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Is energy risk scale Invariant? evidence from crude oil futures

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This study diverges from earlier research by utilizing power-law functions to model realized variances in crude oil prices and analyzing these functions across various time scales. The findings reveal several key insights. First, uncertainty in crude oil markets exhibits fractal-like properties, manifested in scale invariant power-law behavior. Second, the estimated power-law exponent demonstrates invariance in the intertemporal dimension, a result confirmed through the test for total invariance, which did not reject the hypothesis of total invariance in power-law behavior. Third, the study provides evidence that the variance of crude oil price variance is statistically infinite, rendering sample variance estimates inherently context-dependent. Fourth, in contrast to earlier literature supporting the lognormal model, the findings decisively reject the lognormal model as a valid data-generating process for realized crude oil price variances across all time scales. These results have significant theoretical and practical implications. The fractal properties and infinite variance challenge conventional assumptions about crude oil market dynamics, while the rejection of the lognormal model highlights the need for alternative frameworks in modeling and risk management.

1. Introduction

The behavior of volatility in crude oil markets has long posed a challenge to traditional econometric modeling. Price series in energy markets often exhibit irregularities that escape the confines of Gaussian assumptions and conditional heteroskedastic frameworks. Models such as those in the GARCH family, while widely adopted, remain ultimately constrained by parametric formulations that fail to capture the full breadth of empirical irregularity and temporal complexity observed in high-frequency oil price data.

Narayan and Narayan (2007) emphasized the significance of volatility modeling in the context of macroeconomic stability, particularly for oil-exporting and importing economies. Their investigation, utilizing the EGARCH specification, revealed both asymmetry and persistence in volatility responses to shocks, a result which suggests that volatility in crude oil markets cannot be understood merely as a stationary Gaussian process. Expanding upon this insight, Wei, Wang, and Huang (2010) evaluated a broad array of GARCH-type models—including FIGARCH, APARCH, and HYGARCH—finding no consistent superiority among them, thereby casting doubt on the robustness of conventional volatility forecasting approaches. Earlier contributions by Kang et al. (2009), Cheong (2009), and Mohammadi and Su (2010) similarly sought to establish model hierarchies within this class, but yielded mixed outcomes.

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This paper adopts a different perspective. Rather than pursuing further refinements within the GARCH paradigm, we inquire whether range-based variance of crude oil prices exhibits statistical properties of scale invariance and total invariance—properties that, if verified, would signal the presence of deeper symmetries in the stochastic nature of market fluctuations. This inquiry extends beyond model comparison; it seeks to characterize the fundamental structure of volatility itself.

To this end, we construct range-based measures of crude oil price variance and employ maximum likelihood estimation to infer the scaling exponent characterizing their distributional behavior. This empirical strategy allows us to rigorously test for the presence of power-law scaling and invariance properties. Notably, our approach builds upon Grobys (2024), who introduced a formal test for total invariance in the context of range-based foreign exchange rate variances. We extend this methodology to range-based variances in crude oil prices, thereby contributing to a broader understanding of invariance properties in financial volatility.

Our empirical foundation is further supported by the framework of Andersen et al. (2003), particularly their treatment of realized variance as a consistent estimator of quadratic variation in continuous-time price processes. The relevance of realized variance to volatility modeling has been firmly established by Barndorff-Nielsen and Shephard (2002), who introduced it as a robust alternative to parametric volatility proxies. However, despite its statistical advantages, realized variance has rarely been analyzed for deeper invariance properties. By investigating this dimension, we introduce a novel extension to its empirical characterization. Further theoretical context is provided by foundational studies such as Engle (1982) and Bollerslev et al. (1994), which highlighted the dynamic and persistent nature of volatility. Yet their frameworks—predicated on finite variance assumptions and mean-reverting structures—are not designed to detect or accommodate infinite variance behavior or scaling laws.

Our results suggest that range-based variances in crude oil prices may follow statistical laws akin to those observed in fractal phenomena: heavy tails, self-similarity, and scaling invariance. This observation leads us to reassess the appropriateness of Gaussian-based lognormal models in favor of more generalized forms governed by power-law distributions. Indeed, our empirical analysis decisively rejects the lognormal hypothesis as a viable description for range-based crude oil price variance across scales.

The remainder of the paper is organized as follows. Section 2 provides a literature review. Section 3 introduces the data and the methodological framework. Section 4 presents the empirical findings on scale and total invariance. Section 5 discusses implications for econometric modeling and financial practice. Section 6 concludes.

2. Literature review

The modeling of crude oil price volatility continues to evolve across an expanding landscape of empirical and theoretical innovation. Among the more tenacious traditions are the GARCH-based econometric frameworks, extended in recent years to account for both jump dynamics and long-memory behavior. Shu and Luo (2025), for instance, introduce a Realized EGARCH model that incorporates extreme-value data and stochastic jumps, yielding improvements in the empirical fit and predictive sharpness of volatility estimates. Dutta and Bouri (2024) offer a complementary extension of the Heterogeneous Autoregressive (HAR) model by embedding jump components, thus further enhancing the model's performance in turbulent regimes. Likewise, Fang et al. (2023) deploy a GARCH-MIDAS approach to incorporate global economic policy uncertainty as a predictor of crude oil futures volatility. These works, inheriting the structural form of Engle (1982) and Bollerslev et al. (1994), maintain the conditional heteroskedasticity paradigm while probing its empirical frontiers.

A parallel, and increasingly influential, line of inquiry exploits the algorithmic power of machine learning. Kim et al. (2025), Lin et al. (2025), and Huang et al. (2024) implement neural architectures—ranging from attention-based mechanisms and Transformers to composite models blending gated recurrent units with ensemble empirical mode decomposition. These contributions, though largely data-driven and agnostic to underlying statistical structure, nonetheless offer superior performance in capturing nonlinear dependencies and abrupt regime shifts. However, they are often opaque and reliant on implicit distributional premises, an issue that limits their interpretive robustness.

In a broader systems context, oil market volatility has been studied for its interdependencies with macro-financial and sustainability-linked assets. Olasehinde-Williams and Akadiri (2025), applying entropy-based measures within a DCC-GARCH framework, reveal significant spillovers from oil to sustainable markets. McGillivray and Swishchuk (2024), meanwhile, utilize Hawkes processes to uncover feedback dynamics between crude oil and gasoline futures, suggesting latent systemic risk features embedded within microstructure behavior. Though these approaches treat volatility as an exogenous driver, they implicitly underscore its structural significance.

Another cluster of research addresses volatility through structural and multi-factor modeling. He et al. (2024) relate crude oil price fluctuations to climate policy uncertainty, with the salient finding that it is not the level but the change in policy uncertainty that most affects volatility. Gao et al. (2023) propose a multifactor pricing model incorporating convenience yield, interest rate dynamics, and spot price processes to improve alignment with observed futures prices.

In contrast to these predictive and interaction-based frameworks, the present study proceeds from a foundational interrogation of volatility's statistical architecture.[2] Specifically, we examine whether the realized variance of crude oil prices conforms to a power-law

[2] Earlier studies on crude oil volatility focus on volatility prediction (Sadorsky, 2006), analyzing the link between oil price volatility and the macroeconomy (Ferderer, 1996; Lee et al., 1995; Yang et al., 2002; Chen and Chen, 2007), explore the links between oil price volatility and equity prices (Huang et al., 1996; Sadorsky, 1999, Sadorsky, 2003), investigate in a comparative manner the volatility of crude oil, refined petroleum and natural gas prices (Regnier, 2006; Plourde and Watkins, 1998; Pindyck, 1999), or analyzed the asymmetry of the impact of oil price shocks on economic activities (Huang et al., 2005).

distribution, and whether this structure exhibits scale invariance and total invariance. The method adopted—employing range-based variance estimates and maximum likelihood estimation of scaling exponents—follows from Grobys (2024), who introduced a test for total invariance in the context of foreign exchange markets. Our extension of this methodology to the crude oil domain allows us to uncover fractal properties and distributional heavy tails, challenging conventional Gaussian assumptions.

This study offers three primary contributions. First, it introduces a methodological innovation by extending the test for total invariance into a new empirical domain. Second, it provides a theoretical enrichment to the modeling of range-based variance by situating it within a power-law framework. Third, it articulates a diagnostic instrument that may inform or constrain econometric and machine learning models alike, contingent upon the statistical properties of the underlying volatility process.

