Journal of Theoretical Probability (2025) 38:51
https://doi.org/10.1007/s10959-025-01422-z

Discretization of integrals driven by multifractional Brownian motions with discontinuous integrands

Kostiantyn Ralchenko[1,2] · **Foad Shokrollahi**[1] · **Tommi Sottinen**[1]

Received: 8 August 2024 / Revised: 11 March 2025 / Accepted: 1 May 2025

Abstract
We establish the rate of convergence in the L^1-norm for equidistant approximations of stochastic integrals with discontinuous integrands driven by multifractional Brownian motion. Our findings extend the known results for the case when the driver is a fractional Brownian motion.

Keywords Approximation of stochastic integral · Discontinuous integrands · Rate of convergence · Multifractional Brownian motions

Mathematics Subject Classification (2020) 60G15 · 60G22 · 62F12 · 62M09

1 Introduction

We consider equidistant approximations of stochastic integrals driven by multifractional Brownian motion with discontinuous integrands. Specifically, we establish the rate of convergence for equidistant approximations of pathwise stochastic integrals:

$$\int_0^1 \Psi'(X_s)dX_s \approx \sum_{k=1}^{n} \Psi'\left(X_{t_{k-1}}\right)\left(X_{t_k} - X_{t_{k-1}}\right),$$

✉ Kostiantyn Ralchenko
kostiantyn.ralchenko@uwasa.fi

Foad Shokrollahi
foad.shokrollahi@uwasa.fi

Tommi Sottinen
tommi.sottinen@uwasa.fi

1 School of Technology and Innovations, University of Vaasa, P.O. Box 700, Vaasa FIN-65101, Finland

2 Department of Probability, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, Volodymyrska St., Kyiv 01601, Ukraine

Published online: 23 May 2025

where $t_k = \frac{k}{n}$, $k = 0, 1, \ldots, n$. Here, Ψ represents a difference of convex functions, and X denotes a multifractional Brownian motion (see Sect. 2 for details). The integral is interpreted as a pathwise Stieltjes integral, following the integration theory for discontinuous integrands developed in [6] using a modification of Zähle's fractional integration theory [19, 20].

Recently, a similar problem was addressed in [2] for the case when the driving process X is centered, Gaussian and Hölder continuous of order $H > \frac{1}{2}$. Additionally, in [2], X satisfies the following conditions: its variance function $V(t)$ is non-decreasing on $[0, 1]$, $V(1) = 1$, and its variogram function is represented as:

$$\mathbb{E}(X_t - X_s)^2 = \sigma^2 |t-s|^{2H} + o\left(|t-s|^{2H}\right), \quad \text{as } |t-s| \to 0.$$

Examples of such processes include fractional, bifractional, and sub-fractional Brownian motions, the fractional Ornstein–Uhlenbeck process, and normalized multi-mixed fractional Brownian motion, among others. In [2], the exact rate of convergence for approximations of stochastic integrals in the L^1-distance is found to be proportional to n^{1-2H}, which corresponds to the known rate in the case of smooth integrands (see [10] and references therein). Notably, for the case of fractional Brownian motion, this problem was studied earlier in [1]. For other related studies on stochastic integrals with discontinuous integrands, see also [11, 12, 16, 18].

In this paper, we focus on approximating integrals driven by multifractional Brownian motion. This process generalizes fractional Brownian motion by allowing the Hurst index to vary over time. Such a generalization enables the modeling of stochastic processes whose path regularity and "memory depth" evolve over time. In this case, the variance function of the process is $V(t) = t^{2H_t}$, which is generally non-monotone. Consequently, the direct application of results from [2] is infeasible, as the proofs there rely on the monotonicity of $V(t)$. However, by exploiting the specific form of the variance function, we can address these challenges and establish a rate of convergence proportional to n^{1-2H} with $H = \min\{\min_t H_t, \alpha\}$, where α is a Hölder exponent of H_t. To achieve this, we adapt the general proof scheme from [2], but significantly modify and generalize the auxiliary results to accommodate a process with non-monotone variance.

The paper is organized as follows: In Sect. 2, we review various definitions of multifractional Brownian motion and outline its properties necessary for the subsequent sections. Section 3 presents the statement of our main result. The proofs are provided in Sect. 4.

2 Multifractional Brownian motion: Definition and examples

Let $H\colon [0, 1] \to (\frac{1}{2}, 1)$ be a continuous function satisfying the following assumptions:

(A1) $H_{\min} := \min_{t\in[0,1]} H_t > \frac{1}{2}$ and $H_{\max} := \max_{t\in[0,1]} H_t < 1$.

(A2) There exist constants $C > 0$ and $\alpha \in (\frac{1}{2}, 1]$ such that for all $t, s \in [0, 1]$

$$|H_t - H_s| \le C\,|t-s|^{\alpha}.$$

There exist several generalizations of fractional Brownian motion to the case where the Hurst index H is varying with time. These generalizations are based on different integral representations of fractional Brownian motion. It is important to emphasize that distinct definitions can result in processes with substantially different covariance structures, as demonstrated in [8] and [17].

Example 2.1 (Moving-average multifractional Brownian motion [14]) Multifractional Brownian motion was first introduced by Peltier and Lévy Véhel [14]. Their definition is based on the Mandelbrot–van Ness representation for fractional Brownian motion (see, for example, [6, Chapter 1.3]). The *moving-average multifractional Brownian motion* is defined by

$$X_t = C_1\left(H_t\right) \int_{-\infty}^{t} \left[(t-s)_+^{H_t-\frac{1}{2}} - (-s)_+^{H_t-\frac{1}{2}} \right] dW_s, \tag{2.1}$$

where $W = \{W_t, t \in \mathbb{R}\}$ is a two-sided Wiener process, $x_+ = \max\{x, 0\}$, and

$$C_1(H) = \left(\frac{2H\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)} \right)^{1/2} = \frac{(2H\Gamma(2H)\sin(\pi H))^{1/2}}{\Gamma\left(H+\frac{1}{2}\right)}.$$

Example 2.2 (Multifractional Volterra-type Brownian motion [5, 15]) The next definition of a multifractional Brownian motion is based on the integral representation of the fractional Brownian motion through a Brownian motion on a finite interval developed in [13]. The *multifractional Volterra-type Brownian motion* is the process

$$X_t = \int_0^t K_{H_t}(t,s)\, dW_s, \tag{2.2}$$

where $W = \{W_t, t \ge 0\}$ is a Wiener process, and $K_H(t,s)$ is the Molchan kernel defined by

$$K_H(t,s) = C_2(H) s^{\frac{1}{2}-H} \int_s^t (v-s)^{H-\frac{3}{2}} v^{H-\frac{1}{2}}\, dv, \quad H \in (\tfrac{1}{2}, 1),$$

with $C_2(H) = C_1(H)(H - \frac{1}{2})$.

Example 2.3 (Harmonizable multifractional Brownian motion [3, 7]) Consider another generalization, introduced in [3]. Let $W(\cdot)$ be a complex random measure on $\mathbb{R}$ such that

1) for all $A, B \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{E} W(A)\overline{W(B)} = \lambda(A \cap B),$$

where λ is the Lebesgue measure;

2) for an arbitrary sequence $\{A_1, A_2, \dots\} \subset \mathcal{B}(\mathbb{R})$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, we have

$$W\left(\bigcup_{i\geq 1} A_i\right) = \sum_{i\geq 1} W(A_i),$$

(here $\{W(A_i), i \geq 1\}$ are centered normal random variables);
3) for all $A \in \mathcal{B}(\mathbb{R})$,

$$W(A) = \overline{W(-A)},$$

4) for all $\theta \in \mathbb{R}$,

$$\left\{e^{i\theta} W(A), A \in \mathcal{B}(\mathbb{R})\right\} \stackrel{d}{=} \{W(A), A \in \mathcal{B}(\mathbb{R})\}.$$

The *harmonizable multifractional Brownian motion* is defined by

$$X_t = C_3(H_t) \int_{\mathbb{R}} \frac{e^{itx} - 1}{|x|^{\frac{1}{2}+H_t}} W(dx), \tag{2.3}$$

where $C_3(H) = (H\Gamma(2H)\sin(\pi H)/\pi)^{1/2}$.

In the sequel, we consider a generalization of the fractional Brownian motion defined by $X_t = B_t^{H_t}$, $t \in [0, 1]$, where $\left\{B_t^H, t \in [0, 1], H \in \left(\frac{1}{2}, 1\right)\right\}$ is a family of random variables such that

(B1) for a fixed $H \in \left(\frac{1}{2}, 1\right)$, the process $\left\{B_t^H, t \in [0, 1]\right\}$ is a fractional Brownian motion with the Hurst parameter H;
(B2) for all $t \in [0, 1]$ and all $H_1, H_2 \in [H_{\min}, H_{\max}]$,

$$\mathbb{E}\left(B_t^{H_1} - B_t^{H_2}\right)^2 \leq C(H_1 - H_2)^2, \tag{2.4}$$

where C is a constant that may depend on $H_{\min}$ and $H_{\max}$.

