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AN EXTENSION AND REFINEMENT OF THE THEOREMS OF DOUGLAS AND SEBESTYÉN FOR UNBOUNDED OPERATORS

YOSRA BARKAOU AND SEPPO HASSI

ABSTRACT. For a closed densely defined operator T from a Hilbert space \mathfrak{H} to a Hilbert space \mathfrak{K} , necessary and sufficient conditions are established for the factorization of T with a bounded nonnegative operator X on \mathfrak{K} . This result yields a new extension and a refinement of a well-known theorem of R.G. Douglas, which shows that the operator inequality $A^*A \leq \lambda^2 B^*B$, $\lambda \geq 0$, is equivalent to the factorization $A = CB$ with $\|C\| \leq \lambda$. The main results give necessary and sufficient conditions for the existence of an intermediate selfadjoint operator $H \geq 0$, such that $A^*A \leq \lambda H \leq \lambda^2 B^*B$. The key results are proved by first extending a theorem of Z. Sebestyén to the setting of unbounded operators.

1. INTRODUCTION

Let $(\mathfrak{H}, (\cdot, \cdot)_{\mathfrak{H}})$ be a Hilbert space and denote by $\mathbf{B}(\mathfrak{H})$ the class of bounded everywhere defined operators on \mathfrak{H} . For $T, B \in \mathbf{B}(\mathfrak{H})$, R.G. Douglas [7, Theorem 1] showed, in 1966, that the following equivalence holds for some $\lambda \geq 0$:

$$(1.1) \quad TT^* \leq \lambda^2 BB^* \Leftrightarrow T = BC, C \in \mathbf{B}(\mathfrak{H}) \Leftrightarrow \text{ran } T \subseteq \text{ran } B.$$

Later, in 1983, Z. Sebestyén [13] established the following characterization for a related problem:

$$(1.2) \quad T^*T \leq \lambda T^*B \text{ for some } \lambda \geq 0 \Leftrightarrow T = XB, X \in \mathbf{B}^+(\mathfrak{H}),$$

where $T, B \in \mathbf{B}(\mathfrak{H})$ and $\mathbf{B}^+(\mathfrak{H})$ stands for the class of bounded nonnegative operators on \mathfrak{H} . In fact, the inequality in (1.2) is closely connected to the one in (1.1), since the identity $T = XB$ implies the following two inequalities:

$$(1.3) \quad T^*T \leq \lambda T^*B \leq \lambda^2 B^*B.$$

Therefore, the existence of a product presentation $T = XB$, where the factor X is not only bounded, but also *nonnegative* involves an intermediate nonnegative self-adjoint operator λT^*B lying between T^*T and $\lambda^2 B^*B$ in the theorem of Douglas.

The study of such factorizations has since been extended to more general settings. In fact, also Douglas in [7] indicated some results for the case when T and B are densely defined closed linear operators. Some further extensions of the results in [7] can be found in the literature; see, for instance, [4, 8, 9] and the references therein. In 2013, D. Popovici and Z. Sebestyén [12, Theorem 2.2] generalized the second equivalence in (1.1) to multivalued linear operators (linear relations) and showed that $T \subseteq BC$ for some linear relation C if and only if $\text{ran } T \subseteq \text{ran } B$. On the other

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hand, the first equivalence in (1.1) was established by S. Hassi and H.S.V. de Snoo [10] for both unbounded linear operators and linear relations in 2015.

The purpose in the present paper is to extend (1.2) to the context of unbounded operators T and B and show its close connection to (1.1). Here the operators T and B are acting from the Hilbert space \mathfrak{H} to another Hilbert space \mathfrak{K} with dense domains $\text{dom } T$ and $\text{dom } B$, respectively, and they are assumed to be closed. If, in addition, B is nonnegative and selfadjoint in \mathfrak{H} , the factorization $T = XB$ is closely connected to the quasi-affinity of T to a nonnegative selfadjoint operator in \mathfrak{H} ; see [3]. Furthermore, this factorization implies several local spectral-theoretic properties for T studied in [2]. In the case of bounded operators Sebestyén's result in (1.2) has also been studied in recent papers by M. L. Arias, G. Corach, and M. C. Gonzalez [1] and by M. Contino, M. A. Dritschel, A. Maestripieri, and S. Marcantognini [6]. In this last paper also some local spectral-theoretic results for $T = XB \in \mathbf{B}(\mathfrak{H})$ have been established.