3. Data

Intraday data on crude oil futures covering the January 2, 1985 to November 20, 2024 are downloaded from investing.com.[3] Data covers open, high, and low prices over 10,150 trading days. To compute annualized daily variances, we employ the range-based variance estimator, proposed by Parkinson (1980), which is given by:

$$\sigma_t^2 = T\frac{1}{4\ln(2)}(\ln(H_t) - \ln(L_t))^2 \tag{1}$$

where H_t and L_t denote the highest and lowest price for crude oil on trading day t, and σ_t^2 denotes the annualized daily variance, where $T = 250$, as 250 trading days per annum are assumed.[4] Weekly or monthly annualized range-based crude oil price variances are computed by summing up five or 22 non-overlapping daily variances multiplied by $T = 52$ or $T = 12$ to obtain annualization. Computing non-overlapping data on realized crude oil variances leaves us with 2029 weekly and 460 monthly realized variance observations for crude oil prices. The descriptive statistics for our data are reported in Table 1, whereas the time series evolutions are visualized in Fig. 1–3. Fig. 1–3 show clearly that the maximum of range-based crude oil variance was reached in March 2020 during the outbreak of COVID-19 pandemic. It becomes also evident that periods of high and low variances alternate—a phenomenon which is typically referred to “volatility clustering” considered a stylized fact of financial markets. From Table 1 we observe that the data on range-based variances exhibit extremely heavy tails as indicated by kurtosis values ranging between 169.88 for monthly and 782.70 for daily data respectively. Unsurprisingly, Jarque-Bera tests reject the normal distribution regardless the data frequency.

4. Statistical analysis

4.1. Power laws

Consistent with earlier research (Grobys, 2023, 2024; Fathi et al., 2025), we model range-based crude oil price variances using the following power-law function:

$$p(x) = Cx^{-\alpha} \tag{2}$$

where $C = (\alpha - 1)x_{MIN}^{\alpha-1}$ with $\alpha \in \{\mathbb{R}_+ | \alpha > 1\}$, $x = \sigma_t^2$ denotes the respective annualized range-based crude oil variance for some time frequency, provided that $x \in \{\mathbb{R}_+ | x_{MIN} \leq x < \infty\}$, x_{MIN} is the minimum realization governed by the power-law, whereas α is the magnitude of the corresponding power-law exponent, which captures via extrapolation the low-probability deviation not seen in the data (Taleb, 2020).[5] It is noteworthy that Cirillo and Taleb (2020, p. 606) highlight that “the more fat-tailed a statistical distribution, the more the ‘tail wags the dog’. That is to say, more statistical information resides in the extremes and less in the ‘bulk’—the events of high frequency—where it becomes almost noise.” Motivated by Cirillo and Taleb (2020), in what follows, we exclusively focus our analysis on the tail of the distribution, that is, the region where the statistical signal resides, that is, $x \geq x_{MIN}$. From Eq. (2), it follows that conditional moments of order k, $E[x|x > x_{MIN}]$ are defined as:

$$E[X^k | x > x_{MIN}] = \frac{(\alpha - 1)}{(\alpha - 1 - k)} x_{MIN}^k \tag{3}$$

From Eq. (3) it becomes evident that the theoretical mean only exists for $\alpha > 2$, whereas the variance of range-based crude oil price variance only exists for $\alpha > 3$. Also, Eq. (3) implies that if $1 < \alpha \leq 2$, $E[x|x > x_{MIN}] = \infty$, and consequently, $E[x] = \infty$—obviously, a situation where the ‘tail wags the dog,’ in the parlance of Cirillo and Taleb (2020).

[3] Note that Grobys (2024) identifies investing.com as reliable data provider and finds that data on foreign exchange rates coincides with data provided from Bloomberg suggesting that both data providers share the same source.

[4] Note that recent studies from Grobys (2023, 2024) employ this variance estimator. Our choice is, first, motivated by Chou et al.’s (2010) findings that range-based variance estimators comprise more information than changes in closing prices, and moreover, Shu and Zhang’s (2006) stress that all types of range-based variance estimators perform very well. Furthermore, Grobys (2023, 2024) documents that the Parkinson variance estimator is less inflated than the Garman-Klass estimator which is manifested in lower maximum variance realizations produced from the Parkinson estimator.

[5] For the sake of notation clarity, we omit the index indicating the data frequency.

Table 1
Descriptive statistics for range-based crude oil price variance and various time frequencies. Intraday data on crude oil futures covering the January 2, 1985 to November 20, 2024 are downloaded from investing.com. Data covers open, high, and low prices over 10,150 trading days. To compute annualized daily variances, we employ the range-based variance estimator, proposed by Parkinson (1980), which is given by: $\sigma_t^2 = T\frac{1}{4\ln(2)}(\ln(H_t) - \ln(L_t))^2$, where H_t and L_t denote the highest and lowest price for crude oil on trading day t, and σ_t^2 denotes the annualized daily variance, where $T = 250$, as 250 trading days per annum are assumed. Weekly or monthly annualized crude oil price variances are computed by summing up five or 22 subsequent daily variances multiplied by $T = 52$ or $T = 12$ to obtain annualization. Computing non-overlapping data on range-based crude oil variances leaves us with 2029 weekly and 460 monthly range-based variance observations for crude oil prices. The descriptive statistics for our data are reported this table.

Frequency	Daily	Weekly	Monthly
Mean	0.1274	0.1325	0.1347
Median	0.0606	0.0789	0.0893
Maximum	18.7101	6.9257	4.3935
Minimum	0.0000	0.0031	0.0067
Std. Dev.	0.3888	0.3045	0.2640
Skewness	22.7903	15.0439	11.6077
Kurtosis	782.7017	298.6062	169.8779
Jarque-Bera (JB)	258,000,000.0000	7,464,038.0000	544,087.5000
(p-value JB)	(0.0000)	(0.0000)	(0.0000)
T	10,150	2,029	460

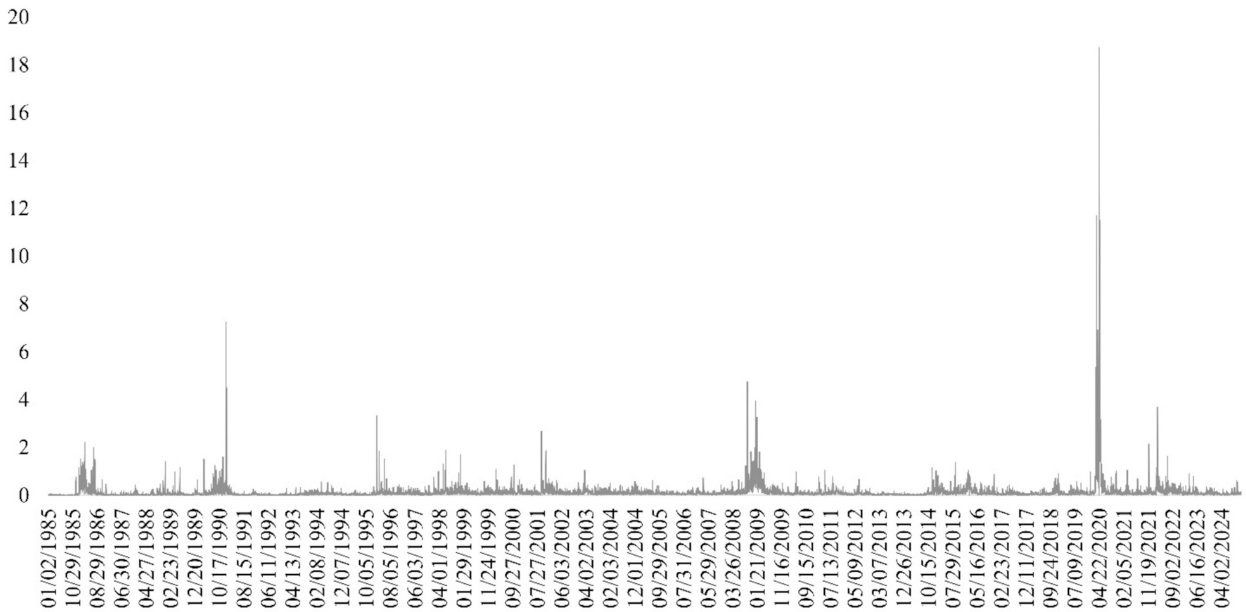

Fig. 1. Evolution of annualized daily range-based crude oil price variance. Intraday data on crude oil futures are downloaded from investing.com. Data covers open, high, and low prices over the January 2, 1985 to November 20, 2024 period. To compute annualized daily variances, we employ the range-based variance estimator, proposed by Parkinson (1980), which is given by: $\sigma_t^2 = T\frac{1}{4\ln(2)}(\ln(H_t) - \ln(L_t))^2$, where H_t and L_t denote the highest and lowest price for crude oil on trading day t, and σ_t^2 denotes the annualized daily variance, where $T = 250$, as 250 trading days per annum are assumed. This figure plots the evolution of daily data on range-based crude variances comprising 10,150 daily observations.