The above conditions are satisfied, for instance, by every one of the generalizations described in Examples 2.1–2.3, since conditions (B1) and (B2) hold for representations (2.1)–(2.3), see [7, 14, 15]. In particular, the bound (2.4) for the Mandelbrot–van Ness representation (2.1) was established in [14, proof of Thm. 4], for the Volterra representation (2.2) it was proved in [15, Eqs. (16)–(17)], and for the harmonizable representation (2.3) it can be found in [9, proof of Lemma 3.1].

For further reference, we collect necessary properties of the variance and variogram functions of multifractional Brownian motion in the following lemma. It is worth noting that, in the specific case of harmonizable multifractional Brownian motion, this result is already known (see [4]). Here, we present it in a more general framework, i.e., for any type of multifractional Brownian motion that satisfies assumptions (B1) and (B2).

Lemma 2.4 *The multifractional Brownian motion* $X = \{X_t, t \in [0, 1]\}$ *has the following properties.*

(i) For all $t \in [0, 1]$

$$V(t) := \mathbb{E} X_t^2 = t^{2H_t}.$$

(ii) For all $t, s \in [0, 1]$

$$\vartheta(t, s) := \mathbb{E}\left(X_t - X_s\right)^2 \le |t - s|^{2H_{\min}} + C\,|t - s|^{H_{\min}+\alpha} + C\,|t - s|^{2\alpha}.$$

Proof According to the assumption (B2), if $H_t = H = \text{const}$, then the process $X_t = B_t^{H_t}$ is a fractional Brownian motion. This implies the statement (i) and the following bound

$$\mathbb{E}\left(B_t^{H_t} - B_s^{H_t}\right)^2 = |t - s|^{2H_t} \le |t - s|^{2H_{\min}}. \tag{2.5}$$

Moreover, the assumptions (B2) and (A2) yield

$$\mathbb{E}\left(B_s^{H_t} - B_s^{H_s}\right)^2 \le C\left(H_t - H_s\right)^2 \le C\,|t - s|^{2\alpha}. \tag{2.6}$$

Furthermore, by the Cauchy–Schwarz inequality we derive from (2.5) and (2.6) that

$$\mathbb{E}\left|\left(B_t^{H_t} - B_s^{H_t}\right)\left(B_s^{H_t} - B_s^{H_s}\right)\right| \le C\,|t - s|^{H_{\min}+\alpha}$$

Thus,

$$\begin{aligned}\mathbb{E}\left(X_t - X_s\right)^2 &= \mathbb{E}\left(B_t^{H_t} - B_s^{H_t}\right)^2 + \mathbb{E}\left(B_s^{H_t} - B_s^{H_s}\right)^2 \\ &\quad + 2\mathbb{E}\left[\left(B_t^{H_t} - B_s^{H_t}\right)\left(B_s^{H_t} - B_s^{H_s}\right)\right] \\ &\le |t - s|^{2H_{\min}} + C\,|t - s|^{2\alpha} + C\,|t - s|^{H_{\min}+\alpha},\end{aligned}$$

and the claim (ii) is proved. □

Remark 2.5 For a convex function Ψ, let Ψ' denote its one-sided derivative. In condition (A1), we assume that the function H_t is bounded away from one. This guarantees that

$$\int_0^1 \frac{1}{\sqrt{V(s)}}\,ds \le \int_0^1 s^{-H_{\max}}\,ds < \infty.$$

Then, by [6] $\int_0^1 \Psi'(X_s)dX_s$ exists as a pathwise Riemann–Stieltjes integral; moreover, it satisfies the following chain rule:

$$\int_0^1 \Psi'(X_s)dX_s = \Psi\left(X_1\right) - \Psi\left(X_0\right). \tag{2.7}$$

3 Main result

Let $t_k = \frac{k}{n}$, $k = 0, 1, \dots, n$, be an equidistant partition of the interval $[0, 1]$. Throughout the article, we use the notation

$$\varphi(a) := \mathbb{E}\left[Y \mathbb{1}_{Y>a}\right] = \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}}, \quad a \in \mathbb{R}. \tag{3.1}$$

In what follows, let C denote a generic constant that may change its value from one occurrence to another.

The following theorem is the main result of the paper.

Theorem 3.1 *Let $X_t = B_t^{H_t}$ be a multifractional Brownian motion with the Hurst function H_t satisfying (A1)–(A2). Let Ψ be a convex function with the left-sided derivative Ψ', and let μ denote the measure associated with the second derivative of Ψ such that $\int_{\mathbb{R}} \varphi(a)\mu(da) < \infty$. Then for any $\widetilde{H} \in (\frac{1}{2}, H_{\min}] \cap (\frac{1}{2}, \alpha)$,*

$$\begin{aligned}
&\mathbb{E}\left|\int_0^1 \Psi'(X_s)dX_s - \sum_{k=1}^n \Psi'\left(X_{t_{k-1}}\right)\left(X_{t_k} - X_{t_{k-1}}\right)\right| \\
&\le \int_{\mathbb{R}} \int_0^1 s^{-H_s} \varphi\left(\frac{a}{s^{H_s}}\right) ds\, \mu(da) \left(\frac{1}{n}\right)^{2\widetilde{H}-1} + \int_{\mathbb{R}} R_n(a)\mu(da),
\end{aligned} \tag{3.2}$$

where the remainder satisfies

$$\int_{\mathbb{R}} R_n(a)\mu(da) \le C n^{-\min\{2\widetilde{H} - H_{\max}, H_{\min}+\alpha-1, 2\alpha-1\}}. \tag{3.3}$$

Remark 3.2 Assumption $\widetilde{H} \in (\frac{1}{2}, H_{\min}] \cap (\frac{1}{2}, \alpha)$ guarantees that that the remainder is negligible compared to the first term in (3.2). Indeed, we have

$$2\widetilde{H} - H_{\max} > 2\widetilde{H} - 1, \quad H_{\min} + \alpha - 1 > 2\widetilde{H} - 1, \quad \text{and} \quad 2\alpha - 1 > 2\widetilde{H} - 1.$$

Hence,

$$\frac{\int_{\mathbb{R}} R_n(a)\mu(da)}{n^{1-2\widetilde{H}}} \to 0, \quad \text{as } n \to \infty.$$

Remark 3.3 One can formulate the statement of Theorem 3.1 more precisely by considering the cases $\alpha > H_{\min}$ and $\alpha \in (\frac{1}{2}, H_{\min}]$ separately. Evidently, in the case $\alpha > H_{\min}$, (3.2) holds with $\widetilde{H} = H_{\min}$. And in the general case, i.e., $\alpha > \frac{1}{2}$, one has

$$\mathbb{E}\left|\int_0^1 \Psi'(X_s)dX_s - \sum_{k=1}^n \Psi'\left(X_{t_{k-1}}\right)\left(X_{t_k} - X_{t_{k-1}}\right)\right| \le C n^{1-2\min\{H_{\min}, \alpha\}}. \tag{3.4}$$

Note that for $\frac{1}{2} < \alpha \le H_{\min}$, the leading term in (3.2) has the same order $n^{1-2\alpha}$ as the remainder; so we cannot obtain more precise rate of convergence than (3.4).

Remark 3.4 When the function H is sufficiently smooth and the difference between $H_{\max}$ and $H_{\min}$ is rather small, one can establish a lower bound in addition to (3.2). Namely, under additional assumptions

$$\alpha > H_{\max} \quad \text{and} \quad 3H_{\max} - 2H_{\min} < 1, \tag{3.5}$$

the following inequality holds

$$\begin{aligned} &\mathbb{E}\left|\int_0^1 \Psi'(X_s)dX_s - \sum_{k=1}^n \Psi'\left(X_{t_{k-1}}\right)\left(X_{t_k} - X_{t_{k-1}}\right)\right| \\ &\ge \int_{\mathbb{R}}\int_0^1 s^{-H_s}\varphi\left(\frac{a}{s^{H_s}}\right)ds\,\mu(da)\left(\frac{1}{n}\right)^{2H_{\max}-1} + \int_{\mathbb{R}} R_n(a)\mu(da), \end{aligned} \tag{3.6}$$

where the same remainder that satisfies (3.3). Due to assumptions (3.5), the remainder in (3.6) is negligible compared to the leading term.

The proof of the lower bound (3.6) is conducted similarly to that of Theorem 3.1, but one uses the inequality

$$\vartheta(t,s) \ge |t-s|^{2H_{\max}} + g(t,s), \quad \text{where } |g(t,s)| \le C\,|t-s|^{H_{\min}+\alpha}, \tag{3.7}$$

instead of Lemma 2.4 (ii). The bound (3.7) is derived similarly to Lemma 2.4; the remainder function $g(t,s)$ is the same, namely

$$g(t,s) = \mathbb{E}(B_s^{H_t} - B_s^{H_s})^2 + 2\mathbb{E}[(B_t^{H_t} - B_s^{H_t})(B_s^{H_t} - B_s^{H_s})].$$

Remark 3.5 In particular, the assumptions (3.5) hold in the case $H_t = H = \text{const}$ (i.e., when X is a fractional Brownian motion). Indeed, in this case one can take $\alpha = 1$, $H_{\min} = H_{\max} = H$, and the bounds (3.2) and (3.6) imply that

$$\begin{aligned} &\mathbb{E}\left|\int_0^1 \Psi'(X_s)dX_s - \sum_{k=1}^n \Psi'\left(X_{t_{k-1}}\right)\left(X_{t_k} - X_{t_{k-1}}\right)\right| \\ &= \int_{\mathbb{R}}\int_0^1 s^{-H}\varphi\left(\frac{a}{s^{H}}\right)ds\,\mu(da)\left(\frac{1}{n}\right)^{2H-1} + \widetilde{R}_n(a), \end{aligned}$$

with $\widetilde{R}_n(a) \le Cn^{-H}$. This coincides with the result of [2] for the case of fractional Brownian motion.

