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A common component of Fama and French factor variances[☆]

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ABSTRACT

This is the first study that explicitly explores the risk of the Fama and French equity factors in terms of their realized variances. Our results show that realized factor variances exhibit strong power-law behavior. A striking commonality is that the power-law exponents are close to $\alpha \approx 2$ regardless of which factor variance is analyzed. Notably, our novel joint test designed to test Mandelbrot's infinite variance hypothesis in the cross-section of realized factor variances shows that the null hypothesis of $\alpha = 1.9$ cannot be rejected, which further corroborates the evidence that (a) there exist a common component governing factor variance risk, and (b) factor variance risk is statistically undefined. Further evidence derived from co-fractality analysis shows that (c) risk diversification appears to be very limited as factor variances tend to exhibit power-law behavior coincidently. We argue that our study has several theoretical and practical implications—especially due to the fact that factor investing reached $5 trillion in assets under management.

1. Introduction

In their well-known and often-cited 1992 paper, published in the well-respected *Journal of Finance*, Fama and French (1992, p. 464) declared the death of the capital asset pricing model (CAPM) by making the following bold statement:

> We are forced to conclude that the SLB model does not describe the last 50 years of average stock returns.[1] (Fama and French, 1992, p. 464)

In their paper, Fama and French (1992) provided strong evidence for a negative relation between size and average return and a positive relation between book-to-market equity and average return. Based on their findings, Fama and French (1993) proposed a three-factor model consisting of the excess market factor, a size factor that is a zero-cost portfolio long on small stocks and short on big stocks, and a value factor that is a zero-cost portfolio long on high book-to-market ratio stocks and short on low high book-to-market ratio stocks. This study has been cited more than 30,000 times and was considered 'groundbreaking research'.

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[1] Note that this model has been widely-used since the 1960s and was proposed in highly-influential works by Sharpe (1964), Treynor (1962), Lintner (1965a,b) and Mossin (1966).

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In 2015, Fama and French (2015) proposed an expanded asset pricing model consisting of five factors; that is, a profitability factor and an investment factor on top of the previous three equity factors. Unsurprisingly, linear factor models such as the capital asset pricing model (CAPM) or Fama and French factor models have become the cornerstone of empirical asset pricing research and are also widely-used in corporate finance studying capital budgeting, for instance.

What do we actually know about the first moments (e.g., the "factor risk premiums") of the often-used Fama and French factors? Whereas Dichev (1998) and Chan et al. (2000) document that the relative performance of small and large firms has been much smaller and often even negative since the early 1980 s, Hirshleifer (2001) argues that the U.S. small firm effect appears to have disappeared. Similarly, Schwert (2003) highlights that the value effect seems to have disappeared after the papers that highlighted them were published. Even Fama and French (2015) concluded that the value factor is redundant after accounting for the investment factor in their five-factor model. In this regard, Grobys and Kolari (2022) perform international tests on Fama and French factor models and conclude:

> Surprisingly, employing FF (2018) test for nested models, the three-factor model did not outperform the CAPM in NA, Europe, and Japan. This finding is interesting in view of the fact that approximately 30 years ago FF proposed the three-factor model in lieu of the CAPM's inability to describe the cross section of average returns. (Grobys and Kolari, 2022, p. 644)

Interestingly, 30 years after Fama and French (1992) declared the death of the CAPM, amounting research provided evidence for that their own proposed factor models can be queried. Moreover, a recent study of Dong Li Rapach & Zhou (2022) explores the information content of cross-sectional anomaly strategies. Employing a large pool of factor returns and various model specifications, the authors find that past returns on long–short anomaly portfolios help to forecast the market risk premium within the US market. On the other hand, another recent study from Cakici, Fieberg, Metko, and Zaremba (2023) investigates the same issue using new data from both US and international markets, and explores hundreds of anomalies in forty-two countries around the world. The authors rely on machine-learning techniques to forecast market equity premia based on anomaly portfolio returns. Contrary to Dong et al. (2022), Cakici et al. (2023) conclude that equity anomalies cannot predict market returns:

> Any apparent predictability lacks external validity in two critical aspects: stock market selection and anomaly sample. While some evidence may be spotted in individual markets—such as the USA—it originates from a handful of specific anomalies and depends heavily on seemingly unimportant methodological choices. (Cakici et al., 2023)

Overall, recent evidence confirms that we do not seem to know much about the reliability of factor premiums which should be—according to theory—a compensation for risk. What about factor risk?

This is the first study that explicitly explores the risk of the popular Fama and French equity factors in terms of their realized variances. Using realized variances offers a more accurate representation of the risk dynamics associated with asset returns, as they capture both the magnitude and the temporal dependencies of return fluctuations (Andersen et al., 2001). Further, realized variances have been shown to provide a more precise measure of volatility, as they are less susceptible to biases and measurement errors that can affect other volatility estimators (Barndorff-Nielsen & Shephard, 2002). We compute the realized variances for the excess market factor (MKT), size factor (SMB), value factor (HML), investment factor (CMA), profitability factor (RMW), and momentum factor (MOM). The sample period for MKT, SMB, and HML is from June 1926 to September 2022, whereas the sample period for RMW and CMA is from June 1963 to September 2022. The sample for MOM factor encompasses the period from November 1926 to September 2022. Following Mandelbrot (1963a) and Taleb (2020), who advocated the usage of power laws, a novel aspect of this study is that it models realized variances as power laws. We estimate factor risk-specific power-law exponents using the maximum-likelihood estimation (MLE) procedure as proposed by Clauset et al. (2009). Also, we test the power-law models using the goodness-of-fit test derived from Clauset et al. (2009). Since previous literature documented that realized volatility of financial assets is close to log-normal (e.g., Andersen, Bollerslev, Diebold, & Ebens, 2001; Andersen, Bollerslev, Diebold, and Labys, 2001a & 2001b), another important aspect of this study is that we perform empirical tests to clarify the plausibility of other standard distributions including the well-established lognormal.

This study contributes to the current literature in some important ways. For instance, this the first study that explores whether the risk of well-established equity factors is subject to power-law behavior. Lux and Alfarano (2016) provide an extensive literature review on the literature on power laws in financial economics. The authors argued that the pertinent literature gradually converged to the insight that financial asset returns are governed by power-law exponents significantly larger than 2 and mostly close to 3. As a result, the presumed existence of theoretical variances motivated the vast majority of finance scholars to use standard econometrics models such as ordinary least squares (OLS) which require the theoretical variance to exist. The question arises, if the theoretical variances of asset returns exists, why does the vast majority of asset pricing studies fail scientific replication, as pointed out in a recent study of Hou, Xue, and Zhang (2020)?[2] Further, one may wonder why does the literature on corporate bond factors, as documented by Dick-Nielsen,

[2] Note that Hou et al. (2020) is not the only study putting emphasis on replication failure in financial research. A recent study of Smith and Timmermann (2022) that uses a novel methodology for identifying pervasive and discrete changes in cross-sectional risk premia documents that size, value, and investment risk premia are insignificantly different from zero in the most recent sample. Furthermore, Chen and Velikov (2023) analyze 204 stock market anomalies by accounting for effective bid–ask spreads, post-publication effects, and the modern era of trading technology that began in the early 2000s. Remarkably, Chen and Velikov (2023) are forced to conclude that "expected returns are negligible." Another recent study of Huang, Li, Wang, and Zhou (2020) explore return predictability in terms of time series momentum. The authors argue that the performance of time series momentum is virtually the same as that of a strategy that is based on historical sample mean and does not require any predictability.

Feldhütter, Pedersen, and Stolborg (2023), suffer from replication failures or inconsistent methodological choices? In addition, why do firm size, industrial sector, geographical region, or investment incentives not correlate with the patterns of errors in capital budgeting, as reported by Soares, Coutinho, and Martins (2007)?[3]

In an attempt to understand the abovementioned issues, one novel aspect of this study is that it takes a new perspective by analyzing realized variances because realized variances contain information that standard models such as generalized auto-regressive conditional heteroskedasticity (GARCH) models – which are typically derived under the assumption of normality – cannot reveal (e.g., Bubák, Kočenda, and Žikeš, 2011; Andersen, Bollerslev, Diebold, and Labys, 2003; Andersen, Bollerslev, and Meddahi, 2004). Hence, our approach may reveal novel insights concerning the risk dynamics of the popular Fama and French equity factors. We hypothesize that a potential reason for why finance studies suffer from the aforementioned issues could be that the variance of equity factors is infinite—a result which would be in stark contrast to the pertinent literature (Lux and Alfarano, 2016).

Next, some recent literature has emerged that explores power-law behavior of asset market variances. For instance, Grobys (2021) investigates the dynamics of annualized daily realized variances for the S&P 500, gold, crude oil, the GBP/USD exchange rate, and Bitcoin. His findings indicate that the power-law null model cannot be rejected for any of the realized variances. Moreover, the estimated power-law exponents are within the range $2 < \widehat{\alpha} < 3$, indicating that the variance of those realized asset market variances is not defined. Grobys (2021) argues that a manifestation of this issue is that estimated *t*-statistics are subject to sample-specificity in finite samples. Moreover, Grobys (2021) concludes that the potential reason for replication failures in financial economics could be that research methodologies often employed in financial economics are not designed for that specific research environment. Moreover, in another recent study, Grobys (2023a) explores the dynamics of annualized daily realized variances for the G10 currencies. His findings indicate that the power-law null model cannot be rejected for the vast majority of realized foreign exchange rate variances. Strikingly, estimated power-law exponents are within the range $2 < \widehat{\alpha} < 3$ for all realized foreign exchange rate variances and estimated power-law exponents exhibit a high level of stability across various subsamples.

The current study extends this emerging field of literature in some important aspects. First, it is the first that explores realized variance risk for equity market factors which are long-short portfolios and therefore could exhibit very different risk dynamics as opposed to single assets. Given that factor investing reached about $5 trillion assets under management, this is an important issue that needs to be investigated for enabling us to accurately managing factor risks. Second, some well-established literature argues that realized asset volatility is close to lognormally distributed (e.g., Andersen, Bollerslev, Diebold, & Ebens, 2001; Andersen, Bollerslev, Diebold, and Labys, 2001a & 2001b). Whereas the studies of Grobys (2021; 2023a) do not explicitly test the lognormal distribution against proposed power-law processes, this is the first study to test this issue for realized equity factor variances.

Using a sample from June 1926 to September 2022 for MKT, SMB, and HML, June 1963 to September 2022 for RMW and CMA, and November 1926 to September 2022 for MOM, we compute the monthly realized factor variances in line with Merton (1980). Q-Q plots show that the tails of the quantiles associated with the factor's realized variance distributions do not align with the theoretical quantile derived from the log-normal. Regardless of which realized factor variance we consider, the mean excess plot shows an upward linear trend that is mainly linear in the first part, which typically indicates a heavy-tailed distribution such as a power law. Using maximum likelihood estimation and selecting the cutoffs in line with Clauset et al.'s (2009) approach—which is based on the optimal Kolmogoroff-Smirnov distance—estimated power-law exponents vary between $\widehat{\alpha} = 1.96$ for the realized variance of the value factor, and $\widehat{\alpha} = 2.41$ for the realized variance of the excess market factor. Goodness-of-fit tests provide strong evidence for that lognormal or exponential distributions are rejected as underlying distributions generating realized factor variances, whereas the power-law null model cannot be rejected for any of the realized factor variances. This result is in stark contrast to the literature on realized volatility (e.g., Andersen, Bollerslev, Diebold, & Ebens, 2001; Andersen, Bollerslev, Diebold, and Labys, 2001a & 2001b).

Since $\alpha \approx 2$ for all realized factor variances, we test Mandelbrot's (1963b) infinite theoretical variance hypothesis. To do so, we use blocks bootstraps to compute robust standard deviations of the estimated power-law exponents. Our findings strongly indicate that the infinite theoretical variance hypothesis cannot be rejected for any of the realized factor variances because estimated *p*-values are within the range $0.06 \leq p \leq 0.48$. While these findings are contrary to the pertinent literature—as documented in Lux and Alfarano (2016)—they are consistent with the Mandelbrot's (1963b) infinite theoretical variance hypothesis.

Further evidence suggests that the dynamics of realized factor variances can be priced by Grobys' (2023b) recently proposed multifractal model of asset invariances. Specifically, the corresponding multifractal model which is capable of generating the factor variances for all Fama and French equity factor incorporates a multiplicative cascade derived from binominal bending with probability $p = 0.70$. Grobys (2023b) showed that the same multifractal model is capable of pricing the realized asset variances for crude oil and the S&P 500. Hence, an interesting commonality is here that realized factor variances appear to share a common power-law exponent manifested in $\alpha \approx 2$. Finally, we explore co-dependencies across realized factor variances measured in terms of co-fractality. Our findings indicate that for 11 out of 15 the co-fractality parameter is estimated at $\widehat{\lambda}^{CF} > 0.80$, suggesting a high level of (weak) co-fractality among realized factor variances. This implies, in turn, that there is not much room for risk diversification because power-law behavior often coincides; that is, extreme events are likely to occur within the same parts of time series paths.

From a more practical point of view, our results imply that the factor investing industry is exposed to a considerably higher level of extreme risk than earlier believed because the power law exponent captures via extrapolation low-probability deviations not seen in

[3] In a recent study, Dessaint et al. (2021, p. 35) commented on valuation errors in capital budgeting that "acquirers experience significantly lower cumulative abnormal returns when announcing bids for low rather than high beta targets and [the authors] estimate that using the CAPM in this context leads to valuation errors (relative to the market's view) that correspond, on average, to 12% to 33% of the deal values.".

the data (Taleb, 2020). From a more theoretical perspective, the results of our study have some serious implications. As pointed out in Fama (1963):

> From a purely statistical standpoint, if the population variance of the distribution of first differences is infinite, the sample variance is probably a meaningless measure of dispersion. Moreover, if the variance is infinite, other statistical tools (e.g., least-squares regression) which are based on the assumption of finite variance will, at best, be considerably weakened and may in fact give very misleading answers. (Fama, 1963, p. 421)

In line with Fama's (1963) implications derived from infinite variances, we hypothesize that both documented replication failures and inconsistent methodological choices (Hou et al., 2020; Dick-Nielsen et al., 2023) could be manifestations of undefined risk in terms of factor variances because statistical inference derived from OLS or any other methodology that requires the variance to exist will inevitably result in misleading inference.

This study is organized as follows. The next section describes the data and provides some preliminary analysis. The third section provides the methodology, whereas the fourth section discusses the findings. The last section concludes.