4.2. Maximum likelihood estimation and goodness-of-fit test

Since maximum likelihood estimation (MLE) is considered the most accurate method for estimating power-law exponents (White et al., 2008; Clauset et al., 2009), the power-law exponents for the range-based crude oil price variances are estimated as follows:

$$\widehat{\alpha} = 1 + N\left(\sum_{i=1}^{N} \ln\left(\frac{x_i}{x_{MIN}}\right)\right)^{-1} \tag{4}$$

where $\widehat{\alpha}$ denotes the MLE estimator, N is the number of observations exceeding x_{MIN}, and other notations are as previously defined. Furthermore, Clauset et al. (2009) derive the corresponding standard deviation as:

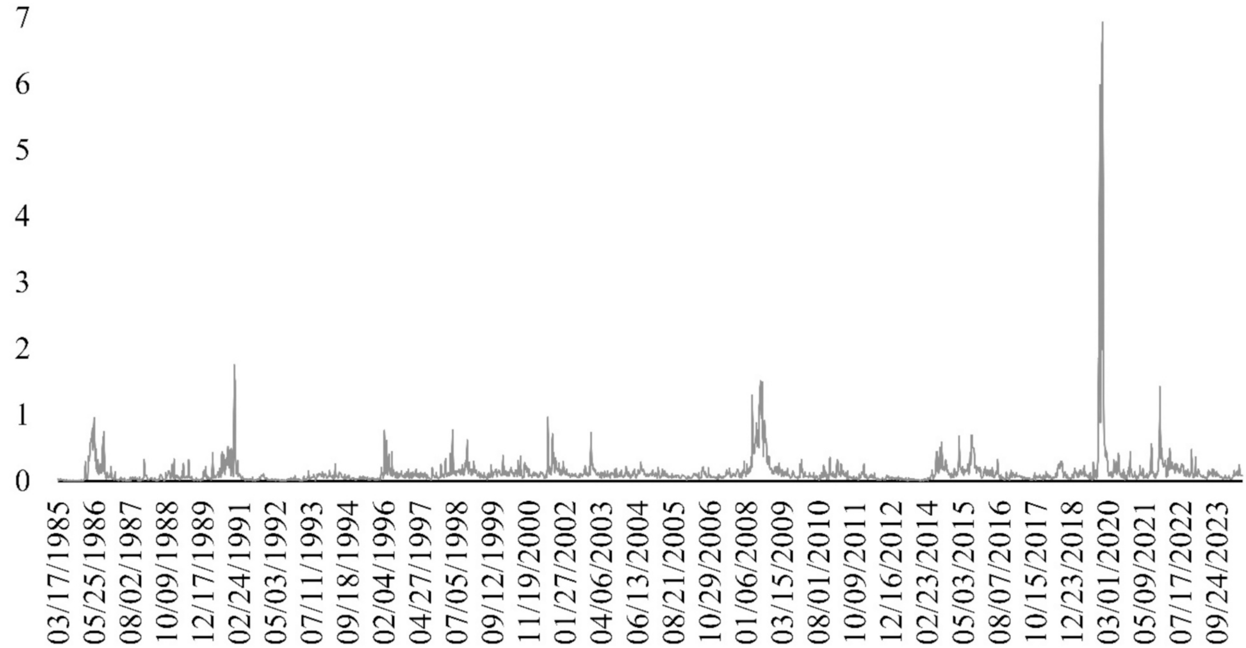

Fig. 2. Evolution of annualized weekly range-based crude oil price variance. Intraday data on crude oil futures are downloaded from investing.com. Data covers open, high, and low prices over the January 2, 1985 to November 20, 2024 period. To compute daily range-based variances, we employ the range-based variance estimator, proposed by Parkinson (1980), which is given by: $\sigma_t^2 = \frac{1}{4\ln(2)}(\ln(H_t) - \ln(L_t))^2$, where H_t and L_t denote the highest and lowest price for crude oil on trading day t, and σ_t^2 denotes the annualized daily variance. Weekly annualized crude oil price variances are computed by summing up five daily variances multiplied by $T = 52$ obtain annualization. Computing non-overlapping data on range-based weekly crude oil variances leaves us with 2029 weekly observations. This figure plots the evolution of weekly data on range-based crude variances.

$$\widehat{\sigma}_{\widehat{\alpha}} = \frac{\widehat{\alpha} - 1}{\sqrt{N}} + O\left(\frac{1}{N}\right) \tag{5}$$

Because the MLE estimator depends on the chosen x_{MIN}, the question arises which candidate should be chosen for x_{MIN}. To select x_{MIN}, Clauset et al. (2009) propose an approach based on the optimal Kolmogorov-Smirnov (KS) distance D, which measures the maximum distance between the cumulative density functions (CDFs) of the data and the fitted power-law model as defined by:

$$D = MAX_{x \geq x_{MIN}} |S(x) - P(x)| \tag{6}$$

where $S(x)$ is the CDF of the data for the observation with a value of at least x_{MIN}, and $P(x)$ is the CDF for the power-law model that best fits the data in the region $x \geq x_{MIN}$. The optimal $\widehat{x}_{MIN}$ is then the value of x_{MIN} that minimizes D. Clauset et al.'s (2009) findings suggest that their proposed approach to select the optimal x_{MIN} outperforms traditional log–log regressions.

To analyze whether the power-law model given by the parameter vector $(\widehat{\alpha}, \widehat{x}_{MIN})$ is plausible, this study uses the goodness-of-fit test (GoF), as derived from Clauset et al. (2009). This GoF produces a p-value that quantifies the plausibility of the power-law null model by comparing D from Eq. (6) with distance measurements for comparable synthetic data sets drawn from the hypothesized model, generated by the function $p(x) = (\widehat{\alpha} - 1)\widehat{x}_{MIN}^{\widehat{\alpha}-1} x^{-\widehat{\alpha}}$. The p-value is then calculated as the fraction of synthetic distances that exceed the empirical distance. Consistent with Clauset et al. (2009), we make the conservative choice that the power law is ruled out if $p \leq 0.1$; that is, it is ruled out if there is a probability of 1 in 10 or less that we would merely by chance get data that agree as poorly with the model as the data we have.[6]

4.3. Computing robust standard deviations via modern blocks bootstraps

Grobys (2024) notes that $\widehat{\sigma}_{\widehat{\alpha}}$, as defined in Eq. (5), is derived under the assumption of independently distributed observations (Clauset et al., 2009). In the presence of dependency structures such as volatility clustering—a stylized fact of financial markets—the estimator $\widehat{\sigma}_{\widehat{\alpha}}$ will be biased. To address this issue, Grobys (2024) proposes the usage of modern blocks bootstraps (BB). Denoting the selected block length as m, a BB procedure is chosen such that $E[m] = \sqrt{T}$. Let us define that data vector $\boldsymbol{y} \in \mathbb{R}^{Tx1}$ consisting of crude oil variances for some given time scale. We draw blocks of the dimension m randomly from the vector $\boldsymbol{y}$. These blocks are governed by a

[6] The GoF test is detailed in Clauset et al. (2009).

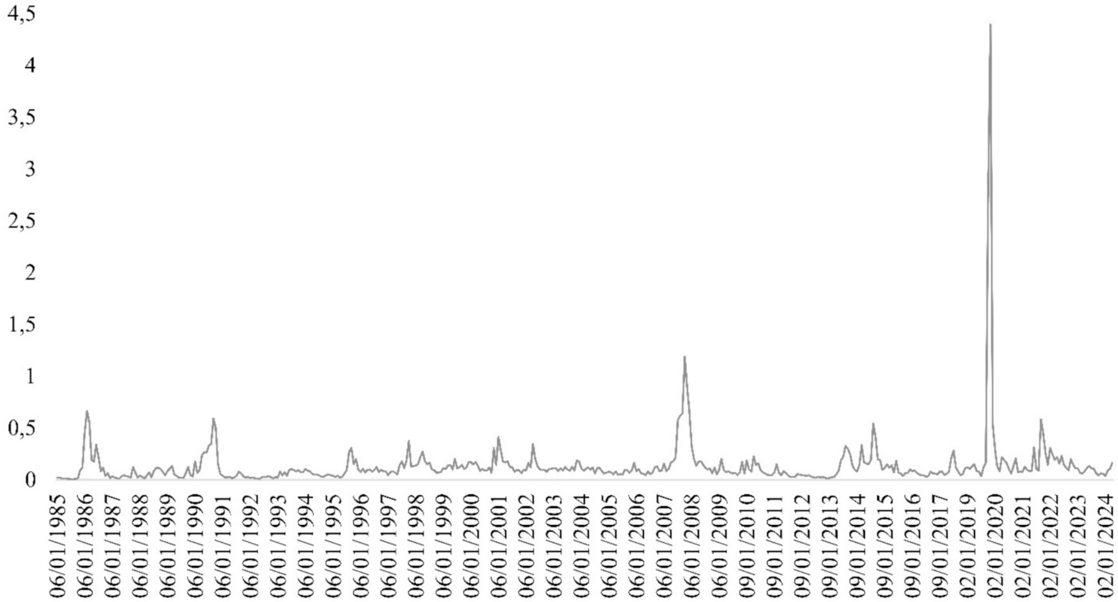

Fig. 3. Evolution of annualized monthly range-based crude oil price variance. Intraday data on crude oil futures are downloaded from investing.com. Data covers open, high, and low prices over the January 2, 1985 to November 20, 2024 period. To compute daily range-based variances, we employ the range-based variance estimator, proposed by Parkinson (1980), which is given by: $\sigma_t^2 = \frac{1}{4\ln(2)}(\ln(H_t) - \ln(L_t))^2$, where H_t and L_t denote the highest and lowest price for crude oil on trading day t, and σ_t^2 denotes the annualized daily variance. Weekly annualized crude oil price variances are computed by summing up 22 daily variances multiplied by $T = 12$ obtain annualization. Computing non-overlapping data on range-based monthly crude oil variances leaves us with 460 monthly observations. This figure plots the evolution of monthly data on range-based crude variances.

