



# Local Spectral Theory for Unbounded Product of Nonnegative Selfadjoint Operators

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## Abstract

An unbounded product  $T = AB$  of two nonnegative selfadjoint operators  $A$  and  $B$ , where  $\rho(T) \neq \emptyset$  and  $A$  is bounded, is proved to have the single valued extension property (SVEP) and, more significantly, to be a  $\mathbb{C}$ -generalized scalar operator. This property plays a central role as, it not only implies the Dunford's property for the product  $AB$ , but it also allows any operator  $T$  that is quasi-affine to a nonnegative selfadjoint operator  $S$  to satisfy  $\sigma(T) = \sigma(S)$ . These results have required a deeper study of the spectral connection between operators  $S$  having SVEP and Dunford properties and operators  $T$  that are quasi-affine to  $S$ .

**Keywords** Selfadjoint operator · Nonnegative operator · Quasi-affinity · Quasi-similarity · Single valued extension property · Dunford's property

**Mathematics Subject Classification** 47B65 · 47B25 · 47A05 · 47A11

## 1 Introduction

The study of operator products of nonnegative selfadjoint operators on a complex Hilbert space  $\mathcal{H}$  has always been a source of motivation for many mathematicians [9, 15]. In particular, the class

$$\mathcal{O}^{+2}(\mathcal{H}) = \left\{ T = AB; A = A^* \geq 0 \text{ and } B = B^* \geq 0 \right\} \quad (1.1)$$

This paper is dedicated to our great friend and colleague Henk de Snoo on the occasion of his 80th birthday, wishing we can share many new happy moments in the coming years, too.

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has been considered by P.Y.Wu [32], in 1988, in the finite-dimensional setting where the following equivalence holds for a matrix  $M$  :

$$M \in \mathcal{O}^{+2}(\mathfrak{H}) \Leftrightarrow M \text{ is similar to a positive matrix.} \tag{1.2}$$

In 2020, M. Contino, M. A. Dritschel, A. Maestripieri, and S. Marcantognini [8] considered this problem for infinite-dimensional spaces, notably, when the elements  $A, B$  of (1.1) belong to the class  $B^+(\mathfrak{H})$  of bounded nonnegative operators on  $\mathfrak{H}$ . Based on the following observation

$$T \text{ is similar to } S \in B^+(\mathfrak{H}) \Leftrightarrow \sigma(T) \subseteq \mathbb{R}^+ \text{ and } T \text{ is a scalar operator,} \tag{1.3}$$

they proved that, although  $\sigma(T) \subseteq \mathbb{R}^+$  for any  $T \in \mathcal{O}^{+2}(\mathfrak{H})$ , similarity to a bounded nonnegative operator is no longer sufficient to cover the class  $\mathcal{O}^{+2}(\mathfrak{H})$ , as in (1.2). Even weaker conditions than similarity such as quasi-affinity to nonnegative operators are also shown to be insufficient to characterize that class. However, one has the following implication:

$$T \in \mathcal{O}^{+2}(\mathfrak{H}) \text{ and } T \text{ is quasi-affine to } S \in B^+(\mathfrak{H}) \Rightarrow \sigma(T) = \sigma(S), \tag{1.4}$$

which shows that quasi-affinity behaves like similarity in  $\mathcal{O}^{+2}(\mathfrak{H})$ . Motivated by (1.3), instead of scalar operators, a larger class would be considered and shown to include the class  $\mathcal{O}^{+2}(\mathfrak{H})$ , which is the class of generalized scalar operators. These operators play a crucial role, specially in the unbounded case, since they will give rise to different spectral properties of  $\mathcal{O}^{+2}(\mathfrak{H})$ , already proved in [8].

On the other hand, the latter class was characterized via Sebestyén theorem [23] as follows:

$$T \in \mathcal{O}^{+2}(\mathfrak{H}) \Leftrightarrow T^*T \leq XT \text{ has a bounded nonnegative solution } X \tag{1.5}$$

$$\Leftrightarrow T \text{ has an optimal pair,} \tag{1.6}$$

where a pair  $(A, B) \in B^+(\mathfrak{H}) \times B^+(\mathfrak{H})$  is called optimal for  $T = AB$ , if

$$\ker T = \ker B \text{ and } \overline{\text{ran}} T = \overline{\text{ran}} A. \tag{1.7}$$

In the present paper, the above results will be extended to the general setting of closed unbounded operators. In particular, the emphasis will be on the following subclass of closed operators of  $\mathcal{O}^{+2}(\mathfrak{H})$

$$\mathcal{L}_l^{+2}(\mathfrak{H}) = \left\{ T = AB; A \in B^+(\mathfrak{H}) \text{ and } B = B^* \geq 0 \right\}, \tag{1.8}$$

where  $B$  is in general a closed unbounded operator on  $\mathfrak{H}$ . Note that this class has already been introduced in [3] and characterized through an inequality similar to (1.5). However, unlike (1.6), a further condition is needed to characterize the optimality as

shown in the equivalence below:

$$T \in \mathcal{L}_{op}^{+2}(\mathfrak{H}) \Leftrightarrow T^*T \leq T^*Y \text{ has a solution } Y = Y^* \geq 0 \text{ such that}$$

$$\text{dom } T \subseteq \text{dom } Y \subseteq \text{dom } (T^*Y)^{\frac{1}{2}} \text{ and } \ker T = \ker Y.$$

Here  $\mathcal{L}_{op}^{+2}(\mathfrak{H})$  is the class of operators in  $\mathcal{L}_l^{+2}(\mathfrak{H})$  with optimal pairs, which will be considered in detail in Section 2.1. These operators are in fact needed to conclude properties shared by normal operators, such as having finite ascent and possessing the following description of the analytic spectral subspace of  $T \in \mathcal{L}_{op}^{+2}(\mathfrak{H})$  at any  $\lambda \in \mathbb{C}$  :

$$\ker(T^* - \lambda I) = \mathfrak{H}_{T^*}(\{\lambda\}).$$

This will be seen in Section 3.1, where an operator  $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$  with  $\rho(T) \neq \emptyset$  is shown to satisfy  $\sigma(T) \subseteq \mathbb{R}^+$ , and this gives rise to the first local spectral property: the single valued extension property (SVEP). More significantly, such an operator is proved to be a  $\mathbb{C}$ -generalized scalar operator, exactly as in the bounded case. The latter property is the core of Section 3.2 and constitutes a useful key for the study of further local spectral properties of  $T$  namely property (k) which is, by definition, a combination of the SVEP and Dunford properties; cf [25]. This property appears to be very important in the last part of this paper, which will be devoted to the quasi-affinity to nonnegative selfadjoint operators.

Recall from [21, 22] that a linear operator  $T_1$  is said to be *G-quasi-affine*, shortly *quasi-affine*, to some linear operator  $T_2$  if there exists a quasi-affinity  $G \in B(\mathfrak{H})$  such that  $GT_1 \subseteq T_2G$ . Here  $B(\mathfrak{H})$  denotes the class of bounded operators from  $\mathfrak{H}$  to  $\mathfrak{H}$  and a quasi-affinity is a bounded injective operator with dense range. In Section 3.3, the implication in (1.4) is improved and generalized not only for unbounded operators but also for operators that are not necessarily in  $\mathcal{L}_l^{+2}(\mathfrak{H})$ . For this, one first establishes that once a closed operator  $T$  is quasi-affine to an operator  $S$  with property (k) then

$$\sigma(S) \subseteq \sigma(T);$$

see Lemma 3.16. Note that this result is a generalization of [8, Lemma 3.7], where a stronger condition than property (k) condition was considered for  $S$ , i.e.  $S \in B^+(\mathfrak{H})$ . This is a useful tool to finally conclude that any operator  $T$  such that  $T$  and  $T^*$  have property (k) satisfies the following implication

$$T \text{ is quasi-affine to } S = S^* \geq 0 \Rightarrow \sigma(T) = \sigma(S) \subseteq \mathbb{R}^+;$$

see Corollary 3.17. In this case, the Hilbert space  $\mathfrak{H}$  can even be described via the local analytic spectral subspace of both  $S$  and  $T^*$  as follows:

$$\mathfrak{H} = \mathfrak{H}_S(\sigma(T)) = \mathfrak{H}_{T^*}(\sigma(S^*)). \tag{1.9}$$

Note that in the general case where  $T$  is quasi-affine to a closed operator  $T_2$ , only the first equality in (1.9) is proved to hold for  $S = T_2$ ; see Lemma 3.16.

## 2 Preliminaries

In this section  $T \in LO(\mathfrak{H}, \mathfrak{K})$  stands for a linear operator from  $\mathfrak{H}$  to a complex Hilbert space  $\mathfrak{K}$  with domain  $\text{dom } T$  and range  $\text{ran } T$ . If  $\mathfrak{H} = \mathfrak{K}$  then  $LO(\mathfrak{H}, \mathfrak{K}) = LO(\mathfrak{H})$ . In addition, one writes  $T \in CO(\mathfrak{H}, \mathfrak{K})$  if  $T$  is closed and  $T \in CO(\mathfrak{H})$  if  $\mathfrak{K} = \mathfrak{H}$ . In this case, the resolvent set  $\rho(T)$  of  $T$  is the set of all  $\mu \in \mathbb{C}$  for which  $R(\mu, T) := (T - \mu I)^{-1} \in B(\mathfrak{H})$ . The set  $\sigma(T) = \mathbb{C} \setminus \rho(T)$  is called the *spectrum* of  $T$ . The *surjective spectrum* of  $T \in CO(\mathfrak{H})$  is  $\sigma_{sur}(T) := \mathbb{C} \setminus \rho_{sur}(T)$ , where  $\rho_{sur}(T)$  is the set of all  $\mu \in \mathbb{C}$  for which  $\text{ran } (T - \mu I) = \mathfrak{H}$ .

