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A finite-time singularity in the dynamics of the US equity market: Will the US equity market eventually collapse?

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ABSTRACT

Fitting Dow Jones 30 index data for the 1790–1999 period into a log-periodic power-law singularity (LPPLS) model, the seminal paper by Johansen and Sornette (2001) was the first to show that the US equity growth rate is *accelerating* such that the market is growing as a power law toward a spontaneous singularity. Their model suggests that the US equity market will reach this critical point in the year 2052 ± 10 years, signaling an abrupt transition to a new regime. This study re-examines this important issue using (*i*) a novel approach to calibrate the LPPLS model and (*ii*) a different data set including >20 years of additional data. The extended data account for the dot.com bubble burst (2000), the Global Financial Crisis period (2008–2009), the COVID−19 crisis (2020−2022), and the ongoing Russian–Ukrainian war (starting in 2022), which are all events with severe consequences for the global economy. The calibrated LPPLS model suggests that the US equity market will reach a singularity condition by June 2050.

"*The world has gone mad, and the system is broken.*"
(Ray Dalio, hedge fund manager and founder of Bridgewater Associates)

1. Introduction

A well-established fact, one documented in the literature on physics, is that faster-than-exponential growth is not sustainable in the long term and will eventually result in what physicists typically term a "finite-time singularity."[2] In a series of research articles, the well-known physicist Didier Sornette and his research collaborators provided extensive evidence that the recurring build-up of stock market bubbles manifests itself as an overall super-exponential, power-law acceleration in the price decorated by log-periodic precursors, a concept related to fractals.[3] Building the theory for his model on complex systems, Sornette (2017) argues as follows:

> Financial markets are not only systems with extreme events. Financial markets constitute one among many other systems exhibiting a complex organization and dynamics with similar behavior. Systems with a large number of mutually interacting parts, often open to their environment, self-organize their internal structure and their dynamics with novel and sometimes macroscopic ('emergent') properties. The complex system approach, which involves 'seeing' interconnections and relationships, that is, the whole picture as well as the component parts, is nowadays pervasive in modern control of engineering devices and business management. (p. 15).

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2 An interesting discussion on this subject is, for example, provided in West (2017, Chapter 10).

3 See Johansen, Ledoit, and Sornette (1999, 2000), Johansen and Sornette (1999), or Sornette and Johansen, (1997). Moreover, Sornette (2017) provides a detailed overview of the relevant literature.

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Viewing stock markets as complex self-organizing systems, crashes are caused by the slow build-up of long-range correlations, resulting in the market's global cooperative behavior and eventual collapse (namely, a finite-time singularity) within a short, critical time interval. An empirical manifestation of this issue is that drawdowns are governed by power-law processes (Filimonov & Sornette, 2015). In their 2001 paper, Johansen and Sornette propose a log-period power-law singularity (LPPLS) model to estimate finite-time singularities in the dynamics of the world population and some financial indices. The authors argue that the growth rates of both the world population and financial indices are compatible with a spontaneous singularity occurring at the *same* critical time (2052 ± 10 years), signaling an abrupt transition to a new regime. It is interesting to note that Johansen and Sornette's (2001) findings are in line with Godley's (2012) prediction of a period of consolidation due to the unsustainable growth of the US economy in the last decade of the twentieth century based on their investigation of fiscal policy, foreign trade, and private income expenditure and borrowing.[4] The arrival of a finite-time singularity in the near future may have an enormous impact on the global financial ecosystem.

Sornette (2017) argues that the Dow Jones index's faster-than-exponential growth is manifested in an acceleration of growth rates, where the growth rate of the return of the Dow Jones Index was, on average, about 3% per year between 1780 and 1930 and shifted to an average of about 7% per year between 1930 and 2000.[5] Because the arrival of a finite-time singularity is the result of an accelerated growth rate that is not sustainable, a question arises regarding what has happened in the US financial system since the publication of Johansen and Sornette's (2001) study?[6] Did the acceleration of the growth rate of the US equity market in the last 20 years slow down, perhaps averting the risk of a finite-time singularity and, thus, a severe collapse of the financial ecosystem? In the wake of the Global Financial Crisis (2008–2009), the US Federal Reserve Bank made use of considerable interventions to avert a total collapse of the financial ecosystem. Unfortunately, the monetary policy chosen proved to be a double-edged sword.

In this regard, Ray Dalio, founder of the company Bridgewater Associates, which manages the world's largest hedge fund, termed these interventions on behalf of the central banks "money printing," which inevitably resulted in dramatically "inflated stock prices."[7] In fact, in the Global Financial Crisis period, the Dow Jones Index reached its local minimum in February 2009, at 9881.04. In December 2919, however, the index quotation reached 32,961.45; that is, in one decade, the average return on the Dow Jones Index was >20% per year, which strongly suggests an additional acceleration in growth rates. In this regard, it is noteworthy that central banks continued "intervening" by means of printing money to support the financially stricken global economy in the wake of the COVID−19 pandemic (2020–2022). Indeed, the government of the United States−followed by many other governments of Western countries−has printed money in an attempt to support deficit spending over many years. In this regard, Ray Dalio states the following:

> Large government deficits exist and will almost certainly increase substantially, which will require huge amounts of more debt to be sold by governments–amounts that cannot naturally be absorbed without driving up interest rates at a time when an interest rate rise would be devastating for markets and economies because the world is so leveraged long.[8]

Unsurprisingly, in a CNBC interview that took place in November 2019, he summed up his view by stating, "The world has gone mad, and the [monetary] system is broken."[9]

Due to recent developments in the financial ecosystem involving the Global Financial Crisis period (2008–2009), the COVID−19 crisis (2020–2022), and the ongoing Russian–Ukrainian war (starting in 2022)–all events with dramatic impacts on the global economy−it is important to re-assess the relevance of Johansen and Sornette's (2001) apocalyptic view of the future of the financial sphere. The purpose of this study is to re-examine Johansen and Sornette (2001) study by adopting a coarse-grained perspective on the US equity market to analyze whether the LPPLS model provides evidence for an expected finite-time singularity in the year 2052. While Johansen and Sornette (2001) calibrate the LPPLS model in their main analysis of monthly Dow Jones Index data using a sample from 1790 to 2000, the current research uses monthly data on the S&P 500 for the 1870 to 2022 period, which were obtained from Robert Shiller's online library. As a result, the most recent challenging economic periods, such as the Global Financial Crisis, which is perhaps the most severe crisis in recent financial history, are accounted for in the sample. The LPPLS model is calibrated using what we term a "worst-case-scenario model" as the first-order approximation for the first stage. The model is tested by implementing a standard augmented Dickey–fuller (ADF) test for the residuals (Lin, Ren, & Sornette, 2014). The accuracy of the proposed approach when used for the first-order approximation model is explored by calibrating it on daily data to predict the well-known October 1929, October 1987 and October 2008 crashes.

This study contributes to the literature in various important ways. First, as mentioned above, given the previously mentioned extraordinary economic challenges in the last two decades, it is important to re-assess the relevance of a potential regime switch in the financial ecosystem, as first predicted in Johansen and Sornette's (2001) study. As Sornette (2017) points out, the LPPLS model's prediction capability becomes more accurate as we approach the estimated critical time. It is important to note that Johansen and Sornette's (2001) research used a data range that ended in December 1999. Because the predicted critical time in their study is 2052, >40% of the waiting time, which refers to the time between the end of the original sample and the predicted finite-time singularity, has already elapsed. Moreover, while the data used in Johansen and Sornette's (2001) research do not account for the extraordinarily challenging economic periods that occurred in the most recent two decades. This is the first study examining the potential arrival

[4] Similarly, Kapitza (1996) documented the super-exponential acceleration of human activity, which is consistent with a power-law singularity, as mentioned in Sornette (2017, p. 378).

[5] See Figure 2.1 in Sornette (2017, p. 28).

[6] In this regard, West (2017, p. 414) argues, "This kind of growth behavior is clearly unsustainable because it requires an unlimited, ever-increasing, and eventually infinite supply of energy and resources at some finite time in the future in order to maintain it. Left unchecked, the theory predicts that it triggers a transition to a phase that leads to stagnation and eventual collapse, […] This scenario sounds like a rehash of the Malthusian argument that has been summarily dismissed by generations of economists: namely, that we won't be able to keep up with demand and that open-ended growth will eventually lead to catastrophe."

[7] In his lecture entitled "The Economic Machine," he details the mechanism via which stock prices can be inflated due to monetary policies created by the central bank (see https://www.youtube.com/watch?v=PHe0bXAIuk0). Specifically, he argues that the central bank is forced to print money upon reaching a regime of zero interest rates. Unlike cutting spending, debt reduction, and wealth redistribution, printing money is, in general, inflationary and stimulative. However, the central bank uses the printed money to buy financial assets and government bonds, which drives up asset prices, making people more creditworthy.

[8] See https://www.cnbc.com/2019/11/06/ray-dalio-says-the-economy-isnt-growing-because-the-world-is-mad.html.

[9] See https://www.cnbc.com/2019/11/08/bridgewaters-ray-dalio-on-economy-worlds-gone-mad-system-is-broken.html.

of a finite-time singularity using S&P 500 data for an extended sample period comprising these times of enormous economic stress.[10] Next, reassessing whether prior documented results are still relevant ex-post publication is an important issue per se. In this regard, Hou, Xue, and Zhang (2020), who attempted to replicate 452 cross-sectional asset pricing anomalies, concluded that at least 65% of these anomalies could not be replicated. The authors argue, "The crux is that unlike natural sciences, economics, finance, and accounting are mostly observational in nature. As such, it is critical to evaluate the reliability of published results against 'similar, but not identical,' specifications" (Hou et al., 2020, p. 2022). Interestingly, the authors also found that, even for replicated anomalies, their economic magnitudes are much smaller than those documented in the original studies. The current research is the first project to provide a sound replication of Johansen and Sornette's (2001) research that meets the requirements for a scientific replication in line with Hamermesh (2007) because we employ (*i*) a different population sample period and (*ii*) a similar but not identical approach to estimating the LPPLS model parameters.

Moreover, Sornette (2017, p. 334) points out that his proposed approach to calibrating LPPLS models serves as "first-order approximations, and novel improved methods have been developed that are not published." Indeed, LPPLS models are not standard techniques taught at business schools. Perhaps the reason for this is that these models were originally developed in the physics literature (the mathematical and statistical physics of bifurcations and phase transitions) and, as a consequence, can be difficult to grasp for most finance scholars, who lack the required mathematical background. Thus, a gap in the literature is an intuitive LPPLS model set-up that can be implemented using standard software.[11] The current research remedies this gap by proposing a very simple yet powerful three-stage estimation procedure for estimating the LPPLS model, in which the first stage corresponds to what we term a "worst-case-scenario model." In providing a simple approach to calibrating LPPLS models, the current research lives up to Mandelbrot's (2008, p. 125) argument: "Contrary to popular opinion, mathematics is about simplifying life, not complicating it." Finally, the current research adds to the literature on bubble diagnoses and predictions using quantitative methodologies. Jiang et al. (2010) and Sornette (2017) provide detailed literature reviews on this topic.

Using monthly data on the S&P 500, the results of the current research show that the LPPLS model predicts the arrival of a finite-time singularity in 2050. A 95% confidence interval for the critical time is estimated between December 2043 and December 2050. This figure is close to the critical time derived in Johansen and Sornette (2001), corresponding to 2052 ± 10 years. However, it is noteworthy that the point estimate for the critical time derived in Johansen and Sornette's (2001) original study is outside the *upper* bound of the confidence interval derived in the current research. This means that, after adding more recent data, the estimated arrival of a finite-time singularity is statistically significantly *earlier* than previously expected by Johansen and Sornette (2001). The reason for this issue could be that the extreme monetary policies adopted by the central bank inflated stock prices in an unforeseen manner, which accelerated the arrival of the expected critical time. Finally, robustness checks show that the proposed approach to estimating the LPPLS model based on the "worst-case-scenario model" is capable of successfully predicting both of the aforementioned well-known crashes.

This paper is organized as follows. The next section provides an overview of the data. The third section provides the empirical analysis, and the final section offers our conclusions.

2. Data

Data for this study's main analysis were downloaded from Robert Shiller's data library, which is accessible for free at www.econ.yale.edu/~shiller/data.htm. Specifically, monthly data on the S&P 500 were downloaded from January 1871 to November 2022. Moreover, for robustness checks, additional daily data on the S&P 500 were obtained for the periods from January 2, 1980, until December 31, 1986, and from October 9, 2002, until December 31, 2007, from www.finance.yahoo.com. Finally, daily data on the Dow Jones Index for the period from June 1, 1921, until December 31, 1928, were retrieved from https://stooq.com/q/d/?s=%5Edji.