geometric distribution, that is, $m\ GEO(p)$ with $E[m] = \frac{(1-p)}{p}$. Drawing geometrically-distributed random blocks ensures that the data are stationarity (Godfrey, 2009). For example, our daily data comprises $T = 10150$, and because $\sqrt{T} \approx 101$, it follows that $p = 0.0098$. Note that the blocks drawn from $\boldsymbol{y}$ vary in lengths. The randomly drawn blocks m are stacked in vector $\boldsymbol{y}_b$ as:

$$\boldsymbol{y}_b = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \end{bmatrix} \tag{7}$$

The procedure is stopped when the length of the synthetic data vector $\boldsymbol{y}_b$ exhibits a length exceeding T. Observations exceeding T are cut off to ensure that the synthetic data vector $\boldsymbol{y}_b$ exhibits the same length as the original data vector $\boldsymbol{y}$. This process corresponds to one iteration b of the BB procedure. Employing this BB, for each iteration $b = 1, \cdots, B$, the MLE estimators are obtained in accordance with Eq. (4), yielding, $[\,\widehat{\alpha}_1 \quad \widehat{\alpha}_2 \quad \cdots \quad \widehat{\alpha}_B\,]$. Using $B = 1,000$ iterations, the bootstrapped point estimates, the corresponding bootstrapped (robust) standard deviations $\widehat{\sigma}_{\widehat{\alpha}_{BOOT}}$ are then given by:

$$\widehat{\sigma}_{\widehat{\alpha}_{BOOT}} = \sqrt{\frac{1}{B}\sum_{b=1}^{B}\left(\widehat{\alpha}_b - \overline{\widehat{\alpha}}\right)^2} \tag{8}$$

where $\overline{\widehat{\alpha}} = \sum_{b=1}^{B}\frac{\widehat{\alpha}_b}{B}$. Note that Grobys (2024) highlights that this approach is robust to unknown dependency structures in the data, providing a reliable estimator for the standard deviation of the estimated power-law exponent.

4.4. Testing for scale invariance

To explore whether crude oil variances are time scale invariant, we adopt the test procedure proposed by Grobys (2024), that is:

$$\widehat{\lambda} = (\dot{\boldsymbol{\alpha}} - q\mathbf{1})' \,\widehat{\boldsymbol{\Sigma}}_{\dot{\alpha}}^{-1} (\dot{\boldsymbol{\alpha}} - q\mathbf{1}) \tag{9}$$

with $\dot{\boldsymbol{\alpha}} = (\widehat{\alpha}_D, \widehat{\alpha}_W, \widehat{\alpha}_M)$, where $\widehat{\alpha}_D$, $\widehat{\alpha}_W$, and $\widehat{\alpha}_M$ denote point estimators for the power-law exponents derived from daily (D), weekly (W), and monthly (M) range-based crude oil variance data, $\mathbf{1} \in \mathbb{R}^{3x1}$ is a vector consisting of ones, q is the hypothesized scale invariant power-law exponent, and $\widehat{\boldsymbol{\Sigma}}_{\widehat{\boldsymbol{\alpha}}}$ is defined as:

$$\widehat{\boldsymbol{\Sigma}}_{\widehat{\boldsymbol{\alpha}}} = \begin{pmatrix} \widehat{\sigma}^2_{\widehat{\alpha}_{BB,D}} & 0 & 0 \\ 0 & \widehat{\sigma}^2_{\widehat{\alpha}_{BB,W}} & 0 \\ 0 & 0 & \widehat{\sigma}^2_{\widehat{\alpha}_{BB,M}} \end{pmatrix} \tag{10}$$

where $\widehat{\sigma}^2_{\widehat{\alpha}_{BB,D}}$, $\widehat{\sigma}^2_{\widehat{\alpha}_{BB,W}}$, and $\widehat{\sigma}^2_{\widehat{\alpha}_{BB,M}}$ denote robust variances for the estimated power-law exponents derived from BB. The test statistic $\widehat{\lambda}$ can be re-written as:

$$\widehat{\lambda} = \sum_i \frac{(\widehat{\alpha}_i - q)^2}{\widehat{\sigma}^2_{\widehat{\alpha}_{BB,i}}} \tag{11}$$

where $i = \{D, W, M\}$ indicates the corresponding time frequency. The estimated test statistic, denoted as $\widehat{\lambda}$, is under the null hypothesis distributed as $\chi^2(3)$. Following Grobys (2024), the test statistic is iteratively computed for various hypothesized values of the common exponent q. Here, we examine the economically relevant interval $q = (2, 2.1, \cdots, 3.1)$. Using a statistical significance level of 5 % level, the null hypothesis is not rejected for $\widehat{\lambda} < 7.81$. Note that rejecting the null hypothesis $\boldsymbol{\alpha}' = q\mathbf{1}$ for all $q = (2, 2.1, \cdots, 3.1)$ would indicate that crude oil uncertainty exhibits time scale-specific power-law behavior.

4.5. Testing for total invariance

Time scale invariance does not necessarily imply invariance over time. For instance, some studies have documented that tail risk is time-varying (Galbraith and Zernov, 2004; Quintos et al., 2001; Wagner, 2003; Werner and Upper, 2004). Conversely, recent studies by Grobys (2023, 2024) and Fathi, Grobys, & Äijö, 2025 do not support this view, documenting that power-law behavior in foreign exchange rate variance or Fama-French equity factor variances remains stable over time. Therefore, to examine whether the power-law behavior of crude oil price variances is not only invariant across time scales but also invariant over time, we employ the test for total invariance proposed by Grobys (2024), defined as:

$$\widehat{\lambda} = \sum_i \sum_j \frac{(\widehat{\alpha}_{i,j} - q)^2}{\widehat{\sigma}^2_{\widehat{\alpha}_{BB,i,j}}} \tag{12}$$

where $i = \{D, W, M\}$ denotes the corresponding time frequency, and $j = \{1, 2\}$ indicates subsample 1 or subsample 2. Again, point estimates for the power-law exponents are obtained via MLE (Clauset et al., 2009), while robust standard deviations are computed using the BB procedure for each subsample and data frequency. The estimated test statistic, denoted as $\widehat{\lambda}$, is under the null hypothesis distributed as $\chi^2(6)$. Again, the test statistic is iteratively estimated for the economically important interval $q = (2, 2.1, \cdots, 3.1)$. Using a statistical significance level of 5 % level, the null hypothesis is not rejected for $\widehat{\lambda} < 12.59$. Rejection of the null hypothesis $\boldsymbol{\alpha}' = q\mathbf{1}$ for all $q = (2, 2.1, \cdots, 3.1)$ would indicate that power-law behavior of crude oil price uncertainty is varying over time or varying across time scales.

4.6. Robustness checks

With regard to the main analysis, a reader might question whether the observed power-law behavior is truly genuine, as time-varying volatility could induce power law-like characteristics in the tails of the distribution. A potential consequence is that conventional statistical methods for estimating power laws might erroneously indicate the presence of a power law when none actually exists. Furthermore, it may be noteworthy that the GoF test proposed in Clauset et al. (2009) is derived under the assumption of independently distributed observations. Therefore, to address these issues, we follow Fathi, Grobys, & Äijö, 2025 by fitting autoregressive models of order p (AR(p)) to the realized variance data. The order p is chosen with respect to the partial autocorrelation function of the realized variance data for a given time frequency:

$$x_{i,t} = b_{i,0} + b_{i,1}x_{i,t-1} + b_{i,2}x_{i,t-2} + \cdots + b_{i,p}x_{i,t-p} + \epsilon_{i,t} \tag{13}$$

where $x_{i,t}$ denotes the respective annualized range-based crude oil variance for some time frequency $i = \{D, W, M\}$, $b_{i,0}, b_{i,1}, \cdots, b_{i,p}$ are autoregressive parameters to be estimated, whereas $\epsilon_{i,t}$ is the innovation process that is independently-distributed via construction. We hypothesize that if range-based crude oil variances are subject to genuine power-law behavior, the innovation process $\epsilon_{i,t}$ should be governed by a power law too. To test this issue, we carry out Clauset et al.'s (2009) GoF test, as described in section 4.2, while making use of the absolute amount of the estimated innovation process, that is, $|\widehat{\epsilon}_{i,t}|$.

