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ON KREĬN'S EXTENSION THEORY OF NONNEGATIVE OPERATORS

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Abstract. The famous M.G. Kreĭn's extension theory of nonnegative operators is being presented in elementary terms, building on the completion of nonnegative operator blocks. The present treatment includes the refinements to the theory due to T. Ando and K. Nishio. Moreover, the extension theory not only includes the case of operator extensions but also the case of relation (multivalued-operator) extensions.

1. Introduction

Let S be a densely defined nonnegative operator in a Hilbert space $\mathfrak{H}$. A description of all nonnegative selfadjoint extensions of S is due to M.G. Kreĭn, see [26]. In particular, it was shown that there are two extremal extensions of S, the Friedrichs (hard) extension and the Kreĭn-von Neumann (soft) extension. Further results were obtained by T. Ando and K. Nishio [7], who considered the case that S need not be densely defined.

In order to obtain such results several approaches have been used by various authors. Kreĭn [26] employed the Cayley transform to reduce the extension of nonnegative operators to the extension of symmetric contractive operators. He also used semibounded sesquilinear forms, cf. [23]; a technique which can be extended to the case of linear relations along the lines of [15]. Extensions of symmetric contractive operators has been the subject of many papers, see [9], [13], and the references in those papers. An approach in terms of "boundary conditions" to the extensions of a positive operator S was proposed by M.I. Vishik [35] and M.S. Birman [10]. (See also the later exposition of this theory based on the investigation of quadratic forms in [2].) This approach was subsequently formalized in the concept of "boundary triplet" and was further developed by many authors (see for instance [17] and the references therein).

In this paper the main results of Kreĭn's work and later refinements on extensions of nonnegative operators are presented in an elementary manner starting from the solution to a simple completion problem. The systematic use of block representations of the extreme contractive extensions of a not everywhere defined symmetric contraction makes it possible to simplify and unify the proofs and to make the exposition quite brief.

The contents of this paper are as follows. In Section 2 certain incomplete block operators are completed to nonnegative operators. There is a close connection to shorted operators, which were introduced originally by M.G. Kreĭn [26]. Later shorted operators, or Schur complements were used in system theory, see for instance [3], [4], [5], [33]. The results of Section 2 are translated in Section 3 to the case of selfadjoint contractive extensions of symmetric contractions. The nonemptyness of the set of such extensions follows from the

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existence of solutions to associated completion problems. All the main results are then presented with quite elementary arguments, for instance, Kreĭn's uniqueness criterion about the equality of the extreme extensions. The actual extension theory of nonnegative operators is given in Section 4. The connections between the completion problems treated in this paper and the extension theory are briefly explained in an Appendix. There somewhat more sophisticated tools than elsewhere in this paper will be used; namely the presentation is based on the notions of abstract boundary conditions and Weyl functions.

The following notations will be used throughout the paper. For Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ the bounded linear operators from $\mathfrak{H}$ to $\mathfrak{K}$ will be denoted by $[\mathfrak{H},\mathfrak{K}]$, or by $[\mathfrak{H}]$ when $\mathfrak{K}=\mathfrak{H}$. A (closed) linear relation A in a Hilbert space $\mathfrak{H}$ is just a (closed) linear subspace of $\mathfrak{H}\oplus\mathfrak{H}$. For the convenience of the reader some facts concerning linear relations will be briefly reviewed. Recall the following notions

$$\operatorname{dom} A=\{\, f\in\mathfrak{H} : \{f,g\}\in A \text{ for some } g\in\mathfrak{H}\,\},\quad \ker A=\{\, f\in\mathfrak{H} : \{f,0\}\in A\,\},$$
$$\operatorname{ran} A=\{\, g\in\mathfrak{H} : \{f,g\}\in A \text{ for some } f\in\mathfrak{H}\,\},\quad \operatorname{mul} A=\{\, g\in\mathfrak{H} : \{0,g\}\in A\,\}.$$

With $A^{-1}=\{\{g,f\} : \{f,g\}\in A\}$ it is clear that $\operatorname{dom} A^{-1}=\operatorname{ran} A$, $\operatorname{ran} A^{-1}=\operatorname{dom} A$, $\ker A^{-1}=\operatorname{mul} A$, and $\operatorname{mul} A^{-1}=\ker A$. A linear relation A is (the graph of) an operator precisely when $\operatorname{mul} A=\{0\}$. The (Moore-Penrose) pseudo-inverse of a relation A is the operator $A^{(-1)}$ defined by

$$A^{(-1)}=\{\,\{g,f\}\in\mathfrak{H}^2 : \{f,g\}\in A,\ \ f\perp\ker A\,\},$$

which coincides with the usual notion when A is an operator. For a closed linear relation A define $A_s=\{\,\{f,g\}\in A : g\perp\operatorname{mul} A\,\}$. Clearly A_s is a closed linear operator; it is called the (orthogonal) operator part of A. Note that $\operatorname{dom} A_s=\operatorname{dom} A$ and $\overline{\operatorname{dom}}\, A=(\operatorname{mul} A)^{\perp}$. The adjoint A^* of a linear relation A is defined by

$$A^*=\{\,\{f,g\}\in\mathfrak{H}^2 : (g,h)=(f,k) \text{ for all } \{h,k\}\in A\,\}.$$

The adjoint is always linear and closed. A linear relation A is symmetric if $A\subset A^*$, and selfadjoint if $A=A^*$. A linear relation A is nonnegative if $(f',f)\geq 0$ for all $\{f,f'\}\in A$, or equivalently, if $(A_s f,f)\geq 0$ for all $f\in\operatorname{dom} A$. A nonnegative relation is symmetric. Let A be an operator which is semibounded from below $A\geq\beta$. Then $\operatorname{dom}[A]$ stands for the closure of $\operatorname{dom} A$ with respect to the norm $\|f\|_A^2=(1-\beta)\|f\|^2+(Af,f)$, $f\in\operatorname{dom} A$. The closure of the form (Af,f) is denoted by $\mathfrak{t}[f]$ or by $A[f]=A[f,f]$. It is well-known [1], [23] that $\operatorname{dom}[A]=\operatorname{dom}(A-\beta)^{1/2}$ when A is selfadjoint. If A is a semibounded linear relation then define $\operatorname{dom}[A]=\operatorname{dom}[A_s]$. The notation $\operatorname{ran}[A]$ stands for $\operatorname{dom}[A^{-1}]$. Let A be a symmetric relation in a Hilbert space. If A has a nontrivial multivalued part, then then every selfadjoint extension of A has a nontrivial multivalued part. If A is a densely defined operator, then $\operatorname{mul} A^*=(\operatorname{dom} A)^{\perp}$ is trivial, and A and all its selfadjoint extensions are operators. If A is an operator which is nondensely defined, then $\operatorname{mul} A^*=(\operatorname{dom} A)^{\perp}$ is nontrivial, and there exist selfadjoint extensions with a nontrivial multivalued part. The componentwise sum or equivalently the linear span of the graphs of linear relations A and B is denoted by $A\,\widehat{+}\,B$.

2. Nonnegative operator blocks and shorted operators

In this section incomplete operator blocks of a special form are completed to nonnegative operators. The solutions to the completion problem are described. The connection to the notion of shorted operators [26] and the relation between completions and extensions are explained.

2.1. **Completion to nonnegative operator blocks.** Let $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ be an orthogonal decomposition of the Hilbert space $\mathfrak{H}$ and let A^0 be an incomplete block operator of the form

$$A^0 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & * \end{pmatrix}, \tag{2.1}$$

where $A_{ij} \in [\mathfrak{H}_j, \mathfrak{H}_i]$, $i, j = 1, 2$. If the set

$$\{\, A_{22} \in [\mathfrak{H}_2] : \; A = (A_{ij})_{i,j=1}^2 \geq 0 \,\} \tag{2.2}$$

is nonempty, then $A_{11} \geq 0$ and $A_{21} = A_{12}^*$. The converse question will now be addressed.

Proposition 2.1. *Let A^0 be the incomplete block operator in the Hilbert space $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, given by* (2.1)*. Assume that $A_{11} \geq 0$ and $A_{21} = A_{12}^*$. Then:*

- (i) *There exists a completion $A \in [\mathfrak{H}]$ of A^0 with some operator A_{22} if and only if* $\operatorname{ran} A_{12} \subset \operatorname{ran} A_{11}^{1/2}$.
- (ii) *In this case the operator $S = A_{11}^{(-1/2)} A_{12}$ is well defined and $S \in [\mathfrak{H}_2, \mathfrak{H}_1]$. Moreover, S^*S is the smallest operator in the set* (2.2)*.*

Proof. (i) Assume that there exists a completion A_{22} belonging to the set (2.2). Then $0 \in \rho(A_{11} + \varepsilon)$ for all $\varepsilon > 0$, and therefore

$$\begin{aligned} &\begin{pmatrix} I & 0 \\ -A_{21}(A_{11}+\varepsilon)^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11}+\varepsilon & A_{12} \\ A_{21} & A_{22}+\varepsilon \end{pmatrix} \begin{pmatrix} I & -(A_{11}+\varepsilon)^{-1}A_{12} \\ 0 & I \end{pmatrix} \\ &\qquad = \begin{pmatrix} A_{11}+\varepsilon & 0 \\ 0 & A_{22}+\varepsilon - A_{21}(A_{11}+\varepsilon)^{-1}A_{12} \end{pmatrix}. \end{aligned} \tag{2.3}$$

The operator in the right side of (2.3) is nonnegative if and only if

$$A_{21}(A_{11}+\varepsilon)^{-1}A_{12} \leq A_{22} + \varepsilon, \tag{2.4}$$

or equivalently

$$\int_0^{\|A_{11}\|} (t+\varepsilon)^{-1}\, d\|E_t A_{12} f\|^2 \leq \varepsilon \|f\|^2 + \|A_{22}^{\frac{1}{2}} f\|^2, \quad \varepsilon > 0, \tag{2.5}$$

where E_t denotes the spectral family of A_{11}. Letting $\varepsilon \downarrow 0$ in (2.5), the monotone convergence theorem implies that $A_{12} f \in \operatorname{ran} A_{11}^{1/2}$ for all $f \in \mathfrak{H}_2$. Thus, $\operatorname{ran} A_{12} \subset \operatorname{ran} A_{11}^{1/2}$.

Conversely, if $\operatorname{ran} A_{12} \subset \operatorname{ran} A_{11}^{1/2}$, then the operator $S := A_{11}^{(-1/2)} A_{12}$ is well defined and $S \in [\mathfrak{H}_2, \mathfrak{H}_1]$. Since $A_{12} = A_{11}^{1/2} S$, it follows from $A_{21} = S^* A_{11}^{1/2}$ and

$$A = \begin{pmatrix} A_{11}^{1/2} \\ S^* \end{pmatrix} \begin{pmatrix} A_{11}^{1/2} & S \end{pmatrix} \geq 0, \tag{2.6}$$

that the operator $A_{22} = S^*S$ gives a completion for A^0.

(ii) According to (i) $A_{21} = S^* A_{11}^{1/2}$, and $S^*S \in [\mathfrak{H}_2]$ gives a solution to the completion problem (2.2). Now

$$s - \lim_{\varepsilon \downarrow 0} A_{21}(A_{11}+\varepsilon)^{-1}A_{12} = s - \lim_{\varepsilon \downarrow 0} S^* A_{11}^{1/2}(A_{11}+\varepsilon)^{-1} A_{11}^{1/2} S = S^*S,$$

and if A_{22} is an arbitrary operator in the set (2.2), then by letting $\varepsilon \downarrow 0$ in (2.4) one concludes that $S^*S \le A_{22}$. Therefore, S^*S satisfies the desired minimality property. □

The above proposition establishes the existence of a minimal solution $A = A_{min}$ to the completion problem (2.1) in block form via (2.6):

$$A_{min} = \begin{pmatrix} A_{11}^{1/2} \\ S^* \end{pmatrix} \begin{pmatrix} A_{11}^{1/2} & S \end{pmatrix}. \tag{2.7}$$

Now every other completion $A \in [\mathfrak{H}]$ can be represented in a similar block form as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11}^{1/2} \\ S^* \end{pmatrix} \begin{pmatrix} A_{11}^{1/2} & S \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A_{22} - S^*S \end{pmatrix}. \tag{2.8}$$

Corollary 2.2. *Let $A = (A_{ij})_{i,j=1}^2$ be a block representation of $A \in [\mathfrak{H}]$ with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Then A is nonnegative precisely when:*

(i) $A_{11} \ge 0$, $A_{21} = A_{12}^*$;
(ii) $\operatorname{ran} A_{12} \subset \operatorname{ran} A_{11}^{1/2}$;
(iii) $A_{22} \ge S^*S$, $S = A_{11}^{(-1/2)} A_{12}$.

Moreover, if $\operatorname{ran} A_{12} \subset \operatorname{ran} A_{11}$ *then the condition* (iii) *takes the form* $A_{22} \ge A_{21} A_{11}^{(-1)} A_{12}$.

Proof. The first assertion follows immediately from Proposition 2.1 and the above block representations. As to the last statement it is enough to notice that if $\operatorname{ran} A_{12} \subset \operatorname{ran} A_{11}$, then S^*S can be rewritten as $S^*S = A_{21} A_{11}^{(-1)} A_{12}$. □

Observe, that the inclusion $\operatorname{ran} A_{12} \subset \operatorname{ran} A_{11}$ holds if $\operatorname{ran} A_{11}$ is closed, which is so in particular if $0 \in \rho(A_{11})$, i.e., if A_{11} is invertible. In this last case the difference $A_{22} - S^*S = A_{22} - A_{21} A_{11}^{-1} A_{12}$ is called a Schur complement of $A = (A_{ij})_{i,j=1}^2$. In this sense Corollary 2.2 contains Sylvester's criterion: the operator $A = (A_{ij})_{i,j=1}^2$ with $0 \in \rho(A_{11})$ is nonnegative if and only if $A_{11} \ge 0$, $A_{21} = A_{12}^*$, and $A_{22} - A_{21} A_{11}^{-1} A_{12} \ge 0$.

For each solution $A \in [\mathfrak{H}]$ of the completion problem (2.1) the identities (2.7) and (2.8) give the following result concerning the kernels $\ker A$ and $\ker A_{min}$:

$$\ker A = \ker A_{min} \cap \ker \begin{pmatrix} 0 & 0 \\ 0 & A_{22} - S^*S \end{pmatrix}, \tag{2.9}$$

as, in general, nonnegative bounded linear operators C_1 and C_2 satisfy the following identity $\ker (C_1 + C_2) = \ker C_1 \cap \ker C_2$. Hence, among all solutions $A \in [\mathfrak{H}]$ of the completion problem (2.1) the minimal solution A_{min} has the largest kernel: $\ker A \subset \ker A_{min}$.

Proposition 2.3. *Let A_{min} be the minimal solution to the completion problem* (2.1)*. Then*

$$\ker A_{min} = \left\{ \begin{pmatrix} -A_{11}^{-1/2}(S\varphi) \\ \varphi \end{pmatrix} : S\varphi \in \operatorname{ran} A_{11}^{1/2}, \quad \varphi \in \mathfrak{H}_2 \right\}, \tag{2.10}$$

where $A_{11}^{-1/2}$ stands for the preimage. Moreover, if $A \in [\mathfrak{H}]$ is a solution to the completion problem (2.1) *given in the block form* (2.8) *and if $R := A_{22} - S^*S$, then*

(i) $\ker A = \ker A_{min}$ *if and only if* $S^{-1}(\operatorname{ran} A_{11}^{1/2}) \subset \ker R$.
(ii) *If* $\operatorname{ran} S \cap \operatorname{ran} A_{11}^{1/2} = \{0\}$, *then* $\ker A = \ker A_{min}$ *if and only if* $\ker S \subset \ker R$.

(iii) *The minimal solution to the completion problem* (2.1) *is the only solution* $A \in [\mathfrak{H}]$ *for which* $\ker A = \ker A_{min}$ *if and only if* $\operatorname{ran} A_{12} \subset \operatorname{ran} A_{11}$. *This condition is satisfied in particular if* $\operatorname{ran} A_{11}$ *is closed.*

Proof. It follows from (2.7) that

$$\ker A_{min} = \left\{ \begin{pmatrix} \psi \\ \varphi \end{pmatrix} : A_{11}^{1/2}\psi + S\varphi = 0 \right\},$$

which leads to the identity (2.10). In order to prove (i), (ii), and (iii), observe that $\ker A = \ker A_{min}$ if and only if

$$\ker A_{min} \subset \ker \begin{pmatrix} 0 & 0 \\ 0 & A_{22} - S^*S \end{pmatrix}, \tag{2.11}$$

which follows from (2.9).

(i) According to (2.10) and (2.11) the identity $\ker A = \ker A_{min}$ holds if and only if $R\varphi = 0$ for all $\varphi \in S^{-1}(\operatorname{ran} A_{11}^{1/2})$.

(ii) If $\operatorname{ran} S \cap \operatorname{ran} A_{11}^{1/2} = \{0\}$, then $S^{-1}(\operatorname{ran} A_{11}^{1/2}) = \ker S$, and the assertion follows from the result in (i).