In this paper, we start by establishing first a complete unbounded analog of (1.2). At this point, recall that the ordering $0 \leq H_1 \leq H_2$ in the case of unbounded operators acting on a Hilbert space \mathfrak{H} is defined for a pair of nonnegative selfadjoint operators H_1 and H_2 by

$$(1.4) \quad 0 \leq H_1 \leq H_2 \Leftrightarrow \text{dom } H_2^{\frac{1}{2}} \subseteq \text{dom } H_1^{\frac{1}{2}} \text{ and } \|H_1^{\frac{1}{2}} f\| \leq \|H_2^{\frac{1}{2}} f\|, \quad f \in \text{dom } H_2^{\frac{1}{2}},$$

cf. [5, Definition 5.2.3]. Inspired by the work of Sebestyén, our approach hinges on the construction of an auxiliary Hilbert space by means of the following sesquilinear form:

$$\tau_{T,B}[f, g] := (Tf, Bg)_{\mathfrak{K}}, \quad f, g \in \text{dom } \tau_{T,B} = \text{dom } B \subseteq \text{dom } T.$$

In the first step, the following equivalences are proved in Theorem 2.1:

$$(1.5) \quad XB \subseteq T \Leftrightarrow \tau_T \leq \lambda \tau_{T,B} \Leftrightarrow \|Tf\|_{\mathfrak{K}}^2 \leq \lambda (Tf, Bf)_{\mathfrak{K}},$$

where the form τ_T is defined by $\tau_T[f] = (Tf, Tf)_{\mathfrak{K}}$. An important further result is that the form $\tau_{T,B}$ is closable and, therefore, its closure gives rise to a nonnegative selfadjoint operator H , and the inequality in (1.5) can be described explicitly by means of H , X and B as follows:

$$(1.6) \quad T^*T \leq \lambda H, \quad H = B^* X^{\frac{1}{2}} \overline{X^{\frac{1}{2}} B};$$

see Theorem 2.2. An essential difference here is that in (1.2) the nonnegative operator T^*B is selfadjoint, while (1.6) shows that T^*B is in general just a symmetric restriction of $H = H^* \geq 0$ due to the unboundedness of the operators T and B . A further study of the inequality $T^*T \leq \lambda H$, where H is only assumed to be a selfadjoint operator (i.e. without the specific formula for H in (1.6)), is carried out and yields further equivalent conditions for (1.5) in Proposition 2.5, for instance:

$$XB \subseteq T \Leftrightarrow T^*T \leq \lambda H, \text{ where } H \subseteq B^*T \text{ and } \text{dom } B \subseteq \text{dom } H^{\frac{1}{2}}.$$

This not only completes the extension of Sebestyén's result to the present unbounded framework, but the above mentioned results motivate the investigation of the *reversed* version of the above inequalities to be studied in Section 3. Namely, also in the present case of unbounded operators, the inequality $T^*T \leq \lambda H$ implies the inequality $H \leq \lambda B^*B$; see Corollary 2.3. This second inequality will be characterized in Theorem 3.1 with some further results, completing the study of the inequalities (1.1)–(1.3) in the case of unbounded operators.

2. AN EXTENSION OF SEBESTYÉN'S THEOREM FOR UNBOUNDED OPERATORS

The main purpose of this article is to solve and extend the above factorization problem for unbounded operators and present analogous characterizations for the factorization of T as in (1.2). Our approach is inspired by Sebestyén's theorem [13], which is extended to the unbounded case as follows.

Theorem 2.1. *Let $(\mathfrak{H}, (\cdot, \cdot)_{\mathfrak{H}}), (\mathfrak{K}, (\cdot, \cdot)_{\mathfrak{K}})$ be two complex Hilbert spaces and let $T, B : H \rightarrow K$ be closed densely defined operators. Then the following statements are equivalent for some $\lambda \geq 0$:*

- (1) $\|Tf\|_{\mathfrak{K}}^2 \leq \lambda (Tf, Bf)_{\mathfrak{K}}$ for all $f \in \text{dom } B \subseteq \text{dom } T$;
- (2) there exists $X \in \mathbf{B}^+(\mathfrak{K})$ such that $\|X\|_{\mathfrak{K}} \leq \lambda$ and $XB \subseteq T$.

In this case, X can be selected such that $\text{ran } X \subseteq \overline{\text{ran } T}$.

Proof. Assume that (1) holds. Then $(Tf, Bg)_{\mathfrak{K}}, f, g \in \text{dom } B$, defines a nonnegative sesquilinear form in the Hilbert space \mathfrak{H} . Observe, that

$$(2.1) \quad (Tf, Bf)_{\mathfrak{K}} = 0 \quad \Leftrightarrow \quad Tf = 0, \quad f \in \text{dom } B.$$

By completing the quotient space $[\text{dom } B / (\ker T \cap \text{dom } B)]$ one obtains a Hilbert space \mathfrak{K}_B whose inner product is denoted by $\langle \tilde{f}, \tilde{g} \rangle_{\mathfrak{K}_B}, \tilde{f}, \tilde{g} \in \mathfrak{K}_B$, such that

$$(2.2) \quad \langle \tilde{f}, \tilde{g} \rangle_{\mathfrak{K}_B} = (Tf, Bg)_{\mathfrak{K}}, \quad f, g \in \text{dom } B.$$