Finally, it is well documented in the literature that the realized volatility of financial assets is approximately lognormal (e.g., Andersen et al., 2001; Andersen et al., 2001a; Andersen et al., 2001b). Conversely, Renò and Rizza (2003), who examine the unconditional volatility distribution of the Italian futures market, conclude that the standard assumption of lognormal unconditional volatility must be rejected, and that a power-law model provides a much better description. Therefore, to assess the validity of the lognormality assumption, we explicitly test for lognormality. To do so, we first transform the data by taking the natural logarithm of the realized variances, and then we carry out the Jarque-Bera (JB) test, a commonly used test for normality based on the skewness and kurtosis of the data. The JB test is defined as:

$$\lambda_{JB} = \frac{T}{6}\left(S^2 + \frac{(K-3)^2}{4}\right) \tag{14}$$

where T is the sample size, S denotes the sample skewness $S = \frac{1/T\sum_{t=1}^{T}(y_{i,t}-\bar{y}_i)^3}{\left(1/T\sum_{t=1}^{T}(y_{i,t}-\bar{y}_i)^2\right)^{3/2}}$, and K is the sample kurtosis defined as $K = \frac{1/T\sum_{t=1}^{T}(y_{i,t}-\bar{y}_i)^4}{\left(1/T\sum_{t=1}^{T}(y_{i,t}-\bar{y}_i)^2\right)^2}$. Furthermore, $y_{i,t} = \ln(x_{i,t})$ is either the natural logarithm of range-based crude oil price variance for a given frequency i, or $y_{i,t} = \ln(|\widehat{\epsilon}_{i,t}|)$, as defined earlier. Note that $\lambda_{JB} \sim \chi^2(2)$ and using a conventional significance level of 5 %, we do not reject the null hypothesis of lognormality if $\widehat{\lambda}_{JB} < 5.99$.

5. Results

Table 2 presents the estimated power-law exponents for realized crude oil variances across different time frequencies, derived using the Parkinson estimator and MLE, as outlined in Eq. (4). The sample period spans from January 2, 1985, to November 20, 2024. The results indicate that the estimated power-law exponents range from $\widehat{\alpha} = 2.5503$ for (annualized) daily range-based variances to $\widehat{\alpha} = 2.7030$ for (annualized) monthly range-based variances. Furthermore, the proportion of observations governed by power-law processes varies between 11.64 % for daily data and 47.83 % for monthly data. This suggests that at lower time frequencies, a larger fraction of sample observations adheres to a power-law distribution. These point estimates align closely with those reported by Grobys (2023), who examined power-law behavior in annualized daily range-based variances of G10 currency pairs over the period from May 16, 2006, to November 19, 2021. Specifically, Grobys (2023) documented power-law exponents ranging from $\widehat{\alpha} = 2.25$ for annualized daily AUD/USD range-based variances to $\widehat{\alpha} = 2.78$ for annualized daily EUR/USD range-based variances.

Moreover, implementing Clauset et al.'s (2009) GoF tests yields p-values ranging from 0.3330 (daily data) to 0.9480 (weekly data), indicating robust support for the power-law null model across all time scales.[7] These findings corroborate recent studies, such as Grobys (2023, 2024) and Fathi (2025), which report consistent evidence for power-law behavior in realized variances of foreign exchange rates and Fama-French equity factors across varying time scales.

While the point estimates for power-law exponents derived via MLE are unbiased, the associated standard deviations may underestimate parameter uncertainty in the presence of dependency structures, such as volatility clustering. Given the long-range dependence observed in crude oil price variances, the BB procedure is employed to obtain robust standard deviations. Consistent with Grobys (2024), modern BB is implemented with an expected block length of $\sqrt{T}$ and defined by a geometric distribution, governed by parameters $p = 0.0098$, $p = 0.0217$, and $p = 0.0455$ for daily, weekly, and monthly data, respectively. The results of $B = 1,000$ BB replications, reported in Table 3, reveal that the average point estimates for bootstrapped power-law exponents align closely with those in Table 2. For instance, the MLE estimate for daily data is $\widehat{\alpha} = 2.5503$, while the point estimate obtained via BB is estimated at $\overline{\widehat{\alpha}} = \sum_{b=1}^{B}\frac{\widehat{\alpha}_b}{B} = 2.5618$, both unbiased. However, notable differences emerge in standard deviations: Table 2 reports a standard deviation of $\widehat{\sigma}_{\widehat{\alpha}} = 0.0460$, whereas Table 3 indicates a robust standard deviation of $\widehat{\sigma}_{\widehat{\alpha}_{BB}} = 0.1875$, approximately four times higher. This highlights the importance of implementing BB to derive robust estimators for parameter uncertainty.

Using the point estimates from Table 2 and the robust standard deviations from Table 3, we test for scale invariance using the framework detailed in Eqs. (9)–(11). Key results, reported in Table 4, include the following: First, the null hypothesis is rejected for $\alpha < 2.3$. Because Eq. (3) implies that the theoretical mean does not exist for $\alpha \leq 2$, rejection of a joint exponent $\alpha < 2.3$ means, in turn, that the theoretical mean of range-based crude oil variances does exist. Second, the null hypothesis is also rejected for $\alpha > 2.9$. Because Eq. (3) implies that the theoretical variance does not exist for $\alpha \leq 3$, rejection of a joint exponent $\alpha > 2.9$ means, in turn, that the theoretical variance of range-based crude oil variances is infinite. Third, the null hypothesis cannot be rejected for $2.3 \leq \alpha \leq 2.9$ which suggests that range-based crude oil price variance is (a) scale invariant, and (b) because $2 < \alpha < 3$, the population mean of this joint power law process is defined, whereas the theoretical variance is infinite. Remarkably, the optimal exponent is estimated at $\alpha = 2.6$—identical to the universal power-law exponent governing range-based foreign exchange rate variances (Grobys, 2024). The associated p-value of 0.9691 suggests a high level of consistency between the observed data and the null hypothesis, providing no statistically significant evidence to reject it and thereby reinforcing the robustness of these findings.

While scale invariance implies consistent power-law behavior across time scales, it does not necessarily guarantee temporal invariance. To test for Grobys' (2024) concept of total invariance, encompassing both scale and intertemporal invariance, the data are split into two non-overlapping subsamples of equal length. Descriptive statistics for these subsamples are provided in Appendix Table A.1. Table 5 reports the estimated power-law exponents for subsamples across various time scales, derived using

[7] To implement the GoF tests, we used 250 iterations for each hypothesis test.

Table 2
Estimated power-law models for various range-based crude oil price variance frequencies. This table reports power-law exponents for various time frequencies estimated as: $\hat{\alpha} = 1 + N\left(\sum_{i=1}^{N} \ln\left(\frac{x_i}{x_{MIN}}\right)\right)^{-1}$, where $x = \sigma_t^2$ denotes the respective annualized crude oil variance for some time frequency, provided that $x \in \{\mathbb{R}_+ | x_{MIN} \leq x < \infty\}$, $\hat{\alpha}$ denotes the MLE estimator, N is the number of observations exceeding x_{MIN}. The corresponding standard deviation of $\hat{\alpha}$ is estimated as: $\hat{\sigma}_{\hat{\alpha}} = \frac{\hat{\alpha}-1}{\sqrt{N}} + O\left(\frac{1}{N}\right)$. The optimal value for x_{MIN} is selected via using the optimal Kolmogorov-Smirnov (KS) distance D, which measures the maximum distance between the cumulative density functions (CDFs) of the data and the fitted power-law model as defined by: $D = MAX_{x \geq x_{MIN}} |S(x) - P(x)|$, where $S(x)$ is the CDF of the data for the observation with a value of at least x_{MIN}, and $P(x)$ is the CDF for the power-law model that best fits the data in the region $x \geq x_{MIN}$. The optimal $\hat{x}_{MIN}$ is then the value of x_{MIN} that minimizes D. To analyze the plausibility of the power-law model, the estimated parameter vector $(\hat{\alpha}, \hat{x}_{MIN})$ employed in a goodness-of-fit test (GoF) producing a p-value that quantifies the plausibility of the power-law null model by comparing D with distance measurements for comparable synthetic data sets drawn from the hypothesized model, generated by the function $p(x) = (\hat{\alpha} - 1)\hat{x}_{MIN}^{\hat{\alpha}-1} x^{-\hat{\alpha}}$. The p-value is calculated as the fraction of synthetic distances that exceed the empirical distance.

