \quad T^*X = \overline{XT} \in B^+(\mathfrak{H}).$$

Proof. (i) \Rightarrow (i) Observe that the inclusion $GT \subseteq SG$ implies that $GTG^{-1} \subseteq S$. Since $S \in B(\mathfrak{H})$ and $\overline{\text{dom}} GTG^{-1} = \mathfrak{H}$ one concludes that

$$(4.15) \quad S = (GTG^{-1})^* = (G^{-1})^*T^*G^* = \overline{GTG^{-1}} \in B^+(\mathfrak{H}).$$

(ii) \Rightarrow (i) Fix $S_0 := \overline{GTG^{-1}}$. Then, $S_0 \in B^+(\mathfrak{H})$ and $GT = GTG^{-1}G \subseteq \overline{GTG^{-1}}G \subseteq S_0G$, as required.

The equivalence (ii) \Leftrightarrow (iii) follows directly from Lemma 2.13.

Now, assume that T is G -quasi-affine to $S \in B^+(\mathfrak{H})$. Then $G^*GT \subseteq G^*SG =: A \in B^+(\mathfrak{H})$, and hence $T \subseteq BA$ with $B^{-1} := G^*G \in B^+(\mathfrak{H})$. To see (4.14), observe from (iii) that for $M := X^{-\frac{1}{2}}T^*X^{\frac{1}{2}} \in B^+(\mathfrak{H})$ one has $X^{\frac{1}{2}}MX^{\frac{1}{2}} \subseteq T^*X$, which yields that

$$\overline{XT} = (T^*X)^* = (X^{\frac{1}{2}}MX^{\frac{1}{2}})^* = X^{\frac{1}{2}}MX^{\frac{1}{2}} \in B^+(\mathfrak{H}). \quad \square$$

The following theorem is the optimal analogue of Corollary 4.5 in the context of the reversed inequality.

Theorem 4.11. *Let $T \in CO(\mathfrak{H})$ be a densely defined operator. If T is G -quasi-affine to some $S \in B^+(\mathfrak{H})$ then exists $A \in B^+(\mathfrak{H})$ such that*

$$(4.16) \quad T^*T \geq \frac{1}{\lambda} \overline{AT}$$

for some $\lambda > 0$.

Proof. Observe from the inclusion $GT \subseteq SG$ that $T \subseteq G^{-1}SG$ and $G^*GT \subseteq G^*SG =: A \in B^+(\mathfrak{H})$. Hence $A = (G^*GT)^* = T^*G^*G$ and one has

$$AT \subseteq (G^*SG)G^{-1}SG \subseteq G^*S^2G = G^*S(G^*S)^* \in B^+(\mathfrak{H}).$$

This implies that $G^*S^2G = (AT)^* = \overline{AT} \in B^+(\mathfrak{H})$ and for all $x \in \text{dom } T = \text{dom } AT = \text{dom } T^*G^*GT \subseteq \text{dom } \overline{AT}$ one has

$$\begin{aligned} \langle \overline{AT}^{\frac{1}{2}}x, \overline{AT}^{\frac{1}{2}}x \rangle &= \langle ATx, x \rangle = \langle T^*G^*GTx, x \rangle = \langle G^*GTx, Tx \rangle \\ &\leq \|G^*G\| \langle Tx, Tx \rangle \\ &= \|G^*G\| \langle (T^*T)^{\frac{1}{2}}x, (T^*T)^{\frac{1}{2}}x \rangle. \end{aligned}$$

This completes the argument. \square

Note that in the particular case where $AT = \overline{AT}$, Theorem 4.11 ultimately reduces to the bounded operator setting since $\text{dom } T = \text{dom } AT = \mathfrak{H}$, thereby justifying the inequality (4.16). This section is ended with a recapitalization joining the classes $\mathcal{L}_l^{+2}(\mathfrak{H})$ and $\mathcal{L}_l^{+2}(\mathfrak{H})$ together with quasi-similarity, and quasi-affinity to a bounded nonnegative operator.

Proposition 4.12. *Let $T \in CO(\mathfrak{H})$ be densely defined. Then the following assertions are equivalent:*

- (i) T is quasi-similar to $S \in B^+(\mathfrak{H})$.
- (ii) T and T^* are quasi-affine to $S \in B^+(\mathfrak{H})$.
- (iii) $\overline{X_1^{\frac{1}{2}}T^*X_1^{-\frac{1}{2}}} = X_1^{-\frac{1}{2}}TX_1^{\frac{1}{2}} \in B^+(\mathfrak{H})$ and $\overline{X_2^{\frac{1}{2}}TX_2^{-\frac{1}{2}}} = X_2^{-\frac{1}{2}}T^*X_2^{\frac{1}{2}} \in B^+(\mathfrak{H})$ for some quasi-affinities $X_1, X_2 \in B^+(\mathfrak{H})$.

In this case, there exist $A_i, B_i^{-1} \in B^+(\mathfrak{H})$, $i = 1, 2$ such that $T_1 = A_1B_1$, $T_2 = B_2A_2$ and

$$T_1 \subseteq T \subseteq T_2.$$

Moreover, $\overline{X_2T}, \overline{X_1T^*} \in B^+(\mathfrak{H})$.

Proof. The proof follows immediately from a combination of Lemma 4.8 and Proposition 4.10. \square

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Article II



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Three equivalent conditions for spectral decomposable linear relations

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Abstract The spectral decomposability of a closed linear relation T on a complex Banach space is demonstrated through three new characterisations: The first two are expressed in terms of the extended Bishop and decomposition properties while the third one is given by means of the coinduced operator of T and its local spectral subspaces. This has been achieved through the intensive study of the properties of the last mentioned subspaces as well as the ER-SVEP.

Mathematics Subject Classification 47A06 · 47A10 · 47A11 · 47A25 · 47B40

1 Introduction

The concept of decomposability plays a crucial role in the local spectral theory of linear operators on complex Banach spaces. It was defined for the first time for bounded linear operators by C.Foiaş [9] in 1963. Then, in 1968 it was generalized for closed linear operators by F.- H. Vasilescu [15] to become "S-residual decomposability" where S is a subset of $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. It was only in 2022 that this concept was extended to include a more general case, the case of multi-valued linear operators or linear relations, and was then defined as the concept of "extended spectral decomposability" in [3].

Of special interest are the different characterisations of decomposability. For example, in 2000, K.B.Laursen and M.M.Neumann [12] managed to characterise this concept for a bounded linear operator T_0 on a complex Banach space X in Theorem 1.2.29 as follows:

$$T_0 \text{ is decomposable} \Leftrightarrow T_0 \text{ has } (\delta) \text{ and } (C) \Leftrightarrow T_0 \text{ has } (\beta) \text{ and } (\delta), \quad (1.1)$$

where (δ) , (C) , and (β) stand for the properties of decomposition, Dunford and Bishop, respectively. They also established the following equivalence in Theorem 1.2.23:

$$T_0 \text{ is decomposable} \Leftrightarrow T_0 \text{ has } (C) \text{ and } \sigma(T_0/X_{T_0}(F)) \subseteq \overline{\sigma(T_0) \setminus F} \text{ for every} \\ \text{closed subset } F \text{ of } \mathbb{C}, \quad (1.2)$$

where $X_{T_0}(F)$ and $T_0/X_{T_0}(F)$ denote the local spectral subspace of T_0 with respect to F and the induced operator by T_0 on the quotient space $X/X_{T_0}(F)$, respectively.

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Our aim in this paper is to set the latest equivalences in a more general framework, the framework of linear relations. In other words, we are going to reformulate the equations (1.1) and (1.2) for a closed linear relation T and obtain the following new equivalences:

$$\begin{aligned}
 T \text{ is extended spectral decomposable} &\Leftrightarrow T \text{ has the properties } (\mathcal{E}_C) \text{ and } (\mathcal{E}_\delta) \\
 &\Leftrightarrow T \text{ has the properties } (\mathcal{E}_\beta) \text{ and } (\mathcal{E}_\delta)
 \end{aligned}
 \tag{1.3}$$

and,

$$\begin{aligned}
 &T \text{ is extended spectral decomposable} \\
 &\quad \Updownarrow \\
 &T \text{ has the property } (\mathcal{E}_C) \text{ and } \widetilde{\sigma}(T/\widetilde{X}_T(F)) \subseteq \overline{\widetilde{\sigma}(T) \setminus F} \text{ for all closed subsets } F \text{ of } \mathbb{C}_\infty.
 \end{aligned}
 \tag{1.4}$$

Having such multiple equivalent descriptions ensures flexibility in choosing the most convenient framework or approach when describing the decomposability of a linear relation. This leads to a deeper understanding of the relation’s structure and properties.

Regarding (1.3), it is worth mentioning that the properties (\mathcal{E}_C) and (\mathcal{E}_β) stand for the extended Dunford and Bishop properties, respectively, and they have actually been proven to be two necessary conditions for the extended spectral decomposability of T ; See [3]. However, the property (\mathcal{E}_δ) is new and it will be introduced as the extended decomposition property and proven as a necessary condition for the extended spectral decomposability as well. In this way, the two direct implications of (3) will be guaranteed, and it only remains to deal with the indirect ones.

As for equivalence (1.4), $\widetilde{X}_T(F)$, $T/\widetilde{X}_T(F)$ as well as $\widetilde{\sigma}(T)$ denote the extended local spectral subspace of T at F , the coinduced operator by T on $\widetilde{X}_T(F)$ as well as the extended spectrum of T , respectively. Here, it is very important to note that the converse direction of (1.4) was the purpose of [4]. So, our task will be to focus on how far we can solve the remaining direction.

Our paper is organized as follows. In Sect. 2, we will first recall some results related to the theory of linear relations already studied in [5,7,8]. Next, we are going to develop several properties of one of the necessary conditions of extended spectral decomposability, called the ER-SVEP, which will be quite useful for the proof of our main theorems. Finally, we will present several new results on the extended local spectral subspaces of T , which will be further intensified once connected with the ER-SVEP. As for Sect. 3, it will be the core of our work. We will divide it into two sub-parts: the purpose of Sect. 3.1 is to settle the first two characterizations of the extended spectral decomposability mentioned in (1.3), while Sect. 3.2 aims to reach the second characterization in (1.4).

All this illustrates how perfectly the concept of spectral decomposability has been generalized and well-developed for the field of linear relations.

2 Preliminaries

In this section, we shall start with some general notions on linear relations by referring to the works of R.Cross [8] and A.Baskakov [7]. Then, we will refer to [3] and focus on the local spectral theory of a closed linear relation T which is fundamental for the understanding of the concept of the extended spectral decomposability. More importantly, we will study different several properties of T involving the extended local spectral subspaces as well as the ER-SVEP.

Let X be a complex Banach space. According to R.Cross [8], a mapping T from X to the collection of nonempty subsets of X , whose domain is $\mathcal{D}(T) := \{x \in X; Tx \neq \emptyset\}$, is called a *linear relation* if it satisfies $\alpha Tx + \beta Ty = T(\alpha x + \beta y)$ for all $x, y \in \mathcal{D}(T)$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Note that T is entirely determined by its graph given by

$$G(T) = \{(u, v) \in X \times Y \mid u \in \mathcal{D}(T) \text{ and } v \in Tx\}.$$

Throughout this paper, the class of all linear relations and all bounded linear operators from X to X is denoted by $LR(X)$ and $End(X)$, respectively. Let us notice that $T \in End(X)$ whenever $T(0) = \{0\}$ and $\mathcal{D}(T) = X$.

The *inverse* of T is the relation T^{-1} defined by $G(T^{-1}) = \{(y, x) \in Y \times X \mid (x, y) \in G(T)\}$. We denote by $Ker(T) := T^{-1}(0)$ the *Kernel* of T . If $Ker(T) = \{0\}$, we say that T is *injective*. By $T(X_0) := \bigcup_{x_0 \in \mathcal{D}(T) \cap X_0} Tx_0$ and $Im(T) := T(X)$, we denote the range of a set $X_0 \subseteq X$ and the range of T , respectively. If $Im(T) = X$, then T is called *surjective*.

T is said to be *continuous* if $\|T\| := \|Q_T T\| < \infty$, where $Q_T : X \rightarrow X/\overline{T(0)}$ denotes the *natural quotient map* with domain X and kernel $\text{Ker}(Q_T) := \overline{T(0)}$. If $G(T)$ is closed, we say that T is *closed* and we write $T \in CR(X)$. We also write $T \in CR_c(X)$ if T is closed and continuous. When $T \in CR(X)$ is surjective and injective, it is called *invertible*.

The *product* and the *sum* of T and $U \in LR(X)$ are given by $G(TU) := \{(x, z) \in X \times X; (x, u) \in G(U) \text{ and } (u, z) \in G(T) \text{ for some } u \in X\}$ and $G(T + U) := \{(x, t + u) \mid (x, t) \in G(T), (x, u) \in G(U)\}$, respectively.

Let $T \in CR(X)$. Recall from [7] that the *resolvent set* of T is denoted by $\rho(T)$ and defined as the set of all $\eta \in \mathbb{C}$ for which $T - \eta I$ is invertible. For $\eta \in \rho(T)$, the *resolvent function* of T is the function $R(\eta, T)$ such that $R(\eta, T) = (T - \eta I)^{-1}$. The *spectrum* of T is the set $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

A linear operator T_0 on X is called a *selection* of T if $\mathcal{D}(T_0) = \mathcal{D}(T)$ and $T = T_0 + T - T$. In the next proposition, we will show that the spectrum of a linear relation may coincide with that of its selection in a particular case.

Proposition 2.1 *Let $T \in CR_c(X)$ be such that $X = \mathcal{D}(T) \oplus T(0)$ and let \tilde{P} be the bounded linear projection with the property that $\text{Ker}(\tilde{P}) = T(0)$ and $\text{Im}(\tilde{P}) = \mathcal{D}(T)$. Then, $\tilde{P}T \in \text{End}(\mathcal{D}(T))$ is a continuous linear selection of T . Moreover, we have $\sigma(T) = \sigma(\tilde{P}T)$.*

Proof Let $A = \tilde{P}T$. Then, $\mathcal{D}(A) = \mathcal{D}(T)$ and, for all $x \in \mathcal{D}(T)$, we have $Tx = Tx + T(0) = Tx + \tilde{P}^{-1}(0) = \tilde{P}^{-1}\tilde{P}(Tx) = \tilde{P}^{-1}(\tilde{P}\tilde{P})(Tx) = \tilde{P}^{-1}\tilde{P}(\tilde{P}T(x)) = \tilde{P}T(x) + \tilde{P}^{-1}(0) = Ax + T(0)$. Hence, A is a selection of T . Moreover, since T is continuous, it follows from [8, Corollary II.3.13] that $\|A\| = \|\tilde{P}T\| \leq \|\tilde{P}\| \|T\| < +\infty$. This proves the continuity of A . On the other hand, since $T \in CR_c(X)$, it follows from the closed graph theorem [8, Theorem III.4.2] that $\mathcal{D}(T)$ is closed, and therefore $A \in \text{End}(\mathcal{D}(T))$.

Proving $\sigma(T) = \sigma(\tilde{P}T)$ is equivalent to prove that $\rho(T) = \rho(\tilde{P}T)$. For doing so, let $\mu \in \rho(T)$ and consider $x \in \text{Ker}(\tilde{P}T - \mu I)$. Since $\tilde{P}T$ is a selection of T , it then follows that $0 = (\tilde{P}T - \mu)x \in (T - \mu I)x$, so $x \in \text{Ker}(T - \mu I) = \{0\}$. Now, consider $t \in \mathcal{D}(T) \subseteq X$. Then, there exists $t_0 \in \mathcal{D}(T)$ such that $t \in (T - \mu I)t_0 = (\tilde{P}T - \mu I)t_0 + T(0)$, and therefore $t - (\tilde{P}T - \mu I)t_0 \in T(0) \cap \mathcal{D}(T) = \{0\}$. Consequently, $t = (\tilde{P}T - \mu I)t_0 \in \text{Im}(\tilde{P}T - \mu I)$, which means that $\tilde{P}T - \mu I$ is surjective. We then conclude that $\mu \in \rho(\tilde{P}T)$. To see the other inclusion, let $\mu \in \rho(\tilde{P}T)$ and consider $y \in X$. Then, $y = y_0 + y_1$, where $(y_0, y_1) \in \mathcal{D}(T) \times T(0)$, and hence $y_0 \in \text{Im}(\tilde{P}T - \mu I)$. Thus, there exists $x_0 \in \mathcal{D}(T)$ such that $y_0 = (\tilde{P}T - \mu I)x_0$. This implies that

$$y = (\tilde{P}T - \mu I)x_0 + y_1 \in (\tilde{P}T - \mu I)x_0 + T(0) = (T - \mu I)x_0 \in \text{Im}(T - \mu I).$$

Consequently, $(T - \mu I)$ is surjective. Since $\text{Ker}(T - \mu I) \subseteq \text{Ker}(\tilde{P}T - \mu I) = \{0\}$, we then conclude that $T - \mu I$ is injective, and hence invertible. This means that $\mu \in \rho(T)$. \square

Given a relation $T \in CR(X)$ and a closed linear subspace Y_0 of X , we refer to [7] and define Y_0 (respectively, Y_1) as a *T -strongly invariant subspace* (respectively, *T -weakly invariant subspace*) if $R(\eta, T)Y_0 \subseteq Y_0$ for all $\eta \in \rho(T)$ (respectively, $T y_0 \cap Y_0 \neq \emptyset$ for all $y_0 \in \mathcal{D}(T) \cap Y_0$). The *strong restriction* of T to Y_0 is the relation $T|_s Y \in CR(Y)$ whose resolvent is $R(\cdot, T|_s Y) : \rho(T) \rightarrow \text{End}(Y)$ which satisfies $R(\cdot, T|_s Y)(\eta) := R(\eta, T)|_Y$ for all $\eta \in \rho(T)$. In general, a subspace $M \subseteq X$ is said to be *T -invariant* if $T(M) \subseteq M$.