Now let $V : \mathfrak{K}_B \rightarrow \mathfrak{K}$ be defined by

$$V\tilde{f} = Tf \quad \text{for all } f \in \text{dom } B.$$

Then V is a well-defined linear operator by (2.1) and (2.2), and it follows from (1) that it is bounded by $\sqrt{\lambda}$. It is claimed that

$$V^*Bf = \tilde{f} \quad \text{for all } f \in \text{dom } B.$$

To see this, let $f, g \in \text{dom } B$ and $\tilde{g} \in \mathfrak{K}_B$. Then,

$$\langle \tilde{g}, V^*Bf \rangle_{\mathfrak{K}_B} = (V\tilde{g}, Bf)_{\mathfrak{K}} = (Tg, Bf)_{\mathfrak{K}} = \langle \tilde{g}, \tilde{f} \rangle_{\mathfrak{K}_B},$$

and therefore $V^*Bf = \tilde{f}$, as claimed. Consequently, $X := VV^* \in \mathbf{B}^+(\mathfrak{K})$ and one has $\|X\|_{\mathfrak{K}} \leq \lambda$, $\overline{\text{ran } X} \subseteq \overline{\text{ran } T}$, by construction, and

$$XBf = V\tilde{f} = Tf \quad \text{for all } f \in \text{dom } B.$$

This proves that $XB \subseteq T$.

For the converse, assume (2) and let $f \in \text{dom } B \subseteq \text{dom } T$. Then,

$$\|Tf\|^2 = \|XBf\|^2 \leq \|X^{\frac{1}{2}}\|^2 \|X^{\frac{1}{2}}Bf\|^2 = \lambda (Tf, Bf)_{\mathfrak{K}},$$

with $\lambda = \|X\|$. This completes the proof of (1). \square

The proof shows that if (2) of Theorem 2.1 holds then $\lambda = \|X\|$ satisfies the inequality in (1). Notice also that if (1) of Theorem 2.1 is satisfied, then $T \neq 0$ implies that $\lambda > 0$. In this case, the form

$$(2.3) \quad \tau_{T,B}[f, g] := (Tf, Bg)_{\mathfrak{K}}, \quad f, g \in \text{dom } B \subseteq \text{dom } T,$$

is nonnegative. The next theorem shows that $\tau_{T,B}$ is closable and identifies its closure.

Theorem 2.2. *Let $T, B : \mathfrak{H} \rightarrow \mathfrak{K}$ be closed densely defined linear operators such that condition (1) or, equivalently, (2) of Theorem 2.1 holds. Then, the form in (2.3) is closable and its closure is given by*

$$(2.4) \quad \overline{\tau_{T,B}}[f, g] := \overline{(X^{\frac{1}{2}}Bf, X^{\frac{1}{2}}Bg)}_{\mathfrak{K}} = (H^{\frac{1}{2}}f, H^{\frac{1}{2}}g)_{\mathfrak{K}} \quad f, g \in \text{dom } \overline{\tau_{T,B}},$$

where $H = B^*X^{\frac{1}{2}}X^{\frac{1}{2}}B = H^* \geq 0$ and X is as in Theorem 2.1. Here H is the unique representing selfadjoint nonnegative operator of the form $\overline{\tau_{T,B}}$ with $\text{dom } \overline{\tau_{T,B}} = \text{dom } H^{\frac{1}{2}} \subseteq \text{dom } T$.

Furthermore, with $\tau_T[f, g] := (Tf, Tg)_{\mathfrak{K}}$, $f, g \in \text{dom } T$, each of the following statements is equivalent to the items (1) and (2) of Theorem 2.1:

- (1) $\tau_T \leq \lambda \tau_{T,B}$ for some $\lambda \geq 0$;
- (2) $\tau_T \leq \lambda \overline{\tau_{T,B}}$ for some $\lambda \geq 0$;
- (3) $T^*T \leq \lambda H$ for some $\lambda \geq 0$.

In particular, if $\text{dom } T = \text{dom } B$ then $T = XB$, $T^*B = H$ is selfadjoint, and the form in (2.3) is closed.