Frequency	Daily	Weekly	Monthly
$\hat{\alpha}$	2.5503	2.5843	2.7030
$\hat{\sigma}_{\hat{\alpha}}$	0.0460	0.0628	0.1194
$\hat{x}_{min}$	0.2233	0.1114	0.0936
N (%)	1,181 (11.64 %)	668 (32.92 %)	220 (47.83 %)
T	10,150	2,029	460
p-value (GoF test)[a]	0.3320	0.9480	0.7160

[a] The p-value derived from Clauset et al.'s (2009) GoF test uses 250 synthetic data sets.

Table 3
Descriptive statistics for bootstrapped exponents for the overall samples. Denoting the selected block length as m, a blocks bootstrap procedure is chosen such that $E[m] = \sqrt{T}$. Defining $\boldsymbol{y}$ as a Tx1 data vector of crude oil variances for some given time scale, we draw blocks of the dimension m randomly from the vector $\boldsymbol{y}$. These blocks are governed by a geometric distribution, that is, m $GEO(p)$ with $E[m] = \frac{(1-p)}{p}$. Drawing geometrically-distributed random blocks ensures that the data are stationarity (Godfrey, 2009). For example, our daily data comprises $T = 10150$, and because $\sqrt{T} \approx 101$, it follows that $p = 0.0098$. Note that the blocks drawn from $\boldsymbol{y}$ vary in lengths. The randomly drawn blocks m are stacked in vector $\boldsymbol{y}_b$ as: $\boldsymbol{y}_b = [m_1, m_2, m_3, ...]'$. The procedure is stopped when the length of the synthetic data vector $\boldsymbol{y}_b$ exhibits a length exceeding T. Observations exceeding T are cut off to ensure that the synthetic data vector $\boldsymbol{y}_b$ exhibits the same length as the original data vector $\boldsymbol{y}$. This process corresponds to one iteration b of the blocks bootstrap procedure. Employing this blocks bootstraps, for each iteration b, the MLE estimators are estimated using the procedure described in the previous section, giving us: $[\hat{\alpha}_1 \quad \hat{\alpha}_2 \quad \cdots \quad \hat{\alpha}_B]$, with $B = 1,000$ iterations. Using the bootstrapped point estimates, the corresponding bootstrapped (robust) standard deviations $\hat{\sigma}_{\hat{\alpha}_{BOOT}}$ are then given by: $\hat{\sigma}_{\hat{\alpha}_{BOOT}} = \sqrt{\frac{1}{B}\sum_{b=1}^{B}(\hat{\alpha}_b - \bar{\hat{\alpha}})^2}$, where $\bar{\hat{\alpha}} = \sum_{b=1}^{B} \frac{\hat{\alpha}_b}{B}$. This table reports the descriptive statistics for the estimated power-law exponents derived from blocks bootstraps for various time frequencies.

Frequency	Daily	Weekly	Monthly
Mean	2.5618	2.6086	2.7775
Median	2.5505	2.5836	2.7456
Maximum	3.6524	3.6789	3.8054
Minimum	2.0613	2.1285	2.2441
Std. Dev.	0.1875	0.2109	0.2441
Skewness	0.7931	0.9844	0.8549
Kurtosis	4.9705	5.2190	4.3041
Jarque-Bera (JB)	266.6232	366.6744	192.6755
(p-value JB)	0.0000	0.0000	0.0000
B	1,000	1,000	1,000

Clauset et al.'s (2009) MLE approach. The estimates range from $\hat{\alpha} = 2.4267$ for daily data (Subsample 2) to $\hat{\alpha} = 2.9035$ for monthly data (Subsample 1). Robust standard deviations, obtained via BB and reported in Table 6, are larger for subsamples than for the full sample, reflecting increased parameter uncertainty, consistent with Grobys (2024).

Using these estimates, total invariance is tested as per Eq. (9). Results in Table 7 reveal the following: We observe that the null hypothesis is rejected for $\alpha < 2.4$, reinforcing the robustness of our findings. Again, this result means that the theoretical mean of realized crude oil price variances does exist. Second, the null hypothesis is also rejected for $\alpha > 3$, implying that the theoretical variance of range-based crude oil variances is infinite. Third, the null hypothesis cannot be rejected for $2.4 \leq \alpha \leq 3$ which suggests that range-based crude oil price variance is subject to total invariance. Whereas this test suggests that the optimum is reached for $\alpha \approx 2.7$ (p-value 0.9064), the earlier result $\alpha = 2.6$ remains robust, as indicated by the p-value of 0.8751.

Given the autocorrelation observed in partial autocorrelation functions (unreported), range-based crude oil price variances are further whitened using AR(p) models. Lag orders of $p = 3$ for daily and monthly data, and $p = 6$ for weekly data, are determined from

Table 4
Testing for scale-invariance. To examine whether range-based crude oil variance is scale invariant, we carry out the following statistical test: $\widehat{\lambda} = \sum_i \frac{(\widehat{\alpha}_i - q)^2}{\widehat{\sigma}^2_{\widehat{\alpha}_{BOOT},i}}$, where $i = \{D, W, M\}$ indicates the corresponding time frequency. The estimated test statistic, denoted as $\widehat{\lambda}$, is under the null hypothesis distributed as $\chi^2(3)$. The test statistic is iteratively estimated using the economically important interval $q = (2, 2.1, \cdots, 3.1)$. Using a statistical significance level of 5 % level, the null hypothesis is not rejected for $\widehat{\lambda} < 7.81$. **Bold** figures denote that the null hypothesis is rejected on a 5 % level.

α	$\widehat{\lambda}$	p-value
2	**24.58**	**0.0000**
2.1	**17.14**	**0.0001**
2.2	**11.06**	**0.0114**
2.3	6.32	0.0970
2.4	2.95	0.3994
2.5	0.92	0.8206
2.6	0.25	0.9691
2.7	0.94	0.8158
2.8	2.98	0.3947
2.9	6.37	0.0949
3	**11.12**	**0.0111**
3.1	**17.22**	**0.0006**

Table 5
Estimated power-law models for various subsamples. This table reports power-law exponents for various time frequencies and two non-overlapping subsamples estimated as: $\widehat{\alpha} = 1 + N\left(\sum_{i=1}^{N} \ln\left(\frac{x_i}{x_{MIN}}\right)\right)^{-1}$, where $x = \sigma_t^2$ denotes the respective annualized crude oil variance for some time frequency and subsample, provided that $x \in \{\mathbb{R}_+ | x_{MIN} \leq x < \infty\}$, $\widehat{\alpha}$ denotes the MLE estimator, N is the number of observations exceeding x_{MIN}. The corresponding standard deviation of $\widehat{\alpha}$ is estimated as: $\widehat{\sigma}_{\widehat{\alpha}} = \frac{\widehat{\alpha} - 1}{\sqrt{N}} + O\left(\frac{1}{N}\right)$. The optimal value for x_{MIN} is selected via using the optimal Kolmogorov-Smirnov (KS) distance D, which measures the maximum distance between the cumulative density functions (CDFs) of the data and the fitted power-law model as defined by: $D = MAX_{x \geq x_{MIN}} |S(x) - P(x)|$, where $S(x)$ is the CDF of the data for the observation with a value of at least x_{MIN}, and $P(x)$ is the CDF for the power-law model that best fits the data in the region $x \geq x_{MIN}$. The optimal $\widehat{x}_{MIN}$ is then the value of x_{MIN} that minimizes D. To analyze the plausibility of the power-law model, the estimated parameter vector $(\widehat{\alpha}, \widehat{x}_{MIN})$ employed in a goodness-of-fit test (GoF) producing a p-value that quantifies the plausibility of the power-law null model by comparing D with distance measurements for comparable synthetic data sets drawn from the hypothesized model, generated by the function $p(x) = (\widehat{\alpha} - 1)\widehat{x}_{MIN}^{\widehat{\alpha}-1} x^{-\widehat{\alpha}}$. The p-value is calculated as the fraction of synthetic distances that exceed the empirical distance.