Lemma 2.2 *Let $T \in CR(X)$ and let M be a linear subspace of X . Then, M is a T -weakly invariant linear subspace if and only if $T(M) \subseteq M + T(0)$.*

Let $T \in CR(X)$. Then, we notice from [5] that whenever $T(0) \neq \{0\}$ or, equivalently, T is not a linear operator, the point ∞ always belongs to the extended spectrum of T . Hence, it becomes very important to deal with the topology of the Riemann sphere $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ in order to study the spectral theory of T . Let us recall that the *extended spectrum* of T is a subset of \mathbb{C}_∞ defined as

$$\tilde{\sigma}(T) := \begin{cases} \sigma(T) & \text{if } T \in \text{End}(X) \\ \sigma(T) \cup \{\infty\} & \text{otherwise.} \end{cases}$$

Following [10], the neighborhoods in \mathbb{C}_∞ correspond exactly to the usual definitions of neighborhoods in \mathbb{C} adding the neighborhoods of ∞ which are of the form $V_\infty = \{v \in \mathbb{C} \mid |v| > r_0\} \cup \{\infty\}$, where $r_0 > 0$. In addition, according to [11], the open sets in \mathbb{C}_∞ are the usual open sets in \mathbb{C} and those of the form $V \cup \{\infty\}$, where $\mathbb{C} \setminus V$ is a compact set in \mathbb{C} . For an open set $U_\infty \subseteq \mathbb{C}_\infty$, we denote by $H^\sharp(U_\infty, X)$ the space of all analytic functions $f : U_\infty \rightarrow X$ and we define $P(U_\infty, X) := \{f \in H^\sharp(U_\infty, X) \mid f(\infty) = 0 \text{ if } \infty \in U_\infty\}$.



Now, being rather interested in the local spectral theory of T , we will turn to [3] and define the *extended local resolvent set* of T at a point w_0 , denoted by $\tilde{\rho}_T(w_0)$, as the set of all $\lambda \in \mathbb{C}_\infty$ for which there are an open neighborhood U_λ in \mathbb{C}_∞ of λ and an analytic function $\tilde{f}_{w_0,\lambda} \in P(U_\lambda, \mathcal{D}(T))$ such that $w_0 \in (\mu I - T)\tilde{f}_{w_0,\lambda}(\mu)$ for all $\mu \in U_\lambda \cap \mathbb{C}$. The *extended local spectrum* of T at $w_0 \in \mathbb{C}_\infty$ is given by

$$\tilde{\sigma}_T(w_0) := \mathbb{C}_\infty \setminus \tilde{\rho}_T(w_0).$$

For a set $F \subseteq \mathbb{C}_\infty$, the *extended local spectral subspace* of T at F is defined by

$$\tilde{X}_T(F) := \{w \in X \mid \tilde{\sigma}_T(w) \subseteq F\}.$$

If we suppose in addition that F is closed, then we define the *extended glocal spectral subspace* of T at F as the set $\tilde{\mathcal{X}}_T(F)$ of all $x \in X$ for which there exists $\tilde{\Psi} \in P(\mathbb{C}_\infty \setminus F, \mathcal{D}(T))$ such that $x \in (T - \mu I)\tilde{\Psi}(\mu)$ for all $\mu \in (\mathbb{C}_\infty \setminus F) \cap \mathbb{C}$.

Lemma 2.3 *Let $T \in CR(X)$ and X_0 be a T -strongly invariant linear subspace of X . If $T|_s X_0 \in End(X_0)$ and $\sigma(T|_s X_0) \subseteq F$, then $X_0 \subseteq \tilde{X}_T(F)$.*

Proof Let $x_0 \in X_0$ and consider $\lambda \in \mathbb{C}_\infty \setminus F$. By hypothesis, we obtain that $\lambda \in \tilde{\rho}(T|_s X_0)$, and therefore there exists a function \tilde{f}_{x_0} given by

$$\tilde{f}_{x_0} : \tilde{\rho}(T|_s X_0) \rightarrow \begin{matrix} \mathcal{D}(T|_s X_0) \\ \mu \rightarrow \begin{cases} R(\mu, T|_s X_0)x_0 & \text{if } \mu \neq \infty, \\ 0 & \text{if } \mu = \infty \in \tilde{\rho}(T|_s X_0) \end{cases} \end{matrix}$$

and verifies $x_0 = (T|_s X_0 - \mu I)\tilde{f}_{x_0}(\mu)$ for all $\mu \in \tilde{\rho}(T|_s X_0) \cap \mathbb{C}$. Since X_0 is a T -strongly invariant linear subspace, it follows from [3, Proposition 2.2] that $x_0 \in (T - \mu I)\tilde{f}_{x_0}(\mu)$ for all $\mu \in \tilde{\rho}(T|_s X_0) \cap \mathbb{C}$.

On the other hand, $T|_s X_0 \in End(X_0)$, so $\tilde{\sigma}(T|_s X_0) = \sigma(T|_s X_0)$. As $\sigma(T|_s X_0)$ is compact in \mathbb{C} , we deduce that $\tilde{\rho}(T|_s X_0)$ is open in \mathbb{C}_∞ . Hence, $\tilde{\rho}(T|_s X_0)$ becomes an open neighborhood of λ and $\tilde{f}_{x_0} \in P(\tilde{\rho}(T|_s X_0), \mathcal{D}(T|_s X_0))$. Moreover, we have $\mathcal{D}(T|_s X_0) \subseteq \mathcal{D}(T)$, which entails that $\tilde{f}_{x_0} \in P(\tilde{\rho}(T|_s X_0), \mathcal{D}(T))$. We then conclude that $\lambda \in \tilde{\rho}_T(x_0)$, and therefore $x_0 \in \tilde{X}_T(F)$. \square

Remark 2.4 Let $T \in CR(X)$ and let $F \subseteq \mathbb{C}_\infty$ be such that $T(\tilde{X}_T(F)) \subseteq \tilde{X}_T(F)$. Then,

$$(T - \lambda I)(\tilde{X}_T(F)) = \tilde{X}_T(F) \quad \text{for all } \lambda \in (\mathbb{C}_\infty \setminus F) \cap \mathbb{C}.$$

It is worth pointing out that the subsets mentioned above have already been well studied using the topology of \mathbb{C} by M. Mnif [13], which can be seen in the following definition.

Definition 2.5 Let $T \in CR(X)$ and let $w_0 \in X$.

- (i) The *local resolvent set* of T at w_0 is the set of all $\eta_0 \in \mathbb{C}$ for which there exist an open neighborhood U_{η_0} of η_0 in \mathbb{C} and a function $f_{w_0,\eta_0} \in H(U_{\eta_0}, \mathcal{D}(T))$ verifying

$$w_0 \in (T - \mu I)f_{a,\eta_0}(\mu) \quad \text{for all } \mu \in U_{\eta_0},$$

where $H(U, Y)$ denotes the space of all analytic functions from an open set $U \subseteq \mathbb{C}$ to a complex Banach space Y . The set $\mathbb{C}_{\rho_T}(w_0)$ is denoted by $\sigma_T(w_0)$ and called the *local spectrum* of T at w_0 .

- (ii) The *local spectral subspace* of T at a subset F of \mathbb{C} is given by $X_T(F) := \{x \in X \mid \sigma_T(x) \subseteq F\}$. The *glocal spectral subspace* of T at a closed set $S \subseteq \mathbb{C}$ is the set of all $x \in X$ for which there exists $h \in H(U, X)$ such that

$$x \in (T - \mu I)h(\mu) \quad \text{for all } \mu \in \mathbb{C} \setminus S.$$

There exists a very strong connection between the glocal spectral subspaces of $T \in CR(X)$ at a closed subset F of \mathbb{C} and the extended glocal ones at $F \cup \{\infty\}$. This can be translated in the following lemma.

Lemma 2.6 *Let $T \in CR(X)$ and let $F \subseteq \mathbb{C}$ be a closed set. Then,*

$$\tilde{\mathcal{X}}_T(F \cup \{\infty\}) = \tilde{\mathcal{X}}_T(F).$$

Proof For $F_\infty := F \cup \{\infty\}$, we have the following equivalences:

$$\begin{aligned} x \in \tilde{\mathcal{X}}_T(F) &\Leftrightarrow \exists f \in H(\mathbb{C} \setminus F, \mathcal{D}(T)); x \in (T - \mu I)f(\mu) \forall \mu \in \mathbb{C} \setminus F \\ &\Leftrightarrow \exists f \in H(\mathbb{C}_\infty \setminus F_\infty, \mathcal{D}(T)); x \in (T - \mu I)f(\mu) \forall \mu \in (\mathbb{C}_\infty \setminus F_\infty) \cap \mathbb{C} \\ &\Leftrightarrow \exists f \in P(\mathbb{C}_\infty \setminus F_\infty, \mathcal{D}(T)); x \in (T - \mu I)f(\mu) \forall \mu \in (\mathbb{C}_\infty \setminus F_\infty) \cap \mathbb{C} \\ &\Leftrightarrow x \in \tilde{\mathcal{X}}_T(F_\infty). \end{aligned}$$

□

It is worth mentioning that the result of the previous lemma also remains true for local spectral subspaces. In fact, this stems from the important link existing between the local spectral theory of T and its extended one, which will be expressed in the remaining part of this section.

Definition 2.7 [3, Definition 3.4] A relation $T \in CR(X)$ is said to have the *extended relatively single-valued extension property*, abbreviated ER-SVEP, at $\eta_0 \in \mathbb{C}_\infty$ if for every open connected neighborhood $U_{\eta_0} \subseteq \mathbb{C}_\infty$ of η_0 and for all functions $\varrho \in P(U_{\eta_0}, \mathcal{D}(T))$ which satisfy

$$0 \in (T - \mu I)\varrho(\mu) \text{ for all } \mu \in U_{\eta_0} \cap \mathbb{C}, \quad (2.1)$$

we have $\varrho(\mu) \in T(0) \cap Ker(T)$ for all $\mu \in U_{\eta_0}$. The relation T is said to have the ER-SVEP if T has the ER-SVEP at every $\eta \in \mathbb{C}_\infty$.

In the following, we will present further important results helpful for the characterization of extended spectral decomposability (see Theorem 3.10). We will in fact show that Proposition 1.2.20 of [12] can be generalized even for a linear relation $T \in CR_c(X)$. To this end, we shall start by studying the connection between the ER-SVEP and the extended spectrum of a strong restriction of T on an extended local spectral subspace.

Proposition 2.8 Let $T \in CR_c(X)$ be such that $\rho(T) \neq \emptyset$ and let S be a closed subset of \mathbb{C}_∞ such that $\tilde{\mathcal{X}}_T(S)$ is a T -invariant closed linear subspace. If T has the ER-SVEP, then

$$\tilde{\sigma}(T|_S \tilde{\mathcal{X}}_T(S)) \subseteq S \cup \{\infty\}. \quad (2.2)$$

Proof Let $S \subseteq \mathbb{C}_\infty$. Proving (2.2) is equivalent to proving that $\sigma(T|_S \tilde{\mathcal{X}}_T(S)) \subseteq S \cap \mathbb{C}$ or, equivalently, $\mathbb{C} \setminus (S \cap \mathbb{C}) \subseteq \rho(T|_S \tilde{\mathcal{X}}_T(S))$. To see this, let $\lambda \in \mathbb{C} \setminus (S \cap \mathbb{C}) = (\mathbb{C}_\infty \setminus S) \cap \mathbb{C}$ and consider an arbitrary $y \in \tilde{\mathcal{X}}_T(S)$. Then, in light of Remark 2.4, there exists an element $x \in \mathcal{D}(T) \cap \tilde{\mathcal{X}}_T(S)$ such that $y \in (T - \lambda I)x$. On the other hand, it follows from [4, Proposition 3.3] that $\tilde{\mathcal{X}}_T(S)$ is a T -strongly invariant linear subspace, so, by [3, Proposition 2.2], we have

$$\mathcal{D}(T) \cap \tilde{\mathcal{X}}_T(S) = \mathcal{D}(T|_S \tilde{\mathcal{X}}_T(S)). \quad (2.3)$$

Furthermore, [4, Proposition 3.1] implies that $T(0) \subseteq \tilde{\mathcal{X}}_T(S)$, which yields, by [3, Proposition 2.2], that $T|_S \tilde{\mathcal{X}}_T(S)(x) = T(x)$ and also

$$(T - \lambda I)x = (T|_S \tilde{\mathcal{X}}_T(S) - \lambda I)x. \quad (2.4)$$

This means that $y \in (T|_S \tilde{\mathcal{X}}_T(S) - \lambda I)x \subseteq Im(T|_S \tilde{\mathcal{X}}_T(S) - \lambda I)$, and hence $T|_S \tilde{\mathcal{X}}_T(S) - \lambda I$ is surjective. In view of [4, Lemma 3.1], proving the invertibility of $T|_S \tilde{\mathcal{X}}_T(S) - \lambda I$ then requires proving that this map belongs to $CR_c(\tilde{\mathcal{X}}_T(S))$, has the ER-SVEP and that $(T|_S \tilde{\mathcal{X}}_T(S) - \lambda I)(0) \cap Ker(T|_S \tilde{\mathcal{X}}_T(S) - \lambda I) = \{0\}$.

To this end, we first observe from the fact that $T|_S \tilde{\mathcal{X}}_T(S)$ is closed and from [1, Lemma 1] that $T|_S \tilde{\mathcal{X}}_T(S) - \lambda I \in CR(\tilde{\mathcal{X}}_T(S))$. Moreover, since $\tilde{\mathcal{X}}_T(S)$ is closed, a combination of (2.3) and [8, Theorem III.4.2] yields that $T|_S \tilde{\mathcal{X}}_T(S) - \lambda I \in CR_c(\tilde{\mathcal{X}}_T(S))$. On the other hand, as $\rho(T) \neq \emptyset$, it follows from [14, Lemma 6.1] that $Ker(T) \cap T(0) = \{0\}$. This entails, by (2.4), that

$$\begin{aligned} Ker(T|_S \tilde{\mathcal{X}}_T(S) - \lambda I) \cap (T|_S \tilde{\mathcal{X}}_T(S) - \lambda I)(0) &= Ker(T|_S \tilde{\mathcal{X}}_T(S) - \lambda I) \cap T(0) \\ &\subseteq Ker(T - \lambda I) \cap T(0) \subseteq Ker(T) \cap T(0) \\ &= \{0\}. \end{aligned}$$

As T has the ER-SVEP and $\tilde{\mathcal{X}}_T(S)$ is a T -strongly invariant linear subspace, we infer from assertions (i) and (iii) of [4, Lemma 3.1] that $T|_S \tilde{\mathcal{X}}_T(S) - \lambda I$ has the ER-SVEP. We then conclude that $T|_S \tilde{\mathcal{X}}_T(S) - \lambda I$ is invertible, as desired. Consequently, $\mathbb{C} \setminus (S \cap \mathbb{C}) \subseteq \rho(T|_S \tilde{\mathcal{X}}_T(S))$. □



Let us note that in [12, Proposition 1.2.20], the authors have also given a property that connects the SVEP with the operator $T_0/X_{T_0}(F)$ where $T_0 \in \text{End}(X)$ and $F \in \mathbb{C}$. In order to generalize this property for a closed linear relation $T \in CR(X)$, let us recall from [3] that, for a T -invariant closed linear subspace $M \subseteq X$, the *coinduced* linear operator $T/M : X/M \rightarrow X/M$ induced by T on the quotient space X/M is defined by:

$$T/M\tilde{x} := \tilde{T}\alpha = Q_M T(\alpha) \quad \text{for all } \tilde{x} \in \mathcal{D}(T/M) := \frac{\mathcal{D}(T) + M}{M},$$

where $x = \alpha + m$, $(\alpha, m) \in \mathcal{D}(T) \times M$ and $Q_M : X \rightarrow X/M$ is the natural quotient map. In the remaining work, we denote by $CR_c^\oplus(X)$ the space of all relations $T \in CR_c(X)$ for which $X = \mathcal{D}(T) \oplus T(0)$.

Corollary 2.9 *Let $T \in CR_c^\oplus(X)$. If T has the ER-SVEP then, for every closed set $S \subseteq \mathbb{C}_\infty$ for which $\tilde{X}_T(S)$ is closed, we have*

- (i) $\tilde{\sigma}(T|_S \tilde{X}_T(S)) \subseteq S \cup \{\infty\}$;
- (ii) $T/\tilde{X}_T(S)$ has the ER-SVEP.