Proof. Let X be the nonnegative operator in item (2) of Theorem 2.1 and let $f, g \in \text{dom } \tau_{T,B}$. Then, (2.3) yields

$$(2.5) \quad \tau_{T,B}[f, g] = (Tf, Bg)_{\mathfrak{K}} = (XBf, Bg)_{\mathfrak{K}} = (X^{\frac{1}{2}}Bf, X^{\frac{1}{2}}Bg)_{\mathfrak{K}}$$

and therefore to prove the closability of $\tau_{T,B}$ is equivalent to prove the closability of the associated operator $X^{\frac{1}{2}}B$ to $\tau_{T,B}$, by [11, VI, Example 1.23]). To see this, let $(f_n)_{n \in \mathbb{N}} \subseteq \text{dom } B$ such that $f_n \xrightarrow{n \rightarrow +\infty} 0$ and $X^{\frac{1}{2}}Bf_n \xrightarrow{n \rightarrow +\infty} g$. Since $X^{\frac{1}{2}} \in \mathbf{B}(\mathfrak{H})$, it follows that $XBf_n \xrightarrow{n \rightarrow +\infty} X^{\frac{1}{2}}g$. On the other hand, the inclusion $XB \subseteq T$ together with the fact that T is closed yields that $X^{\frac{1}{2}}g = 0$. Since $g \in \overline{\text{ran } X^{\frac{1}{2}}}$, one concludes that $g = 0$. Thus $X^{\frac{1}{2}}B$ is closable. Consequently, $\overline{X^{\frac{1}{2}}B}$ is a densely defined operator such that $X^{\frac{1}{2}}X^{\frac{1}{2}}B \subseteq \overline{XB} \subseteq T$ and, in particular,

$$\text{dom } \overline{X^{\frac{1}{2}}B} \subseteq \text{dom } T.$$

Furthermore, the operator $H := (X^{\frac{1}{2}}B)^* \overline{X^{\frac{1}{2}}B} \geq 0$ is selfadjoint and it follows from (2.5) that

$$\overline{\tau_{T,B}}[f, g] = \overline{(X^{\frac{1}{2}}Bf, X^{\frac{1}{2}}Bg)}_{\mathfrak{K}} = (H^{\frac{1}{2}}f, H^{\frac{1}{2}}g)_{\mathfrak{K}} \quad \text{for all } f, g \in \text{dom } \overline{\tau_{T,B}}.$$

One concludes that $\text{dom } \overline{\tau_{T,B}} = \text{dom } H^{\frac{1}{2}} = \text{dom } \overline{X^{\frac{1}{2}}B} \subseteq \text{dom } T$.

To see the stated equivalences, observe first from (2.3) that for all $f \in \text{dom } \tau_{T,B} = \text{dom } B \subseteq \text{dom } T$ and for a fixed $\lambda \geq 0$, one has

$$(2.6) \quad \|Tf\|_{\mathfrak{K}}^2 \leq \lambda (Tf, Bf)_{\mathfrak{K}} \Leftrightarrow \tau_T[f] \leq \lambda \tau_{T,B}[f].$$

On the other hand, it is clear that τ_T is closed, since T is closed; cf. [11, VI, Example 1.13]. Hence, item (v) of [5, Lemma 5.2.2] gives

$$(2.7) \quad \tau_T \leq \lambda \tau_{T,B} \Rightarrow \tau_T \leq \lambda \overline{\tau_{T,B}}.$$

Furthermore, $\tau_{T,B} \subseteq \overline{\tau_{T,B}}$ so again by [5, Lemma 5.2.2] one has $\overline{\tau_{T,B}} \leq \tau_{T,B}$. This together with (2.7) implies that

$$(2.8) \quad \tau_T \leq \lambda \tau_{T,B} \Leftrightarrow \tau_T \leq \lambda \overline{\tau_{T,B}} \Leftrightarrow T^*T \leq \lambda H,$$

see [5, Theorem 5.2.4] (or, [11, VI, Remark 2.29]). One concludes the equivalences (1) – (3) by a combination of Theorem 2.1 with (2.6) and (2.8).

As to the last statement observe that if $\text{dom } T = \text{dom } B$ then clearly $T = XB$ and $T^*B = B^*XB = B^*T$. Moreover, in this case the operator $X^{\frac{1}{2}}B$ is closed or, equivalently, the form in (2.3) is closed as follows from (2.5), and in this case $H = B^*XB$. \square

Corollary 2.3. *Let the operators $T, B : \mathfrak{H} \rightarrow \mathfrak{K}$ satisfy the conditions (1) and (2) in Theorem 2.1 for some $\lambda \geq 0$ and $X \in \mathbf{B}^+(\mathfrak{H})$. Then the following statements hold:*

- (1) $0 \leq (Tf, Bf)_{\mathfrak{K}} \leq \lambda \|Bf\|^2$ for all $f \in \text{dom } B$;
- (2) the operator T^*B is symmetric and, moreover,

$$(2.9) \quad T^*B \subseteq H \subseteq B^*T;$$

- (3) $T^*T \leq \lambda H \leq \lambda^2 B^*B$;

- (4) if T^*B is selfadjoint, then equalities hold in (2.9) and one has

$$(2.10) \quad T^*T \leq \lambda T^*B = \lambda B^*XB \leq \lambda^2 B^*B.$$

Proof. (1) Since $XB \subseteq T$ one has for all $f \in \text{dom } B$,

$$(2.11) \quad (Tf, Bf)_{\mathfrak{K}} = (XBf, Bf)_{\mathfrak{K}} = \|X^{\frac{1}{2}}Bf\|_{\mathfrak{K}}^2 \leq \|X^{\frac{1}{2}}\|^2 \|Bf\|_{\mathfrak{K}}^2,$$

so that the inequality holds for $\lambda = \|X\|_{\mathfrak{K}}$.

