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Author(s): Barkaoui, Yosra; Mnif, Maher

Title: Three equivalent conditions for spectral decomposable linear relations

Year: 2025

Version: Accepted manuscript

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<https://doi.org/10.1007/s40065-025-00503-5>

Please cite the original version:

Barkaoui, Y., & Mnif, M. (2025). Three equivalent conditions for spectral decomposable linear relations. *Arabian Journal of Mathematics* 14, 15-27. <https://doi.org/10.1007/s40065-025-00503-5>

Three equivalent conditions for spectral decomposable linear relations

Yosra Barkaoui ^{a1} and Maher Mnif ^{b2}

¹Department of Mathematics and Statistics, University of Vaasa, Finland

²Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax, Tunisia

Abstract The spectral decomposability of a closed linear relation T on a complex Banach space is demonstrated through three new characterisations: The first two are expressed in terms of the extended Bishop and decomposition properties while the third one is given by means of the coinduced operator of T and its local spectral subspaces. This has been achieved through the intensive study of the properties of the last mentioned subspaces as well as the ER-SVEP.

Mathematics Subject Classification. MSC(2020) 47A06. 47A10. 47A11. 47A25. 47B40.

1 Introduction

The concept of decomposability plays a crucial role in the local spectral theory of linear operators on complex Banach spaces. It was defined for the first time for bounded linear operators by C.Foiaş [9] in 1963. Then, in 1968 it was generalized for closed linear operators by F.- H. Vasilescu [15] to become "S-residual decomposability" where S is a subset of $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. It was only in 2022 that this concept was extended to include a more general case, the case of multi-valued linear operators or linear relations, and was then defined as the concept of "extended spectral decomposability" in [2].

Of special interest are the different characterisations of decomposability. For example, in 2000, K.B.Laursen and M.M.Neumann [12] managed to characterise this concept for a bounded linear operator T_0 on a complex Banach space X in Theorem 1.2.29 as follows:

$$T_0 \text{ is decomposable} \Leftrightarrow T_0 \text{ has } (\delta) \text{ and } (C) \Leftrightarrow T_0 \text{ has } (\beta) \text{ and } (\delta), \quad (1.1)$$

where (δ) , (C) , and (β) stand for the properties of decomposition, Dunford and Bishop, respectively. They also established the following equivalence in Theorem 1.2.23:

$$T_0 \text{ is decomposable} \Leftrightarrow T_0 \text{ has } (C) \text{ and } \sigma(T_0/X_{T_0}(F)) \subseteq \overline{\sigma(T_0) \setminus F} \text{ for every} \\ \text{closed subset } F \text{ of } \mathbb{C}, \quad (1.2)$$

where $X_{T_0}(F)$ and $T_0/X_{T_0}(F)$ denote the local spectral subspace of T_0 with respect to F and the induced operator by T_0 on the quotient space $X/X_{T_0}(F)$, respectively.

^abarkaoyo@uwasa.fi

^bmaher.mnif@gmail.com

Our aim in this paper is to set the latest equivalences in a more general framework, the framework of linear relations. In other words, we are going to reformulate the equations (1.1) and (1.2) for a closed linear relation T and obtain the following new equivalences:

$$\begin{aligned} T \text{ is extended spectral decomposable} &\Leftrightarrow T \text{ has the properties } (\mathcal{E}_C) \text{ and } (\mathcal{E}_\delta) \\ &\Leftrightarrow T \text{ has the properties } (\mathcal{E}_\beta) \text{ and } (\mathcal{E}_\delta) \end{aligned} \quad (1.3)$$

and,

$$\begin{aligned} T \text{ is extended spectral decomposable} \\ \Updownarrow \end{aligned} \quad (1.4)$$

T has the property (\mathcal{E}_C) and $\tilde{\sigma}(T/\widetilde{X}_T(F)) \subseteq \overline{\tilde{\sigma}(T) \setminus F}$ for all closed subsets F of \mathbb{C}_∞ .

Having such multiple equivalent descriptions ensures flexibility in choosing the most convenient framework or approach when describing the decomposability of a linear relation. This leads to a deeper understanding of the relation's structure and properties.

Regarding (1.3), it is worth mentioning that the properties (\mathcal{E}_C) and (\mathcal{E}_β) stand for the extended Dunford and Bishop properties, respectively, and they have actually been proven to be two necessary conditions for the extended spectral decomposability of T ; See [2]. However, the property (\mathcal{E}_δ) is new and it will be introduced as the extended decomposition property and proven as a necessary condition for the extended spectral decomposability as well. In this way, the two direct implications of (3) will be guaranteed, and it only remains to deal with the indirect ones.

As for equivalence (1.4), $\widetilde{X}_T(F)$, $T/\widetilde{X}_T(F)$ as well as $\tilde{\sigma}(T)$ denote the extended local spectral subspace of T at F , the coinduced operator by T on $\widetilde{X}_T(F)$ as well as the extended spectrum of T , respectively. Here, it is very important to note that the converse direction of (1.4) was the purpose of [4]. So, our task will be to focus on how far we can solve the remaining direction.

Our paper is organized as follows. In Section 2, we will first recall some results related to the theory of linear relations already studied in [5], [7] and [8]. Next, we are going to develop several properties of one of the necessary conditions of extended spectral decomposability, called the ER-SVEP, which will be quite useful for the proof of our main theorems. Finally, we will present several new results on the extended local spectral subspaces of T , which will be further intensified once connected with the ER-SVEP. As for Section 3, it will be the core of our work. We will divide it into two sub-parts: the purpose of Section 3.1 is to settle the first two characterizations of the extended spectral decomposability mentioned in (1.3), while Section 3.2 aims to reach the second characterization in (1.4).

All this illustrates how perfectly the concept of spectral decomposability has been generalized and well-developed for the field of linear relations.

2 Preliminaries

In this section, we shall start with some general notions on linear relations by referring to the works of R.Cross [8] and A.Baskakov [7]. Then, we will refer to [2] and focus on the local spectral theory of a closed linear relation T which is fundamental for the understanding of the concept of the extended spectral decomposability. More importantly, we will study different several properties of T involving the extended local spectral subspaces as well as the ER-SVEP.

Let X be a complex Banach space. According to R.Cross [8], a mapping T from X to the collection of nonempty subsets of X , whose domain is $\mathcal{D}(T) := \{x \in X; Tx \neq \emptyset\}$, is called a

linear relation if it satisfies $\alpha Tx + \beta Ty = T(\alpha x + \beta y)$ for all $x, y \in \mathcal{D}(T)$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Note that T is entirely determined by its graph given by

$$G(T) = \{(u, v) \in X \times Y ; u \in \mathcal{D}(T) \text{ and } v \in Tx\}.$$

Throughout this paper, the class of all linear relations and all bounded linear operators from X to X is denoted by $LR(X)$ and $End(X)$, respectively. Let us notice that $T \in End(X)$ whenever $T(0) = \{0\}$ and $\mathcal{D}(T) = X$.

The *inverse* of T is the relation T^{-1} defined by $G(T^{-1}) = \{(y, x) \in Y \times X ; (x, y) \in G(T)\}$. We denote by $Ker(T) := T^{-1}(0)$ the *Kernel* of T . If $Ker(T) = \{0\}$, we say that T is *injective*. By $T(X_0) := \bigcup_{x_0 \in \mathcal{D}(T) \cap X_0} Tx_0$ and $Im(T) := T(X)$, we denote the range of a set $X_0 \subseteq X$ and the range of T , respectively. If $Im(T) = X$, then T is called *surjective*.

T is said to be *continuous* if $\|T\| := \|Q_T T\| < \infty$, where $Q_T : X \rightarrow X/\overline{T(0)}$ denotes the *natural quotient map* with domain X and kernel $Ker(Q_T) := \overline{T(0)}$. If $G(T)$ is closed, we say that T is *closed* and we write $T \in CR(X)$. We also write $T \in CR_c(X)$ if T is closed and continuous. When $T \in CR(X)$ is surjective and injective, it is called *invertible*.

The *product* and the *sum* of T and $U \in LR(X)$ are given by $G(TU) := \{(x, z) \in X \times X ; (x, u) \in G(U) \text{ and } (u, z) \in G(T) \text{ for some } u \in X\}$ and $G(T + U) := \{(x, t + u) ; (x, t) \in G(T), (x, u) \in G(U)\}$, respectively.

Let $T \in CR(X)$. Recall from [7] that the *resolvent* set of T is denoted by $\rho(T)$ and defined as the set of all $\eta \in \mathbb{C}$ for which $T - \eta I$ is invertible. For $\eta \in \rho(T)$, the *resolvent function* of T is the function $R(\cdot, T)$ such that $R(\eta, T) = (T - \eta I)^{-1}$. The *spectrum* of T is the set $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

A linear operator T_0 on X is called a *selection* of T if $\mathcal{D}(T_0) = \mathcal{D}(T)$ and $T = T_0 + T - T_0$. In the next proposition, we will show that the spectrum of a linear relation may coincide with that of its selection in a particular case.