An operator  $T \in LO(\mathfrak{H})$  is said to be *symmetric* if  $\langle Tx, y \rangle = \langle x, Ty \rangle$  for all  $x, y \in \text{dom } T$ . If  $\langle Tx, x \rangle \geq 0$  for all  $x \in \text{dom } T$ , then  $T$  is *nonnegative*. It is *selfadjoint* when  $\overline{\text{dom } T} = \mathfrak{H}$  and  $T^* = T$ . Note that if  $T$  is nonnegative and selfadjoint, then it admits a unique nonnegative selfadjoint square root which will be denoted by  $T^{\frac{1}{2}}$ ; cf. [24, 31]. If  $0 \leq T = T^* \in B(\mathfrak{H})$  then one writes  $T \in B^+(\mathfrak{H})$ .

Similar to the bounded case, the class  $\mathcal{L}_l^{+2}(\mathfrak{H})$  has been treated and characterized in [3, Theorem 2.12] through Sebestyén inequality involving unbounded operators as follows.

**Theorem 2.1** *Let  $T$  be a closed densely defined operator. Then,  $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$  if and only if  $T^*T \leq T^*Y = YT$  admits a solution  $Y = Y^* \geq 0$  such that  $\text{dom } T \subseteq \text{dom } Y \subseteq \text{dom } (T^*Y)^{\frac{1}{2}}$ .*

The following result appears in [3, Proposition 2.5] and shows that, exactly as in the finite-dimensional case, any element  $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$  has positive spectrum. It is revisited here for its relevance and applicability.

**Corollary 2.2** *If  $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$  then there exists a nonnegative selfadjoint operator  $S$  such that  $T$  is pre-similar to  $S$ , i.e.,  $TG = GS$  for some  $G \in B(\mathfrak{H})$ . Moreover, if  $\rho(T) \neq \emptyset$ , then*

$$\sigma(T) \subseteq \mathbb{R}^+.$$

**Lemma 2.3** ([3, Lemma 2.1]) *Let  $A \in B^+(\mathfrak{K})$  and  $B \in CO(\mathfrak{H}, \mathfrak{K})$ , and let  $\alpha \in [0, 1]$ . If  $AB^{**}$  is closed (closable), then  $A^\alpha B^{**}$  is closed (closable) and*

$$(B^*A^\alpha)^* = A^\alpha B^{**}. \tag{2.1}$$

*Analogously, if  $\ker A = \{0\}$  and  $B^{**}A^{-1}$  is closed, then  $B^{**}A^{-\alpha}$  is closed and*

$$(A^{-\alpha}B^*)^* = B^{**}A^{-\alpha}. \tag{2.2}$$

### 2.1 Optimal pair

Recall that in the bounded case, an operator  $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$  can be factorized as  $T = AB$ , where  $A, B \in B^+(\mathfrak{H})$ ,  $\ker B = \ker T$  and  $\overline{\text{ran } T} = \overline{\text{ran } A}$ . Such a pair  $(A, B)$  is called

optimal; see [8]. In the case of bounded operators,  $T$  and  $T^*$  play a symmetric role since  $A$  and  $B$  both are bounded. Hence one can conclude the following equivalences:

$$\begin{aligned}
 T \text{ has an optimal pair} &\Leftrightarrow T^*T \leq TX \text{ admits a solution } X \in B^+(\mathfrak{H}) \\
 &\Leftrightarrow T \in \mathcal{L}_l^{+2}(\mathfrak{H}).
 \end{aligned}
 \tag{2.3}$$

see [8, Theorem 4.5]. The objective of this section is to develop suitable approaches for extending the equivalence (2.3) for unbounded operators; see Theorem 2.8.

**Definition 2.4** If  $T = AB \in \mathcal{L}_l^{+2}(\mathfrak{H})$ , then the pair  $(A, B)$  is called *optimal* if

$$\overline{\text{ran}} T = \overline{\text{ran}} A \quad \text{and} \quad \ker T = \ker B.
 \tag{2.4}$$

The subclass of operators  $T = AB \in \mathcal{L}_l^{+2}(\mathfrak{H})$  consisting of optimal pairs  $(A, B)$ , where  $A \in B^+(\mathfrak{H})$  and  $B = B^* \geq 0$ , is denoted by  $\mathcal{L}_{op}^{+2}(\mathfrak{H})$ .

**Example 2.5** Let

$$\begin{aligned}
 T : l^2(\mathbb{N}^*) &\rightarrow l^2(\mathbb{N}^*) \\
 x = (x_1, x_2, \dots) &\mapsto (0, 2x_2, 0, 4x_4, \dots, 2nx_{2n}, \dots).
 \end{aligned}$$

Then, for  $A(x_n) := (0, x_2, 0, x_4, \dots, x_{2n}, \dots)$  and  $B(x_n) := (nx_n)_{n \in \mathbb{N}^*}$ , one observes that  $T = AB \in \mathcal{L}_l^{+2}(l^2(\mathbb{N}^*))$  although  $(A, B)$  is not an optimal pair for  $T$ , since  $\ker T \not\subseteq \ker B = \{0\}$ . However,  $T = AT$  and  $T$  itself is also nonnegative and selfadjoint, so  $(A, T)$  can be chosen to be an optimal pair for  $T$ . Here,  $\sigma(T) = \sigma(AB) = \sigma(BA) = 2\mathbb{N} \subseteq \mathbb{R}^+$ .

The first step towards the construction of an optimal pair in (2.4) relies on the following theorem, where Sebestyén’s inequality is characterized in the framework of unbounded operators. Note that the proof is already established in [3, Proposition 2.10] under the additional condition  $\text{dom } T = \text{dom } (T^*B)^{\frac{1}{2}}$ . Here, the latter condition is omitted and one still get the same result. For a completeness, the proof of (i)  $\Rightarrow$  (ii) is given.

**Theorem 2.6** Let  $T, B : \mathfrak{H} \rightarrow \mathfrak{K}$  be closed densely defined linear operators such that  $T^*B = B^*T$  is selfadjoint. Then the following assertions are equivalent for some  $0 \leq \lambda (= \|X\|)$ :

- (i)  $T^*T \leq \lambda T^*B$  and  $\text{dom } B \subseteq \text{dom } (T^*B)^{\frac{1}{2}}$ ;
- (ii)  $XB \subseteq T$  has a solution  $X \in B^+(\mathfrak{K})$ .

In this case  $B^*XB = T^*B = B^*T$

and, moreover,  $X$  can be chosen such that  $\ker T^* = \ker X$  with  $\|X\| \leq \lambda$ .

**Proof** Assume that (i) holds and rewrite the operator inequality in terms of forms as follows:

$$(Tf, Tf) \leq \lambda((T^*B)^{\frac{1}{2}}f, (T^*B)^{\frac{1}{2}}f) \quad \text{for all } f \in \text{dom } (T^*B)^{\frac{1}{2}}.
 \tag{2.5}$$

On the other hand, since

$$\text{dom } T^*B \subseteq \text{dom } B \subseteq \text{dom } (T^*B)^{\frac{1}{2}} \subseteq \text{dom } T, \tag{2.6}$$

(2.5) implies that

$$(Tf, Tf) \leq \lambda(T^*Bf, f) = \lambda(Bf, Tf) \text{ for all } f. \tag{2.7}$$

Hence the form  $(T^*Bf, g)$ ,  $f, g \in \text{dom } T^*B$ , is a restriction of the closable form  $(Bf, Tg)$ ,  $f, g \in \text{dom } B$ ; see [2, Theorem 2.2]. In fact, (2.6) shows that  $\text{dom } B$  is a core for the closed form on the righthand side on (2.5) as well as for the closure  $(H^{\frac{1}{2}}f, H^{\frac{1}{2}}g)$ ,  $f, g \in \text{dom } H^{\frac{1}{2}}$  of the form  $(Bf, Tg)$ ,  $f, g \in \text{dom } B$ , so that these two closed forms coincide and, in particular,  $T^*B = H$ . The statement (ii) is now clear from [2, Theorem 2.1].  $\square$

A consequence of Theorem 2.6 yields immediately the first condition in (2.4). The next proposition treats this even in a more general case, where  $T = AB \notin \mathcal{L}_t^{+2}(\mathfrak{H})$ .

**Proposition 2.7** *Let  $T = AB$  be such that  $A \in B^+(\mathfrak{H})$  and  $B \in CO(\mathfrak{H})$ . Then,  $T^*B = B^*T$  is a nonnegative selfadjoint operator such that*

$$\text{dom } (T^*B)^{\frac{1}{2}} = \text{dom } T = \text{dom } B. \tag{2.8}$$

Moreover, there exists  $A_0 \in B^+(\mathfrak{H})$  such that  $T = A_0B$  and  $\overline{\text{ran}} A_0 = \overline{\text{ran}} T$ . For this pair,  $\text{ran } A_0 \cap \ker B^* = \{0\}$ .