(iii) According to (i) A_{min} is the only solution A to the completion problem (2.1) for which equality holds in (2.10) if and only if $S^{-1}(\operatorname{ran} A_{11}^{1/2}) = \mathfrak{H}_2$. But this is equivalent to $\operatorname{ran} S \subset \operatorname{ran} A_{11}^{1/2}$ and hence also to $\operatorname{ran} A_{12} \subset \operatorname{ran} A_{11}$. The last statement in (iii) follows from $\operatorname{ran} A_{12} \subset \operatorname{ran} A_{11}^{1/2} \subset \overline{\operatorname{ran}}\, A_{11} = \operatorname{ran} A_{11}$. □

Remark 2.4. Let $A = (A_{ij})_{i,j=1}^2$ be a block representation of $A \geq 0$, $A \in [\mathfrak{H}]$, with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Observe, that

$$\ker \begin{pmatrix} A_{11} & A_{12} \end{pmatrix} = \ker \begin{pmatrix} A_{11}^{1/2} & S \end{pmatrix}. \tag{2.12}$$

It follows from (2.12) that the identity (2.10) can be rewritten as

$$\ker A_{min} = \left\{ \begin{pmatrix} -A_{11}^{-1}(A_{12}\,\varphi) \\ \varphi \end{pmatrix} : A_{12}\varphi \in \operatorname{ran} A_{11}, \quad \varphi \in \mathfrak{H}_2 \right\}, \tag{2.13}$$

where A_{11}^{-1} stands for the preimage.

Since $A_{11} \geq 0$, Proposition 2.1 shows that $\ker A_{11} = \ker A_{11}^{1/2} \subset \ker A_{21}$. In particular, $\ker A_{11} \oplus \{0\} \subset \ker A$. Therefore the completion problem (2.1) can be reduced initially. With respect to the decomposition $\mathfrak{H}_1 = \ker A_{11} \oplus \overline{\operatorname{ran}}\, A_{11}$ the operators A_{11} and A_{21} can be decomposed as $A_{11} = 0 \oplus A'_{11}$ with $\ker A'_{11} = \{0\}$ and $A_{21} = 0 \oplus A'_{21}$. Hence, the completion problem (2.1) can be reduced to a completion problem in $\mathfrak{H} \ominus \ker A_{11}$ via

$$A^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A'_{11} & A'_{12} \\ 0 & A'_{21} & * \end{pmatrix}, \tag{2.14}$$

where the decomposition is with respect to $\mathfrak{H} = \ker A_{11} \oplus \overline{\operatorname{ran}}\, A_{11} \oplus \mathfrak{H}_2$. Therefore, without loss of generality, it may be assumed in the sequel that the incomplete block operator A^0 in (2.1) satisfies $\ker A_{11} = \{0\}$. In this case the preimages in (2.10) and (2.13) coincide with the usual inverses.

2.2. **Shorted operators.** Let $\mathfrak{H}$ be a Hilbert space, let $\mathfrak{N}$ be a closed linear subspace of $\mathfrak{H}$, and let $A \in [\mathfrak{H}]$ be a nonnegative operator. Then the set

$$\{ D \in [\mathfrak{H}] : 0 \le D \le A, \operatorname{ran} D \subset \mathfrak{N} \} \tag{2.15}$$

contains a unique maximal element, which is called the *shortening* $A_{\mathfrak{N}}$ of A to the subspace $\mathfrak{N}$. The operator $A_{\mathfrak{N}}$ was introduced by M.G. Kreĭn [26]; the mapping $A \to A_{\mathfrak{N}}$ is called the *Kreĭn transformation*. It will be shown that the existence and uniqueness of the operator $A_{\mathfrak{N}}$ are consequences of Proposition 2.1 and the corresponding decomposition (2.8) with $\mathfrak{H}_2 = \mathfrak{N}$.

Proposition 2.5. (Cf. [26], [34], [27]) *Let $A = (A_{ij})_{i,j=1}^2$ be a block representation of $A \ge 0$, $A \in [\mathfrak{H}]$, with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{N}$. The set* (2.15) *contains a unique maximal element $A_{\mathfrak{N}}$ given by*

$$A_{\mathfrak{N}} = \begin{pmatrix} 0 & 0 \\ 0 & A_{22} - S^*S \end{pmatrix}, \quad S = A_{11}^{(-1/2)} A_{12}. \tag{2.16}$$

Moreover, each element $D \in [\mathfrak{H}]$ in (2.15) *satisfies the inequality*

$$(Df, f) \le \inf_{g \in \mathfrak{H}_1} (A(f-g), f-g), \quad f \in \mathfrak{H}, \tag{2.17}$$

and $A_{\mathfrak{N}}$ is the only element in (2.15) *for which equality holds:*

$$(A_{\mathfrak{N}} f, f) = \inf_{g \in \mathfrak{H}_1} (A(f-g), f-g), \quad f \in \mathfrak{H}. \tag{2.18}$$

Proof. According to Corollary 2.2 $A_{\mathfrak{N}}$ given by (2.16) belongs to the set in (2.15). Now let D be any operator in the set in (2.15). Since $D \ge 0$ and $\operatorname{ran} D \subset \mathfrak{N}$, it has the representation

$$D = \begin{pmatrix} 0 & 0 \\ 0 & D_{22} \end{pmatrix}, \quad D_{22} \ge 0. \tag{2.19}$$

Moreover, it follows from $A - D \ge 0$ and Proposition 2.1 that $A_{22} - D_{22} \ge S^*S$, or in other words, that $D \le A_{\mathfrak{N}}$. Therefore, $A_{\mathfrak{N}}$ is the unique maximal element in (2.15).

Now let $f = f_1 \oplus f_2 \in \mathfrak{H}_1 \oplus \mathfrak{N}$ and let $g \in \mathfrak{H}_1$. Then it follows from (2.8) and (2.16) that

$$\begin{aligned} (A(f-g), f-g) &= \|A_{11}^{1/2}(f_1 - g) + Sf_2\|^2 + ((A_{22} - S^*S)f_2, f_2) \\ &= \|A_{11}^{1/2}(f_1 - g) + Sf_2\|^2 + (A_{\mathfrak{N}} f_2, f_2) \\ &= \|A_{11}^{1/2}(f_1 - g) + Sf_2\|^2 + (A_{\mathfrak{N}} f, f). \end{aligned}$$

Recall that $\operatorname{ran} S \subset (\ker A_{11}^{1/2})^\perp = \overline{\operatorname{ran}}\, A_{11}^{1/2}$, which leads to (2.18).

Since $A_{\mathfrak{N}}$ is the unique maximal element in (2.15), (2.18) implies that every operator D in the set in (2.15) satisfies (2.17). Assume that for some $D \in [\mathfrak{H}]$ with $0 \le D \le A$, $\operatorname{ran} D \subset \mathfrak{N}$:

$$(Df, f) = \inf_{g \in \mathfrak{H}_1} (A(f-g), f-g), \quad f \in \mathfrak{H}. \tag{2.20}$$

Then $(Df, f) = (A_{\mathfrak{N}} f, f)$, $f \in \mathfrak{H}$, and consequently $D = A_{\mathfrak{N}}$. □

Example 2.6. Let $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{N}$, $R \in [\mathfrak{N}]$, $R \ge 0$, let $\mathfrak{H}_3$ be a Hilbert space, $X \in [\mathfrak{H}_1, \mathfrak{H}_3]$, $Y \in [\mathfrak{N}, \mathfrak{H}_3]$, and let $\operatorname{ran} Y \subset \overline{\operatorname{ran}}\, X$. Let the operator $A \in [\mathfrak{H}]$ be given by

$$A = \begin{pmatrix} X^* \\ Y^* \end{pmatrix} (X, Y) + \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix} \ge 0. \tag{2.21}$$

It follows from the definition (2.21) of A that

$$(A(f-g),f-g)=\|X(f_1-g_1)+Yf_2\|^2+(Rf_2,f_2),\quad f_1,g_1\in\mathfrak{H}_1,\quad f_2\in\mathfrak{N},$$

where $f=f_1\oplus f_2$ and $g=g_1\oplus 0$. Hence, (2.18) and $\operatorname{ran} Y\subset\overline{\operatorname{ran}}\,X$ imply that

$$(A_{\mathfrak{N}}f,f)=\left(\begin{pmatrix}0&0\\0&R\end{pmatrix}f,f\right),\quad f\in\mathfrak{H}.$$

The operator in the right side is nonnegative, is majorized by A, and has its range in $\mathfrak{N}$, so that the shortening of A to the subspace $\mathfrak{N}$ is given by:

$$A_{\mathfrak{N}}=\begin{pmatrix}0&0\\0&R\end{pmatrix}. \tag{2.22}$$

In particular, $A_{\mathfrak{N}}=0$ if and only if $R=0$.

The identity (2.18) provides a characterization of shorted operators from which various well-known monotonicity and continuity properties follow, cf. [3], [4], [33].

Proposition 2.7. *Let $A,B\in[\mathfrak{H}]$ be nonnegative operators, let $\mathfrak{M},\mathfrak{N}$ be closed linear subspaces of $\mathfrak{H}$, and let $\lambda\geq 0$. Then:*

(i) $(\lambda A)_{\mathfrak{N}}=\lambda A_{\mathfrak{N}}$;
(ii) $A_{\mathfrak{N}}+B_{\mathfrak{N}}\leq(A+B)_{\mathfrak{N}}$, *in particular* $A\leq B\Rightarrow A_{\mathfrak{N}}\leq B_{\mathfrak{N}}$;
(iii) $A_{\mathfrak{M}\cap\mathfrak{N}}=(A_{\mathfrak{M}})_{\mathfrak{N}}$, *in particular* $\mathfrak{M}\subset\mathfrak{N}\Rightarrow A_{\mathfrak{M}}\leq A_{\mathfrak{N}}$.

Moreover, if $A_n\in[\mathfrak{H}]$ and $A_n\geq A$, then

$$A=w-\lim_{n\to\infty}A_n\Rightarrow A_{\mathfrak{N}}=s-\lim_{n\to\infty}(A_n)_{\mathfrak{N}}. \tag{2.23}$$

Proof. The statements (i)–(iii) follow easily from (2.18). As to the last statement, let $f\in\mathfrak{H}$. According to (2.18) for each $\varepsilon>0$ there exists $g_\varepsilon\in\mathfrak{H}_1$, such that

$$(A_{\mathfrak{N}}f,f)\leq(A(f-g_\varepsilon),f-g_\varepsilon)\leq(A_{\mathfrak{N}}f,f)+\varepsilon/2.$$

Since $w-\lim_{n\to\infty}A_n=A$ and $A_n\geq A$, there exists some $n_0=n_0(g_\varepsilon)$, such that

$$(A_{\mathfrak{N}}f,f)\leq(A_n(f-g_\varepsilon),f-g_\varepsilon)\leq(A_{\mathfrak{N}}f,f)+\varepsilon,\quad n\geq n_0,$$

and, therefore, also

$$(A_{\mathfrak{N}}f,f)\leq((A_n)_{\mathfrak{N}}f,f)\leq(A_{\mathfrak{N}}f,f)+\varepsilon,\quad n\geq n_0.$$

As $\varepsilon>0$ is arbitrary, it follows that $\lim_{n\to\infty}((A_n)_{\mathfrak{N}}f,f)=(A_{\mathfrak{N}}f,f)$, $f\in\mathfrak{H}$; and hence $A_{\mathfrak{N}}=w-\lim_{n\to\infty}(A_n)_{\mathfrak{N}}$. Now $A_n\geq A$ implies that $(A_n)_{\mathfrak{N}}\geq A_{\mathfrak{N}}$. For the nonnegative operators $B_n:=(A_n)_{\mathfrak{N}}-A_{\mathfrak{N}}$ the weak convergence to zero implies the strong convergence: $w-\lim B_n=0\Rightarrow s-\lim B_n=0$. □

Remark 2.8. Items (i) and (ii) express the sublinear property and item (iii) expresses the multiplicative property of shortening. The implication (2.23) need not be true without the condition $A_n\geq A$, even if the strong limit is replaced with the weak limit $A_{\mathfrak{N}}=w-\lim_{n\to\infty}(A_n)_{\mathfrak{N}}$. For instance, let $X\in[\mathfrak{H}_1,\mathfrak{H}_3]$ be such that $\overline{\operatorname{ran}}\,X=\mathfrak{H}_3$, let $Y\in[\mathfrak{N},\mathfrak{H}_3]$ be nontrivial, and let $\varepsilon_n\downarrow 0$ as $n\to\infty$. Then

$$A_n=\begin{pmatrix}\varepsilon_nX^*\\Y^*\end{pmatrix}\begin{pmatrix}\varepsilon_nX&Y\end{pmatrix}\to\begin{pmatrix}0&0\\0&Y^*Y\end{pmatrix}=A,$$

in the strong sense as $n \to \infty$. However, Example 2.6 shows that $(A_n)_{\mathfrak{N}} = 0$ for all $n \in \mathbb{N}$, and $A_{\mathfrak{N}} = Y^*Y \neq 0$. In addition, notice that strong convergence of nonnegative operators does not follow from their weak convergence. For instance, when $C_n = (I + U^{n*})(I + U^n)$, where U denotes the unilaterial shift, then $C_n \to 2I$ in the weak sense, whereas the limit in the strong sense does not exist.

Clearly, with $\mathfrak{N} = \mathfrak{H}_2$, a solution A to (2.1) is minimal precisely when $A_{\mathfrak{N}}$ is trivial.

Corollary 2.9. *Let $A \in [\mathfrak{H}]$ be a solution to the completion problem* (2.1) *with a block representation $A = (A_{ij})_{i,j=1}^2$. Then A is equal to the minimal solution if and only if*

$$\inf_{g \in \mathfrak{H}_1} (A(f-g), f-g) = 0, \quad f \in \mathfrak{H}.$$

This result can be stated in a different, but an equivalent, manner. Let $A \in [\mathfrak{H}]$ be a solution to the completion problem (2.1). Provide the Hilbert space $\mathfrak{H}$ with the scalar product (Af, g), $f, g \in \mathfrak{H}$, and denote by $\mathfrak{H}_A$ the semi-Hilbert space obtained by the completion of $\mathfrak{H}$ with respect to (Af, g). Then A is minimal if and only if $\mathfrak{H}_1$ is dense in $\mathfrak{H}_A$, or equivalently, if and only if $\mathfrak{H}_1/\ker A$ is dense in the quotient space $\mathfrak{H}_A/\ker A$.

2.3. **Completions and extensions.** Let $A \in [\mathfrak{H}]$ be a solution to the completion problem (2.1). The restriction of this bounded linear operator to the subspace $\mathfrak{H}_1 \oplus \{0\}$ is a bounded symmetric operator A_1:

$$A_1 = A \upharpoonright \mathfrak{H}_1 = \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}.$$

The completion problem is equivalent to the description of all bounded selfadjoint extensions in $[\mathfrak{H}]$ of A_1, see the Appendix. As the operator A_1 is not defined everywhere, its adjoint A_1^* in $\mathfrak{H}$ is a linear relation (multivalued operator). In order to determine this adjoint observe that, in the language of relations,

$$A_1 = A \cap Z^*, \quad Z = \{0\} \oplus \mathfrak{H}_2. \tag{2.24}$$

Lemma 2.10. *Let $A \in [\mathfrak{H}]$ be a solution to the completion problem* (2.1)*. Then*

$$A_1^* = A \,\widehat{+}\, (\{0\} \oplus \mathfrak{H}_2). \tag{2.25}$$

Proof. The representation (2.24) implies that $A_1^* = \operatorname{clos}(A \,\widehat{+}\, Z)$. In fact $A \,\widehat{+}\, Z$ is closed, which implies (2.25). To see that $A \,\widehat{+}\, Z$ is indeed closed, assume that $\{f_n, g_n\} \to \{f, g\}$ as $n \to \infty$ for a sequence $\{f_n, g_n\} \in A \,\widehat{+}\, (\{0\} \oplus \mathfrak{H}_2)$, where $g_n = Af_n + h_n$ and $h_n \in \mathfrak{H}_2$. Then $Af_n \to Af$, since A is bounded, and $h_n = g_n - Af_n \to g - Af =: h \in \mathfrak{H}_2$ as $n \to \infty$. Therefore, $\{f, g\} = \{f, Af\} \,\widehat{+}\, \{0, h\} \in A \,\widehat{+}\, (\{0\} \oplus \mathfrak{H}_2)$. □

Lemma 2.11. *Let $A \in [\mathfrak{H}]$ be a solution to the completion problem* (2.1)*. Then*

$$\operatorname{ran} A \subset \operatorname{ran} A_1^* \cap \operatorname{ran} A^{1/2}. \tag{2.26}$$

Moreover, equality holds in (2.26) *if $A = A_{min}$:*

$$\operatorname{ran} A_{min} = \operatorname{ran} A_1^* \cap \operatorname{ran} A_{min}^{1/2}.$$

Proof. Since A is a selfadjoint extension of A_1, one has $A \subset A_1^*$ and therefore (2.26) holds.