Frequency	Daily		Weekly		Monthly	
Sample	Sample 1	Sample 2	Sample 1	Sample 2	Sample 1	Sample 2
$\widehat{\alpha}$	2.7975	2.4267	2.7072	2.4819	2.9035	2.5574
$\widehat{x}_{min}$	0.2411	0.2134	0.0889	0.1165	0.0874	0.1030
T	5,075	5,075	1,015	1,014	230	230

these functions. Table A.2 in the appendix reports the fitted AR(p) models, showing statistically significant parameters (at least a 1 % significance level) and coefficients of determination ranging from $R^2 = 0.3542$ for daily data to $R^2 = 0.5175$ for weekly data. Using absolute residuals from AR(p) models (viz., $|\widehat{\epsilon}_{i,t}|$), GoF tests are conducted, as summarized in Table A.4. The results, with p-values between 0.2500 (weekly data) and 0.6560 (monthly data), confirm that the power-law null model cannot be rejected for $|\widehat{\epsilon}_{i,t}|$, consistent with Fathi, Grobys, & Äijö, 2025. Finally, to test for lognormality, range-based crude oil price variances are log-transformed, and the Jarque-Bera (JB) test is applied. Results in Table A.5 clearly reject lognormality across all time scales ($\widehat{\lambda}_{JB} > 5.99$). As a robustness check, the absolute residuals from AR(p) models are subjected to JB tests, which also reject lognormality, further corroborating these findings.

6. Conclusion

Standard approaches to modeling crude oil price volatility, rooted largely in Gaussian or lognormal assumptions, remain conceptually and empirically inadequate. These models, though mathematically tractable, impose finite-variance constraints and disregard the heavy-tailed behavior and long-range dependencies often observed in actual market data. As such, they fall short in

Table 6
Descriptive statistics for bootstrapped power-law exponents for various subsamples. Denoting the selected block length as m, a blocks bootstrap procedure is chosen such that $E[m] = \sqrt{T}$. Defining $\boldsymbol{y}$ as a Tx1 data vector of crude oil variances for some given time scale and subsample, we draw blocks of the dimension m randomly from the vector $\boldsymbol{y}$. These blocks are governed by a geometric distribution, that is, m $GEO(p)$ with $E[m] = \frac{(1-p)}{p}$. Drawing geometrically-distributed random blocks ensures that the data are stationarity (Godfrey, 2009). For example, our daily subsample data comprises $T = 5075$, and because $\sqrt{T} \approx 71$, it follows that $p = 0.0139$. Note that the blocks drawn from $\boldsymbol{y}$ vary in lengths. The randomly drawn blocks m are stacked in vector $\boldsymbol{y}_b$ as: $\boldsymbol{y}_b = [m_1, m_2, m_3, \ldots]$'. The procedure is stopped when the length of the synthetic data vector $\boldsymbol{y}_b$ exhibits a length exceeding T. Observations exceeding T are cut off to ensure that the synthetic data vector $\boldsymbol{y}_b$ exhibits the same length as the original data vector $\boldsymbol{y}$. This process corresponds to one iteration b of the blocks bootstrap procedure. Employing this blocks bootstraps, for each iteration b, the MLE estimators are estimated using the procedure described in the previous section, giving us: $[\widehat{\alpha}_1 \quad \widehat{\alpha}_2 \quad \cdots \quad \widehat{\alpha}_B]$, with $B = 1000$ iterations. Using the bootstrapped point estimates, the corresponding bootstrapped (robust) standard deviations $\widehat{\sigma}_{\widehat{\alpha}_{BOOT}}$ are then given by: $\widehat{\sigma}_{\widehat{\alpha}_{BOOT}} = \sqrt{\frac{1}{B}\sum_{b=1}^{B}(\widehat{\alpha}_b - \overline{\widehat{\alpha}})^2}$, where $\overline{\widehat{\alpha}} = \sum_{b=1}^{B}\frac{\widehat{\alpha}_b}{B}$. This table reports the descriptive statistics for the estimated power-law exponents derived from blocks bootstraps for various time frequencies.

Sample	Sample 1			Sample 2		
Frequency	Daily	Weekly	Monthly	Daily	Weekly	Monthly
Mean	2.4348	2.7593	3.0198	2.4824	2.5277	2.5847
Median	2.4198	2.7354	2.9857	2.4321	2.4740	2.5010
Maximum	3.8014	3.9557	4.8444	4.2770	4.3121	5.2415
Minimum	1.8853	2.1985	1.7693	1.9767	1.9546	1.9851
Std. Dev.	0.1989	0.2481	0.3530	0.2881	0.3041	0.3642
Skewness	0.9262	0.7278	0.6901	1.4458	1.4410	2.0463
Kurtosis	6.7062	4.3478	4.5781	6.9501	6.8694	10.0502
Jarque-Bera (JB)	715.3041	163.9698	183.1428	998.5420	969.9163	2,768.9700
(p-value JB)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
B	1,000	1,000	1,000	1,000	1,000	1,000

Table 7
Testing for total invariance. To examine whether range-based crude oil variance is scale invariant, we carry out the following statistical test: $\widehat{\lambda} = \sum_i\sum_j \frac{(\widehat{\alpha}_{i,j}-q)^2}{\widehat{\sigma}^2_{\widehat{\alpha}_{BOOT},i,j}}$, where $i = \{D, W, M\}$ indicates the corresponding time frequency, and $j \in \{1, 2\}$ indicates subsample 1 or 2. The estimated test statistic, denoted as $\widehat{\lambda}$, is under the null hypothesis distributed as $\chi^2(6)$. The test statistic is iteratively estimated for the economically important interval $q = (2, 2.1, \cdots, 3.1)$. Using a statistical significance level of 5 % level, the null hypothesis is not rejected for $\widehat{\lambda} < 12.59$. **Bold** figures denote that the null hypothesis is rejected on a 5 % level.

α	$\widehat{\lambda}$	p-value
2	**37.80**	**0.0000**
2.1	**27.91**	**0.0001**
2.2	**19.62**	**0.0032**
2.3	**12.92**	**0.0443**
2.4	7.83	0.2508
2.5	4.33	0.6321
2.6	2.44	0.8751
2.7	2.14	0.9064
2.8	3.44	0.7519
2.9	6.34	0.3862
3	10.84	0.0935
3.1	**16.94**	**0.0095**

periods of market stress, where extreme fluctuations dominate the empirical record and elude conventional estimation.

This study advances an alternative by examining the statistical form of realized variance through the lens of power-law distributions. Employing range-based variance measures and maximum likelihood estimation of scaling exponents, we document that crude oil price variance exhibits both scale invariance and total invariance. These findings signal the presence of fractal-like structure and infinite-variance behavior, incompatible with the assumptions underpinning much of the volatility modeling literature.

By extending the total invariance test originally developed for exchange rate markets (Grobys, 2024) into the domain of energy volatility, this paper introduces a methodological and empirical refinement. The discovery of stable scaling behavior provides a foundation for rethinking the nature of volatility in financial markets—one that is not contingent on conditional means or variance dynamics, but rather grounded in more fundamental statistical regularities. The implications are both theoretical and practical. For econometricians, the results suggest that predictive models should accommodate infinite variance regimes and scaling laws, rather than rely solely on mean-reverting structures. For practitioners and policy-makers, the findings imply that tail risks in crude oil markets may be more persistent and structurally embedded than previously acknowledged.

Future research may consider whether these invariance properties extend across asset classes, or how they can be embedded within machine learning frameworks that remain agnostic to parametric distributional forms. The present findings offer a point of departure—a statistical diagnosis of volatility that precedes prescription.

CRediT authorship contribution statement

Klaus Grobys: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

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The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

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Table A1
Descriptive statistics for range-based crude oil price variance for various frequencies and subsamples. Intraday data on crude oil futures covering the January 2, 1985 to November 20, 2024 are downloaded from investing.com. Data covers open, high, and low prices over 10,150 trading days. To compute annualized daily variances, we employ the range-based variance estimator, proposed by Parkinson (1980), which is given by: $\sigma_t^2 = T\frac{1}{4\ln(2)}(\ln(H_t) - \ln(L_t))^2$, where H_t and L_t denote the highest and lowest price for crude oil on trading day t, and σ_t^2 denotes the annualized daily variance, where $T = 250$, as 250 trading days per annum are assumed. Weekly or monthly annualized crude oil price variances are computed by summing up five or 22 subsequent daily variances multiplied by $T = 52$ or $T = 12$ to obtain annualization. Computing non-overlapping data on range-based crude oil variances leaves us with 2029 weekly and 460 monthly range-based variance observations for crude oil prices. The overall sample is split into two non-overlapping subsamples of equal length. The descriptive statistics for our subsample data are reported this table.