**Proof** Let  $T = AB$ , where  $A \in B^+(\mathfrak{H})$  and  $B \in CO(\mathfrak{H})$ . Since  $AB = T$  is closed and  $A \in B^+(\mathfrak{H})$ , it follows from Lemma 2.3 that  $A^{\frac{1}{2}}B$  is closed, and hence  $T^*B = B^*AB = B^*T = (A^{\frac{1}{2}}B)^*(A^{\frac{1}{2}}B)$  is a nonnegative selfadjoint operator such that

$$\text{dom } (T^*B)^{\frac{1}{2}} = \text{dom } A^{\frac{1}{2}}B = \text{dom } B = \text{dom } T,$$

which completes the proof of (2.8). Now clearly

$$T^*T = (AB)^*(AB) \leq \|A\|B^*AB = \|A\|T^*B.$$

Together with Theorem 2.6 this leads to the existence of  $A_0 \in B^+(\mathfrak{H})$  with the property that  $T = A_0B$  and  $\ker T^* = \ker A_0$ . Thus,

$$\overline{\text{ran}} T = (\ker T^*)^\perp = (\ker A_0)^\perp = \overline{\text{ran}} A_0.$$

Now, let  $x \in \text{ran } A_0 \cap \ker B^*$ . Then, there exists  $y \in \mathfrak{H}$  such that  $x = A_0y$  and  $0 = B^*x = B^*A_0y = T^*y$ . Thus,  $y \in \ker T^* = \ker A_0$  and  $x = A_0y = 0$ . Consequently,  $\text{ran } A_0 \cap \ker B^* = \{0\}$ .  $\square$

For the next step related to the second condition in (2.4), it is important to note that Theorem 2.1 yields the following implication

$$T \in \mathcal{L}_{op}^{+2}(\mathfrak{H}) \Rightarrow T^*T \leq T^*Y = YT \text{ has a solution } Y = Y^* \geq 0 \text{ such that} \\ \text{dom } T \subseteq \text{dom } Y \subseteq \text{dom } (T^*Y)^{\frac{1}{2}}.$$

Contrary to the bounded case, the reversed implication is not true in the unbounded case. This is illustrated in the following theorem where the class  $\mathcal{L}_{op}^{+2}(\mathfrak{H})$  is characterized.

**Theorem 2.8** *Let  $T \in CO(\mathfrak{H})$  be a densely defined linear operator. Then, the following statements are equivalent:*

- (i)  $T \in \mathcal{L}_{op}^{+2}(\mathfrak{H})$ ;
- (ii)  $T^*T \leq T^*Y$  has a solution  $Y = Y^* \geq 0$  such that  $\text{dom } T \subseteq \text{dom } Y \subseteq \text{dom } (T^*Y)^{\frac{1}{2}}$  and  $\ker T = \ker Y$ .

**Proof** The implication (i)  $\Rightarrow$  (ii) follows from Theorem 2.1. Now, assume (ii). Then, using Theorem 2.6, one constructs  $X \in B^+(\mathfrak{H})$  such that  $T = XY$  and  $\overline{\text{ran}} T = \overline{\text{ran}} X$ . Since  $\ker Y = \ker T$ , one concludes that  $(X, Y)$  is an optimal pair for  $T$  and  $T \in \mathcal{L}_{op}^{+2}(\mathfrak{H})$ . □

A different approach to construct optimal pairs is treated in the next proposition. It addresses the general setting in which a closed product of closed operators  $A_0B_0$  where  $A_0 \in B(\mathfrak{H})$  gives rise to a new product  $AB$  which satisfies the second condition of optimality stated in (2.4).

**Proposition 2.9** *Let  $T = AB$  be a closed densely defined linear operator on  $\mathfrak{H}$  such that  $A \in B(\mathfrak{H})$  and  $B \in CO(\mathfrak{H})$ . Then, there exist  $A_1 \in B(\mathfrak{H})$  and  $B_1 = B_1^* \geq 0$  such that*

$$T = A_1B_1 \quad \text{and} \quad \ker T = \ker B_1.$$

*In this case  $\text{dom } B_1 = \text{dom } B$ . In particular, if  $A \in \mathcal{L}_1^{+2}(\mathfrak{H})$  and  $\ker B_1 = \ker T$  then  $B_1 = B$  and  $A_1 = A$ . In this case,  $T \in \mathcal{L}_{op}^{+2}(\mathfrak{H})$ .*

**Proof** Set  $L_0 := \ker T \subseteq \text{dom } T = \text{dom } B$ . Then the closed subspace  $L_0$  decomposes  $\text{dom } B = \text{dom } B \cap L_0^\perp \oplus L_0$ , where  $\oplus$  stands for the orthogonal sum of the involved linear subspaces. Define  $\widehat{B} := B \upharpoonright L_0^\perp$ . Then,  $\text{dom } \widehat{B} = \text{dom } B \cap L_0^\perp$  and  $B = (\widehat{B}; B \upharpoonright L_0)$ . Moreover,  $\widehat{B}$  and  $B \upharpoonright L_0$  are closed and since  $L_0$  is closed,  $B \upharpoonright L_0$  is bounded by the closed graph theorem. Now, let  $\widetilde{B} := (\widehat{B}; 0 \upharpoonright L_0)$ . Then,  $\text{dom } \widetilde{B} = \text{dom } \widehat{B} \oplus L_0 = \text{dom } B$  and the following equalities hold:

$$i) T = A\widetilde{B} \quad \text{and} \quad ii) \ker \widetilde{B} = \ker T.$$

To see i), one observes that

$$\underline{T} = A(\widehat{B}; B \upharpoonright L_0) = (A\widehat{B}; AB \upharpoonright L_0) = (A\widehat{B}; T \upharpoonright \ker T) = (A\widehat{B}; 0) = A(\widetilde{B}; 0 \upharpoonright L_0) = A\widetilde{B}.$$

For the proof of *ii*), one concludes from *i*) that  $\ker \tilde{B} \subseteq \ker T$ ,

and other inclusion follows immediately from the construction of  $\tilde{B}$ .

On the other hand,  $\tilde{B}$  is closed and  $\overline{\text{dom } \tilde{B}} = \overline{\text{dom } B} = \mathfrak{H}$ . Hence, there exists a partial isometry  $U : \mathfrak{H} \rightarrow \overline{\text{ran } \tilde{B}}$ , where  $U \upharpoonright \ker \tilde{B} = 0$  and  $U : \ker \tilde{B}^\perp \rightarrow \overline{\text{ran } \tilde{B}}$  is an isometry, such that  $\tilde{B} = U|\tilde{B}|$ , which corresponds to the polar decomposition of  $\tilde{B}$ ; cf. [16]. As a result,  $T = A_1 B_1$  with

$$A_1 = AU \in B(\mathfrak{H}) \quad \text{and } B_1 = |\tilde{B}| \geq 0.$$

Furthermore,  $\ker B_1 = \ker \tilde{B} = \ker T$ . □

**Corollary 2.10** *Let  $T = A_1 B_0 \in CO(\mathfrak{H})$ , where  $A_1 \in B(\mathfrak{H})$  and  $B_0 = B_0^* \geq 0$  with  $\ker T = \ker B_0$ . Then,  $T \in \mathcal{L}_{op}^{+2}(\mathfrak{H})$  if and only if*

$$\text{dom } T \subseteq \text{dom } B_0 \subseteq \text{dom } (T^* B_0)^{\frac{1}{2}} \quad \text{and } T^* T \leq \lambda T^* B_0 \text{ for some } \lambda \geq 0,$$

in which case  $T = A_0 B_0$  for some  $A_0 \in B^+(\mathfrak{H})$  and one can assume that  $\overline{\text{ran } A_0} = \overline{\text{ran } T}$ .

In this case,  $\ker A_0 \cap \text{ran } B_0 = \ker B_0 \cap \text{ran } A_0 = \{0\}$ , and hence

$$\ker T \cap \text{ran } T = \ker T^* \cap \text{ran } T^* = \{0\}. \tag{2.9}$$

### 3 Local Spectral Theory in $\mathcal{L}_l^{+2}(\mathfrak{H})$ and Quasi-Affinity to $S = S^* \geq 0$

In the framework of bounded operators, it is shown in [8, Proposition 7.2] that if  $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$  is quasi-affine to a bounded nonnegative operator, then their spectra coincide. The objective of this section is to extend such a result not only to the general case of unbounded operators  $T$  but also for  $T \notin \mathcal{L}_l^{+2}(\mathfrak{H})$ . This will be treated in Section 3.3 which requires the study of some local spectral properties for  $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ , carried out in Section 3.1. A key idea in this approach is to deal with the class of unbounded generalized scalar operators which will be proved to lie in  $\mathcal{L}_l^{+2}(\mathfrak{H})$  and provide an important connection to local spectral theory; see Sect. 3.2.

#### 3.1 Local Spectral Theory in $\mathcal{L}_l^{+2}(\mathfrak{H})$

The single valued extension property (SVEP) will be the first local spectral property established for  $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$ . This property was first introduced by J.Finch [13] in the case of bounded operators and, in the present unbounded setting, the definition is recalled from [1].

**Definition 3.1** An operator  $T \in CO(\mathfrak{H})$  is said to have the *single valued extension property* (abbreviated SVEP) at  $\alpha_0 \in \mathbb{C}$  if, for every neighborhood  $U_{\alpha_0}$  of  $\alpha_0$ , the only analytic  $f_{\alpha_0} : U_{\alpha_0} \rightarrow \text{dom } T$ , which satisfies

$$0 = (T - \mu I)f_{\alpha_0}(\mu) \quad \text{for all } \mu \in U_{\alpha_0}$$

is the function  $f_{\alpha_0} \equiv 0$ . The operator  $T$  is said to have the SVEP if it has the SVEP at every  $\alpha_0 \in \mathbb{C}$ .