2. Data and preliminary analysis

The data used in this study include daily observations of the MKT, SMB, HML, RMW, CMA, and MOM factors, obtained from the data library of Kenneth R. French. Following Merton (1980), monthly realized variances are computed as the sum of the squared deviation of daily returns from their monthly means according to Equation (1):

$$RV_{j,t} = \sigma^2_{j,t} = \sum_{i=1}^{n} \left(R_{i,j,t} - \overline{R}_{j,t}\right)^2, \tag{1}$$

where, $\text{RV}_{\text{j,t}}$ ($\sigma^2_{j,t}$) denotes the monthly realized variance of factor j in month t, $R_{i,j,t}$ denotes the return of factor j on day i of a specific month t, $\overline{\text{R}}_{\text{j,t}}$ denotes the corresponding average daily return of factor j in month t, and n denotes the number of trading days per month. Note that this or similar measures for monthly realized variances or volatilities derived from daily data are frequently used in finance research (e.g., Moreira and Muir, 2017; Cederburg, O'Doherty, Wang, and Yan, 2020; Barroso and Detzel, 2021). Due to data availability, the sample period for the factor variances of MKT, SMB, and HML is from June 1926 to September 2022 (1,155 monthly observations), whereas the sample period for RMW realized variance and CMA realized variance is from June 1963 to September 2022 (711 monthly observations). The sample for MOM factor variance encompasses the period from November 1926 to September 2022 (1,151 monthly observations).

Table 1 presents the descriptive statistics for the monthly realized variances of the Fama and French factors. The mean values show that MKT has the highest average realized variance, followed by MOM, HML, SMB, RMW, and CMA. The median values reveal a similar pattern, with MKT realized variance exhibiting the highest median value and CMA realized variance the lowest. In addition, the maximum and minimum values indicate a wide range of realized variances across all factors, with the largest variation occurring for MKT realized variance. The standard deviations highlight the high level of uncertainty in realized factor variances, with MKT realized variance having the highest standard deviation, and CMA realized variance the lowest. All realized factor variances show extremely high kurtosis values ranging from 27.34 for HML realized variance to 66.32 for MKT realized variance, which indicates the presence of extremely fat tails in the distribution. Furthermore, among all those factor variances, the skewness is found to be most profound for MKT realized variance.

Table 1
Summary statistics for the realized Fama and French factor variances.

	$\widehat{\sigma}^2_{MKT}$	$\widehat{\sigma}^2_{SMB}$	$\widehat{\sigma}^2_{HML}$	$\widehat{\sigma}^2_{RMW}$	$\widehat{\sigma}^2_{CMA}$	$\widehat{\sigma}^2_{MOM}$
Mean	24.41	7.27	8.01	3.05	2.74	12.44
Median	10.11	3.20	2.89	1.25	1.45	4.39
Maximum	717.39	208.01	153.89	78.34	54.56	389.96
Minimum	0.54	0.25	0.30	0.16	0.15	0.18
Std. Dev	51.04	16.59	16.36	6.00	4.36	26.99
Skewness	6.84	6.41	4.55	6.33	5.58	6.40
Kurtosis	66.32	53.81	27.34	58.28	47.13	61.84
Observations	1,155	1,155	1,155	711	711	1,151

Data used to compute factor variances for the Fama and French factors are collected from the Data Library of Kenneth R. French and cover the excess market return (MKT), size factor (SMB), value factor (HML), profitability factor (RMW), investment factor (CMA), and momentum factor (MOM). The first three factors are the original Fama and French three factors model (Fama and French, 1993), while RMW and CMA are from the original Fama and French five-factor model (Fama and French, 2015). The MOM factor represents the momentum factor as proposed by Carhart (1997). The sample period for the MKT, SMB, and HML realized variance data (e.g., $\widehat{\sigma}^2_{MKT}$, $\widehat{\sigma}^2_{SMB}$, $\widehat{\sigma}^2_{HML}$) spans the period July 1926 to September 2022 (1,155 monthly observations). The RMW and CMA realized variance data (e.g., $\widehat{\sigma}^2_{RMW}$, $\widehat{\sigma}^2_{CMA}$) cover the period from July 1963 to September 2022 (711 monthly observations), whereas the MOM factor variance data (e.g., $\widehat{\sigma}^2_{MOM}$) encompass the period from November 1926 to September 2022 (1,151 monthly observations).

Next, we continue to visually investigate the time series graphs of each factor's realized variance to gain a comprehensive understanding of their behavior over time. Fig. 1 shows the evolution of monthly realized variance for each factor over the sample period. The time series graphs are particularly informative, as they demonstrate that—although the data typically fluctuates around the sample mean—there are several sudden shifts in each of the figures. These sudden shifts serve as indicators for extreme events, such as the Great Depression from 1929 to 1939, the recession in the early 1990 s, the 2007–2008 financial crisis, the European sovereign debt crisis spanning from 2009 to 2010, and the COVID-19 pandemic in 2020.

In addition, Fig. 2 depicts the discrepancy between the theoretical log-normal distribution (black curve), and the distribution obeyed by each factor's monthly realized variance (red curve). Self-evidently, Fig. 2 shows that the log-normal distribution does not accurately interpolate the density function of the realized factor variance series'. In this regard, Taleb (2020, p. 23), points out that "as we fatten the tails we get higher peaks, smaller shoulders, and a higher incidence of a very large deviation." We clearly observe this issue in Fig. 2, where the peaks of the realized factor variance distributions exceed the theoretical peaks predicted by the log-normal distribution (red curve). We observe the same pattern for each realized factor variance.

Further visual evidence is provided by investigating the log-normal Q-Q plots, shown in Fig. 3. Fig. 3 reveals that the tails of the quantiles associated with the factor's realized variance distributions do not align with the theoretical quantile derived from the log-normal (red line). From Fig. 3 we observe that both ends of the Q-Q plots for all factor's variances deviate from the theoretical log-normal quantiles, whereas the center follows the red line. This pattern is indicative of a fat-tailed distribution implying a higher frequency of extreme events than what would be anticipated under a log-normal distribution.

As a final tool for visual inspection, we employ the mean excess plot (ME plot) as additional potentially indicative tool for identifying power-law processes. For a given threshold u, the mean excess of a random variable X is defined as,

$$E(X - u|X > u). \tag{2}$$

An upward linear trend in the mean excess plot signifies a power law, where the slope of the plot has a positive derivative with respect to the tail index parameter alpha (α). A larger value of α corresponds to a steeper slope in the mean excess plot. As depicted in Fig. 4, an upward linear trend that is mainly linear in the first part may indicate a heavy-tailed distribution such as a power law. However, further analysis is necessary to distinguish between these distributions and identify the specific distribution.

3. Methodology

3.1. Estimating the power-law exponents using maximum-likelihood estimation

Following Grobys (2021; 2023a), we model realized Fama and French factor variances using the following power-law function:

$$p(x) = Cx^{-\alpha}, \tag{3}$$

where $C = (\alpha - 1)x_{MIN}^{\alpha-1}$ with $\alpha \in \{\mathbb{R}_+|\alpha > 1\}$, $x = \sigma_{j,t}^2$ denotes the respective realized Fama and French factor variance provided $x \in \{\mathbb{R}_+|x_{MIN} \le x < \infty\}$, x_{MIN} is the minimum value of realized Fama and French factor variance that is governed by a power-law process, and α is the magnitude of the specific tail exponent.[4] Choosing power laws for modeling the extremely fat tails is in line with Mandelbrot (1963a) who argued that there are strong pragmatic reasons to begin the study of economic distributions and time series by those that satisfy the law of Pareto. Moreover, Taleb (2020) highlighted that the tail exponent α of a power law function captures via extrapolation the low-probability deviation not seen in the data which plays a disproportionately large role in determining the mean of the theoretical distribution.

The expectation for the realized factor variance, provided $X > x_{MIN}$, defined as $E[X|X > x_{MIN}]$, is given by:

$$E[X|X > x_{MIN}] = \int_{x_{MIN}}^{\infty} xp(x)dx = \frac{(\alpha - 1)}{(\alpha - 2)}x_{MIN}. \tag{4}$$

The second moment, provided $X > x_{MIN}$, defined as $E\left[X^2|X > x_{MIN}\right]$ (viz., the variance of the variance), is defined as:

$$E\left[X^2|X > x_{MIN}\right] = \int_{x_{MIN}}^{\infty} x^2p(x)dx = \frac{(\alpha - 1)}{(\alpha - 3)}x_{MIN}^2. \tag{5}$$

Higher moments of order k are analogously defined as:

[4] This study follows the notation in Clauset et al. (2009). To simplify notation, the index i denoting the respective realized factor variance j is dropped. Note that some other studies parametrize the *tail probability* as $P(|X| > x) \propto x^{-\alpha}$ as opposed to parametrizing the *density* function as in Equation (3) (see, for instance, Lux and Alfrano, 2016). In our study, we use the maximum-likelihood estimation as described in detail in the often-cited study of Clauset et al. (2009) which gives us point estimates for α and corresponding goodness-of-fits tests derived from the function $p(x) = Cx^{-\alpha}$. Hence, in our study we parametrize the *density* function instead of the *tail probability*.

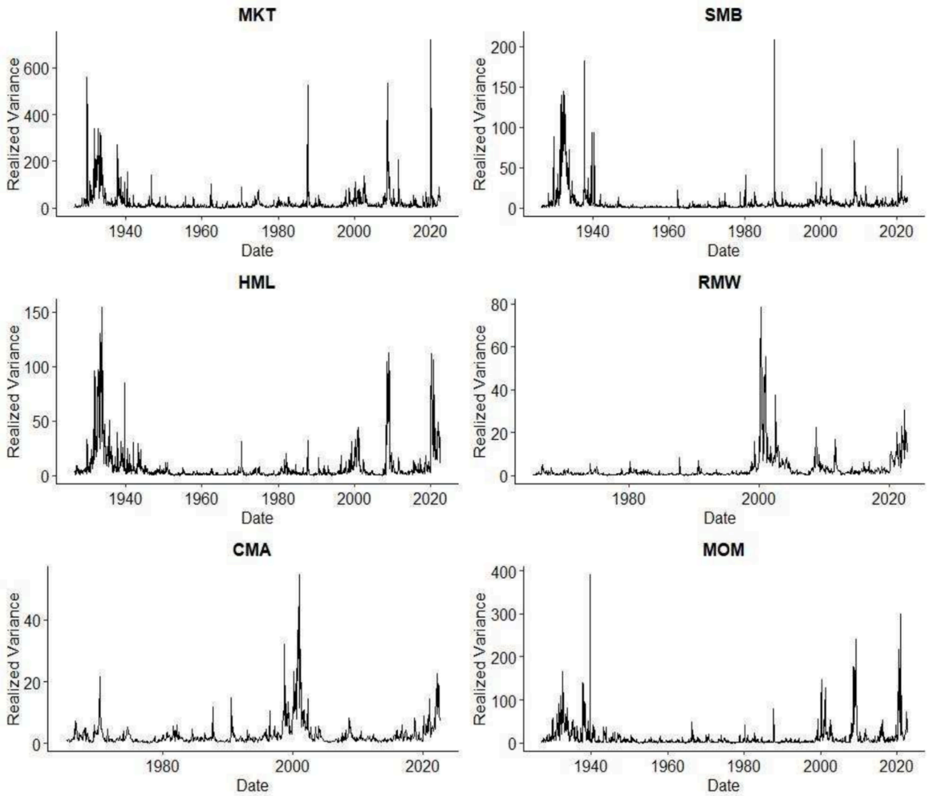

Fig. 1. Time series plot. The time series graph of monthly realized variance for excess market return (MKT), size (SMB) and value (HML) factors from July 1926 to September 2022 (1,155 monthly observations), profitability (RMW), investment (CMA) factors from July 1963 to September 2022 (711 monthly observations) and momentum factor (MOM) from November 1926 to September 2022 (1,151 monthly observations).

$$E\left[X^k|X > x_{MIN}\right] = \frac{(\alpha-1)}{(\alpha-1-k)}x_{MIN}^k. \tag{6}$$

From Equations (4) and (5), we observe that the theoretical mean for realized factor variances only exists for $\alpha > 2$, whereas the variance of realized factor variance requires $\alpha > 3$.

Following White et al. (2008) and Clauset et al. (2009), who argued that maximum-likelihood estimation (MLE) performs best for estimating power law exponents, tail exponents are estimated as:

$$\widehat{\alpha} = 1 + N\left(\sum\nolimits_{i=1}^{N} \ln\left(\frac{x_i}{x_{MIN}}\right)\right)^{-1}, \tag{7}$$

where $\widehat{\alpha}$ denotes the MLE estimator, $N \leq T$ is the number of observations exceeding x_{MIN}, and other notation is as before. As noted in Clauset et al. (2009), it is necessary to determine the relevant value for x_{MIN} to accurately estimate the most adequate tail exponent. More specifically, as shown in Equation (7), the MLE estimator depends on the chosen x_{MIN} and, hence, there are different MLE estimators from which to choose. In this regard, Clauset et al. (2009) observed that it is common practice to employ the $\widehat{\alpha}/x_{MIN}$-plot and choose the value for x_{MIN} beyond which $\widehat{\alpha}$ is stable.