(2) Under the conditions of Theorem 2.1 one has $XB \subseteq T$ for $X \in \mathbf{B}^+(\mathfrak{K})$, and since $\overline{XB} = \overline{X^{\frac{1}{2}}X^{\frac{1}{2}}B}$ one concludes that

$$(2.12) \quad T^*B \subseteq B^*XB \subseteq B^*X^{\frac{1}{2}}\overline{X^{\frac{1}{2}}B} = H \subseteq \overline{B^*X^{\frac{1}{2}}X^{\frac{1}{2}}B} = B^*\overline{XB} \subseteq B^*T.$$

Since H is selfadjoint, T^*B is symmetric and the proof of (2.9) is completed.

(3) Observe from Theorem 2.2 that $\text{dom } B \subseteq \text{dom } H^{\frac{1}{2}} \subseteq \text{dom } T$ and let $f \in \text{dom } B$. Then,

$$\|H^{\frac{1}{2}}f\|_{\mathfrak{K}}^2 = \|\overline{X^{\frac{1}{2}}Bf}\|_{\mathfrak{K}}^2 = \|X^{\frac{1}{2}}Bf\|_{\mathfrak{K}}^2 \leq \|X^{\frac{1}{2}}\|_{\mathfrak{K}}^2 \|Bf\|_{\mathfrak{K}}^2,$$

which shows that $H \leq \lambda B^*B$ with $\lambda = \|X\|_{\mathfrak{K}}$. The other inequality was proved in Theorem 2.2.

(4) Assume that T^*B is selfadjoint. Then $B^*T \subseteq (T^*B)^* = T^*B$, so from (2.12) one concludes that

$$(2.13) \quad T^*B = B^*XB = H = B^*T.$$

This proves the equalities in (2.9) and by item (3) completes the proof. \square

Inspired by Corollary 2.3, a natural question arises as to whether items (1) and (3) can also be regarded as sufficient conditions. As a first step, item (3) will be examined in the next lemma and Proposition 2.5 in a more general framework, where the nonnegative operator $H = H^*$ is assumed to be independent from X . The second step deals with item (1), in particular with the question when the following implication holds for some $\lambda \geq 0$:

$$(2.14) \quad 0 \leq (Tf, Bf)_{\mathfrak{K}} \leq \lambda \|Bf\|^2 \Rightarrow \|Tf\|_{\mathfrak{K}}^2 \leq \lambda (Tf, Bf)_{\mathfrak{K}}$$

for all $f \in \text{dom } B \subseteq \text{dom } T$. This question induces the study of a reversed version of Sebestyén's inequality appearing in the left-hand side of (2.14) which will be investigated in Section 3.

Lemma 2.4. *Let $H = H^* \geq 0$ and $T, B : \mathfrak{H} \rightarrow \mathfrak{K}$ be closed densely defined operators such that $\text{dom } B \subseteq \text{dom } H^{\frac{1}{2}}$ and $H \subseteq B^*T$. Then the following implication holds:*

$$(2.15) \quad T^*T \leq \lambda H \Rightarrow \|Tf\|_{\mathfrak{K}}^2 \leq \lambda(Tf, Bf)_{\mathfrak{K}} \text{ for all } f \in \text{dom } B,$$

in particular, $\text{dom } B \subseteq \text{dom } T$.

Proof. Assume that

$$(2.16) \quad T^*T \leq \lambda H.$$

Then $\text{dom } H^{\frac{1}{2}} \subseteq \text{dom } (T^*T)^{\frac{1}{2}} = \text{dom } T$, see (1.4). Now let $f \in \text{dom } B \subseteq \text{dom } H^{\frac{1}{2}}$. Since $\text{dom } H$ is a core for $H^{\frac{1}{2}}$, there exists $(f_n)_{n \in \mathbb{N}} \subseteq \text{dom } H \subseteq \text{dom } B^*T$, see (2.9), such that $f_n \xrightarrow[n \rightarrow +\infty]{} f$ and $H^{\frac{1}{2}}f_n \xrightarrow[n \rightarrow +\infty]{} H^{\frac{1}{2}}f$. By (2.16) this implies that $\|T(f_n - f)\|_{\mathfrak{K}} \xrightarrow[n \rightarrow +\infty]{} 0$ and hence $Tf_n \xrightarrow[n \rightarrow +\infty]{} Tf$. Thus

$$\begin{aligned} (Tf, Bf)_{\mathfrak{K}} &= \lim_{n \rightarrow +\infty} (Tf_n, Bf)_{\mathfrak{K}} = \lim_{n \rightarrow +\infty} (B^*Tf_n, f)_{\mathfrak{K}} = \lim_{n \rightarrow +\infty} (Hf_n, f)_{\mathfrak{K}} \\ &= \lim_{n \rightarrow +\infty} (H^{\frac{1}{2}}f_n, H^{\frac{1}{2}}f)_{\mathfrak{K}} = (H^{\frac{1}{2}}f, H^{\frac{1}{2}}f)_{\mathfrak{K}} = \|H^{\frac{1}{2}}f\|_{\mathfrak{K}}^2. \end{aligned}$$