Now let $A = A_{min}$. To prove the reverse inclusion, let $g \in \operatorname{ran} A_1^* \cap \operatorname{ran} A_{min}^{1/2}$. Since $g \in \operatorname{ran} A_1^*$, it follows from Lemma 2.24 with $A = A_{min}$, that $g = A_{min} f + h$ for some $f \in \mathfrak{H}$ and $h \in \mathfrak{H}_2$. Moreover, since $g \in \operatorname{ran} A_{min}^{1/2}$ one obtains

$$h \in \operatorname{ran} A_{min}^{1/2} = \operatorname{ran} \begin{pmatrix} A_{11}^{1/2} \\ A_{21} A_{11}^{(-1/2)} \end{pmatrix}, \tag{2.27}$$

cf. (2.7). Since $h \in \mathfrak{H}_2$, (2.27) implies that $h = 0$ and therefore $g \in \operatorname{ran} A_{min}$. □

It follows from Lemma 2.10 that the defect subspace of A_1 at $z \in \rho(A_{11})$ is given by

$$\mathfrak{N}_z(A_1^*) = \ker(A_1^* - z) = \left\{ \begin{pmatrix} -(A_{11} - z)^{-1} A_{21}^* f_2 \\ f_2 \end{pmatrix} : f_2 \in \mathfrak{H}_2 \right\}. \tag{2.28}$$

Let $\widehat{\mathfrak{N}}_z(A_1^*) = \{ \{f, zf\} : f \in \mathfrak{N}_z(A_1^*) \}$ and define a special extension A_z of A_1 by

$$A_z = A_1 \widehat{+} \widehat{\mathfrak{N}}_z(A_1^*). \tag{2.29}$$

Lemma 2.12. *For $z \in \rho(A_{11})$ the extension A_z in* (2.29) *has the block representation:*

$$A_z = \begin{pmatrix} A_{11} & A_{21}^* \\ A_{21} & A_{22}(z) \end{pmatrix}, \tag{2.30}$$

where

$$A_{22}(z) = zI + A_{21}(A_{11} - z)^{-1} A_{21}^*. \tag{2.31}$$

Proof. Assume that $z \in \rho(A_{11})$, so that (2.28) holds. Let

$$\left\{ \begin{pmatrix} 0 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\} \in A_z.$$

Then $f_1 = (A_{11} - z)^{-1} A_{21}^* f_2$ and consequently $g_1 = A_{11} f_1 - z(A_{11} - z)^{-1} A_{21}^* f_2 = A_{21}^* f_2$, $g_2 = A_{21} f_1 + z f_2 = (zI + A_{21}(A_{11} - z)^{-1} A_{21}^*) f_2$. Therefore, A_z has the form (2.30), (2.31). □

If $z \in \rho(A_{11})$, then the block representation (2.30) shows that $(A_z)^* = A_{\bar{z}}$. In particular, if $z \in \rho(A_{11}) \cap \mathbb{R}$, then A_z is selfadjoint. Since $A_{11} \geq 0$, each $z < 0$ belongs to $\rho(A_{11})$.

Lemma 2.13. *Let A_z, $z < 0$, be defined by* (2.29). *Then*

$$s - \lim_{z \uparrow 0} A_z = A_{min}.$$

Proof. The statement follows from (2.31) and $s - \lim_{z \uparrow 0} (zI + A_{21}(A_{11} - z)^{-1} A_{21}^*) = S^* S$; cf. the proof of Proposition 2.1. □

3. Selfadjoint contractive extensions of symmetric contractions

Let $\mathfrak{H}_1$ be a closed linear subspace of the Hilbert space $\mathfrak{H}$ and let $T_1 \in [\mathfrak{H}_1, \mathfrak{H}]$ be a symmetric contraction. The set of all selfadjoint contractive extensions of T_1 is denoted by $\operatorname{Ext}_{T_1}(-1, 1)$:

$$\operatorname{Ext}_{T_1}(-1, 1) = \{ T \in [\mathfrak{H}] : T_1 \subset T = T^*, \ \|T\| \leq 1 \}.$$

It was shown by Kreĭn [26], cf. also [1], that the set $\operatorname{Ext}_{T_1}(-1, 1)$ is nonempty. Here Kreĭn's result and a description of $\operatorname{Ext}_{T_1}(-1, 1)$ is obtained via the preceding completion problem. An important tool in the description is provided by the notion of defect operator.

3.1. **Defect operators.** Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces and let $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$ be a contraction. The corresponding defect operators D_T and D_{T^*} are defined by

$$D_T = (I - T^*T)^{1/2}, \quad D_{T^*} = (I - TT^*)^{1/2}.$$

They satisfy the well-known commutation properties

$$TD_T = D_{T^*}T, \quad T^*D_{T^*} = D_TT^*, \tag{3.1}$$

which follow from the spectral theorem.

Corollary 3.1. *Let $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$ be a contraction. Then*

$$T(\ker D_T) = \ker D_{T^*}, \quad T^*(\ker D_{T^*}) = \ker D_T. \tag{3.2}$$

In particular,

$$\ker D_T = \{0\} \textit{ if and only if } \ker D_{T^*} = \{0\}. \tag{3.3}$$

Proof. It suffices to show the first identity in (3.2). Let $\varphi \in \ker D_T$, then according to (3.1) $T\varphi \in \ker D_{T^*}$. Hence, $T(\ker D_T) \subset \ker D_{T^*}$. Conversely, let $\varphi \in \ker D_{T^*}$. Then $\varphi = TT^*\varphi$, and $T^*\varphi \in \ker D_T$ according to (3.1), which shows the reverse inclusion. □

The ranges $\operatorname{ran} T$ and $\operatorname{ran} D_{T^*}$ are complementary subspaces in the sense of [11]: $\mathfrak{H} = \operatorname{ran} T + \operatorname{ran} D_{T^*}$ with overlapping space $\operatorname{ran} T \cap \operatorname{ran} D_{T^*}$, cf. also [6]. The intersection $\operatorname{ran} T \cap \operatorname{ran} D_{T^*}$ is in general nontrivial; it can be described as follows.

Lemma 3.2. *Let $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$ be a contraction. Then*

$$\operatorname{ran} T \cap \operatorname{ran} D_{T^*} = \operatorname{ran} TD_T = \operatorname{ran} D_{T^*}T.$$

Proof. The commutation relations (3.1) show that $\operatorname{ran} TD_T = \operatorname{ran} D_{T^*}T \subset \operatorname{ran} T \cap \operatorname{ran} D_{T^*}$. Hence, it suffices to prove the inclusion

$$\operatorname{ran} T \cap \operatorname{ran} D_{T^*} \subset \operatorname{ran} TD_T.$$

Suppose that $\varphi \in \operatorname{ran} T \cap \operatorname{ran} D_{T^*}$. Then it follows from (3.1) that $D_{T^*}\varphi \in \operatorname{ran} T$ and $T^*\varphi \in \operatorname{ran} D_T$. Hence, $D_{T^*}\varphi = Tf$ and $T^*\varphi = D_Tg$ for some $f, g \in \mathfrak{H}$. Therefore, $\varphi = (I - TT^*)\varphi + TT^*\varphi = D_{T^*}Tf + TD_Tg = TD_T(f + g)$. □

3.2. **Reduction to a completion problem.** Let $\mathfrak{H}_1$ be a closed linear subspace of the Hilbert space $\mathfrak{H}$ and let $T_1 \in [\mathfrak{H}_1, \mathfrak{H}]$ be a symmetric contraction. Write T_1 in the block form $T_1 = \binom{T_{11}}{T_{21}}$ with respect to the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. In the next theorem it is shown that the completion problems associated with

$$A_\pm^0 = \begin{pmatrix} I \pm T_{11} & \pm T_{12} \\ \pm T_{21} & * \end{pmatrix} \tag{3.4}$$

have solutions. Moreover, the corresponding minimal solutions A_+ and A_- are shown to be connected with two extreme selfadjoint contractive extensions of T_1.

Theorem 3.3. *Let $T_1 = \binom{T_{11}}{T_{21}} \in [\mathfrak{H}_1, \mathfrak{H}]$ be a symmetric contraction from the Hilbert space $\mathfrak{H}_1$ into the Hilbert space $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Then:*

(i) *The completion problems for $A_\pm^0$ in (3.4) have minimal solutions $A_\pm$.*
(ii) *The operators $T_m := A_+ - I$ and $T_M := I - A_-$ belong to* $\operatorname{Ext}_{T_1}(-1, 1)$.

(iii) *The operators* T_m *and* T_M *have the block form*

$$T_m = \begin{pmatrix} T_{11} & D_{T_{11}}V^* \\ VD_{T_{11}} & -I + V(I-T_{11})V^* \end{pmatrix}, \quad T_M = \begin{pmatrix} T_{11} & D_{T_{11}}V^* \\ VD_{T_{11}} & I - V(I+T_{11})V^* \end{pmatrix}, \tag{3.5}$$

where V *is given by* $V := T_{21}D_{T_{11}}^{(-1)}$.

(iv) *The operators* T_m *and* T_M *are extremal extensions of* T_1*:*

$$T \in \operatorname{Ext}_{T_1}(-1,1) \quad \text{if and only if} \quad T = T^* \in [\mathfrak{H}], \quad T_m \le T \le T_M. \tag{3.6}$$

(v) *The operators* T_m *and* T_M *are connected via*

$$(-T)_m = -T_M, \quad (-T)_M = -T_m. \tag{3.7}$$

Proof. (i) The condition $\|T_1\| \le 1$ is equivalent to $T_{21}^*T_{21} \le I - T_{11}^2 = D_{T_{11}}^2$, where $D_{T_{11}}$ is the (selfadjoint) defect operator of T_{11}, so that

$$\|T_{21}f\| \le \|D_{T_{11}}f\|, \quad f \in \mathfrak{H}_1. \tag{3.8}$$

This implies the existence of a contraction $V \in [\mathfrak{H}_1, \mathfrak{H}_2]$ such that $VD_{T_{11}}f = T_{21}f$, $f \in \mathfrak{H}_1$, cf. [19], [20]. Moreover, V is uniquely determined if the extra condition $\ker V \supset \ker D_{T_{11}}$ is assumed. Since

$$T_{21}^* = D_{T_{11}}V^* = (I+T_{11})^{1/2}(I-T_{11})^{1/2}V^*, \tag{3.9}$$

the range inclusions

$$\operatorname{ran} T_{21}^* \subset \operatorname{ran}(I \pm T_{11})^{1/2} \tag{3.10}$$

hold. By Proposition 2.1 the following operators

$$S_+ = (I+T_{11})^{(-1/2)}T_{21}^*, \quad S_- = (I-T_{11})^{(-1/2)}T_{21}^*. \tag{3.11}$$

are well defined. According to Proposition 2.1 there are minimal solutions $A_\pm$ to the completion problems for $A_\pm^0$ in (3.4).

(ii)&(iii) The operators $T_m = A_+ - I$ and $T_M = I - A_-$ can be rewritten as follows:

$$T_m = A_+ - I = \begin{pmatrix} T_{11} & T_{21}^* \\ T_{21} & -I + S_+^*S_+ \end{pmatrix}, \quad T_M = I - A_- = \begin{pmatrix} T_{11} & T_{21}^* \\ T_{21} & I - S_-^*S_- \end{pmatrix}. \tag{3.12}$$

Observe that

$$S_\pm = (I \pm T_{11})^{(-1/2)}D_{T_{11}}V^* = P_\pm(I \mp T_{11})^{1/2}V^* = (I \mp T_{11})^{1/2}P_\pm V^*, \tag{3.13}$$

where $P_\pm$ are the orthogonal projections onto

$$(\ker(I \pm T_{11})^{1/2})^\perp = (\ker(I \pm T_{11}))^\perp = \overline{\operatorname{ran}}(I \pm T_{11}) = \overline{\operatorname{ran}}(I \pm T_{11})^{1/2}.$$

Since $\ker V \supset \ker D_{T_{11}}$ implies $\overline{\operatorname{ran}}\, V^* \subset \overline{\operatorname{ran}}\, D_{T_{11}} \subset \overline{\operatorname{ran}}(I \pm T_{11})^{1/2}$, it follows that

$$S_+ = (I-T_{11})^{1/2}V^*, \quad S_- = (I+T_{11})^{1/2}V^*. \tag{3.14}$$

Consequently,

$$S_+^*S_+ = V(I-T_{11})V^*, \quad S_-^*S_- = V(I+T_{11})V^*, \tag{3.15}$$

which implies the representations for T_m and T_M in (iii). Clearly, T_m and T_M are selfadjoint extensions of T_1, which satisfy the inequalities $I + T_m \geq 0$ and $I - T_M \geq 0$. Moreover, it follows from (3.5) that

$$T_M - T_m = \begin{pmatrix} 0 & 0 \\ 0 & 2(I - VV^*) \end{pmatrix}. \tag{3.16}$$

Hence, $T_M \geq T_m$ and consequently $I - T_m \geq I - T_M \geq 0$ and $I + T_M \geq I + T_m \geq 0$. Therefore, T_m and T_M are contractions, which proves (ii).

(iv) Observe, that $T \in \operatorname{Ext}_{T_1}(-1,1)$ if and only if $T = T^* \supset T_1$ and $I \pm T \geq 0$. By Proposition 2.1 this is equivalent to

$$S_+^* S_+ - I \leq T_{22} \leq I - S_-^* S_-. \tag{3.17}$$

The inequalities (3.17) are equivalent to (3.6).

(v) The relations (3.7) follow from (3.11) and (3.12). □

The formulas for T_m and T_M in (3.5) were obtained by V. Kolmanovich and M.M. Malamud, see [24, p.5], [25]. Apparently they are a part of mathematical folklore.

3.3. **Selfadjoint contractive extensions of symmetric contractions.** The first result in this subsection is the well-known result of M.G. Kreĭn concerning the nonemptyness of $\operatorname{Ext}_{T_1}(-1,1)$, which is now just an immediate consequence of the previous theorem. Notice, that Theorem 3.3 also gives an explicit structure for the set $\operatorname{Ext}_{T_1}(-1,1)$.

Theorem 3.4. ([26], cf. also [1]) *Let the Hilbert space $\mathfrak{H}$ have the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and let $T_1 = \binom{T_{11}}{T_{21}} \in [\mathfrak{H}_1, \mathfrak{H}]$ be a symmetric contraction. Then $\operatorname{Ext}_{T_1}(-1,1)$ is a nonempty operator interval with endpoints T_m and T_M; i.e. the equivalence (3.6) holds.*

The result in Theorem 3.4 can also be stated in terms of operator balls. The defect operator $D_{V^*} \in [\mathfrak{H}_2]$ gives rise to an orthogonal decomposition of the Hilbert space $\mathfrak{H}_2$:

$$\mathfrak{H}_2 = \overline{\operatorname{ran}}\, D_{V^*} \oplus \ker D_{V^*}.$$

Denote the set of all selfadjoint contractions on a Hilbert space $\mathcal{H}$ by

$$C_{\mathcal{H}} = \{\, K \in [\mathcal{H}] : \, K = K^*, \; \|K\| \leq 1 \,\}.$$

Corollary 3.5. [26, 27] *Let the Hilbert space $\mathfrak{H}$ have the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, let $T_1 = \binom{T_{11}}{T_{21}} \in [\mathfrak{H}_1, \mathfrak{H}]$, and let $T_0 = (T_m + T_M)/2$, where T_m and T_M are as in Theorem 3.3. Then the formula*

$$T = T_0 + 2^{-1}(T_M - T_m)^{1/2} \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} (T_M - T_m)^{1/2}, \tag{3.18}$$

or, equivalently, the formula

$$T = \begin{pmatrix} T_{11} & D_{T_{11}} V^* \\ V D_{T_{11}} & -V T_{11}^* V^* \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & D_{V^*} K D_{V^*} \end{pmatrix}, \tag{3.19}$$

establishes a one-to-one correspondence between the set $\operatorname{Ext}_{T_1}(-1,1)$ and the operator ball $C_{\mathcal{H}}$, where $\mathcal{H} = \overline{\operatorname{ran}}\, D_{V^*}$.

Proof. The inequality in (3.6) can be rewritten in the form

$$-(T_M - T_m)/2 = T_m - T_0 \le T - T_0 \le T_M - T_0 = (T_M - T_m)/2.$$

These inequalities are equivalent to the existence of a selfadjoint contraction $\widetilde{K}$ in $\mathfrak{H}$ with $\operatorname{ran}\widetilde{K} \subset \mathcal{H} = \overline{\operatorname{ran}}\, D_{V^*}$, such that $2(T - T_0) = (T_M - T_m)^{1/2}\widetilde{K}(T_M - T_m)^{1/2}$, cf. (3.16). This implies the representation (3.18) for T. The one-to-one correspondence between $T \in \operatorname{Ext}_{T_1}(-1,1)$ and $K \in C_{\mathcal{H}}$ via (3.18) is obvious from the given identities. To see the equivalence of (3.18) and (3.19) observe that (3.5) implies

$$T_0 = \begin{pmatrix} T_{11} & D_{T_{11}}V^* \\ VD_{T_{11}} & -VT_{11}V^* \end{pmatrix}.$$

The form of the second term in the right side of (3.19) is clear from (3.16) and (3.18). □

The extensions T_m and T_M can be characterized in a different manner, cf. Corollary 2.9.