Proposition 2.1. *Let $T \in CR_c(X)$ be such that $X = \mathcal{D}(T) \oplus T(0)$ and let \tilde{P} be the bounded linear projection with the property that $Ker(\tilde{P}) = T(0)$ and $Im(\tilde{P}) = \mathcal{D}(T)$. Then, $\tilde{P}T \in End(\mathcal{D}(T))$ is a continuous linear selection of T . Moreover, we have $\sigma(T) = \sigma(\tilde{P}T)$.*

Proof. Let $A = \tilde{P}T$. Then, $\mathcal{D}(A) = \mathcal{D}(T)$ and, for all $x \in \mathcal{D}(T)$, we have $Tx = Tx + T(0) = Tx + \tilde{P}^{-1}(0) = \tilde{P}^{-1}\tilde{P}(Tx) = \tilde{P}^{-1}(\tilde{P}\tilde{P})(Tx) = \tilde{P}^{-1}\tilde{P}(\tilde{P}T(x)) = \tilde{P}T(x) + \tilde{P}^{-1}(0) = Ax + T(0)$. Hence, A is a selection of T . Moreover, since T is continuous, it follows from [8, Corollary II.3.13] that $\|A\| = \|\tilde{P}T\| \leq \|\tilde{P}\|\|T\| < +\infty$. This proves the continuity of A . On the other hand, since $T \in CR_c(X)$, it follows from the closed graph theorem [8, Theorem III.4.2] that $\mathcal{D}(T)$ is closed, and therefore $A \in End(\mathcal{D}(T))$.

Proving $\sigma(T) = \sigma(\tilde{P}T)$ is equivalent to prove that $\rho(T) = \rho(\tilde{P}T)$. For doing so, let $\mu \in \rho(T)$ and consider $x \in Ker(\tilde{P}T - \mu I)$. Since $\tilde{P}T$ is a selection of T , it then follows that $0 = (\tilde{P}T - \mu I)x \in (T - \mu I)x$, so $x \in Ker(T - \mu I) = \{0\}$. Now, consider $t \in \mathcal{D}(T) \subseteq X$. Then, there exists $t_0 \in \mathcal{D}(T)$ such that $t \in (T - \mu I)t_0 = (\tilde{P}T - \mu I)t_0 + T(0)$, and therefore $t - (\tilde{P}T - \mu I)t_0 \in T(0) \cap \mathcal{D}(T) = \{0\}$. Consequently, $t = (\tilde{P}T - \mu I)t_0 \in Im(\tilde{P}T - \mu I)$, which means that $\tilde{P}T - \mu I$ is surjective. We then conclude that $\mu \in \rho(\tilde{P}T)$. To see the other inclusion, let $\mu \in \rho(\tilde{P}T)$ and consider $y \in X$. Then, $y = y_0 + y_1$, where $(y_0, y_1) \in \mathcal{D}(T) \times T(0)$, and hence $y_0 \in Im(\tilde{P}T - \mu I)$. Thus, there exists $x_0 \in \mathcal{D}(T)$ such that $y_0 = (\tilde{P}T - \mu I)x_0$. This implies that

$$y = (\tilde{P}T - \mu I)x_0 + y_1 \in (\tilde{P}T - \mu I)x_0 + T(0) = (T - \mu I)x_0 \in Im(T - \mu I).$$

Consequently, $(T - \mu I)$ is surjective. Since $Ker(T - \mu I) \subseteq Ker(\tilde{P}T - \mu I) = \{0\}$, we then conclude that $T - \mu I$ is injective, and hence invertible. This means that $\mu \in \rho(T)$. □

Given a relation $T \in CR(X)$ and a closed linear subspace Y_0 of X , we refer to [7] and define Y_0 (respectively, Y_1) as a T -strongly invariant subspace (respectively, T -weakly invariant subspace) if $R(\eta, T)Y_0 \subseteq Y_0$ for all $\eta \in \rho(T)$ (respectively, $Ty_0 \cap Y_0 \neq \emptyset$ for all $y_0 \in \mathcal{D}(T) \cap Y_0$). The strong restriction of T to Y_0 is the relation $T|_s Y \in CR(Y)$ whose resolvent is $R(\cdot, T|_s Y) : \rho(T) \rightarrow \text{End}(Y)$ which satisfies $R(\cdot, T|_s Y_0)(\eta) := R(\eta, T)|_Y$ for all $\eta \in \rho(T)$. In general, a subspace $M \subseteq X$ is said to be T -invariant if $T(M) \subseteq M$.

Lemma 2.2. *Let $T \in CR(X)$ and let M be a linear subspace of X . Then, M is a T -weakly invariant linear subspace if and only if $T(M) \subseteq M + T(0)$.*

Let $T \in CR(X)$. Then, we notice from [5] that whenever $T(0) \neq \{0\}$ or, equivalently, T is not a linear operator, the point ∞ always belongs to the extended spectrum of T . Hence, it becomes very important to deal with the topology of the Riemann sphere $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ in order to study the spectral theory of T . Let us recall that the extended spectrum of T is a subset of \mathbb{C}_∞ defined as

$$\tilde{\sigma}(T) := \begin{cases} \sigma(T) & \text{if } T \in \text{End}(X) \\ \sigma(T) \cup \{\infty\} & \text{otherwise.} \end{cases}$$

Following [10], the neighborhoods in \mathbb{C}_∞ correspond exactly to the usual definitions of neighborhoods in \mathbb{C} adding the neighborhoods of ∞ which are of the form $V_\infty = \{v \in \mathbb{C} ; |v| > r_0\} \cup \{\infty\}$, where $r_0 > 0$. In addition, according to [11], the open sets in \mathbb{C}_∞ are the usual open sets in \mathbb{C} and those of the form $V \cup \{\infty\}$, where $\mathbb{C} \setminus V$ is a compact set in \mathbb{C} . For an open set $U_\infty \subseteq \mathbb{C}_\infty$, we denote by $H^\sharp(U_\infty, X)$ the space of all analytic functions $f : U_\infty \rightarrow X$ and we define $P(U_\infty, X) := \{f \in H^\sharp(U_\infty, X) \mid f(\infty) = 0 \text{ if } \infty \in U_\infty\}$.

Now, being rather interested in the local spectral theory of T , we will turn to [2] and define the extended local resolvent set of T at a point w_0 , denoted by $\tilde{\rho}_T(w_0)$, as the set of all $\lambda \in \mathbb{C}_\infty$ for which there are an open neighborhood U_λ in \mathbb{C}_∞ of λ and an analytic function $\tilde{f}_{w_0, \lambda} \in P(U_\lambda, \mathcal{D}(T))$ such that $w_0 \in (\mu I - T)\tilde{f}_{w_0, \lambda}(\mu)$ for all $\mu \in U_\lambda \cap \mathbb{C}$. The extended local spectrum of T at $w_0 \in \mathbb{C}_\infty$ is given by

$$\tilde{\sigma}_T(w_0) := \mathbb{C}_\infty \setminus \tilde{\rho}_T(w_0).$$

For a set $F \subseteq \mathbb{C}_\infty$, the extended local spectral subspace of T at F is defined by

$$\tilde{X}_T(F) := \{w \in X ; \tilde{\sigma}_T(w) \subseteq F\}.$$

If we suppose in addition that F is closed, then we define the extended global spectral subspace of T at F as the set $\tilde{\mathcal{X}}_T(F)$ of all $x \in X$ for which there exists $\tilde{\Psi} \in P(\mathbb{C}_\infty \setminus F, \mathcal{D}(T))$ such that $x \in (T - \mu I)\tilde{\Psi}(\mu)$ for all $\mu \in (\mathbb{C}_\infty \setminus F) \cap \mathbb{C}$.

Lemma 2.3. *Let $T \in CR(X)$ and X_0 be a T -strongly invariant linear subspace of X . If $T|_s X_0 \in \text{End}(X_0)$ and $\sigma(T|_s X_0) \subseteq F$, then $X_0 \subseteq \tilde{X}_T(F)$.*

Proof. Let $x_0 \in X_0$ and consider $\lambda \in \mathbb{C}_\infty \setminus F$. By hypothesis, we obtain that $\lambda \in \tilde{\rho}(T|_s X_0)$, and therefore there exists a function \tilde{f}_{x_0} given by

$$\begin{aligned} \tilde{f}_{x_0} : \tilde{\rho}(T|_s X_0) &\rightarrow \mathcal{D}(T|_s X_0) \\ \mu &\rightarrow \begin{cases} R(\mu, T|_s X_0)x_0 & \text{if } \mu \neq \infty, \\ 0 & \text{if } \mu = \infty \in \tilde{\rho}(T|_s X_0) \end{cases} \end{aligned}$$

and verifies $x_0 = (T|_s X_0 - \mu I)\tilde{f}_{x_0}(\mu)$ for all $\mu \in \tilde{\rho}(T|_s X_0) \cap \mathbb{C}$. Since X_0 is a T -strongly invariant linear subspace, it follows from [2, Proposition 2.2] that $x_0 \in (T - \mu I)\tilde{f}_{x_0}(\mu)$ for all

$\mu \in \tilde{\rho}(T|_s X_0) \cap \mathbb{C}$.