Combining this with (2.16) completes the argument. \square

The next proposition gives some further operator-theoretic criteria which are equivalent to the conditions in Theorem 2.1. The proof is directly obtained from a combination of Theorem 2.1, Corollary 2.3, and Lemma 2.4.

Proposition 2.5. *Let $T, B : \mathfrak{H} \rightarrow \mathfrak{K}$ be closed densely defined operators. Then, the following statements are equivalent:*

- (1) $\|Tf\|_{\mathfrak{K}}^2 \leq \lambda(Tf, Bf)_{\mathfrak{K}}$ for all $f \in \text{dom } B \subseteq \text{dom } T$;
- (2) $\|Tf\|_{\mathfrak{K}}^2 \leq \lambda(Tf, Bf)_{\mathfrak{K}} \leq \lambda^2 \|Bf\|_{\mathfrak{K}}^2$ for all $f \in \text{dom } B \subseteq \text{dom } T$;
- (3) $T^*T \leq \lambda H \leq \lambda^2 B^*B$ for some $\lambda \geq 0$ and some $0 \leq H = H^* \subseteq B^*T$;
- (4) $T^*T \leq \lambda H$ for some $\lambda \geq 0$ and $0 \leq H = H^* \subseteq B^*T$ with $\text{dom } B \subseteq \text{dom } H^{\frac{1}{2}}$.

*In particular, if B^*T is selfadjoint then $H = B^*T$.*

Proposition 2.5 is, in fact, a useful tool to cover Sebestyén's theorem in the general case of unbounded operators. In particular, one has the following result.

Corollary 2.6. *Let $T, B : \mathfrak{H} \rightarrow \mathfrak{K}$ be closed densely defined operators. Then, the following statements are equivalent:*

- (1) $T = XB$ has a solution $X \in \mathbf{B}^+(\mathfrak{K})$;
- (2) $\|Tf\|_{\mathfrak{K}}^2 \leq \lambda(Tf, Bf)_{\mathfrak{K}}$ for all $f \in \text{dom } B = \text{dom } T$;
- (3) $T^*T \leq \lambda B^*T \leq \lambda^2 B^*B$ for some $\lambda \geq 0$ and $\text{dom } T \subseteq \text{dom } B$;
- (4) $T^*T \leq \lambda B^*T$ for some $\lambda \geq 0$ and $\text{dom } T \subseteq \text{dom } B \subseteq \text{dom } (B^*T)^{\frac{1}{2}}$.

Proof. The equivalence of (1) and (2) follows immediately from Theorem 2.1. Next observe that the equality $\text{dom } T = \text{dom } B$ in items (1) and (2) implies that $B^*T = H$ is selfadjoint and nonnegative; see Theorem 2.2. On the other hand, in items (3) and (4) B^*T is selfadjoint because of the assumed operator inequality $T^*T \leq \lambda B^*T$, see (1.4), and therefore it coincides with H in Theorem 2.1 by Corollary 2.3. Now the domain inclusion assumed in (3) and (4) combined with the standard domain inclusions by the stated operator inequalities, see (1.4), yield the equality $\text{dom } T = \text{dom } B$. This completes the proof. \square

3. CHARACTERIZATION OF THE REVERSED INEQUALITY

The second step involving (2.14) is now considered. Analogous to Theorem 2.1, the following result characterizes a reversed inequality.

Theorem 3.1. *Let $(\mathfrak{H}, (\cdot, \cdot)_{\mathfrak{H}})$ and $(\mathfrak{K}, (\cdot, \cdot)_{\mathfrak{K}})$ be complex Hilbert spaces and $T, B : \mathfrak{H} \rightarrow \mathfrak{K}$ be closed densely defined operators. Then, the following statements are equivalent for some $m > 0$:*

- (1b) $\|Tf\|_{\mathfrak{K}}^2 \geq m(Tf, Bf)_{\mathfrak{K}} \geq 0$ for all $f \in \text{dom } T \subseteq \text{dom } B$;
- (2b) *there exists $Y \in \mathbf{B}^+(\mathfrak{K})$ such that $YT \subseteq P_T B$, where P_T stands for the orthogonal projection onto $\overline{\text{ran } T}$.*