Since this approach is somewhat subjective and can be sensitive to noise or fluctuations in the tail of the distribution, the authors proposed to select x_{MIN} based on minimizing the distance between the power-law model and the empirical data: Specifically, the Kolmogorov–Smirnov distance (D) is the maximum distance between the cumulative density functions (CDFs) of the data and the fitted model as defined by:

$$D = MAX_{x \geq x_{MIN}} |S(x) - P(x)|, \tag{8}$$

where $S(x)$ is the CDF of the data for the observation with value at least x_{MIN}, and $P(x)$ is the CDF for the power-law model that best fits the data in the region $x \geq x_{MIN}$. The estimate $\widehat{x}_{MIN}$ is then the value of x_{MIN} that minimizes D. Further, the standard deviation for estimated power-law exponents is according to Clauset et al. (2009) given by:

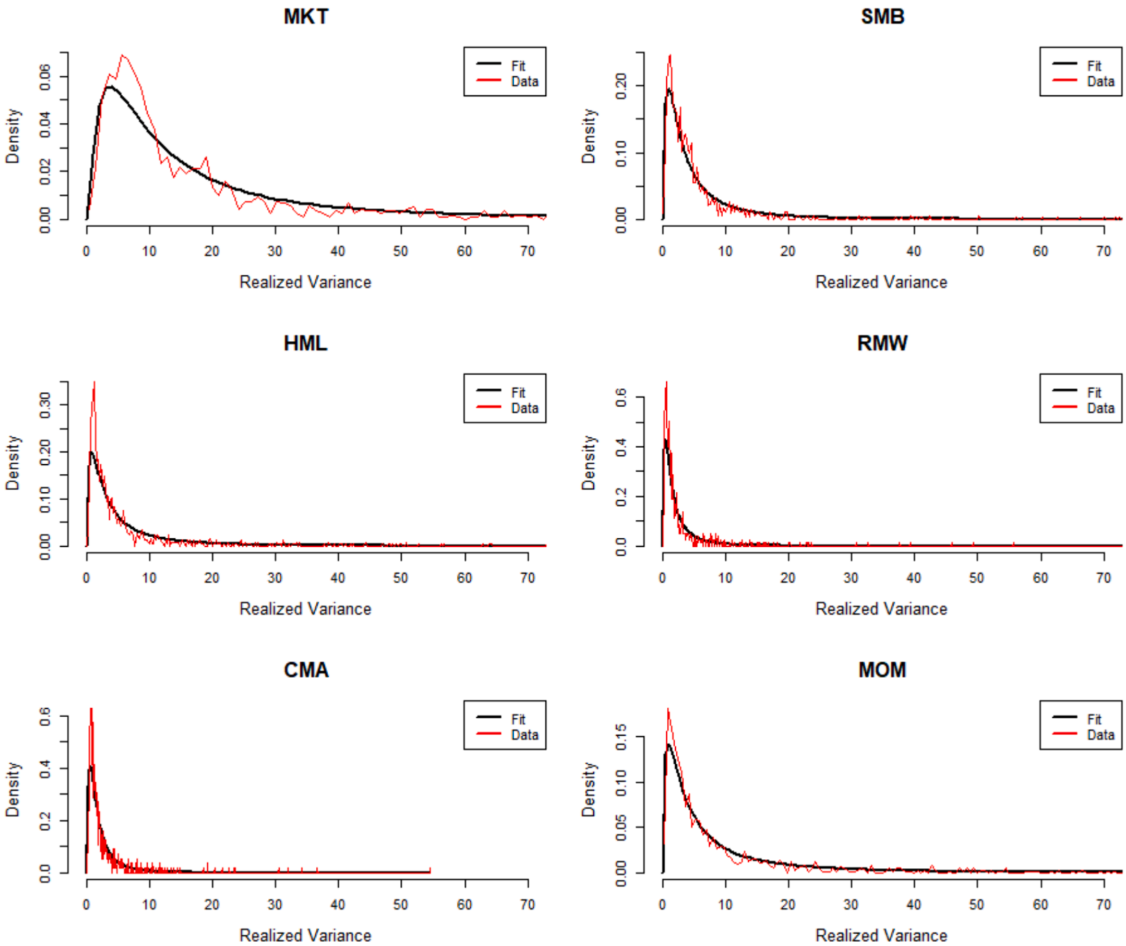

Fig. 2. Histogram plot. Theoretical Log-normal distribution (black curve) versus distribution followed by the factor's variance (red curve) for the monthly realized variance of the excess market return (MKT), size factor (SMB), value factor (HML), profitability factor (RMW), investment factor (CMA), and momentum factor (MOM). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$\sigma = \frac{\widehat{\alpha} - 1}{\sqrt{N}} + O\left(\frac{1}{N}\right). \tag{9}$$

The descriptive statistics for estimated power law exponents are reported in Table 2. From Table 2 we observe that the tail exponent is estimated at $\widehat{\alpha} < 3$ for all Fama and French factor variances, indicating that the variance of realized factor variance does not exist for any of the factor's variances. This implies, in turn, that the distributions of Fama and French realized factor variances has no theoretically defined variance. In the context of realized asset variances, this result further implies that extreme market movements and tail risk might be more prevalent than assumed when modeling realized factor variance via the lognormal distribution.

Furthermore, the most striking result from Table 2 is that the estimated alpha for the HML realized variance is estimated at $\widehat{\alpha} < 2$, which indicates that not only the variance for this factor variance is undefined, but also the theoretical mean is undefined. This is an issue that has serious implications which will be discussed later.[5]

[5] In addition, from the results presented in last row of Table 2, one might question the limited number of observations governed by the power law distribution, suggesting that the power law explains only a small fraction of the time series observations. However, it is important to note that the shares of the top 20 percent of the cumulative total of the distributions are 68.01, 69.00, 72.98, 68.10, 60.84, and 72.59 percent for the realized factor variances of the MKT, SMB, HML, RMW, CMA, and MOM, respectively. This indicates that a small fraction of observations has a large impact on the entire dataset. These numbers are closely aligned with the well-known Pareto 80/20 rule, where the top 20 percent of the cumulative total accounts for 80 percent of the cumulative total of the distribution. This means that the vast majority of the distribution of realized factor variances can essentially be considered as noise—simply because it has minimal impact compared to the significant influence employed by the minority segment that is governed by a power law process.

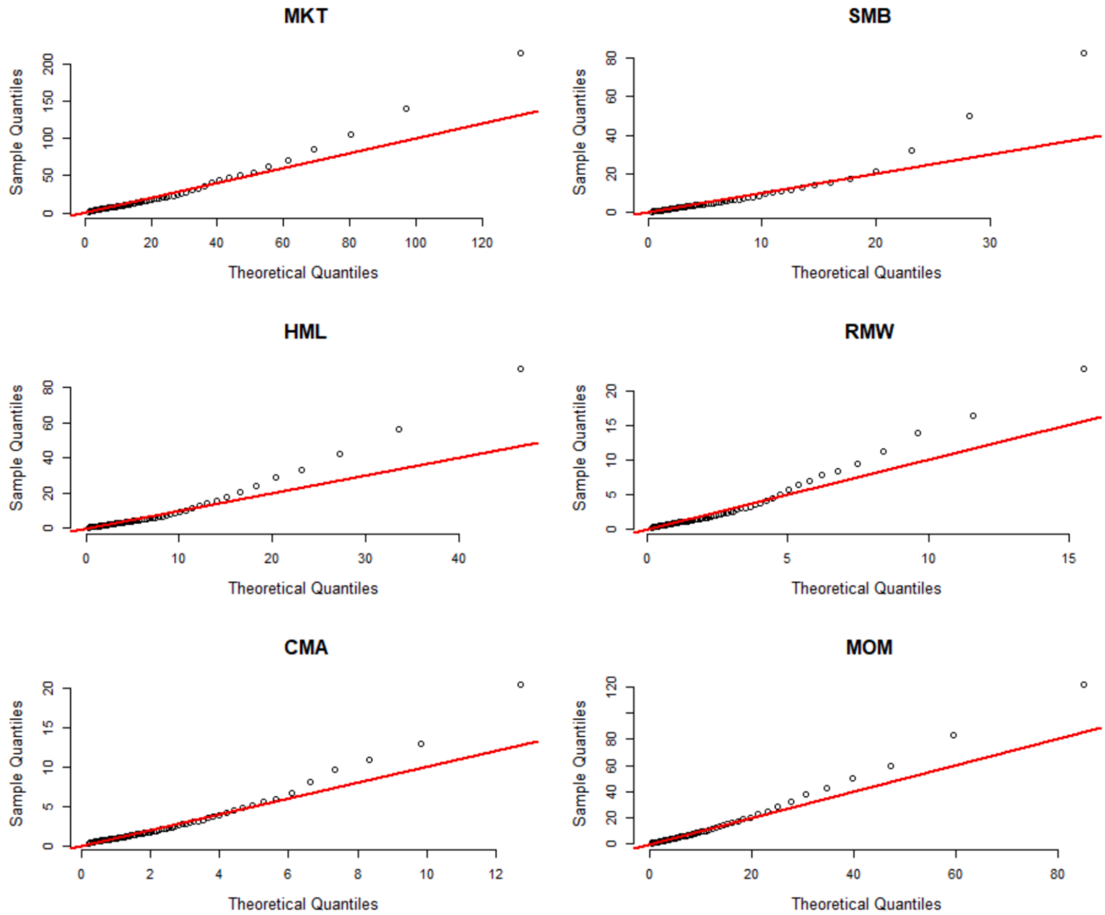

Fig. 3. Log-normal QQ Plot. The theoretical log-normal quantile (red line) versus factor's variance quantile for the monthly realized variance of excess market return (MKT), size factor (SMB), value factor (HML), profitability factor (RMW), investment factor (CMA), and momentum factor (MOM). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

3.2. Is the power-law null model plausible?

The findings from the previous section should be approached with caution, as a tail exponent within the range $2 < \alpha < 3$ does not necessarily imply that the power law is an appropriate fit for the data. Therefore, it is important to verify whether the observed dataset genuinely exhibits a power-law distribution. Following Clauset et al. (2009), we test this hypothesis by goodness-of-fit test, derived from simulations using the Kolmogorov-Smirnoff distance D, as given in Equation (8), to determine whether the empirical data and the generated data from the power law distribution with specific x_{MIN}, and α values belong to the same distribution. The null hypothesis of this test asserts that the empirical data and the generated synthetic data from the power-law distribution, with specific values for x_{MIN} and α, belong to the same distribution, implying that the power law distribution provides a plausible fit to the empirical data.

The alternative hypothesis posits that the empirical data and the generated synthetic data do not belong to the same distribution, suggesting that the power-law distribution does not provide a plausible fit to the empirical data and other possible distributions should be considered. To perform this analysis, we generate large synthetic datasets of power-law distribution using the estimated parameters x_{MIN} and α from the previous section for each realized factor variance. Specifically, the p-value of this test is defined as the fraction $\#(D_S > D)/K$, where D_S is the estimated Kolmogorov-Smirnoff distance, as defined in Equation (8), for some synthetic dataset for given x_{MIN} and α values (as estimated for the original factor variances), D is the estimated Kolmogorov-Smirnoff distance for the original dataset, and K is the number of synthetic datasets generated.[6]

We choose a significance level of 5 %, that is, we do not reject the power law model if $\#\frac{D_S>D}{K} > 0.05$. To ensure the reliability and

[6] That is, each synthetic data set is individually fit to its corresponding power-law model, and the Kolmogorov-Smirnov distance D is computed for each model. Then, we compute the p-value by counting the fraction of the time that the resulting statistic exceeds the value for the empirical data.

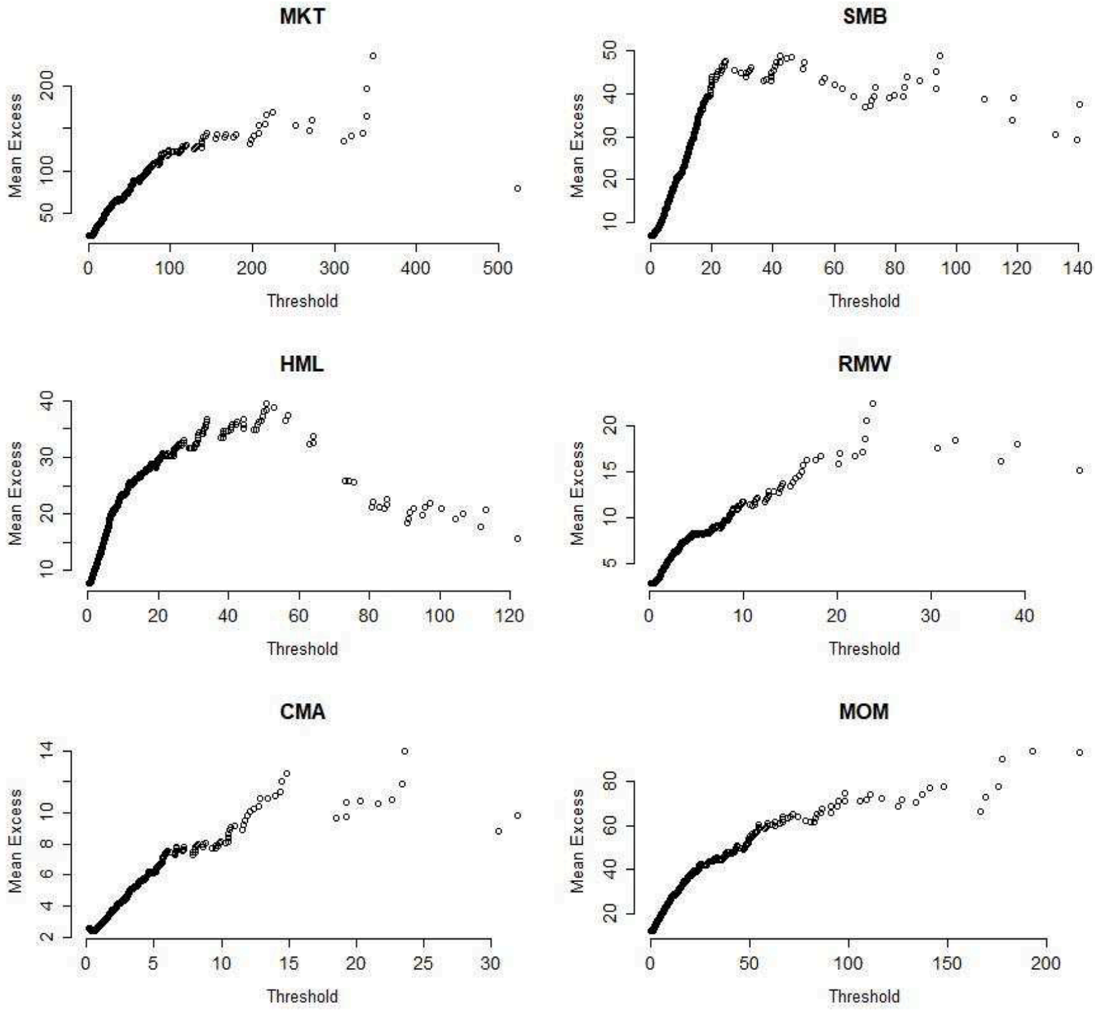

Fig. 4. Mean Excess Plot. This figure plots the sorted data on the *x*-axis against the corresponding values of the mean excess function for the monthly realized variance of the excess market return (MKT), size factor (SMB), value factor (HML), profitability factor (RMW), investment factor (CMA), and momentum factor (MOM). For a given threshold u of a random variable X, the mean excess is defined as $E(X - u|X > u)$. An upward linear trend in the mean excess plot is indicative of a Pareto distribution, where the slope of the plot has a positive derivative with respect to the tail index parameter alpha (α).

robustness of our findings, we simulated 2,500 synthetic datasets to examine whether the empirical data and the generated dataset belong to the same distribution.[7] Table 3 provides the results of the goodness-of-fit tests using our realized factor variance data.[8] The second column of Table 3 reports the *p*-value for testing the power-law null hypothesis. We observe that for all realized factor variances, the *p*-values exceed 5 %, which provides strong evidence for that power laws appear to be indeed plausible distributions for modeling realized factor variances.