On the other hand, $T|_s X_0 \in \text{End}(X_0)$, so $\tilde{\sigma}(T|_s X_0) = \sigma(T|_s X_0)$. As $\sigma(T|_s X_0)$ is compact in \mathbb{C} , we deduce that $\tilde{\rho}(T|_s X_0)$ is open in \mathbb{C}_∞ . Hence, $\tilde{\rho}(T|_s X_0)$ becomes an open neighborhood of λ and $\tilde{f}_{x_0} \in P(\tilde{\rho}(T|_s X_0), \mathcal{D}(T|_s X_0))$. Moreover, we have $\mathcal{D}(T|_s X_0) \subseteq \mathcal{D}(T)$, which entails that $\tilde{f}_{x_0} \in P(\tilde{\rho}(T|_s X_0), \mathcal{D}(T))$. We then conclude that $\lambda \in \tilde{\rho}_T(x_0)$, and therefore $x_0 \in \widetilde{X}_T(F)$. \square

Remark 2.4. Let $T \in CR(X)$ and let $F \subseteq \mathbb{C}_\infty$ be such that $T(\widetilde{X}_T(F)) \subseteq \widetilde{X}_T(F)$. Then,

$$(T - \lambda I)(\widetilde{X}_T(F)) = \widetilde{X}_T(F) \quad \text{for all } \lambda \in (\mathbb{C}_\infty \setminus F) \cap \mathbb{C}.$$

It is worth pointing out that the subsets mentioned above have already been well studied using the topology of \mathbb{C} by M.Mnif [13], which can be seen in the following definition.

Definition 2.5. Let $T \in CR(X)$ and let $w_0 \in X$.

- (i) The *local resolvent set* of T at w_0 is the set of all $\eta_0 \in \mathbb{C}$ for which there exist an open neighborhood U_{η_0} of η_0 in \mathbb{C} and a function $f_{w_0, \eta_0} \in H(U_{\eta_0}, \mathcal{D}(T))$ verifying

$$w_0 \in (T - \mu)f_{a, \eta_0}(\mu) \quad \text{for all } \mu \in U_{\eta_0},$$

where $H(U, Y)$ denotes the space of all analytic functions from an open set $U \subseteq \mathbb{C}$ to a complex Banach space Y . The set $\mathbb{C} \setminus \rho_T(w_0)$ is denoted by $\sigma_T(w_0)$ and called the *local spectrum* of T at w_0 .

- (ii) The *local spectral subspace* of T at a subset F of \mathbb{C} is given by $X_T(F) := \{x \in X ; \sigma_T(x) \subseteq F\}$. The *glocal spectral subspace* of T at a closed set $S \subseteq \mathbb{C}$ is the set of all $x \in X$ for which there exists $h \in H(U, X)$ such that

$$x \in (T - \mu I)h(\mu) \quad \text{for all } \mu \in \mathbb{C} \setminus S.$$

There exists a very strong connection between the glocal spectral subspaces of $T \in CR(X)$ at a closed subset F of \mathbb{C} and the extended glocal ones at $F \cup \{\infty\}$. This can be translated in the following lemma.

Lemma 2.6. Let $T \in CR(X)$ and let $F \subseteq \mathbb{C}$ be a closed set. Then,

$$\widetilde{\mathcal{X}}_T(F \cup \{\infty\}) = \mathcal{X}_T(F).$$

Proof. For $F_\infty := F \cup \{\infty\}$, we have the following equivalences:

$$\begin{aligned} x \in \mathcal{X}_T(F) &\Leftrightarrow \exists f \in H(\mathbb{C} \setminus F, \mathcal{D}(T)) ; x \in (T - \mu I)f(\mu) \quad \forall \mu \in \mathbb{C} \setminus F \\ &\Leftrightarrow \exists f \in H(\mathbb{C}_\infty \setminus F_\infty, \mathcal{D}(T)) ; x \in (T - \mu I)f(\mu) \quad \forall \mu \in (\mathbb{C}_\infty \setminus F_\infty) \cap \mathbb{C} \\ &\Leftrightarrow \exists f \in P(\mathbb{C}_\infty \setminus F_\infty, \mathcal{D}(T)) ; x \in (T - \mu I)f(\mu) \quad \forall \mu \in (\mathbb{C}_\infty \setminus F_\infty) \cap \mathbb{C} \\ &\Leftrightarrow x \in \widetilde{\mathcal{X}}_T(F_\infty). \end{aligned}$$

\square

It is worth mentioning that the result of the previous lemma also remains true for local spectral subspaces. In fact, this stems from the important link existing between the local spectral theory of T and its extended one, which will be expressed in the remaining part of this section.

Definition 2.7. [2, Definition 3.4] A relation $T \in CR(X)$ is said to have the extended relatively single-valued extension property, abbreviated ER-SVEP, at $\eta_0 \in \mathbb{C}_\infty$ if for every open connected neighborhood $U_{\eta_0} \subseteq \mathbb{C}_\infty$ of η_0 and for all functions $\varrho \in P(U_{\eta_0}, \mathcal{D}(T))$ which satisfy

$$0 \in (T - \mu I)\varrho(\mu) \text{ for all } \mu \in U_{\eta_0} \cap \mathbb{C}, \quad (2.1)$$

we have $\varrho(\mu) \in T(0) \cap \text{Ker}(T)$ for all $\mu \in U_{\eta_0}$. The relation T is said to have the ER-SVEP if T has the ER-SVEP at every $\eta \in \mathbb{C}_\infty$.

In the following, we will present further important results helpful for the characterization of extended spectral decomposability (see Theorem 3.10). We will in fact show that Proposition 1.2.20 of [12] can be generalized even for a linear relation $T \in CR_c(X)$. To this end, we shall start by studying the connection between the ER-SVEP and the extended spectrum of a strong restriction of T on an extended local spectral subspace.

Proposition 2.8. Let $T \in CR_c(X)$ be such that $\rho(T) \neq \emptyset$ and let S be a closed subset of \mathbb{C}_∞ such that $\widetilde{X}_T(S)$ is a T -invariant closed linear subspace. If T has the ER-SVEP, then

$$\widetilde{\sigma}(T|_s \widetilde{X}_T(S)) \subseteq S \cup \{\infty\}. \quad (2.2)$$

Proof. Let $S \subseteq \mathbb{C}_\infty$. Proving (2.2) is equivalent to proving that $\sigma(T|_s \widetilde{X}_T(S)) \subseteq S \cap \mathbb{C}$ or, equivalently, $\mathbb{C} \setminus (S \cap \mathbb{C}) \subseteq \rho(T|_s \widetilde{X}_T(S))$. To see this, let $\lambda \in \mathbb{C} \setminus (S \cap \mathbb{C}) = (\mathbb{C}_\infty \setminus S) \cap \mathbb{C}$ and consider an arbitrary $y \in \widetilde{X}_T(S)$. Then, in light of Remark 2.4, there exists an element $x \in \mathcal{D}(T) \cap \widetilde{X}_T(S)$ such that $y \in (T - \lambda I)x$. On the other hand, it follows from [4, Proposition 3.3] that $\widetilde{X}_T(S)$ is a T -strongly invariant linear subspace, so, by [2, Proposition 2.2], we have

$$\mathcal{D}(T) \cap \widetilde{X}_T(S) = \mathcal{D}(T|_s \widetilde{X}_T(S)). \quad (2.3)$$

Furthermore, [4, Proposition 3.1] implies that $T(0) \subseteq \widetilde{X}_T(S)$, which yields, by [2, Proposition 2.2], that $T|_s \widetilde{X}_T(S)(x) = T(x)$ and also

$$(T - \lambda I)x = (T|_s \widetilde{X}_T(S) - \lambda I)x. \quad (2.4)$$

This means that $y \in (T|_s \widetilde{X}_T(S) - \lambda I)x \subseteq \text{Im}(T|_s \widetilde{X}_T(S) - \lambda I)$, and hence $T|_s \widetilde{X}_T(S) - \lambda I$ is surjective. In view of [4, Lemma 3.1], proving the invertibility of $T|_s \widetilde{X}_T(S) - \lambda I$ then requires proving that this map belongs to $CR_c(\widetilde{X}_T(S))$, has the ER-SVEP and that $(T|_s \widetilde{X}_T(S) - \lambda I)(0) \cap \text{Ker}(T|_s \widetilde{X}_T(S) - \lambda I) = \{0\}$.