Proof. Consider the sesquilinear form (Tf, Bg) , $f, g \in \text{dom } T$. By assumption the quadratic form (Tf, Bf) is nonnegative for all $f \in \text{dom } T$. Therefore, it satisfies the Cauchy-Schwarz inequality, i.e.,

$$(3.1) \quad |(Tg, Bf)_{\mathfrak{K}}| \leq (Tf, Bf)_{\mathfrak{K}}^{\frac{1}{2}} (Tg, Bg)_{\mathfrak{K}}^{\frac{1}{2}} \quad \text{for all } f, g \in \text{dom } T.$$

Hence, if $(Tf, Bf)_{\mathfrak{K}} = 0$ for some $f \in \text{dom } T$ then by (3.1) $(Tg, Bf)_{\mathfrak{K}} = 0$ holds for all $g \in \text{dom } T$, i.e., $Bf \in (\text{ran } T)^{\perp} = \ker T^*$. The converse is also true and, therefore, for $f \in \text{dom } T$ one has $(Tf, Bf)_{\mathfrak{K}} = 0$ if and only if $f \in \ker T^* B$.

Next observe that $P_T B(\text{dom } T) = \{0\}$ if and only if $(Tf, Bf) = 0$ for all $f \in \text{dom } T$, i.e., (Tf, Bg) is the 0-form on $\text{dom } T$; cf. (3.1). In this case (1b) holds trivially and $Y = 0$ satisfies the inclusion in (2b), and the equivalence of (1b) and (2b) holds in this case.

(1b) \Rightarrow (2b) Assume that $P_T B(\text{dom } T) \neq \{0\}$ or, equivalently, that for some $f \in \text{dom } T$ one has $(Tf, Bf)_{\mathfrak{K}} > 0$. In particular, in this case also $T \neq 0$.

Now introduce the Hilbert space $(\mathfrak{K}_T, \langle \cdot, \cdot \rangle_{\mathfrak{K}_T})$ by completing the factor space $[\text{dom } T / (\text{dom } T \cap \ker T^* B)]$ with respect to the inner product

$$(3.2) \quad \langle \tilde{f}, \tilde{g} \rangle_{\mathfrak{K}_T} := (Tf, Bg)_{\mathfrak{K}}, \quad f, g \in \text{dom } T,$$

where \tilde{f}, \tilde{g} represent the corresponding equivalence classes.

Next, let the mapping $Z : \text{ran } T \rightarrow \mathfrak{K}_T$ be defined by

$$ZTf = \tilde{f} \quad \text{for all } f \in \text{dom } T.$$

Then (3.2) shows that Z is a well-defined linear operator which is bounded by $1/\sqrt{m}$, since the assumption in (1b) implies that

$$\|ZTf\|_{\mathfrak{K}_T}^2 = \langle \tilde{f}, \tilde{f} \rangle_{\mathfrak{K}_T} = (Tf, Bf)_{\mathfrak{K}} \leq \frac{1}{m} \|Tf\|_{\mathfrak{K}}^2 \quad \text{for all } f \in \text{dom } T.$$

By continuity Z can be extended to a bounded operator from $\overline{\text{ran } T}$ to \mathfrak{K}_T and with a zero continuation to $(\text{ran } T)^{\perp}$ one gets a bounded operator $\mathfrak{K} \rightarrow \mathfrak{K}_T$, which is still denoted by Z . It is claimed that

$$Z^* \tilde{f} = P_T Bf \quad \text{for all } f \in \text{dom } T.$$

To see this, let $h = Tt$, $t \in \text{dom } T$ and $f \in \text{dom } T$. Then,

$$(h, Z^* \tilde{f})_{\mathfrak{K}} = (ZTt, \tilde{f})_{\mathfrak{K}_T} = (\tilde{t}, \tilde{f})_{\mathfrak{K}_T} = (Tt, Bf)_{\mathfrak{K}} = (h, Bf)_{\mathfrak{K}},$$

which proves that $Z^* \tilde{f} - Bf \perp \text{ran } T$. By construction, $(\text{ran } T)^{\perp} \subseteq \ker Z$ and hence $\text{ran } Z^* \subseteq \overline{\text{ran } T}$. Therefore, $Z^* \tilde{f} = P_T Z^* \tilde{f} = P_T Bf$ as claimed. Thus, for $Y := Z^* Z \in \mathbf{B}^+(\mathfrak{K})$ one has $\|Y\| \leq \frac{1}{m}$ and

$$(3.3) \quad YTf = Z^* ZTf = Z^* \tilde{f} = P_T Bf \quad \text{for all } f \in \text{dom } T,$$

which means that $YT \subseteq P_T B$.