3.3. Estimating robust standard errors and testing Mandelbrot's infinite variance hypothesis

Since the estimated power-law exponents are close to $\alpha \approx 2$ for all factor variances, a natural question that arises is whether the theoretical means of realized factor variances are statistically defined. This is an important issue to clarify because undefined theo-

[7] The choice of 2,500 simulations are based on Clauset et al. (2009) recommendation, which is intended to achieve a balance between computational feasibility and the precision of the *p*-value estimation.

[8] In this study, we utilized the code provided in the paper 'Fitting Power Law Distributions to Data' by Willy Lai. We would like to thank Willy Lai for their valuable contribution and for making the code available, which greatly facilitated our research and analysis. The paper can be accessed at: https://www.stat.berkeley.edu/~aldous/Research/Ugrad/Willy_Lai.pdf.

Table 2
Estimating power law exponents.

	$\widehat{\sigma}^2_{MKT}$	$\widehat{\sigma}^2_{SMB}$	$\widehat{\sigma}^2_{HML}$	$\widehat{\sigma}^2_{RMW}$	$\widehat{\sigma}^2_{CMA}$	$\widehat{\sigma}^2_{MOM}$
$\widehat{\alpha}$	2.41	2.16	1.96	2.00	2.39	2.10
$\widehat{\sigma}$	0.11	0.05	0.04	0.05	0.10	0.07
x_{MIN}	40.14	3.38	3.31	0.97	2.50	11.44
D	0.03	0.02	0.04	0.04	0.04	0.06
% observations for which $x \geq x_{MIN}$	13.85 %	48.31 %	44.85 %	64.14 %	28.83 %	24.07 %

This table reports the results of estimating power law exponents for the following realized factor variances: Excess market factor (MKT), size factor (SMB), value factor (HML), profitability factor (RMW), investment factor (CMA), and momentum factor (MOM). The $\widehat{\alpha}$ represents the estimated tail exponent and $\widehat{\sigma}$ is the estimated standard deviation. The lower threshold x_{MIN} is estimated using the optimized Kolmogorov-Smirnov distance (D) distance as proposed by Clauset et al. (2009). The estimated x_{MIN} is the value that corresponds to the optimal distance D. Additionally, for each factor variance, the fraction of observations governed by power law process is reported. The sample period for the MKT, SMB, and HML realized variances span the period July 1926 to September 2022 (1,155 monthly observations). The RMW and CMA realized variances cover the period from July 1963 to September 2022 (711 monthly observations), whereas the MOM factor variance encompasses the period from November 1926 to September 2022 (1,151monthly observations).

retical means would imply that standard econometric models based on the concept of correlation—such as ordinary least squares (OLS) or generalized methods of moments (GMM)—cannot be used to draw conclusions, as they would inevitably generate misleading statistical results (Fama, 1963).

It is important to note that the standard errors documented in Clauset et al. (2009) are derived under the assumption of *independence*. It is, however, a stylized fact that financial market data typically exhibit volatility clustering which suggest some type of dependence structure.[9] As a result, the standard deviations reported in Table 3 are underestimated, and hence, inadequate for hypothesis testing. To address this issue, we follow Grobys (2023c) and employ blocks bootstraps in line with Grobys and Juntilla (2021) for estimating robust standard deviations.

Denoting the selected block length as m, we implement a blocks bootstrap procedure such that $E[m] = T^{1/3}$ which is the suggested block length for variance estimation in Godfrey (2009). Then from a given data vector $\boldsymbol{x}$, blocks m are randomly drawn which are distributed as a geometric distribution $m \sim GEO(p)$ with $E[m] = \frac{(1-p)}{p}$. For example, for the overall data sample, for the realized variances of the MKT, SMB, HML, and MOM factor, we employ $p = 0.0909$, whereas we employ $p = 0.1111$ for the realized variances of the RMW and CMA factor. Employing this bootstrap approach, the blocks drawn from a given data vector $\boldsymbol{x}$ vary in lengths. Randomly drawn blocks m from data vector $\boldsymbol{x}$ are stacked in vector $\boldsymbol{x}^b$ as:

$$\boldsymbol{x}_i^b = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \end{bmatrix}.$$

Note again that the length of the blocks $m_1, m_2, \ldots$ varies. The procedure is stopped when the length of the artificial vector $\boldsymbol{x}^b$ exhibits a length that exceeds T. Observations exceeding T are cut off; that is, every artificial data vector $\boldsymbol{x}^b$ has the same length as the original data vector $\boldsymbol{x}$. Using this blocks bootstrap procedure, for each data vector, $B = 1,000$ artificial data vectors are constructed,

$$\begin{bmatrix} \boldsymbol{x}^1 & \boldsymbol{x}^2 & \cdots & \boldsymbol{x}^B \end{bmatrix},$$

and point estimates for α are obtained for each bootstrapped data vector $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^B$ using Clauset et al.'s (2009). For each realized factor variance, the estimated tail exponents of the bootstrap data are stacked in a 1xB vector,

$$\begin{bmatrix} \widehat{\alpha}^1 & \widehat{\alpha}^2 & \cdots & \widehat{\alpha}^B \end{bmatrix}.$$

Finally, the corresponding bootstrapped standard error $\widehat{\sigma}_{BOOT}$ is computed for the bootstrapped $\widehat{\alpha}^1, \widehat{\alpha}^2, \ldots, \widehat{\alpha}^B$ for each given realized factor variance. The descriptive statistics for the blocks bootstrapped tail exponents are reported for each realized factor variance in Table 4. Comparing the bootstrapped standard deviations with the ones reported in Table 2, it becomes evident that once (unknown) dependence structures (i.e., volatility clustering) are accounted for, standard deviations are substantially larger. As an example, the ordinary standard deviation for the realized HML realized variance is estimated at $\widehat{\sigma} = 0.05$, whereas the bootstrapped counterpart is estimated at $\widehat{\sigma}^{boot}_{HML} = 0.16$; that is, once dependence structures are accounted for, the estimation uncertainty is about three times larger.

[9] While traditional finance research typically refers to the concept of autocorrelation in this context, we would rather like to refer to dependency as the concept of correlation requires the variance to be defined.

Table 3
Goodness-of-fit test.

Realized factor variance	Power law	Log-normal	Exponential
$\widehat{\sigma}^2_{MKT}$	0.84	0.01	0.00
$\widehat{\sigma}^2_{SMB}$	0.59	0.00	0.00
$\widehat{\sigma}^2_{HML}$	0.69	0.00	0.00
$\widehat{\sigma}^2_{RMW}$	0.42	0.03	0.00
$\widehat{\sigma}^2_{CMA}$	0.50	0.03	0.00
$\widehat{\sigma}^2_{MOM}$	0.06	0.00	0.00

This table reports the results from goodness-of-fit test, by Kolmogorov Smirnoff test to examine if our empirical data and the generated data from the power law distribution with particular x_{min} and α belong to the same distribution in the second column. The table also represents the goodness-of-fit test for log- normal and exponential distributions in the third and fourth columns respectively. The null hypothesis of this test asserts that the empirical data and the generated synthetic data from the power law distribution, with specific x_{MIN} and α values, belong to the same distribution, implying that the power law distribution provides a plausible fit to the empirical data.

Moreover, it becomes evident that the means of the bootstrapped power-law exponents are close to the estimates for the original data. However, this should not come as a great surprise because the point estimates for the exponents obtained via Clauset et al.'s (2009) MLE approach should be unbiased.

Because the point estimates for the tail exponents of the realized factor variances are close to $\alpha \approx 2$, a natural question that arises is: Can we reject the infinite mean hypothesis? Recall that Fama (1963) argued that correlation-based methodologies will inevitably result in very misleading statistical inference if the theoretical variance is undefined. Hence, it may be not surprising that Mandelbrot's (1963b) seminal study on cotton price changes—ascertaining that cotton price changes do not exhibit a defined theoretical variance—resulted in an enormous amount of follow-up research. Interestingly, in a literature review covering power laws in financial economics, Lux and Alfarano (2016) argued that other studies raised doubts over the validity of the infinite variance hypothesis by questioning the stability-under-aggregation property of these estimates, and the pertinent literature gradually converged to the insight of an exponent $\alpha \geq 3$. These studies, however, do not employ realized variances. As pointed out from Barndorff-Nielsen and Shephard (2002) realized variances provide a more precise measure of volatility, as they are less susceptible to biases and measurement errors that can affect other volatility estimators. As a result, we test the infinite theoretical variance hypothesis for all realized factor variances as follows:

$$H_0 : \alpha \leq 2 \text{ versus } H_1 : \alpha > 2 \,.$$

In Panel B of Table 4 we report the p-values based on the empirical distribution of the bootstrapped power-law exponents. The p-value is given by the fraction of power-law exponents for which $\widehat{\alpha} \leq 2$ is satisfied. We observe from Panel B of Table 4 that the p-values are ≥ 0.05 for all realized factor variances. In Panel C we report the p-values derived from t-tests based on the bootstrapped standard deviations. The p-values are virtually the same, and hence, strongly confirm this evidence. Overall, it appears that realized factor variances are governed by a power-law process and regardless of which realized factor variance we consider, the hypothesis $\alpha \approx 2$ cannot be rejected which suggest a remarkable commonality across realized factor variances.

To further strengthen our results, we introduce a joint test which is able to assess whether Fama and French factor variances share a common component manifested in a common power-law exponent. First, we note that due to our bootstrapping approach $COV\left(\widehat{\alpha}_i, \widehat{\alpha}_j\right) = 0 \; \forall \; i \neq j$. As a consequence, we can define the following covariance matrix:

$$\widehat{\Sigma}_{\hat{\alpha}} = \begin{pmatrix} \widehat{\sigma}^2_{\hat{\alpha}_{boot},MKT} & 0 & 0 & 0 & 0 & 0 \\ 0 & \widehat{\sigma}^2_{\hat{\alpha}_{boot},SMB} & 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{\sigma}^2_{\hat{\alpha}_{boot},HML} & 0 & 0 & 0 \\ 0 & 0 & 0 & \widehat{\sigma}^2_{\hat{\alpha}_{boot},RMW} & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\sigma}^2_{\hat{\alpha}_{boot},CMA} & 0 \\ 0 & 0 & 0 & 0 & 0 & \widehat{\sigma}^2_{\hat{\alpha}_{boot},MOM} \end{pmatrix},$$

where $\widehat{\sigma}^2_{\hat{\alpha}_{boot},MKT}, \ldots, \widehat{\sigma}^2_{\hat{\alpha}_{boot},MOM}$ denote the bootstrapped variances obtained via block bootstrap as outlined earlier. Using the vector of estimated power-law exponents from Table 2, we define the following test statistic:

$$\widehat{\lambda} = (\widehat{\boldsymbol{\alpha}} - q1)' \widehat{\Sigma}_{\hat{\alpha}}^{-1} (\widehat{\boldsymbol{\alpha}} - q1), \tag{10}$$

where the covariance matrix $\widehat{\Sigma}_{\hat{\alpha}}$ has the dimension 6x6, $\widehat{\boldsymbol{\alpha}}$ is the 6x1 vector of estimated power-law exponents, 1 is a 6x1 vector of ones

Table 4
Summary statistics for blocks bootstrapped power law exponents.

Panel A. Descriptive statistics for the bootstrapped tail exponents.						
	$\hat{\alpha}^{boot}_{MKT}$	$\hat{\alpha}^{boot}_{SMB}$	$\hat{\alpha}^{boot}_{HML}$	$\hat{\alpha}^{boot}_{RMW}$	$\hat{\alpha}^{boot}_{CMA}$	$\hat{\alpha}^{boot}_{MOM}$
Mean	2.25	2.22	2.01	2.23	2.49	2.21
Median	2.21	2.20	1.99	2.13	2.45	2.13
Maximum	3.22	3.18	3.14	4.00	4.39	4.01
Minimum	1.78	1.84	1.68	1.67	1.79	1.73
Std. Dev.	0.24	0.18	0.16	0.33	0.32	0.33
Skewness	0.80	0.81	1.43	1.75	1.09	1.44
Kurtosis	3.50	4.56	8.59	7.34	5.99	5.24
Jarque-Bera	117.52	210.45	1,642.99	1,296.71	571.39	553.84
Probability	0.00	0.00	0.00	0.00	0.00	0.00
Observations	1,000	1,000	1,000	1,000	1,000	1,000
Panel B. Estimated *p*-values for testing the infinite mean hypothesis using the empirical distribution of bootstrapped power law exponents.						
p-value	0.14	0.11	0.54	0.24	0.05	0.36
Panel C. Estimated *p*-values for testing the infinite mean hypothesis using bootstrapped standard deviations.						
p-value	0.15	0.11	0.48	0.24	0.06	0.26

Denoting the selected block length as m, we implement a blocks bootstrap procedure such that $E[m] = T^{1/3}$. From a given data vector $\boldsymbol{x}$, blocks m_j are randomly drawn which are distributed as a geometric distribution $m_j \sim GEO(p)$ with $E[m_j] = \frac{(1-p)}{p}$. Employing this bootstrap approach, the blocks drawn from a given data vector $\boldsymbol{x}$ vary in lengths. Randomly drawn blocks m_j from data vector $\boldsymbol{x}$ are stacked in vector $\boldsymbol{x}^b$ as: $\boldsymbol{x}^b_i = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \end{bmatrix}$. The procedure is stopped when the length of the artificial vector $\boldsymbol{x}^b$ exhibits a length that exceeds T. Observations exceeding T are cut off; that is, every artificial data vector $\boldsymbol{x}^b$ has the same length as the original data vector $\boldsymbol{x}$. Using this blocks bootstrap procedure, for each data vector, $B = 1,000$ artificial data vectors are constructed, $\begin{bmatrix} \boldsymbol{x}^1 & \boldsymbol{x}^2 & \dots & \boldsymbol{x}^B \end{bmatrix}$, and point estimates for α are obtained for each bootstrapped data vector $\boldsymbol{x}^1, \boldsymbol{x}^2, \dots, \boldsymbol{x}^B$ using Clauset et al.'s (2009) approach. For each factor variance, the estimated tail exponents of the bootstrap data are stacked in a 1xB vector, $\begin{bmatrix} \hat{\alpha}^1 & \hat{\alpha}^2 & \dots & \hat{\alpha}^B \end{bmatrix}$. Finally, the corresponding bootstrapped standard error $\hat{\sigma}_{BOOT}$ is computed for the bootstrapped $\hat{\alpha}^1, \hat{\alpha}^2, \dots, \hat{\alpha}^B$ for each given realized factor variance. The descriptive statistics for the blocks bootstrapped tail exponents are reported in Panel A, whereas Panel B and C report the *p*-values for testing the infinite mean hypothesis.

and q is the hypothesized common power-law exponent. The estimated test statistic, denoted as $\hat{\lambda}$, is under the null hypothesis distributed as $\chi^2(6)$. The test statistic is iteratively estimated covering the economically important interval $q = (1.2, 1.3, \dots, 3.1, 3.2)$. Note acceptance of the null hypothesis would suggest that a common power-law exponent exists that governs the realized factor variance risk of the Fama and French factors.