3. Methodology

3.1. Implementing the LPPLS model using a novel three-stage estimation approach

According to Sornette (2017), a simple power-law model for financial log-prices is given by

$$ln[p(t)] = A + B(t_c - t)^{\beta}, \quad (1)$$

where $ln[p(t)]$ is the logarithm of the S&P 500 at time t, t_c is the critical time, A is the expected value of the S&P 500 in the logarithm, B measures the exposure to faster-than-exponential growth, and β is the power-law exponent controlling faster-than-exponential price growth. The critical time t_c indicates a regime change. In this regard, Zhang, Sornette, Balcilar, Gupta, Ozdemir, and Yetkiner (2016, p. 129) point out that "The regime change may be the time of the crash, the burst of the bubble, or the fading of the bubble. Not all bubbles end with a crash and therefore the change in regime is not necessarily a crash. The regime change, however, will be a change from super-exponential growth to a lower growth and the end of the accelerating oscillations." Note that, for this model specification, $A > 0$, $B < 0$, and $0.1 \leq \beta \leq 0.9$ must hold. Sornette (2017) points out that the simple power-law model of eq. (1) must be expanded by accounting for periodic oscillations:

$$ln[p(t)] = A + B(t_c - t)^{\beta}[1 + Ccos(\omega ln(t_c - t) + \phi)], \quad (2)$$

where C measures the exposure the log-periodic oscillations around the power-law singular growth, ω denotes the angular log-frequency of oscillations during the bubble, ϕ is the phase parameter, and all other notations are as previously stated. Whereas Sornette (2017) documents that ϕ cannot be meaningfully restricted, this study follows Bothmer & Meister (2003) and requires $|C| < 1$ while also imposing the constraint $5 \leq \omega \leq 15$.

The implementation of the LPPLS model of Eq. (2) requires several model parameters $\mathbf{\Phi} = (A, B, t_c, \beta, C, \phi)$ be estimated using a highly non-linear model. Using any non-linear solver to find optimal values for all parameters at once will inevitably generate spurious fits, in which different initial chosen values for some parameters in the parameter vector, $\mathbf{\Phi}$, will result in different values for some other parameters. To address this issue, the calibration of the model is typically based on some combination of finding suggested solutions for parameters, freezing some of the parameters, and using some non-linear solver to find solutions for the free parameters. Note that some search algorithms, such as the often-used Levenberg–Marquardt algorithm, are sensible for the selected initial values and may suggest local instead of global minimums when initial solutions are less optimal.

[10] Note also that the S&P 500 index is a considerably broader index than the Dow Jones Index and consists of firms often recognized as "industry leaders." In this regard, Gnabo, Hvozdyk, and Lahaye (2014) highlight that the S&P 500 index is widely considered an important gauge of the US equity market and prominently quoted in stock markets around the world.

[11] It is interesting to note that Didier Sornette revealed, in a recent podcast from February 3, 2023, that banks and other financial institutions asked him if he could share his code with them, which clearly highlights that the LPPLS model implementation remains a "blackbox" for the wider audience–even for highly educated specialists working in the modern finance industry. The podcast is available here: https://www.youtube.com/watch?v=ssuaIxLBtIE.

This study proposes an intuitively plausible approach to choosing some initial values for parameters when employing non-linear solvers, such as Microsoft Excel's solver.[12] Because A is the expected value of the S&P 500 in the logarithm, $A = ln[p(T)]$ in association with $t_c = T+1$ means that the last price notation of the S&P 500 is the expected price for the critical time t_c and the critical event is suggested to occur in the subsequent time period. Next, Sornette (2017) points out that the exponent β must be between 0 and 1 for the price to accelerate and remain finite. Moreover, Sornette (2017, p. 335) emphasizes that "The more stringent criterion $0.2 \leq \beta \leq 0.8$ has been found useful to avoid the pathologies associated with the endpoints 0 and 1 of the interval."

Thus, treating β as "fixed," where $\overline{\beta} \in \{0.2, 0.4, 0.6, 0.8\}$, and setting $B = -1$, Eq. (1) simplifies to the following[13]:

$$ln[p(t)] = ln[p(T)] - (t_c - t)^{\overline{\beta}} \tag{3}$$

Panel A of Table 1 reports the corresponding initial model specifications. Note that $ln[p(T)] = 8.23$ corresponds to the natural logarithm of the last price quotation of the S&P 500 in the sample. Because we use monthly data, one unit of time corresponds to 1/12, and because the sample ends at time $T = 151.9167$, it follows that $T+1 = 152$. We see that using $\overline{\beta} = 0.4$ generates the minimum sum of squared residuals (SSR). Note that the initial model specifications can be regarded as "worst-case scenarios" because these models suggest that the finite-time singularity occurs "immediately," implying that there are no further precautionary actions possible. Next, when optimizing the first model, the following constraints are accounted for:

$$t_c \geq T+1$$

$$0.1 \leq \beta \leq 0.9$$

The first constraint allows for the possibility that the finite-time singularity arrives at some unknown time point in the future, whereas the second constraint is needed for the price to accelerate and remain finite. Employing Microsoft Excel's non-linear solver to compute optimal values for the parameter vector $\mathbf{\Phi}_1^* = (A^*, B^*, t_c^*, \beta^*)$, the optimized values for the first model's (Model 1) parameters are reported in Panel B of Table 1. We see that model specification 2 of Model 1 provides parameter values generating the minimum SSR with $\mathbf{\Phi}_1^* = (33.24, -19.45, 152, 0.10)$. It is interesting to note that, regardless of the model specification, all optimized models suggest that the finite-time singularity occurs at the same time, that is, $t_c^* = T+1$. Fig. 1 plots the optimized Model 1, along with the natural logarithm of the S&P 500, denoted as $ln[p(t)]$. A visual inspection of Fig. 1 shows that this model properly captures the faster-than-exponential growth property of the S&P 500.

Using $\mathbf{\Phi}_1^*$ as initial values for Model 2 in Eq. (2) in association with $C = \phi = 0$, Model 2 is then optimized for varying values of ω, that is, $\omega \in \{5, 6, \ldots, 14, 15\}$.[14] The rationale for this approach rests upon the argument in Sornette (2017) documenting that, in practice, $5 < \omega < 15$ is often used. Note that Sornette (2017, p. 232) derives the relationship $\lambda = exp\left(\frac{2\pi}{\omega}\right)$ and argues, "For the October 1987 crash, we find $\lambda \cong 1.5-1.7$ (this value is remarkably universal and is found to be approximately the same for other crashes…)." Despite this empirical evidence, it is interesting to note that the optimal values for ω can, at times, exhibit

Table 1
Calibrating the power-law model for the S&P 500 using monthly data from 1871 to 2022.

Panel A. Initial parameter values for Model 1.

Specification	A	B	$\overline{\beta}$	$t_c = T+1$	SSR
1	8.23	−1	0.2	152	13,960.61
2	8.23	−1	0.4	152	1256.66
3	8.23	−1	0.6	152	139,262.30
4	8.23	−1	0.8	152	1,586,913.00

Panel B. Optimized parameter values for Model 1.

Specification	A^*	B^*	β^*	t_c^*	SSR
1	16.05	−4.57	0.24	152	192.75
2	**33.24**	**−19.45**	**0.10**	**152**	**164.81**
3	9.73	−0.83	0.47	152	257.22
4	9.72	−0.80	0.48	152	258.68

To model financial log-prices we use a simple power-law model which is given by,

$$ln[p(t)] = A + B(t_c - t)^{\beta}$$

where $ln[p(t)]$ is the logarithm of the S&P 500 at time t, t_c is the critical time, A is the expected value of the S&P 500 in logarithm, B measures the exposure to faster-than-exponential growth and β is the power-law exponent controlling faster than exponential price growth. Note that for this model specification $A > 0$, $B < 0$, and $0.1 \leq \beta \leq 0.9$ must hold. Treating β as "fixed", where $\overline{\beta} \in \{0.2, 0.4, 0.6, 0.8\}$, and setting $B = -1$, Eq. (1) simplifies to:

$$ln[p(t)] = ln[p(T)] - (t_c - t)^{\overline{\beta}}$$

Note that A is the expected value of the S&P 500 in logarithm, and $A = ln[p(T)]$ in association with $t_c = T+1$ means the last price notation of the S&P 500 is the expected price for the critical time t_c and the critical event is suggested to occur in the subsequent time period. Panel A of Table 1 reports the corresponding initial model specifications. Note that $ln[p(T)] = 8.23$ corresponds to the natural logarithm of the last price quotation of the S&P 500 in the sample. Next, the model is optimized allowing for the following constraints:

$$t_c \geq T+1$$

$$0.1 \leq \beta \leq 0.9$$

The first constraint allows for the possibility that the finite time singularity arrives at some unknown time point in the future, whereas the second constraint is needed for the price to accelerate and to remain finite. Employing Microsoft Excel's non-linear solver to compute optimal values for the parameter vector $\mathbf{\Phi}_1^* = (A^*, B^*, t_c^*, \beta^*)$, the optimized values are reported in Panel B of Table 1. The sample period is from January 1871 to November 2022 comprising 1823 monthly observations.

[12] It is important to note that the methodological discussion that follows is intended to provide an intuitive "first-order approximation," which is applicable using standard software.

[13] Note that the parameter B is typically considerably smaller than A in its economic magnitude. As an example, Sornette (2017, p. 345) reports in Table 9.9 the results of attempting to predict the stock market crash that occurred on August 1998 using S&P 500 data. Using 1991 as *first date* with *last dates* varying between 97.444 and 97.8904 in the model specifications, the corresponding parameters for A range between 6.06 and 6.51, whereas the corresponding parameters for B range between −0.338 and − 0.584. Thus, using $B = -1$, as suggested in the model simplification, is perhaps a somewhat arbitrary but not unreasonable choice for initial values, one that is, however, in line with Lin, Ren, and Sornette (2014), who require $B < 0$.

[14] Note that this approach is similar to that suggested by Liberatore (2010) when proposing to fit a simple model with $C = 0$ and $\beta = 1$ in the first step and then using the obtained optimized parameter values as a seed for the second estimation step; that is, Liberatore (2010) optimizes the function $ln[p(t)] = A - (t_c - t)$ in the first step.

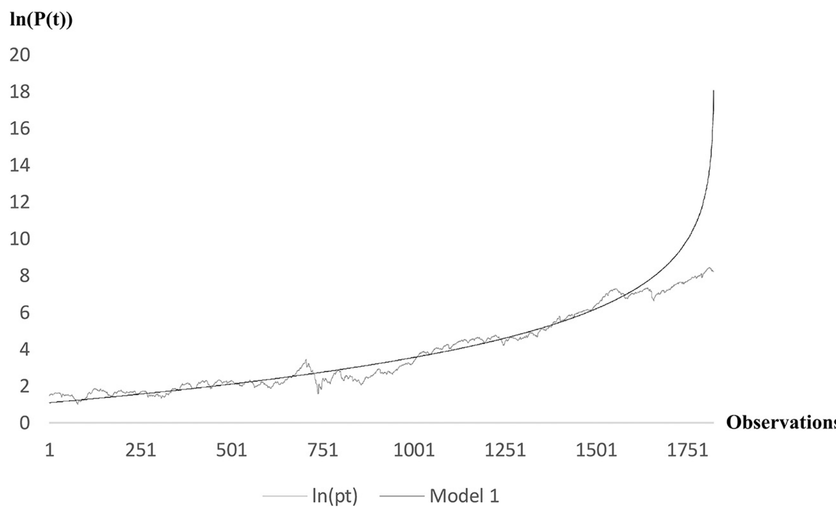

Fig. 1. The S&P 500 in logarithms and optimized power-law Model 1.
This figure plots the optimized Model 1 along with the natural logarithm of the S&P 500 denoted as $ln[p(t)]$.

considerable variation, depending perhaps on the distance between the end of the data sample used and the expected critical time t_c, for example.[15] Therefore and because $1.5 < \lambda < 3.5 \Leftrightarrow 5 < \omega < 15$, it seems reasonable to examine various initial values of ω in the process of optimizing Model 2. Furthermore, when optimizing Model 2, the following constraints are accounted for:

$t_c \geq T+1$

$0.1 \leq \beta \leq 0.9$

$5 \leq \omega \leq 15$

Note that the parameter ϕ remains unconstrained, which is in line with Sornette (2017), who points out that the phase parameter ϕ cannot be meaningfully restricted. Again, this is done using Microsoft Excel's non-linear solver to compute the optimal values for the parameter vector $\mathbf{\Phi}_2^{**} = (A^{**}, B^{**}, t_c^{**}, \beta^{**}, C^{**}, \omega^{**}, \phi^{**})$. Given the input parameter vector $\mathbf{\Phi}_2^{*} = (\mathbf{\Phi}_1^{*}, C, \omega, \phi) = (A^{*}, B^{*}, t_c^{*}, \beta^{*}, C, \omega, \phi)$, using $A^{*} = 33.24$, $B^{*} = -19.45$, $t_c^{*} = 152$, $\beta^{*} = 0.1$, as obtained from the optimization in the previous step, as well as choosing the initial values $C = \phi = 0$, only ω is varied in this step for the reasons discussed previously. Using $\omega \in \{5, 6, \ldots, 14, 15\}$ successively, Panel A of Table 2 reports the input data vectors, whereas Panel B of Table 2 reports the optimized parameter vectors $\mathbf{\Phi}_2^{**}$ for each run. We observe from Panel A of Table 2 that the input parameterization given by $\mathbf{\Phi}_2^{*} = (A^{*}, B^{*}, t_c^{*}, \beta^{*}, C, \omega, \phi) = (33.24, -19.45, 152, 0.1, 0, 6, 0)$ results in the optimized parameterization for Model 2, yielding the lowest SSR (e.g., $SSR = 140.32$), where $A^{**} = 36.08$, $B^{**} = -17.65$, $t_c^{**} = 179.53$, $\beta^{**} = 0.13$, $C^{**} = 0.01$, $\omega^{**} = 5.00$, and $\phi^{**} = 2.04$.