Proof (i) Let $S \subseteq \mathbb{C}_\infty$ be such that $\tilde{X}_T(S)$ is closed. Then, it follows from [4, Proposition 3.4], assertion (2), that $\tilde{X}_T(S)$ is a T -invariant linear subspace. On the other hand, we remark from Proposition 2.1 that $\rho(T) \neq \emptyset$. Applying Proposition 2.8, we then conclude the desired result.
 (ii) Since $X = \mathcal{D}(T) \oplus T(0)$, it follows from assertion (i) of [4, Lemma 4.1] that $T/\tilde{X}_T(S) \in \text{End}(X/\tilde{X}_T(S))$. In addition, there is a bounded linear projection \tilde{P} such that $\text{Im}(\tilde{P}) = \mathcal{D}(T)$ and $\text{Ker}(\tilde{P}) = T(0)$. To prove that $T/\tilde{X}_T(S)$ has the ER-SVEP, let $\lambda_0 \in \mathbb{C}_\infty$, U_{λ_0} be an open neighborhood of λ_0 in \mathbb{C}_∞ and let $f \in P(U_{\lambda_0}, X/\tilde{X}_T(S))$ be such that

$$\bar{0} = (T/\tilde{X}_T(S) - \mu I) f(\mu) \quad \text{for all } \mu \in U_{\lambda_0} \cap \mathbb{C}. \tag{2.5}$$

Thus, $f|_{U_{\lambda_0} \cap \mathbb{C}} \in H(U_{\lambda_0} \cap \mathbb{C}, X/\tilde{X}_T(S))$. This ensures, by the surjectivity of the operator $Q_{\tilde{X}_T(S)} : X \rightarrow X/\tilde{X}_T(S)$ and by [12, Proposition 2.1.4], the existence of a function $\psi \in H(U_{\lambda_0} \cap \mathbb{C}, X)$ with the property that $f|_{U_{\lambda_0} \cap \mathbb{C}}(\mu) = Q_{\tilde{X}_T(S)} \psi(\mu) = Q_{\tilde{X}_T(S)} (\tilde{P}\psi(\mu) + (I - \tilde{P})\psi(\mu)) = Q_{\tilde{X}_T(S)} (\tilde{P}\psi)(\mu) = \tilde{P}\psi(\mu)$ for every $\mu \in U_{\lambda_0} \cap \mathbb{C}$. We then infer from (2.5) that

$$\bar{0} = (T/\tilde{X}_T(S) - \mu I) \overline{(\tilde{P}\psi)(\mu)} = Q_{\tilde{X}_T(S)} (T - \mu) \tilde{P}\psi(\mu) \quad \text{for all } \mu \in U_{\lambda_0} \cap \mathbb{C}.$$

Consequently,

$$(T - \mu) \tilde{P}\psi(\mu) \subseteq \tilde{X}_T(S) \quad \text{for all } \mu \in U_{\lambda_0} \cap \mathbb{C}. \tag{2.6}$$

Next, we claim that $\tilde{P}\psi(\mu) \in \tilde{X}_T(S)$ for all $\mu \in U_{\lambda_0} \cap \mathbb{C}$. To verify this, let us separate two cases.
 If $v \in U_{\lambda_0} \cap \mathbb{C} \cap S$, then we know from assertion (2) of [4, Proposition 3.4] that (2.6) implies immediately that $\tilde{P}\psi(v) \in \tilde{X}_T(S)$.
 If $v \in U_{\lambda_0} \cap \mathbb{C} \cap (\mathbb{C}_\infty \setminus S)$, then it follows from (2.2) that $v \in \rho(T|_S \tilde{X}_T(S))$. Hence, thanks to (2.6), the function

$$\tilde{h} : U_{\lambda_0} \cap \mathbb{C} \cap (\mathbb{C}_\infty \setminus S) \rightarrow \tilde{X}_T(S) \\ w \mapsto R(w, T|_S \tilde{X}_T(S))(T - wI) \tilde{P}\psi(w)$$

is well defined and belongs to $P(U_{\lambda_0} \cap \mathbb{C} \cap (\mathbb{C}_\infty \setminus S), \tilde{X}_T(S))$. On the other hand, as noted before, $\tilde{X}_T(S)$ is a T -strongly invariant linear subspace containing $T(0)$, so, by [3, Proposition 2.3], we obtain that

$$\begin{aligned} (T - vI)\tilde{h}(v) &= (T - vI) [R(v, T|_S \tilde{X}_T(S))(T - vI) \tilde{P}\psi(v)] \\ &= (T|_S \tilde{X}_T(S) - vI) R(v, T|_S \tilde{X}_T(S))(T - vI) \tilde{P}\psi(v) \\ &= (T - vI) \tilde{P}\psi(v). \end{aligned}$$

Consequently, $0 \in (T - vI)\tilde{g}(v)$, where $\tilde{g} := \tilde{h} - \tilde{P}\psi|_{U_{\lambda_0} \cap \mathbb{C} \cap (\mathbb{C}_\infty \setminus S)}$. Since v is arbitrary, $U_{\lambda_0} \cap (\mathbb{C}_\infty \setminus S) \cap \mathbb{C}$ is an open subset of \mathbb{C}_∞ and T has the ER-SVEP, we conclude that $\tilde{g} = 0$ locally in $U_{\lambda_0} \cap (\mathbb{C}_\infty \setminus S) \cap \mathbb{C}$. Thus, $\tilde{P}\psi(v) = \tilde{h}(v) \in \tilde{X}_T(S)$.

We then conclude that $\tilde{P}\psi(\mu) \in \tilde{X}_T(S)$ for all $\mu \in U_{\lambda_0} \cap \mathbb{C}$, as claimed. This entails that $f(\mu) = \overline{\tilde{P}\psi(\mu)} = \bar{0}$ for all $\mu \in U_{\lambda_0} \cap \mathbb{C}$. As $f \in P(U_{\lambda_0}, X/\tilde{X}_T(S))$, we obtain that $f = 0$, so $T/\tilde{X}_T(S)$ has the ER-SVEP. \square

It was proved in [3, Lemma 3.4] that, once T has the ER-SVEP, the local and the glocal spectral subspaces coincide under the condition $\rho(T) \neq \emptyset$. But even if we weaken that condition, we can also obtain the same result, which will be expressed in the following lemma.

Lemma 2.10 *Let $T \in CR(X)$ be a relation with ER-SVEP such that $\text{Ker}(T) \cap T(0) = \{0\}$. Then, for every closed subset F of \mathbb{C}_∞ , we have*

$$\tilde{X}_T(F) = \tilde{\mathcal{X}}_T(F).$$

3 Equivalent conditions for extended spectral decomposable linear relations

The remainder of this paper is devoted to the characterisation of the spectral decomposability for closed linear relations. More precisely, we are going to establish the equivalences (1.3) together with the equivalences (1.4). It is important to mention that the direct sense of (1.3) (respectively the indirect sense of (1.4)) has already been studied in [3] (respectively in [4]). Let us recall from [3, Definition 3.1] that a relation $T \in CR_c(X)$ with the property that $\rho(T) \neq \emptyset$ is called *extended spectral decomposable* if for every open cover $\{U, V\}$ of the extended complex plane \mathbb{C}_∞ by an open bounded subset U of \mathbb{C} and an open subset V of \mathbb{C}_∞ , there exist two T -strongly invariant closed linear subspaces M and N of X with the following properties:

- (i) $\tilde{\sigma}(T|_s M) = \sigma(T|_s M) \subseteq U$, $\tilde{\sigma}(T|_s N) \subseteq V$;
- (ii) $T(N) \subseteq N$ and $\mathcal{P}(N) \subseteq N$ for some bounded linear projection $\mathcal{P} : X \rightarrow X$ verifying $\text{Im}(\mathcal{P}) = \mathcal{D}(T)$;
- (iii) $X = M + N$.

3.1 Proof of the first two equivalent conditions (1.3)

To achieve the equivalences (1.3), we shall start by introducing the extended decomposition property (\mathcal{E}_δ) for closed linear relations. Note that this property has already been defined in the framework of bounded linear operators by means of glocal spectral subspaces; see for instance [12, Definition 1.2.28]. In our case, we will introduce it using the tools of the extended local spectral theory of linear relations, more precisely, by means of the extended glocal spectral subspaces.

Definition 3.1 A linear relation $T \in CR(X)$ has the *extended decomposition property* (\mathcal{E}_δ) if for every open cover $\{U, V\}$ of \mathbb{C}_∞ , where U is an open bounded subset of \mathbb{C} and V is an open subset of \mathbb{C}_∞ , we have

$$X = \mathcal{X}_T(\bar{U}) + \tilde{\mathcal{X}}_T(\bar{V}).$$

Thanks to Lemma 2.6, one can reformulate the above definition using only closed subsets of \mathbb{C}_∞ that contain the point ∞ or by using only closed subsets of \mathbb{C} . This can be translated in the following corollary.

Corollary 3.2 *Let $T \in CR(X)$. Then, T has the decomposition property (\mathcal{E}_δ) if and only if for every open cover $\{U, V\}$ of \mathbb{C}_∞ , where U is an open bounded subset of \mathbb{C} and V is an open subset of \mathbb{C}_∞ , we have*

$$X = \tilde{\mathcal{X}}_T(\bar{U} \cup \{\infty\}) + \tilde{\mathcal{X}}_T(\bar{V}) = \mathcal{X}_T(\bar{U}) + \mathcal{X}_T(\overline{V \cap \mathbb{C}}).$$

Now, we are close to showing that the property (\mathcal{E}_δ) is a necessary condition for extended spectral decomposability. To do this, we will first give a useful result linking the extended local spectral subspace of a relation T at an arbitrary subset of \mathbb{C}_∞ with the spectrum of its strong restriction $T|_s X_0$ on a strongly invariant linear subspace X_0 . More precisely, we are going to extend assertion (e) of [12, Proposition 1.2.16] from the case of bounded linear operators to that of multi-valued linear ones in order to obtain the following lemma.

Lemma 3.3 *Let $T \in CR(X)$ and let $S \subseteq \mathbb{C}$. If X_0 is a T -strongly invariant linear subspace of X such that $\sigma(T|_s X_0) \subseteq S$, then $X_0 \subseteq \tilde{X}_T(S \cup \{\infty\})$.*

Proof Let $x_0 \in X_0$ and let $\lambda \in \mathbb{C}_\infty \setminus (S \cup \{\infty\})$. Then, $\lambda \in \rho(T|_S X_0)$ and, for $U_\lambda := \rho(T|_S X_0)$ and $f_{\lambda, x_0} := R(\cdot, T|_S X_0)x_0$, we have

$$x_0 \in (T|_S X_0 - \mu)f_{\lambda, x_0}(\mu) \quad \text{for all } \mu \in U_\lambda.$$

In light of [3, Proposition 2.3], this yields that

$$x_0 \in (T - \mu)f_{\lambda, x_0}(\mu) \quad \text{for all } \mu \in U_\lambda.$$

As $U_\lambda = \rho(T|_S X_0)$ is an open neighborhood of λ in \mathbb{C} and $f_{\lambda, x_0} \in H(U_\lambda, \mathcal{D}(T))$, we conclude, by [4, Remark 3.1.], that $\lambda \in \rho_T(x_0) \subseteq \widetilde{\rho}_T(x_0)$. Hence, $x_0 \in \widetilde{X}_T(S \cup \{\infty\})$, which implies the desired inclusion. \square

Let us recall that we have already proved in [3] that the ER-SVEP as well as the extended Bishop’s property and the extended Dunford’s property are three necessary conditions for a relation $T \in CR_c(X)$ to be extended spectral decomposable. In the next theorem, we will prove that even the extended decomposition property is a necessary condition for T to become extended spectral decomposable.

Theorem 3.4 *Let $T \in CR(X)$ be such that $\rho(T)$ has an unbounded connected component. If T is an extended spectral decomposable linear relation, then T has the property (\mathcal{E}_δ) .*

Proof Suppose that $T \in CR_c(X)$ is an extended spectral decomposable linear relation, and let U_1 be an open bounded subset of \mathbb{C} and U_2 be an open subset of \mathbb{C}_∞ such that $\mathbb{C}_\infty = U_1 \cup U_2$. Then, the extended spectral decomposability of T implies the existence of two T -strongly invariant linear subspaces X_0 and X_1 of X verifying the following:

- (i) $X = X_0 + X_1$;
- (ii) $\sigma(T|_S X_0) \subseteq U_1 \subseteq \overline{U_1}$ and $\sigma(T|_S X_1) \subseteq \overline{U_2} \cap \mathbb{C}$.

Combining this with Lemma 3.3, we obtain that $X = \widetilde{X}_T(\overline{U_1} \cup \{\infty\}) + \widetilde{X}_T(\overline{U_2})$.

On the other hand, we know from [3, Theorem 3.3] that T has the ER-SVEP, so [3, Lemma 3.4] yields that $\widetilde{\mathcal{X}}_T(\overline{U_1} \cup \{\infty\}) = \widetilde{X}_T(\overline{U_1} \cup \{\infty\})$ and $\widetilde{\mathcal{X}}_T(\overline{U_2}) = \widetilde{X}_T(\overline{U_2})$. Thus, we have

$$X = \widetilde{\mathcal{X}}_T(\overline{U_1} \cup \{\infty\}) + \widetilde{\mathcal{X}}_T(\overline{U_2}).$$

We then conclude from Corollary 3.2 that T has the property (\mathcal{E}_δ) . \square

Remark 3.5 In the framework of bounded linear operators, we only have to add the Dunford’s property or the Bishop’s property in order to obtain the converse of Theorem 3.4. Yet, this is not that obvious in the general framework of closed linear relations. For this reason, we opted for a specific type of linear relations, belonging to the space $CR_c^\oplus(X)$. This will enable us to successfully extend the equivalences (a) \Leftrightarrow (c) of [12, Theorem 2.5.19] to the case of linear relations (see Theorem 3.10 for more details). Before doing so, we first have to go through some steps that can be expressed in Lemma 3.6, Lemma 3.7 and Lemma 3.8.

Recall from [3] that a relation $T \in CR_c(X)$ is said to have *extended Dunford’s property*, shortly property (\mathcal{E}_C) , if $\widetilde{X}_T(F)$ is closed for every closed subset $F \subseteq \mathbb{C}_\infty$. Throughout the rest of this paper, whenever $T \in CR_c^\oplus(X)$, we denote by P_T the bounded linear projection on X for which $\text{Ker}(P_T) = T(0)$ and $\text{Im}(P_T) = \mathcal{D}(T)$.

Lemma 3.6 *If $T \in CR_c^\oplus(X)$, then the following properties hold:*

- (i) $\sigma_T(x) = \sigma_{P_T T}(P_T x)$ for all $x \in X$;
- (ii) *If T has the property (\mathcal{E}_C) then T has the ER-SVEP.*

Proof We first observe from by Proposition 2.1 that $P_T T$ is a bounded linear selection of T on $\mathcal{D}(T)$.

- (i) Let $x \in X$. Then, we have the following equivalences:

$$\begin{aligned} \lambda_0 \in \rho_T(x) &\Leftrightarrow \exists \text{ an open neighborhood } V_0 \text{ of } \lambda_0 \text{ in } \mathbb{C} \text{ and a function } f_0 \in H(V_0, \mathcal{D}(T)) \\ &\quad \text{such that } x \in (T - \mu I)f_0(\mu) \text{ for all } \mu \in V_0 \\ &\Leftrightarrow \exists \text{ an open neighborhood } V_0 \text{ of } \lambda_0 \text{ in } \mathbb{C} \text{ and a function } f_0 \in H(V_0, \mathcal{D}(T)) \\ &\quad \text{such that } x = P_T x + (x - P_T x) \in (P_T T - \mu I)f_0(\mu) + T(0) \text{ for all } \mu \in V_0 \\ &\Leftrightarrow \exists \text{ an open neighborhood } V_0 \text{ of } \lambda_0 \text{ in } \mathbb{C} \text{ and a function } f_0 \in H(V_0, \mathcal{D}(T)) \\ &\quad \text{such that } P_T x = (P_T T - \mu I)f_0(\mu) \text{ for all } \mu \in V_0 \\ &\Leftrightarrow \lambda_0 \in \rho_{P_T T}(P_T x). \end{aligned}$$

This establishes that $\sigma_T(x) = \sigma_{P_T T}(P_T x)$.

- (ii) We first claim that $P_T T$ has the property (C), i.e. $X_{P_T T}$ is closed for every closed set $F \subseteq \mathbb{C}$. To see this, let S be a closed subset of \mathbb{C} and $(x_n)_{n \in \mathbb{N}} \subseteq X_{P_T T}(S)$ be a sequence for which $x_n \xrightarrow{n \rightarrow +\infty} x$ in X . This means, by (i) that

$$\sigma_T(x_n) = \sigma_{P_T T}(P_T x_n) = \sigma_{P_T T}(x_n) \subseteq S \text{ for all } n \in \mathbb{N}.$$

Hence, $(x_n)_{n \in \mathbb{N}} \subseteq X_T(S)$. On the other hand, we infer from [2, Remark 5.2] that $X_T(S) = \widetilde{X}_T(S \cup \{\infty\})$, which yields that $(x_n)_{n \in \mathbb{N}} \subseteq \widetilde{X}_T(S \cup \{\infty\})$. Since T has the property (\mathcal{E}_C) and $x_n \xrightarrow{n \rightarrow +\infty} x$, we obtain that $x \in \widetilde{X}_T(S \cup \{\infty\}) = X_T(S)$. Using (i), we then conclude that $P_T x \in X_{P_T T}(S)$. Furthermore, as $T \in CR_c(X)$, $(x_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}(T)$ and $x_n \xrightarrow{n \rightarrow +\infty} x$, the closed graph theorem [8, Theorem III.4.2] implies that $x \in \mathcal{D}(T)$. Thus, $x = P_T x \in X_{P_T T}(S)$, and therefore $P_T T$ has the property (C), as claimed. Consequently, $P_T T$ has the SVEP, by [12, Proposition 1.2.19]. Now, to prove that T has the ER-SVEP, consider $\eta_0 \in \mathbb{C}_\infty$, and let U_{η_0} be an open connected neighborhood of η_0 in \mathbb{C}_∞ and $f_{\eta_0} \in P(U_{\eta_0}, \mathcal{D}(T))$ be such that

$$0 \in (T - \lambda I)f_{\eta_0}(\lambda) \text{ for all } \lambda \in U_{\eta_0} \cap \mathbb{C}. \tag{3.1}$$

Then, $f_{\eta_0}|_{U_{\eta_0} \cap \mathbb{C}} \in H(U_{\eta_0} \cap \mathbb{C}, \mathcal{D}(T))$ and, for all $\lambda \in U_{\eta_0} \cap \mathbb{C}$, we have $0 = (P_T T - \lambda I)f_{\eta_0}(\lambda)$. Since $P_T T$ has the SVEP, it follows that $f_{\eta_0}|_{U_{\eta_0} \cap \mathbb{C}} = 0$. Moreover, as $f_{\eta_0} \in P(U_{\eta_0}, \mathcal{D}(T))$, we conclude that $f_{\eta_0} \equiv 0$, and therefore T has the ER-SVEP. \square

The following lemma demonstrates the first sufficient condition of the extended spectral decomposability quoted in (1.3).