The results are reported in Table 5. Note that using a 5 percent significance level, the corresponding critical value for the chi-square distribution is $\chi^2_{0.95}(6) = 12.59$. Strikingly, the null hypothesis cannot be rejected for $1.7 < \alpha < 2.5$ which strongly confirms the presence of a common exponent governing the realized factor variances. Since the *p*-value for testing the null hypotheses $H_0 : \alpha = 1.9$ or $H_0 : \alpha = 2.0$ exceeds 5 percent, Mandelbrot's infinite variance hypothesis is confirmed because both hypotheses assume that the theoretical mean of the realized factor variances is undefined. Furthermore, the optimal α governing Fama and French realized factor variances is $\alpha = 2.1$ as this test statistic generates the largest *p*-value corresponding to 0.74. Finally, because the null hypothesis is clearly rejected for all joint tests for which $\alpha > 2.4$, we can clearly rule out that the variance of realized factor variances exists. Overall, a key finding is here that realized factor variances have a common component that is manifested in sharing the same power-law behavior—that is, the common component is manifested here in a common power-law exponent governing Fama and French factor variances.

3.4. Testing alternative distributions against power laws

Our results derived from Clauset et al.'s (2009) goodness-of-fit test do not guarantee that power-law distributions are ideal matches for the data, as there might be other distributions that, perhaps, could fit the data better. Therefore, for each realized factor variance, we compare the corresponding power-law distribution with the log-normal distribution to identify the most appropriate fit for the data. It is important to note that earlier research postulated that realized volatility is close to lognormally distributed (e.g., Andersen, Bollerslev, Diebold, & Ebens, 2001; Andersen, Bollerslev, Diebold, and Labys, 2001a & 2001b). Therefore, in this section we examine whether the lognormal distribution is a plausible fit to the data. According to Clauset et al. (2009), we test this hypothesis using a similar goodness-of-fit test with the same significance level of 5 percent like outlined earlier. Thereby, the null model which is assumed is the lognormal distribution. The results from the goodness-of-fit tests for lognormal distributions are presented in column 3 of

Table 5
Testing for a common power-law exponent governing Fama and French factor variance risk.

q	$\widehat{\lambda}$	p-value
1.2	**103.80**	**0.00**
1.3	**83.43**	**0.00**
1.4	**65.36**	**0.00**
1.5	**49.60**	**0.00**
1.6	**36.15**	**0.00**
1.7	**25.00**	**0.00**
1.8	**16.17**	**0.01**
1.9	9.64	0.14
2	5.43	0.49
2.1	3.52	0.74
2.2	3.91	0.69
2.3	6.62	0.36
2.4	11.64	0.07
2.5	**18.96**	**0.00**
2.6	**28.59**	**0.00**
2.7	**40.53**	**0.00**
2.8	**54.78**	**0.00**
2.9	**71.34**	**0.00**
3	**90.21**	**0.00**
3.1	**111.38**	**0.00**
3.2	**134.86**	**0.00**

To explore whether the exists a common component governing power-law behavior of the realized Fama and French factor variances, the following estimated test statistic is proposed: $\widehat{\lambda} = (\widehat{\boldsymbol{\alpha}} - q\mathbf{1})' \widehat{\Sigma}_{\widehat{\alpha}}^{-1} (\widehat{\boldsymbol{\alpha}} - q\mathbf{1})$, where the covariance matrix $\widehat{\Sigma}_{\widehat{\alpha}}$ has the dimension 6x6, $\widehat{\boldsymbol{\alpha}}$ is the 6x1 vector of estimated power-law exponents, 1 is a 6x1 vector of ones and q is the hypothesized common power-law exponent. The estimated test statistic denoted as $\widehat{\lambda}$ is under the null hypothesis distributed as $\chi^2(6)$. The test statistic is iteratively estimated covering the economically important interval $q = (1.2, 1.3, \ldots, 3.1, 3.2)$. The corresponding test statistic is under the null hypothesis distributed as $\chi^2(6)$. **Bold** figures indicate statistical significance on a 5 % level.

Table 3. Strikingly, we observe that the p-values for all of the six realized factor variances are less than 0.05. We interpret this as strong evidence for rejecting the assumption of time-honored lognormality of realized factor variances. Consequently, these findings suggest that finance researchers may need to move beyond the lognormal assumption to more accurately understand and predict the behavior of factor risk. In addition, to further strengthen our analysis, we also conduct goodness-of-fit tests using the exponential distribution as the null model. The exponential distribution serves as alternative model for fitting data exhibiting a considerably fatter tail than the normal distribution. The results of the goodness-of-fit tests for exponential distributions are reported in column 4 of Table 3. The documented findings suggest that p-values are zero in all cases, indicating that the exponential distribution does not provide an adequate fit for the realized factor variances series.

3.5. Additional analysis

3.5.1. Further evidence from one-sigma tests

While the goodness-of-fit tests used in previous sections have the assumed model as the null model, a recent study from Grobys (2023c) proposes a test where the power-law model is under the alternative allowing us to have control over the type-1 error. Grobys' (2023c) notes that Taleb (2020) argues that a common feature of fat-tailed distributions is that more observations are within one-standard deviation than predicted by the normal distribution. Specifically, a stylized fact for the normal distribution is that, as $T \longrightarrow \infty$, we would expect that a fraction of 0.6827 of the observations are within one standard deviation from the mean. The rationale of Grobys' (2023c) one-sigma test is as follows: Distributions that exhibit theoretically defined mean and variance can be standardized which allows us to compute the fraction of observation that can be expected to occur within one standard deviation from the mean, as $T \longrightarrow \infty$. It can be shown that a fraction of 0.8189 is within one standard deviation from the mean for a standardized lognormal distribution (LGN). Hence, the following test statistic can be used:

$$\lambda = \frac{(x_{\leq \pm 1\sigma} - m_{\leq \pm 1\sigma})^2}{m_{\leq \pm 1\sigma}} + \frac{(x_{> \pm 1\sigma} - m_{> \pm 1\sigma})^2}{m_{> \pm 1\sigma}}, \tag{11}$$

where $m_{\leq \pm 1\sigma}$ denotes the expected number of observations occurring within one standard deviation from the mean of the lognormal,

$m_{>\pm 1\sigma}$, denotes the expected number of observations exceeding one standard deviation from the mean, whereas $x_{\leq \pm 1\sigma}$ and $x_{>\pm 1\sigma}$ are corresponding values for the observed distribution. In an early study, Pearson (1900) showed that this type of test statistic is distributed as chi-square with one degree of freedom. In Table A1 in the appendix we report the corresponding results derived from one-sigma tests. We observe from Table A1 that $\widehat{\lambda}$ exceeds the critical value of $\chi^2_{0.95}(1) = 3.84$ which is the corresponding critical value using a significance level of 5 percent. The *p*-values are equal to zero for all factor variances indicating that the well-established lognormal distribution can be ruled out as underlying data-generating process. These results strongly confirm our earlier results derived from Clauset et al.'s (2009) goodness-of-fit tests.

3.5.2. Evidence from a multifractal model of asset invariances

Next, recent work from Grobys (2023b) extends Mandelbrot's (2008) multifractal model of asset returns and proposed a multifractal model of asset invariances. Grobys (2023b) shows that his model is capable of pricing various realized asset market variances (e.g., S&P 500, Bitcoin, gold, crude oil, GBP/USD foreign exchange rate). We hypothesize that if realized factor variances exhibit the same power-law behavior, they should be generated by the same multifractal model. To explore whether the Fama and French factor variances can be explained by Grobys' (2023b) multifractal model of asset invariances, we first estimate the Hurst exponents for the realized factor variances using detrended fluctuation analysis (DFA). Specifically, the DFA used in this study can be summarized as follows: First, a given data series x_t is converted to the mean-centered cumulative sum:

$$\widetilde{x}_t = \sum_{t=1}^{T} x_t \,. \tag{12}$$

In the present research context,x_t is a vector of realized factor variances. Different time scales k are defined, that is, $k \in \{4, 8, 16, 32, 64, 128, 256, 512\}$. Depending on the defined time scale, data is split into epochs and for each epoch s, a time series regression is used to detrend the data. For example, employing $k = 512$ means that monthly data for $\widetilde{x}_t$ is split into two non-overlapping epochs. For each epoch s, the following regression is employed:

$$\widetilde{x}_t = \gamma_0 + \gamma_1 t + e_t \,, \tag{13}$$

where $t = 1, \cdots, 512$ for the first epoch and $t = 513, \cdots, 1024$ for the second epoch. Then for each respective epoch s, the root mean squared error (RMSE) is computed as:

$$RMSE_S = \sqrt{\frac{1}{T_s} \sum_{t}^{T_s} \widehat{e}_t} \,, \tag{14}$$

where $T_s = 512$. Finally, the estimates for $RMSE_S$ are averaged for each time scale k, giving us $\overline{RMSE}_k$. According to theory, the following relation holds:

$$\overline{RMSE}_k = ck^H. \tag{15}$$

The corresponding Hurst exponent is then estimated by implementing a linear fit between log-scales and log-$\overline{RMSE}_k$. If the data were independent, the ratio between numerator and denominator should be, according to the theory, 1:2, corresponding to a Hurst exponent of $H = 0.50$. Moreover, $H > 0.50$ is in line with long-term dependence, that is, a long memory of the stochastic process in which the data are persistent; however, $H < 0.50$ implies antipersistence, which is characterized by the tendency to revert back. Grobys (2023b) shows that the realized asset variances for the S&P 500 and crude oil can be priced by a multifractal model derived from a multiplicative cascade using binominal bending with a probability of $p = 0.70$. Specifically, this model generates power-law exponents with expectation $\overline{\widehat{\alpha}} = 1.9187$ $(\widehat{\sigma}_\alpha = 0.2916)$ and $\overline{\widehat{H}} = 0.9176$ $(\widehat{\sigma}_H = 0.0557)$. In Table A2, the point estimates for estimated power-law exponents and estimated Hurst exponents are reported. We see that the estimated power-law exponents are less than 1.96 standard deviations from $\alpha = 1.9187$ suggesting that the realized factor variances are in line with predictions generated from the multifractal model. While Hurst exponents suggest an even higher level of persistence than predicted from the multifractal model, a robustness check in Grobys (2023b) provides evidence for that for smaller samples, Hurst exponents corresponding to $H \approx 1.20$ are still in the support of the multifractal model. Overall, the evidence suggests the presence of a common underlying mechanism that generates the power-law behavior which we empirically observe for the realized Fama and French factor variances.

3.5.3. Evidence from co-fractality analysis

To which extent do realized factor variances co-move? Note that a common power-law exponent governing realized factor variances does not necessarily imply that factor variances are in power-law regimes at the same time. Since correlations are undefined if variances are infinite, co-dependencies between realized factor variances can be investigated using Grobys's (2023a) recently proposed concept of co-fractality.[10] Since co-fractality in its strong form is perhaps less relevant for empirical data, we focus here solely on co-fractality in its weak form. Specifically, let us define n_1 and n_2 as the number of observations that are governed by power laws for

[10] The concept of co-fractality is related to the concept of tail dependence and discussed in detail in Grobys' (2023a) study.

processes $x_{i,t}$ and $x_{j,t}$. Then, the (weak) co-fractality statistic, defined as λ^{CF}, is $\lambda^{CF} = 0$ if for two processes $x_{i,t}$ and $x_{j,t}$, none of the power-law observations coincide, whereas $\lambda^{CF} = 1$ if all of the observations (e.g., $MIN\{n_1, n_2\}$) which are governed by power laws coincide.[11] Self-evidently, as $\lambda^{CF} \rightarrow 1$, there is not much room for risk diversification because power-law behavior coincides; in other words, extreme events are likely to occur within the same parts of time series paths.

To explore co-fractality for realized factor variances, we consider the sample from July 1963 to September 2022 because we do not have observation for the realized variances of RMW and CMA in the ex-ante July 1963 period. As a result, our sample consists of 711 monthly observations. The estimates $\widehat{\lambda}^{CF}$ are reported in Table A3 in the appendix. We see that the number of sample observations which are governed by a power-law process varies between 77 months for the realized variance of the MKT factor and 456 for the realized variance of the RMW factor. Next, we observe from Table A3 that for 7 out of 15 realized factor variance pairs, $\widehat{\lambda}^{CF} > 0.90$, whereas for 11 out of 15 $\widehat{\lambda}^{CF} > 0.80$, suggesting a high level of (weak) co-fractality among realized factor variances. Overall, our results suggest a high level of co-movement in the tails of the realized factor variances.

This commonality aligns well with the theory of the market model. According to this model, the return of an asset or portfolio can be expressed as a function of market return and a unique idiosyncratic component, formulated as:

$$R_t = \alpha + \beta M_t + \varepsilon_t . \tag{16}$$

When applying this model to SMB (Small Minus Big) factor returns, we get:

$$SMB_t = (\alpha_s - \alpha_B) + (\beta_s - \beta_B)M_t + \left(\varepsilon_{s,t} - \varepsilon_{B,t}\right), \tag{17}$$

where S is the portfolio of small stocks and B is the portfolio of big stocks. Consequently, the variance of SMB is expressed as:

$$Var(SMB_t) = \underbrace{(\beta_s - \beta_B)^2}_{>0} var(M_t) + var(\varepsilon_s - \varepsilon_B). \tag{18}$$

This suggests that the SMB factor's variance is essentially a scaled version of market variance—influenced by the difference in market sensitivities (betas) of small and big stocks, alongside the variance of differential idiosyncratic terms. The correspondence of our empirical findings with the market model implies a deeper, systematic behavior in factor variances, suggesting that the commonality observed in factor variances—including that of all factors analyzed—is not an anomaly but rather a reflection of fundamental market dynamics, as captured by the market model. This insight highlights the significance of the observed common component in factor variance risk. However, it is essential to emphasize that this model is applicable only for values of $x < x_{min}$ because variance is undefined for $x \geq x_{min}$. The interesting thing is here that it turns out that even the observations $x \geq x_{min}$ exhibit a high level of co-dependence measured in terms of co-fractality.[12]

3.5.4. *Are estimated power-law exponents reliable? Evidence from sample-split tests*

Next, one might wonder whether the results reported in this study are subject to sample-specificity themselves, which is, of course, a valid concern. To explore this issue, we split each subsample into two non-overlapping subsamples of equal length. Specifically, the first subsample spans from July 1926 to October 1972 for the realized MKT, SMB, and HML realized variances, July 1963 to February1993 for the realized RMW and CMA realized variances, and November 1926 to September 1974 for the realized MOM factor variance.