Strikingly, the optimal Model 2 suggests that the finite-time singularity occurs at $t_c^{**} = 179.53$, which corresponds, in the notation used here, to 331 units of time (e.g., months) in the future. Given that the data sample ends in November 2022, the critical time is reached in June 2050, which is very close to Johansen and Sornette (2001) prediction, which was made using different data and a different sample, one that ended in December 1999. Fig. 2 plots the optimized Model 2, along with the natural logarithm of the S&P 500, which is denoted as $ln[p(t)]$, whereas Fig. 3 plots the residuals covering the in-sample time window (e.g., January 1871–November 2022). Comparing Fig. 2 in the current research with Fig. 10.8 in Sornette (2017) shows that the models are virtually indistinguishable from one another despite using (*i*) different data, (*ii*) different samples, and (*iii*) different estimation approaches. Next, a visual inspection of Fig. 3 shows that the residuals do not appear to be integrated, despite being highly autocorrelated.

3.2. *Residuals tests*

Analyzing the residuals of the LPPLS is an important issue in evaluating the model fit to the data. Non-stationary residuals may indicate spurious regression and make statistical reasoning cloudy. Lin et al. (2014) argue that the LPPLS model residuals should follow a stationary mean-reverting process. The stationarity of the residual series is tested using the ADF test, where calibrations within the 99% confidence level for the stationarity of the residuals are considered "successful;" that is, the LPPLS signature is considered statistically significant (e.g., Jiang et al., 2010). Implementing the ADF test requires running the following test regression:

$$\Delta u_t = \delta_0 + \delta_1 t + \delta_2 u_{t-1} + \gamma_1 \Delta u_{t-1} + \ldots + \gamma_p \Delta u_{t-p} + \epsilon_t \quad (4)$$

where $\delta_0 = \delta_1 = 0$ for the model specification in which no deterministic terms are accounted for and $\delta_1 = 0$ for the model specification accounting for only a constant term. Moreover, the parameters $\gamma_1, \ldots, \gamma_p$ are related to the autoregressive component of this model. Note that the model using $\delta_0 = \delta_1 = 0$ corresponds to modeling a random walk under the null hypothesis, whereas using $\delta_1 = 0$ corresponds to modeling a random walk with a drift under the null hypothesis. In consequence, there are three main versions of the ADF test. It is carried out under the null hypothesis $\widehat{\delta}_2 = 0$ against the alternative hypothesis of $\widehat{\delta}_2 < 0$. The estimated test statistic $\widehat{\lambda}$ is then the estimated *t*-statistic associated with the point estimate $\widehat{\delta}_2$. Table 3 reports the estimated test statistics for various ADF tests, the critical values for the 5% and 1% levels, and the *p*-value. Note that all model specifications employ a lag-order of $p = 2$, as suggested by the Schwarz Criterion. Based on Table 3, it becomes evident that the null model is rejected at the 1% level for the model specification that does not account for any deterministic terms (Model 1) and the model specification that only accounts for a constant term

[15] For example, Sornette (2017, p. 331) reports, in Tables 9.2 and 9.3, the results of attempting to predict the well-known stock market crash that occurred in October 1987 using S&P 500 data. Using varying *end-dates* in the model specifications (86.88–87.65) provides varying values for the optimal ω, ranging from 4.1 (for 87.04 as *end-date*) to 12.3 (for 87.52 as *end-date*).

Table 2
Calibrating the LPPLS model for the S&P 500 using monthly data from 1871 to 2022.

Panel A. Initial parameter values for Model 2.

Specification	1	2	3	4	5	6	7	8	9	10	11
A^*	33.24	33.24	33.24	33.24	33.24	33.24	33.24	33.24	33.24	33.24	33.24
B^*	−19.45	−19.45	−19.45	−19.45	−19.45	−19.45	−19.45	−19.45	−19.45	−19.45	−19.45
t_c^*	152	152	152	152	152	152	152	152	152	152	152
β^*	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
C	0	0	0	0	0	0	0	0	0	0	0
ω	5	6	7	8	9	10	11	12	13	14	15
ϕ	0	0	0	0	0	0	0	0	0	0	0
SSR	1722.65	1722.65	1722.65	1722.65	1722.65	1722.65	1722.65	1722.65	1722.65	1722.65	1722.65

Panel B. Optimized parameter values for Model 2.

Specification	1	2	3	4	5	6	7	8	9	10	11
A^{**}	37.31	**36.08**	49.53	34.07	37.27	30.04	30.61	39.42	30.18	39.04	33.80
B^{**}	−20.16	**−17.65**	−28.91	−16.55	−19.26	−14.20	−13.69	−20.18	−14.66	−19.79	−17.85
t_c^{**}	170.66	**179.53**	184.44	175.40	176.17	168.60	174.45	181.98	167.66	182.38	168.21
β^{**}	0.12	**0.13**	0.10	0.13	0.12	0.14	0.15	0.12	0.13	0.13	0.12
C^{**}	−0.01	**0.01**	−0.01	−0.01	0.01	0.01	−0.01	−0.01	0.00	−0.01	0.00
ω^{**}	5.00	**5.00**	6.16	9.93	10.03	8.11	9.80	11.18	15.00	11.23	15.00
ϕ^{**}	−0.05	**2.04**	−0.48	−12.31	−3.42	−0.03	0.96	−0.02	−4.88	−0.33	1.27
SSR	178.36	**140.32**	157.65	185.60	185.16	205.75	186.22	192.86	241.73	193.85	241.72

In line with Sornette (2017, p. 335), the log-period power-law singularity (LPPLS) model is given by,

$$ln[p(t)] = A + B(t_c - t)^{\beta}[1 + Ccos(\omega ln(t_c - t) + \phi)]$$

with $A > 0$, $B < 0$, and $0.1 \le \beta \le 0.9$, and A is the expected value of the S&P 500 in logarithm, B measures the exposure to faster-than-exponential growth, β is the power-law exponent controlling faster than exponential price growth and C measures the exposure responsible for periodic oscillations, ω is the angular frequency of the log-periodic oscillations during the bubble formation and ϕ is the phase parameter. Whereas ϕ cannot be meaningful restricted, here we require $|C| < 1$ and impose the constraint $5 \le \omega \le 15$. We choose the optimal values for $\mathbf{\Phi}_1^* = (33.24, -19.45, 152, 0.10)$ from the first estimation step. Using in addition $C = \phi = 0$, the LPPLS model (Model 2) is then optimized for varying values ω, that is, $\omega \in \{5, 6, \dots, 14, 15\}$. Whereas Panel A of Table 2 reports the corresponding input parameters, Panel B of Table 2 reports the optimized parameters using the following constraints:

$$t_c \ge T + 1$$

$$0.1 \le \beta \le 0.9$$

$$5 \le \omega \le 15$$

Note that the parameter ϕ remains unconstrained which is in line with Sornette (2017, p. 336). The parameters are obtained using Microsoft Excel's non-linear solver. The sample period is from January 1871 to November 2022 comprising 1823 monthly observations.

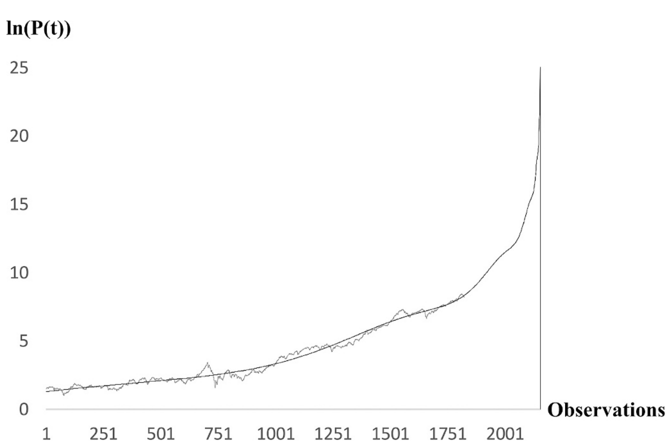

Fig. 2. The S&P 500 in logarithms and optimized Model 2 (LPPLS model) using monthly data for the 1871–2022 period.
This figure shows the optimized Model 2 along with the natural logarithm of the S&P 500 denoted as $ln[p(t)]$.

(Model 2). Unreported results show that neither $\widehat{\delta}_0$ for Model 2 nor $\widehat{\delta}_0$ and $\widehat{\delta}_1$ for Model 3 are statistically significant.[16] Thus, it is inferred that Model 1 correctly specifies the residuals in the ADF test regression. In consequence, the LPPLS model residuals follow a stationary mean-reverting process, implying that the LPPLS signature is statistically significant.

Can an LPPLS signature in financial market data occur purely as a matter of chance? Johansen, Ledoit, and Sornette (2000) tested the null hypothesis that a GARCH(1,1) model accounting for fat-tailed *t*-distributions could generate such LPPLS signatures. Using 1000 artificial datasets that were 400 weeks in length, only two exhibited LPPLS-signature-like properties that correspond to a confidence level of 99.8% for rejecting the hypothesis that a GARCH(1,1) model—which is, according to Sornette (2017, p. 326), "one of the standard benchmarks of financial time series used intensively by both academics and practitioners"—could generate meaningful log-periodicity in the data.[17] Thus,

[16] The corresponding *t*-statistics for $\widehat{\delta}_0$ ($\widehat{\delta}_0$ and $\widehat{\delta}_1$) for Model 2 (Model 3) are estimated at −0.17 (−0.15 and 0.07).

[17] A more comprehensive discussion of this issue is provided by Sornette (2017, Chapter 9).

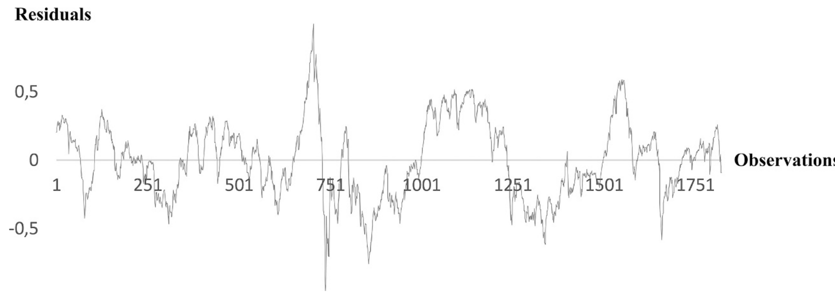

Fig. 3. Residuals of the calibrated LPPLS model.
This figure plots the residuals of Model 2 covering the in-sample time window (e.g., January 1871–November 2022).

Table 3
Testing the residuals of the LPPLS model.

Model specification	$\hat{\lambda}$	Critical values	
		5%	1%
No deterministic terms	−3.89*** (0.0001)	−1.94	−2.56
Constant	−3.89*** (0.0021)	−2.86	−3.43
Constant and trend	−3.89** (0.0126)	−3.41	−3.96

Using the residuals from the LPPLS model, we implement ADF tests using the following test regression,

$$\Delta u_t = \delta_0 + \delta_1 t + \delta_2 u_{t-1} + \gamma_1 \Delta u_{t-1} + \ldots + \gamma_p \Delta u_{t-p} + \epsilon_t$$

where $\delta_0 = \delta_1 = 0$ for the model specification where no deterministic terms are accounted for, and $\delta_1 = 0$ for the model specification accounting only for a constant term. The parameters $\gamma_1, \ldots, \gamma_p$ are related to the autoregressive part of the model. The model using $\delta_0 = \delta_1 = 0$ corresponds to modeling a random walk under the null hypothesis, whereas using $\delta_1 = 0$ corresponds to modeling a random walk with a drift under the null hypothesis. As a consequence, there are three main versions of the ADF test. The ADF test is then carried out under the null hypothesis $\hat{\delta}_2 = 0$ against the alternative hypothesis of $\hat{\delta}_2 < 0$. The estimated test statistic $\hat{\lambda}$ is then the estimated *t*-statistic associated with the point estimate $\hat{\delta}_2$. Table 3 reports the estimated test statistics for various ADF tests, the critical values for the 5% and 1% level and the *p*-value. Note that all model specifications employ a lag-order of $p = 2$ as suggested by the Schwarz-Criterion. The sample period is from January 1871 to November 2022 comprising 1823 monthly observations.

we can rule out that the manifestation of a statistically significant LPPLS signature would be a matter of chance.

3.3. Estimating confidence intervals

In general, point estimates derived from different calibrations could be used and, according to the central limit theorem, the sample average should be distributed as normal. In Table 4, the descriptive statistics are reported for the average parameter estimates for calibrations reported in Panel B of Table 2. It becomes evident that the Jarque–Bera (JB) test cannot reject the null hypothesis of normally distributed parameter estimates.[18] There is, however, one exception, that is, the average point estimate for the phase parameter ϕ, which appears to be non-normally distributed, at least based on the sample of model calibrations. The reason for this is perhaps the nature of this parameter, which cannot, according to Sornette (2017), be meaningfully constrained.[19] Given the descriptive statistics, a 95% confidence interval (CI) for the critical time is then CI = [173; 180], corresponding to the period from December 2043 to December 2050. The critical time and confidence interval are close to the figures derived in Johansen and Sornette's (2001), corresponding to 2052 ± 10 years. However, there are two issues that are worth mentioning. First, note that the 95% confidence interval derived using the intuitive first-order model approximation proposed here delivers a considerably tighter confidence interval than the 70% confidence interval derived in Johansen and Sornette's (2001) research. Second, the point estimate for the critical time derived in Johansen and Sornette's (2001) original study is outside the *upper* bound of the 95% confidence interval derived in the current research. This means that, after adding the more recent data, the estimated arrival of a finite-time singularity is statistically significantly *earlier* than previously expected.