Lemma 3.7 *If $T \in CR_c^\oplus(X)$ has the properties (\mathcal{E}_δ) and (\mathcal{E}_C) , then T is an extended spectral decomposable linear relation.*

Proof We first observe from Proposition 2.1 that $\rho(T) = \rho(P_T T) \neq \emptyset$. Now, let $\{U_1, U_2\}$ be an open cover of \mathbb{C}_∞ such that U_1 is a bounded subset of \mathbb{C} . Then, there exists an other open cover $\{U, V\}$ of \mathbb{C}_∞ such that $\overline{U} \subseteq U_1$ and $\overline{V} \subseteq U_2$. Since T has the property (\mathcal{E}_δ) , it follows from Corollary 3.2 that

$$X = \widetilde{X}_T(\overline{U} \cup \{\infty\}) + \widetilde{X}_T(\overline{V}).$$

In addition, as T has the property (\mathcal{E}_C) , it then follows from Lemma 3.6 that T has the ER-SVEP. Combining this with Lemma 2.10 and with the fact that $\text{Ker}(T) \cap T(0) = \{0\}$, we obtain that $X = \widetilde{X}_T(\overline{U} \cup \{\infty\}) + \widetilde{X}_T(\overline{V})$. On the other hand, Corollary 2.9 leads to the following inclusions

$$\tilde{\sigma}(T|_s \widetilde{X}_T(\overline{U} \cup \{\infty\})) \subseteq \overline{U} \cup \{\infty\} \subseteq U_1 \cup \{\infty\} \text{ and } \tilde{\sigma}(T|_s \widetilde{X}_T(\overline{V})) \subseteq \overline{V} \subseteq U_2. \tag{3.2}$$

Next, set $Y := \mathcal{D}(T) \cap \widetilde{X}_T(\overline{U} \cup \{\infty\})$ and $Z := \widetilde{X}_T(\overline{V})$, and claim that Y and Z verify assertions (i), (ii) and (iii) of the definition of the spectral decomposability mentioned in the beginning of Sect. 3 with $\mathcal{P} = P_T$. To do this, we first observe from the property (\mathcal{E}_C) , [4, Proposition 3.3] and [4, Proposition 3.4] that $\widetilde{X}_T(\overline{U} \cup \{\infty\})$ and $\widetilde{X}_T(\overline{V})$ are both T -invariant and T -strongly invariant linear subspaces of X . This easily implies that Y and Z are two T -strongly invariant linear subspaces with the property that $T(Z) \subseteq Z$. Furthermore, we have

$$X = Z + \widetilde{X}_T(\overline{U} \cup \{\infty\}) \cap (T(0) + \mathcal{D}(T)) = Z + T(0) + Y = Z + Y.$$

In view of (3.2), what remains to be shown are the following statements:

- (I) $P(Z) \subseteq Z$;
- (II) $\tilde{\sigma}(T|_s Y) = \sigma(T|_s Y) \subseteq U_1$.

In order to prove assertion (I), we notice that $Z = T(0) + Z \cap \mathcal{D}(T)$, so $P(Z) \subseteq P(T(0)) + P(Z \cap \mathcal{D}(T)) = Z \cap \mathcal{D}(T) \subseteq Z$. To establish assertion (II), we first observe, by [3, Proposition 2.3], that

$$T|_s Y(0) \subseteq T(0) \cap Y \subseteq T(0) \cap \mathcal{D}(T) = \{0\}.$$

On the other hand, we have

$$T(Y) \subseteq T(\widetilde{X}_T(\overline{U} \cup \{\infty\})) \subseteq \widetilde{X}_T(\overline{U} \cup \{\infty\}) \cap (\mathcal{D}(T) + T(0)) \subseteq Y + T(0),$$

so, by Lemma 2.2, Y is a T -weakly invariant linear subspace. Applying [3, Proposition 2.2], we then obtain that $\mathcal{D}(T|_s Y) = Y \cap \mathcal{D}(T) = Y$, and hence $T|_s Y \in \text{End}(Y)$. This means, by definition, that $\tilde{\sigma}(T|_s Y) = \sigma(T|_s Y)$. It remains to be shown that $\sigma(T|_s Y) \subseteq U_1$.

Claim. $\sigma(T|_s Y) \subseteq \sigma(T|_s \tilde{X}_T(\bar{U} \cup \{\infty\}))$.

To do so, let $\lambda \in \rho(T|_s \tilde{X}_T(\bar{U} \cup \{\infty\}))$ and consider $y \in Y \subseteq \tilde{X}_T(\bar{U} \cup \{\infty\})$. Then, there exists $x \in \mathcal{D}(T|_s \tilde{X}_T(\bar{U} \cup \{\infty\}))$ such that $y \in (T|_s \tilde{X}_T(\bar{U} \cup \{\infty\}) - \lambda I)x$. In view of [3, Proposition 2.3], we obtain that

$$y \in (T|_s \tilde{X}_T(\bar{U} \cup \{\infty\}) - \lambda I)x \subseteq (T - \lambda I)x. \tag{3.3}$$

Moreover, since $\tilde{X}_T(\bar{U} \cup \{\infty\})$ is a T -strongly and weakly invariant linear subspace, it follows from [3, Proposition 2.2] that $\mathcal{D}(T|_s \tilde{X}_T(\bar{U} \cup \{\infty\})) = \mathcal{D}(T) \cap \tilde{X}_T(\bar{U} \cup \{\infty\}) = Y$. This implies, by a combination of (3.3) and [3, Proposition 2.2], that

$$y \in (T - \lambda I)x = (T|_s Y - \lambda I)x + T(0).$$

Consequently, $y - (T|_s Y - \lambda I)x \in T(0) \cap Y \subseteq T(0) \cap \mathcal{D}(T) = \{0\}$, which proves that $y \in \text{Im}(T|_s Y - \lambda I)$. Hence $T|_s Y - \lambda I$ is surjective.

On the other hand, T has the ER-SVEP, $\text{Ker}(T) \cap T(0) = \{0\}$ and Y is a T -strongly invariant linear subspace, so, by assertion (iii) of [4, Lemma 3.1], we deduce that $T|_s Y$ has the ER-SVEP too. As $T|_s Y \in \text{End}(Y)$, we then conclude that $T|_s Y - \lambda I$ has the ER-SVEP, again from [4, Lemma 3.1]. In addition, we have

$$\text{Ker}(T|_s Y - \lambda I) \cap (T|_s Y - \lambda I)(0) = \text{Ker}(T|_s Y - \lambda I) \cap \{0\} = \{0\}.$$

Hence, we obtain from part (ii) of [4, Lemma 3.1] that $T|_s Y - \lambda I$ is invertible, and therefore $\lambda \in \rho(T|_s Y)$, as claimed.

A simple combination of the previous claim and (3.2) leads to the inclusion $\sigma(T|_s Y) \subseteq \sigma(T|_s \tilde{X}_T(\bar{U} \cup \{\infty\})) \subseteq U_1$, which completes the proof of (II). \square

Now, we are very close to reach the characterisation of the extended spectral decomposability of linear relations and establish the equivalence (1.3). Before getting into the details, we will start by establishing the link between the extended Bishop and Dunford properties in Lemma 3.8. For doing so, some terminology is recalled from [3]. Let $U \subseteq \mathbb{C}_\infty$ be an open set and let $T \in CR(X)$. Then, the relation T_U is defined as follows:

$$\begin{aligned} T_U : P(U, \mathcal{D}(T)) &\longrightarrow H^\sharp(U, X) \\ f &\longrightarrow T_U(f) = \{\varphi \in H^\sharp(U, X) \mid \varphi(\mu) \in (T - \mu I)f(\mu) \forall \mu \in U \cap \mathbb{C}\}. \end{aligned}$$

If $T \in CR_c(X)$, then it is said to have the *extended Bishop's property*, shortly the property (\mathcal{E}_β) , if, for every open set $O \subseteq \mathbb{C}_\infty$ and every $(t_n)_{n \in \mathbb{N}} \subseteq P(U, \mathcal{D}(T))$, $(h_n)_{n \in \mathbb{N}} \subseteq H^\sharp(U, X)$ satisfying $h_n(\eta) \in (T - \eta I)f_n(\eta)$, for all $\lambda \in U \cap \mathbb{C}$, following implication holds

$$h_n \xrightarrow{n \rightarrow +\infty} 0 \text{ in } H^\sharp(U, X) \implies t_n \xrightarrow{n \rightarrow +\infty} 0 \text{ in } P(U, \mathcal{D}(T)).$$

Lemma 3.8 *Let $T \in CR(X)$ be a relation such that $\text{Ker}(T) \cap T(0) = \{0\}$ and $\text{Im}(T_U)$ is closed for every open subset U of \mathbb{C}_∞ . If T has the property (\mathcal{E}_β) , then T has the property (\mathcal{E}_C) .*

Proof Since T has (\mathcal{E}_β) , it then follows from [3, Theorem 3.1] that T_U is injective. Thus, T has the ER-SVEP. Combining this with the fact that $\text{Ker}(T) \cap T(0) = \{0\}$ and Lemma 2.10, we deduce that $\tilde{X}_T(F) = \tilde{X}_T(F)$ for every closed set $F \subseteq \mathbb{C}_\infty$. As a consequence, one may proceed exactly as in the proof of [3, Theorem 3.3] and use the fact that $\text{Im}(T_U)$ is closed to complete the argument. \square

A simple combination of Lemma 3.8 and assertion (ii) of [3, Theorem 3.1] leads to the following result.

Corollary 3.9 *If $T \in CR_c^\oplus(X)$ has the property (\mathcal{E}_β) , then T has the property (\mathcal{E}_C) .*

We are finally in a position to present the main theorem of this section and solve (1.3). This is actually a generalization of the equivalences (a) \Leftrightarrow (c) \Leftrightarrow (d) of [12, Theorem 2.5.19] that described the decomposability of bounded linear operators.

Theorem 3.10 *For every relation $T \in CR_c^\oplus(X)$, the following statements are equivalent:*

- (i) T is extended spectral decomposable;
- (ii) T has the properties (\mathcal{E}_β) and (\mathcal{E}_δ) ;
- (iii) T has the properties (\mathcal{E}_C) and (\mathcal{E}_δ) .

Proof In light of Proposition 2.1, we deduce that $P_T T$ is a continuous linear selection of T verifying $P_T T \in \text{End}(\mathcal{D}(T))$ and $\rho(T) = \rho(P_T T)$, which implies that $\rho(T)$ contains an unbounded connected component. Hence, the implication (i) \Rightarrow (ii) is immediate from [3, Theorem 3.2] and Theorem 3.4. The implication (ii) \Rightarrow (iii) is clear from Lemma 3.8 and the remaining implication (iii) \Rightarrow (i) follows directly from Lemma 3.7. \square

The following example illustrates how, although a linear relation has both the extended Bishop and Dunford properties, it lacks the extended spectral decomposability property through the lack of (\mathcal{E}_δ) .

Example 3.11 Let $X = l^2(\mathbb{Z})$, S_r be the unilateral right shift operator on $l^2(\mathbb{N})$, and let $T \in LR(X)$ given by

$$G(T) = G(S_r) + (\{0\} \times l^2(\mathbb{N})^\perp).$$

Then, T has the properties (\mathcal{E}_β) and (\mathcal{E}_C) . Moreover, T is not extended spectral decomposable and does not have (\mathcal{E}_δ) .

Indeed, we first observe that $l^2(\mathbb{N})$ is a closed subspace of $l^2(\mathbb{Z})$ since $l^2(\mathbb{N}) = \{(x_n)_{n \in \mathbb{N}} \subseteq l^2(\mathbb{Z}); x_n = 0 \text{ for all } n < 0\}$. Moreover, from the construction of T one has $\mathcal{D}(T) = \mathcal{D}(S_r) = l^2(\mathbb{N}) = T(0)^\perp$, and since $S_r \in \text{End}(l^2(\mathbb{N}))$, it follows that T is closed. Hence, $T \in CR_c^\oplus(X)$. Now, to establish the property (\mathcal{E}_β) , let $U \subseteq \mathbb{C}_\infty$ be an open set, $(f_n)_{n \in \mathbb{N}} \subseteq P(U, \mathcal{D}(T))$, and $(g_n)_{n \in \mathbb{N}} \subseteq H^\sharp(U, X)$ be such that $g_n \xrightarrow{n \rightarrow +\infty} 0$ in $H^\sharp(U, X)$ and

$$g_n(\mu) \in (T - \mu I)f_n(\mu) \quad \text{for all } \mu \in U \cap \mathbb{C}. \quad (3.4)$$

Then, $g_n|_{U \cap \mathbb{C}} \xrightarrow{n \rightarrow +\infty} 0$ in $H(U \cap \mathbb{C}, \mathcal{D}(T))$ and $P_T g_n|_{U \cap \mathbb{C}}(\mu) = P_T(T - \mu I)f_n|_{U \cap \mathbb{C}}(\mu) = (S_r - \mu I)f_n|_{U \cap \mathbb{C}}(\mu)$ for all $n \in \mathbb{N}$ and $\mu \in U \cap \mathbb{C}$. On the other hand, S_r is an isometry, so it has the Bishop's property in the sense of [12, Definition 1.2.5] by [12, Proposition 1.6.7], and therefore $f_n|_{U \cap \mathbb{C}} \xrightarrow{n \rightarrow +\infty} 0$ in $H(U \cap \mathbb{C}, \mathcal{D}(T))$. Since $f \in P(U, \mathcal{D}(T))$, we then conclude that $f_n \xrightarrow{n \rightarrow +\infty} 0$ in $P(U, \mathcal{D}(T))$ and hence T has the property (\mathcal{E}_β) . A direct application of Corollary 3.9 yields that T has the property (\mathcal{E}_C) .

Next, suppose on the contrary that T has (\mathcal{E}_δ) . Then, for all open cover $\{U, V\}$ of \mathbb{C}_∞ , one has

$$X = \mathcal{X}_T(\overline{U}) + \widetilde{\mathcal{X}}_T(\overline{V}) = X_T(\overline{U}) + \widetilde{X}_T(\overline{V}).$$

Combining this with item (i) of Lemma 3.6, we conclude by Corollary 3.2 that

$$l^2(\mathbb{N}) = P_T X = X_{S_r}(\overline{U}) + X_{S_r}(\overline{V \cap \mathbb{C}}) \quad (3.5)$$

for all open cover $\{U, V\}$ of \mathbb{C}_∞ . But S_r has the Bishop's property as mentioned above, so [12, Propositions 1.2.19], [12, Proposition 3.3.2] together with (3.5) implies that

$$l^2(\mathbb{N}) = \mathcal{X}_{S_r}(\overline{U}) + \mathcal{X}_{S_r}(\overline{V \cap \mathbb{C}})$$

all open cover $\{U, V\}$ of \mathbb{C}_∞ . As a consequence, S_r has the decomposition property in the sense of [12, Definition 1.2.28]. Using again [12, Proposition 1.6.7], one concludes that S_r is invertible, contradiction.

Finally, the fact that S_r is an isometry together with Proposition 2.1 shows that

$$\rho(T) = \rho(S_r) = \overline{\mathbb{C} \setminus B_{\mathbb{C}}(0, 1)},$$

which means that $\rho(T)$ has an unbounded connected component. Applying Theorem 3.4, we then conclude that T is not an extended spectral decomposable linear relation.



3.2 The proof of the third equivalent condition (1.4)

As we have already done in the previous section, we will devote this part to giving a different characterization of the extended spectral decomposability, using this time coinduced linear operators. To better see things, we are going to seek the best way to extend the equivalence (a) \Leftrightarrow (b) described in [12, Theorem 1.2.23] from the case of bounded linear operators to that of multi-valued linear ones in order to reach the equivalences (1.4). It is worth recalling that the opposite implication of (1.4) was the purpose of [4], so all we have to do is to prove the other implication.

Lemma 3.12 *If $T \in CR_c^\oplus(X)$ is extended spectral decomposable then $\tilde{\sigma}(T/\tilde{X}_T(F)) \subseteq \overline{\tilde{\sigma}(T)\backslash F}$ for every closed subset F of \mathbb{C}_∞ .*

Proof We first observe from Theorem 3.10 that T has the property (\mathcal{E}_C) , so, by [4, Lemma 4.1], $T/\tilde{X}_T(F) \in \text{End}(X/\tilde{X}_T(F))$. This ensures that $\tilde{\sigma}(T/\tilde{X}_T(F)) = \sigma(T/\tilde{X}_T(F))$. Hence, we only need to prove that $\tilde{\sigma}(T/\tilde{X}_T(F)) \subseteq \overline{\tilde{\sigma}(T)\backslash F} \cap \mathbb{C}$. To do this, let $\lambda \in (\mathbb{C}_\infty \setminus \overline{\tilde{\sigma}(T)\backslash F}) \cap \mathbb{C}$ and choose an open cover $\mathbb{C}_\infty = \{U, V_\infty\}$ where U is an open bounded subset of \mathbb{C} and V_∞ is an open subset of \mathbb{C}_∞ such that $\lambda \in \mathbb{C}_\infty \setminus \overline{V_\infty}$ and $\tilde{\sigma}(T)\backslash F \subseteq \mathbb{C}_\infty \setminus \overline{U}$. From the extended spectral decomposability of T , we obtain T -strongly invariant linear subspaces X_0 and X_∞ for which

$$X = X_0 + X_\infty, \quad \tilde{\sigma}(T|_s X_0) = \sigma(T|_s X_0) \subseteq U \quad \text{and} \quad \tilde{\sigma}(T|_s X_\infty) \subseteq V_\infty.$$

It then follows from [6, Remark 2.7] that $\sigma(T|_s X_0) \subseteq \overline{U} \cap \sigma(T) \subseteq F$, so, by Lemma 2.3, we obtain $X_0 \subseteq \tilde{X}_T(F)$. Moreover, Lemma 3.3 ensures that $X_1 \subseteq \tilde{X}_T(\overline{V_\infty})$, and therefore

$$X/\tilde{X}_T(F) = Q_{\tilde{X}_T(F)}(X_0 + X_1) = Q_{\tilde{X}_T(F)}\tilde{X}_T(\overline{V_\infty}). \quad (3.6)$$

On the other hand, we have $\lambda \in \mathbb{C}_\infty \setminus \overline{V_\infty} = (\mathbb{C}_\infty \setminus \overline{V_\infty}) \cap \mathbb{C}$. We then obtain from [4, Proposition 3.4] that $(T - \lambda I)\tilde{X}_T(\overline{V_\infty}) = \tilde{X}_T(\overline{V_\infty})$. This entails, by (3.6) that

$$\begin{aligned} X/\tilde{X}_T(F) &= Q_{\tilde{X}_T(F)}(T - \lambda I)\tilde{X}_T(\overline{V_\infty}) = (T/\tilde{X}_T(F) - \lambda I)Q_{\tilde{X}_T(F)}\tilde{X}_T(\overline{V_\infty}) \\ &= (T/\tilde{X}_T(F) - \lambda I)(X/\tilde{X}_T(F)). \end{aligned}$$

Hence, $(T/\tilde{X}_T(F) - \lambda I)$ is surjective. In view of [4, Lemma 3.1], it suffices to prove that $(T/\tilde{X}_T(F) - \lambda I)$ has the ER-SVEP, since $(T/\tilde{X}_T(F) - \lambda I)(\overline{0}) \cap \text{Ker}(T/\tilde{X}_T(F) - \lambda I) = \{\overline{0}\}$. To do this, we first observe from Proposition 2.1 that $\rho(T) = \rho(P_T T)$, so $\rho(T)$ has an unbounded connected component. Since T is extended spectral decomposable, it then follows from [3, Theorem 3.3] that T has the ER-SVEP. Moreover, as T has the property (\mathcal{E}_C) , we infer from Corollary 2.9 that $T/\tilde{X}_T(F)$ has the ER-SVEP. Assertion (i) of [4, Lemma 3.1] then yields that $(T/\tilde{X}_T(F) - \lambda I)$ has the ER-SVEP, as desired. Consequently, $\lambda \in \rho(T/\tilde{X}_T(F))$. \square

We end this section by stating the third equivalent condition for the extended spectral decomposability of a closed linear relation in the following theorem.