The second subsample spans from November 1972 to September 2022 for the realized MKT SMB, and HML realized variances, March 1993 to September 2022 for the realized RMW and CMA realized variances, and October 1974 to September 2022 for the realized MOM factor variance. For each subsample and realized factor variance, we implement the blocks bootstrap procedure as detailed in section 3.3. The descriptive statistics for the estimated power-law exponents are reported in Panels A and B of Table A4 in the appendix, whereas Panel C reports the results from z-tests for testing parameter equality. The two-sample z-tests are given by,

$$\widehat{z} = \frac{(\widehat{\alpha}_{1i} - \widehat{\alpha}_{2i})}{\sqrt{\left(0.5\widehat{\sigma}^2_{\widehat{\alpha}_{1i}} + 0.5\widehat{\sigma}^2_{\widehat{\alpha}_{2i}}\right)}} , \tag{19}$$

where $\widehat{\alpha}_{1i}$ ($\widehat{\alpha}_{2i}$) denotes the estimate of α for sample 1 (sample 2) and factor variance i, and $\widehat{\sigma}^2_{\widehat{\alpha}_{1i}}$ ($\widehat{\sigma}^2_{\widehat{\alpha}_{2i}}$) denotes the corresponding estimated sample variance obtained from blocks bootstraps. Using a standard significance level of 5 percent, we observe from Panel C of Table A4 that for five out of six realized factor variances, we cannot reject the null hypothesis that the point estimates $\widehat{\alpha}_{1i}$ and $\widehat{\alpha}_{2i}$ are statistically the same. The realized SMB variance appears to be the only exception here. Overall, our results suggest that power-law exponents exhibit a high level of reliability, which appears to be contrary to GARCH-type models because Mandelbrot (2008)

[11] Recall that observations $x_{i,t} \geq x_{MIN,i,t}$ and $x_{j,t} \geq x_{MIN,j,t}$ are governed by power-law processes.

[12] Note that we use the market model only to show that the market model predicts co-movement of factor variances too; however, the market model is not designed for non-linearities. Since power-law behavior strongly indicate non-linearities, the tails of the realized variance distributions need to be investigated separately using, for instance, co-fractality analysis.

criticized the problem of changing parameters retrieved from GARCH-type models, arguing that "… many recent models of price variation try to explain the obviously shifting pattern of volatility by inserting parameters that change by the day, hour, and second; such are the GARCH family mentioned earlier." Our evidence suggests that unlike GARCH-type models, power-law models work with just a set of a few consistent parameters that typically remain constant over time and place. This is also in line with Calvet and Fisher (2004) or Lux, Morales-Arias, and Sattarhoff (2014), who found that power-law models usually outperform GARCH models in this regard.

3.5.5. Does the tail index change after controlling for autocorrelation in realized variances?

A reader could be concerned about the existence of a genuine power law in the factor variance distributions. It is well-known that time-varying volatility can induce power law-like behavior in the tails of asset returns. That implies that common statistical methods for estimating power laws will erroneously produce estimates indicating a power law when there is none; that is, the tails are thick but it is due to the fact that they are generated by a mixture of exponential-tail distributions rather than a genuine power law. Thus, a natural question that arises is whether the same problem arises for estimating the power-law behavior of realized factor variances. To explore this issue, we estimate autoregressive models of order p for each realized factor variance such as:

$$\widehat{\sigma}^2_{i,t} = \beta_{0,i} + \beta_{1,i}\widehat{\sigma}^2_{i,t-1} + \beta_{2,i}\widehat{\sigma}^2_{i,t-2} + \cdots + \beta_{p,i}\widehat{\sigma}^2_{i,t-p} + \varepsilon_{i,t}\,,$$

where $\varepsilon_{i,t}$ is the innovation process, p is the lag order, and $i \in \{MKT, SMB, HML, RMW, CMA, MOM\}$. The optimal lag-order p is obtained by inspection of the partial autocorrelation function.[13] Note that this type of model can be considered a version of an ARCH-type model where the equation modeling the conditional variance is modeled directly by using the own lags of the realized factor variances. Then, we use the absolute amount of the innovation processes, $|\widehat{\varepsilon}_{MKT}|$, $|\widehat{\varepsilon}_{SMB}|$, …, $|\widehat{\varepsilon}_{MOM}|$, and estimate the power-law exponents and implement Clauset et al.'s (2009) goodness-of-fit tests using the procedure outlined in section 3.

In Table 6 we report the point estimates for estimated autoregressive models of realized Fama and French factor variances. The results from Table 6 show that there are strong patterns of autocorrelation in the data-generating processes of realized factor variances. The order of autocorrelation varies between $p = 2$ (e.g., $\widehat{\sigma}^2_{MOM,t}$) and $p = 4$ (e.g., $\widehat{\sigma}^2_{SMB,t}$). The descriptive statistics of the innovation processes are reported in Table 7. From Table 7 it becomes evident that even after controlling for higher order autocorrelation, the innovation processes of realized factor variances exhibit extremely heavy tails, as indicated by kurtosis values varying between 34.11 (e.g., $\widehat{\varepsilon}_{HML}$) and 111.40 (e.g., $\widehat{\varepsilon}_{MKT}$), respectively. Finally, in Table 8 we report the estimated power-law exponents for the innovation processes of realized factor variances using the absolute amount, $|\widehat{\varepsilon}_{MKT}|$, $|\widehat{\varepsilon}_{SMB}|$, …, $|\widehat{\varepsilon}_{MOM}|$. Strikingly, the estimated power-law exponents are virtually the same which strongly supports our earlier finding of genuine power-law behavior.[14] Also, Clauset et al.'s (2009) GoF tests do not reject the power-law null models for the vast majority of realized factor variance innovations which further strengthens our main findings.

3.5.6. Are our results subject to discretization errors?

To examine the impact of different aggregation periods on the reliability of our findings suggesting power-law behavior of realized factor variances, we conduct a robustness check by using quarterly realized variance data derived from squared daily return data. This approach allows us to address potential concerns related to discretization errors that may influence the tail properties of the realized factor variance distributions across varying data frequencies. Thus, instead of using the 22-day window of non-overlapping squared daily returns to compute monthly realized variance, we employ a 66-day window of non-overlapping squared daily returns to calculate quarterly realized variance data.

The summary statistics for the quarterly realized variance are reported in Table A5. in the appendix. The results show that quarterly data exhibits similar statistical characteristics like monthly data. From Table A5, the mean values indicate that—as with the monthly realized factor variances—the MKT realized variance exhibits the highest sample average, followed by momentum MOM, HML, SMB, RMW, and CMA realized variances. Similarly, the median values reinforce this pattern, with MKT realized variance showing the highest median value and CMA realized variance the lowest. Additionally, the maximum and minimum values reveal a broad range of realized variances across all factors, with the most significant variation observed for MKT realized variance. The standard deviation is highest for realized MKT realized variance and lowest for realized CMA realized variance. All realized factor variances display extremely high kurtosis values, ranging from 17.12 for HML realized variance to 39.62 for RMW realized variance, which indicates the presence of exceptionally fat tails in the distribution. Moreover, the skewness is most pronounced for the realized RMW realized variance, suggesting notable asymmetry in its distribution.

Next, we estimate the power-law exponents for the quarterly realized variance data. The results are reported in Panel A of Table A6. Similar to the monthly data reported in Table 2, the tail exponent for all quarterly realized factor variances is estimated to be $\widehat{\alpha} < 3$ for all realized factor variances, indicating that the variance of factor variance does not exist for any of the factor's quarterly variances.

[13] Results are unreported to save space but are available upon request.

[14] As an example, let us consider the estimated power-law exponent for the market factor variance which is, according, to Table 2, $\widehat{\alpha} = 2.41$ with a corresponding 95% confidence interval $\alpha \in [1.94, 2.88]$ because the robust estimate for the standard deviation is, as reported in Table 4, $\widehat{\sigma} = 0.24$. The point estimate for the power-law exponent for the innovations process of the market factor variance is, according to Table 8, $\widehat{\alpha} = 2.19$ which is clearly within the 95% confidence interval for the power-law exponent for the market factor variance. That is, statistically, the point estimates are not different from each other. The same argument holds for the power-law exponents of the remaining realized factor variances too.

Table 6
Point estimates for autoregressive models of realized Fama and French factor variances.

	$\widehat{\beta}_{0,i}$	$\widehat{\beta}_{1,i}$	$\widehat{\beta}_{2,i}$	$\widehat{\beta}_{3,i}$	$\widehat{\beta}_{4,i}$	R-square
$\widehat{\sigma}^2_{MKT,t}$	9.02***	0.48***	0.02	0.13***		0.31
	(6.17)	(16.39)	(0.61)	(4.55)		
$\widehat{\sigma}^2_{SMB,t}$	1.52***	0.34***	0.15***	0.15***	0.15***	0.43
	(3.60)	(11.71)	(5.12)	(4.91)	(5.01)	
$\widehat{\sigma}^2_{HML,t}$	1.18***	0.44***	0.15***	0.27***		0.59
	(3.36)	(15.45)	(4.79)	(9.40)		
$\widehat{\sigma}^2_{RMW,t}$	0.63***	0.60***	0.02	0.18***		0.54
	(3.56)	(16.23)	(0.36)	(4.87)		
$\widehat{\sigma}^2_{CMA,t}$	0.44***	0.35***	0.18***	0.32***		0.55
	(3.24)	(9.64)	(4.89)	(8.83)		
$\widehat{\sigma}^2_{MOM,t}$	4.43***	0.38***	0.27***			0.32
	(5.96)	(13.34)	(9.33)			

*** Statistically significant on a 1 % level

Data used to compute realized Fama and French factor variances are collected from the Data Library of Kenneth R. French and cover the excess market return (MKT), size factor (SMB), value factor (HML), profitability factor (RMW), investment factor (CMA), and momentum factor (MOM). The first three factors are the original Fama and French three factors model (Fama and French, 1993), while RMW and CMA are from the original Fama and French five-factor model (Fama and French, 2015). The MOM factor represents the momentum factor as proposed by Carhart (1997). The sample period for the MKT, SMB, and HML realized variance data spans the period July 1926 to September 2022 (1,155 monthly observations). The RMW and CMA realized variance data cover the period from July 1963 to September 2022 (711 monthly observations), whereas the MOM factor variance data encompasses the period from November 1926 to September 2022 (1,151 monthly observations). The innovation process of realized Fama and French factor variances is estimated by running autoregressive models of order p: $\widehat{\sigma}^2_{i,t} = \beta_{0,i} + \beta_{1,i}\widehat{\sigma}^2_{i,t-1} + \beta_{2,i}\widehat{\sigma}^2_{i,t-2} + \cdots + \beta_{p,i}\widehat{\sigma}^2_{i,t-p} + \varepsilon_{i,t}$, where $\varepsilon_{i,t}$ is the innovation process, p is the lag order, and $i \in \{MKT, SMB, HML, RMW, CMA, MOM\}$. The optimal lag-order p is obtained by inspection of the partial autocorrelation function. This table reports the point estimates of the regressions as well as the coefficients of determination (R-square). The *t*-statistics are given in parentheses.

Table 7
Summary statistics for the innovations of realized Fama and French factor variances.

	$\widehat{\varepsilon}_{MKT}$	$\widehat{\varepsilon}_{SMB}$	$\widehat{\varepsilon}_{HML}$	$\widehat{\varepsilon}_{RMW}$	$\widehat{\varepsilon}_{CMA}$	$\widehat{\varepsilon}_{MOM}$
Mean	0.00	0.00	0.00	0.00	0.00	0.00
Median	−6.58	−1.30	−1.04	−0.45	−0.35	−3.50
Maximum	686.67	201.29	88.07	44.54	26.90	376.10
Minimum	−217.76	−63.99	−60.95	−27.50	−20.39	−151.42
Std. Dev	42.58	12.58	10.44	4.10	2.94	22.28
Skewness	8.39	7.60	3.72	4.92	3.31	7.06
Kurtosis	111.40	97.73	34.11	54.17	34.86	101.42
Observations	1,152	1,151	1,152	708	708	1,149

Data used to compute realized Fama and French factor variances are collected from the Data Library of Kenneth R. French and cover the excess market return (MKT), size factor (SMB), value factor (HML), profitability factor (RMW), investment factor (CMA), and momentum factor (MOM). The first three factors are the original Fama and French three factors model (Fama and French, 1993), while RMW and CMA are from the original Fama and French five-factor model (Fama and French, 2015). The MOM factor represents the momentum factor as proposed by Carhart (1997). The sample period for the MKT, SMB, and HML realized variance data (e.g., $\widehat{\sigma}^2_{MKT}$, $\widehat{\sigma}^2_{SMB}$, $\widehat{\sigma}^2_{HML}$) spans the period July 1926 to September 2022 (1,155 monthly observations). The RMW and CMA realized variance data (e.g., $\widehat{\sigma}^2_{RMW}$, $\widehat{\sigma}^2_{CMA}$) cover the period from July 1963 to September 2022 (711 monthly observations), whereas the MOM factor variance data (e.g., $\widehat{\sigma}^2_{MOM}$) encompass the period from November 1926 to September 2022 (1,151 monthly observations). The innovation process of realized Fama and French factor variances is estimated by running autoregressive models of order p: $\widehat{\sigma}^2_{i,t} = \beta_{0,i} + \beta_{1,i}\widehat{\sigma}^2_{i,t-1} + \beta_{2,i}\widehat{\sigma}^2_{i,t-2} + \cdots + \beta_{p,i}\widehat{\sigma}^2_{i,t-p} + \varepsilon_{i,t}$, where $\varepsilon_{i,t}$ is the innovation process, p is the lag-order, and $i \in \{MKT, SMB, HML, RMW, CMA, MOM\}$. The optimal lag-order p is obtained by inspection of the partial autocorrelation function. This table reports the descriptive statistics for the innovation process $\varepsilon_{i,t}$.

Additionally, Panel B of Table A6 reports the 95 percent confidence intervals for the estimated power-law exponents for the monthly realized variance data. The analysis shows that the point estimates for the estimated power-law exponents derived from quarterly realized variance data consistently fall within the confidence intervals derived from monthly data. This consistency across different sampling frequencies strengthens the reliability of our findings and supports the hypothesis of invariance in the relationship between realized variances at different data frequencies.