Corollary 3.6. ([26]) *Let $T \in \operatorname{Ext}_{T_1}(-1,1)$. Then*

(i) $T = T_m$ *if and only if* $\inf_{g\in\mathfrak{H}_1}((I + T)(f - g), f - g) = 0$ *for every* $f \in \mathfrak{H}$*;*
(ii) $T = T_M$ *if and only if* $\inf_{g\in\mathfrak{H}_1}((I - T)(f - g), f - g) = 0$ *for every* $f \in \mathfrak{H}$*.*

Let $T \in [\mathfrak{H}]$ be a contraction, in general, with $\ker(I \pm T) \neq \{0\}$. Denote by $\mathfrak{H}_\pm$ the semi-Hilbert spaces obtained by the completion of $\mathfrak{H}$ with respect to the seminorms $\|\cdot\|_{I\pm T}$,

$$(f, g)_{I\pm T} = ((I \pm T)f, g), \quad \|f\|_{I\pm T} = ((I \pm T)f, f)^{1/2}. \tag{3.20}$$

Since (3.20) define only seminorms, quotient spaces $\mathfrak{H}/\ker(I \pm T)$ have to be considered to obtain Hilbert spaces. This yields another result due to M.G. Kreĭn.

Corollary 3.7. ([26], cf. also [1]) *Assume $T \in \operatorname{Ext}_{T_1}(-1,1)$. Then $T = T_m$ if and only if $\mathfrak{H}_1$ is dense in $\mathfrak{H}_+$, and $T = T_M$ if and only if $\mathfrak{H}_1$ is dense in $\mathfrak{H}_-$.*

Remark 3.8. Let $T_1 = \binom{T_{11}}{T_{21}} \in [\mathfrak{H}_1, \mathfrak{H}]$ be a symmetric contraction presented in the block form with respect to $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. If $T \in \operatorname{Ext}_{T_1}(-1,1)$, then $f_1 \in \ker(I \pm T_{11})$ implies that $f_1 \oplus 0 \in \ker(I \pm T)$, cf. Remark 2.4. This indicates that the extension problem for $T_1 = \binom{T_{11}}{T_{21}} \in [\mathfrak{H}_1, \mathfrak{H}]$ could have been reduced to an extension problem with $\ker(I \pm T_{11}) = \{0\}$. In fact, let $\mathfrak{K}_1$ and $\mathfrak{K}_2$ be defined by

$$\mathfrak{K}_1 := \ker(I + T_{11}), \quad \mathfrak{K}_2 := \ker(I - T_{11}). \tag{3.21}$$

It follows from (3.8) that T_1 and T, when written in the block form with respect to the decomposition $\mathfrak{H} = \mathfrak{K}_1 \oplus \mathfrak{K}_2 \oplus \mathfrak{H}_1' \oplus \mathfrak{H}_2$, where $\mathfrak{H}_1' = \mathfrak{H}_1 \ominus (\mathfrak{K}_1 \oplus \mathfrak{K}_2)$, have the form

$$T_{11} = (-I_{\mathfrak{K}_1}) \oplus I_{\mathfrak{K}_2} \oplus T_{11}', \quad T_{21} = (0, 0, T_{21}'), \quad T_{12} = T_{21}^*,$$

with $T_{11}' \in [\mathfrak{H}_1']$, $T_{21}' \in [\mathfrak{H}_1', \mathfrak{H}_2]$. Hence, the description of the selfadjoint contractive extensions T of T_1 in $\mathfrak{H}$ is reduced to the description of the selfadjoint contractive extensions T' of $T_1' = \binom{T_{11}'}{T_{21}'}$ in $\mathfrak{H}' = \mathfrak{H}_1' \oplus \mathfrak{H}_2$.

Proposition 2.3 gives rise to a full analog for the contractions T_m and T_M. Here only a useful description for the kernels $\ker(I + T_m)$ and $\ker(I - T_M)$ will be given.

Lemma 3.9. *If* $\ker(I + T_{11}) = \{0\}$*, then*

$$\ker(I + T_m) = \left\{ \begin{pmatrix} (I + T_{11})^{-1}D_{T_{11}}V^* f_2 \\ -f_2 \end{pmatrix} : \, V^*f_2 \in \operatorname{ran}(I + T_{11})^{1/2}, \, f_2 \in \mathfrak{H}_2 \right\}.$$

If $\ker(I-T_{11})=\{0\}$, *then*

$$\ker(I-T_M)=\left\{\begin{pmatrix}(I-T_{11})^{-1}D_{T_{11}}V^*f_2\\ f_2\end{pmatrix} : V^*f_2\in\operatorname{ran}(I-T_{11})^{1/2},\ f_2\in\mathfrak{H}_2\right\}.$$

Proof. It follows from (3.11), (3.12), and Proposition 2.3 (or directly from (3.5)) that

$$\ker(I+T_m)=\left\{\begin{pmatrix}(I+T_{11})^{-1/2}S_+f_2\\ -f_2\end{pmatrix} : S_+f_2\in\operatorname{ran}(I+T_{11})^{1/2},\ f_2\in\mathfrak{H}_2\right\}.$$

Here $S_+=(I-T_{11})^{1/2}V^*$, so that $(I+T_{11})^{-1/2}S_+f_2=(I+T_{11})^{-1}D_{T_{11}}V^*f_2$. It is easy to check that

$$(I-T_{11})^{1/2}V^*f_2\in\operatorname{ran}(I+T_{11})^{1/2}\quad\Longleftrightarrow\quad V^*f_2\in\operatorname{ran}(I+T_{11})^{1/2}.$$

This proves the first statement. The proof of the second statement is similar. □

As a consequence one also obtains a description for the kernels $\ker(I+T_M)$ and $\ker(I-T_m)$.

Lemma 3.10. *Let* $\ker(I\pm T_{11})=\{0\}$ *and let* $f=f_1\oplus f_2\in\mathfrak{H}_1\oplus\mathfrak{H}_2$. *Then the following statements hold:*

(i) $f\in\ker(I+T_M)$ *if and only if* $f\in\ker(I+T_m)$ *and* $D_{V^*}f_2=0$;
(ii) $f\in\ker(I-T_m)$ *if and only if* $f\in\ker(I-T_M)$ *and* $D_{V^*}f_2=0$.

Moreover

(iii) $\ker(I+T_M)=\{0\}$ *if and only if* $\ker D_V\cap\operatorname{ran}(I+T_{11})^{1/2}=\{0\}$;
(iv) $\ker(I-T_m)=\{0\}$ *if and only if* $\ker D_V\cap\operatorname{ran}(I-T_{11})^{1/2}=\{0\}$.

Proof. (i) According to (3.16)

$$I+T_M=I+T_m+\begin{pmatrix}0&0\\0&2(I-VV^*)\end{pmatrix},$$

where the bounded linear operators $I+T_M$ and $I+T_m$ are nonnegative, and V^* is a contraction. Hence the statement follows from $\ker(I-VV^*)=\ker D_{V^*}$.

(iii) It follows from Lemma 3.9 and part (i) that $\ker(I+T_M)=\{0\}$ if and only if

$$\{f_2\in\ker D_{V^*} : V^*f_2\in\operatorname{ran}(I+T_{11})^{1/2}\}=\{0\}. \tag{3.22}$$

The assertion now follows from the identity $V^*(\ker D_{V^*})=\ker D_V$, cf. Corollary 3.1.

The proofs of (ii) and (iv) are similar to the proofs of (i) and (iii). □

3.4. **Completions and extensions.** Assume that $T_1=\binom{T_{11}}{T_{21}}\in[\mathfrak{H}_1,\mathfrak{H}]$ is a symmetric contraction presented in the block form with respect to $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$. Therefore its adjoint T_1^* is a closed linear relation. Let $T\in\operatorname{Ext}_{T_1}(-1,1)$, so that T is a contractive selfadjoint extension of $T_1\in[\mathfrak{H}_1,\mathfrak{H}]$. The following result is straightforward, cf. Lemma 2.10.

Lemma 3.11. *Let* $T\in\operatorname{Ext}_{T_1}(-1,1)$, *then*

$$T_1^*=T\,\widehat{+}\,(\{0\}\oplus\mathfrak{H}_2).$$

The next lemma contains some useful facts, which follow directly from Lemma 2.11.

Lemma 3.12. *Let* T_1, T_m *and* T_M *be as in Theorem 3.3. Then:*

(i) $\operatorname{ran}(I+T_m)=\operatorname{ran}(I+T_1)^*\cap\operatorname{ran}(I+T_m)^{1/2}$;
(ii) $\operatorname{ran}(I-T_M)=\operatorname{ran}(I-T_1)^*\cap\operatorname{ran}(I-T_M)^{1/2}$.

It follows from Lemma 3.11 that the defect subspace $\mathfrak{N}_z(T_1^*)$ at $z \in \rho(T_{11})$ is given by

$$\mathfrak{N}_z(T_1^*) = \ker(T_1^* - z) = \left\{ \begin{pmatrix} -(T_{11}-z)^{-1}T_{21}^* f_2 \\ f_2 \end{pmatrix} : \; f_2 \in \mathfrak{N} = (\operatorname{dom} T_1)^\perp \right\}.$$

Let $\widehat{\mathfrak{N}}_z(T_1^*) = \{ \{f, zf\} : f \in \mathfrak{N}_z(T_1^*) \}$ and define the special extension T_z of T_1 by

$$T_z = T_1 \,\widehat{+}\, \widehat{\mathfrak{N}}_z(T_1^*). \tag{3.23}$$

Lemma 3.13. *For $z \in \rho(T_{11})$ the extension T_z in* (3.23) *has the block representation*

$$T_z = \begin{pmatrix} T_{11} & T_{21}^* \\ T_{21} & T_{22}(z) \end{pmatrix}, \tag{3.24}$$

where

$$T_{22}(z) = zI + T_{21}(T_{11}-z)^{-1}T_{21}^*. \tag{3.25}$$

If $z \in \rho(T_{11})$, then $(T_z)^* = T_{\bar z}$, and if $z \in \rho(T_{11}) \cap \mathbb{R}$, then T_z is selfadjoint. Since T_{11} is a contraction each $z < -1$ or $z > 1$ belongs to $\rho(T_{11})$. A translation for Lemma 2.13 is obtained by using the following limits, cf. (3.15):

(i) $s-\lim_{r\uparrow 1} T_{21}(I + rT_{11})^{-1}T_{21}^* = V(I - T_{11})V^* = S_+^* S_+$;
(ii) $s-\lim_{r\uparrow 1} T_{21}(I - rT_{11})^{-1}T_{21}^* = V(I + T_{11})V^* = S_-^* S_-$.

Lemma 3.14. *Let T_z, $z \in \rho(T_{11})$, be given by* (3.24), (3.25). *Then*

$$s - \lim_{z\uparrow -1} T_z = T_m, \quad s - \lim_{z\downarrow 1} T_z = T_M. \tag{3.26}$$

If $T_1 \in [\mathfrak{H}_1, \mathfrak{H}]$ is a contraction, then rT_1 with $0 \le r < 1$ is also a contraction. The selfadjoint contractive extensions of rT_1 form a nonempty operator interval whose endpoints are denoted by $(rT)_m$ and $(rT)_M$. The next result concerns the limiting behaviour of these extensions as $r \to 1$.

Proposition 3.15. *Let $\mathfrak{H}_1$ be a closed linear subspace of $\mathfrak{H}$ and let $T_1 \in [\mathfrak{H}_1, \mathfrak{H}]$ be a symmetric contraction. Then*

(i) $s - \lim_{r\uparrow 1}(rT)_m = T_m$;
(ii) $s - \lim_{r\uparrow 1}(rT)_M = T_M$.

Proof. According to (3.5) the extension $(rT)_m$ of rT_1 has the block representation:

$$(rT)_m = \begin{pmatrix} rT_{11} & D_{rT_{11}}V^*(r) \\ V(r)D_{rT_{11}} & -I + V(r)(I - rT_{11})V^*(r) \end{pmatrix},$$

where the operator $V(r)$ is given by

$$V(r) = rT_{21}D_{rT_{11}}^{-1} = rVD_{T_{11}}D_{rT_{11}}^{-1},$$

cf. Theorem 3.3. An application of the Lebesgue dominated convergence theorem shows that for all $f \in \mathfrak{H}_1$,

$$\lim_{r\uparrow 1} \|rD_{T_{11}}D_{rT_{11}}^{-1}f - f\|^2 = \lim_{r\uparrow 1} \int_{-1}^{1} \left| r\sqrt{\frac{1-t^2}{1-r^2t^2}} - 1 \right|^2 d(E_t f, f) = 0,$$

where E_t stands for the spectral family of T_{11}. Consequently,

$$s - \lim_{r\uparrow 1} V(r) = V, \quad s - \lim_{r\uparrow 1} V^*(r) = V^*.$$

In view of (3.5) these equalities imply (i).

Part (ii) can be proved similarly. □

According to B.N. Parlett [32, p.252], W.M. Kahan has obtained the following formula

$$T_0 = \lim_{z\downarrow 1} \begin{pmatrix} T_{11} & T_{21}^* \\ T_{21} & -T_{21}(zI - T_{11}^2)^{-1}T_{11}T_{21}^* \end{pmatrix}$$

for one of the elements of the class $\mathrm{Ext}_{T_1}(-1,1)$ in the case that $\mathfrak{H}$ is finite-dimensional. It follows from (3.5) that $T_0 = (T_m + T_M)/2$.

3.5. **Kreĭn's uniqueness criterion.** Although the set $\mathrm{Ext}_{T_1}(-1,1)$ is a nonempty operator interval it may degenerate when the endpoints T_m and T_M coincide. This situation has been characterized by M.G. Kreĭn [26]. Here a geometric approach to obtain Kreĭn's criterion is presented.

The next result characterizes isometric operators in the class of all contractions.

Proposition 3.16. *For a contraction* $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$ *the following statements are equivalent:*

(i) T *is isometric;*

(ii) $\ker T = \{0\}$ *and* $\operatorname{ran} T \cap \operatorname{ran} D_{T^*} = \{0\}$*;*

(iii) *for some, and equivalently for every, subspace* $\mathfrak{L}$ *with* $\operatorname{ran} T \subset \overline{\mathfrak{L}}\,(=\operatorname{clos}\mathfrak{L})$ *one has*

$$\sup_{f\in\mathfrak{L}} \frac{|(f, T\varphi)|}{\|D_{T^*}f\|} = \infty \quad \text{for every } \varphi \in \mathfrak{H}_1\backslash\{0\}. \tag{3.27}$$

Proof. (i) ⇒ (iii) Let $\mathfrak{L}$ be an arbitrary subspace with $\operatorname{ran} T \subset \overline{\mathfrak{L}}$. Assume that the supremum in (3.27) is finite for some $\varphi \in \mathfrak{H}_1$. Then there exists $C > 0$, such that

$$|(f, T\varphi)| \le C\|D_{T^*}f\| \quad \text{for every } f \in \mathfrak{L}.$$

Since $\operatorname{ran} T \subset \overline{\mathfrak{L}}$, also the following inequality holds:

$$\|\varphi\|^2 = \|T\varphi\|^2 \le C\|D_{T^*}T\varphi\|. \tag{3.28}$$

Here $D_{T^*}^2 T\varphi = TD_T^2\varphi = 0$, since T is isometric. Therefore, (3.28) implies $\varphi = 0$. Consequently (3.27) holds for every $\varphi \neq 0$.

(iii) ⇒ (ii) Assume that (3.27) is satisfied with some subspace $\mathfrak{L}$. If (ii) does not hold, then either $\ker T \neq \{0\}$, in which case (3.27) does not hold for $0 \neq \varphi \in \ker T$, or $\operatorname{ran} T \cap \operatorname{ran} D_{T^*} \neq \{0\}$. However, then with $0 \neq T\varphi = D_{T^*}h$ the supremum in (3.27) is finite even if f varies over the whole space $\mathfrak{H}_2$. Thus, if (ii) does not hold then (3.27) fails to be true.

(ii) ⇒ (i) Let $\operatorname{ran} T \cap \operatorname{ran} D_{T^*} = \{0\}$. Then by Lemma 3.2 $TD_T = 0$ and it follows from $\ker T = \{0\}$ that $D_T = 0$, i.e., T is isometric. This completes the proof. □

As a direct consequence one obtains Kreĭn's uniqueness criterion.