Moreover, since in the case $H_t = H = \text{const}$ we have an exact rate of convergence n^{1-2H}, we see that the result of Theorem 3.1 cannot be improved substantially.

4 Proofs

4.1 Some auxiliary bounds

In what follows, we will often use the following simple upper bound for small a.

Lemma 4.1 *Let $\mu \in \mathbb{R}$. Then for all $|a| \le 1$ and $s > 0$*

$$\varphi\left(\frac{a}{s^{\mu}}\right) \le Ca^{-2}s^{2\mu}\varphi(a),$$

where $C = 2e^{-1/2}$ is an absolute constant.

Proof Denote $h(x) = xe^{-x}$. The derivative of $h(x)$ equals $h'(x) = e^{-x}(1-x)$, whence $\max_{x\in\mathbb{R}} h(x) = h(1) = e^{-1}$. Therefore, for any $a \in \mathbb{R}$,

$$\frac{a^2}{s^{2\mu}}\varphi\left(\frac{a}{s^{\mu}}\right) = \frac{2}{\sqrt{2\pi}}h\left(\frac{a^2}{2s^{2\mu}}\right) \le \frac{2}{e\sqrt{2\pi}}.$$

Note that $\varphi(a)$ decreases when $|a|$ decreases. Hence, for $|a| \le 1$ one has

$$\varphi(a) \ge \varphi(1) = \frac{1}{\sqrt{2\pi e}}.$$

Combining two obtained inequalities, we conclude the proof. □

The next auxiliary result provides an upper bound for an integral for specific power-exponential integrands. Such integrals often arise in subsequent proofs.

Lemma 4.2 *Let $\lambda < -1$ and $\mu > 0$. Then for all $|a| \ge 1$,*

$$\int_0^1 s^{\lambda}\,\varphi\left(\frac{a}{s^{\mu}}\right)ds \le Ca^{-2}\varphi(a).$$

The constant C may depend on λ and μ.

Proof Denote $a^2 = x$,

$$F(x) := \frac{\int_0^1 s^{\lambda}\varphi\left(\frac{\sqrt{x}}{s^{\mu}}\right)ds}{x^{-1}\varphi(\sqrt{x})} = \frac{\int_0^1 s^{\lambda}\exp\left\{-\frac{x}{2s^{2\mu}}\right\}ds}{x^{-1}\exp\left\{-\frac{x}{2}\right\}}.$$

We need to show that F is bounded on $[1,\infty)$. By substitution $\frac{x}{2s^{2\mu}} = z$, we have

$$F(x) = \frac{\frac{1}{2\mu}\left(\frac{x}{2}\right)^{\frac{\lambda+1}{2\mu}}\int_{x/2}^{\infty} z^{-\frac{\lambda+1}{2\mu}-1}e^{-z}dz}{x^{-1}e^{-x/2}} = 2^{-\frac{\lambda+1}{2\mu}-1}\frac{\int_{x/2}^{\infty} z^{-\frac{\lambda+1}{2\mu}-1}e^{-z}dz}{x^{-\frac{\lambda+1}{2\mu}-1}e^{-\frac{x}{2}}}.$$

As $x \to \infty$, we get by l'Hôpital's rule

$$\lim_{x\to\infty} F(x)=2^{-\frac{\lambda+1}{2\mu}-1} \lim_{x\to\infty} \frac{-\frac{1}{2}(\frac{x}{2})^{-\frac{\lambda+1}{2\mu}-1} e^{-\frac{x}{2}}}{-\frac{1}{2}x^{-\frac{\lambda+1}{2\mu}-1} e^{-\frac{x}{2}} - \left(\frac{\lambda+1}{2\mu}+1\right) x^{-\frac{\lambda+1}{2\mu}-2} e^{-\frac{x}{2}}}=2^{-\frac{\lambda+1}{2\mu}-1}.$$

Taking into account the continuity of the function $F(x)$, we derive its boundedness for all $x \geq 1$. □

4.2 Approximation estimates

In this section, we present upper bounds for various terms appearing in the proof of the main result.

Lemma 4.3 *For all $a \in \mathbb{R}$,*

$$\sum_{k=2}^{n} \frac{\left(t_k^{H_{t_k}} - t_{k-1}^{H_{t_{k-1}}}\right)^2}{t_{k-1}^{H_{t_{k-1}}}} \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) \leq C\varphi(a) n^{-\min\{H_{\min},2\alpha-1\}}.$$

Proof Evidently, $\varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) \leq \varphi(a)$. Hence, it suffices to estimate the following sum:

$$\begin{aligned}\sum_{k=2}^{n} \frac{\left(t_k^{H_{t_k}} - t_{k-1}^{H_{t_{k-1}}}\right)^2}{t_{k-1}^{H_{t_{k-1}}}} &\leq 2\sum_{k=2}^{n} \frac{\left(t_k^{H_{t_k}} - t_{k}^{H_{t_{k-1}}}\right)^2}{t_{k-1}^{H_{t_{k-1}}}} + 2\sum_{k=2}^{n} \frac{\left(t_k^{H_{t_{k-1}}} - t_{k-1}^{H_{t_{k-1}}}\right)^2}{t_{k-1}^{H_{t_{k-1}}}} \\ &=:2(A_1 + A_2).\end{aligned} \tag{4.1}$$

First, let us bound A_1. Using the mean value theorem and assumption (A2), we obtain

$$\begin{aligned}\left|t_k^{H_{t_k}} - t_k^{H_{t_{k-1}}}\right| &\leq \left|t_k^{H_{\min}} \log t_k\right| \left|H_{t_k} - H_{t_{k-1}}\right| \leq C t_k^{H_{\min}} \left|\log t_k\right| \left|t_k - t_{k-1}\right|^{\alpha} \\ &= C n^{-\alpha} t_k^{H_{\min}} \left|\log t_k\right|.\end{aligned} \tag{4.2}$$

It is well known that for any $\delta > 0$ there exists a constant $C = C(\delta) > 0$ such that $|\log s| \leq C s^{-\delta}$ for all $s \in (0, 1]$. Fix any $0 < \delta < H_{\min} - \frac{1}{2}H_{\max}$. (This is possible because $H_{\min} > \frac{1}{2} > \frac{1}{2}H_{\max}$.) Then,

$$\left|t_k^{H_{t_k}} - t_k^{H_{t_{k-1}}}\right| \leq C n^{-\alpha} t_k^{H_{\min}-\delta}.$$

Therefore,

$$A_1 \le Cn^{-2\alpha} \sum_{k=2}^{n} \frac{t_k^{2H_{\min}-2\delta}}{t_{k-1}^{H_{\max}}} = Cn^{-2\alpha} \sum_{k=2}^{n} \left(\frac{t_k}{t_{k-1}}\right)^{2H_{\min}-2\delta} t_{k-1}^{2H_{\min}-H_{\max}-2\delta}$$
$$\le Cn^{1-2\alpha} \cdot \frac{1}{n} \sum_{k=1}^{n} t_{k-1}^{2H_{\min}-H_{\max}-2\delta},$$

where we have used the bounds

$$\left(\frac{t_k}{t_{k-1}}\right)^{2H_{\min}-2\delta} = \left(\frac{k}{k-1}\right)^{2H_{\min}-2\delta} = \left(1 + \frac{1}{k-1}\right)^{2H_{\min}-2\delta} \le 2^{2H_{\min}-2\delta}, \quad k \ge 2.$$

Further, the term $\frac{1}{n} \sum_{k=1}^{n} t_{k-1}^{2H_{\min}-H_{\max}-2\delta}$ is bounded as a convergent Riemann sum. Indeed,

$$\frac{1}{n} \sum_{k=1}^{n} t_{k-1}^{2H_{\min}-H_{\max}-2\delta} \to \int_0^1 s^{2H_{\min}-H_{\max}-2\delta} ds < \infty, \quad \text{as } n \to \infty.$$

Hence,

$$A_1 \le Cn^{1-2\alpha}. \tag{4.3}$$

Next, we estimate A_2. Note that the numerators can be bounded as follows:

$$t_k^{H_{t_{k-1}}} - t_{k-1}^{H_{t_{k-1}}} = H_{t_{k-1}} \int_{t_{k-1}}^{t_k} x^{H_{t_{k-1}}-1} dx \le H_{t_{k-1}} t_{k-1}^{H_{t_{k-1}}-1} (t_k - t_{k-1}) \le \frac{t_{k-1}^{H_{t_{k-1}}-1}}{n}.$$

Therefore,

$$A_2 \le \frac{1}{n^2} \sum_{k=2}^{n} t_{k-1}^{H_{t_{k-1}}-2} \le \frac{1}{n^2} \sum_{k=2}^{n} t_{k-1}^{H_{\min}-2} = n^{-H_{\min}} \sum_{k=2}^{n} (k-1)^{H_{\min}-2} \le Cn^{-H_{\min}}, \tag{4.4}$$

since the series $\sum_{k=2}^{\infty} (k-1)^{H_{\min}-2}$ converges.