**Proposition 3.2** *Let  $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$  be such that  $\rho(T) \neq \emptyset$ . Then  $T$  and  $T^*$  have the SVEP.*

**Proof** Let  $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$  be such that  $\rho(T) \neq \emptyset$  and, suppose to the contrary that  $T$  does not have the SVEP. Then there exist  $\lambda_0 \in \mathbb{C}$ , an open neighborhood  $U_0$  of  $\lambda_0$  in  $\mathbb{C}$  and a nonzero analytic function  $f : U_0 \rightarrow \text{dom } T$  such that

$$(T - \mu I)f(\mu) = 0 \quad \text{for all } \mu \in U_0. \tag{3.1}$$

Hence there exists  $\mu_1 \in U_0$  and an open set  $U_1 \subseteq U_0$  such that  $f(\mu_1) \neq 0$  and (3.1) holds for all  $\mu \in U_1$ . This implies that  $\mu_1 \in \sigma_p(T)$ . Without loss of generality, one may assume that  $U_1 \subseteq \sigma_p(T)$ , which shows that  $\mu_1 \in \text{int}(\sigma_p(T))$ , where  $\text{int}(\sigma_p(T))$  denotes the interior of  $\sigma_p(T)$ .

On the other hand, by Corollary 2.2  $\sigma(T) \subseteq \mathbb{R}^+$  and thus  $\text{int}(\sigma_p(T)) = \emptyset$ , contradiction. A similar argument show that  $T^*$  has the SVEP, since  $\sigma(T^*) \subseteq \mathbb{R}^+$ .  $\square$

The upcoming results are devoted to outline the main substeps required to prove the second local spectral property of an operator in  $\mathcal{L}_l^{+2}(\mathfrak{H})$ , which concerns the local spectral subspaces defined below; cf. [1, 11].

**Definition 3.3** Let  $T \in CO(\mathfrak{H})$ .

- (i) The *local resolvent set* of  $T$  at a point  $x \in \mathfrak{H}$  is the set  $\rho_T(x)$  of all  $\lambda_0 \in \mathbb{C}$  for which there exist an open neighborhood  $U_{\lambda_0}$  of  $\lambda_0$  in  $\mathbb{C}$  and  $f_{x,\lambda_0} \in \mathfrak{H}(U_{\lambda_0}, \text{dom } T)$ , i.e., an analytic function  $f_{x,\lambda_0} : U_{\lambda_0} \rightarrow \text{dom } T$  which satisfies

$$x = (T - \mu)f_{x,\lambda_0}(\mu) \quad \text{for all } \mu \in U_{\lambda_0}.$$

The set  $\mathbb{C} \setminus \rho_T(x)$  is called the *local spectrum* of  $T$  at the point  $x$  and denoted by  $\sigma_T(x)$ .

- (ii) The *analytic spectral subspace* of  $T$  at a subset  $F$  of  $\mathbb{C}$  is given by

$$\mathfrak{H}_T(F) = \{x \in \mathfrak{H} ; \widetilde{\sigma}_T(x) \subseteq F\}.$$

If  $\mathfrak{H}_T(F)$  is closed for all closed  $F \subseteq \mathbb{C}$ , then  $T$  has the *Dunford's property*, shortly property (C).

- (iii)  $\mathcal{H}_T(F)$  stands for the *glocal spectral subspace* of  $T$  at a closed set  $F \subseteq \mathbb{C}$  and it is defined as the set of all  $x \in \mathfrak{H}$  for which there exists an analytic function  $\ell : \mathbb{C} \setminus F \rightarrow \text{dom } T$  such that

$$x = (T - \mu I)\ell(\mu) \quad \text{for all } \mu \in \mathbb{C} \setminus F.$$

It is easily seen that  $\mathcal{H}_T(F) \subseteq \mathfrak{H}_T(F)$  for all closed sets  $F \subseteq \mathbb{C}$ . Moreover, the converse inclusion holds when  $T$  has the SVEP.

**Definition 3.4** ([25], Definition 2.2) The operator  $T$  is said to have property  $(k)$  if it has both SVEP and  $(C)$ .

Recall from [19] that the quasi-nilpotent part of an operator  $T \in CO(\mathfrak{H})$  is defined by:

$$\mathfrak{H}_0(T) := \{x \in \text{dom } {}^\infty T ; \lim_{n \rightarrow \infty} \|T^n x\|^{\frac{1}{n}} = 0\},$$

where  $\text{dom } {}^\infty T = \bigcap_{n=1}^{+\infty} \text{dom } T^n$ . In particular,  $\ker T^n \subseteq \mathfrak{H}_0(T)$  for all  $n > 0$ .

The following lemma is a collection of some properties related to SVEP and local spectrum and will be a quite useful key for the remaining parts. Item (i) and (ii) are already studied in the general case of linear relations respectively in [20, Proposition 3.1] and [4] and, the proof of item (iv) follows immediately from [1, Corollary 4.9].

**Lemma 3.5** *Let  $T \in CO(\mathfrak{H})$ . Then, the following properties hold:*

- (i)  $\sigma_{T-\lambda_0 I}(x) \subseteq \{0\}$  if and only if  $\sigma_T(x) \subseteq \{\lambda_0\}$  for every  $\lambda_0 \in \mathbb{C}$ .
- (ii) If  $T$  has the SVEP, then  $T - \lambda_0 I$  has the SVEP for every  $\lambda_0 \in \mathbb{C}$ .
- (iii) If  $T$  has the SVEP, then  $\mathfrak{H}_T(\{\lambda\}) = \mathfrak{H}_0(T - \lambda I)$  for every  $\lambda \in \mathbb{C}$ .
- (iv) If  $T - \lambda I$  has a finite ascent for every  $\lambda \in \mathbb{C}$ , then  $T$  has the SVEP.
- (v) If  $T$  is normal, in particular selfadjoint, then  $T$  has the SVEP.

**Proof** To see (iii), assume that  $T$  has the SVEP and let  $\lambda \in \mathbb{C}$ . Then  $T - \lambda I$  has the SVEP, by (ii) and [19, Proposition 1.3] ensures that  $\mathfrak{H}_0(T - \lambda I) = \{x \in \text{dom } {}^\infty T ; \lim_{n \rightarrow \infty} \|(T - \lambda I)^n x\|^{\frac{1}{n}} = 0\} = \{x \in \mathfrak{H} ; \sigma_{T-\lambda I}(x) \subseteq \{0\}\}$ . A direct application of (i) yields

$$\mathfrak{H}_0(T - \lambda I) = \{x \in \mathfrak{H} ; \sigma_T(x) \subseteq \{\lambda\}\} = \mathfrak{H}_T(\{\lambda\}).$$

For the proof of (v), assume that  $T$  is normal. Then  $T - \lambda I$  is normal for all  $\lambda \in \mathbb{C}$ , and hence it has finite ascent by [14, Theorem 2.1]. One then concludes the result using item (iv). □

**Lemma 3.6** *Let  $T \in CO(\mathfrak{H})$ . Then the following statements hold:*

- (i) If  $x \in \mathfrak{H}$ ,  $F \subseteq \mathbb{C}$  and  $(T - \lambda I)x \in \mathfrak{H}_T(F)$  for some  $\lambda \in F$ , then  $x \in \mathfrak{H}_T(F)$ .
- (ii)  $\ker(T - \lambda I) \subseteq \mathfrak{H}_T(\{\lambda\})$  for all  $\lambda \in \mathbb{C}$ .

**Proof** The statement in (i) is proved by the same arguments as in the proof of [17, Proposition 1.2.16] in the context of bounded operators. To see (ii), let  $x \in \ker(T - \lambda I)$ . Then  $(T - \lambda I)x = 0 \in \mathfrak{H}_T(\{\lambda\})$  and a direct application of item (i) yields that  $x \in \mathfrak{H}_T(\{\lambda\})$ . □

A particular case, where the reverse inclusion in item (ii) of Lemma 3.6 holds, is the case of bounded normal operators; see [8, Lemma 5.1]. In order to extend such a result to the unbounded setting, some notions of local spectral theory in the extended complex plane  $\mathbb{C}_\infty$  are required. For more details see [11, 29, 30] for the unbounded setting of operators and [5] for the multivalued one.

**Definition 3.7** Let  $T \in CO(\mathfrak{H})$ . Then,

- (i) The *extended local resolvent set* of  $T$  at a point  $x \in \mathfrak{H}$  is the set  $\widetilde{\rho}_T(x)$  of all  $\lambda \in \mathbb{C}_\infty$  for which there exist an open neighborhood  $W_\infty \subseteq \mathbb{C}_\infty$  of  $\lambda$  and analytic function  $f_{x,\lambda} : W_\infty \rightarrow \text{dom } T$  such that

$$(T - \mu I)f_{x,\lambda}(\mu) = x \quad \text{for all } \mu \in W_\infty \cap \mathbb{C}.$$

The *extended local spectrum* of  $T$  is  $\widetilde{\sigma}_T(x) := \mathbb{C}_\infty \setminus \widetilde{\rho}_T(x)$ .