3.4. Other robustness checks

3.4.1. Predicting the stock market crash of October 19, 1987

One may wonder how the approach proposed in this study to implement a first-order model approximation for estimating LPPLS parameters would have performed in predicting other singularities, specifically crashes. The most prominent stock market crash ever witnessed in the history of financial markets is perhaps the stock market crash of October 19, 1987. In line with traditional finance theory, the −22.6% return of the Dow Jones Index corresponded to a 20-sigma event. The odds that such an event could have occurred were, according to Mandelbrot (2008), <1 in $10.^{50}$ In this regard, Mandelbrot (2008, p. 4) highlighted that "It is a number outside the scale of nature. You could span the powers of ten from the smallest subatomic particle to the breadth of the measurable universe—and still never meet such a number." Moreover, Sornette (2017) emphasizes the following:

> A lot of work has been carried out to unravel the origin(s) of the crash, notably in the properties of trading and the structure of markets; however, no clear cause has been singled out. It is noteworthy that the strong market decline during October 1987 followed what for many countries had been an unprecedented market increase during the first nine months of the year and even before. In the US market, for instance, stock prices advanced 31.4% over those nine months. Some commentators have suggested that the real cause of October's decline was that overinflated prices generated a speculative bubble during the earlier period. (p. 5)

Overall, the October 1987 crash remains an intellectual curiosity for at least three reasons. First, the economic magnitude of a more-than-

[18] Because the JB hypothesizes, under the null hypothesis, that the random variable is normally distributed, a non-rejected JB test implies that, statistically, we cannot reject the hypothesis that the random variable follows a normal distribution.

[19] Note that the simplest model specification for the LPPLS model, which is discussed by Sornette (2017, Chapter 7), does not even account for such a phase parameter.

Table 4
Descriptive statistics of average parameter estimates for various model calibrations.

	A	B	t_c	β	C	ω	ϕ	SSR
Mean	36.36	−18.51	176.13	0.13	0.00	9.14	−1.85	186.75
Median	36.67	−18.45	175.79	0.13	−0.01	9.87	−0.19	185.91
Maximum	49.53	−13.69	184.44	0.15	0.01	15.00	2.04	241.73
Minimum	30.04	−28.91	167.66	0.10	−0.01	5.00	−12.31	140.32
Std. Dev.	5.85	4.43	5.92	0.01	0.01	3.14	4.19	26.98
Skewness	0.96	−1.18	−0.08	−0.49	0.47	0.19	−1.70	0.29
Kurtosis	3.63	4.10	1.69	3.07	1.38	2.38	4.95	3.41
Jarque-Bera (JB) test	1.71	2.83	0.72	0.40	1.47	0.22	6.42	0.21
p-value (JB test)	0.4253	0.2427	0.6966	0.8183	0.4799	0.8967	0.0403	0.9002

This table reports the descriptive statistics for the average parameter estimates for calibrations reported in Panel B of Table 2.

20% reduction in equity market capitalization on just a single day of trading was spectacular; second, the probabilistic occurrence of that event was beyond finance models, as it was "a number outside the scale of nature" (Mandelbrot, 2008, p. 4); and third, the crash occurred without any preceding warning signals. Thus, to evaluate how the proposed first-order model approximation for estimating LPPLS parameters would have performed in predicting this extraordinary crash, the model of Eq. (3) is fit with an initial parameter $A = 5.49$ (i.e., $ln(p_T)$ with $p_T = 242.17$, the price quotation of the S&P 500 on December 31, 1986); $t_c = T+1$, treating β as fixed, where $\overline{\beta} \in \{0.2, 0.4, 0.6, 0.8\}$; and setting $B = -1$.

Panel A of Table A.1 reports the corresponding initial model specifications. Note that we use daily data covering a sample from January 2, 1980 until December 31, 1986 and corresponding to 1170 observations.[20] Thus, in this model, one unit of time is equal to 0.004.[21] The sample ends at time $T = 7.08$, and in consequence, $T+1 = 7.084$. We see that using $\overline{\beta} = 0.2$ generates the minimum *SSR*. Implementing the constraints $t_c \geq T+1$ and $0.1 \leq \beta \leq 0.9$ and employing Microsoft Excel's non-linear solver to compute optimal values for the parameter vector $\mathbf{\Phi}_1^* = (A^*, B^*, t_c^*, \beta^*)$, the optimized values for the first model's (Model 1's) parameters are reported in Panel B of Table A.1. We see that model specification 1 of Model 1 provides parameter values generating the minimum *SSR*, with $\mathbf{\Phi}_1^* = (7.16, -1.60, 8.02, 0.20)$. Using $\mathbf{\Phi}_1^*$ as the initial value for Model 2 of Eq. (2) in association with $C = \phi = 0$, Model 2 is then optimized for varying values ω; that is, $\omega \in \{5, 6, ..., 14, 15\}$. Optimizing Model 2 requires accounting for the constraints $t_c \geq T+1$, $5 \leq \omega \leq 15$, and $0.1 \leq \beta \leq 0.9$, whereas the parameter ϕ remains unconstrained.

Once again employing Microsoft Excel's non-linear solver to compute the optimal values for the parameter vector $\mathbf{\Phi}_2^{**} = (A^{**}, B^{**}, t_c^{**}, \beta^{**}, C^{**}, \omega^{**}, \phi^{**})$ given the input parameter vector $\mathbf{\Phi}_2^* = (\mathbf{\Phi}_1^*, C, \omega, \phi) = (A^*, B^*, t_c^*, \beta^*, C, \omega, \phi)$ with $A^* = 7.16$, $B^* = -1.60$, $t_c^* = 8.02$, and $\beta^* = 0.20$, as obtained from the optimization in the previous step, while $C = \phi = 0$, only ω is varied in this step. Using $\omega \in \{5, 6, ..., 14, 15\}$ successively, Panel A of Table A.2 reports the input data vectors, whereas Panel B reports the optimized parameter vectors $\mathbf{\Phi}_2^{**}$ for each run. We observe, from Panel A of Table A.2, that the input parameterization given by $\mathbf{\Phi}_2^* = (A^*, B^*, t_c^*, \beta^*, C, \omega, \phi) = (7.16, -1.60, 8.02, 0.20, 0, 13, 0)$ results in the optimized parameterization for Model 2, yielding the lowest *SSR* (e.g., $SSR = 4.10$), with $A^{**} = 6.07$, $B^{**} = -0.50$, $t_c^{**} = 8.26$, $\beta^{**} = 0.50$, $C^{**} = -0.10$, $\omega^{**} = 13.63$, and $\phi^{**} = -1.91$. Thus, the optimal Model 2 suggests that a finite-time singularity will occur on March 1, 1988, which corresponds, in the notation used here, to 294 units of time (e.g., days) in the future. The corresponding model is plotted in Fig. A.1 in the Appendix. This means that the model deviates from the real crash that occurred on October 19, 1987, by 92 days, which should not come as a surprise, as Sornette points out that "…we expect that fits will give values of t_c which are in general close to but *systematically* later than the real time of the crash: the critical time t_c is included in the log-periodic power law structure of the bubble, whereas the crash is randomly triggered with a biased probability increasing strongly close to t_c" (Sornette, 2017, p. 332).

Interestingly, using the same data sample, Sornette (2017) documented that, in his model set-up where the realized critical time is, according to his notation, 87.80 (October 19, 1987),[22] his LPPLS model calibration produced two optimums, 88.35 and 87.68; that is, the former expected critical time is 201 days *later* than October 19, 1987, whereas the later data predicts the crash 44 days too *early*, corresponding to an average deviation from the factual event ("error") of 122.50 days. On the other hand, the first-order approximation model proposed in the current research produces one optimum that deviates from the real crash by 92 days, representing an error reduction of 25%.[23]

3.4.2. Predicting the crash of October 23, 1929

Next, another prominent stock market crash is that of October 1929. According to Sornette (2017), the crash occurred on October 23, 1929, which should serve as the critical time in order to make the models comparable in our case. However, the October 23, 1929 crash consisted of a series of subsequent drawdowns. Starting with a return of −6.31% on October 23, 1929, the Dow Jones Index fell by 31.45% after only 14 trading days (in the October 23, to November 12, 1929, period). It is interesting to note that Sornette (2017, p. 240) points out that "The similarity between the situations in 1929 and 1987 was in fact noticed at a qualitative level in an article in the *Wall Street Journal* on October 19, 1987, the very morning of the day of the stock market crash…" On the other hand, the LPPLS methodology is designed to capture such similarities in a *quantitative* way. Thus, to evaluate how the proposed first-order model approximation for estimating LPPLS parameters would have performed in predicting the October 23, 1929 crash, the model of Eq. (3) is fit with the initial parameter $A = 5.70$ (i.e., $ln(p_T)$ with $p_T = 300.00$, the price quotation of the Dow Jones Index on December 31, 1928); $t_c = T+1$; treating β as "fixed," where $\overline{\beta} \in \{0.2, 0.4, 0.6, 0.8\}$; and setting $B = -1$.

Panel A of Table A.3 reports the corresponding initial model specifications. Note that we use daily data covering a sample from June 1, 1921, until December 31, 1928, corresponding to 1942 observations.[24] In this model, the unit of time is 0.0038. Thus, the sample ends at time $T = 7.3062$, and as a consequence, $T+1 = 7.3100$. We see that using

[20] Note that Sornette (2017, p. 231) uses the same sample (January 1980 to December 1986) in some parts of his LPPLS analyses.

[21] As we use 7 years of daily data and 1770 daily observations, it follows that $0.004 \bullet 1,770 = 7.084 \cong 7$.

[22] See Table 9.5 in Sornette (2017, p. 331).

[23] Note that the difference between the average errors is (122.50–92) = 30.5 and 30.5 corresponds to 24.90% of 122.50, which is the average error of Sornette's (2017) calibrated model.

[24] Note that Sornette (2017, p. 231ff) uses the same sample (June 1921 to December 1928) in certain parts of his LPPLS analyses.

$\overline{\beta} = 0.2$ generates the minimum *SSR*. Implementing the constraints $t_c \geq T+1$ and $0.1 \leq \beta \leq 0.9$ and employing Microsoft Excel's non-linear solver to compute optimal values for the parameter vector $\mathbf{\Phi}_1^* = (A^*, B^*, t_c^*, \beta^*)$, the optimized values for the first model's (Model 1's) parameters are reported in Panel B of Table A.3. We see that model specification 3 of Model 1 provides parameter values generating the minimum *SSR,* with $\mathbf{\Phi}_1^* = (6.11, -0.59, 8.12, 0.53)$. Using $\mathbf{\Phi}_1^*$ as the initial values for Model 2 of Eq. (2) in association with $C = \phi = 0$, Model 2 is then optimized for varying values of ω, that is, $\omega \in \{5, 6, \ldots, 14, 15\}$. Optimizing Model 2 requires accounting for the constraints $t_c \geq T+1$, $5 \leq \omega \leq 15$ and $0.1 \leq \beta \leq 0.9$, whereas the parameter ϕ remains unconstrained.

Employing Microsoft Excel's non-linear solver to compute the optimal values for the parameter vector $\mathbf{\Phi}_2^{**} = (A^{**}, B^{**}, t_c^{**}, \beta^{**}, C^{**}, \omega^{**}, \phi^{**})$, given the input parameter vector $\mathbf{\Phi}_2^* = (\mathbf{\Phi}_1^*, C, \omega, \phi) = (A^*, B^*, t_c^*, \beta^*, C, \omega, \phi)$ with $A^* = 6.11$, $B^* = -0.59$, $t_c^* = 8.12$, $\beta^* = 0.53$, as obtained from the optimization in the previous step while $C = \phi = 0$, only ω is varied in the second step. Using $\omega \in \{5, 6, \ldots, 14, 15\}$ successively, Panel A of Table A.4 reports the input data vectors, whereas Panel B reports the optimized parameter vectors $\mathbf{\Phi}_2^{**}$ for each run. We observe, from Panel A of Table A.4, that the input parameterization given by $\mathbf{\Phi}_2^* = (A^*, B^*, t_c^*, \beta^*, C, \omega, \phi) = (6.11, -0.59, 8.12, 0.53, 0, 11, 0)$ results in the optimized parameterization for Model 2, which produces the lowest SSR (e.g., SSR $= 4.01$), with $A^{**} = 6.04$, $B^{**} = -0.35$, $t_c^{**} = 8.97$, $\beta^{**} = 0.75$, $C^{**} = 0.09$, $\omega^{**} = 12.68$, and $\phi^{**} = -4.03$. Thus, the optimal Model 2 suggests that a finite-time singularity will arrive on June 11, 1930, which corresponds, in the notation used here, to 434 units of time (e.g., days) in the future. The corresponding model is plotted along with the Dow Jones Index in terms of its natural logarithms in Fig. A.2 in the Appendix.