To this end, we first observe from the fact that $T|_s \widetilde{X}_T(S)$ is closed and from [1, Lemma 1] that $T|_s \widetilde{X}_T(S) - \lambda I \in CR(\widetilde{X}_T(S))$. Moreover, since $\widetilde{X}_T(S)$ is closed, a combination of (2.3) and [8, Theorem III.4.2] yields that $T|_s \widetilde{X}_T(S) - \lambda I \in CR_c(\widetilde{X}_T(S))$. On the other hand, as $\rho(T) \neq \emptyset$, it follows from [14, Lemma 6.1] that $\text{Ker}(T) \cap T(0) = \{0\}$. This entails, by (2.4), that

$$\begin{aligned} \text{Ker}(T|_s \widetilde{X}_T(S) - \lambda I) \cap (T|_s \widetilde{X}_T(S) - \lambda I)(0) &= \text{Ker}(T|_s \widetilde{X}_T(S) - \lambda I) \cap T(0) \\ &\subseteq \text{Ker}(T - \lambda I) \cap T(0) \subseteq \text{Ker}(T) \cap T(0) \\ &= \{0\}. \end{aligned}$$

As T has the ER-SVEP and $\widetilde{X}_T(S)$ is a T -strongly invariant linear subspace, we infer from assertions (i) and (iii) of [4, Lemma 3.1] that $T|_s \widetilde{X}_T(S) - \lambda I$ has the ER-SVEP. We then conclude that $T|_s \widetilde{X}_T(S) - \lambda I$ is invertible, as desired. Consequently, $\mathbb{C} \setminus (S \cap \mathbb{C}) \subseteq \rho(T|_s \widetilde{X}_T(S))$. \square

Let us note that in [12, Proposition 1.2.20], the authors have also given a property that connects the SVEP with the operator $T_0/X_{T_0}(F)$ where $T_0 \in \text{End}(X)$ and $F \in \mathbb{C}$. In order to generalize this property for a closed linear relation $T \in CR(X)$, let us recall from [2] that, for a T -invariant closed linear subspace $M \subseteq X$, the *coinduced* linear operator $T/M : X/M \rightarrow X/M$ induced by T on the quotient space X/M is defined by:

$$T/M\tilde{x} := \widetilde{T\alpha} = Q_M T(\alpha) \quad \text{for all } \tilde{x} \in \mathcal{D}(T/M) := \frac{\mathcal{D}(T) + M}{M},$$

where $x = \alpha + m$, $(\alpha, m) \in \mathcal{D}(T) \times M$ and $Q_M : X \rightarrow X/M$ is the natural quotient map. In the remaining work, we denote by $CR_c^\oplus(X)$ the space of all relations $T \in CR_c(X)$ for which $X = \mathcal{D}(T) \oplus T(0)$.

Corollary 2.9. *Let $T \in CR_c^\oplus(X)$. If T has the ER-SVEP then, for every closed set $S \subseteq \mathbb{C}_\infty$ for which $\widetilde{X}_T(S)$ is closed, we have*

i/ $\tilde{\sigma}(T|_s \widetilde{X}_T(S)) \subseteq S \cup \{\infty\}$;

ii/ $T/\widetilde{X}_T(S)$ has the ER-SVEP.

Proof. (i) Let $S \subseteq \mathbb{C}_\infty$ be such that $\widetilde{X}_T(S)$ is closed. Then, it follows from [4, Proposition 3.4], assertion (2), that $\widetilde{X}_T(S)$ is a T -invariant linear subspace. On the other hand, we remark from Proposition 2.1 that $\rho(T) \neq \emptyset$. Applying Proposition 2.8, we then conclude the desired result.

(ii) Since $X = \mathcal{D}(T) \oplus T(0)$, it follows from assertion (i) of [4, Lemma 4.1] that $T/\widetilde{X}_T(S) \in \text{End}(X/\widetilde{X}_T(S))$. In addition, there is a bounded linear projection \tilde{P} such that $\text{Im}(\tilde{P}) = \mathcal{D}(T)$ and $\text{Ker}(\tilde{P}) = T(0)$. To prove that $T/\widetilde{X}_T(S)$ has the ER-SVEP, let $\lambda_0 \in \mathbb{C}_\infty$, U_{λ_0} be an open neighborhood of λ_0 in \mathbb{C}_∞ and let $f \in P(U_{\lambda_0}, X/\widetilde{X}_T(S))$ be such that

$$\bar{0} = (T/\widetilde{X}_T(S) - \mu I) f(\mu) \quad \text{for all } \mu \in U_{\lambda_0} \cap \mathbb{C}. \quad (2.5)$$

Thus, $f|_{U_{\lambda_0} \cap \mathbb{C}} \in H(U_{\lambda_0} \cap \mathbb{C}, X/\widetilde{X}_T(S))$. This ensures, by the surjectivity of the operator $Q_{\widetilde{X}_T(S)} : X \rightarrow X/\widetilde{X}_T(S)$ and by [12, Proposition 2.1.4], the existence of a function $\psi \in H(U_{\lambda_0} \cap \mathbb{C}, X)$ with the property that $f|_{U_{\lambda_0} \cap \mathbb{C}}(\mu) = Q_{\widetilde{X}_T(S)} \psi(\mu) = Q_{\widetilde{X}_T(S)} (\tilde{P}\psi(\mu) + (I - \tilde{P})\psi(\mu)) = Q_{\widetilde{X}_T(S)} (\tilde{P}\psi)(\mu) = \overline{\tilde{P}\psi}(\mu)$ for every $\mu \in U_{\lambda_0} \cap \mathbb{C}$. We then infer from (2.5) that

$$\bar{0} = (T/\widetilde{X}_T(S) - \mu I) \overline{(\tilde{P}\psi)(\mu)} = Q_{\widetilde{X}_T(S)} (T - \mu) \tilde{P}\psi(\mu) \quad \text{for all } \mu \in U_{\lambda_0} \cap \mathbb{C}.$$

Consequently,

$$(T - \mu) \tilde{P}\psi(\mu) \subseteq \widetilde{X}_T(S) \quad \text{for all } \mu \in U_{\lambda_0} \cap \mathbb{C}. \quad (2.6)$$

Next, we claim that $\tilde{P}\psi(\mu) \in \widetilde{X}_T(S)$ for all $\mu \in U_{\lambda_0} \cap \mathbb{C}$. To verify this, let us separate two cases.

If $\nu \in U_{\lambda_0} \cap \mathbb{C} \cap S$, then we know from assertion (2) of [4, Proposition 3.4] that (2.6) implies immediately that $\tilde{P}\psi(\nu) \in \widetilde{X}_T(S)$.

If $\nu \in U_{\lambda_0} \cap \mathbb{C} \cap (\mathbb{C}_\infty \setminus S)$, then it follows from (2.2) that $\nu \in \rho(T|_s \widetilde{X}_T(S))$. Hence, thanks to (2.6), the function

$$\begin{aligned} \tilde{h} : U_{\lambda_0} \cap \mathbb{C} \cap (\mathbb{C}_\infty \setminus S) &\rightarrow \widetilde{X}_T(S) \\ w &\mapsto R(w, T|_s \widetilde{X}_T(S))(T - wI) \tilde{P}\psi(w) \end{aligned}$$

is well defined and belongs to $P(U_{\lambda_0} \cap \mathbb{C} \cap (\mathbb{C}_\infty \setminus S), \widetilde{X}_T(S))$. On the other hand, as noted before, $\widetilde{X}_T(S)$ is a T -strongly invariant linear subspace containing $T(0)$, so, by [2, Proposition 2.3], we obtain that

$$\begin{aligned} (T - \nu I)\tilde{h}(\nu) &= (T - \nu I) \left[R(\nu, T|_s \widetilde{X}_T(S))(T - \nu I)\tilde{P}\psi(\nu) \right] \\ &= (T|_s \widetilde{X}_T(S) - \nu I) R(\nu, T|_s \widetilde{X}_T(S))(T - \nu I)\tilde{P}\psi(\nu) \\ &= (T - \nu I)\tilde{P}\psi(\nu). \end{aligned}$$

Consequently, $0 \in (T - \nu I)\tilde{g}(\nu)$, where $\tilde{g} := \tilde{h} - \tilde{P}\psi|_{U_{\lambda_0} \cap \mathbb{C} \cap (\mathbb{C}_\infty \setminus S)}$. Since ν is arbitrary, $U_{\lambda_0} \cap (\mathbb{C}_\infty \setminus S) \cap \mathbb{C}$ is an open subset of \mathbb{C}_∞ and T has the ER-SVEP, we conclude that $\tilde{g} = 0$ locally in $U_{\lambda_0} \cap (\mathbb{C}_\infty \setminus S) \cap \mathbb{C}$. Thus, $\tilde{P}\psi(\nu) = \tilde{h}(\nu) \in \widetilde{X}_T(S)$.