Theorem 3.13 *Let $T \in CR_c^\oplus(X)$. Then, the following assertions are equivalent:*

- (i) T has the property (\mathcal{E}_C) and $\tilde{\sigma}(T/\tilde{X}_T(F)) \subseteq \overline{\tilde{\sigma}(T)\backslash F}$ for all closed subsets F of \mathbb{C}_∞ ;
- (ii) T is an extended decomposable linear relation.

Proof One may conclude the proof by combining [4, Theorem 4.1] with Lemma 3.6 and Lemma 3.12. \square

Remark 3.14 For the relation $T \in l^2(\mathbb{Z})$ described in Example 3.11, Theorem 3.13 shows that there exists a set F_0 in \mathbb{C}_∞ for which $\tilde{\sigma}(T/\tilde{X}_T(F_0)) \not\subseteq \overline{\tilde{\sigma}(T)\backslash F_0}$.

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Article III



Local Spectral Theory for Unbounded Product of Nonnegative Selfadjoint Operators

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Abstract

An unbounded product $T = AB$ of two nonnegative selfadjoint operators A and B , where $\rho(T) \neq \emptyset$ and A is bounded, is proved to have the single valued extension property (SVEP) and, more significantly, to be a \mathbb{C} -generalized scalar operator. This property plays a central role as, it not only implies the Dunford's property for the product AB , but it also allows any operator T that is quasi-affine to a nonnegative selfadjoint operator S to satisfy $\sigma(T) = \sigma(S)$. These results have required a deeper study of the spectral connection between operators S having SVEP and Dunford properties and operators T that are quasi-affine to S .

Keywords Selfadjoint operator · Nonnegative operator · Quasi-affinity · Quasi-similarity · Single valued extension property · Dunford's property

Mathematics Subject Classification 47B65 · 47B25 · 47A05 · 47A11

1 Introduction

The study of operator products of nonnegative selfadjoint operators on a complex Hilbert space \mathfrak{H} has always been a source of motivation for many mathematicians [9, 15]. In particular, the class

$$\mathcal{O}^{+2}(\mathfrak{H}) = \left\{ T = AB; A = A^* \geq 0 \text{ and } B = B^* \geq 0 \right\} \quad (1.1)$$

This paper is dedicated to our great friend and colleague Henk de Snoo on the occasion of his 80th birthday, wishing we can share many new happy moments in the coming years, too.

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has been considered by P.Y.Wu [32], in 1988, in the finite-dimensional setting where the following equivalence holds for a matrix M :

$$M \in \mathcal{O}^{+2}(\mathfrak{H}) \Leftrightarrow M \text{ is similar to a positive matrix.} \quad (1.2)$$

In 2020, M. Contino, M. A. Dritschel, A. Maestripieri, and S. Marcantognini [8] considered this problem for infinite-dimensional spaces, notably, when the elements A, B of (1.1) belong to the class $B^+(\mathfrak{H})$ of bounded nonnegative operators on \mathfrak{H} . Based on the following observation

$$T \text{ is similar to } S \in B^+(\mathfrak{H}) \Leftrightarrow \sigma(T) \subseteq \mathbb{R}^+ \text{ and } T \text{ is a scalar operator,} \quad (1.3)$$

they proved that, although $\sigma(T) \subseteq \mathbb{R}^+$ for any $T \in \mathcal{O}^{+2}(\mathfrak{H})$, similarity to a bounded nonnegative operator is no longer sufficient to cover the class $\mathcal{O}^{+2}(\mathfrak{H})$, as in (1.2). Even weaker conditions than similarity such as quasi-affinity to nonnegative operators are also shown to be insufficient to characterize that class. However, one has the following implication:

$$T \in \mathcal{O}^{+2}(\mathfrak{H}) \text{ and } T \text{ is quasi-affine to } S \in B^+(\mathfrak{H}) \Rightarrow \sigma(T) = \sigma(S), \quad (1.4)$$

which shows that quasi-affinity behaves like similarity in $\mathcal{O}^{+2}(\mathfrak{H})$. Motivated by (1.3), instead of scalar operators, a larger class would be considered and shown to include the class $\mathcal{O}^{+2}(\mathfrak{H})$, which is the class of generalized scalar operators. These operators play a crucial role, specially in the unbounded case, since they will give rise to different spectral properties of $\mathcal{O}^{+2}(\mathfrak{H})$, already proved in [8].

On the other hand, the latter class was characterized via Sebestyén theorem [23] as follows:

$$T \in \mathcal{O}^{+2}(\mathfrak{H}) \Leftrightarrow T^*T \leq XT \text{ has a bounded nonnegative solution } X \quad (1.5)$$

$$\Leftrightarrow T \text{ has an optimal pair,} \quad (1.6)$$

where a pair $(A, B) \in B^+(\mathfrak{H}) \times B^+(\mathfrak{H})$ is called optimal for $T = AB$, if

$$\ker T = \ker B \text{ and } \overline{\text{ran}} T = \overline{\text{ran}} A. \quad (1.7)$$

In the present paper, the above results will be extended to the general setting of closed unbounded operators. In particular, the emphasis will be on the following subclass of closed operators of $\mathcal{O}^{+2}(\mathfrak{H})$

$$\mathcal{L}_l^{+2}(\mathfrak{H}) = \left\{ T = AB; A \in B^+(\mathfrak{H}) \text{ and } B = B^* \geq 0 \right\}, \quad (1.8)$$

where B is in general a closed unbounded operator on \mathfrak{H} . Note that this class has already been introduced in [3] and characterized through an inequality similar to (1.5). However, unlike (1.6), a further condition is needed to characterize the optimality as

shown in the equivalence below:

$$T \in \mathcal{L}_{op}^{+2}(\mathfrak{H}) \Leftrightarrow T^*T \leq T^*Y \text{ has a solution } Y = Y^* \geq 0 \text{ such that} \\ \text{dom } T \subseteq \text{dom } Y \subseteq \text{dom } (T^*Y)^{\frac{1}{2}} \text{ and } \ker T = \ker Y.$$

Here $\mathcal{L}_{op}^{+2}(\mathfrak{H})$ is the class of operators in $\mathcal{L}_l^{+2}(\mathfrak{H})$ with optimal pairs, which will be considered in detail in Section 2.1. These operators are in fact needed to conclude properties shared by normal operators, such as having finite ascent and possessing the following description of the analytic spectral subspace of $T \in \mathcal{L}_{op}^{+2}(\mathfrak{H})$ at any $\lambda \in \mathbb{C}$:

$$\ker(T^* - \lambda I) = \mathfrak{H}_{T^*}(\{\lambda\}).$$

This will be seen in Section 3.1, where an operator $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ with $\rho(T) \neq \emptyset$ is shown to satisfy $\sigma(T) \subseteq \mathbb{R}^+$, and this gives rise to the first local spectral property: the single valued extension property (SVEP). More significantly, such an operator is proved to be a \mathbb{C} -generalized scalar operator, exactly as in the bounded case. The latter property is the core of Section 3.2 and constitutes a useful key for the study of further local spectral properties of T namely property (k) which is, by definition, a combination of the SVEP and Dunford properties; cf [25]. This property appears to be very important in the last part of this paper, which will be devoted to the quasi-affinity to nonnegative selfadjoint operators.

Recall from [21, 22] that a linear operator T_1 is said to be *G-quasi-affine*, shortly *quasi-affine*, to some linear operator T_2 if there exists a quasi-affinity $G \in B(\mathfrak{H})$ such that $GT_1 \subseteq T_2G$. Here $B(\mathfrak{H})$ denotes the class of bounded operators from \mathfrak{H} to \mathfrak{H} and a quasi-affinity is a bounded injective operator with dense range. In Section 3.3, the implication in (1.4) is improved and generalized not only for unbounded operators but also for operators that are not necessarily in $\mathcal{L}_l^{+2}(\mathfrak{H})$. For this, one first establishes that once a closed operator T is quasi-affine to an operator S with property (k) then

$$\sigma(S) \subseteq \sigma(T);$$

see Lemma 3.16. Note that this result is a generalization of [8, Lemma 3.7], where a stronger condition than property (k) condition was considered for S , i.e. $S \in B^+(\mathfrak{H})$. This is a useful tool to finally conclude that any operator T such that T and T^* have property (k) satisfies the following implication

$$T \text{ is quasi-affine to } S = S^* \geq 0 \Rightarrow \sigma(T) = \sigma(S) \subseteq \mathbb{R}^+;$$

see Corollary 3.17. In this case, the Hilbert space \mathfrak{H} can even be described via the local analytic spectral subspace of both S and T^* as follows:

$$\mathfrak{H} = \mathfrak{H}_S(\sigma(T)) = \mathfrak{H}_{T^*}(\sigma(S^*)). \quad (1.9)$$

Note that in the general case where T is quasi-affine to a closed operator T_2 , only the first equality in (1.9) is proved to hold for $S = T_2$; see Lemma 3.16.

2 Preliminaries

In this section $T \in LO(\mathfrak{H}, \mathfrak{K})$ stands for a linear operator from \mathfrak{H} to a complex Hilbert space \mathfrak{K} with domain $\text{dom } T$ and range $\text{ran } T$. If $\mathfrak{H} = \mathfrak{K}$ then $LO(\mathfrak{H}, \mathfrak{K}) = LO(\mathfrak{H})$. In addition, one writes $T \in CO(\mathfrak{H}, \mathfrak{K})$ if T is closed and $T \in CO(\mathfrak{H})$ if $\mathfrak{K} = \mathfrak{H}$. In this case, the resolvent set $\rho(T)$ of T is the set of all $\mu \in \mathbb{C}$ for which $R(\mu, T) := (T - \mu I)^{-1} \in B(\mathfrak{H})$. The set $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T . The *surjective spectrum* of $T \in CO(\mathfrak{H})$ is $\sigma_{sur}(T) := \mathbb{C} \setminus \rho_{sur}(T)$, where $\rho_{sur}(T)$ is the set of all $\mu \in \mathbb{C}$ for which $\text{ran}(T - \mu I) = \mathfrak{H}$.

An operator $T \in LO(\mathfrak{H})$ is said to be *symmetric* if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in \text{dom } T$. If $\langle Tx, x \rangle \geq 0$ for all $x \in \text{dom } T$, then T is *nonnegative*. It is *selfadjoint* when $\overline{\text{dom } T} = \mathfrak{H}$ and $T^* = T$. Note that if T is nonnegative and selfadjoint, then it admits a unique nonnegative selfadjoint square root which will be denoted by $T^{\frac{1}{2}}$; cf. [24, 31]. If $0 \leq T = T^* \in B(\mathfrak{H})$ then one writes $T \in B^+(\mathfrak{H})$.

Similar to the bounded case, the class $\mathcal{L}_l^{+2}(\mathfrak{H})$ has been treated and characterized in [3, Theorem 2.12] through Sebestyén inequality involving unbounded operators as follows.

Theorem 2.1 *Let T be a closed densely defined operator. Then, $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ if and only if $T^*T \leq T^*Y = YT$ admits a solution $Y = Y^* \geq 0$ such that $\text{dom } T \subseteq \text{dom } Y \subseteq \text{dom}(T^*Y)^{\frac{1}{2}}$.*

The following result appears in [3, Proposition 2.5] and shows that, exactly as in the finite-dimensional case, any element $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ has positive spectrum. It is revisited here for its relevance and applicability.

Corollary 2.2 *If $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ then there exists a nonnegative selfadjoint operator S such that T is pre-similar to S , i.e., $TG = GS$ for some $G \in B(\mathfrak{H})$. Moreover, if $\rho(T) \neq \emptyset$, then*

$$\sigma(T) \subseteq \mathbb{R}^+.$$

Lemma 2.3 ([3], Lemma 2.1) *Let $A \in B^+(\mathfrak{K})$ and $B \in CO(\mathfrak{H}, \mathfrak{K})$, and let $\alpha \in [0, 1]$. If AB^{**} is closed (closable), then $A^\alpha B^{**}$ is closed (closable) and*

$$(B^*A^\alpha)^* = A^\alpha B^{**}. \quad (2.1)$$

*Analogously, if $\ker A = \{0\}$ and $B^{**}A^{-1}$ is closed, then $B^{**}A^{-\alpha}$ is closed and*

$$(A^{-\alpha}B^*)^* = B^{**}A^{-\alpha}. \quad (2.2)$$

2.1 Optimal pair

Recall that in the bounded case, an operator $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ can be factorized as $T = AB$, where $A, B \in B^+(\mathfrak{H})$, $\ker B = \ker T$ and $\overline{\text{ran } T} = \overline{\text{ran } A}$. Such a pair (A, B) is called

optimal; see [8]. In the case of bounded operators, T and T^* play a symmetric role since A and B both are bounded. Hence one can conclude the following equivalences:

$$\begin{aligned} T \text{ has an optimal pair } &\Leftrightarrow T^*T \leq TX \text{ admits a solution } X \in B^+(\mathfrak{H}) \\ &\Leftrightarrow T \in \mathcal{L}_l^{+2}(\mathfrak{H}). \end{aligned} \quad (2.3)$$

see [8, Theorem 4.5]. The objective of this section is to develop suitable approaches for extending the equivalence (2.3) for unbounded operators; see Theorem 2.8.

Definition 2.4 If $T = AB \in \mathcal{L}_l^{+2}(\mathfrak{H})$, then the pair (A, B) is called *optimal* if

$$\overline{\text{ran}} T = \overline{\text{ran}} A \quad \text{and} \quad \ker T = \ker B. \quad (2.4)$$

The subclass of operators $T = AB \in \mathcal{L}_l^{+2}(\mathfrak{H})$ consisting of optimal pairs (A, B) , where $A \in B^+(\mathfrak{H})$ and $B = B^* \geq 0$, is denoted by $\mathcal{L}_{op}^{+2}(\mathfrak{H})$.

Example 2.5 Let

$$\begin{aligned} T : l^2(\mathbb{N}^*) &\rightarrow l^2(\mathbb{N}^*) \\ x = (x_1, x_2, \dots) &\mapsto (0, 2x_2, 0, 4x_4, \dots, 2nx_{2n}, \dots). \end{aligned}$$

Then, for $A(x_n) := (0, x_2, 0, x_4, \dots, x_{2n}, \dots)$ and $B(x_n) := (nx_n)_{n \in \mathbb{N}^*}$, one observes that $T = AB \in \mathcal{L}_l^{+2}(l^2(\mathbb{N}^*))$ although (A, B) is not an optimal pair for T , since $\ker T \not\subseteq \ker B = \{0\}$. However, $T = AT$ and T itself is also nonnegative and selfadjoint, so (A, T) can be chosen to be an optimal pair for T . Here, $\sigma(T) = \sigma(AB) = \sigma(BA) = 2\mathbb{N} \subseteq \mathbb{R}^+$.

The first step towards the construction of an optimal pair in (2.4) relies on the following theorem, where Sebestyén's inequality is characterized in the framework of unbounded operators. Note that the proof is already established in [3, Proposition 2.10] under the additional condition $\text{dom } T = \text{dom } (T^*B)^{\frac{1}{2}}$. Here, the latter condition is omitted and one still get the same result. For a completeness, the proof of (i) \Rightarrow (ii) is given.

Theorem 2.6 Let $T, B : \mathfrak{H} \rightarrow \mathfrak{K}$ be closed densely defined linear operators such that $T^*B = B^*T$ is selfadjoint. Then the following assertions are equivalent for some $0 \leq \lambda (= \|X\|)$:

- (i) $T^*T \leq \lambda T^*B$ and $\text{dom } B \subseteq \text{dom } (T^*B)^{\frac{1}{2}}$;
- (ii) $XB \subseteq T$ has a solution $X \in B^+(\mathfrak{K})$.

In this case $B^*XB = T^*B = B^*T$

and, moreover, X can be chosen such that $\ker T^* = \ker X$ with $\|X\| \leq \lambda$.

Proof Assume that (i) holds and rewrite the operator inequality in terms of forms as follows:

$$(Tf, Tf) \leq \lambda((T^*B)^{\frac{1}{2}}f, (T^*B)^{\frac{1}{2}}f) \quad \text{for all } f \in \text{dom } (T^*B)^{\frac{1}{2}}. \quad (2.5)$$

On the other hand, since

$$\operatorname{dom} T^*B \subseteq \operatorname{dom} B \subseteq \operatorname{dom} (T^*B)^{\frac{1}{2}} \subseteq \operatorname{dom} T, \quad (2.6)$$

(2.5) implies that

$$(Tf, Tf) \leq \lambda(T^*Bf, f) = \lambda(Bf, Tf) \text{ for all.} \quad (2.7)$$

Hence the form (T^*Bf, g) , $f, g \in \operatorname{dom} T^*B$, is a restriction of the closable form (Bf, Tg) , $f, g \in \operatorname{dom} B$; see [2, Theorem 2.2]. In fact, (2.6) shows that $\operatorname{dom} B$ is a core for the closed form on the righthand side on (2.5) as well as for the closure $(H^{\frac{1}{2}}f, H^{\frac{1}{2}}g)$, $f, g \in \operatorname{dom} H^{\frac{1}{2}}$ of the form (Bf, Tg) , $f, g \in \operatorname{dom} B$, so that these two closed forms coincide and, in particular, $T^*B = H$. The statement (ii) is now clear from [2, Theorem 2.1]. \square

A consequence of Theorem 2.6 yields immediately the first condition in (2.4). The next proposition treats this even in a more general case, where $T = AB \notin \mathcal{L}_1^{+2}(\mathfrak{H})$.