Combining (4.1)–(4.4), we conclude the proof. □

Now we are ready to establish two key lemmas concerning approximations. The following result is a counterpart of [2, Lemma 4.10].

Lemma 4.4 *For all $a \in \mathbb{R}$,*

$$\left| \int_0^1 s^{-H_s} \varphi\left(\frac{a}{s^{H_s}}\right) ds - \frac{1}{n} \sum_{k=2}^{n} t_{k-1}^{-H_{t_{k-1}}} \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) \right| \le C\varphi(a) n^{H_{\max}-1}.$$

Proof We start by writing

$$
\begin{aligned}
&\left|\int_0^1 s^{-H_s}\varphi\left(\frac{a}{s^{H_s}}\right)ds - \frac{1}{n}\sum_{k=2}^{n} t_{k-1}^{-H_{t_{k-1}}}\varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right)\right| \\
&= \left|\int_0^{t_1} s^{-H_s}\varphi\left(\frac{a}{s^{H_s}}\right)ds + \sum_{k=2}^{n}\int_{t_{k-1}}^{t_k}\left(s^{-H_s}\varphi\left(\frac{a}{s^{H_s}}\right) - t_{k-1}^{-H_{t_{k-1}}}\varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right)\right)ds\right| \\
&\le B_0 + B_1 + B_2, \qquad (4.5)
\end{aligned}
$$

where

$$
\begin{aligned}
B_0 &:= \int_0^{t_1} s^{-H_s}\varphi\left(\frac{a}{s^{H_s}}\right)ds, \\
B_1 &:= \sum_{k=2}^{n}\int_{t_{k-1}}^{t_k}\left|s^{-H_s}\varphi\left(\frac{a}{s^{H_s}}\right) - s^{-H_{t_{k-1}}}\varphi\left(\frac{a}{s^{H_{t_{k-1}}}}\right)\right|ds, \\
B_2 &:= \sum_{k=2}^{n}\int_{t_{k-1}}^{t_k}\left|s^{-H_{t_{k-1}}}\varphi\left(\frac{a}{s^{H_{t_{k-1}}}}\right) - t_{k-1}^{-H_{t_{k-1}}}\varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right)\right|ds.
\end{aligned}
$$

The term B_0 can be bounded as follows:

$$
B_0 = \int_0^{\frac{1}{n}} s^{-H_s}\varphi\left(\frac{a}{s^{H_s}}\right)ds \le \varphi(a)\int_0^{\frac{1}{n}} s^{-H_{\max}}ds = C\varphi(a)n^{H_{\max}-1}. \qquad (4.6)
$$

Let us consider B_1.

$$
\begin{aligned}
B_1 &\le \sum_{k=2}^{n}\int_{t_{k-1}}^{t_k}\varphi\left(\frac{a}{s^{H_s}}\right)\left|s^{-H_s} - s^{-H_{t_{k-1}}}\right|ds \\
&\quad + \sum_{k=2}^{n}\int_{t_{k-1}}^{t_k} s^{-H_{t_{k-1}}}\left|\varphi\left(\frac{a}{s^{H_s}}\right) - \varphi\left(\frac{a}{s^{H_{t_{k-1}}}}\right)\right|ds \\
&=: B_{11} + B_{12}. \qquad (4.7)
\end{aligned}
$$

Since $\varphi\left(\frac{a}{s^{H_s}}\right) \le \varphi(a)$, we see that

$$
B_{11} \le \varphi(a)\sum_{k=2}^{n}\int_{t_{k-1}}^{t_k}\left|s^{-H_s} - s^{-H_{t_{k-1}}}\right|ds.
$$

Using the mean value theorem and the assumption (A2), we can bound the integrand as follows:

$$
\left|s^{-H_s} - s^{-H_{t_{k-1}}}\right| \le s^{-H_{\max}}\left|\log s\right|\left|H_s - H_{t_{k-1}}\right|
$$

$$\leq C s^{-H_{\max}} |\log s| \, |s - t_{k-1}|^{\alpha} \leq C s^{-H_{\max}} |\log s| \, n^{-\alpha}.$$

Then,

$$B_{11} \leq C\varphi(a) n^{-\alpha} \int_0^1 s^{-H_{\max}} |\log s| \, ds \leq C\varphi(a) n^{-\alpha}, \tag{4.8}$$

because the function $s \mapsto s^{-H_{\max}} |\log s|$ is integrable on $[0, 1]$.

In order to estimate B_{12}, note that

$$\partial_x \varphi\left(\frac{a}{s^x}\right) = -\frac{a}{s^x} \varphi\left(\frac{a}{s^x}\right) \partial_x \left(\frac{a}{s^x}\right) = \frac{a^2}{s^{2x}} \varphi\left(\frac{a}{s^x}\right) \log s.$$

Therefore,

$$\varphi\left(\frac{a}{s^{H_s}}\right) - \varphi\left(\frac{a}{s^{H_{t_{k-1}}}}\right) = a^2 \log s \int_{H_{t_{k-1}}}^{H_s} s^{-2x} \varphi\left(\frac{a}{s^x}\right) dx, \tag{4.9}$$

whence

$$B_{12} \leq a^2 \sum_{k=2}^{n} \int_{t_{k-1}}^{t_k} s^{-H_{\max}} |\log s| \left| \int_{H_{t_{k-1}}}^{H_s} s^{-2x} \varphi\left(\frac{a}{s^x}\right) dx \right| ds. \tag{4.10}$$

Let us consider two cases separately.

(i) Case $|a| \leq 1$. By Lemma 4.1, $a^2 s^{-2x} \varphi\left(\frac{a}{s^x}\right) \leq C\varphi(a)$. Using this bound and Assumption (A2), we get

$$a^2 \left| \int_{H_{t_{k-1}}}^{H_s} s^{-2x} \varphi\left(\frac{a}{s^x}\right) dx \right| \leq C\varphi(a) \left| H_s - H_{t_{k-1}} \right| \leq C\varphi(a) \, |s - t_{k-1}|^{\alpha}$$
$$\leq C\varphi(a) n^{-\alpha}.$$

We insert this inequality into (4.10) and obtain

$$B_{12} \leq C\varphi(a) n^{-\alpha} \int_0^1 s^{-H_{\max}} |\log s| \, ds \leq C\varphi(a) n^{-\alpha}.$$

(ii) Case $|a| > 1$. The inner integral in (4.10) can be estimated as follows:

$$\left| \int_{H_{t_{k-1}}}^{H_s} s^{-2x} \varphi\left(\frac{a}{s^x}\right) dx \right| \leq s^{-2H_{\max}} \varphi\left(\frac{a}{s^{H_{\min}}}\right) \left| H_s - H_{t_{k-1}} \right|$$
$$\leq C n^{-\alpha} s^{-2H_{\max}} \varphi\left(\frac{a}{s^{H_{\min}}}\right).$$

by Assumption (A2). Then, it follows from (4.10) that

$$B_{12} \le Cn^{-\alpha}a^2 \int_0^1 s^{-3H_{\max}} |\log s| \, \varphi\left(\frac{a}{s^{H_{\min}}}\right) ds,$$

Choosing arbitrary $\delta > 0$ and applying again the bound $|\log s| \le C_\delta s^{-\delta}$, we get

$$B_{12} \le Cn^{-\alpha}a^2 \int_0^1 s^{-3H_{\max}-\delta} \varphi\left(\frac{a}{s^{H_{\min}}}\right) ds \le C\varphi(\alpha)n^{-\alpha},$$

where the last inequality follows from Lemma 4.2.

Thus, in both cases, $B_{12} \le C\varphi(\alpha)n^{-\alpha}$. Combining this result with (4.7) and (4.8), we see that

$$B_1 \le C\varphi(\alpha)n^{-\alpha} \le C\varphi(\alpha)n^{H_{\max}-1} \tag{4.11}$$

(because $-\alpha < -\frac{1}{2} < H_{\max} - 1$).

Now, let us consider B_2. We have

$$\begin{aligned} B_2 \le &\sum_{k=2}^{n} \int_{t_{k-1}}^{t_k} s^{-H_{t_{k-1}}} \left|\varphi\left(\frac{a}{s^{H_{t_{k-1}}}}\right) - \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right)\right| ds \\ &+ \sum_{k=2}^{n} \int_{t_{k-1}}^{t_k} \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) \left|s^{-H_{t_{k-1}}} - t_{k-1}^{-H_{t_{k-1}}}\right| ds \\ =:& B_{21} + B_{22}. \end{aligned}$$

Let us estimate B_{21}. Using the relation $\varphi'(x) = -x\varphi(x)$, we compute

$$\partial_u \varphi\left(\frac{a}{u^{H_{t_{k-1}}}}\right) = H_{t_{k-1}} a^2 u^{-2H_{t_{k-1}}-1} \varphi\left(\frac{a}{u^{H_{t_{k-1}}}}\right). \tag{4.12}$$

Then, B_{21} can be rewritten as follows:

$$B_{21} = a^2 \sum_{k=2}^{n} H_{t_{k-1}} \int_{t_{k-1}}^{t_k} s^{-H_{t_{k-1}}} \int_{t_{k-1}}^{s} u^{-2H_{t_{k-1}}-1} \varphi\left(\frac{a}{u^{H_{t_{k-1}}}}\right) du\, ds. \tag{4.13}$$

Let us consider two cases.