Proposition 3.17. [26] *Let* $\mathfrak{H}_1$ *be a closed linear subspace of the Hilbert space* $\mathfrak{H}$ *and let* $T_1 \in [\mathfrak{H}_1, \mathfrak{H}]$ *be a symmetric contraction. Then* $T_m = T_M$ *if and only if*

$$\sup_{f\in\mathfrak{H}_1} \frac{|(T_1 f, \varphi)|^2}{\|f\|^2 - \|T_1 f\|^2} = \infty \quad \text{for every } \varphi \in \mathfrak{H}_2 \setminus \{0\}. \tag{3.29}$$

Proof. The definition of V in the proof of Theorem 3.3 (see (3.8), (3.9)) implies that

$$(T_1 f, \varphi) = (T_{21} f, \varphi) = (D_{T_{11}} f, V^* \varphi), \qquad \|f\|^2 - \|T_1 f\|^2 = \|D_V D_{T_{11}} f\|^2.$$

In view of (3.16) $T_m = T_M$ if and only if V^* is an isometry. Since $\operatorname{ran} V^* \subset \overline{\operatorname{ran}}\, D_{T_{11}}$, the assertion follows from the equivalence of (i) and (iii) in Proposition 3.16 with $\mathfrak{L} = \operatorname{ran} D_{T_{11}}$. □

3.6. **Extremal selfadjoint contractive extensions.** The next result contains a characterization for the extensions $T = T_m$ and $T = T_M$ among the class of all selfadjoint contractive extensions of T_1, which goes back to [26] and [27].

Proposition 3.18. [26, 27] *Let $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{N}$ and let $T \in \operatorname{Ext}_{T_1}(-1, 1)$. Then:*

(i) $T = T_m + (I + T)_{\mathfrak{N}}, \quad T = T_M - (I - T)_{\mathfrak{N}}$;
(ii) $T_M - T_m = (I + T_M)_{\mathfrak{N}} = (I - T_m)_{\mathfrak{N}}$;
(iii) $(D^2_{T_m})_{\mathfrak{N}} = 0, \quad (D^2_{T_M})_{\mathfrak{N}} = 0$.

Proof. (i) Let $T \in \operatorname{Ext}_{T_1}(-1, 1)$, so that $I \pm T \geq 0$. It follows from (2.8) and Proposition 2.5 that $I + T = I + T_m + (I + T)_{\mathfrak{N}}$ and $I - T = I - T_M + (I - T)_{\mathfrak{N}}$.

(ii) Apply the first equality in (i) to $T = T_M$ and the second equality in (i) to $T = T_m$.

(iii) Observe, that $D^2_{T_m} \leq 2(I + T_m)$ and $D^2_{T_M} \leq 2(I - T_M)$. Hence, by Proposition 2.7,

$$0 \leq (D^2_{T_m})_{\mathfrak{N}} \leq 2(I + T_m)_{\mathfrak{N}}, \quad 0 \leq (D^2_{T_M})_{\mathfrak{N}} \leq 2(I - T_M)_{\mathfrak{N}}. \tag{3.30}$$

By part (i) $(I + T_m)_{\mathfrak{N}} = (I - T_M)_{\mathfrak{N}} = 0$. Hence, (3.30) gives $(D^2_{T_m})_{\mathfrak{N}} = 0$, $(D^2_{T_M})_{\mathfrak{N}} = 0$. □

Part (i) of Proposition 3.18 gives the following characterization for T_m and T_M: if $T \in \operatorname{Ext}_{T_1}(-1, 1)$, then $T = T_m$ if and only if $(I+T)_{\mathfrak{N}} = 0$, and $T = T_M$ if and only if $(I-T)_{\mathfrak{N}} = 0$. Part (iii) deserves some further attention.

Lemma 3.19. *Let $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{N}$ and let $T_1 = \binom{T_{11}}{T_{21}} \in [\mathfrak{H}_1, \mathfrak{H}]$ be a symmetric contraction with* $\ker(I \pm T_{11}) = \{0\}$. *Let $T \in \operatorname{Ext}_{T_1}(-1, 1)$ and $K \in C_{\mathcal{H}}$ with $\mathcal{H} = \overline{\operatorname{ran}}\, D_{V^*}$ be connected via* (3.18) *or* (3.19). *Then*

$$(D^2_T)_{\mathfrak{N}} = \begin{pmatrix} 0 & 0 \\ 0 & D_{V^*} D^2_K D_{V^*} \end{pmatrix}.$$

Proof. A simple calculation using the correspondence (3.19), the commutation relations (3.1), and the selfadjointness of T_{11} and K leads to

$$D^2_T = I - T^* T = \begin{pmatrix} X^* \\ Y^* \end{pmatrix} (X\, Y) + \begin{pmatrix} 0 & 0 \\ 0 & D_{V^*} D^2_K D_{V^*} \end{pmatrix}, \tag{3.31}$$

where $X = D_V D_{T_{11}}$ and $Y = -(D_V T_{11} V^* + V^* K D_{V^*})$. Since $\ker(I \pm T_{11}) = \{0\}$, one has $\overline{\operatorname{ran}}\, D_{T_{11}} = \mathfrak{H}_1$ and consequently $\overline{\operatorname{ran}}\, X = \overline{\operatorname{ran}}\, D_V$. Therefore, $\operatorname{ran} D_V T_{11} V^* \subset \overline{\operatorname{ran}}\, X$. Furthermore, with $K \in [\mathcal{H}]$ and $\mathcal{H} = \overline{\operatorname{ran}}\, D_{V^*}$, it follows that

$$\operatorname{ran} V^* K D_{V^*} \subset V^*(\mathcal{H}) \subset \overline{\operatorname{ran}}\, V^* D_{V^*} = \overline{\operatorname{ran}}\, D_V V^* \subset \overline{\operatorname{ran}}\, D_V.$$

Hence, $\operatorname{ran} Y \subset \overline{\operatorname{ran}}\, X$. The result is now obtained from Example 2.6. □

The nonempty operator interval $\operatorname{Ext}_{T_1}(-1, 1)$ of all selfadjoint contractive extensions of T_1 is a closed convex set; the set of extreme points of $\operatorname{Ext}_{T_1}(-1, 1)$ will be denoted by $\operatorname{Ext}^E_{T_1}(-1, 1)$. According to (3.18) and (3.19) $T \in \operatorname{Ext}^E_{T_1}(-1, 1)$ if and only if $K = K^* = K^{-1}$, i.e. K is selfadjoint and unitary in $\mathcal{H}$. Clearly, $K = -I_{\mathcal{H}}$ and $K = I_{\mathcal{H}}$ correspond to the extensions T_m and T_M, respectively. In fact, one obtains from Lemma 3.19 the following characterization for the class of extremal selfadjoint contractive extensions of T_1.

Proposition 3.20. *Let* $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{N}$ *and let* $T \in \operatorname{Ext}_{T_1}(-1,1)$. *Assume that* $\ker(I \pm T_{11}) = \{0\}$. *Then the following statements are equivalent:*

(i) $T \in \operatorname{Ext}_{T_1}^{E}(-1,1)$;
(ii) $(D_T^2)_{\mathfrak{N}} = 0$;
(iii) $\inf_{g\in\mathfrak{H}_1} \|D_T(f-g)\| = 0$ *for every* $f \in \mathfrak{H}$.

Proof. It follows from (3.18) and (3.19) that $T \in \operatorname{Ext}_{T_1}^{E}(-1,1)$ if and only if $D_K = 0$. Hence, the equivalence of (i) and (ii) is obtained from Lemma 3.19.

The equivalence of (ii) and (iii) is a consequence of Proposition 2.5. □

4. Extensions of nonnegative operators and relations

In this section the main results of M.G. Kreĭn's work [26] on the extension theory of nonnegative densely defined operators in a Hilbert space are presented. In fact, the general case of nonnegative nondensely defined operators and multivalued extensions is considered, cf. [15]. Also some complements due to T. Ando and K. Nishio [7] are treated.

4.1. **Linear fractional transformations.** The extension theory of nonnegative relations can be connected to the extension theory of symmetric contractive operators. Define the linear fractional transformation X, taking a linear relation A into a linear relation $X(A)$, by

$$X(A) = \{\, \{f+f', f-f'\} :\ \widehat{f} = \{f,f'\} \in A \,\} = -I + 2(I+A)^{-1}. \tag{4.1}$$

Clearly, X maps the (closed) linear relations one-to-one onto themselves, $X^2 = I$, and

$$X(A)^{-1} = X(-A), \tag{4.2}$$

for every linear relation A. Moreover,

$$\begin{aligned} &\operatorname{dom} X(A) = \operatorname{ran}(I+A), \quad \operatorname{ran} X(A) = \operatorname{ran}(I-A),\\ &\ker(X(A)-I) = \ker A, \quad \ker(X(A)+I) = \operatorname{mul} A. \end{aligned} \tag{4.3}$$

In addition, X preserves closures, adjoints, componentwise sums, orthogonal sums, intersections, and inclusions. The relation $X(A)$ is symmetric if and only if A is symmetric. It follows from (4.1) and

$$\|f+f'\|^2 - \|f-f'\|^2 = 4\mathrm{Re}\,(f',f)$$

that X gives a one-to-one correspondence between nonnegative linear relations and symmetric contractions. Moreover, X provides a one-to-one correspondence between nonnegative selfadjoint relations and selfadjoint contractions.

4.2. **Nonnegative selfadjoint extensions of nonnegative relations.** Let A be a closed nonnegative relation in the Hilbert space $\mathfrak{H}$, and let the set of all nonnegative selfadjoint extensions $\widetilde{A} = \widetilde{A}^*$ of A be denoted by $\operatorname{Ext}_A(0,\infty)$. The linear fractional transformation $T_1 = X(A)$ of A defined by (4.1) is a symmetric contraction with $\operatorname{dom} T_1 = \operatorname{ran}(I+A)$. Moreover, the formula (4.1) gives a bijective correspondence between the contractive selfadjoint extensions $T \in \operatorname{Ext}_{T_1}(-1,1)$ of T_1 and the nonnegative selfadjoint extensions $\widetilde{A} =$

$\widetilde{A}^* \in \operatorname{Ext}(0,\infty)$ of $A \geq 0$. Let T_m and T_M be the selfadjoint contractive extensions of T_1 as in Theorem 3.3, and define the selfadjoint relations A_F and A_K by

$$A_F = X^{-1}(T_m) = -I + 2(I+T_m)^{-1}, \quad A_K = X^{-1}(T_M) = -I + 2(I+T_M)^{-1}. \tag{4.4}$$

The following theorem is a translation of the corresponding facts for selfadjoint contractive extensions of a contractive symmetric operator.

Theorem 4.1. *Let A be a closed nonnegative relation in $\mathfrak{H}$. Then* $\operatorname{Ext}_A(0,\infty)$ *is nonempty; in fact A_F and A_K belong to* $\operatorname{Ext}_A(0,\infty)$. *If $\widetilde{A} \in \operatorname{Ext}_A(0,\infty)$, then*

$$(A_F + a)^{-1} \leq (\widetilde{A} + a)^{-1} \leq (A_K + a)^{-1}, \quad a > 0. \tag{4.5}$$

Moreover, the selfadjoint extensions A_F and A_K of A are connected via

$$(A^{-1})_F = (A_K)^{-1}, \quad (A^{-1})_K = (A_F)^{-1}. \tag{4.6}$$

Proof. The fact that A_F and A_N are selfadjoint extensions of A follows from the preservation of adjoints and inclusions of the linear fractional transformation X. Moreover, since T_m and T_M are contractive, the extensions A_F and A_K are nonnegative, so that $\operatorname{Ext}_A(0,\infty)$ is nonempty. Let $\widetilde{A} \in \operatorname{Ext}_A(0,\infty)$ and let $T := X(\widetilde{A})$. Then $T \in \operatorname{Ext}_{T_1}(-1,1)$ and

$$(\widetilde{A} + a)^{-1} = \frac{1}{a-1} I - \frac{2}{(a-1)^2} \left(T + \frac{a+1}{a-1} \right)^{-1}, \quad a > 0,$$

where $|(a-1)^{-1}(a+1)| > 1$ for $a > 0$. Hence, (4.5) follows from (3.6). The relations (4.6) follow from (3.7), (4.2), and (4.4). □

The selfadjoint extensions A_F and A_K of A are called the Friedrichs (hard) and the Kreĭn-von Neumann (soft) extension, respectively. The extremal properties (4.5) of the Friedrichs and Kreĭn-von Neumann extensions have been discovered by Kreĭn [26] in the case when A is a densely defined operator. The case when A is not densely defined was considered by T. Ando and K. Nishio [7], and E.A. Coddington and H.S.V. de Snoo [15] (who allowed relation extensions). The formulas (4.6) can be found in [7] and [15].

4.3. **The Friedrichs and the Kreĭn-von Neumann extension.** The Friedrichs and the Kreĭn-von Neumann extension can be characterized via semibounded sesquilinear forms. The form domain generated by $A \geq 0$ is denoted by $\operatorname{dom}[A]$. It coincides with the completion of $\operatorname{dom} A$ with respect to the inner product $(f,g)_A = (f,g) + (Af,g)$, $f,g \in \operatorname{dom} A$, when A is an operator, and with respect to the inner product $(\widehat{f}, \widehat{g})_A = (f,g) + (f',g)$, where $\widehat{f} = \{f,f'\}$, $\widehat{g} = \{g,g'\} \in A$, when A is a linear relation. The corresponding form is denoted by $\widetilde{A}[\cdot,\cdot] =: \widetilde{A}[\cdot]$. Observe, that if $T = X(\widetilde{A})$ and $\widetilde{A} = \widetilde{A}^* \geq 0$ then

$$\operatorname{ran}(I+T)^{1/2} = \operatorname{dom} \widetilde{A}^{1/2}, \qquad \operatorname{ran}(I-T)^{1/2} = \operatorname{ran} \widetilde{A}^{1/2}. \tag{4.7}$$

For the Friedrichs extension A_F of A one has the following result.

Theorem 4.2. *Let A be a closed nonnegative relation in $\mathfrak{H}$. Then:*

(i) $\operatorname{dom}[A] = \operatorname{dom}[A_F]$;
(ii) $\widetilde{A}[\cdot] \leq A_F[\cdot]$ *and in particular* $\operatorname{dom}[A_F] \subset \operatorname{dom}[\widetilde{A}]$ *for every* $\widetilde{A} \in \operatorname{Ext}_A(0,\infty)$;
(iii) $A_F = \{\, \{f,f'\} \in A^* : \, f \in \operatorname{dom}[A] \,\}$, *in particular* $\operatorname{mul} A_F = \operatorname{mul} A^*$.

Moreover, A_F is the only extension $\widetilde{A} \in \operatorname{Ext}_A(0,\infty)$ for which $\operatorname{dom} \widetilde{A} \subset \operatorname{dom}[A]$.

Proof. Let $T = X(\widetilde{A}) = -I+2(I+A)^{-1}$. Then $\widehat{f} = \{f, f'\} \in \widetilde{A}$ is equivalent to $\{f+f', 2f\} \in I+T$. Therefore,

$$2\| \widehat{f} \|^2_{\widetilde{A}} = (f+f', 2f) = (g, (I+T)g) = \|g\|^2_{I+T}, \tag{4.8}$$

where $g = f + f' \in \operatorname{dom} T$. Now (i) and the last statement of the theorem follow from Corollary 3.7 and the identities $T_1 = X(A)$, $T_m = X(A_F)$.

(ii) As it is well known, cf. [26], [23] (see also [15]) this assertion is equivalent to the first inequality in (4.5).

(iii) Part (i) shows that $A_F \subset L := \{ \{f, f'\} \in A^* : f \in \operatorname{dom}[A] \}$. To prove the reverse inclusion assume that $\{f, f'\} \in L$. Then $f \in \operatorname{dom} A^* = \operatorname{ran}(I + T_1^*)$ and in addition $f \in \operatorname{dom} A_F^{1/2} = \operatorname{ran}(I+T_m)^{1/2}$ by (4.7). Now Lemma 3.12 gives $f \in \operatorname{ran}(I+T_m) = \operatorname{dom} A_F$. Therefore, $\{f, k\} \in A_F \subset A^*$ for some $k \in \mathfrak{H}$. This implies $\{0, f' - k\} \in A^*$, i.e. $f' - k \in \operatorname{mul} A^*$. It remains to prove that $\operatorname{mul} A^* = \operatorname{mul} A_F$, which then finally gives $\{f, f'\} \in A_F$.

By construction $\ker(I + T_m) = (\operatorname{ran}(I + T_1))^{\perp}$ and moreover $\operatorname{mul} A_F = \ker(I + T_m)$ and $\operatorname{dom} A = \operatorname{ran}(I + T_1)$, cf. (4.3). Therefore, $\operatorname{mul} A_F = (\operatorname{dom} A)^{\perp} = \operatorname{mul} A^*$. □

The Kreĭn-von Neumann extension A_K of A can be characterized analogously, by interchanging the roles of ranges and domains, cf. [26], [7], [15].

Theorem 4.3. *Let A be a closed nonnegative relation in $\mathfrak{H}$. Then:*

(i) $\operatorname{ran}[A] = \operatorname{ran}[A_K]$;

(ii) $A_K[\cdot] \le \widetilde{A}[\cdot]$ *and in particular* $\operatorname{dom}[\widetilde{A}] \subseteq \operatorname{dom}[A_K]$ *for every* $\widetilde{A} \in \operatorname{Ext}_A(0, \infty)$;

(iii) $A_K = \{ \{f, f'\} \in A^* : f' \in \operatorname{ran}[A] \}$, *in particular* $\ker A_K = \ker A^*$.