Proposition 2.7 *Let $T = AB$ be such that $A \in B^+(\mathfrak{H})$ and $B \in CO(\mathfrak{H})$. Then, $T^*B = B^*T$ is a nonnegative selfadjoint operator such that*

$$\operatorname{dom} (T^*B)^{\frac{1}{2}} = \operatorname{dom} T = \operatorname{dom} B. \quad (2.8)$$

Moreover, there exists $A_0 \in B^+(\mathfrak{H})$ such that $T = A_0B$ and $\overline{\operatorname{ran}} A_0 = \overline{\operatorname{ran}} T$. For this pair, $\operatorname{ran} A_0 \cap \ker B^* = \{0\}$.

Proof Let $T = AB$, where $A \in B^+(\mathfrak{H})$ and $B \in CO(\mathfrak{H})$. Since $AB = T$ is closed and $A \in B^+(\mathfrak{H})$, it follows from Lemma 2.3 that $A^{\frac{1}{2}}B$ is closed, and hence $T^*B = B^*AB = B^*T = (A^{\frac{1}{2}}B)^*(A^{\frac{1}{2}}B)$ is a nonnegative selfadjoint operator such that

$$\operatorname{dom} (T^*B)^{\frac{1}{2}} = \operatorname{dom} A^{\frac{1}{2}}B = \operatorname{dom} B = \operatorname{dom} T,$$

which completes the proof of (2.8). Now clearly

$$T^*T = (AB)^*(AB) \leq \|A\|B^*AB = \|A\|T^*B.$$

Together with Theorem 2.6 this leads to the existence of $A_0 \in B^+(\mathfrak{H})$ with the property that $T = A_0B$ and $\ker T^* = \ker A_0$. Thus,

$$\overline{\operatorname{ran}} T = (\ker T^*)^\perp = (\ker A_0)^\perp = \overline{\operatorname{ran}} A_0.$$

Now, let $x \in \operatorname{ran} A_0 \cap \ker B^*$. Then, there exists $y \in \mathfrak{H}$ such that $x = A_0y$ and $0 = B^*x = B^*A_0y = T^*y$. Thus, $y \in \ker T^* = \ker A_0$ and $x = A_0y = 0$. Consequently, $\operatorname{ran} A_0 \cap \ker B^* = \{0\}$. \square

For the next step related to the second condition in (2.4), it is important to note that Theorem 2.1 yields the following implication

$$T \in \mathcal{L}_{op}^{+2}(\mathfrak{H}) \Rightarrow T^*T \leq T^*Y = YT \text{ has a solution } Y = Y^* \geq 0 \text{ such that} \\ \text{dom } T \subseteq \text{dom } Y \subseteq \text{dom } (T^*Y)^{\frac{1}{2}}.$$

Contrary to the bounded case, the reversed implication is not true in the unbounded case. This is illustrated in the following theorem where the class $\mathcal{L}_{op}^{+2}(\mathfrak{H})$ is characterized.

Theorem 2.8 *Let $T \in CO(\mathfrak{H})$ be a densely defined linear operator. Then, the following statements are equivalent:*

- (i) $T \in \mathcal{L}_{op}^{+2}(\mathfrak{H})$;
- (ii) $T^*T \leq T^*Y$ has a solution $Y = Y^* \geq 0$ such that $\text{dom } T \subseteq \text{dom } Y \subseteq \text{dom } (T^*Y)^{\frac{1}{2}}$ and $\ker T = \ker Y$.

Proof The implication (i) \Rightarrow (ii) follows from Theorem 2.1. Now, assume (ii). Then, using Theorem 2.6, one constructs $X \in B^+(\mathfrak{H})$ such that $T = XY$ and $\overline{\text{ran}} T = \overline{\text{ran}} X$. Since $\ker Y = \ker T$, one concludes that (X, Y) is an optimal pair for T and $T \in \mathcal{L}_{op}^{+2}(\mathfrak{H})$. \square

A different approach to construct optimal pairs is treated in the next proposition. It addresses the general setting in which a closed product of closed operators A_0B_0 where $A_0 \in B(\mathfrak{H})$ gives rise to a new product AB which satisfies the second condition of optimality stated in (2.4).

Proposition 2.9 *Let $T = AB$ be a closed densely defined linear operator on \mathfrak{H} such that $A \in B(\mathfrak{H})$ and $B \in CO(\mathfrak{H})$. Then, there exist $A_1 \in B(\mathfrak{H})$ and $B_1 = B_1^* \geq 0$ such that*

$$T = A_1B_1 \quad \text{and} \quad \ker T = \ker B_1.$$

In this case $\text{dom } B_1 = \text{dom } B$. In particular, if $A \in \mathcal{L}_1^{+2}(\mathfrak{H})$ and $\ker B_1 = \ker T$ then $B_1 = B$ and $A_1 = A$. In this case, $T \in \mathcal{L}_{op}^{+2}(\mathfrak{H})$.

Proof Set $L_0 := \ker T \subseteq \text{dom } T = \text{dom } B$. Then the closed subspace L_0 decomposes $\text{dom } B = \text{dom } B \cap L_0^\perp \oplus L_0$, where \oplus stands for the orthogonal sum of the involved linear subspaces. Define $\widehat{B} := B \upharpoonright L_0^\perp$. Then, $\text{dom } \widehat{B} = \text{dom } B \cap L_0^\perp$ and $B = (\widehat{B}; B \upharpoonright L_0)$. Moreover, \widehat{B} and $B \upharpoonright L_0$ are closed and since L_0 is closed, $B \upharpoonright L_0$ is bounded by the closed graph theorem. Now, let $\widetilde{B} := (\widehat{B}; 0 \upharpoonright L_0)$. Then, $\text{dom } \widetilde{B} = \text{dom } \widehat{B} \oplus L_0 = \text{dom } B$ and the following equalities hold:

$$i) T = A\widetilde{B} \quad \text{and} \quad ii) \ker \widetilde{B} = \ker T.$$

To see i), one observes that

$$T = A(\widehat{B}; B \upharpoonright L_0) = (A\widehat{B}; AB \upharpoonright L_0) = (A\widehat{B}; T \upharpoonright \ker T) = (A\widehat{B}; 0) = A(\widehat{B}; 0 \upharpoonright L_0) = A\widetilde{B}.$$

For the proof of *ii*), one concludes from *i*) that $\ker \tilde{B} \subseteq \ker T$, and other inclusion follows immediately from the construction of \tilde{B} .

On the other hand, \tilde{B} is closed and $\overline{\text{dom } \tilde{B}} = \overline{\text{dom } B} = \mathfrak{H}$. Hence, there exists a partial isometry $U : \mathfrak{H} \rightarrow \overline{\text{ran } \tilde{B}}$, where $U \upharpoonright \ker \tilde{B} = 0$ and $U : \ker \tilde{B}^\perp \rightarrow \text{ran } \tilde{B}$ is an isometry, such that $\tilde{B} = U|\tilde{B}|$, which corresponds to the polar decomposition of \tilde{B} ; cf. [16]. As a result, $T = A_1 B_1$ with

$$A_1 = AU \in B(\mathfrak{H}) \quad \text{and} \quad B_1 = |\tilde{B}| \geq 0.$$

Furthermore, $\ker B_1 = \ker \tilde{B} = \ker T$. □

Corollary 2.10 *Let $T = A_1 B_0 \in CO(\mathfrak{H})$, where $A_1 \in B(\mathfrak{H})$ and $B_0 = B_0^* \geq 0$ with $\ker T = \ker B_0$. Then, $T \in \mathcal{L}_{op}^{+2}(\mathfrak{H})$ if and only if*

$$\text{dom } T \subseteq \text{dom } B_0 \subseteq \text{dom } (T^* B_0)^{\frac{1}{2}} \quad \text{and} \quad T^* T \leq \lambda T^* B_0 \text{ for some } \lambda \geq 0,$$

in which case $T = A_0 B_0$ for some $A_0 \in B^+(\mathfrak{H})$ and one can assume that $\overline{\text{ran } A_0} = \overline{\text{ran } T}$.

In this case, $\ker A_0 \cap \text{ran } B_0 = \ker B_0 \cap \text{ran } A_0 = \{0\}$, and hence

$$\ker T \cap \text{ran } T = \ker T^* \cap \text{ran } T^* = \{0\}. \quad (2.9)$$

3 Local Spectral Theory in $\mathcal{L}_l^{+2}(\mathfrak{H})$ and Quasi-Affinity to $S = S^* \geq 0$

In the framework of bounded operators, it is shown in [8, Proposition 7.2] that if $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ is quasi-affine to a bounded nonnegative operator, then their spectra coincide. The objective of this section is to extend such a result not only to the general case of unbounded operators T but also for $T \notin \mathcal{L}_l^{+2}(\mathfrak{H})$. This will be treated in Section 3.3 which requires the study of some local spectral properties for $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$, carried out in Section 3.1. A key idea in this approach is to deal with the class of unbounded generalized scalar operators which will be proved to lie in $\mathcal{L}_l^{+2}(\mathfrak{H})$ and provide an important connection to local spectral theory; see Sect. 3.2.

3.1 Local Spectral Theory in $\mathcal{L}_l^{+2}(\mathfrak{H})$

The single valued extension property (SVEP) will be the first local spectral property established for $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$. This property was first introduced by J.Finch [13] in the case of bounded operators and, in the present unbounded setting, the definition is recalled from [1].

Definition 3.1 An operator $T \in CO(\mathfrak{H})$ is said to have the *single valued extension property* (abbreviated SVEP) at $\alpha_0 \in \mathbb{C}$ if, for every neighborhood U_{α_0} of α_0 , the only analytic $f_{\alpha_0} : U_{\alpha_0} \rightarrow \text{dom } T$, which satisfies

$$0 = (T - \mu I)f_{\alpha_0}(\mu) \quad \text{for all } \mu \in U_{\alpha_0}$$

is the function $f_{\alpha_0} \equiv 0$. The operator T is said to have the SVEP if it has the SVEP at every $\alpha_0 \in \mathbb{C}$.

Proposition 3.2 *Let $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ be such that $\rho(T) \neq \emptyset$. Then T and T^* have the SVEP.*

Proof Let $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ be such that $\rho(T) \neq \emptyset$ and, suppose to the contrary that T does not have the SVEP. Then there exist $\lambda_0 \in \mathbb{C}$, an open neighborhood U_0 of λ_0 in \mathbb{C} and a nonzero analytic function $f : U_0 \rightarrow \text{dom } T$ such that

$$(T - \mu I)f(\mu) = 0 \quad \text{for all } \mu \in U_0. \quad (3.1)$$

Hence there exists $\mu_1 \in U_0$ and an open set $U_1 \subseteq U_0$ such that $f(\mu_1) \neq 0$ and (3.1) holds for all $\mu \in U_1$. This implies that $\mu_1 \in \sigma_p(T)$. Without loss of generality, one may assume that $U_1 \subseteq \sigma_p(T)$, which shows that $\mu_1 \in \text{int}(\sigma_p(T))$, where $\text{int}(\sigma_p(T))$ denotes the interior of $\sigma_p(T)$.

On the other hand, by Corollary 2.2 $\sigma(T) \subseteq \mathbb{R}^+$ and thus $\text{int}(\sigma_p(T)) = \emptyset$, contradiction. A similar argument show that T^* has the SVEP, since $\sigma(T^*) \subseteq \mathbb{R}^+$. \square

The upcoming results are devoted to outline the main substeps required to prove the second local spectral property of an operator in $\mathcal{L}_l^{+2}(\mathfrak{H})$, which concerns the local spectral subspaces defined below; cf. [1, 11].

Definition 3.3 Let $T \in CO(\mathfrak{H})$.

- (i) The *local resolvent set* of T at a point $x \in \mathfrak{H}$ is the set $\rho_T(x)$ of all $\lambda_0 \in \mathbb{C}$ for which there exist an open neighborhood U_{λ_0} of λ_0 in \mathbb{C} and $f_{x,\lambda_0} \in \mathfrak{H}(U_{\lambda_0}, \text{dom } T)$, i.e., an analytic function $f_{x,\lambda_0} : U_{\lambda_0} \rightarrow \text{dom } T$ which satisfies

$$x = (T - \mu)f_{x,\lambda_0}(\mu) \quad \text{for all } \mu \in U_{\lambda_0}.$$

The set $\mathbb{C} \setminus \rho_T(x)$ is called the *local spectrum* of T at the point x and denoted by $\sigma_T(x)$.

- (ii) The *analytic spectral subspace* of T at a subset F of \mathbb{C} is given by

$$\mathfrak{H}_T(F) = \{x \in \mathfrak{H} ; \tilde{\sigma}_T(x) \subseteq F\}.$$

If $\mathfrak{H}_T(F)$ is closed for all closed $F \subseteq \mathbb{C}$, then T has the *Dunford's property*, shortly property (C).

- (iii) $\mathcal{H}_T(F)$ stands for the *glocal spectral subspace* of T at a closed set $F \subseteq \mathbb{C}$ and it is defined as the set of all $x \in \mathfrak{H}$ for which there exists an analytic function $\ell : \mathbb{C} \setminus F \rightarrow \text{dom } T$ such that

$$x = (T - \mu I)\ell(\mu) \quad \text{for all } \mu \in \mathbb{C} \setminus F.$$

It is easily seen that $\mathcal{H}_T(F) \subseteq \mathfrak{H}_T(F)$ for all closed sets $F \subseteq \mathbb{C}$. Moreover, the converse inclusion holds when T has the SVEP.

Definition 3.4 ([25], Definition 2.2) The operator T is said to have property (k) if it has both SVEP and (C) .

Recall from [19] that the quasi-nilpotent part of an operator $T \in CO(\mathfrak{H})$ is defined by:

$$\mathfrak{H}_0(T) := \{x \in \text{dom}^\infty T ; \lim_{n \rightarrow \infty} \|T^n x\|^{\frac{1}{n}} = 0\},$$

where $\text{dom}^\infty T = \bigcap_{n=1}^{+\infty} \text{dom} T^n$. In particular, $\ker T^n \subseteq \mathfrak{H}_0(T)$ for all $n > 0$.

The following lemma is a collection of some properties related to SVEP and local spectrum and will be a quite useful key for the remaining parts. Item (i) and (ii) are already studied in the general case of linear relations respectively in [20, Proposition 3.1] and [4] and, the proof of item (iv) follows immediately from [1, Corollary 4.9].

Lemma 3.5 *Let $T \in CO(\mathfrak{H})$. Then, the following properties hold:*

- (i) $\sigma_{T-\lambda_0 I}(x) \subseteq \{0\}$ if and only if $\sigma_T(x) \subseteq \{\lambda_0\}$ for every $\lambda_0 \in \mathbb{C}$.
- (ii) If T has the SVEP, then $T - \lambda_0 I$ has the SVEP for every $\lambda_0 \in \mathbb{C}$.
- (iii) If T has the SVEP, then $\mathfrak{H}_T(\{\lambda\}) = \mathfrak{H}_0(T - \lambda I)$ for every $\lambda \in \mathbb{C}$.
- (iv) If $T - \lambda I$ has a finite ascent for every $\lambda \in \mathbb{C}$, then T has the SVEP.
- (v) If T is normal, in particular selfadjoint, then T has the SVEP.

Proof To see (iii), assume that T has the SVEP and let $\lambda \in \mathbb{C}$. Then $T - \lambda I$ has the SVEP, by (ii) and [19, Proposition 1.3] ensures that $\mathfrak{H}_0(T - \lambda I) = \{x \in \text{dom}^\infty T ; \lim_{n \rightarrow \infty} \|(T - \lambda I)^n x\|^{\frac{1}{n}} = 0\} = \{x \in \mathfrak{H} ; \sigma_{T-\lambda I}(x) \subseteq \{0\}\}$. A direct application of (i) yields

$$\mathfrak{H}_0(T - \lambda I) = \{x \in \mathfrak{H} ; \sigma_T(x) \subseteq \{\lambda\}\} = \mathfrak{H}_T(\{\lambda\}).$$

For the proof of (v), assume that T is normal. Then $T - \lambda I$ is normal for all $\lambda \in \mathbb{C}$, and hence it has finite ascent by [14, Theorem 2.1]. One then concludes the result using item (iv). \square

Lemma 3.6 *Let $T \in CO(\mathfrak{H})$. Then the following statements hold:*

- (i) If $x \in \mathfrak{H}$, $F \subseteq \mathbb{C}$ and $(T - \lambda I)x \in \mathfrak{H}_T(F)$ for some $\lambda \in F$, then $x \in \mathfrak{H}_T(F)$.
- (ii) $\ker(T - \lambda I) \subseteq \mathfrak{H}_T(\{\lambda\})$ for all $\lambda \in \mathbb{C}$.

Proof The statement in (i) is proved by the same arguments as in the proof of [17, Proposition 1.2.16] in the context of bounded operators. To see (ii), let $x \in \ker(T - \lambda I)$. Then $(T - \lambda I)x = 0 \in \mathfrak{H}_T(\{\lambda\})$ and a direct application of item (i) yields that $x \in \mathfrak{H}_T(\{\lambda\})$. \square

A particular case, where the reverse inclusion in item (ii) of Lemma 3.6 holds, is the case of bounded normal operators; see [8, Lemma 5.1]. In order to extend such a result to the unbounded setting, some notions of local spectral theory in the extended complex plane \mathbb{C}_∞ are required. For more details see [11, 29, 30] for the unbounded setting of operators and [5] for the multivalued one.

Definition 3.7 Let $T \in CO(\mathfrak{H})$. Then,

- (i) The *extended local resolvent set* of T at a point $x \in \mathfrak{H}$ is the set $\widetilde{\rho}_T(x)$ of all $\lambda \in \mathbb{C}_\infty$ for which there exist an open neighborhood $W_\infty \subseteq \mathbb{C}_\infty$ of λ and analytic function $f_{x,\lambda} : W_\infty \rightarrow \text{dom } T$ such that

$$(T - \mu I)f_{x,\lambda}(\mu) = x \quad \text{for all } \mu \in W_\infty \cap \mathbb{C}.$$

The *extended local spectrum* of T is $\widetilde{\sigma}_T(x) := \mathbb{C}_\infty \setminus \widetilde{\rho}_T(x)$.