(i) Case $|a| \le 1$. By Lemma 4.1, $a^2 u^{-2H_{t_{k-1}}} \exp\{-\frac{a^2}{2u^{2H_{t_{k-1}}}}\} \le C\varphi(a)$. Hence, (4.13) yields

$$B_{21} \le C\varphi(a) \sum_{k=2}^{n} \int_{t_{k-1}}^{t_k} s^{-H_{t_{k-1}}} \int_{t_{k-1}}^{s} u^{-1} du\, ds$$

$$
\begin{aligned}
&\le C\varphi(a)\sum_{k=2}^{n}\int_{t_{k-1}}^{t_k} s^{-H_{t_{k-1}}}\, t_{k-1}^{-1}\,(s-t_{k-1})\,ds\\
&\le C\varphi(a)\sum_{k=2}^{n} t_{k-1}^{-H_{t_{k-1}}-1}\int_{t_{k-1}}^{t_k}(s-t_{k-1})\,ds \le C\varphi(a)\frac{1}{n^2}\sum_{k=2}^{n} t_{k-1}^{-H_{t_{k-1}}-1}\\
&\le C\varphi(a)\frac{1}{n^2}\sum_{k=2}^{n} t_{k-1}^{-H_{\max}-1} = C\varphi(a)n^{H_{\max}-1}\sum_{k=2}^{n}(k-1)^{-H_{\max}-1}\\
&\le C\varphi(a)n^{H_{\max}-1},
\end{aligned}
$$

because $\sum_{k=2}^{n}(k-1)^{-H_{\max}-1} \le \sum_{k=2}^{\infty}(k-1)^{-H_{\max}-1} < \infty$.

(ii) Case $|a| > 1$. Changing the order of integration in the right-hand side of (4.13), we obtain

$$
B_{21} \le a^2\sum_{k=2}^{n}\int_{t_{k-1}}^{t_k} u^{-2H_{t_{k-1}}-1}\varphi\left(\frac{a}{u^{H_{t_{k-1}}}}\right)\int_u^{t_k} s^{-H_{t_{k-1}}}\,ds\,du.
$$

The inner integral can be bounded as follows:

$$
\int_u^{t_k} s^{-H_{t_{k-1}}}\,ds \le u^{-H_{t_{k-1}}}\,(t_k-u) \le \frac{1}{n}u^{-H_{t_{k-1}}}.
$$

Then,

$$
\begin{aligned}
B_{21} &\le \frac{1}{n}a^2\sum_{k=2}^{n}\int_{t_{k-1}}^{t_k} u^{-3H_{t_{k-1}}-1}\varphi\left(\frac{a}{u^{H_{t_{k-1}}}}\right)du\\
&\le \frac{1}{n}a^2\sum_{k=2}^{n}\int_{t_{k-1}}^{t_k} u^{-3H_{\max}-1}\varphi\left(\frac{a}{u^{H_{\min}}}\right)du\\
&\le \frac{1}{n}a^2\int_0^1 u^{-3H_{\max}-1}\varphi\left(\frac{a}{u^{H_{\min}}}\right)du.
\end{aligned}
$$

Applying Lemma 4.2, we get

$$
B_{21} \le C\varphi(a)n^{-1} \le C\varphi(a)n^{H_{\max}-1},
$$

that is, we have for $|a| > 1$ the same upper bound for B_{21} as in the case $|a| \le 1$. Now let us consider B_{22}. Using the evident bound

$$
\varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) \le \varphi(a),
$$

we get

$$B_{22} \le C\varphi(a) \sum_{k=2}^{n} \int_{t_{k-1}}^{t_k} \left(t_{k-1}^{-H_{t_{k-1}}} - s^{-H_{t_{k-1}}} \right) ds. \tag{4.14}$$

By the mean value theorem, we obtain

$$t_{k-1}^{-H_{t_{k-1}}} - s^{-H_{t_{k-1}}} \le H_{t_{k-1}} t_{k-1}^{-H_{t_{k-1}}-1} (s - t_{k-1}) \le \frac{1}{n} t_{k-1}^{-H_{\max}-1} = \frac{n^{H_{\max}}}{(k-1)^{H_{\max}+1}}.$$

Substituting this bound into (4.14), we arrive at

$$B_{22} \le C\varphi(a) n^{H_{\max}-1} \sum_{k=2}^{n} (k-1)^{-H_{\max}-1} \le C\varphi(a) n^{H_{\max}-1}.$$

Combining the above bounds for B_{21} and B_{22}, we conclude that

$$B_2 \le C\varphi(a) n^{H_{\max}-1}. \tag{4.15}$$

Finally, taking into account the representation (4.5) and the inequalities (4.6), (4.11) and (4.15), we complete the proof. □

Lemma 4.5 *Let $Y \sim \mathcal{N}(0, 1)$. Then for all $a \in \mathbb{R}$,*

$$\begin{aligned} &\left| \sum_{k=2}^{n} \left(t_k^{H_{t_k}} \left[\varphi\left(\frac{a}{t_k^{H_{t_k}}} \right) - \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}} \right) \right] \right. \right. \\ &\left. \left. - a \left[\mathbb{P}\left(Y > \frac{a}{t_k^{H_{t_k}}} \right) - \mathbb{P}\left(Y > \frac{a}{t_{k-1}^{H_{t_{k-1}}}} \right) \right] \right) \right| \le C\varphi(a) n^{-\min\{H_{\min}, 2\alpha-1\}}. \end{aligned}$$

Proof We decompose the left-hand side of the desired inequality as follows:

$$\begin{aligned} &\left| \sum_{k=1}^{n} \left(t_k^{H_{t_k}} \left[\varphi\left(\frac{a}{t_k^{H_{t_k}}} \right) - \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}} \right) \right] \right. \right. \\ &\left. \left. - a \left[\mathbb{P}\left(Y > \frac{a}{t_k^{H_{t_k}}} \right) - \mathbb{P}\left(Y > \frac{a}{t_{k-1}^{H_{t_{k-1}}}} \right) \right] \right) \right| \\ &\le D_0 + D_1 + D_2, \end{aligned} \tag{4.16}$$

where

$$D_0 = \left| t_1^{H_{t_1}} \varphi\left(\frac{a}{t_1^{H_{t_1}}} \right) - a\mathbb{P}\left(Y > \frac{a}{t_1^{H_{t_1}}} \right) \right|,$$

$$\begin{aligned} D_1 = \Bigg| \sum_{k=2}^{n} \Bigg(t_k^{H_{t_k}} \left[\varphi\left(\frac{a}{t_k^{H_{t_k}}}\right) - \varphi\left(\frac{a}{t_k^{H_{t_{k-1}}}}\right) \right] \\ - a \left[\mathbb{P}\left(Y > \frac{a}{t_k^{H_{t_k}}}\right) - \mathbb{P}\left(Y > \frac{a}{t_k^{H_{t_{k-1}}}}\right) \right] \Bigg) \Bigg|, \end{aligned} \tag{4.17}$$

$$\begin{aligned} D_2 = \Bigg| \sum_{k=2}^{n} \Bigg(t_k^{H_{t_k}} \left[\varphi\left(\frac{a}{t_k^{H_{t_{k-1}}}}\right) - \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) \right] \\ - a \left[\mathbb{P}\left(Y > \frac{a}{t_k^{H_{t_{k-1}}}}\right) - \mathbb{P}\left(Y > \frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) \right] \Bigg) \Bigg|. \end{aligned} \tag{4.18}$$

Let us estimate each term separately. In order to bound D_0, we observe that

$$t_1^{H_{t_1}} \varphi\left(\frac{a}{t_1^{H_{t_1}}}\right) \geq a \mathbb{P}\left(Y > \frac{a}{t_1^{H_{t_1}}}\right). \tag{4.19}$$

Indeed, denoting $x = \frac{a}{t_1^{H_{t_1}}}$, we get by (3.1)

$$x \mathbb{P}(Y > x) = \mathbb{E}[x \mathbb{1}_{Y > x}] \leq \mathbb{E}[Y \mathbb{1}_{Y > x}] = \varphi(x),$$

whence (4.19) follows. Then, taking into account (4.19), we may write

$$D_0 \leq t_1^{H_{t_1}} \varphi\left(\frac{a}{t_1^{H_{t_1}}}\right) \leq t_1^{H_{\min}} \varphi(a) = n^{-H_{\min}} \varphi(a). \tag{4.20}$$

Now let us consider the term D_1. Similarly to (4.9), we have

$$\varphi\left(\frac{a}{t_k^{H_{t_k}}}\right) - \varphi\left(\frac{a}{t_k^{H_{t_{k-1}}}}\right) = a^2 \log t_k \int_{H_{t_{k-1}}}^{H_{t_k}} t_k^{-2x} \varphi\left(\frac{a}{t_k^x}\right) dx. \tag{4.21}$$