Frequency	Daily		Weekly		Monthly	
Sample	Sample 1	Sample 2	Sample 1	Sample 2	Sample 1	Sample 2
Mean	0.1009	0.1539	0.1049	0.1602	0.1066	0.1628
Median	0.0516	0.0689	0.0735	0.0833	0.0870	0.0921
Maximum	7.2414	18.7101	1.7554	6.9257	0.6661	4.3935
Minimum	0.0000	0.0000	0.0031	0.0093	0.0067	0.0123
Std. Dev.	0.2053	0.5087	0.1339	0.4076	0.1008	0.3577
Skewness	13.8818	19.3059	5.0603	12.1981	2.5563	9.0885
Kurtosis	363.3762	520.6331	44.3458	182.4651	11.5636	98.2850
Jarque-Bera (JB)	27625309.0000	56974253.0000	76628.3300	1385922.0000	953.2824	90175.6400
(*p*-value JB)	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Observations	5,075	5,075	1,015	1,014	230	230

Table A2
Estimating autoregressive models for range-based crude oil price variances at various frequencies. We fit autoregressive models of order p (AR(p)) to the range-based variance data. The order p is chosen with respect to the partial autocorrelation function of the range-based variance data for a given time frequency: $x_t = b_0 + b_1 x_{t-1} + b_2 x_{t-2} + \cdots + b_p x_{t-p} + \epsilon_t$, where $x = \sigma_t^2$ denotes the respective annualized crude oil variance for some time frequency, $b_0, b_1, \cdots, b_p$ are autoregressive parameters to be estimated, whereas ϵ_t is the innovation process that is independently-distributed via construction. This table reports the corresponding point estimates for fitted AR(p)-models. The corresponding t-statistics are given in parentheses. The sample period is from January 2, 1985 to November 20, 2024.

	$\widehat{b}_0$	$\widehat{b}_1$	$\widehat{b}_2$	$\widehat{b}_3$	$\widehat{b}_4$	$\widehat{b}_5$	$\widehat{b}_6$	$\widehat{b}_7$	R^2
Daily	0.0367***	0.3890***	0.1186***	0.2047***					0.3542
(t-statistic)	(10.9732)	(40.0260)	(11.4134)	(21.0607)					

(*continued on next page*)

Table A2 (*continued*)

	$\hat{b}_0$	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$	$\hat{b}_4$	$\hat{b}_5$	$\hat{b}_6$	$\hat{b}_7$	R^2
Weekly	0.0300***	0.5355***	0.2077***	−0.2000***	0.2436***	0.1687***	−0.1789***		0.5175
(t-statistic)	(5.4936)	(24.4400)	(8.4219)	(−8.1497)	(9.9441)	(6.8401)	(−8.1622)		
Monthly	0.0605***	0.7249***	−0.3199***	0.1499***					0.3704
(t-statistic)	(5.1074)	(15.6066)	(−5.7534)	(3.2272)					

*** Statistically significant on a 1 % level.

Table A3

Descriptive statistics for the innovation processes of fitted autoregressive models. We fit autoregressive models of order p (AR(p)) to the range-based variance data. The order p is chosen with respect to the partial autocorrelation function of the range-based variance data for a given time frequency: $x_t = b_0 + b_1 x_{t-1} + b_2 x_{t-2} + \cdots + b_p x_{t-p} + \epsilon_t$, where $x = \sigma_t^2$ denotes the respective annualized crude oil variance for some time frequency, $b_0, b_1, \cdots, b_p$ are autoregressive parameters to be estimated, whereas ϵ_t is the innovation process that is independently-distributed via construction. The sample period is from January 2, 1985 to November 20, 2024. This table reports the descriptive statistics for the absolute amount of the innovations processes $|\hat{\epsilon}_t|$.

Frequency	Daily	Weekly	Monthly
Mean	0.0888	0.0714	0.0706
Median	0.0442	0.0337	0.0418
Maximum	17.6621	4.7508	2.6046
Minimum	0.0000	0.0000	0.0006
Std. Dev.	0.2996	0.1994	0.1978
Skewness	28.9431	13.4692	10.0205
Kurtosis	1342.1070	245.2404	115.1659
Jarque-Bera (JB)	760,000,000.0000	5,007,440.0000	247,214.8000
(p-value JB)	(0.0000)	(0.0000)	(0.0000)
Observations	10,147	2,023	457

Table A4

Estimated power law models for the innovation processes of fitted autoregressive models. We fit autoregressive models of order p (AR(p)) to the range-based variance data. The order p is chosen with respect to the partial autocorrelation function of the range-based variance data for a given time frequency: $x_t = b_0 + b_1 x_{t-1} + b_2 x_{t-2} + \cdots + b_p x_{t-p} + \epsilon_t$, where $x = \sigma_t^2$ denotes the respective annualized crude oil variance for some time frequency, $b_0, b_1, \cdots, b_p$ are autoregressive parameters to be estimated, whereas ϵ_t is the innovation process that is independently-distributed via construction. The sample period is from January 2, 1985 to November 20, 2024. Using the absolute amount of the innovation processes $|\hat{\epsilon}_t|$, power-law exponents for various time frequencies are estimated as: $\hat{\alpha} = 1 + N\left(\sum_{i=1}^{N} \ln\left(\frac{x_i}{x_{MIN}}\right)\right)^{-1}$, where $x = |\hat{\epsilon}_t|$ denotes the respective innovation process for some annualized crude oil variance and a given time frequency, provided that $x \in \{\mathbb{R}_+ | x_{MIN} \le x < \infty\}$, $\hat{\alpha}$ denotes the MLE estimator, N is the number of observations exceeding x_{MIN}. The corresponding standard deviation of $\hat{\alpha}$ is estimated as: $\hat{\sigma}_{\hat{\alpha}} = \frac{\hat{\alpha} - 1}{\sqrt{N}} + O\left(\frac{1}{N}\right)$. The optimal value for x_{MIN} is selected via using the optimal Kolmogorov-Smirnov (KS) distance D, which measures the maximum distance between the cumulative density functions (CDFs) of the data and the fitted power-law model as defined by: $D = MAX_{x \ge x_{MIN}} |S(x) - P(x)|$, where $S(x)$ is the CDF of the data for the observation with a value of at least x_{MIN}, and $P(x)$ is the CDF for the power-law model that best fits the data in the region $x \ge x_{MIN}$. The optimal $\hat{x}_{MIN}$ is then the value of x_{MIN} that minimizes D. To analyze the plausibility of the power-law model, the estimated parameter vector $(\hat{\alpha}, \hat{x}_{MIN})$ employed in a goodness-of-fit test (GoF) producing a p-value that quantifies the plausibility of the power-law null model by comparing D with distance measurements for comparable synthetic data sets drawn from the hypothesized model, generated by the function $p(x) = (\hat{\alpha} - 1)\hat{x}_{MIN}^{\hat{\alpha}-1} x^{-\hat{\alpha}}$. The p-value is calculated as the fraction of synthetic distances that exceed the empirical distance.

Frequency	Daily	Weekly	Monthly
$\hat{\alpha}$	2.3765	2.2223	2.1922
$\hat{\sigma}_{\hat{\alpha}}$	0.0266	0.0404	0.2135
$\hat{x}_{min}$	0.0737	0.0356	0.1054
N (%)	2,761 (27.21 %)	966 (47.75 %)	40 (8.75 %)
T	10,148	2,023	457
p-value (GoF test)[a]	0.4320	0.2500	0.6560

[a]The p-value derived from Clauset et al.'s (2009) GoF test uses 250 synthetic data sets.

Table A5
Testing for lognormality. We test for lognormality by transforming the data by taking the natural logarithm. This is, we use the plain data on range-based variances for crude oil in terms of natural logarithms and the natural logarithms of the absolute amount of the innovation process of fitted AR(p) models as described in section 3.6. Then we carry out Jarque-Bera tests defined as: $\lambda_{JB} = \frac{T}{6}\left(S^2 + \frac{(K-3)^2}{4}\right)$, where T is the sample size, and S denotes the sample skewness $S = \frac{1/T\sum_{t=1}^{T}(y_t - \bar{y})^3}{\left(1/T\sum_{t=1}^{T}(y_t - \bar{y})^2\right)^{3/2}}$, and K is the sample kurtosis defined as $K = \frac{1/T\sum_{t=1}^{T}(y_t - \bar{y})^4}{\left(1/T\sum_{t=1}^{T}(y_t - \bar{y})^2\right)^2}$. Furthermore, $y_t = \ln(x_t)$ is either the logarithm of range-based crude oil price variance for a given frequency or $y_t = \ln(|\widehat{\epsilon}_t|)$, as defined in section 3.6. Note that $\lambda_{JB}\ \chi^2(2)$ and using a significance level of 5 %, we do not reject the null hypothesis of lognormality if $\widehat{\lambda}_{JB} < 5.99$. If the null hypothesis is accepted, lognormality holds.

Data type	Plain range-based variances			Innovations of AR(p) models		
Frequency	Daily	Weekly	Monthly	Daily	Weekly	Monthly
$\widehat{\lambda}_{JB}$	324.3692	123.6921	39.2323	5,013.6790	624.4179	103.6728
(p-value)	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.najef.2025.102476.

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