- (ii) The *extended local spectral subspace* of T at a set $F \subseteq \mathbb{C}_\infty$ is $\mathfrak{H}_T(F) := \{x \in \mathfrak{H} ; \widetilde{\sigma}_T(x) \subseteq F\}$.
- (iii) The *extended global spectral subspace* of T at a closed set $F \subseteq \mathbb{C}_\infty$ is the set $\widetilde{\mathcal{H}}_T(F)$ of all $x \in \mathfrak{H}$ for which there exists an analytic function $f : \mathbb{C}_\infty \setminus F \rightarrow \text{dom } T$ such that $f(\infty) = 0$ and

$$x = (T - \mu I)f(\mu) \quad \text{for all } \mu \in (\mathbb{C}_\infty \setminus F) \cap \mathbb{C}.$$

Lemma 3.8 Let $T \in CO(\mathfrak{H})$ be an operator with $\rho(T) \neq \emptyset$ and let U be an open neighborhood of $\widetilde{\sigma}(T)$. Then, for $\alpha \in \rho(T)$, the function

$$f_\alpha : U \rightarrow \mathbb{C} \\ z \mapsto \frac{1}{z - \alpha}$$

is non-constant in any connected component of U , belongs to \mathcal{A}_T [11, 30] and admits the following analytic functional calculus

$$f_\alpha(T) = f_\alpha(\infty) + \frac{1}{2i\pi} \int_\gamma f_\alpha(z)R(z, T)dz = R(\alpha, T). \quad (3.2)$$

Here, γ is a rectifiable curve surrounding $\sigma(T)$ in \mathbb{C} and \mathcal{A}_T is the class of complex analytic functions on a neighborhood of $\sigma(T)$.

Corollary 3.9 If $T \in CO(\mathfrak{H})$ is a normal operator then

$$\mathfrak{H}_T(\lambda) = \ker(T - \lambda I) \quad \text{for all } \lambda \in \mathbb{C}. \quad (3.3)$$

Proof The direct inclusion in (3.3) is immediate from Lemma 3.6. For the converse, fix $\alpha \in \rho(T)$ and observe from [8, Lemma 5.1] that

$$\mathfrak{H}_{R(\alpha, T)} \left(\left\{ \frac{1}{\lambda - \alpha} \right\} \right) \cap \text{dom } T = \ker \left(R(\alpha, T) - \frac{1}{\lambda - \alpha} I \right) \cap \text{dom } T \\ = \ker(T - \lambda I) \quad (3.4)$$

for all $\lambda \in \sigma(T)$. On the other hand, [30, Theorem 2.1] yields that

$$f_\alpha(\widetilde{\sigma}_T(x)) = \sigma_{f_\alpha(T)}(x), \quad (3.5)$$

which implies that

$$\begin{aligned} \mathfrak{H}_{R(\alpha,T)}(\{\frac{1}{\lambda-\alpha}\}) \cap \text{dom } T &= \left\{ x \in \mathfrak{H} ; \sigma_{R(\alpha,T)}(x) \subseteq \left\{ \frac{1}{\lambda-\alpha} \right\} \right\} \\ &= \left\{ x \in \mathfrak{H} ; f_\alpha(\widetilde{\sigma}_T(x)) \subseteq \left\{ \frac{1}{\lambda-\alpha} \right\} \right\} \\ &\subseteq \left\{ x \in \mathfrak{H} ; \widetilde{\sigma}_T(x) \subseteq f_\alpha^{-1}\left(\left\{ \frac{1}{\lambda-\alpha} \right\}\right) \right\} \\ &\subseteq \{x \in \mathfrak{H} ; \widetilde{\sigma}_T(x) \subseteq \{\lambda\} \cup \{\infty\}\} \\ &= \widetilde{\mathfrak{H}}_T(\{\lambda\} \cup \{\infty\}). \end{aligned}$$

On the other hand, one obtains from a combination of [6, Lemma 2.6] and [6, Lemma 2.10] that

$$\widetilde{\mathfrak{H}}_T(\{\lambda\} \cup \{\infty\}) = \mathfrak{H}_T(\{\lambda\}),$$

and therefore $\mathfrak{H}_{R(\alpha,T)}(\{\frac{1}{\lambda-\alpha}\}) \cap \text{dom } T \subseteq \mathfrak{H}_T(\{\lambda\})$, as desired. □

The second local spectral property of $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ is described in the following result, which also shows properties similar to normal operators.

Proposition 3.10 *Let $T = AB \in \mathcal{L}_{op}^{+2}(\mathfrak{H})$ be an operator such that $\rho(T) \neq \emptyset$. Then, for all $\lambda \in \mathbb{C}$, one has*

$$\ker(T^* - \lambda I) = \mathfrak{H}_{T^*}(\{\lambda\}). \tag{3.6}$$

In this case,

$$\bigcap_{p \in \mathbb{N}} \overline{\text{ran}}(T - zI)^p = \overline{\text{ran}}(T - zI) \quad \text{for all } z \in \mathbb{C}. \tag{3.7}$$

Proof The inclusion $\ker(T^* - \lambda I) \subseteq \mathfrak{H}_{T^*}(\{\lambda\})$ holds by Lemma 3.6. To see the reverse inclusion, fix $x \in \mathfrak{H}_{T^*}(\{\lambda\})$, and observe from Proposition 3.2 that T^* has the SVEP. Then, item (iii) of Lemma 3.5 ensures that

$$\mathfrak{H}_{T^*}(\{\lambda\}) = \mathfrak{H}_0(T^* - \lambda I) = \{y \in \text{dom } {}^\infty T^* ; \lim_{n \rightarrow \infty} \|(T^* - \lambda I)^n y\|^{\frac{1}{n}} = 0\}.$$

This implies that $x \in \text{dom } {}^\infty T^*$ and

$$\lim_{n \rightarrow \infty} \|(T^* - \lambda I)^n x\|^{\frac{1}{n}} = 0. \tag{3.8}$$

Next, fix

$C = A^{\frac{1}{2}} B A^{\frac{1}{2}}$. Then, $C = C^* \geq 0$, by [3, Proposition 2.5]. Furthermore,

$$\begin{aligned} (C - \lambda I)A^{\frac{1}{2}} &= (A^{\frac{1}{2}} B A^{\frac{1}{2}} - \lambda I)A^{\frac{1}{2}} = (A^{\frac{1}{2}} B A - \lambda A^{\frac{1}{2}}) = A^{\frac{1}{2}}(B A - \lambda I) \\ &= A^{\frac{1}{2}}(T^* - \lambda I). \end{aligned}$$

By induction, one concludes that $(C - \lambda I)^n A^{\frac{1}{2}} = A^{\frac{1}{2}}(T^* - \lambda I)^n$ for all $n > 0$. As $x \in \text{dom } {}^\infty T^*$, it follows that $x \in \text{dom } A^{\frac{1}{2}}(T^* - \lambda I)^n$ and

$$\|(C - \lambda I)^n A^{\frac{1}{2}} x\|^{\frac{1}{n}} = \|A^{\frac{1}{2}}(T^* - \lambda I)^n x\| \leq \|A^{\frac{1}{2}}\|^{\frac{1}{n}} \|(T^* - \lambda I)^n x\|^{\frac{1}{n}}$$

for all $n > 0$. Using (3.8), one obtains

$\|(C - \lambda I)^n A^{\frac{1}{2}} x\|^{\frac{1}{n}} \xrightarrow{n \rightarrow +\infty} 0$, and hence $A^{\frac{1}{2}} x \in \mathfrak{H}_0(C - \lambda I)$. Combining this with the fact that $C = C^*$ and with assertions (v) and (iii) of Lemma 3.5, one concludes that $A^{\frac{1}{2}} x \in \mathfrak{H}_C(\{\lambda\})$. But $\mathfrak{H}_C(\{\lambda\}) = \ker(C - \lambda I)$, in view of Lemma 3.9, so

$$0 = (C - \lambda I)A^{\frac{1}{2}} x = A^{\frac{1}{2}}(T^* - \lambda I)x.$$

One then distinguishes the following two cases:

Case 1 : $x \in \ker(T^* - \lambda I)$. This completes the proof of the desired inclusion.

Case 2 : $(T^* - \lambda I)x \in \ker A^{\frac{1}{2}}$. In this case, $x \in \ker A = \ker T^*$, by Proposition 2.7. Consequently,

$$0 = T^*(T^* - \lambda I)x = (T^* - \lambda I + \lambda I)(T^* - \lambda I)x = (T^* - \lambda I)^2 x + \lambda(T^* - \lambda I)x,$$

which shows that $(T^* - \lambda I)^2 x = -\lambda(T^* - \lambda I)x$. Proceeding by induction, one obtains

$$(T^* - \lambda I)^n x = (-1)^{n-1} \lambda^{n-1} (T^* - \lambda I)x \quad \text{for all } n > 0.$$

Using (3.8), one then concludes that

$$\lim_{n \rightarrow \infty} \|(T^* - \lambda I)^n x\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} |\lambda|^{\frac{n-1}{n}} \|(T^* - \lambda I)x\|^{\frac{1}{n}} = 0.$$

Again, two cases appear:

Case 2.1: If $\|(T^* - \lambda I)x\| \neq 0$, then $\lim_{n \rightarrow \infty} |\lambda|^{\frac{n-1}{n}} \|(T^* - \lambda I)x\|^{\frac{1}{n}} \xrightarrow{n \rightarrow +\infty} |\lambda|$, so $\lambda = 0$. Thus,

$$T^* x = (T^* - \lambda I)x \in \text{ran}(T^*) \cap \ker T^* = \{0\},$$

in view of Corollary 2.10. As a consequence, $x \in \ker T^* = \ker(T^* - \lambda I)$.

Case 2.2 : If $\|(T^* - \lambda I)x\| = 0$, then $x \in \ker(T^* - \lambda I)$.

Hence, the proof of (3.6) is completed.

To see (3.7), let $t \in \mathbb{C}$. Then, a combination of Proposition 3.2 with item (iii) of Lemma 3.5 implies that T^* has the SVEP and

$$\mathfrak{H}_{T^*}(\{\bar{t}\}) = \{x \in \text{dom } {}^\infty T^*; \lim_{n \rightarrow \infty} \|(T^* - \bar{t}I)^n x\|^{\frac{1}{n}} = 0\} = \mathfrak{H}_0(T^* - \bar{t}).$$

This shows, by (3.6) that $\ker(T^* - \bar{t}I)^p \subseteq \ker(T^* - \bar{t}I)$ for all $p \in \mathbb{N}$. Hence for all $p \in \mathbb{N}$ one has $\ker(T^* - \bar{t}I)^p = \ker(T^* - \bar{t}I)$ or, equivalently,

$$\overline{\text{ran}}(T - tI)^p = \overline{\text{ran}}(T - tI).$$

□

It will be shown that the Dunford's property is also a necessary condition for an operator to belong to the class $\mathcal{L}_l^{+2}(\mathfrak{H})$. This will be achieved throughout the study of generalized scalar operators in the next subsection, which forms the core of this paper.

3.2 Unbounded Generalized Scalar Operators

Throughout this part, we refer to [27] and denote by $\mathfrak{H}(\Omega)$ the set of all complex-valued functions f defined on $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ that satisfy:

- (i) f is C^∞ on \mathbb{C}_∞ ;
- (ii) f is analytic on a neighbourhood of $\mathbb{C}_\infty \setminus \Omega$.

In particular, $\mathfrak{H}(\rho(T))$ is the set of all C^∞ functions on \mathbb{C}_∞ which are analytic in a neighbourhood of $\sigma(T)$. Note that there exists a continuous homomorphism from $\mathfrak{H}(\rho(T))$ to $B(\mathfrak{H})$ defined by Dunford and Schwartz [10] and which will be denoted by Φ_T .

Definition 3.11 Let $T \in CO(\mathfrak{H})$ be a densely defined linear operator and let Ω be an open subset of \mathbb{C}_∞ such that $\rho(T) \subseteq \Omega$. Then, a continuous homomorphism $\Phi : \mathfrak{H}(\Omega) \rightarrow B(\mathfrak{H})$ for which $\Phi|_{\mathfrak{H}(\rho(T))} = \Phi_T$ is called an $\Omega - C^\infty$ operational calculus. Moreover, T is said to be

- (i) an Ω -generalized scalar operator, if T has an $\Omega - C^\infty$ operational calculus;
- (ii) a generalized scalar operator, if it is a \mathbb{C}_∞ -generalized scalar operator.

The following Lemma is a particular case of [27, Theorem 12].

Lemma 3.12 Let $T \in CO(\mathfrak{H})$ be a densely defined operator such that $\sigma(T) \subseteq \mathbb{R}$. Then, T is \mathbb{C} -generalized scalar if and only if, for every $\mu_0 \in \mathbb{R}$ one has

$$\|R(\mu, T)\| = O(d(\mu, \mathbb{R})^{-n}), \quad \mu \rightarrow \mu_0 \text{ and } \mu \in \mathbb{C} \setminus \mathbb{R}$$

for some $n > 0$. Here, $d(\mu, \mathbb{R})$ denotes the distance from μ to \mathbb{R} .

A general result of Lemma 2.3 for $\alpha = 1$ is worth mentioning separately and it will be a useful tool for the proof of Theorem 3.14. A proof is given for completeness.

Lemma 3.13 Let $B \in LO(\mathfrak{H})$ and $S \in B(\mathfrak{H})$. If SS^*B is closed, then S^*B is closed. Moreover, if $\overline{\text{dom } B} = \mathfrak{H}$ then $(B^*S)^* = S^*B$.

Proof Let $(x_n, y_n)_{n \in \mathbb{N}} \subseteq G(S^*B)$ be such that $(x_n, y_n) \xrightarrow{n \rightarrow +\infty} (x, y)$ in \mathfrak{H} . Then, $y \in \overline{\text{ran } S^*}$ and $(x_n, SS^*Bx_n) \xrightarrow{n \rightarrow +\infty} (x, Sy)$, since S is bounded. As SS^*B is closed, it then follows that $(x, Sy) \in G(SS^*B)$, and therefore $S(S^*Bx - y) = 0$. Thus, $(S^*Bx - y) \in \ker S \cap \overline{\text{ran } S^*} = \{0\}$. Consequently, $y = S^*Bx$, which shows that S^*B is closed. \square

Theorem 3.14 Let $T = AB \in \mathcal{L}_l^{+2}(\mathfrak{H})$ be a linear operator such that $\rho(T) \neq \emptyset$. Then T is a \mathbb{C} -generalized scalar linear operator.

Proof Without loss of generality, one can assume that $\|A\| \leq k < 1$ for a nonnegative constant k and let $\widehat{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{H}$. The proof will be given in three steps. First it will be established that $\sigma(\widehat{T}) \subseteq \mathbb{R}^+$, where $\widehat{T} := \widehat{A}\widehat{B}$ and $\widehat{A}, \widehat{B} : \widehat{\mathfrak{H}} \rightarrow \widehat{\mathfrak{H}}$ are given by $\widehat{A} := \begin{pmatrix} A & A^{\frac{1}{2}}(I - A)^{\frac{1}{2}} \\ (I - A)^{\frac{1}{2}}A^{\frac{1}{2}} & I - A \end{pmatrix}$ and $\widehat{B} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$. In the second step, it will be shown that \widehat{T} is a \mathbb{C} -generalized scalar operator in $\widehat{\mathfrak{H}}$ and in the last one, it will be proved that T is a \mathbb{C} -generalized scalar operator in \mathfrak{H} .

Step 1. By Lemma 2.3 $A^{\frac{1}{2}}B$ is closed, since $A \in B^+(\mathfrak{H})$ and both B and AB are closed. Moreover, as $\|A\| < 1$, it follows that $(I - A)^{\frac{1}{2}}$ is invertible, and therefore $(I - A)^{\frac{1}{2}}A^{\frac{1}{2}}B$ is closed. This shows by [28, Corollary 2.2.11] that the operator

$$\widehat{T} = \begin{pmatrix} T & 0 \\ (I - A)^{\frac{1}{2}}A^{\frac{1}{2}}B & 0 \end{pmatrix}$$

is closed. Now, it will be established that $\sigma(\widehat{T}) \subseteq \mathbb{R}^+$. For this, consider $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$. As $\rho(T) \neq \emptyset$, one then obtains from Corollary 2.2 that $\lambda \in \rho(T)$, which yields to the following factorization:

$$\widehat{T} - \lambda I = \begin{pmatrix} I & 0 \\ (I - A)^{\frac{1}{2}}A^{\frac{1}{2}}BR(\lambda, T) & I \end{pmatrix} \begin{pmatrix} T - \lambda & 0 \\ 0 & -\lambda I \end{pmatrix}, \quad (3.9)$$

where the block operator on the right-hand side of (3.9) is invertible. On the other hand, $T_2 := (I - A)^{\frac{1}{2}}A^{\frac{1}{2}}BR(\lambda, T)$ is closed and $\text{dom } T_2 = \mathfrak{H}$, so by the graph theorem, one concludes that $T_2 \in B(\mathfrak{H})$. This means, by (3.9), that

$$\rho(\widehat{T}) = \rho(T) \setminus \{0\},$$

and therefore

$$\sigma(\widehat{T}) \subseteq \mathbb{R}^+. \quad (3.10)$$

In this case, for all $\eta \in \mathbb{C} \setminus \mathbb{R}$, one has $\eta \in \rho(T) \cap \rho(\widehat{T})$ and it is claimed that

$$(\widehat{T} - \eta I)^{-1} = \begin{pmatrix} R(\eta, T) & 0 \\ \eta^{-1}T_2 & -\eta^{-1} \end{pmatrix} := R_0(\eta). \quad (3.11)$$

In fact, $R_0(\eta) \in B(\widehat{\mathfrak{H}})$ by construction and one has

$$(\widehat{T} - \eta I)R_0(\eta) = I_{\widehat{\mathfrak{H}}} \quad \text{and} \quad R_0(\eta)(\widehat{T} - \eta I) = I_{\widehat{\mathfrak{H}}} \upharpoonright \text{dom } \widehat{T},$$

so $R(\eta, \widehat{T}) = R_0(\eta)$, as claimed.