Furthermore,

$$\partial_x \mathbb{P}\left(Y > \frac{a}{t_k^x}\right) = -\varphi\left(\frac{a}{t_k^x}\right) \partial_x \left(\frac{a}{t_k^x}\right) = \frac{a}{t_k^x} \varphi\left(\frac{a}{t_k^x}\right) \log t_k.$$

Hence,

$$\mathbb{P}\left(Y > \frac{a}{t_k^{H_{t_k}}}\right) - \mathbb{P}\left(Y > \frac{a}{t_k^{H_{t_{k-1}}}}\right) = a \log t_k \int_{H_{t_{k-1}}}^{H_{t_k}} t_k^{-x} \varphi\left(\frac{a}{t_k^x}\right) dx. \tag{4.22}$$

We insert (4.21) and (4.22) into (4.17) and obtain

$$D_1 = \left| \sum_{k=2}^{n} \left(t_k^{H_{t_k}} a^2 \log t_k \int_{H_{t_{k-1}}}^{H_{t_k}} t_k^{-2x} \varphi\left(\frac{a}{t_k^x}\right) dx - a^2 \log t_k \int_{H_{t_{k-1}}}^{H_{t_k}} t_k^{-x} \varphi\left(\frac{a}{t_k^x}\right) dx \right) \right|$$
$$\leq a^2 \sum_{k=2}^{n} |\log t_k| \left| \int_{H_{t_{k-1}}}^{H_{t_k}} \varphi\left(\frac{a}{t_k^x}\right) t_k^{-2x} \left| t_k^{H_{t_k}} - t_k^x \right| dx \right|.$$

Using the mean value theorem, we get similarly to (4.2)

$$\left| t_k^{H_{t_k}} - t_k^x \right| \leq t_k^{H_{\min}} |\log t_k| \left| H_{t_k} - x \right| \leq C n^{-\alpha} t_k^{H_{\min}} |\log t_k| .$$

Consequently,

$$D_1 \leq C n^{-\alpha} a^2 \sum_{k=2}^{n} t_k^{H_{\min}} |\log t_k|^2 \left| \int_{H_{t_{k-1}}}^{H_{t_k}} \varphi\left(\frac{a}{t_k^x}\right) t_k^{-2x} dx \right|$$
$$\leq C n^{-\alpha} a^2 \sum_{k=2}^{n} t_k^{H_{\min}-2\delta} \left| \int_{H_{t_{k-1}}}^{H_{t_k}} \varphi\left(\frac{a}{t_k^x}\right) t_k^{-2x} dx \right|,$$

where we can choose arbitrary $\delta > 0$.

Let us consider two cases.

(i) Case $|a| \leq 1$. Using the bound $a^2 t_k^{-2x} \varphi\left(\frac{a}{t_k^x}\right) \leq C\varphi(a)$ (see Lemma 4.1) and Assumption (A2), we obtain

$$a^2 \left| \int_{H_{t_{k-1}}}^{H_{t_k}} \varphi\left(\frac{a}{t_k^x}\right) t_k^{-2x} dx \right| \leq C\varphi(a) \left| H_{t_k} - H_{t_{k-1}} \right|$$
$$\leq C\varphi(a) |t_k - t_{k-1}|^{\alpha} = C\varphi(a) n^{-\alpha}.$$

Hence,

$$D_1 \leq C n^{1-2\alpha} \left(\frac{1}{n} \sum_{k=1}^{n} t_k^{H_{\min}-2\delta} \right) \leq C n^{1-2\alpha},$$

since $\frac{1}{n} \sum_{k=1}^{n} t_k^{H_{\min}-2\delta} \to \int_0^1 s^{H_{\min}-2\delta} ds$, as $n \to \infty$.

(ii) Case $|a| > 1$. Since

$$\left| \int_{H_{t_{k-1}}}^{H_{t_k}} \varphi\left(\frac{a}{t_k^x}\right) t_k^{-2x} dx \right| \leq \varphi\left(\frac{a}{t_k^{H_{\min}}}\right) t_k^{-2H_{\max}} \left| H_{t_k} - H_{t_{k-1}} \right|$$
$$\leq C n^{-\alpha} \varphi\left(\frac{a}{t_k^{H_{\min}}}\right) t_k^{-2H_{\max}},$$

we see that

$$D_1 \le C n^{1-2\alpha} a^2 \sum_{k=2}^{n} \varphi\left(\frac{a}{t_k^{H_{\min}}}\right) t_k^{H_{\min}-2H_{\max}-2\delta} \frac{1}{n}.$$

Note that

$$\sum_{k=1}^{n} \varphi\left(\frac{a}{t_k^{H_{\min}}}\right) t_k^{H_{\min}-2H_{\max}-2\delta} \frac{1}{n} \to \int_0^1 \varphi\left(\frac{a}{s^{H_{\min}}}\right) s^{H_{\min}-2H_{\max}-2\delta} ds, \quad \text{as } n \to \infty,$$

where the integral is bounded by $Ca^{-2}\varphi(a)$ according to Lemma 4.2 (here we can choose $\delta > (1 + H_{\min} - 2H_{\max})/2$). Thus, in this case we also have

$$D_1 \le C\varphi(a) n^{1-2\alpha}. \tag{4.23}$$

Now it remains to estimate D_2. Using (4.12), we can write

$$\varphi\left(\frac{a}{t_k^{H_{t_{k-1}}}}\right) - \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) = H_{t_{k-1}} a^2 \int_{t_{k-1}}^{t_k} s^{-2H_{t_{k-1}}-1} \varphi\left(\frac{a}{s^{H_{t_{k-1}}}}\right) ds. \tag{4.24}$$

In addition,

$$\partial_s \mathbb{P}\left(Y > \frac{a}{s^{H_{t_{k-1}}}}\right) = \partial_s \int_{\frac{a}{s^{H_{t_{k-1}}}}}^{\infty} \varphi(v) dv = H_{t_{k-1}} a s^{-H_{t_{k-1}}-1} \varphi\left(\frac{a}{s^{H_{t_{k-1}}}}\right)$$

and

$$\mathbb{P}\left(Y > \frac{a}{t_k^{H_{t_{k-1}}}}\right) - \mathbb{P}\left(Y > \frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) = H_{t_{k-1}} a \int_{t_{k-1}}^{t_k} s^{-H_{t_{k-1}}-1} \varphi\left(\frac{a}{s^{H_{t_{k-1}}}}\right) ds. \tag{4.25}$$

After substitution (4.24) and (4.25) into (4.18), we arrive at

$$\begin{aligned} D_2 &= a^2 \left| \sum_{k=2}^{n} H_{t_{k-1}} \int_{t_{k-1}}^{t_k} s^{-2H_{t_{k-1}}-1} \varphi\left(\frac{a}{s^{H_{t_{k-1}}}}\right) \left(t_k^{H_{t_k}} - s^{H_{t_{k-1}}}\right) ds \right| \\ &\le a^2 \sum_{k=2}^{n} H_{t_{k-1}} \int_{t_{k-1}}^{t_k} s^{-2H_{t_{k-1}}-1} \varphi\left(\frac{a}{s^{H_{t_{k-1}}}}\right) \left| t_k^{H_{t_k}} - t_k^{H_{t_{k-1}}} \right| ds \\ &\quad + a^2 \sum_{k=2}^{n} H_{t_{k-1}} \int_{t_{k-1}}^{t_k} s^{-2H_{t_{k-1}}-1} \varphi\left(\frac{a}{s^{H_{t_{k-1}}}}\right) \left| t_k^{H_{t_{k-1}}} - s^{H_{t_{k-1}}} \right| ds \\ &=: D_{21} + D_{22}. \end{aligned} \tag{4.26}$$

In order to bound D_{21}, we write using (4.2)

$$\left| t_k^{H_{t_k}} - t_k^{H_{t_{k-1}}} \right| \le C t_k^{H_{\min}} n^{-\alpha} \log n \le C n^{-\alpha} \log n.$$

Then,

$$D_{21} \le C n^{-\alpha} \log n \sum_{k=2}^{n} a^2 \int_{t_{k-1}}^{t_k} s^{-2H_{t_{k-1}}-1} \varphi\left(\frac{a}{s^{H_{t_{k-1}}}}\right) ds.$$

(i) Case $|a| \le 1$. Since by Lemma 4.1

$$a^2 s^{-2H_{t_{k-1}}} \varphi\left(\frac{a}{s^{H_{t_{k-1}}}}\right) \le C\varphi(a), \tag{4.27}$$

we see that

$$\begin{aligned} D_{21} &\le C\varphi(a) n^{-\alpha} \log n \sum_{k=2}^{n} \int_{t_{k-1}}^{t_k} s^{-1} ds = C\varphi(a) n^{-\alpha} \log n \int_{t_1}^{1} s^{-1} ds \\ &= C\varphi(a) n^{-\alpha} \log^2 n. \end{aligned}$$