We then conclude that $\tilde{P}\psi(\mu) \in \widetilde{X}_T(S)$ for all $\mu \in U_{\lambda_0} \cap \mathbb{C}$, as claimed. This entails that $f(\mu) = \overline{\tilde{P}\psi(\mu)} = \bar{0}$ for all $\mu \in U_{\lambda_0} \cap \mathbb{C}$. As $f \in P(U_{\lambda_0}, X/\widetilde{X}_T(S))$, we obtain that $f = 0$, so $T/\widetilde{X}_T(S)$ has the ER-SVEP. □

It was proved in [2, Lemma 3.4] that, once T has the ER-SVEP, the local and the global spectral subspaces coincide under the condition $\rho(T) \neq \emptyset$. But even if we weaken that condition, we can also obtain the same result, which will be expressed in the following lemma.

Lemma 2.10. *Let $T \in CR(X)$ be a relation with ER-SVEP such that $\text{Ker}(T) \cap T(0) = \{0\}$. Then, for every closed subset F of \mathbb{C}_∞ , we have*

$$\widetilde{X}_T(F) = \widetilde{\mathcal{X}}_T(F).$$

3 Equivalent conditions for extended spectral decomposable linear relations

The remainder of this paper is devoted to the characterisation of the spectral decomposability for closed linear relations. More precisely, we are going to establish the equivalences (1.3) together with the equivalences (1.4). It is important to mention that the direct sense of (1.3) (respectively the indirect sense of (1.4)) has already been studied in [2] (respectively in [4]). Let us recall from [2, Definition 3.1] that a relation $T \in CR_c(X)$ with the property that $\rho(T) \neq \emptyset$ is called *extended spectral decomposable* if for every open cover $\{U, V\}$ of the extended complex plane \mathbb{C}_∞ by an open bounded subset U of \mathbb{C} and an open subset V of \mathbb{C}_∞ , there exist two T -strongly invariant closed linear subspaces M and N of X with the following properties:

- i/ $\tilde{\sigma}(T|_s M) = \sigma(T|_s M) \subseteq U$, $\tilde{\sigma}(T|_s N) \subseteq V$;
- ii/ $T(N) \subseteq N$ and $\mathcal{P}(N) \subseteq N$ for some bounded linear projection $\mathcal{P} : X \rightarrow X$ verifying $\text{Im}(\mathcal{P}) = \mathcal{D}(T)$;
- iii/ $X = M + N$.

3.1 Proof of the first two equivalent conditions (1.3)

To achieve the equivalences (1.3), we shall start by introducing the extended decomposition property (\mathcal{E}_δ) for closed linear relations. Note that this property has already been defined

in the framework of bounded linear operators by means of global spectral subspaces; see for instance [12, Definition 1.2.28]. In our case, we will introduce it using the tools of the extended local spectral theory of linear relations, more precisely, by means of the extended global spectral subspaces.

Definition 3.1. A linear relation $T \in CR(X)$ has the *extended decomposition property* (\mathcal{E}_δ) if for every open cover $\{U, V\}$ of \mathbb{C}_∞ , where U is an open bounded subset of \mathbb{C} and V is an open subset of \mathbb{C}_∞ , we have

$$X = \mathcal{X}_T(\overline{U}) + \widetilde{\mathcal{X}}_T(\overline{V}).$$

Thanks to Lemma 2.6, one can reformulate the above definition using only closed subsets of \mathbb{C}_∞ that contain the point ∞ or by using only closed subsets of \mathbb{C} . This can be translated in the following corollary.

Corollary 3.2. *Let $T \in CR(X)$. Then, T has the decomposition property (\mathcal{E}_δ) if and only if for every open cover $\{U, V\}$ of \mathbb{C}_∞ , where U is an open bounded subset of \mathbb{C} and V is an open subset of \mathbb{C}_∞ , we have*

$$X = \widetilde{\mathcal{X}}_T(\overline{U \cup \{\infty\}}) + \widetilde{\mathcal{X}}_T(\overline{V}) = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V \cap \mathbb{C}}).$$

Now, we are close to showing that the property (\mathcal{E}_δ) is a necessary condition for extended spectral decomposability. To do this, we will first give a useful result linking the extended local spectral subspace of a relation T at an arbitrary subset of \mathbb{C}_∞ with the spectrum of its strong restriction $T|_s X_0$ on a strongly invariant linear subspace X_0 . More precisely, we are going to extend assertion (e) of [12, Proposition 1.2.16] from the case of bounded linear operators to that of multi-valued linear ones in order to obtain the following lemma.

Lemma 3.3. *Let $T \in CR(X)$ and let $S \subseteq \mathbb{C}$. If X_0 is a T -strongly invariant linear subspace of X such that $\sigma(T|_s X_0) \subseteq S$, then $X_0 \subseteq \widetilde{\mathcal{X}}_T(S \cup \{\infty\})$.*

Proof. Let $x_0 \in X_0$ and let $\lambda \in \mathbb{C}_\infty \setminus (S \cup \{\infty\})$. Then, $\lambda \in \rho(T|_s X_0)$ and, for $U_\lambda := \rho(T|_s X_0)$ and $f_{\lambda, x_0} := R(\cdot, T|_s X_0)x_0$, we have

$$x_0 \in (T|_s X_0 - \mu)f_{\lambda, x_0}(\mu) \quad \text{for all } \mu \in U_\lambda.$$

In light of [2, Proposition 2.3], this yields that

$$x_0 \in (T - \mu)f_{\lambda, x_0}(\mu) \quad \text{for all } \mu \in U_\lambda.$$

As $U_\lambda = \rho(T|_s X_0)$ is an open neighborhood of λ in \mathbb{C} and $f_{\lambda, x_0} \in H(U_\lambda, \mathcal{D}(T))$, we conclude, by [4, Remark 3.1.], that $\lambda \in \rho_T(x_0) \subseteq \widetilde{\rho}_T(x_0)$. Hence, $x_0 \in \widetilde{\mathcal{X}}_T(S \cup \{\infty\})$, which implies the desired inclusion. \square

Let us recall that we have already proved in [2] that the ER-SVEP as well as the extended Bishop's property and the extended Dunford's property are three necessary conditions for a relation $T \in CR_c(X)$ to be extended spectral decomposable. In the next theorem, we will prove that even the extended decomposition property is a necessary condition for T to become extended spectral decomposable.

Theorem 3.4. *Let $T \in CR(X)$ be such that $\rho(T)$ has an unbounded connected component. If T is an extended spectral decomposable linear relation, then T has the property (\mathcal{E}_δ).*

Proof. Suppose that $T \in CR_c(X)$ is an extended spectral decomposable linear relation, and let U_1 be an open bounded subset of \mathbb{C} and U_2 be an open subset of \mathbb{C}_∞ such that $\mathbb{C}_\infty = U_1 \cup U_2$. Then, the extended spectral decomposability of T implies the existence of two T -strongly invariant linear subspaces X_0 and X_1 of X verifying the following:

- (i) $X = X_0 + X_1$;
- (ii) $\sigma(T|_s X_0) \subseteq U_1 \subseteq \overline{U_1}$ and $\sigma(T|_s X_1) \subseteq \overline{U_2} \cap \mathbb{C}$.

Combining this with Lemma 3.3, we obtain that $X = \widetilde{X}_T(\overline{U_1} \cup \{\infty\}) + \widetilde{X}_T(\overline{U_2})$.

On the other hand, we know from [2, Theorem 3.3] that T has the ER-SVEP, so [2, Lemma 3.4] yields that $\widetilde{\mathcal{X}}_T(\overline{U_1} \cup \{\infty\}) = \widetilde{X}_T(\overline{U_1} \cup \{\infty\})$ and $\widetilde{\mathcal{X}}_T(\overline{U_2}) = \widetilde{X}_T(\overline{U_2})$. Thus, we have

$$X = \widetilde{\mathcal{X}}_T(\overline{U_1} \cup \{\infty\}) + \widetilde{\mathcal{X}}_T(\overline{U_2}).$$

We then conclude from Corollary 3.2 that T has the property (\mathcal{E}_δ) . □

Remark 3.5. In the framework of bounded linear operators, we only have to add the Dunford's property or the Bishop's property in order to obtain the converse of Theorem 3.4. Yet, this is not that obvious in the general framework of closed linear relations. For this reason, we opted for a specific type of linear relations, belonging to the space $CR_c^\oplus(X)$. This will enable us to successfully extend the equivalences (a) \Leftrightarrow (c) of [12, Theorem 2.5.19] to the case of linear relations (see Theorem 3.10 for more details). Before doing so, we first have to go through some steps that can be expressed in Lemma 3.6, Lemma 3.7 and Lemma 3.8.

Recall from [2] that a relation $T \in CR_c(X)$ is said to have *extended Dunford's property*, shortly property (\mathcal{E}_C) , if $\widetilde{X}_T(F)$ is closed for every closed subset $F \subseteq \mathbb{C}_\infty$. Throughout the rest of this paper, whenever $T \in CR_c^\oplus(X)$, we denote by P_T the bounded linear projection on X for which $\text{Ker}(P_T) = T(0)$ and $\text{Im}(P_T) = \mathcal{D}(T)$.