Moreover, A_K is the only extension $\widetilde{A} \in \operatorname{Ext}_A(0,\infty)$ for which $\operatorname{ran}\widetilde{A} \subset \operatorname{ran}[A]$.

Proof. Part (i) follows from $\operatorname{ran}[A] = \operatorname{dom}[A^{-1}] = \operatorname{dom}((A^{-1})_F)^{1/2} = \operatorname{ran} A_K^{1/2}$.

Part (ii) follows from the second inequality in (4.5), or from (ii) in Theorem 4.2 and the fact that $0 \le H_1 \le H_2$ is equivalent to $0 \le H_2^{-1} \le H_1^{-1}$, cf. [26], [23], and [15, Theorem 1].

Similarly (iii) and the last statement are obtained from the correspondings results in Theorem 4.2 by means of (4.6). □

The proof of the above theorem was based on Theorem 4.2 and the relations in (4.6). When one applies the same method as was used in the proof of Theorem 4.2 and the definitions of A_F and A_K in (4.4), one arrives at the following characterizations for A_F and A_K.

Corollary 4.4. *Let $\widehat{f} = \{f, f'\} \in A^*$ and let $\widehat{f}_A = \{f_A, f'_A\} \in A$. Then:*

(i) $\widehat{f} \in A_F$ *if and only if*

$$\inf\{ \|f - f_A\|^2 + (f' - f'_A, f - f_A) : \widehat{f}_A = \{f_A, f'_A\} \in A \} = 0; \tag{4.9}$$

(ii) $\widehat{f} \in A_K$ *if and only if*

$$\inf\{ \|f' - f'_A\|^2 + (f' - f'_A, f - f_A) : \widehat{f}_A = \{f_A, f'_A\} \in A \} = 0. \tag{4.10}$$

Proof. (i) This is just a reformulation from Theorem 4.2. In fact, (4.8) implies that

$$2\|f - f_A\|^2 + 2(f' - f'_A, f - f_A) = \|g - h\|^2_{I+T},$$

where $g = f + f'$ and $h = f_A + f'_A \in \operatorname{dom} T_1 = \mathfrak{H}_1$. Now (i) follows from Corollary 3.7.

(ii) Observe, that $\widehat{f} = \{f, f'\} \in \widetilde{A}$ if and only if $\{f + f', 2f'\} \in I - T$. Then with $g = f + f'$ one has $2[(f', f) + \|f'\|^2] = ((I - T)g, g)$ and this gives

$$2[(f' - f'_A, f - f_A) + \|f' - f'_A\|^2] = ((I - T)(g - h), g - h) = \|g - h\|^2_{I-T},$$

where $h = f_A + f'_A \in \mathfrak{H}_1$. Hence, the statement is again obtained from Corollary 3.7. □

Corollary 4.4 gives immediately the following descriptions for A_F and A_K.

Corollary 4.5. *Let A be a closed nonnegative relation in $\mathfrak{H}$. Then:*

(i) *The Friedrichs extension A_F admits the decomposition*

$$A_F = (PA)_F \oplus (\{0\} \oplus \operatorname{mul} A^*),$$

where P is the orthogonal projection from $\mathfrak{H}$ onto $\overline{\operatorname{dom}} A$. In particular, if A is an operator then the orthogonal operator part of A_F is $(PA)_F$.

(ii) *The Kreĭn-von Neumann extension A_K admits the decomposition*

$$A_K = \widehat{A}_K \oplus (\ker A^* \oplus \{0\}), \qquad \widehat{A} = \{\{\widehat{P} f, f'\} : \{f, f'\} \in A\},$$

where $\widehat{P}$ is the orthogonal projection from $\mathfrak{H}$ onto $\overline{\operatorname{ran}} A$. In particular, if $\ker A = \{0\}$ then the orthogonal injective part of A_K is $\widehat{A}_K$.

Proof. (i) Let $f' \in \operatorname{mul} A^*$. Then $\{0, f'\} \in A_F$ since

$$\|0 - f_A\|^2 + (f' - f'_A, 0 - f_A) = \|f_A\|^2 + (f'_A, f_A)$$

and the infimum in (4.9) is achieved when $\{f_A, f'_A\} = \{0, 0\}$.

(ii) Similarly for $f \in \ker A^*$ one has

$$\|0 - f'_A\|^2 + (0 - f'_A, f - f_A) = \|f'_A\|^2 + (f'_A, f_A)$$

and the infimum in (4.10) is achieved when $\{f_A, f'_A\} = \{0, 0\}$. □

Remark 4.6. (a) The construction of the extension A_F for a densely defined operator A contained in Theorem 4.2 goes back to K.O. Friedrichs [21]. It was extended to the setting of relations in [14]. The other assertions of Theorem 4.2 have been obtained by M.G. Krein [26] for a densely defined operator A.

(b) Assertions (ii) and (iii) of Theorem 4.3 have also been discovered by M.G. Krein [26] for a densely defined operator A.

(c) The description (4.9) of A_F has been discovered by K.O. Friedrichs [21], while the description (4.10) of A_K has been obtained by T. Ando and K. Nishio [7]. These descriptions are related to the following characterizations of $\operatorname{dom}[A]$ and $\operatorname{ran}[A]$, respectively:

$$f \in \operatorname{dom}[A] \quad \text{if and only if} \quad f_n \to f, \quad (A(f_n - f_m), f_n - f_m) \to 0, \quad m, n \to \infty,$$

$$g \in \operatorname{ran}[A] \quad \text{if and only if} \quad Af_n \to g, \quad (A(f_n - f_m), f_n - f_m) \to 0, \quad m, n \to \infty,$$

for some sequence $f_n \in \operatorname{dom} A$.

(d) Assertion (i) in Corollary 4.5 has been obtained in [14]. In the case of a positive definite operator $A \geq cI > 0$ the equality in part (ii) of Corollary 4.5 as well as the equality $\operatorname{dom}[A_K] = \operatorname{dom}[A] + \mathfrak{N}_0$, $\mathfrak{N}_a := \ker(A^* - a)$, has been discovered in [26]. Both of these identities have been generalized in [29, Corollary 5] to the case of a nonnegative operator $A \geq 0$ with compact inverse A^{-1}.

Note also the following equivalence (see [29]): $\operatorname{dom}[A_K] \supset \operatorname{dom} A^*$ if and only if the extensions A_F and A_K are transversal, i.e., $A^* = A_F \,\widehat{+}\, A_K$.

4.4. **Limiting characterizations of A_F and A_K.** The Friedrichs and the Kreĭn-von Neumann extensions A_F and A_K can be obtained as strong resolvent limits of a sequence of extensions of A, which is useful for instance for spectral considerations. Let

$$\widehat{\mathfrak{N}}_x = \{\, \{f_x, xf_x\} : \, f_x \in \mathfrak{N}_x \,\} \subset A^*, \quad x < 0, \tag{4.11}$$

be a defect subspace of A^* and define a selfadjoint extension of A by

$$\widetilde{A}_x = A \,\widehat{+}\, \widehat{\mathfrak{N}}_x. \tag{4.12}$$

Proposition 4.7. *Let A be a closed nonnegative relation in $\mathfrak{H}$ and let $\widetilde{A}_x$ be defined by* (4.12)*. Then*

$$A_K = s - R - \lim_{x\uparrow 0} \widetilde{A}_x, \quad A_F = s - R - \lim_{x\downarrow -\infty} \widetilde{A}_x. \tag{4.13}$$

Moreover, if $0 \in \widehat{\rho}(A)$ then $A_K = A \,\widehat{+}\, \widehat{\mathfrak{N}}_0$, and if A is a bounded operator then $A_F = A \,\widehat{+}\, (\{0\} \oplus \operatorname{mul} A^*)$.

Proof. Let $T_1 = X(A)$ be a symmetric contraction defined by (4.1) and for $x < 0$ let $\beta = (1-x)(1+x)^{-1} \in \mathbb{R} \setminus [-1,1]$. Then $T_\beta := T_1 \,\widehat{+}\, \widehat{\mathfrak{N}}_\beta(T_1) = X(\widetilde{A}_x)$ is a selfadjoint extension of T_1. Since $I + T_\beta = 2(I+\widetilde{A}_x)^{-1}$, the identities in (4.13) follow from Lemma 3.14:

$$\begin{aligned} s - \lim_{x\uparrow 0} 2(I+\widetilde{A}_x)^{-1} &= s - \lim_{\beta\downarrow 1}(I+T_\beta) = I + T_M = 2(I+A_K)^{-1}, \\ s - \lim_{x\downarrow -\infty} 2(I+\widetilde{A}_x)^{-1} &= s - \lim_{\beta\uparrow -1}(I+T_\beta) = I + T_m = 2(I+A_F)^{-1}. \end{aligned} \tag{4.14}$$

As to the second part, observe that if $0 \in \widehat{\rho}(A_K)$ then $s - \lim_{x\uparrow 0}(I+A_x)^{-1} = (I+A_0)^{-1}$. Similarly, if $A \in [\mathfrak{H}]$ then $s - \lim_{x\downarrow -\infty}(I+A_x)^{-1} = (I + A \,\widehat{+}\, (\{0\} \oplus \operatorname{mul} A^*))^{-1}$. □

Remark 4.8. The limiting results (4.13) in Proposition 4.7 for operator extensions are contained in [7] and the representation of A_K in case $0 \in \widehat{\rho}(A)$ has been discovered by M.G. Kreĭn [26] when A is a densely defined operator.

Observe that, when A is a bounded operator the representation of A_F can be also obtained from Lemma 2.10 by using part (iii) of Theorem 4.2. Moreover, Corollary 4.5 gives also the following orthogonal representations:

$$A_F = PA \oplus (\{0\} \oplus \operatorname{mul} A^*), \qquad A_K = \widehat{A} \oplus \widehat{\mathfrak{N}}_0,$$

if, respectively, A is a bounded operator and $0 \in \widehat{\rho}(A)$. Namely, then PA, respectively $\widehat{A}^{-1} = \widehat{P} A^{-1}$, is automatically selfadjoint.

In [15] it is shown that the linear relation $A \,\widehat{+}\, (\{0\} \oplus \operatorname{mul} A^*)$ is selfadjoint if and only if $\operatorname{dom} A = \overline{\operatorname{dom}}\, A \cap \operatorname{dom} A^*$, in which case this linear relation coincides with the Friedrichs extension A_F of A, and similarly that the linear relation $A \,\widehat{+}\, (\ker A^* \oplus \{0\})$ is selfadjoint if and only if $\operatorname{ran} A = \overline{\operatorname{ran}}\, A \cap \operatorname{ran} A^*$, in which case it coincides with the Kreĭn-von Neumann extension A_N of A.

4.5. **Positively closable operators.** Let A be a nonnegative relation in the Hilbert space $\mathfrak{H}$. Since $\operatorname{mul} A \subset \operatorname{mul} A^* = \operatorname{mul} A_F$, A_F is an operator if and only if A is densely defined. In this case, each $\widetilde{A} \in \operatorname{Ext}_A(0,\infty)$ is an operator. Moreover, if A is a nonnegative relation and some $\widetilde{A} \in \operatorname{Ext}_A(0,\infty)$ is an operator (so that also A is an operator), then A_K is an operator. For, if not then $\ker (A_K + a)^{-1} = \operatorname{mul} A_K$, $a > 0$, is not trivial, but then according to (4.5) also $\ker(\widetilde{A}+a)^{-1} = \operatorname{mul}\widetilde{A}$ is not trivial. Hence, if A is a nonnegative operator, then there exist selfadjoint operator extensions if and only if the Kreĭn-von Neumann extension A_K is an operator. According to [7] a nonnegative operator A is called *positively closable* if

$$\lim_{n\to\infty}(Af_n, f_n) = 0 \text{ and } \lim_{n\to\infty} Af_n = g \text{ imply } g = 0. \tag{4.15}$$

Note that $(Af_n, f_n) \to 0$, $n \to \infty$, implies $(A(f_n - f_m), f_n - f_m) \to 0$, as follows from the Cauchy-Schwarz inequality for $(A\cdot,\cdot)$, i.e., $|(Af_n, f_m)|^2 \le (Af_n, f_n)(Af_m, f_m)$. Recall that the usual characterization associated with A_K involves the limiting conditions $Af_n \to g$ and $(A(f_n - f_m), f_n - f_m) \to 0$, cf. Remark 4.6 (c). Clearly, if A is a nonnegative operator and A' is the operator induced by A in $\mathfrak{H}' = \mathfrak{H} \ominus \ker A$, then A is positively closable if and only if A' is positively closable. The following interpretation for positive closability in (4.16) does not seem to have been noted earlier in the literature.

Proposition 4.9. *Let A be a nonnegative operator in $\mathfrak{H}$. Then for a sequence $f_n \in \operatorname{dom} A$,*

$$\lim_{n\to\infty}(Af_n, f_n) = 0 \text{ and } \lim_{n\to\infty} Af_n = g \text{ if and only if } g \in \operatorname{mul} A_K, \tag{4.16}$$

and the following statements are equivalent:

(i) *A is positively closable;*
(ii) *the Kreĭn-von Neumann extension A_K of A is an operator.*

Moreover, if $\operatorname{ran} A$ *is closed, then (i) and (ii) are equivalent to*

(iii) $(A\varphi, \varphi) = 0 \Longrightarrow A\varphi = 0$.

Proof. Assume that $\lim_{n\to\infty}(Af_n, f_n) = 0$ and $\lim_{n\to\infty} Af_n = g$. Then the Cauchy-Schwarz inequality for $(A\cdot,\cdot)$ implies that for all $h \in \operatorname{dom} A$,

$$|(g,h)|^2 = \lim_{n\to\infty} |(Af_n, h)|^2 \le \lim_{n\to\infty}(Af_n, f_n)(Ah, h) = 0.$$

Hence, $g \perp \operatorname{dom} A$ so that $\{0, g\} \in A^*$. Now part (ii) of Corollary 4.4 shows that $\{0, g\} \in A_K$. Conversely, if $\{0, g\} \in A_K$ then the existence of a sequence $f_n \in \operatorname{dom} A$ with the desired limiting properties follows from (4.10). This proves (4.16).

The equivalence of (i) and (ii) is now clear, since in view of (4.16) the definition (4.15) of positive closability of $A \ge 0$ can be restated as: $\{0, g\} \in A_K$ implies $g = 0$.

Next assume that $\operatorname{ran} A$ is closed. Then $\lim_{n\to\infty} Af_n = g = A\varphi$ for some $\varphi \in \operatorname{dom} A$ and this implies that

$$(A\varphi, \varphi)^2 = \lim_{n\to\infty}(Af_n, \varphi)^2 \le \lim_{n\to\infty}(Af_n, f_n)(A\varphi, \varphi) = 0.$$

Therefore, the implication (4.15) reduces to (iii). □

Similarly one obtains an interpretation of the positive closability of A^{-1}.

Proposition 4.10. *Let A be a closed nonnegative relation in $\mathfrak{H}$ with* $\ker A = \{0\}$. *Then for a sequence $f_n \in \operatorname{dom} A$,*

$$\lim_{n\to\infty}(Af_n, f_n) = 0 \text{ and } \lim_{n\to\infty} f_n = f \text{ if and only if } f \in \ker A_F, \tag{4.17}$$

and the following statements are equivalent:

(i) *A^{-1} is positively closable;*
(ii) *the Friedrichs extension A_F of A satisfies* $\ker A_F = \{0\}$.

Moreover, if $\operatorname{dom} A$ *is closed, then (i) and (ii) are equivalent to*

(iii) $(A\varphi,\varphi)=0 \Longrightarrow \varphi=0$.

Positive closability can be translated for contractions via the transformation (4.1). This yields some further equivalent conditions given for positive closability in Proposition 4.9.

Proposition 4.11. *Let A be a nonnegative operator in $\mathfrak{H}$ and let $T_1=X(A)=\binom{T_{11}}{VD_{T_{11}}}$ be decomposed as in Theorem 3.3. Then the positive closability of A is equivalent to each of the following statements:*

(iv) $\lim_{n\to\infty}\|D_{T_1}\psi_n\|=0,\quad \lim_{n\to\infty}(I-T_1)\psi_n=g \Longrightarrow g=0$;

(v) $\ker D_V\cap \operatorname{ran}(I+T_{11})^{1/2}=\{0\}$.

Moreover, if $\operatorname{ran}(I-T_1)$ *is closed, then the positive closability of A is equivalent to*

(vi) $D_{T_1}\psi=0 \Longrightarrow (I-T_1)\psi=0$.

In particular, A_K is an operator if D_{V^*} in (3.19) *satisfies* $\ker D_{V^*}=\{0\}$.