(ii) Case $|a| > 1$. We have

$$\begin{aligned} D_{21} &\le C n^{-\alpha} \log n \sum_{k=2}^{n} a^2 \int_{t_{k-1}}^{t_k} s^{-2H_{\max}-1} \varphi\left(\frac{a}{s^{H_{\min}}}\right) ds \\ &\le C n^{-\alpha} \log n \, a^2 \int_0^1 s^{-2H_{\max}-1} \varphi\left(\frac{a}{s^{H_{\min}}}\right) ds \le C\varphi(a) n^{-\alpha} \log n, \end{aligned}$$

where the last inequality follows from Lemma 4.2.
Thus, in both cases we have the following upper bound

$$D_{21} \le C\varphi(a) n^{-\alpha} \log^2 n.$$

For any $\delta > 0$, $\log n \le C n^{\delta}$ for some $C = C(\delta)$. Therefore choosing $\delta < \frac{1}{2}(1-\alpha)$, we finally get

$$D_{21} \le C\varphi(a) n^{-\alpha+2\delta} \le C\varphi(a) n^{1-2\alpha}. \tag{4.28}$$

Let us consider D_{22}. By the mean value theorem,

$$t_k^{H_{t_{k-1}}} - s^{H_{t_{k-1}}} \le H_{t_{k-1}} s^{H_{t_{k-1}}-1} (t_k - s) \le s^{H_{\min}-1} \frac{1}{n}.$$

Therefore,

$$D_{22} \leq \frac{a^2}{n} \sum_{k=2}^{n} \int_{t_{k-1}}^{t_k} s^{-2H_{t_{k-1}}+H_{\min}-2} \varphi\left(\frac{a}{s^{H_{t_{k-1}}}}\right) ds.$$

Again, let us consider two cases.

(i) Case $|a| \leq 1$. Applying the bound (4.27), we obtain

$$\begin{aligned} D_{22} &\leq C\varphi(a)n^{-1} \sum_{k=2}^{n} \int_{t_{k-1}}^{t_k} s^{H_{\min}-2} ds = C\varphi(a)n^{-1} \int_{n^{-1}}^{1} s^{H_{\min}-2} ds \\ &= C\varphi(a)n^{-1} \frac{n^{1-H_{\min}} - 1}{1 - H_{\min}} \leq C\varphi(a)n^{-H_{\min}}. \end{aligned}$$

(ii) Case $|a| > 1$.

$$\begin{aligned} D_{22} &\leq \frac{a^2}{n} \sum_{k=2}^{n} \int_{t_{k-1}}^{t_k} s^{-2H_{\max}+H_{\min}-2} \varphi\left(\frac{a}{s^{H_{\min}}}\right) ds \\ &\leq \frac{a^2}{n} \int_{0}^{1} s^{-2H_{\max}+H_{\min}-2} \varphi\left(\frac{a}{s^{H_{\min}}}\right) ds \leq C\varphi(a)n^{-1} \leq C\varphi(a)n^{-H_{\min}}. \end{aligned}$$

Hence, in both cases $D_{22} \leq C\varphi(a)n^{-H_{\min}}$. Combining this bound with representation (4.26) and inequality (4.28), we get

$$D_2 \leq C\varphi(a)n^{-\min\{H_{\min}, 2\alpha-1\}}. \tag{4.29}$$

Now the proof follows from (4.16), (4.20), (4.23), and (4.29). □

4.3 Proof of Theorem 3.1

The proof of the main result will be done in two steps. We start by considering the case $\Psi(x) = (x - a)^+$. Then, the general case will be reduced to that case by application of the following lemma, proved in [2].

Lemma 4.6 ([2, Lemma 4.1]) *Let Ψ be convex and $\psi = \Psi'_-$ be its left-sided derivative. Then, for any $x, y \in \mathbb{R}$ we have*

$$\begin{aligned} \Psi(x) - \Psi(y) - \psi(y)(x - y) &= \int_{\mathbb{R}} [|x - a| - |y - a| - \operatorname{sgn}(y - a)(x - y)]\mu(da) \\ &= 2 \int_{\mathbb{R}} \left[(x - a)^+ - (y - a)^+ - \mathbb{1}_{y>a}(x - y)\right] \mu(da) \\ &\geq 0. \end{aligned}$$

4.3.1 Case $\Psi(x) = (x-a)^+$

Proposition 4.7 *Let X be a multifractional Brownian motion with the Hurst function H_t satisfying (A1)–(A2). Let $\widetilde{H} \in (\frac{1}{2}, H_{\min}] \cap (\frac{1}{2}, \alpha)$. Then for any $a \in \mathbb{R}$,*

$$\begin{aligned}&\mathbb{E}\left|\int_0^1 \mathbb{1}_{X_s>a}\,dX_s - \sum_{k=1}^{n} \mathbb{1}_{X_{t_{k-1}}>a}\left(X_{t_k} - X_{t_{k-1}}\right)\right| \\ &\le \frac{1}{2}\int_0^1 s^{-H_s}\varphi\left(as^{-H_s}\right)ds\left(\frac{1}{n}\right)^{2\widetilde{H}-1} + R_n(a),\end{aligned} \tag{4.30}$$

where the remainder satisfies

$$R_n(a) \le C\varphi(a)\,n^{-\min\{2\widetilde{H}-H_{\max},H_{\min}+\alpha-1,2\alpha-1\}}. \tag{4.31}$$

Proof The proof follows the scheme from [2, Prop. 4.11]. By the chain rule (2.7), we obtain

$$\begin{aligned}\int_0^1 \mathbb{1}_{X_s>a}dX_s &= (X_1-a)^+ - (X_0-a)^+ \\ &= \sum_{k=1}^{n}\left[\left(X_{t_k}-a\right)^+ - \left(X_{t_{k-1}}-a\right)^+\right].\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^1 \mathbb{1}_{X_s>a}\,dX_s - \sum_{k=1}^{n}\mathbb{1}_{X_{t_{k-1}}>a}\left(X_{t_k}-X_{t_{k-1}}\right) \\ &= \sum_{k=1}^{n}\left[\left(X_{t_k}-a\right)^+ - \left(X_{t_{k-1}}-a\right)^+ - \mathbb{1}_{X_{t_{k-1}}>a}\left(X_{t_k}-X_{t_{k-1}}\right)\right] \\ &\ge 0,\end{aligned} \tag{4.32}$$

where the last inequality follows from Lemma 4.6. Further, from $(x-a)^+ = x\mathbb{1}_{x>a} - a\mathbb{1}_{x>a}$, we obtain the following representation for one-interval increment:

$$\begin{aligned}&\left(X_{t_k}-a\right)^+ - \left(X_{t_{k-1}}-a\right)^+ - \mathbb{1}_{X_{t_{k-1}}>a}\left(X_{t_k}-X_{t_{k-1}}\right) \\ &= X_{t_k}\mathbb{1}_{X_{t_k}>a} - X_{t_k}\mathbb{1}_{X_{t_{k-1}}>a} - a\mathbb{1}_{X_{t_k}>a} + a\mathbb{1}_{X_{t_{k-1}}>a}.\end{aligned} \tag{4.33}$$

Evidently, we have from (3.1) for $k = 1,\dots,n$

$$\mathbb{E}\left[X_{t_k}\mathbb{1}_{X_{t_k}>a}\right] = \sqrt{V(t_k)}\varphi\left(\frac{a}{\sqrt{V(t_k)}}\right) = t_k^{H_{t_k}}\varphi\left(\frac{a}{t_k^{H_{t_k}}}\right), \tag{4.34}$$

and

$$\mathbb{E}\mathbb{1}_{X_{t_k}>a} = \mathbb{P}\left(Y > \frac{a}{t_k^{H_{t_k}}}\right), \tag{4.35}$$

where $Y \sim \mathcal{N}(0,1)$. Note that the relations (4.34) and (4.35) remain valid for $k = 0$ under the convention $\varphi(\pm\infty) = 0$, $\mathbb{P}(Y > +\infty) = 0$, $\mathbb{P}(Y > -\infty) = 1$.