The results indicate that the proposed model deviates from the real crash that occurred on October 23, 1929, by 181 days. It is noteworthy that, using the same data sample, Sornette (2017) documented that, in his model set-up, in which the realized critical time corresponds, according to his notation, to 29.81 (October 23, 1929), his proposed LPPLS model calibration produced two optimums: 30.52 and 30.35[25]; that is, the former expected critical time is 259 days *later* than October 23, 1929, and the latter expects the crash 197 days *later*. That is, on average, Sornette (2017, p. 333) documented that LPPLS model calibrations generated an error of 228 days. Because the proposed optimum of the first-order approximation model proposed in the current research deviates by only 181 days from the real crash, it represents an error reduction of 21%.[26]

It is interesting to note that, even though a different approach is adopted here in setting up the LPPLS model, these approaches lead to remarkably similar observations. Sornette (2017) points out that the time windows used to estimate the finite-time singularities manifested in the October 1987 and October 1929 crash (e.g., January 2, 1980, until December 31, 1986, and June 1, 1921, until December 31, 1928) show similar acceleration and oscillatory structures, as quantified by similar exponents and log-periodic angular frequency. Whereas, in his model, these parameters varied between $\beta = 0.33$ and $\beta = 0.45$ and $\omega = 7.4$ and $\omega = 7.9$, for the October 1987 and October 1929 crashes, the corresponding figures here are $\beta = 0.50$ and $\beta = 0.75$ and $\omega = 13.63$ and $\omega = 12.68$. Overall, the evidence supports Sornette (2017) by showing that the LPPLS model is capable of revealing quantitative similarities across changing regimes in financial market data, which may manifest in crashes.

3.4.3. Predicting the crash of October 15, 2008

Finally, a reader might wonder how well the LPPLS model would have performed in predicting the Global Financial Crisis (GFC) of 2008. The bankruptcy of the US investment bank Lehman Brothers on September 15, 2008, is often considered the climax of the subprime mortgage crisis. The bankruptcy triggered a 4.71% one-day drop in the S&P 500 – then the largest decline since the attacks of September 11, 2001. However, on October 15, 2008, the S&P 500 exhibited an additional remarkable one-day drop of 9.03%, corresponding to a ≈ 7.3-sigma event. Because this extreme event is, in its economic magnitude, rather comparable to the events occurring on October 23, 1929, or October 19, 1987, here, October 15, 2008, is used as the target date subject to prediction. Therefore, to evaluate how the proposed first-order model approximation used in estimating LPPLS parameters would have performed in predicting the October 15, 2008, crash, the model of Eq. (3) is fit with the initial parameter $A = 7.29$ (i.e., $ln(p_T)$ with $p_T = 1468.36$, the price quotation of the S&P 500 Index on December 31, 2007); $t_c = T+1$; treating β as "fixed," where $\overline{\beta} \in \{0.2, 0.4, 0.6, 0.8\}$; and setting $B = -1$. To keep the analysis comparable to earlier model calibrations, the data used to calibrate the LPPLS model end on December 31, 2007. The starting point for the data is October 9, 2002, because, on that day, the S&P 500 reached–in the wake of the dot.com burst–a local minimum corresponding to an index quotation of 776.76.[27] The sample comprises 1316 daily observations.

Panel A of Table A.5 reports the corresponding initial model specifications. In this model, the unit of time is 0.00397. Thus, the sample ends at time $T = 5.2245$, and in consequence, $T+1 = 5.2285$. We see that using $\overline{\beta} = 0.2$ generates the minimum SSR. Implementing the constraints $t_c \geq T+1$ and $0.1 \leq \beta \leq 0.9$ and employing Microsoft Excel's non-linear solver to compute the optimal values for the parameter vector $\mathbf{\Phi}_1^* = (A^*, B^*, t_c^*, \beta^*)$, the optimized values for the first model's (Model 1's) parameters are reported in Panel B of Table A.5. We see that model specification 2 of Model 1 provides parameter values generating the minimum SSR with $\mathbf{\Phi}_1^* = (11.59, -1.94, 16.77, 0.32)$. From Panel B of Table A.5, it becomes evident that all optimized models suggest critical times that are far in the future. For example, the optimal model (e.g., Model 2) suggests $t_c^* = 16.77$, which is more than three times the length of the data sample used for model calibration. Following Sornette (2017), who imposes the restriction that the critical event should be observed in the interval $T+1$ and $2T$, we re-optimize the model using the constraints $t_c \geq T+1$, $0.1 \leq \beta \leq 0.9$, and $t_c \leq 2T$. Using these constraints and employing Microsoft Excel's non-linear solver to compute the optimal values for the parameter vector $\mathbf{\Phi}_1^* = (A^*, B^*, t_c^*, \beta^*)$, the optimized values for the first model's (Model 1's) parameters are reported in Panel C of Table A.5. We observe that that, even though all optimized models suggest $t_c^* = 2T = 10.457$, Model 1 generates the minimum SSR with $\mathbf{\Phi}_1^* = (11.39, -2.98, 10.46, 0.18)$.

Next, using $\mathbf{\Phi}_1^*$ as the initial values for Model 2 of Eq. (2) in association with $C = \phi = 0$, Model 2 is then optimized for varying values of ω, that is, $\omega \in \{5, 6, \ldots, 14, 15\}$. Optimizing Model 2 requires accounting for the constraints $t_c \geq T+1$, $5 \leq \omega \leq 15$, $0.1 \leq \beta \leq 0.9$, and $t_c \leq 2T$, whereas the parameter ϕ remains unconstrained. Employing Microsoft Excel's non-linear solver to compute the optimal values for the parameter vector $\mathbf{\Phi}_2^{**} = (A^{**}, B^{**}, t_c^{**}, \beta^{**}, C^{**}, \omega^{**}, \phi^{**})$, given the input parameter vector $\mathbf{\Phi}_2^* = (\mathbf{\Phi}_1^*, C, \omega, \phi) = (A^*, B^*, t_c^*, \beta^*, C, \omega, \phi)$ with $A^* = 11.39$, $B^* = -2.98$, $t_c^* = 10.457$, $\beta^* = 0.18$, as obtained from the optimization in the previous step, while $C = \phi = 0$, only ω is varied in the second step. Using $\omega \in \{5, 6, \ldots, 14, 15\}$ successively, Panel A of Table A.6 reports the input data vectors, whereas Panel B reports the optimized parameter vectors $\mathbf{\Phi}_2^{**}$ for each run. We observe, from Panel A

[25] See Table 9.5 in Sornette (2017, p. 333).

[26] Note that the difference between the average errors is (228–181) = 47, whereas 47 corresponds to 20.61% of 228, which is the average error of Sornette's (2017) calibrated model.

[27] More precisely, October 9, 2002, is the date of the minimum quotation of the S&P 500 between the peaks of 2000 and 2007.

of Table A.4, that the input parameterization given by $\mathbf{\Phi}_2^* = (A^*, B^*, t_c^*, \beta^*, C, \omega, \phi) = (11.39, -2.98, 10.457, 0.18, 0, 12, 0)$ results in the optimized parameterization for Model 2, which produces the minimum SSR (e.g., SSR $= 1.44$), with $A^{**} = 7.76$, $B^{**} = -0.13$, $t_c^{**} = 8.87$, $\beta^{**} = 0.90$, $C^{**} = -0.06$, $\omega^{**} = 12.93$, and $\phi^{**} = 0.07$. Thus, the optimal Model 2 suggests that a finite-time singularity will arrive on August 19, 2011, which corresponds, in the notation used here, to 917 units of time (e.g., days) in the future. The corresponding model is plotted along with the S&P 500 Index in terms of its natural logarithms in Fig. A.3 in the Appendix.

This result implies that the LPPLS model does *not* predict the extreme event that occurred on October 15, 2008. Does this mean that the model's crash prediction fails? Actually, the model predicts a different crash; that is, it predicts the crash of August 2011. It almost perfectly predicts Black Monday 2011, which is August 8, 2011, the day when US and global stock markets crashed due to Standard and Poor's downgrading of the United States' sovereign debt from AAA to AA+, which took place on Friday 5, 2011. This was the first time in the history that United States' sovereign debt was not graded as risk free by Standard and Poor's. On August 8, 2011, the S&P 500 recorded a one-day stop of 6.66%, corresponding to a $\approx$ 5.4-sigma event.[28] The LPPLS model predicts this crash only 8 days too late. Given that this crash is 917 days in the future−because the data sample used to calibrate the model end on December 31, 2007 − this is indeed an intriguing finding.

How does this remarkable result line up with earlier studies? Zhang et al. (2016) used monthly S&P 500 data covering the period from August 1791 to August 2014 to explore the detection of end-of-bubble signals. Comparing the LPPLS method to other approaches, the authors found that the LPPLS model is more accurate in identifying well-known bubble events. According to Zhang et al. (2016), their LPPLS model is indeed capable of predicting the Subprime Mortgage Crisis of 2007–2008. Unfortunately, the results of the current research and those documented in Zhang et al. (2016) are difficult to compare because their study's purpose is to identify positive and negative bubbles in the S&P 500 since the first month of 1814 based on an initial estimation from August 1791 to December 2013, thus employing a rolling window of 269 months.

Zhang et al. (2016, p. 135) acknowledge that "In most cases, the … LPPLS Confidence estimates give bubble signals quite early before the end of the bubble period […] However, the signal comes relatively late and weak for the Subprime Mortgage Crises of 2007–2008." Finally, Zhang et al. (2016, p. 138) point out that "not all stock market crashes or crises are caused by internal bubble mechanisms with positive (negative) feedback mechanisms. Some of these events are not preceded by faster than exponential price growths (declines) and the crashes are caused by various economic and political events, such as the recessions, panics, wars, and external influences." We argue that the GFC of 2008 was not necessarily a manifestation of a self-perpetuating pattern of investment behavior translated into super-exponential growth in stock prices. Rather, it was caused by an external influence, a subprime mortgage crisis, which inevitably had severe spillover effects on the stock market. Whereas the GFC 2008 had external causes, we argue that the bubble formation that our calibrated model detects is perhaps rather endogenous in nature and, consequently, the result of procyclical positive feedback, which eventually burst. Thus, a key difference between Zhang et al.'s (2016) study and the current research is that Zhang et al.'s (2016) calibrated model did not detect the crash of August 8, 2011, whereas our model did. On the other hand, our model did not detect the crash of October 15, 2008, but it detected, with a high level of precision, the crash occurring in August 2011. Because the LPPLS model is designed to detect crashes that are endogenous in nature, as documented in Sornette (2017), we do not consider our result to be a failure of LPPLS model prediction.[29]

3.5. *Other tests*

Note that faster-than-exponential growth in the underlying price series is the core assumption of the LPPLS model. Here, we hypothesize that we can detect faster-than-exponential growth in the data using standard econometric tests. For example, recall the general regression model of Eq. (4),

$$\Delta u_t = \delta_0 + \delta_1 t + \delta_2 u_{t-1} + \gamma_1 \Delta u_{t-1} + \ldots + \gamma_p \Delta u_{t-p} + \epsilon_t, \tag{4}$$

which is used to test whether the residuals of the optimal LPPLS model are stationary. Applying this model directly to the logarithmic prices of the underlying data (stock index), p_t, we accordingly derive the following:

$$\Delta p_t = \varphi_0 + \varphi_1 t + \varphi_2 p_{t-1} + \varphi_1 \Delta p_{t-1} + \ldots + \varphi_p \Delta p_{t-p} + e_t \tag{5}$$

and hypothesize that faster-than-exponential growth should be manifested in $\varphi_2 < 0$, in association with the acceptance of the null hypothesis, corresponding to an explosively growing non-stationary process. In Table A.7 in the Appendix, the corresponding results are reported for the test regressions using (*i*) monthly data on the S&P 500 for the period from January 1871 to November 2022; (*ii*) daily data on the S&P 500 for the period from January 2, 1980, until December 31, 1986; and (*iii*) daily data on the Dow Jones Index for the period from June 1, 1921, until December 31, 1928.[30] First, we see from Table A.5 that the null hypothesis of a non-stationary process cannot be rejected for any of these models because the *p*-values are >5%.[31] Second, the point estimates $\widehat{\varphi}_1$ are statistically significant at a common 5% level for models (*i*) and (*ii*) and at the 10% level for model (*iii*). We infer that p_t exhibited explosiveness in random-walk behavior *conditionally* on the selected samples. It is noteworthy that, while the underlying non-stationary process for a successfully implemented LPPLS model should exhibit explosiveness, the reverse argument cannot be made; that is, explosiveness in some non-stationary processes does not necessarily imply that the process is subject to log-periodic power-law behavior. Overall, the standard econometric test applied here can only serve as an additional test of whether the data sample meets the conditions required for implementing an LPPLS model.

4. Discussion

4.1. *Limitations*

The current research uses LPPLS models to forecast finite-time singularities resulting from faster-than-exponential growth. As mentioned above, different initial values chosen for some parameters in the parameter vector $\mathbf{\Phi}$ will result in different values for some other

[28] Between August 1, 2011, and August 22, 2011, the S&P 500 lost 12.71%.