Finally, we perform Clauset et al.'s (2009) goodness-of-fit tests to the quarterly realized variance data. The results are reported in Table A7. The second column of Table A7 reports the *p*-value for testing the power-law null hypothesis. Similar to the findings derived from monthly data, the *p*-values for all quarterly realized factor variances exceed 5 percent. This not only provides strong evidence

Table 8
Estimating power law exponents for the innovation processes of realized factor variances.

	$\lvert\hat{\varepsilon}_{MKT}\rvert$	$\lvert\hat{\varepsilon}_{SMB}\rvert$	$\lvert\hat{\varepsilon}_{HML}\rvert$	$\lvert\hat{\varepsilon}_{RMW}\rvert$	$\lvert\hat{\varepsilon}_{CMA}\rvert$	$\lvert\hat{\varepsilon}_{MOM}\rvert$
$\hat{\alpha}$	2.19	2.10	1.92	2.15	2.41	2.13
x_{MIN}	11.78	1.36	1.12	0.69	1.72	6.38
p-value (GoF)	0.24	0.05	0.02	0.73	0.14	0.05

This table reports the results of estimating power law exponents for the absolute amount of the innovation processes of the following realized factor variances: Excess market factor (MKT), size factor (SMB), value factor (HML), profitability factor (RMW), investment factor (CMA), and momentum factor (MOM). The $\hat{\alpha}$ represents the estimated tail exponent and $\hat{\sigma}$ is the estimated standard deviation. The lower threshold x_{MIN} is estimated using the optimized Kolmogorov-Smirnov distance (D) distance as proposed by Clauset et al. (2009). The estimated x_{MIN} is the value that corresponds to the optimal distance D. Additionally, for each factor variance, the fraction of observations governed by power law process is reported. The sample period for the MKT, SMB, and HML realized variances span the period July 1926 to September 2022 (1,155 monthly observations). The RMW and CMA realized variances cover the period from July 1963 to September 2022 (711 monthly observations), whereas the MOM factor variance encompasses the period from November 1926 to September 2022 (1,151monthly observations). The innovation process of factor variances is obtained by using the following autoregressive model: $\hat{\sigma}^2_{i,t} = \beta_{0,i} + \beta_{1,i}\hat{\sigma}^2_{i,t-1} + \beta_{2,i}\hat{\sigma}^2_{i,t-2} + \cdots + \beta_{p,i}\hat{\sigma}^2_{i,t-p} + \varepsilon_{i,t}$, where $\varepsilon_{i,t}$ is the innovation process, p is the lag-order, and $i \in \{MKT, SMB, HML, RMW, CMA, MOM\}$. The optimal lag-order p is obtained by inspection of the partial autocorrelation function. This table reports the estimated power-law exponent $\hat{\alpha}$, the corresponding cutoff x_{MIN}, and the p-value for the goodness-of-fit (GoF) test outlined in section 3. To implement the GoF tests we use 2500 simulated data sets.

supporting the plausibility of the power-law null models but also underscores the robustness of our results across different time frequencies; that is, the distributional tail characteristics of realized factor variances are the same regardless of the aggregation period.

In addition, according to the results reported in column 3 of Table A7, the null hypothesis of the lognormal distribution is rejected for all realized factor variances except for SMB realized variance and MOM realized variance. However, according to Clauset et al. (2009), comparing p-values for the power law along with other possible distributions (here: lognormal or exponential) can provide a persuasive argument concerning the suitability of the power-law model. Explicitly, Clauset et al. (2009) stress that a higher p-value for the power law compared to other competing models, effectively eliminates the alternatives distributions. Thus, as reported in Table A7, the notably higher p-values for power laws relative to those for the lognormal distribution for SMB realized variance and MOM realized variance suggest that the power-law model more accurately captures the behavior of these specific realized factor variances too. Furthermore, from column 4 of Table A7 we observe that—consistent with the results retrieved from the analysis of monthly data—the p-values are zero or near to zero in all cases. This indicates that the exponential distribution does not either offer an adequate fit for quarterly factor variance data.

4. Discussion

4.1. Commonalities and contradictions with other studies

Using daily data, earlier studies found that the realized variances of the S&P 500, crude oil, gold, Bitcoin and foreign exchange rates are governed by power-law processes (Grobys, 2021; 2023a). The estimated tail exponents varied in their economic magnitude but were usually $\hat{\alpha} > 2$ implying that that the theoretical mean of realized variances exist. Using monthly data, we test whether the realized variances of the Fama and French factors exhibit finite theoretical means. Statistically, we could not reject the infinite theoretical mean hypothesis for any of the realized equity factor variance. Given that Lux and Alfarano (2016) argue that the pertinent literature converged to the insight that financial asset returns exhibit power-law behavior in line finite variances, this is, perhaps a surprising finding; However, our results are in line with Mandelbrot's (1963b) early study, which used cotton price changes as a case study to demonstrate that returns on some speculative financial assets do not exhibit a theoretically defined variance.

One might wonder if a potential reason could be that we explicitly analyze realized variances as opposed to the absolute amount of asset returns. Note that our approach to investigate realized variances is motivated by well-established earlier literature arguing that realized variances contain information that standard models, which employ simple asset returns, cannot reveal (e.g., Bubák, Kočenda, and Žikeš, 2011; Andersen, Bollerslev, Diebold, and Labys, 2003; Andersen, Bollerslev, and Meddahi, 2004). It is interesting to note that a recent study of Grobys (2023c) uses the absolute amount of S&P 500 and Bitcoin returns covering the period July 28, 2013, until May 14, 2023. Using weekly data, that study cannot either reject the infinite variance hypothesis for both Bitcoin and S&P 500 returns. Therefore, our study is in line with these recent findings.

4.2. Power-law or lognormal?

Next, there is a well-established literature arguing that realized asset volatility is close to lognormally distributed (e.g., Andersen, Bollerslev, Diebold, & Ebens, 2001; Andersen, Bollerslev, Diebold, and Labys, 2001a & 2001b). While the studies of Grobys (2021; 2023a) do not explicitly test the lognormal distribution against proposed power laws, we explicitly test this issue for realized equity factor variances. Using Clauset et al.'s (2009) goodness-of-fit test our findings strongly suggest that the power-law null model cannot be rejected for any realized factor variance. However, one could perhaps argue that this type of test is often considered weak because it imposes the power-law model under the null hypothesis. Therefore, we employ Grobys' (2023c) recently proposed one-sigma test where the lognormal distribution is the null model and the power-law model forms the alternative. Grobys' (2023c) one-sigma test strongly rejects the lognormal distribution, as indicated by *p*-values that are $p = 0.00$ for all tests carried out. Overall, our findings provide strong evidence for significant power-law behavior of realized factor variances. We believe, however, that this should not come as a great surprise, given that Paretian tails are considered stylized facts of financial asset returns. We argue that lognormally distributed variances are an illusion which may result in severe consequences: While the lognormal distribution has defined first and second moment, our analysis reveals that the underlying power-law distributions do not exhibit theoretically defined variances. Hence, we argue that the often-used lognormal model will inevitably underestimate (realized) factor risks by a substantial margin.

4.3. Implications for factor premiums and the validity of earlier research

Furthermore, the study of Grobys and Kolari (2022) found that the popular Fama and French three-factor model did not outperform the CAPM in North America, Europe, and Japan. This finding indicates that size and value factor premiums do not exist in other markets than the U.S. Are only U.S. investor compensated for risk? While Fama and French (2015) concluded that their earlier proposed value factor is redundant after accounting for the investment factor, a recent study of Cakici et al. (2023) uses new data from U.S. and global markets to re-examine market risk premium predictability by using equity anomalies. Employing machine learning, their study shows that (a) anomalies cannot predict aggregate market returns and (b) any ostensible evidence from the U.S. lacks external validity because it cannot be extended internationally nor does it hold for alternative anomaly sets. Moreover, other studies do not even find any evidence for the existence of significant factor premiums in extended samples (Dichev, 1998; Chan, Karceski, and Lakonishok, 2000; Hirshleifer, 2001; Schwert, 2003; Smith and Timmermann, 2022; Chen and Velikov, 2023).

Given our finding of undefined second moments, the finding of sample-specific factor premiums should not come as a great surprise: Note that Fama (1963) in his early study already pointed out that results derived from standard statistical methods based on the concept of correlation will provide very misleading answers if the variance does not exhibit a theoretically defined mean. Therefore, "statistically significant factor premiums", documented in abundance in the finance literature appear to be sample-specific phenomena, and therefore, inappropriate means for predicting returns.

A reader might wonder what are the practical implications of sample-specificity or how likely is it that two independent studies will find different results even though the data generating processes are the same? It is important to note that the most essential requirement for a factor is (a) the existence of positive factor premium which (b) should be statistically significant on at least a 5 percent level. Once these requirements are satisfied, a potential factor could be used for statistical tests. In this regard, it is interesting to recall that, in their seminal 1992 and 1993 studies, Fama and French proposed the size and value factor because the (excess) market factor alone failed to describe historical average stock returns. As mentioned earlier, the results form subsequent research cast doubts on the existence of the size premium in expanded samples (i.e., Dichev, 1998; Chan et al., 2000; Hirshleifer, 2001; Schwert, 2003). Performing international tests of the Fama and French (1992, 1993) three-factor model, Grobys and Kolari (2022) show that neither the size factor nor value factor matter for describing the cross section of expected stock returns. In view of this evidence, we employ the size factor as factor that is supposedly *not* priced; that is, a factor that does not meet the previously defined factor requirements. In fact, over the July 1963 to September 2022 sample, which used in the present study, the sample average of the size factor corresponded to 0.19 % per month with a *t*-statistic of 1.51 indicating that the size premium is statistically not different from zero.

Therefore, to explore the consequences of sample-specificity, we retrieve the size factor over the July 1963 to September 2022 sample from Kenneth French's website and employ blocks bootstraps, as described in section 3.3., to simulate 5,000 size factor evolutions.[15] For each synthetic size factor, we compute the factor premium (e.g., sample average) and the corresponding *t*-statistic. We plot these two metrics in Fig. 5.[16] We observe from Fig. 5 that even though the original size premium is statistically not different from zero, in 36.50 percent of the synthetic size factor samples, the size premium appears to be 'statistically significant' as indicated by *t*-statistics $> |1.96|$. Specifically, 1,824-out-of-5,000 synthetic samples exhibit size premiums of at least 0.18 % per month with corresponding *t*-statistics $> |1.96|$. Using one-sided tests, our findings indicate that in 47.70 percent of the synthetic size factor samples, the size premium appears to be 'statistically significant' as indicated by *t*-statistics > 1.65.

Next, we use trial-and-error to generate a data series of normally distributed random variables that exhibit sample *t*-statistic corresponding to 1.51—which is the same value of the *t*-statistic that we empirically observe for the size factor over our sample. Again, we use the same blocks bootstraps procedure as described above to generate synthetic samples and explore how many synthetic samples exhibit *t*-statistics $> |1.96|$ or > 1.65, respectively. Our findings (unreported) indicate that 26.30 percent of the synthetic samples exhibit *t*-statistics $> |1.96|$, whereas 34.22 percent of the synthetic samples exhibit *t*-statistics > 1.65. Overall, our findings show that in the presence of an infinite variance process, standard statistical inference is severely distorted.

Future research could employ simulations to study the behavior of the standard Fama and French methodology and predictive regressions under the assumptions that the variances of Fama and French factors do not exist and hence standard OLS inference does not apply.[17] This is, however, outside the scope of this study and therefore left for future research.

[15] Note that we implemented the blocks bootstrap using a block length of 60 months to account for unknown dependency structures in the data. For instance, Asness, Moskowitz, and Pedersen (2013) document return dependencies up to five years.

[16] The descriptive statistics are reported in Tables A.8 in the appendix.

[17] For instance, Cont and Tankov (2004) discuss some data generating processes that can be parametrized to either have or not have a finite first moment (α-stable processes).

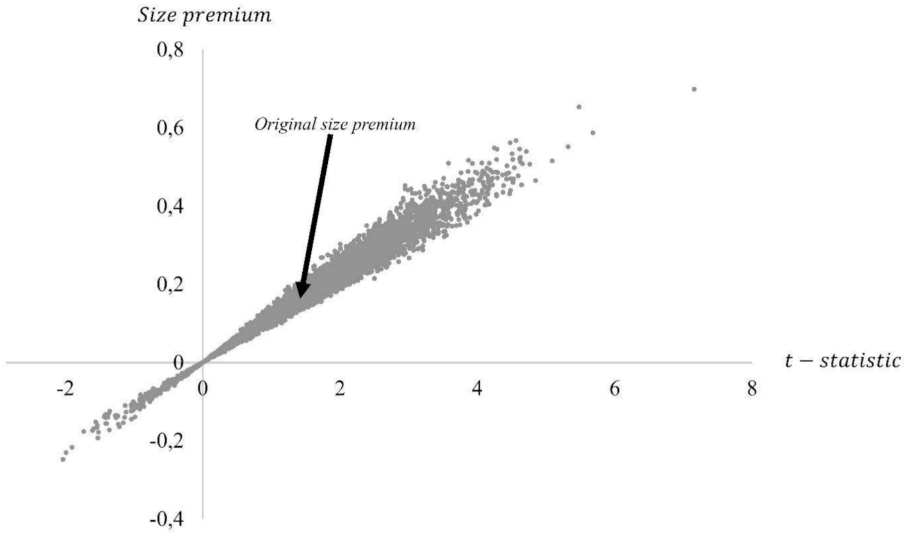

Fig. 5. Synthetic size factor premiums and statistical significances. We retrieve monthly data on Fama and French's size factor over the July 1963 to September 2022 sample from Kenneth French's website and employ blocks bootstraps, as described in section 3.3., to simulate 5,000 size factor evolutions. For each synthetic size factor, we compute the sample average and the corresponding t-statistic and plot these two metrics in this figure. The blocks bootstraps procedure uses an expected block length of 60 months.