Proof. To see that (iv) is equivalent to (4.15), observe that with $\varphi_n=(I+T_1)\psi_n$, $\psi_n\in\mathfrak{H}_1$,

$$\begin{aligned}(A\varphi_n,\varphi_n)&=((-I+2(I+T_1)^{-1})\varphi_n,\varphi_n)=((I+T_1)\psi_n,(I-T_1)\psi_n)=\|D_{T_1}\psi_n\|^2,\\ A\varphi_n&=-\varphi_n+2(I+T_1)^{-1}\varphi_n=(I-T_1)\psi_n.\end{aligned}\tag{4.18}$$

As to (v) observe, that $\operatorname{mul}A_K=\ker(I+T_M)$; cf. (4.3) and (4.4). Moreover, since $\ker(I+T_{11})=\ker(I+T_1)=\operatorname{mul}A=\{0\}$, the equivalence of the positive closability of A to (v) follows from part (iii) of Lemma 3.10.

To see (vi) notice that, if $\operatorname{ran}(I-T_1)=\operatorname{ran}A$ is closed, then it follows from part (iii) of Proposition 4.9 that the condition (iv) simplifies to (vi).

The last statement follows from (v) by taking into account (3.3) in Corollary 3.1. □

Similarly one obtains further equivalent conditions given for the positive closability of A^{-1} in Proposition 4.10.

Proposition 4.12. *Let A be a closed nonnegative relation in $\mathfrak{H}$ with* $\ker A=\{0\}$ *and let $T_1=X(A)=\binom{T_{11}}{VD_{T_{11}}}$ be decomposed as in Theorem 3.3. Then the positive closability of A^{-1} is equivalent to each of the following statements:*

(iv) $\lim_{n\to\infty}\|D_{T_1}\psi_n\|=0,\quad \lim_{n\to\infty}(I+T_1)\psi_n=g \Longrightarrow g=0$;

(v) $\ker D_V\cap \operatorname{ran}(I-T_{11})^{1/2}=\{0\}$.

Moreover, if $\operatorname{ran}(I+T_1)$ *is closed, then the positive closability of A^{-1} is equivalent to*

(vi) $D_{T_1}\psi=0 \Longrightarrow (I+T_1)\psi=0$.

In particular, $\ker A_F=\{0\}$ *if D_{V^*} in* (3.19) *satisfies* $\ker D_{V^*}=\{0\}$.

Remark 4.13. The translation of positive closability for contractions yields another approach to prove Propositions 4.9, 4.10, 4.11, and 4.12. For instance, the equivalence of (ii) in Proposition 4.9 and (iv) in Proposition 4.11 is seen as follows. Let

$$\lim_{n\to\infty}(I-T_1)\psi_n=\lim_{n\to\infty}(I-T_M)\psi_n=g.$$

Then for all $f\in\mathfrak{H}$,

$$\lim_{n\to\infty}((I-T_M^2)\psi_n,f)=\lim_{n\to\infty}((I-T_M)\psi_n,(I+T_M)f)=(g,(I+T_M)f),\tag{4.19}$$

while $\lim\limits_{n\to\infty}D_{T_1}\psi_n=0$ implies

$$\lim_{n\to\infty}|(I-T_M^2)\psi_n,f)|\le\lim_{n\to\infty}\|D_{T_1}\psi_n\|\,\|D_{T_M}f\|=0.\tag{4.20}$$

Together (4.19) and (4.20) show that $g \perp \operatorname{ran}(I+T_M)$, i.e., $g \in \ker(I+T_M)$. In particular, $\ker(I+T_M)$ $(= \operatorname{mul} A_K) = \{0\}$ guarantees $g = 0$.

Conversely, for each $g \in \ker(I+T_M)$ it is easy to construct a sequence $\psi_n \in \mathfrak{H}_1$ satisfying the limiting properties in part (iv) of Proposition 4.11. In fact, in view of Lemmas 3.9, 3.10,

$$g = \begin{pmatrix} (I-T_{11})^{1/2}\psi \\ -Vh \end{pmatrix},$$

where $h = (I+T_{11})^{1/2}\psi \in \ker D_V \cap \operatorname{ran}(I+T_{11})^{1/2}$. Now $\overline{\operatorname{ran}}\,(I-T_{11})^{1/2} = \mathfrak{H}_1$, since $\ker(I-T_{11})$ $(= \ker(I-T_1) = \ker A) = \{0\}$. Therefore,

$$\psi = \lim_{n\to\infty} (I-T_{11})^{1/2}\psi_n$$

for some sequence $\psi_n \in \mathfrak{H}_1$. Then $\lim_{n\to\infty} D_{T_{11}}\psi_n = (I+T_{11})^{1/2}\psi = h$ and consequently

$$\lim_{n\to\infty} \|D_{T_1}\psi_n\|^2 = \lim_{n\to\infty} \|D_V D_{T_{11}}\psi_n\|^2 = \|D_V h\|^2 = 0,$$

$$\lim_{n\to\infty} (I-T_1)\psi_n = \lim_{n\to\infty} \begin{pmatrix} (I-T_{11})\psi_n \\ -VD_{T_{11}}\psi_n \end{pmatrix} = g.$$

4.6. **Kreĭn's uniqueness result.** The following uniqueness criterion results from Proposition 3.17.

Theorem 4.14. ([26]) *Let A be a closed nonnegative relation in $\mathfrak{H}$. Then $A_F = A_K$ (i.e. $\operatorname{Ext}_A(0,\infty)$ consists of one element) if and only if for some (and hence for all) $a > 0$*

$$\sup_{\{f,f'\}\in A} \frac{|(f,\varphi)|^2}{(f',f)} = \infty \quad \textit{for every } \varphi \in \mathfrak{N}_{-a} \setminus \{0\}. \tag{4.21}$$

Proof. The result can be obtained from Proposition 4.14 by using the equivalence $\{f,f'\} \in A$ if and only if $\{f+f', 2f\} \in I + T_1$, and the identities

$$(T_1 g, \varphi) = ((T_1+I)g, \varphi) = 2(f,\varphi), \quad \|g\|^2 - \|T_1 g\|^2 = 4(f',f),$$

where $g = f + f'$ and $\varphi \in (\operatorname{dom} T_1)^\perp = \ker(I + A^*)$. □

Example 4.15. Let A be a symmetric operator in $\mathfrak{H}$, such that $\overline{\operatorname{dom}}\,A^2 = \mathfrak{H}$. Then the set $\operatorname{Ext}_{A^2}(0,\infty)$ consists of more than one element, and hence it is infinite. Actually, if $\varphi \in \mathfrak{N}_i(A)$, i.e. $\{\varphi, i\varphi\} \in A^*$, then clearly $\{\varphi, -\varphi\} \in (A^*)^2$. Since the inclusion $(A^*)^2 \subset (A^2)^*$ is always satisfied, it follows that $\{\varphi, -\varphi\} \in (A^2)^*$. Observe that

$$\frac{|(f,\varphi)|^2}{(A^2 f, f)} = \frac{|(Af,\varphi)|^2}{\|Af\|^2} \le \|\varphi\|^2 \quad \text{for every } f \in \operatorname{dom} A^2.$$

Thus the condition (4.21) is not satisfied and hence $(A^2)_F \ne (A^2)_K$.

It is easily seen that $A^*A,\ AA^* \in \operatorname{Ext}_{A^2}(0,\infty)$, and moreover $A^*A = (A^2)_F$, but in general $A_K \ne AA^*$. In particular, if $A \ge 0$ then $(A_F)^2 \ne (A^2)_F = A^*A$ and $(A^2)_K \ne (A_K)^2$. Moreover, the inequality $m(A^2) \ge m(A)^2$ holds true for the lower bounds $m(A)$ and $m(A^2)$ of the operators A and A^2, and here the equality may also occur. For more information concerning this and similar examples, see also [8].

It should be pointed out that even in the case $\overline{\operatorname{dom}}\,A = \mathfrak{H}$ it is possible that $\operatorname{dom} A^2 = \{0\}$. In [12] it is shown that each symmetric operator A_1 has a restriction A, such that $\overline{\operatorname{dom}}\,A = \mathfrak{H}$ and $\operatorname{dom} A^2 = \{0\}$.

Remark 4.16. The conditions $\overline{\operatorname{dom} A} = \mathfrak{H}$ and $n_+(A) = n_-(A) < \infty$ imply $(A^2)^* = (A^*)^2$. However, if $n_\pm(A) = \infty$, then the inclusion $\operatorname{dom}(A^*)^2 \subset \operatorname{dom}(A^2)^*$ is in general strict. For example, if $A = -\Delta_{\min}$ is the minimal Laplace operator on $L_2(\Omega)$, Ω being a bounded domain in $\mathbb{R}^n$ with a regular boundary, then $A^* = -\Delta_{\max}$ is the maximal operator with $\operatorname{dom} A^* = \{ f \in L_2(\Omega) : \Delta f \in L_2(\Omega) \}$ and the inclusion $(A^*)^2 \subset (A^2)^*$ is strict. Actually, it is clear that $\operatorname{dom}(A^*)^2 \subset \operatorname{dom} A^*$ but $\operatorname{dom}(A^2)^* \not\subset \operatorname{dom} A^*$. Indeed, by the closed graph theorem the inclusion $\operatorname{dom}(A^2)^* \subset \operatorname{dom} A^*$ is equivalent to the validity of the estimate

$$\|\Delta f\|_{L_2(\Omega)} \le c\left[\|\Delta^2 f\|_{L_2(\Omega)} + \|f\|_{L_2(\Omega)}\right] \quad \text{for all } f \in C^\infty(\overline{\Omega}). \tag{4.22}$$

However, according to Hörmander's Theorem [22] the last estimate is impossible. Hence, $\operatorname{dom}(A^2)^* \not\subset \operatorname{dom} A^*$ and consequently $(A^*)^2 \ne (A^2)^*$.

In fact, even the weaker inclusion $\cap_{j=2}^N \operatorname{dom}(A^j)^* \subset \operatorname{dom} A^*$ does not hold, i.e., for an arbitrary N, $\cap_{j=2}^N \operatorname{dom}(A^j)^* \not\subset \operatorname{dom} A^*$. Otherwise one would arrive at the following estimate

$$\|\Delta f\|_{L_2(\Omega)} \le c\left[\sum_{j=2}^N \|\Delta^j f\|_{L_2(\Omega)} + \|f\|_{L_2(\Omega)}\right] \quad \text{for all } f \in C^\infty(\overline{\Omega}), \tag{4.23}$$

which, as it is shown in [30], is not valid.

Note also that similar conclusions are valid for an arbitrary minimal symmetric differential operator $A = P(D)$ in $L^2(\Omega)$ with constant coefficients, which is not of the form $P(D) = P(\xi_1 D_1 + \ldots + \xi_n D_n)$ where $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $D_j := d/dx_j$. Namely, this result is implied by [30, Theorem 2] which guarantees the absence of the estimate (4.23) with Δ replaced by $P(D)$, $P(D)$ not being of the form mentioned above.

4.7. **Extremal selfadjoint extensions.** Denote by $\operatorname{Ext}_A^E(0,\infty)(\subset \operatorname{Ext}_A(0,\infty))$ the set of images of the operators $T \in \operatorname{Ext}_{T_1}^E(-1,1)$ under the transformation (4.1). Since T_m and T_M belong to $\operatorname{Ext}_{T_1}^E(-1,1)$, see Propositions 3.18, 3.20, the extensions A_F and A_K belong to $\operatorname{Ext}_A^E(0,\infty)$. The aim is to describe all extensions $\widetilde{A}$ in the class $\operatorname{Ext}_A^E(0,\infty)$.

The next result is well known, cf. e.g. [6], [7].

Lemma 4.17. *For a closed linear operator T in $\mathfrak{H}$ with $\ker T = \{0\}$ the following equivalence holds:*

$$C_g := \sup_{f \in \operatorname{dom} T} \frac{|(f,g)|}{\|Tf\|} < \infty \quad \Longleftrightarrow \quad g \in \operatorname{ran} T^*.$$

In this case $C_g = \|h\|$ with $h \in (\ker T^*)^\perp$ and $g = T^* h$.

Proposition 4.18. [7] *Let A be a closed nonnegative positively closable operator. Then*

$$\operatorname{dom} A_K^{1/2} = \operatorname{dom}[A_K] = \left\{ h \in \mathfrak{H} : C_h^2 := \sup_{f \in \operatorname{dom} A} \frac{|(Af,h)|^2}{(Af,f)} < \infty \right\}.$$

Moreover, the equality $\|A_K^{1/2} h\| = C_h$ is valid for each $h \in \operatorname{dom}[A_K]$.

Proof. Without loss of generality assume that $\ker A = \{0\}$. Let $B = A^{-1}$ and $\varphi = Af$. Then

$$C_h^2 = \sup_{f \in \operatorname{dom} A} \frac{|(Af,h)|^2}{(Af,f)} = \sup_{\varphi \in \operatorname{dom} B} \frac{|(\varphi,h)|^2}{(B\varphi,\varphi)} = \sup_{\varphi \in \operatorname{dom}[B]} \frac{|(\varphi,h)|^2}{\|B_F^{1/2}\varphi\|^2}. \tag{4.24}$$

By Lemma 4.17 the conditions $h \in \operatorname{ran} B_F^{1/2}$ and $C_h < \infty$ are equivalent and, moreover, for each $h \in \operatorname{ran} B_F^{1/2} = \operatorname{dom} A_K^{1/2}$ the equality $C_h = \|B_F^{-1/2} h\| = \|A_K^{1/2} h\|$ is satisfied. $\square$

It follows from Proposition 4.18 that

$$\inf\{\, \|A_K^{1/2}(f-h)\| : \, h \in \operatorname{dom} A \,\} = 0, \tag{4.25}$$

for every $f \in \operatorname{dom} A_K^{1/2}$. In fact, if A_K is not an operator then (4.25) still holds when $\|A_K^{1/2}(f-h)\|^2$ is replaced by the corresponding quadratic form $A_K[f-h] := A_K[f-h, f-h]$, cf. also (4.10) in Corollary 4.4. Actually, one has the following result.

Proposition 4.19. *For $A \geq 0$ the following statements are equivalent:*

(i) $\widetilde{A} \in \operatorname{Ext}_A^E(0,\infty)$*;*

(ii) *for every* $\{f, f'\} \in \widetilde{A}$*,*

$$\inf\{\, (f' - f'_A, f - f_A) : \, \{f_A, f'_A\} \in A \,\} = 0;$$

(iii) *the form associated with* $\widetilde{A}$ *satisfies* $\widetilde{A}[f,g] = A_K[f,g]$, $f, g \in \operatorname{dom} \widetilde{A}^{1/2}$.

Proof. Let $T = X(\widetilde{A})$ and let $\{f, f'\} \in \widetilde{A}$. Then equivalently $\{f + f', 2f\} \in I + T$, i.e., $(I+T)g = 2f$ for $g = f + f'$, and moreover,

$$\|D_T g\|^2 = (D_T^2 g, g) = 4(f', f).$$

Thus, with $\{f_A, f'_A\} \in A$ and $h = f_A + f'_A$ one can write $\|D_T(g-h)\|^2 = 4(f' - f'_A, f - f_A)$. Therefore, the equivalence of (i) and (ii) is obtained from Proposition 3.20.

To see that (ii) implies (iii) assume that $\{f, f'\} \in \widetilde{A}$. Then it follows from $A_K \leq \widetilde{A}$ that for every $\varepsilon > 0$ there exists $\{f_A, f'_A\} \in A$, such that

$$0 \leq A_K[f-h] \leq (f' - f'_A, f - f_A) < \varepsilon.$$

This implies that $0 \leq (f', f) - A_K[f] < \varepsilon$. Therefore,

$$\widetilde{A}[f] = (f', f) = A_K[f] \tag{4.26}$$

and (iii) follows from the polarization identity.

Conversely, assume that (iii) is satisfied. Then (4.26) holds for every $\{f, f'\} \in \widetilde{A}$ and consequently $(f' - f'_A, f - f_A) = A_K[f - f_A]$ with $\{f_A, f'_A\} \in A$. Now (ii) follows from the fact that the form $A_K[\cdot]$ associated with A_K satisfies the condition in (ii). □

In the case where $A \geq 0$ is a densely defined operator the results in Proposition 4.19 can be found in the literature (cf. e.g. [8, Theorem 4.4] and the references therein).

4.8. **The completion problem revisited.** The extension theory presented above was built on the (minimal) solution to the completion problem (2.1) for A^0. The completion problem for A^0 can be viewed as an extension problem for the bounded nonnegative operator $A_1 = \binom{A_{11}}{A_{21}}$, $\operatorname{dom} A_1 = \mathfrak{H}_1$, in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and can be restated as follows: are there bounded nonnegative selfadjoint extensions for A_1? According to Proposition 2.1 the criterion is $\operatorname{ran} A_{21}^* \subset \operatorname{ran} A_{11}^{1/2}$. In this case, the solution A_{min} in (2.7) is not only minimal in the class of all (bounded nonnegative) completions of A_1, but also minimal in the wider class of all nonnegative selfadjoint extensions of A_1, that is, A_{min} coincides with the Kreĭn-von Neumann extension A_K of A_1. By allowing arbitrary nonnegative selfadjoint (operator) completions, one arrives at a completion problem for A^0 which in terms of A_1 can be restated as follows: when is A_1 positively closable? The next result gives a solution to this problem by means of A_{11} and A_{21}; hence it is a generalization of Proposition 2.1.