Lemma 3.6. *If $T \in CR_c^\oplus(X)$, then the following properties hold:*

- i/ $\sigma_T(x) = \sigma_{P_T T}(P_T x)$ for all $x \in X$;
- ii/ *If T has the property (\mathcal{E}_C) then T has the ER-SVEP.*

Proof. We first observe from Proposition 2.1 that $P_T T$ is a bounded linear selection of T on $\mathcal{D}(T)$.

- i/ Let $x \in X$. Then, we have the following equivalences:

$$\begin{aligned} \lambda_0 \in \rho_T(x) &\Leftrightarrow \exists \text{ an open neighborhood } V_0 \text{ of } \lambda_0 \text{ in } \mathbb{C} \text{ and a function } f_0 \in H(V_0, \mathcal{D}(T)) \\ &\quad \text{such that } x \in (T - \mu I)f_0(\mu) \text{ for all } \mu \in V_0 \\ &\Leftrightarrow \exists \text{ an open neighborhood } V_0 \text{ of } \lambda_0 \text{ in } \mathbb{C} \text{ and a function } f_0 \in H(V_0, \mathcal{D}(T)) \\ &\quad \text{such that } x = P_T x + (x - P_T x) \in (P_T T - \mu I)f_0(\mu) + T(0) \text{ for all } \mu \in V_0 \\ &\Leftrightarrow \exists \text{ an open neighborhood } V_0 \text{ of } \lambda_0 \text{ in } \mathbb{C} \text{ and a function } f_0 \in H(V_0, \mathcal{D}(T)) \\ &\quad \text{such that } P_T x = (P_T T - \mu I)f_0(\mu) \text{ for all } \mu \in V_0 \\ &\Leftrightarrow \lambda_0 \in \rho_{P_T T}(P_T x). \end{aligned}$$

This establishes that $\sigma_T(x) = \sigma_{P_T T}(P_T x)$.

ii/ We first claim that $P_T T$ has the property (C), i.e. $X_{P_T T}$ is closed for every closed set $F \subseteq \mathbb{C}$. To see this, let S be a closed subset of \mathbb{C} and $(x_n)_{n \in \mathbb{N}} \subseteq X_{P_T T}(S)$ be a sequence for which $x_n \xrightarrow{n \rightarrow +\infty} x$ in X . This means, by *i/* that

$$\sigma_T(x_n) = \sigma_{P_T T}(P_T x_n) = \sigma_{P_T T}(x_n) \subseteq S \quad \text{for all } n \in \mathbb{N}.$$

Hence, $(x_n)_{n \in \mathbb{N}} \subseteq X_T(S)$. On the other hand, we infer from [3, Remark 5.2] that $X_T(S) = \widetilde{X}_T(S \cup \{\infty\})$, which yields that $(x_n)_{n \in \mathbb{N}} \subseteq \widetilde{X}_T(S \cup \{\infty\})$. Since T has the property (\mathcal{E}_C) and $x_n \xrightarrow{n \rightarrow +\infty} x$, we obtain that $x \in \widetilde{X}_T(S \cup \{\infty\}) = X_T(S)$. Using *i/*, we then conclude that $P_T x \in X_{P_T T}(S)$. Furthermore, as $T \in CR_c(X)$, $(x_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}(T)$ and $x_n \xrightarrow{n \rightarrow +\infty} x$, the closed graph theorem [8, Theorem III.4.2] implies that $x \in \mathcal{D}(T)$. Thus, $x = P_T x \in X_{P_T T}(S)$, and therefore $P_T T$ has the property (C), as claimed. Consequently, $P_T T$ has the SVEP, by [12, Proposition 1.2.19].

Now, to prove that T has the ER-SVEP, consider $\eta_0 \in \mathbb{C}_\infty$, and let U_{η_0} be an open connected neighborhood of η_0 in \mathbb{C}_∞ and $f_{\eta_0} \in P(U_{\eta_0}, \mathcal{D}(T))$ be such that

$$0 \in (T - \lambda I)f_{\eta_0}(\lambda) \quad \text{for all } \lambda \in U_{\eta_0} \cap \mathbb{C}. \quad (3.1)$$

Then, $f_{\eta_0}|_{U_{\eta_0} \cap \mathbb{C}} \in H(U_{\eta_0} \cap \mathbb{C}, \mathcal{D}(T))$ and, for all $\lambda \in U_{\eta_0} \cap \mathbb{C}$, we have $0 = (P_T T - \lambda I)f_{\eta_0}(\lambda)$. Since $P_T T$ has the SVEP, it follows that $f_{\eta_0}|_{U_{\eta_0} \cap \mathbb{C}} = 0$. Moreover, as $f_{\eta_0} \in P(U_{\eta_0}, \mathcal{D}(T))$, we conclude that $f_{\eta_0} \equiv 0$, and therefore T has the ER-SVEP. □

The following lemma demonstrates the first sufficient condition of the extended spectral decomposability quoted in (1.3).

Lemma 3.7. *If $T \in CR_c^\oplus(X)$ has the properties (\mathcal{E}_δ) and (\mathcal{E}_C) , then T is an extended spectral decomposable linear relation.*

Proof. We first observe from Proposition 2.1 that $\rho(T) = \rho(P_T T) \neq \emptyset$. Now, let $\{U_1, U_2\}$ be an open cover of \mathbb{C}_∞ such that U_1 is a bounded subset of \mathbb{C} . Then, there exists another open cover $\{U, V\}$ of \mathbb{C}_∞ such that $\bar{U} \subseteq U_1$ and $\bar{V} \subseteq U_2$. Since T has the property (\mathcal{E}_δ) , it follows from Corollary 3.2 that

$$X = \widetilde{X}_T(\bar{U} \cup \{\infty\}) + \widetilde{X}_T(\bar{V}).$$

In addition, as T has the property (\mathcal{E}_C) , it then follows from Lemma 3.6 that T has the ER-SVEP. Combining this with Lemma 2.10 and with the fact that $\text{Ker}(T) \cap T(0) = \{0\}$, we obtain that $X = \widetilde{X}_T(\bar{U} \cup \{\infty\}) + \widetilde{X}_T(\bar{V})$. On the other hand, Corollary 2.9 leads to the following inclusions

$$\tilde{\sigma}(T|_s \widetilde{X}_T(\bar{U} \cup \{\infty\})) \subseteq \bar{U} \cup \{\infty\} \subseteq U_1 \cup \{\infty\} \quad \text{and} \quad \tilde{\sigma}(T|_s \widetilde{X}_T(\bar{V})) \subseteq \bar{V} \subseteq U_2. \quad (3.2)$$

Next, set $Y := \mathcal{D}(T) \cap \widetilde{X}_T(\bar{U} \cup \{\infty\})$ and $Z := \widetilde{X}_T(\bar{V})$, and claim that Y and Z verify assertions *i/*, *ii/* and *iii/* of the definition of the spectral decomposability mentioned in the beginning of Section 3 with $\mathcal{P} = P_T$. To do this, we first observe from the property (\mathcal{E}_C) , [4, Proposition 3.3] and [4, Proposition 3.4] that $\widetilde{X}_T(\bar{U} \cup \{\infty\})$ and $\widetilde{X}_T(\bar{V})$ are both T -invariant and T -strongly invariant linear subspaces of X . This easily implies that Y and Z are two T -strongly invariant linear subspaces with the property that $T(Z) \subseteq Z$. Furthermore, we have

$$X = Z + \widetilde{X}_T(\bar{U} \cup \{\infty\}) \cap (T(0) + \mathcal{D}(T)) = Z + T(0) + Y = Z + Y.$$

In view of (3.2), what remains to be shown are the following statements:

(I) $P(Z) \subseteq Z$;

(II) $\tilde{\sigma}(T|_s Y) = \sigma(T|_s Y) \subseteq U_1$.

In order to prove assertion (I), we notice that $Z = T(0) + Z \cap \mathcal{D}(T)$, so $P(Z) \subseteq P(T(0)) + P(Z \cap \mathcal{D}(T)) = Z \cap \mathcal{D}(T) \subseteq Z$. To establish assertion (II), we first observe, by [2, Proposition 2.3], that

$$T|_s Y(0) \subseteq T(0) \cap Y \subseteq T(0) \cap \mathcal{D}(T) = \{0\}.$$

On the other hand, we have

$$T(Y) \subseteq T(\widetilde{X}_T(\overline{U} \cup \{\infty\})) \subseteq \widetilde{X}_T(\overline{U} \cup \{\infty\}) \cap (\mathcal{D}(T) + T(0)) \subseteq Y + T(0),$$

so, by Lemma 2.2, Y is a T -weakly invariant linear subspace. Applying [2, Proposition 2.2], we then obtain that $\mathcal{D}(T|_s Y) = Y \cap \mathcal{D}(T) = Y$, and hence $T|_s Y \in \text{End}(Y)$. This means, by definition, that $\tilde{\sigma}(T|_s Y) = \sigma(T|_s Y)$. It remains to be shown that $\sigma(T|_s Y) \subseteq U_1$.