Step 2. To show that \widehat{T} is a \mathbb{C} -generalized scalar operator, observe first that $\widehat{T} \in L^{+2}(\widehat{\mathfrak{H}})$. Indeed, one can see that $\widehat{A} = SS^*$, where $S = \begin{pmatrix} A^{\frac{1}{2}} \\ (I - A)^{\frac{1}{2}} \end{pmatrix}$ is an isometry on $\widehat{\mathfrak{H}}$, and this implies that $\widehat{A} =: P_{\text{ran } S}$ is a linear projector in

$B^+(\widehat{\mathfrak{H}})$. Furthermore, it follows from [18] that $\widehat{B} = (\widehat{B})^*$, and since $B \geq 0$, one concludes that $\widehat{B} \geq 0$.

On the other hand, one has

$$S^*\widehat{T}S = S^*\widehat{A}\widehat{B}S = S^*(SS^*\widehat{B})S = S^*\widehat{B}S,$$

and hence $(S^*\widehat{T}S)^* = (S^*\widehat{B}S)^* = (\widehat{B}S)^*S = \overline{S^*\widehat{B}S}$. But $SS^*\widehat{B} = \widehat{A}\widehat{B} = \widehat{T}$ is closed, so using Lemma 3.13, one obtains $\overline{S^*\widehat{B}S} = S^*\widehat{B}S$. Consequently,

$$(S^*\widehat{T}S)^* = S^*\widehat{B}S = S^*\widehat{T}S. \quad (3.12)$$

As $\widehat{B} \geq 0$, this implies that $S^*\widehat{T}S \geq 0$ and $\sigma(S^*\widehat{T}S) = \sigma(S^*\widehat{B}S) \subseteq \mathbb{R}^+$. In addition, (3.12) proves that $S^*\widehat{T}S$ is a scalar linear operator, so a generalized scalar one. Applying Lemma 3.12, one concludes that

$$\|R(\mu, S^*\widehat{T}S)\| = O(d(\mu, \mathbb{R})^{-n}), \quad \mu \rightarrow \mu_0 \text{ and } \mu \in \mathbb{C} \setminus \mathbb{R} \quad (3.13)$$

for every $\mu_0 \in \mathbb{R}$ and for some $n > 0$.

Denote $R = \begin{pmatrix} (I - A)^{\frac{1}{2}} \\ -A^{\frac{1}{2}} \end{pmatrix}$. Then R is an isometry, $R^*R = I$ and, moreover, the block operator

$$U = (SR) = \begin{pmatrix} A^{\frac{1}{2}} & (I - A)^{\frac{1}{2}} \\ (I - A)^{\frac{1}{2}} & -A^{\frac{1}{2}} \end{pmatrix}$$

is unitary on $\widehat{\mathfrak{H}}$. Furthermore, observe that

$$U^*\widehat{T}U = \begin{pmatrix} S^* \\ R^* \end{pmatrix} SS^*\widehat{B}(SR) = \begin{pmatrix} S^*\widehat{B}S & S^*\widehat{B}R \\ 0 & 0 \end{pmatrix} \quad (3.14)$$

and, for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ arbitrarily given, one has

$$U^*\widehat{T}U - \lambda I = \begin{pmatrix} S^*\widehat{B}S - \lambda I & S^*\widehat{B}R \\ 0 & -\lambda \end{pmatrix} \quad (3.15)$$

$$= \begin{pmatrix} S^*\widehat{B}S - \lambda I & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} I & \overline{X} \\ 0 & I \end{pmatrix}, \quad (3.16)$$

where $X = (S^*\widehat{B}S - \lambda I)^{-1}S^*\widehat{B}R$. It is claimed that $\overline{X} \in B(\widehat{\mathfrak{H}})$. Indeed,

$$X^* = (S^*\widehat{B}R)^*(S^*\widehat{B}S - \lambda I)^{-1} \supseteq R^*\widehat{B}S(S^*\widehat{B}S - \lambda I)^{-1} \quad (3.17)$$

and since $\text{ran}(S^*\widehat{B}S - \lambda I)^{-1} = \text{dom } S^*\widehat{B}S = \text{dom } \widehat{B}S$, one concludes that $\text{dom } X^* = \widehat{\mathfrak{H}}$. On the other hand, the block operator in (3.14) is densely defined since \widehat{T} and thus also $U^*\widehat{T}U$ is densely defined, so that also the

blocks $S^*\widehat{B}S$ and $S^*\widehat{B}R$ in (3.14) are densely defined operators. As $S^*\widehat{B}R$ is a densely defined operator, its adjoint $(S^*\widehat{B}R)^*$ is a closed and densely defined operator since $(S^*\widehat{B}R)^* \supseteq R^*\widehat{B}S$. This implies, by (3.17), that X^* is also a closed operator with $\text{dom } X^* = \widehat{\mathfrak{H}}$. Therefore, X^* and \overline{X} belong to $B(\widehat{\mathfrak{H}})$, which proves the claim.

Consequently, the block operator $\begin{pmatrix} I & \overline{X} \\ 0 & I \end{pmatrix}$ is invertible on $\widehat{\mathfrak{H}}$. This implies that

$$\begin{aligned} (U^*\widehat{T}U - \lambda I)^{-1} &= \begin{pmatrix} I & -\overline{X} \\ 0 & I \end{pmatrix} \begin{pmatrix} (S^*\widehat{B}S - \lambda I)^{-1} & 0 \\ 0 & -\frac{1}{\lambda} \end{pmatrix} \\ &= U^*R(\lambda, \widehat{T})U. \end{aligned} \quad (3.18)$$

Combining this formula with (3.13), one can conclude that

$$\|R(\mu, \widehat{T})\| = O(d(\mu, \mathbb{R})^{-n}), \quad \mu \rightarrow \mu_0 \text{ and } \mu \in \mathbb{C} \setminus \mathbb{R} \quad (3.19)$$

for every $\mu_0 \in \mathbb{R}$ and for some $n > 0$. Hence, one concludes by Lemma 3.12 that \widehat{T} is a \mathbb{C} -generalized scalar operator.

Step 3. To demonstrate that T is a \mathbb{C} -generalized scalar operator, let $\eta_0 \in \mathbb{R}$ and $\eta \in \mathbb{C} \setminus \mathbb{R}$ be such that $\eta \rightarrow \eta_0$. Then, one obtains from (3.11) that

$$\begin{aligned} \begin{pmatrix} I & 0 \\ 0 & A^{\frac{1}{2}} \end{pmatrix} (\widehat{T} - \eta)^{-1} &= \begin{pmatrix} R(\eta, T) & 0 \\ \frac{1}{\eta}(I - A)^{1/2}TR(\eta, T) & -\frac{1}{\eta}A^{\frac{1}{2}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \frac{1}{\eta}(I - A)^{1/2} & -\frac{1}{\eta}A^{\frac{1}{2}} - I \end{pmatrix} \\ &\quad + \begin{pmatrix} I & 0 \\ (I - A)^{1/2} & I \end{pmatrix} \begin{pmatrix} R(\eta, T) & 0 \\ 0 & I \end{pmatrix}, \end{aligned}$$

which implies that

$$\begin{aligned} \begin{pmatrix} (T - \eta)^{-1} & 0 \\ 0 & I \end{pmatrix} &= \begin{pmatrix} I & 0 \\ -(I - A)^{\frac{1}{2}} & I \end{pmatrix} \left[\begin{pmatrix} I & 0 \\ 0 & A^{\frac{1}{2}} \end{pmatrix} R(\eta, \widehat{T}) \right. \\ &\quad \left. - \begin{pmatrix} 0 & 0 \\ \frac{1}{\eta}(I - A)^{\frac{1}{2}} & -\frac{1}{\eta}A^{\frac{1}{2}} - I \end{pmatrix} \right]. \end{aligned}$$

Combining this with (3.19) one obtains the desired estimate

$$\|R(\eta, T)\| = O(d(\eta, \mathbb{R})^{-n}),$$

which means that T is a \mathbb{C} -generalized scalar operator by Lemma 3.12. \square

The third local spectral property of $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ is stated below.

Proposition 3.15 *If $T \in CO(\mathfrak{H})$ is a \mathbb{C} -generalized scalar operator with $\sigma(T) \subseteq \mathbb{R}$ then T , respectively T^* , has property (k). In particular, every operator in $\mathcal{L}_1^{+2}(\mathfrak{H})$ with $\rho(T) \neq \emptyset$ has property (k).*

Proof Since $\sigma(T) \subseteq \mathbb{R}$, it is clear from the proof of Proposition 3.2 that T the SVEP. On the other hand, $R(\mu, T)$ is a generalized scalar operator for all $\mu \in \rho(T)$, so it has the property (C) by [17]. This means by [17, Proposition 3.3.4] that $R(\mu, T)$ has property (C). It is claimed that also T has (C). To see this, let $F \subseteq \mathbb{C}_\infty$ be a closed set and let $(x_n)_{n \in \mathbb{N}} \subseteq \mathfrak{H}_T(F)$ such that $x_n \xrightarrow{n \rightarrow +\infty} x$ in \mathfrak{H} . Then, for all $n \in \mathbb{N}$, $\sigma_T(x_n) \subseteq F$ and hence $f_\alpha(\tilde{\sigma}_T(x_n)) \subseteq f_\alpha(\sigma_T(x_n) \cup \{\infty\}) \subseteq f_\alpha(F \cup \{\infty\})$. Combining this with (3.2) and (3.5), one concludes that

$$\sigma_{R(\alpha, T)}(x_n) \subseteq f_\alpha(F \cup \{\infty\}) \quad \text{for all } n \in \mathbb{N},$$

and hence $(x_n)_{n \in \mathbb{N}} \subseteq \mathfrak{H}_{R(\alpha, T)}(f_\alpha(F \cup \{\infty\}))$. Since $f_\alpha(F \cup \{\infty\})$ is closed, also the set $\mathfrak{H}_{R(\alpha, T)}(f_\alpha(F \cup \{\infty\}))$ is closed and $x_n \xrightarrow{n \rightarrow +\infty} x$, it then follows that $x \in \mathfrak{H}_{R(\alpha, T)}(f_\alpha(F \cup \{\infty\}))$ and therefore $f_\alpha(\tilde{\sigma}_T(x)) = \sigma_{R(\alpha, T)}(x) \subseteq f_\alpha(F \cup \{\infty\})$. Thus $\sigma_T(x) \subseteq \tilde{\sigma}_T(x) \subseteq f_\alpha^{-1} f_\alpha(\tilde{\sigma}_T(x)) \subseteq F$. This proves that $x \in \mathfrak{H}_T(F)$ and hence T has (C), as claimed.

Now, assume that $T \in \mathcal{L}_1^{+2}(\mathfrak{H})$ such that $\rho(T) \neq \emptyset$. Then, $\sigma(T) \subseteq \mathbb{R}$ by Corollary 2.2 and for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, one has

$$\|R(\lambda, T^*)\| = \|R(\bar{\lambda}, T)\|. \quad (3.20)$$

Moreover, Theorem 3.14 shows that T is a \mathbb{C} -generalized scalar operator, so a combination of (3.20) and Lemma 3.12 yields that T^* is a \mathbb{C} -generalized scalar operator. Hence, the same reasoning as above shows that T^* has property (k). \square

3.3 Quasi-Affinity to Nonnegative Selfadjoint Operators

Recall that in the bounded case, if T is quasi-affine to $S \in B^+(\mathfrak{H})$ then $\sigma(S) \subseteq \sigma(T)$; see [8, Lemma 3.7] (see also [12, 26]). The next result extends this result not only to the unbounded setting but also to operators satisfying property (k), which is a weaker condition than the selfadjointness.

Lemma 3.16 *Let $T \in CO(\mathfrak{H})$ and let S be a an operator with the property (k). If T is quasi-affine to S then*

$$\sigma(S) \subseteq \sigma(T), \quad (3.21)$$

In this case, $\mathfrak{H} = \mathfrak{H}_S(\sigma(T))$.

Proof Since T is quasi-affine to S , there exists a quasi-affinity $G \in B(\mathfrak{H})$ such that $GT \subseteq SG$. The proof is splitted in four steps.

Step 1. $\sigma_S(Gx) \subseteq \sigma(T)$ for all $x \in \mathfrak{H}$.

To see this, let $x \in \mathfrak{H}$ and $\lambda \in \rho(T) \subseteq \rho_T(x)$. Then, there exists an open neighborhood U_λ of λ and an analytic function $f : U_\lambda \rightarrow \text{dom } T$ such that

$x = (T - \mu I)f(\mu)$ for all $\mu \in U_\lambda$, and hence

$$Gx = G(T - \mu I)f(\mu)x = (S - \mu I)Gf(\mu)x \quad \text{for all } \mu \in U_\lambda.$$

As $G \in B(\mathfrak{H})$, it follows that $\mu \mapsto Gf(\mu)$ in an analytic function on U_λ , which implies that $\lambda \in \rho_S(Gx)$. One then concludes that $\rho(T) \subseteq \rho_S(Gx)$.

Step 2. $\mathfrak{H} = \mathfrak{H}_S(\sigma(T))$. For this, fix $x \in \mathfrak{H}$. As $\overline{\text{ran}} G = \mathfrak{H}$, there exists $(x_n) \subseteq \mathfrak{H}$ for which $Gx_n \xrightarrow{n \rightarrow +\infty} x$. On the other hand, Step 1 ensures that $\sigma_S(Gx_n) \subseteq \sigma(T)$ or, equivalently, $Gx_n \subseteq \mathfrak{H}_S(\sigma(T))$. Since S has property (C) one concludes that $x \in \mathfrak{H}_S(\sigma(T))$ and $\mathfrak{H} = \mathfrak{H}_S(\sigma(T))$.

Step 3. $\sigma_{sur}(T) \subseteq \bigcup_{x \in \mathfrak{H}} \sigma_T(x)$. Let $x \in \mathfrak{H}$ be arbitrary and let $\lambda \in \rho_T(x)$. Then, there exists an open neighborhood U_λ of λ in \mathbb{C} and analytic function $f : U_\lambda \rightarrow \text{dom } T$ such that $x = (T - \mu I)f(\mu)$ for all $\mu \in U$, in particular $x = (T - \lambda I)f(\lambda) \in \text{ran}(T - \lambda)$. This proves that $\mathfrak{H} = \text{ran}(T - \lambda)$, and hence $x \in \rho_{sur}(T)$.

Step 4. $\sigma(S) \subseteq \sigma(T)$. First observe from Step 2 that $\sigma_{sur}(x) \subseteq \sigma(T)$ for all $x \in \mathfrak{H}$. On the other hand, S has SVEP, so by [11, Corollary 2.5], $\sigma(S) = \sigma_{sur}(S)$. Applying Step 3, one then concludes that $\sigma(S) \subseteq \bigcup_{x \in \mathfrak{H}} \sigma_T(x) \subseteq \sigma(T)$.

□

In the bounded settings, equality holds in (3.21) for $S \in B^+(\mathfrak{H})$ and $T \in \mathcal{L}_1^{+2}(\mathfrak{H})$; see [8, Proposition 7.2]. This result has been ameliorated to cover the general case where $T \notin \mathcal{L}_1^{+2}(\mathfrak{H})$, which will be seen in the next corollary.

Corollary 3.17 *Let $T \in CO(\mathfrak{H})$ be an operator such that T and T^* have (k). If T is quasi-affine to $S = S^*$ then*

$$\sigma(S) = \sigma(T) = \sigma(T^*) \tag{3.22}$$

and, in this case $\mathfrak{H} = \mathfrak{H}_S(\sigma(T)) = \mathfrak{H}_{T^*}(\sigma(S))$.

Proof Observe from Lemma 3.5 (vi) that S has the SVEP. Moreover, it is clear that S is a scalar operator, and hence generalized scalar. This implies by Proposition 3.15 that S has property (C), and hence property (k). Since T is quasi-affine to S , Lemma 3.16 shows that

$$\sigma(S) \subseteq \sigma(T). \tag{3.23}$$

On the other hand, T^* has property (k) and since S is quasi-affine to T^* one concludes from Lemma 3.16 that

$$\sigma(T^*) \subseteq \sigma(S) \subseteq \mathbb{R}. \tag{3.24}$$

Hence, (3.22) is obtained a combinations of (3.23) and (3.24). The last equalities follow from (3.22) and from the fact that $\sigma_M(x) \subseteq \sigma(M)$ for every $x \in \mathfrak{H}$ and $M \in CO(\mathfrak{H})$. □

Recall from [22] that a linear operator T is said to be *quasi-similar* to a linear operator S if there exist two quasi-affinities $G_1, G_2 \in B(\mathfrak{H})$ such that

$$G_1 T \subseteq S G_1 \quad \text{and} \quad G_2 S \subseteq T G_2.$$

Corollary 3.18 *Let $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ be an operator such that $\rho(T) \neq \emptyset$. If T is quasi-affine, in particular, quasi-similar to a nonnegative selfadjoint operator $S \in CO(\mathfrak{H})$ then*

$$\sigma(S) = \sigma(T).$$

Proof The proof immediately follows from a combination of Theorem 3.14, Proposition 3.15 and Corollary 3.17 and, by definition. \square

It is important to note that, in the particular case where $\overline{\text{ran}} T = \mathfrak{H}$, the quasi-affinity of T^* to a nonnegative selfadjoint operator is enough to prove all the above results. This is based on the following remark stated in [3]:

Remark 3.19 *Let $T \in CO(\mathfrak{H})$ be a densely defined operator with $\overline{\text{ran}} T = \mathfrak{H}$. Then the following statements are equivalent:*

- (i) $T = AB \in \mathcal{L}_l^{+2}(\mathfrak{H})$;
- (ii) T^* is G -quasi-affine to $S = S^* \geq 0$ with $\text{dom } T \subseteq \overline{\text{dom } S^{\frac{1}{2}}(G^{-1})^*}$.

Theorem 3.20 *Let $T \in CO(\mathfrak{H})$ be an operator with $\overline{\text{ran}} T = \mathfrak{H}$ and $\rho(T) \neq \emptyset$. If T^* is quasi-affine to $S = S^* \geq 0$ so that $\text{dom } T \subseteq \overline{\text{dom } S^{\frac{1}{2}}(G^{-1})^*}$ then*

$$\sigma(T) = \sigma(S).$$

Proof The results follows immediately from Remark 3.19 and Corollary 3.18. \square

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