In order to compute $\mathbb{E}[X_{t_k}\mathbb{1}_{X_{t_{k-1}}>a}]$, we denote

$$\gamma_1 = 0, \quad \gamma_k = \frac{\operatorname{Cov}\left(X_{t_k}, X_{t_{k-1}}\right)}{\operatorname{Var} X_{t_{k-1}}}, \ k = 2, \dots, n,$$

and use the representation

$$X_{t_k} = \gamma_k X_{t_{k-1}} + b_k Y_k,$$

where $Y_k \sim \mathcal{N}(0,1)$ is independent of $X_{t_{k-1}}$ and b_k is a normalizing constant (that is, $b_k^2 = \operatorname{Var} X_{t_k} - \gamma_k^2 \operatorname{Var} X_{t_{k-1}}$). Then, we get

$$\mathbb{E}\left[X_{t_k}\mathbb{1}_{X_{t_{k-1}}>a}\right] = \gamma_k \mathbb{E}\left[X_{t_{k-1}}\mathbb{1}_{X_{t_{k-1}}>a}\right] = \gamma_k t_{k-1}^{H_{t_{k-1}}} \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right). \tag{4.36}$$

Combining (4.33)–(4.36) and rearranging terms, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(X_{t_k} - a\right)^+ - \left(X_{t_{k-1}} - a\right)^+ - \mathbb{1}_{X_{t_{k-1}}>a}\left(X_{t_k} - X_{t_{k-1}}\right)\right] \\
&= t_k^{H_{t_k}} \varphi\left(\frac{a}{t_k^{H_{t_k}}}\right) - \gamma_k t_{k-1}^{H_{t_{k-1}}} \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) + a\mathbb{P}\left(Y > \frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) - a\mathbb{P}\left(Y > \frac{a}{t_k^{H_{t_k}}}\right) \\
&= \left[t_k^{H_{t_k}} - \gamma_k t_{k-1}^{H_{t_{k-1}}}\right] \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) + t_k^{H_{t_k}} \left[\varphi\left(\frac{a}{t_k^{H_{t_k}}}\right) - \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right)\right] \\
&\quad + a\mathbb{P}\left(Y > \frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) - a\mathbb{P}\left(Y > \frac{a}{t_k^{H_{t_k}}}\right).
\end{aligned}$$

Therefore,

$$\mathbb{E}\left|\int_0^1 \mathbb{1}_{X_s>a}\, dX_s - \sum_{k=1}^n \mathbb{1}_{X_{t_{k-1}}>a}\left(X_{t_k} - X_{t_{k-1}}\right)\right| = I_{1,n} + I_{2,n} + I_{3,n},$$

where

$$I_{1,n} = \sum_{k=2}^{n} \left[t_k^{H_{t_k}} - \gamma_k t_{k-1}^{H_{t_{k-1}}} \right] \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right),$$

$$I_{2,n} = \sum_{k=1}^{n} t_k^{H_{t_k}} \left[\varphi\left(\frac{a}{t_k^{H_{t_k}}}\right) - \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) \right],$$

$$I_{3,n} = \sum_{k=1}^{n} \left[a\mathbb{P}\left(Y > \frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) - a\mathbb{P}\left(Y > \frac{a}{t_k^{H_{t_k}}}\right) \right].$$

Applying [2, Lemma 4.4], we may write

$$I_{1,n} = I_{1,A,n} + I_{1,B,n},$$

where

$$I_{1,A,n} = -\sum_{k=2}^{n} \frac{\left(t_k^{H_{t_k}} - t_{k-1}^{H_{t_{k-1}}}\right)^2}{2t_{k-1}^{H_{t_{k-1}}}} \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right),$$

$$I_{1,B,n} = \sum_{k=2}^{n} \frac{\vartheta(t_k, t_{k-1})}{2t_{k-1}^{H_{t_{k-1}}}} \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right).$$

By Lemma 4.3,

$$\left|I_{1,A,n}\right| \le C\varphi(a) n^{-\min\{H_{\min}, 2\alpha-1\}}.$$

Observing that $H_{\min} > 2H_{\min} - H_{\max} \ge 2\widetilde{H} - H_{\max}$, we obtain

$$\left|I_{1,A,n}\right| \le C\varphi(a) n^{-\min\{2\widetilde{H}-H_{\max}, 2\alpha-1\}}.$$

Let us consider $I_{1,B,n}$. Applying Lemma 2.4, we get

$$\vartheta(t_k, t_{k-1}) \le n^{-2H_{\min}} + Cn^{-H_{\min}-\alpha} + Cn^{-2\alpha} \le n^{-2\widetilde{H}} + Cn^{-\min\{H_{\min}+\alpha, 2\alpha\}}. \tag{4.37}$$

Moreover, by Lemma 4.4

$$\frac{1}{n}\sum_{k=2}^{n} t_{k-1}^{-H_{t_{k-1}}} \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) = \int_0^1 s^{-H_s} \varphi\left(\frac{a}{s^{H_s}}\right) ds + R_{2,B,n},$$

where $R_{2,B,n} \leq C\varphi(a)n^{H_{\max}-1}$. Hence,

$$\begin{aligned}
I_{1,B,n} &\leq \left(n^{-2\widetilde{H}} + Cn^{-\min\{H_{\min}+\alpha,2\alpha\}}\right) \sum_{k=2}^{n} \frac{1}{2t_{k-1}^{H_{t_{k-1}}}} \varphi\left(\frac{a}{t_{k-1}^{H_{t_{k-1}}}}\right) \\
&= \frac{1}{2}\left(n^{1-2\widetilde{H}} + Cn^{1-\min\{H_{\min}+\alpha,2\alpha\}}\right)\left(\int_0^1 s^{-H_s}\varphi\left(\frac{a}{s^{H_s}}\right) ds + R_{2,B,n}\right) \\
&= \frac{1}{2} n^{1-2\widetilde{H}} \int_0^1 s^{-H_s}\varphi\left(\frac{a}{s^{H_s}}\right) ds + R'_{2,B,n} + R''_{2,B,n},
\end{aligned}$$

where

$$\begin{aligned}
R'_{2,B,n} &= Cn^{1-\min\{H_{\min}+\alpha,2\alpha\}} \int_0^1 s^{-H_s}\varphi\left(\frac{a}{s^{H_s}}\right) ds \\
&\leq Cn^{1-\min\{H_{\min}+\alpha,2\alpha\}}\varphi(a) \int_0^1 s^{-H_{\max}} ds \leq C\varphi(a)n^{-\min\{H_{\min}+\alpha-1,2\alpha-1\}}
\end{aligned}$$

and

$$\begin{aligned}
R''_{2,B,n} &= \left(n^{1-2\widetilde{H}} + Cn^{1-\min\{H_{\min}+\alpha,2\alpha\}}\right) R_{2,B,n} \leq C\varphi(a)n^{H_{\max}-\min\{2\widetilde{H},H_{\min}+\alpha,2\alpha\}} \\
&\leq C\varphi(a)n^{-\min\{2\widetilde{H}-H_{\max},H_{\min}+\alpha-1,2\alpha-1\}}.
\end{aligned}$$

According to Lemma 4.5,

$$\left|I_{2,n} + I_{3,n}\right| \leq C\varphi(a)n^{-\min\{H_{\min},2\alpha-1\}} \leq C\varphi(a)n^{-\min\{2\widetilde{H}-H_{\max},2\alpha-1\}}.$$

Combining all above estimates, we conclude the proof. □

4.3.2 Proof of Theorem 3.1

Using Lemma 4.6 and (2.7), we have

$$\begin{aligned}
&\Psi(X_1) - \Psi(X_0) - \sum_{k=1}^{n} \Psi'\left(X_{t_{k-1}}\right)\left(X_{t_k} - X_{t_{k-1}}\right) \\
&= \sum_{k=1}^{n} \left[\Psi\left(X_{t_k}\right) - \Psi\left(X_{t_{k-1}}\right) - \Psi'\left(X_{t_{k-1}}\right)\left(X_{t_k} - X_{t_{k-1}}\right)\right] \\
&= 2\int_{\mathbb{R}} Z_n^+(a)\mu(da)
\end{aligned}$$

where

$$Z_n^+(a) = \sum_{k=1}^{n} \left[\left(X_{t_k} - a\right)^+ - \left(X_{t_{k-1}} - a\right)^+ - \mathbb{1}_{X_{t_{k-1}}>a}\left(X_{t_k} - X_{t_{k-1}}\right)\right]$$

$$
\begin{aligned}
&= \int_0^1 \mathbb{1}_{X_s > a} dX_s - \sum_{k=1}^{n} \mathbb{1}_{X_{t_{k-1}} > a} \left(X_{t_k} - X_{t_{k-1}} \right) \\
&\geq 0,
\end{aligned}
$$

see (4.32). Taking expectation and using Proposition 4.7 to compute $\mathbb{E}Z_n^+(a)$, we get

$$
\begin{aligned}
&\mathbb{E}\left| \Psi(X_1) - \Psi(X_0) - \sum_{k=1}^{n} \Psi'\left(X_{t_{k-1}}\right)\left(X_{t_k} - X_{t_{k-1}}\right) \right| \\
&= 2 \int_{\mathbb{R}} \mathbb{E}Z_n^+(a)\mu(da) \\
&= \int_{\mathbb{R}} \int_0^1 s^{-H_s} \varphi\left(as^{-H_s}\right) ds\, \mu(da) \left(\frac{1}{n}\right)^{2H_{\min}-1} + 2 \int_{\mathbb{R}} R_n(a)\, \mu(da).
\end{aligned}
$$

Here, the remainder $R_n(a)$ is defined in Proposition 4.7 and satisfies

$$
R_n(a) \leq C\varphi(a)\, n^{-\min\{2\widetilde{H} - H_{\max}, H_{\min} + \alpha - 1, 2\alpha - 1\}}.
$$

which is integrable since $\int_{\mathbb{R}} \varphi(a)\mu(da) < \infty$ by assumption. Similarly, the leading-order term is finite by the fact that

$$
\int_0^1 s^{-H_s} \varphi\left(as^{-H_s}\right) ds \leq \varphi(a) \int_0^1 s^{-H_{\max}}\, ds \leq C\varphi(a).
$$

This completes the proof. □

Acknowledgements Kostiantyn Ralchenko gratefully acknowledges support from the Research Council of Finland, decision number 359815.

Author Contributions All authors contributed equally to this work. All authors reviewed the manuscript.

Funding Open Access funding provided by University of Vaasa.

Data availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no conflict of interest.

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