- (ii) The *extended local spectral subspace* of  $T$  at a set  $F \subseteq \mathbb{C}_\infty$  is  $\widetilde{\mathfrak{H}}_T(F) := \{x \in \mathfrak{H} ; \widetilde{\sigma}_T(x) \subseteq F\}$ .
- (iii) The *extended global spectral subspace* of  $T$  at a closed set  $F \subseteq \mathbb{C}_\infty$  is the set  $\widetilde{\mathcal{H}}_T(F)$  of all  $x \in \mathfrak{H}$  for which there exists an analytic function  $f : \mathbb{C}_\infty \setminus F \rightarrow \text{dom } T$  such that  $f(\infty) = 0$  and

$$x = (T - \mu I)f(\mu) \quad \text{for all } \mu \in (\mathbb{C}_\infty \setminus F) \cap \mathbb{C}.$$

**Lemma 3.8** *Let  $T \in CO(\mathfrak{H})$  be an operator with  $\rho(T) \neq \emptyset$  and let  $U$  be an open neighborhood of  $\widetilde{\sigma}(T)$ . Then, for  $\alpha \in \rho(T)$ , the function*

$$f_\alpha : U \rightarrow \mathbb{C} \\ z \mapsto \frac{1}{z - \alpha}$$

*is non-constant in any connected component of  $U$ , belongs to  $\mathcal{A}_T$  [11, 30] and admits the following analytic functional calculus*

$$f_\alpha(T) = f_\alpha(\infty) + \frac{1}{2i\pi} \int_\gamma f_\alpha(z)R(z, T)dz = R(\alpha, T). \tag{3.2}$$

*Here,  $\gamma$  is a rectifiable curve surrounding  $\sigma(T)$  in  $\mathbb{C}$  and  $\mathcal{A}_T$  is the class of complex analytic functions on a neighborhood of  $\sigma(T)$ .*

**Corollary 3.9** *If  $T \in CO(\mathfrak{H})$  is a normal operator then*

$$\mathfrak{H}_T(\lambda) = \ker(T - \lambda I) \quad \text{for all } \lambda \in \mathbb{C}. \tag{3.3}$$

**Proof** The direct inclusion in (3.3) is immediate from Lemma 3.6. For the converse, fix  $\alpha \in \rho(T)$  and observe from [8, Lemma 5.1] that

$$\begin{aligned} \mathfrak{H}_{R(\alpha, T)} \left( \left\{ \frac{1}{\lambda - \alpha} \right\} \right) \cap \text{dom } T &= \ker(R(\alpha, T) - \frac{1}{\lambda - \alpha} I) \cap \text{dom } T \\ &= \ker(T - \lambda I) \end{aligned} \tag{3.4}$$

for all  $\lambda \in \sigma(T)$ . On the other hand, [30, Theorem 2.1] yields that

$$f_\alpha(\widetilde{\sigma}_T(x)) = \sigma_{f_\alpha(T)}(x), \tag{3.5}$$

which implies that

$$\begin{aligned} \mathfrak{H}_{R(\alpha, T)}(\{\frac{1}{\lambda - \alpha}\}) \cap \text{dom } T &= \left\{x \in \mathfrak{H} ; \sigma_{R(\alpha, T)}(x) \subseteq \left\{\frac{1}{\lambda - \alpha}\right\}\right\} \\ &= \left\{x \in \mathfrak{H} ; f_\alpha(\widetilde{\sigma}_T(x)) \subseteq \left\{\frac{1}{\lambda - \alpha}\right\}\right\} \\ &\subseteq \left\{x \in \mathfrak{H} ; \widetilde{\sigma}_T(x) \subseteq f_\alpha^{-1}\left(\left\{\frac{1}{\lambda - \alpha}\right\}\right)\right\} \\ &\subseteq \{x \in \mathfrak{H} ; \widetilde{\sigma}_T(x) \subseteq \{\lambda\} \cup \{\infty\}\} \\ &= \widetilde{\mathfrak{H}}_T(\{\lambda\} \cup \{\infty\}). \end{aligned}$$

On the other hand, one obtains from a combination of [6, Lemma 2.6] and [6, Lemma 2.10] that

$$\widetilde{\mathfrak{H}}_T(\{\lambda\} \cup \{\infty\}) = \mathfrak{H}_T(\{\lambda\}),$$

and therefore  $\mathfrak{H}_{R(\alpha, T)}(\{\frac{1}{\lambda - \alpha}\}) \cap \text{dom } T \subseteq \mathfrak{H}_T(\{\lambda\})$ , as desired. □

The second local spectral property of  $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$  is described in the following result, which also shows properties similar to normal operators.

**Proposition 3.10** *Let  $T = AB \in \mathcal{L}_{op}^{+2}(\mathfrak{H})$  be an operator such that  $\rho(T) \neq \emptyset$ . Then, for all  $\lambda \in \mathbb{C}$ , one has*

$$\ker(T^* - \lambda I) = \mathfrak{H}_{T^*}(\{\lambda\}). \tag{3.6}$$

In this case,

$$\bigcap_{p \in \mathbb{N}} \overline{\text{ran}}(T - zI)^p = \overline{\text{ran}}(T - zI) \quad \text{for all } z \in \mathbb{C}. \tag{3.7}$$

**Proof** The inclusion  $\ker(T^* - \lambda I) \subseteq \mathfrak{H}_{T^*}(\{\lambda\})$  holds by Lemma 3.6. To see the reverse inclusion, fix  $x \in \mathfrak{H}_{T^*}(\{\lambda\})$ , and observe from Proposition 3.2 that  $T^*$  has the SVEP. Then, item (iii) of Lemma 3.5 ensures that

$$\mathfrak{H}_{T^*}(\{\lambda\}) = \mathfrak{H}_0(T^* - \lambda I) = \{y \in \text{dom } {}^\infty T^* ; \lim_{n \rightarrow \infty} \|(T^* - \lambda I)^n y\|^{\frac{1}{n}} = 0\}.$$

This implies that  $x \in \text{dom } {}^\infty T^*$  and

$$\lim_{n \rightarrow \infty} \|(T^* - \lambda I)^n x\|^{\frac{1}{n}} = 0. \tag{3.8}$$

Next, fix

$C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$ . Then,  $C = C^* \geq 0$ , by [3, Proposition 2.5]. Furthermore,

$$\begin{aligned} (C - \lambda I)A^{\frac{1}{2}} &= (A^{\frac{1}{2}}BA^{\frac{1}{2}} - \lambda I)A^{\frac{1}{2}} = (A^{\frac{1}{2}}BA - \lambda A^{\frac{1}{2}}) = A^{\frac{1}{2}}(BA - \lambda I) \\ &= A^{\frac{1}{2}}(T^* - \lambda I). \end{aligned}$$

By induction, one concludes that  $(C - \lambda I)^n A^{\frac{1}{2}} = A^{\frac{1}{2}}(T^* - \lambda I)^n$  for all  $n > 0$ . As  $x \in \text{dom } {}^\infty T^*$ , it follows that  $x \in \text{dom } A^{\frac{1}{2}}(T^* - \lambda I)^n$  and

$$\|(C - \lambda I)^n A^{\frac{1}{2}} x\|^{\frac{1}{n}} = \|A^{\frac{1}{2}}(T^* - \lambda I)^n x\| \leq \|A^{\frac{1}{2}}\|^{\frac{1}{n}} \|(T^* - \lambda I)^n x\|^{\frac{1}{n}}$$

for all  $n > 0$ . Using (3.8), one obtains

$\|(C - \lambda I)^n A^{\frac{1}{2}} x\|^{\frac{1}{n}} \xrightarrow{n \rightarrow +\infty} 0$ , and hence  $A^{\frac{1}{2}} x \in \mathfrak{H}_0(C - \lambda I)$ . Combining this with the fact that  $C = C^*$  and with assertions (v) and (iii) of Lemma 3.5, one concludes that  $A^{\frac{1}{2}} x \in \mathfrak{H}_C(\{\lambda\})$ . But  $\mathfrak{H}_C(\{\lambda\}) = \ker(C - \lambda I)$ , in view of Lemma 3.9, so

$$0 = (C - \lambda I)A^{\frac{1}{2}} x = A^{\frac{1}{2}}(T^* - \lambda I)x.$$

One then distinguishes the following two cases:

Case 1 :  $x \in \ker(T^* - \lambda I)$ . This completes the proof of the desired inclusion.

Case 2 :  $(T^* - \lambda I)x \in \ker A^{\frac{1}{2}}$ . In this case,  $x \in \ker A = \ker T^*$ , by Proposition 2.7. Consequently,

$$0 = T^*(T^* - \lambda I)x = (T^* - \lambda I + \lambda I)(T^* - \lambda I)x = (T^* - \lambda I)^2 x + \lambda(T^* - \lambda I)x,$$

which shows that  $(T^* - \lambda I)^2 x = -\lambda(T^* - \lambda I)x$ . Proceeding by induction, one obtains

$$(T^* - \lambda I)^n x = (-1)^{n-1} \lambda^{n-1} (T^* - \lambda I)x \text{ for all } n > 0.$$

Using (3.8), one then concludes that

$$\lim_{n \rightarrow \infty} \|(T^* - \lambda I)^n x\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} |\lambda|^{\frac{n-1}{n}} \|(T^* - \lambda I)x\|^{\frac{1}{n}} = 0.$$

Again, two cases appear:

Case 2.1: If  $\|(T^* - \lambda I)x\| \neq 0$ , then  $\lim_{n \rightarrow \infty} |\lambda|^{\frac{n-1}{n}} \|(T^* - \lambda I)x\|^{\frac{1}{n}} \xrightarrow{n \rightarrow +\infty} |\lambda|$ , so  $\lambda = 0$ . Thus,

$$T^* x = (T^* - \lambda I)x \in \text{ran}(T^*) \cap \ker T^* = \{0\},$$

in view of Corollary 2.10. As a consequence,  $x \in \ker T^* = \ker(T^* - \lambda I)$ .

Case 2.2 : If  $\|(T^* - \lambda I)x\| = 0$ , then  $x \in \ker(T^* - \lambda I)$ .

Hence, the proof of (3.6) is completed.