Claim. $\sigma(T|_s Y) \subseteq \sigma(T|_s \widetilde{X}_T(\overline{U} \cup \{\infty\}))$.

To do so, let $\lambda \in \rho(T|_s \widetilde{X}_T(\overline{U} \cup \{\infty\}))$ and consider $y \in Y \subseteq \widetilde{X}_T(\overline{U} \cup \{\infty\})$. Then, there exists $x \in \mathcal{D}(T|_s \widetilde{X}_T(\overline{U} \cup \{\infty\}))$ such that $y \in (T|_s \widetilde{X}_T(\overline{U} \cup \{\infty\}) - \lambda I)x$. In view of [2, Proposition 2.3], we obtain that

$$y \in (T|_s \widetilde{X}_T(\overline{U} \cup \{\infty\}) - \lambda I)x \subseteq (T - \lambda I)x. \quad (3.3)$$

Moreover, since $\widetilde{X}_T(\overline{U} \cup \{\infty\})$ is a T -strongly and weakly invariant linear subspace, it follows from [2, Proposition 2.2] that $\mathcal{D}(T|_s \widetilde{X}_T(\overline{U} \cup \{\infty\})) = \mathcal{D}(T) \cap \widetilde{X}_T(\overline{U} \cup \{\infty\}) = Y$. This implies, by a combination of (3.3) and [2, Proposition 2.2], that

$$y \in (T - \lambda I)x = (T|_s Y - \lambda I)x + T(0).$$

Consequently, $y - (T|_s Y - \lambda I)x \in T(0) \cap Y \subseteq T(0) \cap \mathcal{D}(T) = \{0\}$, which proves that $y \in \text{Im}(T|_s Y - \lambda I)$. Hence $T|_s Y - \lambda I$ is surjective.

On the other hand, T has the ER-SVEP, $\text{Ker}(T) \cap T(0) = \{0\}$ and Y is a T -strongly invariant linear subspace, so, by assertion (iii) of [4, Lemma 3.1], we deduce that $T|_s Y$ has the ER-SVEP too. As $T|_s Y \in \text{End}(Y)$, we then conclude that $T|_s Y - \lambda I$ has the ER-SVEP, again from [4, Lemma 3.1]. In addition, we have

$$\text{Ker}(T|_s Y - \lambda I) \cap (T|_s Y - \lambda I)(0) = \text{Ker}(T|_s Y - \lambda I) \cap \{0\} = \{0\}.$$

Hence, we obtain from part (ii) of [4, Lemma 3.1] that $T|_s Y - \lambda I$ is invertible, and therefore $\lambda \in \rho(T|_s Y)$, as claimed.

A simple combination of the previous claim and (3.2) leads to the inclusion $\sigma(T|_s Y) \subseteq \sigma(T|_s \widetilde{X}_T(\overline{U} \cup \{\infty\})) \subseteq U_1$, which completes the proof of (II). \square

Now, we are very close to reach the characterisation of the extended spectral decomposability of linear relations and establish the equivalence (1.3). Before getting into the details, we will start by establishing the link between the extended Bishop and Dunford properties in Lemma 3.8. For doing so, some terminology is recalled from [2]. Let $U \subseteq \mathbb{C}_\infty$ be an open set and let $T \in CR(X)$. Then, the relation T_U is defined as follows:

$$\begin{aligned} T_U : P(U, \mathcal{D}(T)) &\longrightarrow H^\sharp(U, X) \\ f &\longrightarrow T_U(f) = \{\varphi \in H^\sharp(U, X) ; \varphi(\mu) \in (T - \mu I)f(\mu) \forall \mu \in U \cap \mathbb{C}\}. \end{aligned}$$

If $T \in CR_c(X)$, then it is said to have the *extended Bishop's property*, shortly the property (\mathcal{E}_β) , if, for every open set $O \subseteq \mathbb{C}_\infty$ and every $(t_n)_{n \in \mathbb{N}} \subseteq P(U, \mathcal{D}(T))$, $(h_n)_{n \in \mathbb{N}} \subseteq H^\sharp(U, X)$ satisfying $h_n(\eta) \in (T - \eta I)f_n(\eta)$, for all $\lambda \in U \cap \mathbb{C}$, following implication holds

$$h_n \xrightarrow{n \rightarrow +\infty} 0 \text{ in } H^\sharp(U, X) \implies t_n \xrightarrow{n \rightarrow +\infty} 0 \text{ in } P(U, \mathcal{D}(T)).$$

Lemma 3.8. *Let $T \in CR(X)$ be a relation such that $Ker(T) \cap T(0) = \{0\}$ and $Im(T_U)$ is closed for every open subset U of \mathbb{C}_∞ . If T has the property (\mathcal{E}_β) , then T has the property (\mathcal{E}_C) .*

Proof. Since T has (\mathcal{E}_β) , it then follows from [2, Theorem 3.1] that T_U is injective. Thus, T has the ER-SVEP. Combining this with the fact that $Ker(T) \cap T(0) = \{0\}$ and Lemma 2.10, we deduce that $\widetilde{\mathcal{X}}_T(F) = \widetilde{X}_T(F)$ for every closed set $F \subseteq \mathbb{C}_\infty$. As a consequence, one may proceed exactly as in the proof of [2, Theorem 3.3] and use the fact that $Im(T_U)$ is closed to complete the argument. \square

A simple combination of Lemma 3.8 and assertion *ii/* of [2, Theorem 3.1] leads to the following result.

Corollary 3.9. *If $T \in CR_c^\oplus(X)$ has the property (\mathcal{E}_β) , then T has the property (\mathcal{E}_C) .*

We are finally in a position to present the main theorem of this section and solve (1.3). This is actually a generalization of the equivalences $(a) \Leftrightarrow (c) \Leftrightarrow (d)$ of [12, Theorem 2.5.19] that described the decomposability of bounded linear operators.

Theorem 3.10. *For every relation $T \in CR_c^\oplus(X)$, the following statements are equivalent:*

- (i) T is extended spectral decomposable;
- (ii) T has the properties (\mathcal{E}_β) and (\mathcal{E}_δ) ;
- (iii) T has the properties (\mathcal{E}_C) and (\mathcal{E}_δ) .

Proof. In light of Proposition 2.1, we deduce that $P_T T$ is a continuous linear selection of T verifying $P_T T \in End(\mathcal{D}(T))$ and $\rho(T) = \rho(P_T T)$, which implies that $\rho(T)$ contains an unbounded connected component. Hence, the implication $(i) \Rightarrow (ii)$ is immediate from [2, Theorem 3.2] and Theorem 3.4. The implication $(ii) \Rightarrow (iii)$ is clear from Lemma 3.8 and the remaining implication $(iii) \Rightarrow (i)$ follows directly from Lemma 3.7. \square

The following example illustrates how, although a linear relation has both the extended Bishop and Dunford properties, it lacks the extended spectral decomposability property through the lack of (\mathcal{E}_δ) .

Example 3.11. *Let $X = l^2(\mathbb{Z})$, S_r be the unilateral right shift operator on $l^2(\mathbb{N})$, and let $T \in LR(X)$ given by*

$$G(T) = G(S_r) + (\{0\} \times l^2(\mathbb{N})^\perp).$$

Then, T has the properties (\mathcal{E}_β) and (\mathcal{E}_C) . Moreover, T is not extended spectral decomposable and does not have (\mathcal{E}_δ) .