[29] Another related study is the one of Brée and Joseph (2013), who examine crashes in the Hang Seng stock market index over the period from 1970 to 2008. The authors found that fitted LPPLS models exhibit parameter values within the ranges specified post hoc by Johansen and Sornette (2001) for only seven of 11 crashes. Moreover, their calibrated model predicted the end of the 2007 bubble to occur on October 30, 2007. They argue that, despite these criticisms and due to the partial success in correctly predicting the 2007 crash, it is worth investigating whether fitted LPPL models with critical parameters in acceptable non-independent ranges can be used to give a probabilistic, rather than an all-or-none, prediction of an impending crash.

[30] The lag-order for each model (*i*)-(*iii*) is chosen with respect to the SIC, with a maximum lag length of 24.

[31] Because the ADF hypothesizes, under the null hypothesis, that the random variable exhibits random-walk behavior, a non-rejected ADF test implies that, statistically, we cannot reject the hypothesis that the random variable follows a random walk.

parameters. To address this issue, the calibration of the model is typically based on a combination of finding suggested solutions for parameters, freezing some of the parameters, and using a non-linear solver to find solutions for the free parameters. The approach to selecting the initial values for the three-stage model estimation proposed in the current research is, in the parlance of Sornette (2017, p. 334), only a "first-order approximation." More work is warranted to improve this methodology.

Next, we have shown that the proposed approach to calibrating the LPPLS model yields similar results to the models discussed by Johansen and Sornette (2001) together and Sornette (2017) alone. In fact, the approach proposed here resulted in a substantial decrease in the overestimation of the critical time, as evidenced by the robustness checks. The stock market crashes discussed here have been subject to an enormous amount of research, and the fact that the October 1987 crash, in particular, did not exhibit precursory patterns certainly demands special attention from scholars. However, in the last century, the US economy has faced a considerable amount of economic stress. For instance, from November 9, 1903, to January 19, 1906, the Dow Jones Index increased by 144% (from 30.53 to 74.60 index points), whereas from January 19, 1906, to November 22, 1907, it fell by 48% (from 74.60 to 38.44 points). This stock market crash is often referred to as the Panic of 1907, which had perhaps already begun on March 14, 1907, when the daily return on the Dow Jones Index was −8.29%, statistically corresponding to an 8-sigma event. Overall, there are some similarities between the Panic of 1907 and the October 1987 crash. In the preceding period, stock prices increased by a substantial margin before−in the spirit of Mandelbrot (2008, p. 4)−the "impossible event" suddenly occurred. This is one example of stock market drops that remain undiscussed in the corresponding literature on predicting crashes. This is perhaps not a surprise, because unreported results show that the calibration of the LPPLS model does not work out due to failed convergence in the final model-optimization attempt. This means that the absence of LPPLS signatures does not necessarily imply the absence of sudden crashes.[32] Future research is warranted in order to elaborate on this issue.

4.2. Implications

A question arises regarding how should one interpret the singularity predicted around 2050? As explained in Johansen and Sornette (2001), a singularity will most likely not occur in 2050. The singularity is the mathematical extrapolation of the present trajectory, assuming an infinite Earth and no other mechanisms. On the real Earth and in the human social system, as we approach the singularity, other mechanisms will come to dominate, rounding off the singularity. In consequence, the singularity is a diagnostic that a change of regime will occur. The singularity itself will not occur, because it is impossible from the physical, economic, and social points of views. It is the same with many mathematical models of biological population dynamics, for example, or physical systems that, mathematically, would reach a singularity. When approaching the singularity, processes that have been negligible begin to dominate the dynamics involved.

As an example, let us consider the "Euler disk," that is, a coin that you allow to fall on the floor and that rotates with a characteristic sound, which increases until a climax, when the coin settles within a finite time. It can be shown mathematically, by solving the equations of mechanics, that the number of hits of the coin onto the surface diverges to infinity in finite time − a singularity−as long as we neglect friction due to the surface and the air trapped between the coin and the surface. When approaching the mathematical singularity, friction, which has been negligible, comes to dominate, and the singularity is rounded off. It disappears. However, the mathematical description, in terms of the dynamics regarding a singularity, is sound and signals a change in regime.

5. Conclusion

Using monthly S&P 500 data from 1871 to 2022, this study examined the potential arrival of a finite-time singularity for the US equity market. To do so, this study used (*i*) a novel approach to calibrate the LPPLS model and (*ii*) an extended dataset accounting for >20 years of additional data as compared to earlier relevant research. The extended dataset accounts for the dot.com bubble burst (2000), the Global Financial Crisis period (2008–2009), the COVID−19 crisis (2020–2022), and the ongoing Russian–Ukrainian war (starting in 2022). All these events had and have enormous consequences not only for the US but also for the entire global economy. The proposed calibrated LPPLS model suggests that the US equity market will reach a singularity condition in June 2050, which is in line with earlier research forecasting a regime switch in 2052. What does a finite-time singularity, or regime switch, actually mean? Sornette (2017) discusses various scenarios in which finite-time singularities could manifest themselves. The first and perhaps most obvious potential scenario is a "collapse," the second possible scenario is a "transition to sustainability," and the third is "resuming accelerating growth" by overcoming fundamental barriers. Sornette (2017) argues that social scientists typically take an optimistic point of view and

> expect that the innovative spirit of mankind will be able to solve the problems associated with a continuing increase in the growth rate […] Specifically, […] they believe that world economic development will continue as a successive unfolding of revolutions, for example, the Internet, bio-technological, and other yet unknown innovations replacing the prior agricultural, industrial, medical, and information revolutions of the past. (p. 359).

In this regard, West (2017) points out that.

> [u]nfortunately, however, it's not quite as simple as that. There's yet another major catch, and it's a big one. The theory dictates that to sustain continuous growth the time between successive innovations has to get shorter and shorter. These paradigm-shifting discoveries, adaptions, and innovations must occur at an increasingly accelerated pace. Not only does the general pace of life inevitably quicken, but we must innovate at a faster and faster rate! (p. 417f)

It is noteworthy that the great Benoit Mandelbrot, who developed the theory of roughness and self-similarity, showed that financial markets are subject to fractality, that is, statistical self-similarity. A manifestation of statistical self-similarity in financial markets is that they behave at a higher frequency in the same manner as at a lower frequency, which is a statistical manifestation of power-law behavior.[33] What does this mean for the expected finite-time singularity in 2050? The stock market crashes of October 1987 and October 1929, which were investigated in the current research as robustness checks, may serve as a guide to how such a collapse could evolve. For both events, market participants observed extreme reductions in market capitalization in a very short time. Still, these events were on a *small* scale because we considered relatively short time windows that showed evidence of bubble formation, as predicted by LPPLS models. For the main analysis in this study, however, we adopted a coarse-grained, *large*-scale perspective by exploring the evolution of the US financial market over >150 years.

[32] It is interesting to note that Didier Sornette and the well-known scholar and bestselling author, Nassim Talab, discussed the predictability of crashes using the LPPLS model in an ETH sponsored meeting. It is perhaps for this reason that Taleb terms Sornette's concept of "Dragon Kings" as "Grey Swans" in an attempt to highlight the lack of accuracy in predictability; see https://www.youtube.com/watch?v=vuvbghZuM8U.

[33] Mandelbrot (1963) was the first to show that cotton price changes are governed by a power-law process.

Statistically, we are able to identify the same log-period power-law behavior on a larger scale as we identified for the smaller scales (the crashes of October 1987 and October 1929). Fractality suggests that we can expect to observe a large-scale collapse of the US equity market, as opposed to some other potential scenario.

The presence of statistical self-similarity allows us to calibrate the LPPLS model at various frequencies. However, it is noteworthy that the LPPLS model, in its original form, is based on a hierarchical model of traders exhibiting herd behavior. Specifically, Sornette and Johansen (1998) showed that this hierarchical organization was sufficient to generate log-periodic oscillations and that systemic instability could lead to a crash. Whereas the typical timeframes for bubble formations are periods of approximately 8 years Sornette & Johansen (1997), we follow Johansen and Sornette (2001) in exploring a potential bubble formation over a period of approximately 150 years. While the current research argues that the presence of statistical self-similarity allows us to calibrate the LPPLS model even for this timeframe, a theoretical model is still needed. Such a model could incorporate the concept of learned behavior, as applied to generations of traders building fractal-like structures manifested in self-similar behavior across generations. The development of such a theoretical model is outside the scope of this empirical study and thus left to future research.

It is interesting to note that even Ray Dalio, as a practitioner—working in the finance industry for 50 years—expects a regime switch despite choosing different terminology in expressing this expectation. In his recent book entitled *The Changing World Order*, Dalio (2021) studies major empires and compares the successes and failures of the world's empires throughout history. In studying this issue, he identifies certain critical factors that supposedly result in a regime switch, or a change of the world order. Specifically, Dalio observed the following three factors. First, the confluence of enormous debts and close-to-zero interest rates has led to the massive printing of the world's major currencies, especially the US dollar. Second, due to substantial increases in wealth and political and value gaps in just a century, significant political and social conflicts arose within countries. Third, China, as a rising power, challenges the US, which is, as of today, the existing world power that sets the rules for the world order. Dalio argues that the leading country's (the US's) financial picture begins to change as we approach a regime shift. Having the US dollar as the reserve currency gave the country the extraordinary privilege of being able to borrow more money as compared to other countries, which drove it deeper into debt and accelerated its spending power over the short term but weakened it in the long term. Inevitably, the US began to borrow excessively to finance both domestic overconsumption and international military conflicts. This contributed to the enormous accumulation of US debts, with China serving as the major lender. Finally, when the countries holding the reserve currency and debt of the US lose faith in the US and sell them, this stage will mark the end of the US as a leading empire and constitute the final stage of the regime switch. China, as the major lender, might attain power and set the forthcoming rules for a new world order. Overall, even though Dalio does not explicitly employ the term "finite-time singularity," he unequivocally expects the financial ecosystem in the US to collapse in the near future. In this study, a timeframe for this regime switch is derived.

Finally, a reader may wonder what investors are recommended to do when approaching to the singularity. As Taleb (2012, p. 389) states, "Never ask anyone for their opinion, or recommendations. Just ask them what they have–or don't have–in their portfolio." In this regard, it is interesting to observe that Bridgewater Associates doubled its assets under management in China over the past year to >20 billion yuan, corresponding to 2.93 billion US dollar.[34] According to Reuters, Bridgewater Associates runs the largest foreign hedge fund in Beijing.[35] This should, however, not come as a great surprise, because Dalio expects China to supersede the US as the world's economic leader–a change that corresponds to a regime switch, that is, the finite-time singularity in the current study's parlance.

Data availability

Data will be made available on request.

Appendix A. Appendix

Table A.1
Calibrating the power-law model for the S&P 500 using daily data from 1980 to 1986.

Panel A. Initial parameter values for Model 1.					
Specification	A	B	$\overline{\beta}$	$t_c = T+1$	SSR
1	5.49	-1	0.2	7.084	1091.19
2	5.49	-1	0.4	7.084	2289.70
3	5.49	-1	0.6	7.084	4983.81
4	5.49	-1	0.8	7.084	10,737.59
Panel B. Optimized parameter values for Model 1.					
Specification	A^*	B^*	β^*	t_c^*	SSR
1	**7.16**	**−1.60**	**0.20**	**8.02**	**11.62**
2	6.08	−0.66	0.36	7.54	11.73
3	5.88	−0.50	0.43	7.37	11.81
4	5.93	−0.53	0.41	7.40	11.79

To model financial log-prices we use a simple power-law model which is given by,

$$ln[p(t)] = A + B(t_c - t)^{\beta}$$

where $ln[p(t)]$ is the logarithm of the S&P 500 at time t, t_c is the critical time, A is the expected value of the S&P 500 in logarithm, B measures the exposure to faster-than-exponential growth and β is the power-law exponent controlling faster than exponential price growth. Note that for this model specification $A > 0$, $B < 0$, and

[34] See https://markets.businessinsider.com/news/funds/ray-dalio-bridgewater-associates-china-assets-top-foreign-hedge-fund-2023-1.
[35] See https://www.reuters.com/business/finance/dalios-bridgewater-cements-rank-2022-top-foreign-hedge-fund-china-2023-01-10/.

$0.1 \leq \beta \leq 0.9$ must hold. Treating β as "fixed", where $\overline{\beta} \in \{0.2, 0.4, 0.6, 0.8\}$, and setting $B = -1$, Eq. (1) simplifies to:

$$ln[p(t)] = ln[p(T)] - (t_c - t)^{\overline{\beta}}$$

Note that A is the expected value of the S&P 500 in logarithm, and $A = ln[p(T)]$ in association with $t_c = T + 1$ means the last price notation of the S&P 500 is the expected price for the critical time t_c and the critical event is suggested to occur in the subsequent time period. Panel A of Table A.1 reports the corresponding initial model specifications. Note that $ln[p(T)] = 5.49$ corresponds to the natural logarithm of the last price quotation of the S&P 500 in the sample. Next, the model is optimized allowing for the following constraints:

$$t_c \geq T + 1$$

$$0.1 \leq \beta \leq 0.9$$

The first constraint allows for the possibility that the finite time singularity arrives at some unknown time point in the future, whereas the second constraint is needed for the price to accelerate and to remain finite. Employing Microsoft Excel's non-linear solver to compute optimal values for the parameter vector $\mathbf{\Phi}_1^* = (A^*, B^*, t_c^*, \beta^*)$, the optimized values are reported in Panel B of Table A.1. The sample period from January 2, 1980 until December 31 and comprises 1770 daily observations.