To see (3.7), let  $t \in \mathbb{C}$ . Then, a combination of Proposition 3.2 with item (iii) of Lemma 3.5 implies that  $T^*$  has the SVEP and

$$\mathfrak{H}_{T^*}(\{\bar{t}\}) = \{x \in \text{dom } {}^\infty T^*; \lim_{n \rightarrow \infty} \|(T^* - \bar{t}I)^n x\|^{\frac{1}{n}} = 0\} = \mathfrak{H}_0(T^* - \bar{t}).$$

This shows, by (3.6) that  $\ker(T^* - \bar{t}I)^p \subseteq \ker(T^* - \bar{t}I)$  for all  $p \in \mathbb{N}$ . Hence for all  $p \in \mathbb{N}$  one has  $\ker(T^* - \bar{t}I)^p = \ker(T^* - \bar{t}I)$  or, equivalently,

$$\overline{\text{ran}}(T - tI)^p = \overline{\text{ran}}(T - tI).$$

□

It will be shown that the Dunford’s property is also a necessary condition for an operator to belong to the class  $\mathcal{L}_l^{+2}(\mathfrak{H})$ . This will be achieved throughout the study of generalized scalar operators in the next subsection, which forms the core of this paper.

### 3.2 Unbounded Generalized Scalar Operators

Throughout this part, we refer to [27] and denote by  $\mathfrak{H}(\Omega)$  the set of all complex-valued functions  $f$  defined on  $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$  that satisfy:

- (i)  $f$  is  $C^\infty$  on  $\mathbb{C}_\infty$ ;
- (ii)  $f$  is analytic on a neighbourhood of  $\mathbb{C}_\infty \setminus \Omega$ .

In particular,  $\mathfrak{H}(\rho(T))$  is the set of all  $C^\infty$  functions on  $\mathbb{C}_\infty$  which are analytic in a neighbourhood of  $\sigma(T)$ . Note that there exists a continuous homomorphism from  $\mathfrak{H}(\rho(T))$  to  $B(\mathfrak{H})$  defined by Dunford and Schwartz [10] and which will be denoted by  $\Phi_T$ .

**Definition 3.11** Let  $T \in CO(\mathfrak{H})$  be a densely defined linear operator and let  $\Omega$  be an open subset of  $\mathbb{C}_\infty$  such that  $\rho(T) \subseteq \Omega$ . Then, a continuous homomorphism  $\Phi : \mathfrak{H}(\Omega) \rightarrow B(\mathfrak{H})$  for which  $\Phi|_{\mathfrak{H}(\rho(T))} = \Phi_T$  is called an  $\Omega - C^\infty$  operational calculus. Moreover,  $T$  is said to be

- (i) an  $\Omega$ -generalized scalar operator, if  $T$  has an  $\Omega - C^\infty$  operational calculus;
- (ii) a generalized scalar operator, if it is a  $\mathbb{C}_\infty$ -generalized scalar operator.

The following Lemma is a particular case of [27, Theorem 12].

**Lemma 3.12** Let  $T \in CO(\mathfrak{H})$  be a densely defined operator such that  $\sigma(T) \subseteq \mathbb{R}$ . Then,  $T$  is  $\mathbb{C}$ -generalized scalar if and only if, for every  $\mu_0 \in \mathbb{R}$  one has

$$\|R(\mu, T)\| = O(d(\mu, \mathbb{R})^{-n}), \quad \mu \rightarrow \mu_0 \text{ and } \mu \in \mathbb{C} \setminus \mathbb{R}$$

for some  $n > 0$ . Here,  $d(\mu, \mathbb{R})$  denotes the distance from  $\mu$  to  $\mathbb{R}$ .

A general result of Lemma 2.3 for  $\alpha = 1$  is worth mentioning separately and it will be a useful tool for the proof of Theorem 3.14. A proof is given for completeness.

**Lemma 3.13** Let  $B \in LO(\mathfrak{H})$  and  $S \in B(\mathfrak{H})$ . If  $SS^*B$  is closed, then  $S^*B$  is closed. Moreover, if  $\overline{\text{dom}} B = \mathfrak{H}$  then  $(B^*S)^* = S^*B$ .

**Proof** Let  $(x_n, y_n)_{n \in \mathbb{N}} \subseteq G(S^*B)$  be such that  $(x_n, y_n) \xrightarrow{n \rightarrow +\infty} (x, y)$  in  $\mathfrak{H}$ . Then,  $y \in \overline{\text{ran}} S^*$  and  $(x_n, SS^*Bx_n) \xrightarrow{n \rightarrow +\infty} (x, Sy)$ , since  $S$  is bounded. As  $SS^*B$  is closed, it then follows that  $(x, Sy) \in G(SS^*B)$ , and therefore  $S(S^*Bx - y) = 0$ . Thus,  $(S^*Bx - y) \in \ker S \cap \overline{\text{ran}} S^* = \{0\}$ . Consequently,  $y = S^*Bx$ , which shows that  $S^*B$  is closed. □

**Theorem 3.14** Let  $T = AB \in \mathcal{L}_l^{+2}(\mathfrak{H})$  be a linear operator such that  $\rho(T) \neq \emptyset$ . Then  $T$  is a  $\mathbb{C}$ -generalized scalar linear operator.

**Proof** Without loss of generality, one can assume that  $\|A\| \leq k < 1$  for a nonnegative constant  $k$  and let  $\widehat{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{H}$ . The proof will be given in three steps. First it will be established that  $\sigma(\widehat{T}) \subseteq \mathbb{R}^+$ , where  $\widehat{T} := \widehat{A}\widehat{B}$  and  $\widehat{A}, \widehat{B} : \widehat{\mathfrak{H}} \rightarrow \widehat{\mathfrak{H}}$  are given by  $\widehat{A} := \begin{pmatrix} A & A^{\frac{1}{2}}(I - A)^{\frac{1}{2}} \\ (I - A)^{\frac{1}{2}}A^{\frac{1}{2}} & I - A \end{pmatrix}$  and  $\widehat{B} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ . In the second step, it will be shown that  $\widehat{T}$  is a  $\mathbb{C}$ -generalized scalar operator in  $\widehat{\mathfrak{H}}$  and in the last one, it will be proved that  $T$  is a  $\mathbb{C}$ -generalized scalar operator in  $\mathfrak{H}$ .

*Step 1.* By Lemma 2.3  $A^{\frac{1}{2}}B$  is closed, since  $A \in B^+(\mathfrak{H})$  and both  $B$  and  $AB$  are closed. Moreover, as  $\|A\| < 1$ , it follows that  $(I - A)^{\frac{1}{2}}$  is invertible, and therefore  $(I - A)^{\frac{1}{2}}A^{\frac{1}{2}}B$  is closed. This shows by [28, Corollary 2.2.11] that the operator

$$\widehat{T} = \begin{pmatrix} T & 0 \\ (I - A)^{\frac{1}{2}}A^{\frac{1}{2}}B & 0 \end{pmatrix}$$

is closed. Now, it will be established that  $\sigma(\widehat{T}) \subseteq \mathbb{R}^+$ . For this, consider  $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$ . As  $\rho(T) \neq \emptyset$ , one then obtains from Corollary 2.2 that  $\lambda \in \rho(T)$ , which yields to the following factorization:

$$\widehat{T} - \lambda I = \begin{pmatrix} I & 0 \\ (I - A)^{\frac{1}{2}}A^{\frac{1}{2}}BR(\lambda, T) & I \end{pmatrix} \begin{pmatrix} T - \lambda & 0 \\ 0 & -\lambda I \end{pmatrix}, \tag{3.9}$$

where the block operator on the right-hand side of (3.9) is invertible. On the other hand,  $T_2 := (I - A)^{\frac{1}{2}}A^{\frac{1}{2}}BR(\lambda, T)$  is closed and  $\text{dom } T_2 = \mathfrak{H}$ , so by the graph theorem, one concludes that  $T_2 \in B(\mathfrak{H})$ . This means, by (3.9), that

$$\rho(\widehat{T}) = \rho(T) \setminus \{0\},$$

and therefore

$$\sigma(\widehat{T}) \subseteq \mathbb{R}^+. \tag{3.10}$$

In this case, for all  $\eta \in \mathbb{C} \setminus \mathbb{R}$ , one has  $\eta \in \rho(T) \cap \rho(\widehat{T})$  and it is claimed that

$$(\widehat{T} - \eta I)^{-1} = \begin{pmatrix} R(\eta, T) & 0 \\ \eta^{-1}T_2 & -\eta^{-1} \end{pmatrix} := R_0(\eta). \tag{3.11}$$

In fact,  $R_0(\eta) \in B(\widehat{\mathfrak{H}})$  by construction and one has

$$(\widehat{T} - \eta I)R_0(\eta) = I_{\widehat{\mathfrak{H}}} \quad \text{and} \quad R_0(\eta)(\widehat{T} - \eta I) = I_{\widehat{\mathfrak{H}}} \upharpoonright \text{dom } \widehat{T},$$

so  $R(\eta, \widehat{T}) = R_0(\eta)$ , as claimed.

*Step 2.* To show that  $\widehat{T}$  is a  $\mathbb{C}$ -generalized scalar operator, observe first that  $\widehat{T} \in L^{+2}(\widehat{\mathfrak{H}})$ . Indeed, one can see that  $\widehat{A} = SS^*$ , where  $S = \begin{pmatrix} A^{\frac{1}{2}} \\ (I - A)^{\frac{1}{2}} \end{pmatrix}$  is an isometry on  $\widehat{\mathfrak{H}}$ , and this implies that  $\widehat{A} =: P_{\text{ran } S}$  is a linear projector in

$B^+(\widehat{\mathfrak{H}})$ . Furthermore, it follows from [18] that  $\widehat{B} = (\widehat{B})^*$ , and since  $B \geq 0$ , one concludes that  $\widehat{B} \geq 0$ .