Indeed, we first observe that $l^2(\mathbb{N})$ is a closed subspace of $l^2(\mathbb{Z})$ since $l^2(\mathbb{N}) = \{(x_n)_{n \in \mathbb{N}} \subseteq l^2(\mathbb{Z}); x_n = 0 \text{ for all } n < 0\}$. Moreover, from the construction of T one has $\mathcal{D}(T) = \mathcal{D}(S_r) = l^2(\mathbb{N}) = T(0)^\perp$, and since $S_r \in End(l^2(\mathbb{N}))$, it follows that T is closed. Hence, $T \in CR_c^\oplus(X)$. Now, to establish the property (\mathcal{E}_β) , let $U \subseteq \mathbb{C}_\infty$ be an open set, $(f_n)_{n \in \mathbb{N}} \subseteq P(U, \mathcal{D}(T))$, and $(g_n)_{n \in \mathbb{N}} \subseteq H^\sharp(U, X)$ be such that $g_n \xrightarrow{n \rightarrow +\infty} 0$ in $H^\sharp(U, X)$ and

$$g_n(\mu) \in (T - \mu I)f_n(\mu) \quad \text{for all } \mu \in U \cap \mathbb{C}. \quad (3.4)$$

Then, $g_{n|_{U \cap \mathbb{C}}} \xrightarrow{n \rightarrow +\infty} 0$ in $H(U \cap \mathbb{C}, \mathcal{D}(T))$ and $P_T g_{n|_{U \cap \mathbb{C}}}(\mu) = P_T(T - \mu I)f_{n|_{U \cap \mathbb{C}}}(\mu) = (S_r - \mu I)f_{n|_{U \cap \mathbb{C}}}(\mu)$ for all $n \in \mathbb{N}$ and $\mu \in U \cap \mathbb{C}$. On the other hand, S_r is an isometry, so it has the Bishop's property in the sense of [12, Definition 1.2.5] by [12, Proposition 1.6.7], and therefore $f_{n|_{U \cap \mathbb{C}}} \xrightarrow{n \rightarrow +\infty} 0$ in $H(U \cap \mathbb{C}, \mathcal{D}(T))$. Since $f \in P(U, \mathcal{D}(T))$, we then conclude that $f_n \xrightarrow{n \rightarrow +\infty} 0$ in $P(U, \mathcal{D}(T))$ and hence T has the property (\mathcal{E}_β) . A direct application of Corollary 3.9 yields that T has the property (\mathcal{E}_C) .

Next, suppose on the contrary that T has (\mathcal{E}_δ) . Then, for all open cover $\{U, V\}$ of \mathbb{C}_∞ , one has

$$X = \mathcal{X}_T(\overline{U}) + \widetilde{\mathcal{X}}_T(\overline{V}) = X_T(\overline{U}) + \widetilde{X}_T(\overline{V}).$$

Combining this with item *i/* of Lemma 3.6, we conclude by Corollary 3.2 that

$$l^2(\mathbb{N}) = P_T X = X_{S_r}(\overline{U}) + X_{S_r}(\overline{V \cap \mathbb{C}}) \quad (3.5)$$

for all open cover $\{U, V\}$ of \mathbb{C}_∞ . But S_r has the Bishop's property as mentioned above, so [12, Propositions 1.2.19], [12, Proposition 3.3.2] together with (3.5) implies that

$$l^2(\mathbb{N}) = \mathcal{X}_{S_r}(\overline{U}) + \mathcal{X}_{S_r}(\overline{V \cap \mathbb{C}})$$

all open cover $\{U, V\}$ of \mathbb{C}_∞ . As a consequence, S_r has the decomposition property in the sense of [12, Definition 1.2.28]. Using again [12, Proposition 1.6.7], one concludes that S_r is invertible, contradiction.

Finally, the fact that S_r is an isometry together with Proposition 2.1 shows that

$$\rho(T) = \rho(S_r) = \mathbb{C} \setminus \overline{B_{\mathbb{C}}(0, 1)},$$

which means that $\rho(T)$ has an unbounded connected component. Applying Theorem 3.4, we then conclude that T is not an extended spectral decomposable linear relation.

3.2 The proof of the third equivalent condition (1.4)

As we have already done in the previous section, we will devote this part to giving a different characterization of the extended spectral decomposability, using this time coinduced linear operators. To better see things, we are going to seek the best way to extend the equivalence (a) \Leftrightarrow (b) described in [12, Theorem 1.2.23] from the case of bounded linear operators to that of multi-valued linear ones in order to reach the equivalences (1.4). It is worth recalling that the opposite implication of (1.4) was the purpose of [4], so all we have to do is to prove the other implication.

Lemma 3.12. *If $T \in CR_c^\oplus(X)$ is extended spectral decomposable then $\tilde{\sigma}(T/\widetilde{X}_T(F)) \subseteq \overline{\tilde{\sigma}(T) \setminus F}$ for every closed subset F of \mathbb{C}_∞ .*

Proof. We first observe from Theorem 3.10 that T has the property (\mathcal{E}_C) , so, by [4, Lemma 4.1], $T/\widetilde{X}_T(F) \in \text{End}(X/\widetilde{X}_T(F))$. This ensures that $\tilde{\sigma}(T/\widetilde{X}_T(F)) = \sigma(T/\widetilde{X}_T(F))$. Hence, we only need to prove that $\tilde{\sigma}(T/\widetilde{X}_T(F)) \subseteq \overline{\tilde{\sigma}(T) \setminus F} \cap \mathbb{C}$. To do this, let $\lambda \in (\mathbb{C}_\infty \setminus \overline{\tilde{\sigma}(T) \setminus F}) \cap \mathbb{C}$ and choose an open cover $\mathbb{C}_\infty = \{U, V_\infty\}$ where U is an open bounded subset of \mathbb{C} and V_∞ is an open subset of \mathbb{C}_∞ such that $\lambda \in \mathbb{C}_\infty \setminus \overline{V_\infty}$ and $\tilde{\sigma}(T) \setminus F \subseteq \mathbb{C}_\infty \setminus \overline{U}$. From the extended spectral decomposability of T , we obtain T -strongly invariant linear subspaces X_0 and X_∞ for which

$$X = X_0 + X_\infty, \quad \tilde{\sigma}(T|_s X_0) = \sigma(T|_s X_0) \subseteq U \quad \text{and} \quad \tilde{\sigma}(T|_s X_\infty) \subseteq V_\infty.$$

It then follows from [6, Remark 2.7] that $\sigma(T|_s X_0) \subseteq \overline{U} \cap \sigma(T) \subseteq F$, so, by Lemma 2.3, we obtain $X_0 \subseteq \widetilde{X}_T(F)$. Moreover, Lemma 3.3 ensures that $X_1 \subseteq \widetilde{X}_T(\overline{V_\infty})$, and therefore

$$X/\widetilde{X}_T(F) = Q_{\widetilde{X}_T(F)}(X_0 + X_1) = Q_{\widetilde{X}_T(F)}\widetilde{X}_T(\overline{V_\infty}). \quad (3.6)$$

On the other hand, we have $\lambda \in \mathbb{C}_\infty \setminus \overline{V_\infty} = (\mathbb{C}_\infty \setminus \overline{V_\infty}) \cap \mathbb{C}$. We then obtain from [4, Proposition 3.4] that $(T - \lambda I)\widetilde{X}_T(\overline{V_\infty}) = \widetilde{X}_T(\overline{V_\infty})$. This entails, by (3.6) that

$$\begin{aligned} X/\widetilde{X}_T(F) &= Q_{\widetilde{X}_T(F)}(T - \lambda I)\widetilde{X}_T(\overline{V_\infty}) = (T/\widetilde{X}_T(F) - \lambda I)Q_{\widetilde{X}_T(F)}\widetilde{X}_T(\overline{V_\infty}) \\ &= (T/\widetilde{X}_T(F) - \lambda I)(X/\widetilde{X}_T(F)). \end{aligned}$$

Hence, $(T/\widetilde{X}_T(F) - \lambda I)$ is surjective. In view of [4, Lemma 3.1], it suffices to prove that $(T/\widetilde{X}_T(F) - \lambda I)$ has the ER-SVEP, since $(T/\widetilde{X}_T(F) - \lambda I)(\overline{0}) \cap \text{Ker}(T/\widetilde{X}_T(F) - \lambda I) = \{\overline{0}\}$. To do this, we first observe from Proposition 2.1 that $\rho(T) = \rho(P_T T)$, so $\rho(T)$ has an unbounded connected component. Since T is extended spectral decomposable, it then follows from [2, Theorem 3.3] that T has the ER-SVEP. Moreover, as T has the property (\mathcal{E}_C) , we infer from Corollary 2.9 that $T/\widetilde{X}_T(F)$ has the ER-SVEP. Assertion (i) of [4, Lemma 3.1] then yields that $(T/\widetilde{X}_T(F) - \lambda I)$ has the ER-SVEP, as desired. Consequently, $\lambda \in \rho(T/\widetilde{X}_T(F))$. \square

We end this section by stating the third equivalent condition for the extended spectral decomposability of a closed linear relation in the following theorem.

Theorem 3.13. *Let $T \in CR_c^\oplus(X)$. Then, the following assertions are equivalent:*

- (i) *T has the property (\mathcal{E}_C) and $\tilde{\sigma}(T/\widetilde{X}_T(F)) \subseteq \overline{\tilde{\sigma}(T) \setminus F}$ for all closed subsets F of \mathbb{C}_∞ ;*
- (ii) *T is an extended decomposable linear relation.*

Proof. One may conclude the proof by combining [4, Theorem 4.1] with Lemma 3.6 and Lemma 3.12. \square

Remark 3.14. For the relation $T \in l^2(\mathbb{Z})$ described in Example 3.11, Theorem 3.13 shows that there exists a set F_0 in \mathbb{C}_∞ for which $\tilde{\sigma}(T/\widetilde{X}_T(F_0)) \not\subseteq \overline{\tilde{\sigma}(T) \setminus F_0}$.

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