Table A.2
Calibrating the LPPLS model for the S&P 500 using daily data from 1980 to 1986.

Panel A. Initial parameter values for Model 2.

Specification	1	2	3	4	5	6	7	8	9	10	11
A^*	7.16	7.16	7.16	7.16	7.16	7.16	7.16	7.16	7.16	7.16	7.16
B^*	−1.60	−1.60	−1.60	−1.60	−1.60	−1.60	−1.60	−1.60	−1.60	−1.60	−1.60
t_c^*	8.02	8.02	8.02	8.02	8.02	8.02	8.02	8.02	8.02	8.02	8.02
β^*	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20
C	0	0	0	0	0	0	0	0	0	0	0
ω	5	6	7	8	9	10	11	12	13	14	15
ϕ	0	0	0	0	0	0	0	0	0	0	0
SSR	11.62	11.62	11.62	11.62	11.62	11.62	11.62	11.62	11.62	11.62	11.62

Panel B. Optimized parameter values for Model 2.

Specification	1	2	3	4	5	6	7	8	9	10	11
A^{**}	6.51	9.24	9.25	9.29	9.29	9.17	6.23	6.07	**6.07**	6.07	6.21
B^{**}	−0.83	−2.98	−2.97	−3.03	−3.03	−2.89	−0.62	−0.49	**−0.50**	−0.49	−0.59
t_c^{**}	8.62	10.41	10.49	10.44	10.44	10.53	8.34	8.25	**8.26**	8.25	8.39
β^{**}	0.37	0.18	0.18	0.18	0.18	0.19	0.44	0.50	**0.50**	0.50	0.45
C^{**}	−0.07	0.02	0.02	0.02	0.02	−0.02	0.08	−0.10	**−0.10**	0.10	0.08
ω^{**}	14.99	14.68	14.85	14.74	14.74	14.94	13.92	13.61	**13.63**	13.62	14.12
ϕ^{**}	−17.81	−17.77	−18.27	−17.95	−17.95	−15.39	−5.78	−1.87	**−1.91**	1.26	0.03
SSR	4.21	7.53	7.49	7.52	7.48	7.48	4.11	4.10	**4.10**	4.10	4.11

In line with Sornette (2017, p. 335), the log-period power-law singularity (LPPLS) model is given by,

$$ln[p(t)] = A + B(t_c - t)^{\beta}[1 + Ccos(\omega ln(t_c - t) + \phi)]$$

with $A > 0$, $B < 0$, and $0.1 \leq \beta \leq 0.9$, and A is the expected value of the S&P 500 in logarithm, B measures the exposure to faster-than-exponential growth, β is the power-law exponent controlling faster than exponential price growth and C measures the exposure responsible for periodic oscillations, ω is the angular frequency of the log-periodic oscillations during the bubble formation and ϕ is the phase parameter. Whereas ϕ cannot be meaningful restricted, here we require $|C| < 1$ and impose the constraint $5 \leq \omega \leq 15$. We choose the optimal values for $\mathbf{\Phi}_1^* = (7.16, -1.60, 8.02, 0.20)$ from the first estimation step. Using in addition $C = \phi = 0$, the LPPLS model (Model 2) is then optimized for varying values ω, that is, $\omega \in \{5, 6, \ldots, 14, 15\}$. Whereas Panel A of Table A.2 reports the corresponding input parameters, Panel B of Table A.2 reports the optimized parameters using the following constraints:

$$t_c \geq T + 1$$

$$0.1 \leq \beta \leq 0.9$$

$$5 \leq \omega \leq 15$$

Note that the parameter ϕ remains unconstrained which is in line with Sornette (2017, p. 336). The parameters are obtained using Microsoft Excel's non-linear solver. The sample period from January 2, 1980 until December 31 and comprises 1770 daily observations.

Table A.3
Calibrating the power-law model for the Dow Jones using daily data from 1921 to 1928.

Panel A. Initial parameter values for Model 1.					
Specification	A	B	$\overline{\beta}$	$t_c = T+1$	SSR
1	5.70	−1	0.2	7.3100	352.32
2	5.70	−1	0.4	7.3100	1090.74
3	5.70	−1	0.6	7.3100	3318.12
4	5.70	−1	0.8	7.3100	8814.82

Panel B. Optimized parameter values for Model 1.					
Specification	A^*	B^*	β^*	t_c^*	SSR
1	8.72	−2.41	0.26	10.0960	12.52
2	6.79	−0.99	0.42	8.9479	12.49
3	**6.11**	**−0.59**	**0.53**	**8.1195**	**12.44**
4	6.11	−0.59	0.53	8.0707	12.45

To model financial log-prices we use a simple power-law model which is given by,

$$ln[p(t)] = A + B(t_c - t)^{\beta}$$

where $ln[p(t)]$ is the logarithm of the Dow Jones at time t, t_c is the critical time, A is the expected value of the Dow Jones in logarithm, B measures the exposure to faster-than-exponential growth and β is the power-law exponent controlling faster than exponential price growth. Note that for this model specification $A > 0$, $B < 0$, and $0.1 \le \beta \le 0.9$ must hold. Treating β as "fixed", where $\overline{\beta} \in \{0.2, 0.4, 0.6, 0.8\}$, and setting $B = -1$, Eq. (1) simplifies to:

$$ln[p(t)] = ln[p(T)] - (t_c - t)^{\overline{\beta}}$$

Note that A is the expected value of the Dow Jones in logarithm, and $A = ln[p(T)]$ in association with $t_c = T+1$ means the last price notation of the Dow Jones is the expected price for the critical time t_c and the critical event is suggested to occur in the subsequent time period. Panel A of Table A.3 reports the corresponding initial model specifications. Note that $ln[p(T)] = 5.70$ corresponds to the natural logarithm of the last price quotation of the Dow Jones in the sample. Next, the model is optimized allowing for the following constraints:

$$t_c \ge T+1$$

$$0.1 \le \beta \le 0.9$$

The first constraint allows for the possibility that the finite time singularity arrives at some unknown time point in the future, whereas the second constraint is needed for the price to accelerate and to remain finite. Employing Microsoft Excel's non-linear solver to compute optimal values for the parameter vector $\mathbf{\Phi}_1^* = (A^*, B^*, t_c^*, \beta^*)$, the optimized values are reported in Panel B of Table A.3. The sample period from June 1, 1921 until December 31, 1928 and comprises 1942 daily observations.

Table A.4
Calibrating the LPPLS model for the Dow Jones using daily data from 1921 to 1928.

Panel A. Initial parameter values for Model 2.											
Specification	1	2	3	4	5	6	7	8	9	10	11
A^*	6.11	6.11	6.11	6.11	6.11	6.11	6.11	6.11	6.11	6.11	6.11
B^*	−0.59	−0.59	−0.59	−0.59	−0.59	−0.59	−0.59	−0.59	−0.59	−0.59	−0.59
t_c^*	8.12	8.12	8.12	8.12	8.12	8.12	8.12	8.12	8.12	8.12	8.12
β^*	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.53
C	0	0	0	0	0	0	0	0	0	0	0
ω	5	6	7	8	9	10	11	12	13	14	15
ϕ	0	0	0	0	0	0	0	0	0	0	0
SSR	12.44	12.44	12.44	12.44	12.44	12.44	12.44	12.44	12.44	12.44	12.44

Panel B. Optimized parameter values for Model 2.											
Specification	1	2	3	4	5	6	7	8	9	10	11
A^{**}	5.54	5.58	5.56	5.56	6.04	5.67	**6.04**	6.04	6.05	6.07	6.19
B^{**}	−0.22	−0.25	−0.24	−0.24	−0.35	−0.26	**−0.35**	−0.35	−0.36	−0.36	−0.41
t_c^{**}	7.31	7.40	7.31	7.31	8.97	7.74	**8.97**	8.97	8.99	9.01	9.27
β^{**}	0.90	0.85	0.86	0.86	0.75	0.84	**0.75**	0.75	0.74	0.74	0.70
C^{**}	−0.14	−0.13	0.13	0.13	−0.09	−0.12	**0.09**	0.09	−0.08	−0.08	0.08
ω^{**}	8.90	9.25	8.97	8.96	12.68	9.98	**12.68**	12.68	12.72	12.79	13.61
ϕ^{**}	−3.58	−4.34	−0.55	−0.55	−7.16	0.08	**−4.03**	−4.02	−0.98	−1.14	−0.12
SSR	4.73	4.65	4.69	4.69	4.01	4.74	**4.01**	4.01	4.01	4.01	4.07

In line with Sornette (2017, p. 335), the log-period power-law singularity (LPPLS) model is given by,

$$ln[p(t)] = A + B(t_c - t)^{\beta}[1 + Ccos(\omega ln(t_c - t) + \phi)]$$

with $A > 0$, $B < 0$, and $0.1 \leq \beta \leq 0.9$, and A is the expected value of the Dow Jones in logarithm, B measures the exposure to faster-than-exponential growth, β is the power-law exponent controlling faster than exponential price growth and C measures the exposure responsible for periodic oscillations, ω is the angular frequency of the log-periodic oscillations during the bubble formation and ϕ is the phase parameter. Whereas ϕ cannot be meaningful restricted, here we require $|C| < 1$ and impose the constraint $5 \leq \omega \leq 15$. We choose the optimal values for $\mathbf{\Phi}_1^* = (6.11, -0.59, 8.12, 0.53)$ from the first estimation step. Using in addition $C = \phi = 0$, the LPPLS model (model 2) is then optimized for varying values ω, that is, $\omega \in \{5, 6, \ldots, 14, 15\}$. Whereas Panel A of Table A.4 reports the corresponding input parameters, Panel B of Table A.4 reports the optimized parameters using the following constraints:

$$t_c \geq T + 1$$

$$0.1 \leq \beta \leq 0.9$$

$$5 \leq \omega \leq 15$$

Note that the parameter ϕ remains unconstrained which is in line with Sornette (2017, p. 336). The parameters are obtained using Microsoft Excel's non-linear solver. The sample period from June 1, 1921 until December 31, 1928 comprises 1942 daily observations.

Table A.5
Calibrating the power-law model for the S&P 500 using daily data from 2002 to 2007.

Panel A. Initial parameter values for Model 1.					
Specification	A	B	$\overline{\beta}$	$t_c = T+1$	SSR
1	**7.29**	**−1**	**0.2**	**5.22849**	**2632.11**
2	7.29	−1	0.4	5.22849	3767.88
3	7.29	−1	0.6	5.22849	5654.42
4	7.29	−1	0.8	5.22849	8807.77

Panel B. Optimized parameter values for Model 1.					
Specification	A^*	B^*	β^*	t_c^*	SSR
1	17.44	−7.91	0.11	14.1059	2.7169
2	**11.59**	**−1.94**	**0.32**	**16.7708**	**2.5482**
3	23.64	−12.47	0.10	19.2250	2.5787
4	7.82	−0.49	0.10	54.8941	32.7457

Panel C. Optimized parameter values for Model 1 with additional constraint.					
Specification	A^*	B^*	β^*	t_c^*	SSR
1	**11.39**	**−2.98**	**0.18**	**10.457**	**2.8739**
2	14.69	−6.18	0.10	10.457	2.9401
3	14.54	−6.03	0.10	10.457	2.9392
4	14.69	−6.18	0.10	10.457	2.9401

To model financial log-prices we use a simple power-law model which is given by,

$$ln[p(t)] = A + B(t_c - t)^{\beta}$$

where $ln[p(t)]$ is the logarithm of the S&P 500 at time t, t_c is the critical time, A is the expected value of the S&P 500 in logarithm, B measures the exposure to faster-than-exponential growth and β is the power-law exponent controlling faster than exponential price growth. Note that for this model specification $A > 0$, $B < 0$, and $0.1 \leq \beta \leq 0.9$ must hold. Treating β as "fixed", where $\overline{\beta} \in \{0.2, 0.4, 0.6, 0.8\}$, and setting $B = -1$, Eq. (1) simplifies to:

$$ln[p(t)] = ln[p(T)] - (t_c - t)^{\overline{\beta}}$$

Note that A is the expected value of the S&P 500 in logarithm, and $A = ln[p(T)]$ in association with $t_c = T + 1$ means the last price notation of the S&P 500 is the expected price for the critical time t_c and the critical event is suggested to occur in the subsequent time period. Panel A of Table A.1 reports the corresponding initial model specifications. Note that $ln[p(T)] = 7.29$ corresponds to the natural logarithm of the last price quotation of the S&P 500 in the sample. Next, the model is optimized allowing for the following constraints:

$$t_c \geq T + 1$$

$$0.1 \leq \beta \leq 0.9$$

The first constraint allows for the possibility that the finite time singularity arrives at some unknown time point in the future, whereas the second constraint is needed for the price to accelerate and to remain finite. Employing Microsoft Excel's non-linear solver to compute optimal values for the parameter vector $\mathbf{\Phi}_1^* = (A^*, B^*, t_c^*, \beta^*)$, the optimized values are reported in Panel B of Table A.5. In Panel C of Table A.5 the optimized model estimates are reported accounting for the additional constraint

$t_c \leq 2T$, as detailed in section 3.4.3. The sample is from October 9, 2002 to December 31, 2007 comprising 1316 daily observations.

Table A.6
Calibrating the LPPLS model for the S&P 500 using daily data from 2002-2007.