On the other hand, one has

$$S^*\widehat{T}S = S^*\widehat{A}\widehat{B}S = S^*(SS^*\widehat{B})S = S^*\widehat{B}S,$$

and hence  $(S^*\widehat{T}S)^* = (S^*\widehat{B}S)^* = (\widehat{B}S)^*S = \overline{S^*\widehat{B}S}$ . But  $SS^*\widehat{B} = \widehat{A}\widehat{B} = \widehat{T}$  is closed, so using Lemma 3.13, one obtains  $\overline{S^*\widehat{B}S} = S^*\widehat{B}$ . Consequently,

$$(S^*\widehat{T}S)^* = S^*\widehat{B}S = S^*\widehat{T}S. \tag{3.12}$$

As  $\widehat{B} \geq 0$ , this implies that  $S^*\widehat{T}S \geq 0$  and  $\sigma(S^*\widehat{T}S) = \sigma(S^*\widehat{B}S) \subseteq \mathbb{R}^+$ . In addition, (3.12) proves that  $S^*\widehat{T}S$  is a scalar linear operator, so a generalized scalar one. Applying Lemma 3.12, one concludes that

$$\|R(\mu, S^*\widehat{T}S)\| = O(d(\mu, \mathbb{R})^{-n}), \quad \mu \rightarrow \mu_0 \text{ and } \mu \in \mathbb{C} \setminus \mathbb{R} \tag{3.13}$$

for every  $\mu_0 \in \mathbb{R}$  and for some  $n > 0$ .

Denote  $R = \begin{pmatrix} (I - A)^{\frac{1}{2}} \\ -A^{\frac{1}{2}} \end{pmatrix}$ . Then  $R$  is an isometry,  $R^*R = I$  and, moreover, the block operator

$$U = (SR) = \begin{pmatrix} A^{\frac{1}{2}} & (I - A)^{\frac{1}{2}} \\ (I - A)^{\frac{1}{2}} & -A^{\frac{1}{2}} \end{pmatrix}$$

is unitary on  $\widehat{\mathfrak{H}}$ . Furthermore, observe that

$$U^*\widehat{T}U = \begin{pmatrix} S^* \\ R^* \end{pmatrix} S S^* \widehat{B} (SR) = \begin{pmatrix} S^* \widehat{B} S & S^* \widehat{B} R \\ 0 & 0 \end{pmatrix} \tag{3.14}$$

and, for  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  arbitrarily given, one has

$$U^*\widehat{T}U - \lambda I = \begin{pmatrix} S^* \widehat{B} S - \lambda I & S^* \widehat{B} R \\ 0 & -\lambda \end{pmatrix} \tag{3.15}$$

$$= \begin{pmatrix} S^* \widehat{B} S - \lambda I & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} I & \overline{X} \\ 0 & I \end{pmatrix}, \tag{3.16}$$

where  $X = (S^* \widehat{B} S - \lambda I)^{-1} S^* \widehat{B} R$ . It is claimed that  $\overline{X} \in B(\widehat{\mathfrak{H}})$ . Indeed,

$$X^* = (S^* \widehat{B} R)^* (S^* \widehat{B} S - \lambda I)^{-1} \supseteq R^* \widehat{B} S (S^* \widehat{B} S - \lambda I)^{-1} \tag{3.17}$$

and since  $\text{ran} (S^* \widehat{B} S - \lambda I)^{-1} = \text{dom } S^* \widehat{B} S = \text{dom } \widehat{B} S$ , one concludes that  $\text{dom } X^* = \widehat{\mathfrak{H}}$ . On the other hand, the block operator in (3.14) is densely defined since  $\widehat{T}$  and thus also  $U^*\widehat{T}U$  is densely defined, so that also the

blocks  $S^*\widehat{B}S$  and  $S^*\widehat{B}R$  in (3.14) are densely defined operators. As  $S^*\widehat{B}R$  is a densely defined operator, its adjoint  $(S^*\widehat{B}R)^*$  is a closed and densely defined operator since  $(S^*\widehat{B}R)^* \supseteq R^*\widehat{B}S$ . This implies, by (3.17), that  $X^*$  is also a closed operator with  $\text{dom } X^* = \widehat{\mathfrak{H}}$ . Therefore,  $X^*$  and  $\overline{X}$  belong to  $B(\widehat{\mathfrak{H}})$ , which proves the claim.

Consequently, the block operator  $\begin{pmatrix} I & \overline{X} \\ 0 & I \end{pmatrix}$  is invertible on  $\widehat{\mathfrak{H}}$ . This implies that

$$\begin{aligned} (U^*\widehat{T}U - \lambda I)^{-1} &= \begin{pmatrix} I & -\overline{X} \\ 0 & I \end{pmatrix} \begin{pmatrix} (S^*\widehat{B}S - \lambda I)^{-1} & 0 \\ 0 & -\frac{1}{\lambda} \end{pmatrix} \\ &= U^*R(\lambda, \widehat{T})U. \end{aligned} \tag{3.18}$$

Combining this formula with (3.13), one can conclude that

$$\|R(\mu, \widehat{T})\| = O(d(\mu, \mathbb{R})^{-n}), \quad \mu \rightarrow \mu_0 \text{ and } \mu \in \mathbb{C} \setminus \mathbb{R} \tag{3.19}$$

for every  $\mu_0 \in \mathbb{R}$  and for some  $n > 0$ . Hence, one concludes by Lemma 3.12 that  $\widehat{T}$  is a  $\mathbb{C}$ -generalized scalar operator.

*Step 3.* To demonstrate that  $T$  is a  $\mathbb{C}$ -generalized scalar operator, let  $\eta_0 \in \mathbb{R}$  and  $\eta \in \mathbb{C} \setminus \mathbb{R}$  be such that  $\eta \rightarrow \eta_0$ . Then, one obtains from (3.11) that

$$\begin{aligned} \begin{pmatrix} I & 0 \\ 0 & A^{\frac{1}{2}} \end{pmatrix} (\widehat{T} - \eta)^{-1} &= \begin{pmatrix} R(\eta, T) & 0 \\ \frac{1}{\eta}(I - A)^{1/2}TR(\eta, T) & -\frac{1}{\eta}A^{\frac{1}{2}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \frac{1}{\eta}(I - A)^{1/2} & -\frac{1}{\eta}A^{\frac{1}{2}} - I \end{pmatrix} \\ &\quad + \begin{pmatrix} I & 0 \\ (I - A)^{1/2} & I \end{pmatrix} \begin{pmatrix} R(\eta, T) & 0 \\ 0 & I \end{pmatrix}, \end{aligned}$$

which implies that

$$\begin{aligned} \begin{pmatrix} (T - \eta)^{-1} & 0 \\ 0 & I \end{pmatrix} &= \begin{pmatrix} I & 0 \\ -(I - A)^{\frac{1}{2}} & I \end{pmatrix} \left[ \begin{pmatrix} I & 0 \\ 0 & A^{\frac{1}{2}} \end{pmatrix} R(\eta, \widehat{T}) \right. \\ &\quad \left. - \begin{pmatrix} 0 & 0 \\ \frac{1}{\eta}(I - A)^{\frac{1}{2}} & -\frac{1}{\eta}A^{\frac{1}{2}} - I \end{pmatrix} \right]. \end{aligned}$$

Combining this with (3.19) one obtains the desired estimate

$$\|R(\eta, T)\| = O(d(\eta, \mathbb{R})^{-n}),$$

which means that  $T$  is a  $\mathbb{C}$ -generalized scalar operator by Lemma 3.12.

□

The third local spectral property of  $T \in \mathcal{L}_I^{+2}(\mathfrak{H})$  is stated below.

**Proposition 3.15** *If  $T \in CO(\mathfrak{H})$  is a  $\mathbb{C}$ -generalized scalar operator with  $\sigma(T) \subseteq \mathbb{R}$  then  $T$ , respectively  $T^*$ , has property (k). In particular, every operator in  $\mathcal{L}_l^{+2}(\mathfrak{H})$  with  $\rho(T) \neq \emptyset$  has property (k).*

**Proof** Since  $\sigma(T) \subseteq \mathbb{R}$ , it is clear from the proof of Proposition 3.2 that  $T$  the SVEP. On the other hand,  $R(\mu, T)$  is a generalized scalar operator for all  $\mu \in \rho(T)$ , so it has the property (C) by [17]. This means by [17, Proposition 3.3.4] that  $R(\mu, T)$  has property (C). It is claimed that also  $T$  has (C). To see this, let  $F \subseteq \mathbb{C}_\infty$  be a closed set and let  $(x_n)_{n \in \mathbb{N}} \subseteq \mathfrak{H}_T(F)$  such that  $x_n \xrightarrow{n \rightarrow +\infty} x$  in  $\mathfrak{H}$ . Then, for all  $n \in \mathbb{N}$ ,  $\sigma_T(x_n) \subseteq F$  and hence  $f_\alpha(\tilde{\sigma}_T(x_n)) \subseteq f_\alpha(\sigma_T(x_n) \cup \{\infty\}) \subseteq f_\alpha(F \cup \{\infty\})$ . Combining this with (3.2) and (3.5), one concludes that

$$\sigma_{R(\alpha, T)}(x_n) \subseteq f_\alpha(F \cup \{\infty\}) \quad \text{for all } n \in \mathbb{N},$$

and hence  $(x_n)_{n \in \mathbb{N}} \subseteq \mathfrak{H}_{R(\alpha, T)}(f_\alpha(F \cup \{\infty\}))$ . Since  $f_\alpha(F \cup \{\infty\})$  is closed, also the set  $\mathfrak{H}_{R(\alpha, T)}(f_\alpha(F \cup \{\infty\}))$  is closed and  $x_n \xrightarrow{n \rightarrow +\infty} x$ , it then follows that  $x \in \mathfrak{H}_{R(\alpha, T)}(f_\alpha(F \cup \{\infty\}))$  and therefore  $f_\alpha(\tilde{\sigma}_T(x)) = \sigma_{R(\alpha, T)}(x) \subseteq f_\alpha(F \cup \{\infty\})$ . Thus  $\sigma_T(x) \subseteq \tilde{\sigma}_T(x) \subseteq f_\alpha^{-1} f_\alpha(\tilde{\sigma}_T(x)) \subseteq F$ . This proves that  $x \in \mathfrak{H}_T(F)$  and hence  $T$  has (C), as claimed.