(2b) \Rightarrow (1b) By the first part of the proof, the statement holds trivially if $Y = 0$. Now assume that $Y \neq 0$, so that $M := \|Y\| > 0$. Then by assumption $YT \subseteq P_T B$ and hence for all $f \in \text{dom } T$ one has

$$(Tf, Bf)_{\mathfrak{K}} = (Tf, P_T Bf)_{\mathfrak{K}} = (Tf, Y Tf)_{\mathfrak{K}} = \|Y^{\frac{1}{2}} Tf\|_{\mathfrak{K}}^2 \leq M \|Tf\|_{\mathfrak{K}}^2,$$

which completes the proof of (1b) with $m = 1/M > 0$. \square

The proof shows that one can take $M = 1/m > 0$ in Theorem 3.1 when the form $(T\cdot, B\cdot)$ is nontrivial.

Corollary 3.2. *The inequality (1b) in Theorem 3.1 implies also the following inequality:*

$$(3.4) \quad (Tf, Bf)_{\mathfrak{K}} \geq m \|P_T Bf\|_{\mathfrak{K}}^2 \quad \text{for all } f \in \text{dom } T.$$

If $P_T B$ is closable then also the form $(T\cdot, B\cdot)$ on the domain $\text{dom } T$ is closable, and this holds, in particular, if $\text{ran } B \subseteq \overline{\text{ran } T}$, in which case $P_T B = B$.

Proof. By Theorem 3.1 $YT \subseteq P_T B$, where $\|Y\| \leq 1/m$; cf. (3.3). Therefore, one obtains for all $f \in \text{dom } T$,

$$\|P_T Bf\|_{\mathfrak{K}}^2 = \|YTf\|_{\mathfrak{K}}^2 \leq \frac{1}{m} \|Y^{\frac{1}{2}} Tf\|_{\mathfrak{K}}^2 = \frac{1}{m} (Tf, Y Tf)_{\mathfrak{K}} = \frac{1}{m} (Tf, Bf)_{\mathfrak{K}},$$

which gives the inequality (3.4). Notice also that if $(Tf, Bf) = 0$ for all $f \in \text{dom } T$ then, equivalently, $P_T B(\text{dom } T) = \{0\}$ (cf. the proof of Theorem 3.1), so that (3.4) remains true also in this case.

The second statement can be proved in the same way as the closability was proven in Theorem 2.2. By assumption in Theorem 3.1 B is closed and hence the last statement is clear, since $\text{ran } B \subseteq \overline{\text{ran } T}$ holds precisely when $P_T B = B$. \square

Remark 3.3. Using Theorem 2.1 and switching the roles of T and B in Proposition 2.5 leads to the following equivalent statements (with $m = 1/\lambda$):

- (1) $T^*T \geq mH \geq m^2 B^*B$ for some $0 \leq H = H^* \subseteq T^*B$;
- (2) $\|Tf\|_{\mathfrak{K}}^2 \geq m(Bf, Tf)_{\mathfrak{K}} = m(Tf, Bf)_{\mathfrak{K}} \geq m^2 \|Bf\|_{\mathfrak{K}}^2$ for all $f \in \text{dom } T \subseteq \text{dom } B$;
- (3) $Y_2 T \subseteq B$ for some $Y_2 \in \mathbf{B}^+(\mathfrak{K})$ such that $\|Y_2\|_{\mathfrak{K}} \leq \lambda$ and $\overline{\text{ran } Y_2} \subseteq \overline{\text{ran } B}$.

The inclusion in (3) yields $Y_3 T \subseteq P_{\overline{\text{ran } T}} B$ with $Y_3 := P_{\overline{\text{ran } T}} Y_2 P_{\overline{\text{ran } T}} \in \mathbf{B}^+(\mathfrak{K})$, which is equivalent by Theorem 3.1 and Corollary 3.2 to

$$(3.5) \quad \|Tf\|_{\mathfrak{K}}^2 \geq m(Tf, Bf)_{\mathfrak{K}} \geq m^2 \|P_T Bf\|_{\mathfrak{K}}^2 \quad \text{for all } f \in \text{dom } T.$$

The functional analytic approach in the present paper relies on the nonnegative form $(T\cdot, B\cdot)$ being defined on the domain $\text{dom } B$ (or $\text{dom } T$) and leads to the factorization of the operator T by means of the operator B . The key to get the nonnegative factor X relies on the construction of new suitable Hilbert spaces from the nonnegative form $(T\cdot, B\cdot)$ in each case.

In forthcoming papers the authors are studying quasi-affinity properties of operators T (and T^*) satisfying Sebestyén's inequality in Theorem 2.1 or the reversed inequality in Theorem 3.1 (under some additional assumptions) including the case where T and B are linear relations, as well as the factorizations of T by means of B and a bounded nonnegative operator X as in Theorem 2.1 or the inverse X^{-1} in case of Theorem 3.1; see [3]. Moreover, spectral-theoretic consequences of these results have been investigated in [2] using methods from the local spectral theory.

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