Panel A. Initial parameter values for Model 2.

Specification	1	2	3	4	5	6	7	8	9	10	11
A^*	11.39	11.39	11.39	11.39	11.39	11.39	11.39	11.39	11.39	11.39	11.39
B^*	−2.98	−2.98	−2.98	−2.98	−2.98	−2.98	−2.98	−2.98	−2.98	−2.98	−2.98
t_c^*	10.457	10.457	10.457	10.457	10.457	10.457	10.457	10.457	10.457	10.457	10.457
β^*	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18
C	0	0	0	0	0	0	0	0	0	0	0
ω	5	6	7	8	9	10	11	12	13	14	15
ϕ	0	0	0	0	0	0	0	0	0	0	0
SSR	2.8739	2.8739	2.8739	2.8739	2.8739	2.8739	2.8739	2.8739	2.8739	2.8739	2.8739

Panel B. Optimized parameter values for Model 2.

Specification	1	2	3	4	5	6	7	**8**	9	10	11
A^{**}	10.20	11.38	10.83	11.57	8.87	8.04	10.05	**7.76**	7.81	7.81	7.88
B^{**}	−2.08	−2.44	−2.17	−2.85	−0.63	−0.15	−1.66	**−0.13**	−0.13	−0.13	−0.13
t_c^{**}	8.50	10.46	10.46	10.46	10.46	10.46	10.46	**8.87**	9.36	9.36	9.96
β^{**}	0.24	0.29	0.27	0.23	0.51	0.90	0.28	**0.90**	0.90	0.90	0.90
C^{**}	0.03	−0.03	0.02	0.02	−0.03	0.04	0.01	**−0.06**	0.05	0.05	−0.05
ω^{**}	5	5.66	7.24	7.25	8.77	10.22	10.35	**12.93**	14.04	14.04	15
ϕ^{**}	1.88	0.83	1.96	1.85	1.43	1.15	0.84	**0.07**	0.02	0.02	0.08
SSR	1.95	1.78	1.84	1.85	1.82	1.70	1.99	**1.44**	1.45	1.45	1.46

In line with Sornette (2017, p. 335), the log-period power-law singularity (LPPLS) model is given by,

$$ln[p(t)] = A + B(t_c - t)^{\beta}[1 + Ccos(\omega ln(t_c - t) + \phi)]$$

with $A > 0$, $B < 0$, and $0.1 \leq \beta \leq 0.9$, and A is the expected value of the S&P 500 in logarithm, B measures the exposure to faster-than-exponential growth, β is the power-law exponent controlling faster than exponential price growth and C measures the exposure responsible for periodic oscillations, ω is the angular frequency of the log-periodic oscillations during the bubble formation and ϕ is the phase parameter. Whereas ϕ cannot be meaningful restricted, here we require $|C| < 1$ and impose the constraint $5 \leq \omega \leq 15$. We choose the optimal values for $\mathbf{\Phi}_1^* = (11.39, -2.98, 10.46, 0.18)$ from the first estimation step. Using in addition $C = \phi = 0$, the LPPLS model (model 2) is then optimized for varying values ω, that is, $\omega \in \{5, 6, \ldots, 14, 15\}$. Whereas Panel A of Table A.6 reports the corresponding input parameters, Panel B of Table A.6 reports the optimized parameters using the following constraints:

$$t_c \geq T + 1$$

$$0.1 \leq \beta \leq 0.9$$

$$5 \leq \omega \leq 15$$

$$t_c \leq 2T$$

Note that the parameter ϕ remains unconstrained which is in line with Sornette (2017, p. 336). The parameters are obtained using Microsoft Excel's non-linear solver. The sample is from October 9, 2002 to December 31, 2007 comprising 1316 daily observations.

Table A.7
Testing for explosiveness in random-walk behavior.

Sample	$\widehat{\varphi}_0$	$\widehat{\varphi}_1$	$\widehat{\varphi}_2$	$\widehat{\varphi}_3$	$\widehat{\varphi}_4$	R^2	$\widehat{\lambda}$	*p-value*	*SSR*
(*i*)	0.0008	1.28E-05**	−0.0026*	0.2980***	−0.0803***	0.0858	−1.80	0.7057	2.7476
	(0.44)	(2.27)	(−1.80)	(12.75)	(−3.43)				
(*ii*)	0.0228**	2.26E-06**	−0.0048**	0.0965***		0.0115	−2.15	0.5174	0.1424
	(2.18)	(2.09)	(−2.15)	(4.07)					
(*iii*)	0.0151	2.65E-06*	−0.0035			0.0020	−1.48	0.8363	0.1678
	(1.50)	(1.75)	(−1.48)						

Explosive random-walk behavior corresponding to faster-than-exponential growth is tested using the Augmented Dickey-Fuller (ADF) test, requiring the implementation of the following test regression:

$$\Delta p_t = \varphi_0 + \varphi_1 t + \varphi_2 p_{t-1} + \varphi_1 \Delta p_{t-1} + \ldots + \varphi_p \Delta p_{t-p} + e_t$$

We hypothesize that faster-than-exponential growth should be manifested in $\varphi_2 < 0$ and $\varphi_1 > 0$ implying explosiveness in random-walk behavior. This table reports the parameter estimates for the test regression using (*i*) monthly data on the S&P 500 for the period January 1871 to November 2022, (*ii*) daily data on the S&P 500 for the period January 2, 1980 until December 31, 1986, and (*iii*) daily data on the Dow Jones index for the period June 1, 1921 until December 31, 1928. The lag-order for each model (*i*)-(*iii*) is chosen with respect to the SIC, maximum lag length of 24. The critical values for the ADF tests are −3.96, −3.41, and − 3.13 for the 1%, 5%, and 10% significance level. The ADF test statistic is denoted as $\widehat{\lambda}$ and the corresponding *p*-value are reported too.
***Statistically significant on a 1% level, **Statistically significant on a 5% level, *Statistically significant on a 10% level.

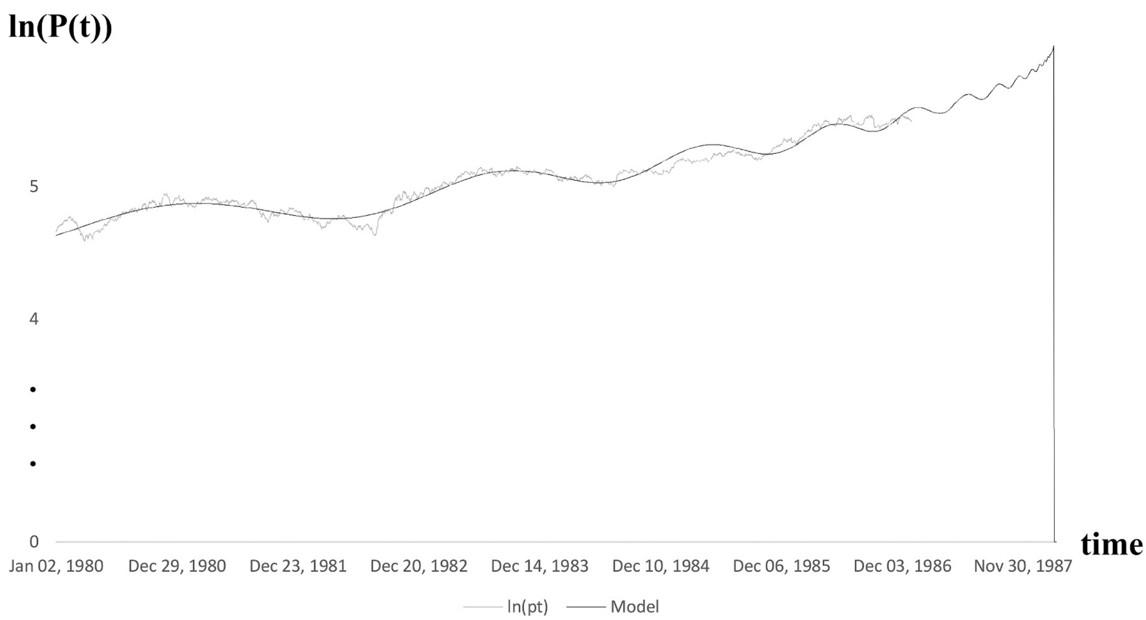

Fig. A.1. The S&P 500 in logarithms and optimized Model 2 (LPPLS model) using daily data for the 1980–1986 period.
This figure shows the optimized Model 2 along with the natural logarithm of the S&P 500 denoted as $ln[p(t)]$.

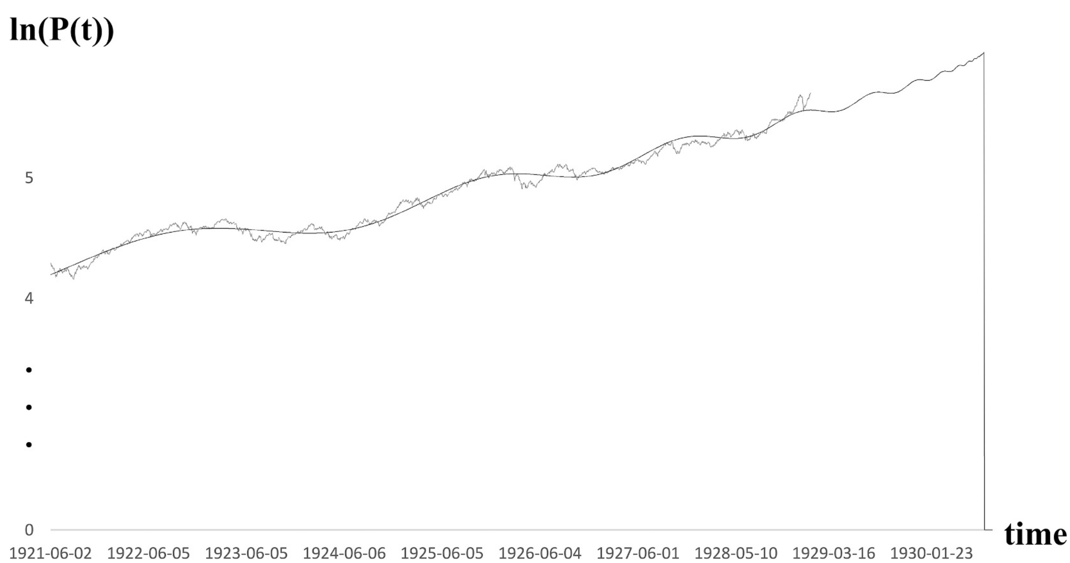

Fig. A.2. The Dow Jones in logarithms and optimized Model 2 (LPPLS model) using daily data for the 1921–1928 period.
This figure shows the optimized Model 2 along with the natural logarithm of the S&P 500 denoted as $ln[p(t)]$.

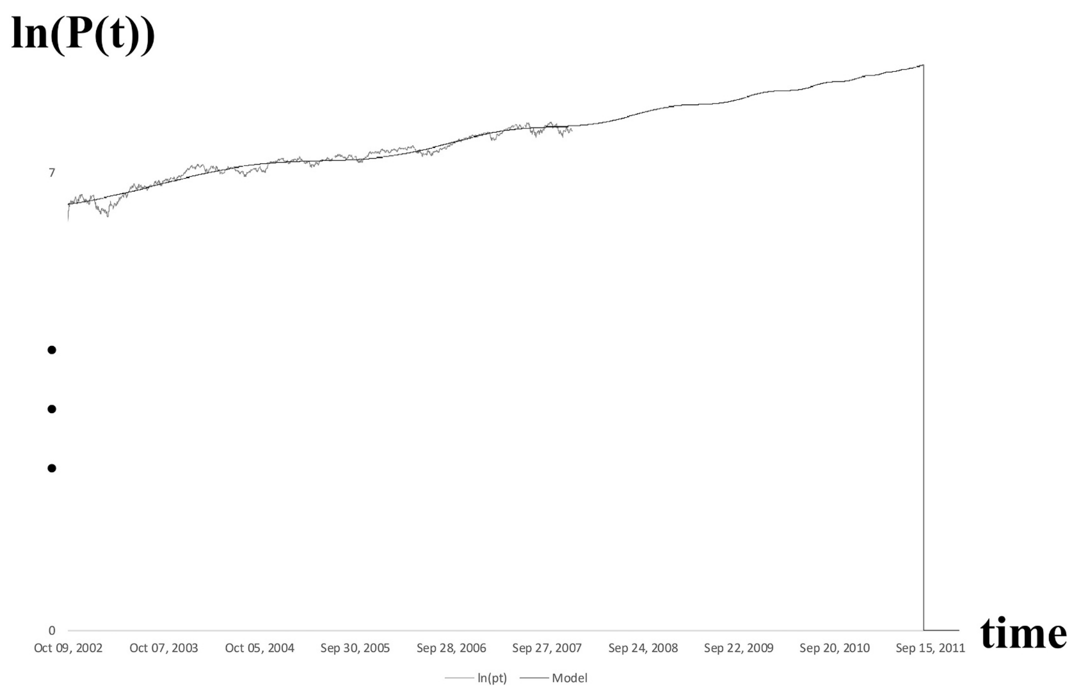

Fig. A.3. The S&P 500 in logarithms and optimized Model 2 (LPPLS model) using daily data for the 2002–2007 period. This figure shows the optimized Model 2 along with the natural logarithm of the S&P 500 denoted as $ln[p(t)]$.

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