Now, assume that  $T \in \mathcal{L}_l^{+2}(\mathfrak{H})$  such that  $\rho(T) \neq \emptyset$ . Then,  $\sigma(T) \subseteq \mathbb{R}$  by Corollary 2.2 and for all  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , one has

$$\|R(\lambda, T^*)\| = \|R(\bar{\lambda}, T)\|. \tag{3.20}$$

Moreover, Theorem 3.14 shows that  $T$  is a  $\mathbb{C}$ -generalized scalar operator, so a combination of (3.20) and Lemma 3.12 yields that  $T^*$  is a  $\mathbb{C}$ -generalized scalar operator. Hence, the same reasoning as above shows that  $T^*$  has property (k).  $\square$

### 3.3 Quasi-Affinity to Nonnegative Selfadjoint Operators

Recall that in the bounded case, if  $T$  is quasi-affine to  $S \in B^+(\mathfrak{H})$  then  $\sigma(S) \subseteq \sigma(T)$ ; see [8, Lemma 3.7] (see also [12, 26]). The next result extends this result not only to the unbounded setting but also to operators satisfying property (k), which is a weaker condition than the selfadjointness.

**Lemma 3.16** *Let  $T \in CO(\mathfrak{H})$  and let  $S$  be a an operator with the property (k). If  $T$  is quasi-affine to  $S$  then*

$$\sigma(S) \subseteq \sigma(T), \tag{3.21}$$

*In this case,  $\mathfrak{H} = \mathfrak{H}_S(\sigma(T))$ .*

**Proof** Since  $T$  is quasi-affine to  $S$ , there exists a quasi-affinity  $G \in B(\mathfrak{H})$  such that  $GT \subseteq SG$ . The proof is splitted in four steps.

Step 1.  $\sigma_S(Gx) \subseteq \sigma(T)$  for all  $x \in \mathfrak{H}$ .

To see this, let  $x \in \mathfrak{H}$  and  $\lambda \in \rho(T) \subseteq \rho_T(x)$ . Then, there exists an open neighborhood  $U_\lambda$  of  $\lambda$  and an analytic function  $f : U_\lambda \rightarrow \text{dom } T$  such that

$x = (T - \mu I)f(\mu)$  for all  $\mu \in U_\lambda$ , and hence

$$Gx = G(T - \mu I)f(\mu)x = (S - \mu I)Gf(\mu)x \quad \text{for all } \mu \in U_\lambda.$$

As  $G \in B(\mathfrak{H})$ , it follows that  $\mu \mapsto Gf(\mu)$  in an analytic function on  $U_\lambda$ , which implies that  $\lambda \in \rho_S(Gx)$ . One then concludes that  $\rho(T) \subseteq \rho_S(Gx)$ .

Step 2.  $\mathfrak{H} = \mathfrak{H}_S(\sigma(T))$ . For this, fix  $x \in \mathfrak{H}$ . As  $\overline{\text{ran } G} = \mathfrak{H}$ , there exists  $(x_n) \subseteq \mathfrak{H}$  for which  $Gx_n \xrightarrow{n \rightarrow +\infty} x$ . On the other hand, Step 1 ensures that  $\sigma_S(Gx_n) \subseteq \sigma(T)$  or, equivalently,  $Gx_n \subseteq \mathfrak{H}_S(\sigma(T))$ . Since  $S$  has property (C) one concludes that  $x \in \mathfrak{H}_S(\sigma(T))$  and  $\mathfrak{H} = \mathfrak{H}_S(\sigma(T))$ .

Step 3.  $\sigma_{sur}(T) \subseteq \bigcup_{x \in \mathfrak{H}} \sigma_T(x)$ . Let  $x \in \mathfrak{H}$  be arbitrary and let  $\lambda \in \rho_T(x)$ . Then, there exists an open neighborhood  $U_\lambda$  of  $\lambda$  in  $\mathbb{C}$  and analytic function  $f : U_\lambda \rightarrow \text{dom } T$  such that  $x = (T - \mu I)f(\mu)$  for all  $\mu \in U$ , in particular  $x = (T - \lambda I)f(\lambda) \in \text{ran } (T - \lambda)$ . This proves that  $\mathfrak{H} = \text{ran } (T - \lambda)$ , and hence  $x \in \rho_{sur}(T)$ .

Step 4.  $\sigma(S) \subseteq \sigma(T)$ . First observe from Step 2 that  $\sigma_{sur}(x) \subseteq \sigma(T)$  for all  $x \in \mathfrak{H}$ . On the other hand,  $S$  has SVEP, so by [11, Corollary 2.5],  $\sigma(S) = \sigma_{sur}(S)$ . Applying Step 3, one then concludes that  $\sigma(S) \subseteq \bigcup_{x \in \mathfrak{H}} \sigma_T(x) \subseteq \sigma(T)$ .

□

In the bounded settings, equality holds in (3.21) for  $S \in B^+(\mathfrak{H})$  and  $T \in \mathcal{L}_I^{+2}(\mathfrak{H})$ ; see [8, Proposition 7.2]. This result has been ameliorated to cover the general case where  $T \notin \mathcal{L}_I^{+2}(\mathfrak{H})$ , which will be seen in the next corollary.

**Corollary 3.17** *Let  $T \in CO(\mathfrak{H})$  be an operator such that  $T$  and  $T^*$  have (k). If  $T$  is quasi-affine to  $S = S^*$  then*

$$\sigma(S) = \sigma(T) = \sigma(T^*) \tag{3.22}$$

and, in this case  $\mathfrak{H} = \mathfrak{H}_S(\sigma(T)) = \mathfrak{H}_{T^*}(\sigma(S))$ .

**Proof** Observe from Lemma 3.5 (vi) that  $S$  has the SVEP. Moreover, it is clear that  $S$  is a scalar operator, and hence generalized scalar. This implies by Proposition 3.15 that  $S$  has property (C), and hence property (k). Since  $T$  is quasi-affine to  $S$ , Lemma 3.16 shows that

$$\sigma(S) \subseteq \sigma(T). \tag{3.23}$$

On the other hand,  $T^*$  has property (k) and since  $S$  is quasi-affine to  $T^*$  one concludes from Lemma 3.16 that

$$\sigma(T^*) \subseteq \sigma(S) \subseteq \mathbb{R}. \tag{3.24}$$

Hence, (3.22) is obtained a combinations of (3.23) and (3.24). The last equalities follow from (3.22) and from the fact that  $\sigma_M(x) \subseteq \sigma(M)$  for every  $x \in \mathfrak{H}$  and  $M \in CO(\mathfrak{H})$ . □

Recall from [22] that a linear operator  $T$  is said to be *quasi-similar* to a linear operator  $S$  if there exist two quasi-affinities  $G_1, G_2 \in B(\mathfrak{H})$  such that

$$G_1 T \subseteq S G_1 \quad \text{and} \quad G_2 S \subseteq T G_2.$$

**Corollary 3.18** *Let  $T \in \mathcal{L}_1^{+2}(\mathfrak{H})$  be an operator such that  $\rho(T) \neq \emptyset$ . If  $T$  is quasi-affine, in particular, quasi-similar to a nonnegative selfadjoint operator  $S \in CO(\mathfrak{H})$  then*

$$\sigma(S) = \sigma(T).$$

**Proof** The proof immediately follows from a combination of Theorem 3.14, Proposition 3.15 and Corollary 3.17 and, by definition.  $\square$

It is important to note that, in the particular case where  $\overline{\text{ran}} T = \mathfrak{H}$ , the quasi-affinity of  $T^*$  to a nonnegative selfadjoint operator is enough to prove all the above results. This is based on the following remark stated in [3]:

**Remark 3.19** Let  $T \in CO(\mathfrak{H})$  be a densely defined operator with  $\overline{\text{ran}} T = \mathfrak{H}$ . Then the following statements are equivalent:

- (i)  $T = AB \in \mathcal{L}_1^{+2}(\mathfrak{H})$ ;
- (ii)  $T^*$  is  $G$ -quasi-affine to  $S = S^* \geq 0$  with  $\text{dom } T \subseteq \overline{\text{dom } S^{\frac{1}{2}}(G^{-1})^*}$ .

**Theorem 3.20** *Let  $T \in CO(\mathfrak{H})$  be an operator with  $\overline{\text{ran}} T = \mathfrak{H}$  and  $\rho(T) \neq \emptyset$ . If  $T^*$  is quasi-affine to  $S = S^* \geq 0$  so that  $\text{dom } T \subseteq \overline{\text{dom } S^{\frac{1}{2}}(G^{-1})^*}$  then*

$$\sigma(T) = \sigma(S).$$

**Proof** The results follows immediately from Remark 3.19 and Corollary 3.18.  $\square$

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