Proposition 4.20. *Let* $A_1 = \binom{A_{11}}{A_{21}}$ *be a bounded nonnegative operator in* $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ *with* $\operatorname{dom} A_1 = \mathfrak{H}_1$ *and assume that* $\ker A_{11} = \{0\}$*. Then the following conditions are equivalent:*

(i) A_1 *is positively closable;*

(ii) $T := A_{21}A_{11}^{-1/2}$ *is a densely defined closable operator in* $\mathfrak{H}_1$*;*

(iii) $S := A_{11}^{-1/2}A_{21}^*$ *is a densely defined operator in* $\mathfrak{H}_2$*.*

In this case, A^0 *in* (2.1) *has a minimal completion* A_{min} *given by*

$$A_{min} = A_K = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & S^*S \end{pmatrix},$$

and moreover the closed form corresponding to A_K *is* $t_K[f,f] = \|A_{11}^{1/2}f_1 + Sf_2\|^2$*.*

Proof. (a) First the equivalence (i) $\Longleftrightarrow$ (ii) will be proved. Assume that (i) holds. Let $\{h_n, g_n\} \in A_{21}A_{11}^{-1/2}$, such that $\{h_n, g_n\} \to \{0, g\}$ as $n \to \infty$. Let $f_n = A_{11}^{-1/2}h_n$. Then $\lim_{n\to\infty} A_{11}^{1/2}f_n = 0$ and $\lim_{n\to\infty} A_{21}f_n = \lim_{n\to\infty} g_n = g$. Consequently the sequence f_n satisfies the limiting conditions in (4.15). Therefore, $g = 0$ by positive closability of A_1. This shows that $A_{21}A_{11}^{-1/2}$ is closable.

Conversely, assume that (ii) holds and let $f_n \in \mathfrak{H}_1$ be a sequence with the limiting properties in (4.15). Let $g_n = A_{21}f_n$ and $h_n = A_{11}^{1/2}f_n$. Then $\{h_n, g_n\} \in A_{21}A_{11}^{-1/2}$. Moreover, $\lim_{n\to\infty} h_n = \lim_{n\to\infty} A_{11}^{1/2}f_n = 0$ and $g = \lim_{n\to\infty} A_{21}f_n$. Hence, $\{0, g\}$ belongs to the closure of $A_{21}A_{11}^{-1/2}$ and therefore $g = 0$ by the closability of $A_{21}A_{11}^{-1/2}$. This proves that A_1 is positively closable.

(b) To prove the equivalence (ii) $\Longleftrightarrow$ (iii) observe that T is densely defined and $T^* = (A_{21}A_{11}^{-1/2})^* = A_{11}^{-1/2}A_{21}^* = S$ since A_{21} is bounded. Thus, if T is closable then $S = T^*$ is densely defined.

Conversely, if S is densely defined then T as a densely defined operator is closable and $\operatorname{clos}(T) = \operatorname{clos}(A_{21}A_{11}^{-1/2}) = S^*$.

(c) Clearly,

$$\widetilde{A} := \begin{pmatrix} A_{11}^{1/2} \\ S^* \end{pmatrix} \begin{pmatrix} A_{11}^{1/2} & S \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & S^*S \end{pmatrix}$$

is a nonnegative selfadjoint extension of A_1. Moreover, it is easy to check that

$$\operatorname{ran}[A_1] = \operatorname{ran}\begin{pmatrix} A_{11}^{1/2} \\ S^* \end{pmatrix} = \operatorname{ran}\widetilde{A}^{1/2}.$$

Hence, $\operatorname{ran}\widetilde{A} \subset \operatorname{ran}[A_1]$. Now Theorem 4.3 shows that $\widetilde{A} = A_K$.

The operator $\begin{pmatrix} A_{11}^{1/2} & S \end{pmatrix}$ is closed, since A_{11} is bounded and S is closed. Therefore, the closed form associated with A_K is $t_K[f,f] = \|A_{11}^{1/2}f_1 + Sf_2\|^2$, $f = f \oplus f_2 \in \mathfrak{H}$. □

Example 4.21. Let the bounded nonnegative operator A_1 in $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ be given by $A_1 = \binom{A_{11}}{I}$ with $\operatorname{dom} A_1 = \mathfrak{H}_1 = \mathfrak{H}_2$. Then $\operatorname{ran} A_1$ is closed, since it follows from $g = \lim_{n\to\infty} A_1 f_n$, $f_n \in \operatorname{dom} A$, that

$$g_2 := \lim_{n\to\infty} A_{21}f_n = \lim_{n\to\infty} f_n, \quad g_1 := \lim A_{11}f_n = A_{11}g_2,$$

i.e., $g = A_1 g_2$. By part (iii) of Proposition 4.9 A_1 is positively closable if and only if $(A_1\varphi,\varphi) = 0$ implies $A_1\varphi = 0$. Clearly, this condition is now equivalent to $\ker A_{11} = \{0\}$. In this case $S = A_{11}^{-1/2} = S^*$ is a closed densely defined operator and A_K is given by

$$A_{min} = A_K = \begin{pmatrix} A_{11}^{1/2} \\ S^* \end{pmatrix} \begin{pmatrix} A_{11}^{1/2} & S \end{pmatrix} = \begin{pmatrix} A_{11} & I \\ I & A_{11}^{-1} \end{pmatrix}.$$

It is bounded if and only if A_{11} is boundedly invertible.

Example 4.22. Let $A_1 = \binom{A_{11}}{(\cdot,g)g}$ be a bounded nonnegative operator in $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ with $g \in \mathfrak{H}_1 = \mathfrak{H}_2 = \operatorname{dom} A_1$ and assume that $\ker A_{11} = \{0\}$. Then $(A_1\varphi,\varphi) = 0$ implies $A_1\varphi = 0$, but A_1 need not be positively closable since $\operatorname{ran} A_1$ is not closed in general (compare with Proposition 4.9).

In fact, Proposition 4.20 shows that A_1 is positively closable if and only if $g \in \operatorname{ran} A_{11}^{1/2}$. In this case $S = (\cdot, g)A_{11}^{-1/2}g \in [\mathfrak{H}_1]$, $S^* = (\cdot, A_{11}^{-1/2}g)g$, and

$$A_{min} = A_K = \begin{pmatrix} A_{11} & (\cdot,g)g \\ (\cdot,g)g & \|A_{11}^{-1/2}g\|^2(\cdot,g)g \end{pmatrix}.$$

If $g \notin \operatorname{ran} A_{11}^{1/2}$ then $S = 0\upharpoonright \{g\}^\perp$ is not densely defined and A_K is a selfadjoint relation.

5. Appendix

The aim of this appendix is to further clarify the connection between the completions of A^0 in (2.1) and the extensions of $A_1 = \binom{A_{11}}{A_{21}}$. For this purpose boundary triplets and Weyl functions will be used. For these notions the reader is referred to [17], [16], and especially to [18, Section 3], where closely related results can be found.

Proposition 5.1. *Let $A_1 = \binom{A_{11}}{A_{21}}$ be a bounded nonnegative operator in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ with $\operatorname{dom} A_1 = \mathfrak{H}_1$ and let*

$$\widetilde{A}_0 = \begin{pmatrix} A_{11} & A_{21}^* \\ A_{21} & A_{22} \end{pmatrix}$$

be a bounded selfadjoint (not necessarily nonnegative) operator extension of A_1. Then:

(i) *A_1 has equal defect numbers (d, d), $d = \dim \mathfrak{H}_2 \le \infty$.*

(ii) *The adjoint linear relation A_1^* of A_1 is given by*

$$A_1^* = \{\, \widehat{f} = \{f, \widetilde{A}_0 f + h\} : f \in \mathfrak{H},\ h \in \mathfrak{H}_2 \,\}.$$

(iii) *A boundary triplet for A_1^* is given by $\Pi = \{\mathfrak{H}_2, \Gamma_0, \Gamma_1\}$, where*

$$\Gamma_0 \widehat{f} = -h, \quad \Gamma_1 \widehat{f} = f_2; \quad f = f_1 \oplus f_2 \in \mathfrak{H}, \quad \widehat{f} = \{f, \widetilde{A}_0 f + h\} \in A_1^*.$$

(iv) *The corresponding γ-field is $\gamma(\lambda) = (\widetilde{A} - \lambda)^{-1}\upharpoonright \mathfrak{H}_2$ and the Weyl function is given by*

$$M(\lambda) = P_2(\widetilde{A}_0 - \lambda)^{-1}\upharpoonright \mathfrak{H}_2 = \left(A_{22} - z - A_{21}(A_{11} - z)^{-1}A_{21}^*\right)^{-1},$$

where P_2 stands for the orthogonal projection onto $\mathfrak{H}_2$.

(v) *The selfadjoint extensions $\widetilde{A}_\tau = \ker(\Gamma_0 + \tau\Gamma_1)$ of A_1 in $\mathfrak{H}$ are in one to one correspondence with the selfadjoint relations τ in $\mathfrak{H}_2$ via*

$$\widetilde{A}_\tau = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} + \tau \end{pmatrix},$$

and their resolvents are connected by

$$(\widetilde{A}_\tau-\lambda)^{-1}=(\widetilde{A}_0-\lambda)^{-1}-\gamma(\lambda)(M(\lambda)+\tau^{-1})^{-1}\gamma(\bar\lambda)^*,\quad \lambda\in\rho(\widetilde{A}_\tau)\cap\rho(\widetilde{A}_0).$$

Proof. (i) It is easy to check that $\mathfrak{N}_\lambda(A_1^*)=\ker(A_1^*-\lambda)$ is equal to $(\widetilde{A}_0-\lambda)^{-1}\mathfrak{H}_2$, $\lambda\in\rho(\widetilde{A}_0)$. Hence, $d_+=d_-=\dim\mathfrak{N}_\lambda(A_1^*)=\dim\mathfrak{H}_2$, $\lambda\in\widehat{\rho}(A_1)$.

(ii) The form of the adjoint A_1^* is obtained from Lemma 2.10.

(iii) Let $\widehat{f}=\{f,f'\}=\{f,\widetilde{A}_0f+h\}\in A_1^*$ and $\widehat{g}=\{g,g'\}=\{g,\widetilde{A}_0g+k\}\in A_1^*$. Then

$$(f',g)-(f,g')=(\widetilde{A}_0f+h,g)-(f,\widetilde{A}_0g+k)=(h,g)-(f,k)=(\Gamma_1\widehat{f},\Gamma_0\widehat{g})-(\Gamma_0\widehat{f},\Gamma_1\widehat{g}),$$

so that the abstract Green's identity holds. The surjectivity of the mapping $\Gamma:=(\Gamma_0,\Gamma_1):A_1^*\to\mathfrak{H}_2\oplus\mathfrak{H}_2$ is clear.

(iv) Each $\widehat{f}_\lambda\in\widehat{\mathfrak{N}}_\lambda(A_1^*)=\{\,\{f_\lambda,\lambda f_\lambda\}:\ f_\lambda\in\mathfrak{N}_\lambda(A_1^*)\,\}$ can be represented as

$$\{f_\lambda,\lambda f_\lambda\}=\{(\widetilde{A}_0-\lambda)^{-1}h,\lambda(\widetilde{A}_0-\lambda)^{-1}h\}=\{(\widetilde{A}_0-\lambda)^{-1}h,\widetilde{A}_0(\widetilde{A}_0-\lambda)^{-1}h-h\}.$$

Hence,

$$\Gamma_0\widehat{f}_\lambda=h,\quad \Gamma_1\widehat{f}_\lambda=P_2(\widetilde{A}_0-\lambda)^{-1}h,$$

which gives the indicated form for $\gamma(\lambda)$ and $M(\lambda)$.

(v) Since τ is a linear relation the set $\ker(\Gamma_0+\tau\Gamma_1)$ is equal to

$$\{\,\widehat{f}\in A_1^*:\ \{\Gamma_1\widehat{f},-\Gamma_0\widehat{f}\}\in\tau\,\}=\{\,\widehat{f}\in A_1^*:\ \{f_2,h\}\in\tau\,\},$$

which implies the desired form for $\widetilde{A}_\tau=\ker(\Gamma_0+\tau\Gamma_1)$. The last statement is just Kreĭn's resolvent formula. □

Passing to the boundary triplet $\widetilde{\Pi}=\{\mathfrak{H}_2,\Gamma_1,-\Gamma_0\}$ one obtains another Weyl function for A_1 which is of the form

$$M_F(\lambda)=-M(\lambda)^{-1}=-A_{22}+z+A_{21}(A_{11}-z)^{-1}A_{21}^*. \tag{5.1}$$

In fact, $M_F(\lambda)$ is associated with the Friedrichs extension of A_1, i.e., $\ker\Gamma_1=A_F$.

Proposition 5.2. *Let the assumptions and notations be as in Proposition 5.1, let M_F be given by* (5.1), *let* $\ker A_{11}=\{0\}$, *and let* $S=A_{11}^{-1/2}A_{21}^*$. *Then:*

(i) *there exists the strong resolvent limit*

$$M_F(0):=s-R-\lim_{x\uparrow 0}M_F(x)=S^*S-A_{22};$$

(ii) *the Friedrichs and the Kreĭn-von Neumann extension of A_1 are given by*

$$A_F=\ker\Gamma_1=\begin{pmatrix}A_{11}&A_{12}\\A_{21}&\{0\}\oplus\mathfrak{H}_2\end{pmatrix},\quad A_K=\ker(\Gamma_0+M_F(0)\Gamma_1)=\begin{pmatrix}A_{11}&A_{12}\\A_{21}&M_F(0)\end{pmatrix}.$$

Proof. (i) It is easy to see that

$$\lim_{x\uparrow 0}(A_{21}(A_{11}+x)^{-1}A_{21}^*h,h)=\lim_{x\uparrow 0}((A_{11}+x)^{-1}A_{21}^*h,A_{21}^*h)=(Sh,Sh).$$

This implies that the linear relation S^*S associated with the closed nonnegative form (Sh,Sh) is the strong resolvent limit $s-R-\lim_{x\uparrow 0}A_{21}(A_{11}+x)^{-1}A_{21}^*=S^*S$, see [18, Proposition 4.1]. Now (5.1) shows that $s-R-\lim_{x\uparrow 0}M_F(x)=S^*S-A_{22}$.

(ii) The statement for A_F follows directly from Proposition 4.7, since $A_F=\ker(\Gamma_0+\tau\Gamma_1)$ must correspond to the selfadjoint relation $\tau=\{0\}\oplus\mathfrak{H}_2$. Moreover, for $x<0$ the proof of Proposition 5.1 shows that $\widehat{f} \in \widetilde{A}_x = A \widehat{+} \widehat{\mathfrak{N}}_x$ if and only if $\{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\} \in M(x)$. Equivalently, $\{\Gamma_1 \widehat{f}, -\Gamma_0 \widehat{f}\} \in M_F(x)$, i.e., $\widehat{f} \in \ker(\Gamma_0 + M_F(x)\Gamma_1)$. Now (4.13) implies

$$A_K = s - R - \lim_{x\uparrow 0} \widetilde{A}_x = \ker(\Gamma_0 + M_F(0)\Gamma_1),$$

where the second identity can be shown with the aid of Kreĭn's formula, cf. [18, p. 172]. □

Corollary 5.3. *Let A_1 be as in Proposition 5.1 with $\ker A_{11} = \{0\}$ and let $S = A_{11}^{-1/2}A_{21}^*$. Then the Friedrichs and the Kreĭn-von Neumann extension A_F and A_K of A_1 satisfy the following statements:*

(i) *$A_K = A_F$ if and only if $M_F(0) = \{0\} \oplus \mathfrak{H}_2$, or equivalently, $\operatorname{dom} S = \{0\}$;*
(ii) *A_K is an operator if and only if $M_F(0)$ is an operator in $\mathfrak{H}_2$, or equivalently, S is a densely defined operator in $\mathfrak{H}_2$;*
(iii) *A_K is a bounded operator if and only if $M_F(0)$ is a bounded operator in $\mathfrak{H}_2$, or equivalently, $S \in [\mathfrak{H}_2]$.*

Observe, that if $A_{22} = 0$ then $M_F(0)$ is associated with the closed nonnegative form

$$\mathfrak{t}[h,h] = (Sh, Sh) = \lim_{x\uparrow 0}(M_F(x)h, h),$$

where

$$\operatorname{dom}\mathfrak{t} = \operatorname{dom} S = \{\, h \in \mathfrak{H}_2 : \lim_{x\uparrow 0}(M_F(x)h,h) < \infty \,\}, \quad (\operatorname{dom}\mathfrak{t})^\perp = \mathfrak{H}_2 \ominus \overline{\operatorname{dom}}\, S = \operatorname{mul} M_F(0).$$

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