4.4. Limitations and future research

It is important to note that our study has some limitations. First, in our study, we follow Merton (1980) in computing realized factor variances. Future studies are encouraged to explore this issue using other approaches to compute realized variances. Moreover, our study explicitly analysis monthly data. However, one could have the view that using intraday data would perhaps result in less noisy estimates for realized factor variances. On the other hand, it is important to note that high frequency data could be polluted with a substantially higher level of noise. Mandelbrot (2008) highlights that high frequency data may suffer from microstructure issues. In analyzing the market for foreign exchange rates, Mandelbrot (2008) argues that data having a higher frequency than two hours or a lower frequency that 180 days are subject to crossovers; that is, points where the mathematical relation does not appear to hold anymore. Also, Mandelbrot (2008) points out that crossovers are common for real as opposed to theoretical fractal data. While this study uses monthly data as monthly variance data is typically used in finance studies (e.g., Moreira and Muir, 2017; Grobys, Ruotsalainen and Äijö, 2018; Grobys and Vähämaa, 2020), future research could extend our study and explore higher or lower frequented factor variance data. Finally, we focus explicitly on exploring the realized variance risk of the well-known Fama and French equity factors. While these factors correspond only to a small subset of potential factor candidates (Feng, Giglio, & Xiu, 2020), one could investigate this issue for other long-short portfolios. This is, however, outside the scope of this paper and therefore left for future research.

5. Conclusion

While earlier studies focused on examining the premiums of equity market factors, this is the first study that explored the realized variances of the well-established Fama and French equity factors. Specifically, we investigated whether realized factor variances would exhibit power-law behavior and if so, to which degree. Contrary to the earlier literature our findings indicated that power laws more appropriately describe the data-generating processes than the time-honored lognormal distribution. The estimated power-law exponents were relatively low; that is, $\alpha \approx 2$ for all realized factor variances. We tested the infinite theoretical mean hypothesis and found that we could not reject this hypothesis. Again, this finding was contrary to the pertinent literature arguing that the theoretical variance exists for most financial assets. In line with earlier literature, we argued that realized variances more reliably reflect the dynamics of asset variance processes and therefore our results would deviate from results retrieved form estimated based on plain returns.

Furthermore, a multifractal model of asset invariances is capable of describing the evolution of the observed realized variances which appear to be generated by the same underlying mechanism manifested in a common power-law exponent $\alpha \approx 2$ resulting from a multifractal cascade based on binominal bending with $p = 0.70$. Our findings indicated that risk diversification is limited because extreme events occurring in the cross section of realized variances tend to coincide. Our findings are important for at least two reasons: From a practical point of view, our findings call for re-evaluations of financial reward-to-risk measures often used for evaluating investment strategies following factor investing which may suffer from sample-specificity. Knowing that often-used metrics such as

Sharpe ratios are misleading in the presence of infinite variances, new methodologies are required. Given that this industry has about \$5 trillion asset under management this is not a trivial issue. From a theoretical point of view, our findings provide evidence that standard econometric methods often used for decision making provide a poor foundation for decision making because they deliver very misleading results due to sample-specificity. In this regard, our results call for a careful re-evaluation of econometric methods used for financial decision making.

Extremely fat-tailed data—which is something that we observe for the Fama and French factor variances—contain extreme observations that can significantly influence the results of statistical analysis. However, traditional parametric methods, which typically assume a normal distribution, may be highly sensitive to these extreme events, eventually resulting in biased estimates or misleading inferences (Fama, 1963). Taleb (2010) has sparked growing interest in non-parametric methods which do not rely on specific distributional assumptions. Such non-parametric methods are less affected by outliers since they do not depend on particular distributional premises (Wasserman, 2006). Results derived from statistical tests using those methodologies provide robustness even when confronted with extreme observations (Huber, 1981). Additionally, given that non-parametric methods are not bounded by distributional assumptions, the resulting estimates and inferences are often more intuitive and straightforward (Efron and Tibshirani, 1994). Moreover, Mandelbrot (2008) underscores the importance of alternative models that can capture the complexities of financial data more robustly. Further, Mandelbrot (1982) introduced the concept of fractal geometry, explaining how financial data often exhibits fractal-like patterns—an inspiration that has encouraged many researchers to seek alternative models for financial data representation. Though the potential of these novel methodologies is evident, a detailed investigation of their applicability and implications lies beyond the scope of this study.

CRediT authorship contribution statement

Masoumeh Fathi: Methodology, Investigation, Writing – review & editing. **Klaus Grobys:** Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis. **Janne Äijö:** Writing – review & editing, Visualization, Supervision.

Appendix

Table A1
Results derived from one-sigma tests.

	$\widehat{\sigma}^2_{MKT}$	$\widehat{\sigma}^2_{SMB}$	$\widehat{\sigma}^2_{HML}$	$\widehat{\sigma}^2_{RMW}$	$\widehat{\sigma}^2_{CMA}$	$\widehat{\sigma}^2_{MOM}$
$prob(x \leq \pm 1\sigma)$	0.9498	0.9550	0.9247	0.9325	0.9241	0.9305
$\widehat{\lambda}$	133.44	144.25	87.17	61.87	53.06	96.58
(p-value)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)

This table reports the results of one-sigma tests as detailed in section 3.5.1. Note that a fraction of 0.8189 is within one standard deviation from the mean for a standardized lognormal distribution (LGN). The reference distribution for the estimated test statistic $\widehat{\lambda}$ is under the null hypothesis distributed as $\chi^2(1)$.

Table A2
Pricing realized factor variances using a multifractal model of asset invariances.

	$\widehat{\sigma}^2_{MKT}$	$\widehat{\sigma}^2_{SMB}$	$\widehat{\sigma}^2_{HML}$	$\widehat{\sigma}^2_{RMW}$	$\widehat{\sigma}^2_{CMA}$	$\widehat{\sigma}^2_{MOM}$
$\widehat{H}$	1.0770	1.1174	1.1838	1.1785	1.1273	1.0912
$\widehat{\sigma}_H$	0.0192	0.0156	0.0214	0.0296	0.0216	0.0245
$\widehat{\alpha}$	2.25	2.22	2.01	2.23	2.49	2.21

This table reports the estimated Hurst exponents using detrended fluctuation analysis. The standard errors are obtained from log-log regressions as detailed in section 3.5.2.

Table A3
Co-fractality matrix for realized factor variances.

	$\widehat{\sigma}^2_{MKT}$	$\widehat{\sigma}^2_{SMB}$	$\widehat{\sigma}^2_{HML}$	$\widehat{\sigma}^2_{RMW}$	$\widehat{\sigma}^2_{CMA}$	$\widehat{\sigma}^2_{MOM}$
#	77	393	272	456	204	149
$\widehat{\sigma}^2_{MKT}$	1	0.96***	0.94***	0.99***	0.66***	0.79***
$\widehat{\sigma}^2_{SMB}$		1	0.82***	0.82***	0.82***	0.91***
$\widehat{\sigma}^2_{HML}$			1	0.94***	0.79***	0.86***

(continued on next page)

Table A3 (*continued*)

	$\widehat{\sigma}^2_{MKT}$	$\widehat{\sigma}^2_{SMB}$	$\widehat{\sigma}^2_{HML}$	$\widehat{\sigma}^2_{RMW}$	$\widehat{\sigma}^2_{CMA}$	$\widehat{\sigma}^2_{MOM}$
$\widehat{\sigma}^2_{RMW}$				1	0.92***	0.98***
$\widehat{\sigma}^2_{CMA}$					1	0.60***
$\widehat{\sigma}^2_{MOM}$						1

*** Statistically significant on a 1 % level.
This table reports the estimated co-fractality matrix using the concept of weak co-fractality as detailed in section 3.5.3. The sample is from July 1963 to September 2022 comprising 711 observations. The symbol # denotes the count of sample observations which are governed by a power-law process.

Table A4
Summary statistics for blocks bootstrapped power-law exponents using sample split tests.

Panel A. Descriptive statistics for the bootstrapped tail exponents for the earlier subsample.						
	$\widehat{\alpha}^{boot}_{MKT}$	$\widehat{\alpha}^{boot}_{SMB}$	$\widehat{\alpha}^{boot}_{HML}$	$\widehat{\alpha}^{boot}_{RMW}$	$\widehat{\alpha}^{boot}_{CMA}$	$\widehat{\alpha}^{boot}_{MOM}$
Mean	1.96	1.84	2.02	3.30	3.00	2.21
Median	1.92	1.81	1.98	3.28	2.97	2.02
Maximum	3.42	2.63	4.57	4.49	5.87	7.22
Minimum	1.54	1.48	1.53	2.53	1.86	1.57
Std. Dev.	0.23	0.16	0.27	0.28	0.52	0.52
Skewness	1.78	1.00	2.69	0.50	0.66	2.19
Kurtosis	9.36	4.71	18.75	3.52	4.28	12.42
Jarque-Bera	2210.40	289.57	1,153.40	53.54	141.32	4,501.60
Probability	0.00	0.00	0.00	0.00	0.00	0.00
Observations	1,000	1,000	1,000	1,000	1,000	1,000
Panel B. Descriptive statistics for the bootstrapped tail exponents for the later subsample.						
	$\widehat{\alpha}^{boot}_{MKT}$	$\widehat{\alpha}^{boot}_{SMB}$	$\widehat{\alpha}^{boot}_{HML}$	$\widehat{\alpha}^{boot}_{RMW}$	$\widehat{\alpha}^{boot}_{CMA}$	$\widehat{\alpha}^{boot}_{MOM}$
Mean	2.39	2.71	2.01	2.51	2.36	2.14
Median	2.38	2.67	2.00	2.42	2.24	2.13
Maximum	3.41	3.99	2.90	6.41	4.87	3.24
Minimum	1.83	2.15	1.61	1.65	1.62	1.60
Std. Dev.	0.20	0.25	0.17	0.53	0.45	0.22
Skewness	0.96	0.96	0.92	1.73	1.60	0.88
Kurtosis	6.45	4.51	5.58	9.26	6.34	5.17
Jarque-Bera	649.96	247.89	416.82	2,127.90	892.48	326.41
Probability	0.00	0.00	0.00	0.00	0.00	0.00
Observations	1,000	1,000	1,000	1,000	1,000	1,000
Panel C. Estimated z-tests.						
$(\widehat{\alpha}_1 - \widehat{\alpha}_2)$	-0.43*	-0.87***	0.01	0.79*	0.64	0.07
$\widehat{\sigma}_{(\widehat{\alpha}_1 - \widehat{\alpha}_2)}$	0.22	0.21	0.23	0.42	0.49	0.40
z-statistic	-1.95	-4.14	0.04	1.88	1.31	0.18

*** Statistically significant on a 1 % level; * statistically significant on a 10 % level.
This table reports the results from a sample split analysis of blocks bootstrapped power law exponents. The first sub-sample spans from July 1926 to October 1972 for the realized MKT, SMB, and HML realized variances, July 1963 to February1993 for the realized RMW and CMA realized variances, and November 1926 to September 1974 for the realized MOM factor variance. The results are reported in Panel A. The second subsample spans from November 1972 to September 2022 for the realized MKT, SMB, and HML realized variances, March 1993 to September 2022 for the realized RMW and CMA realized variances, and October 1974 to September 2022 for the realized MOM factor variance. The results are reported in Panel B. The blocks bootstrap procedure is detailed in section 3.3.

Table A5
Summary statistics for quarterly realized Fama and French factor variance data.

	$\widehat{\sigma}^2_{MKT}$	$\widehat{\sigma}^2_{SMB}$	$\widehat{\sigma}^2_{HML}$	$\widehat{\sigma}^2_{RMW}$	$\widehat{\sigma}^2_{CMA}$	$\widehat{\sigma}^2_{MOM}$
Mean	75.72	22.67	24.99	10.00	8.98	39.27
Median	36.29	11.32	10.11	4.39	4.96	16.48
Maximum	927.75	323.58	325.97	165.27	103.37	573.50

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Table A5 (*continued*)

	$\widehat{\sigma}^2_{MKT}$	$\widehat{\sigma}^2_{SMB}$	$\widehat{\sigma}^2_{HML}$	$\widehat{\sigma}^2_{RMW}$	$\widehat{\sigma}^2_{CMA}$	$\widehat{\sigma}^2_{MOM}$
Minimum	3.62	1.58	1.73	1.10	1.01	1.17
Std. Dev	122.68	40.10	44.96	17.60	12.41	68.74
Skewness	4.24	4.39	3.91	5.47	4.37	4.05
Kurtosis	21.08	21.89	17.12	39.62	24.95	19.93
Observations	384	384	384	226	226	382

This table reports the descriptive statistics for the quarterly realized Fama and French factor variances. Data used in this table are collected from the Data Library of Kenneth R. French and covers the excess market return (MKT), size factor (SMB), value factor (HML), profitability factor (RMW), investment factor (CMA), and momentum factor (MOM). The first three factors are the original Fama and French three factors model (Fama and French, 1993), while RMW and CMA are from the original Fama and French five-factor model (Fama and French, 2015). The MOM factor represents the momentum factor as proposed by Carhart (1997). The sample period for the realized variances of MKT, SMB, and HML spans the period July 1926 to September 2022 (384 quarterly observations). The realized variances for RMW and CMA cover the period from July 1963 to September 2022 (226 quarterly observations), whereas the realized variance of MOM encompasses the period from November 1926 to September 2022 (382 quarterly observations).

Table A6

Estimated power-law exponents for quarterly realized variance data.

Panel A. **Estimate**d power-law **exponents for quarterly realized variance data.**						
	$\widehat{\sigma}^2_{MKT}$	$\widehat{\sigma}^2_{SMB}$	$\widehat{\sigma}^2_{HML}$	$\widehat{\sigma}^2_{RMW}$	$\widehat{\sigma}^2_{CMA}$	$\widehat{\sigma}^2_{MOM}$
$\widehat{\alpha}$	2.12	2.24	2.02	2.05	2.37	2.24
$\widehat{\sigma}$	0.07	0.09	0.07	0.09	0.14	0.13
x_{MIN}	31.20	12.67	10.51	3.47	5.74	43.00
D	0.05	0.04	0.05	0.04	0.04	0.06
% observations for which $x \geq x_{MIN}$	59.64 %	45.31 %	49.22 %	63.72 %	45.57 %	22.51 %
Panel B. 95 % confidence intervals for point estimates derived from monthly data.						
95 % Confidence Interval	(1.94,2.88)	(1.81,2.51)	(1.65,2.27)	(1.35,2.65)	(1.76,3.02)	(1,45,2.75)

This table reports the results of estimating power-law exponents for quarterly realized variances of the following factors: excess market factor (MKT), size factor (SMB), value factor (HML), profitability factor (RMW), investment factor (CMA), and momentum factor (MOM). The $\widehat{\alpha}$ represents the estimated tail exponent and $\widehat{\sigma}$ is the estimated standard deviation. The lower threshold x_{MIN} is estimated using the optimized Kolmogorov-Smirnov distance (D) distance, as detilaed in Clauset et al. (2009). The estimated x_{MIN} is the value that corresponds to the optimal distance D. Additionally, for each realized factor variance, the fraction of observations governed by power-law process is reported. The sample period for MKT, SMB, and HML realized variances spans the period July 1926 to September 2022 (384 quarterly observations). The RMW and CMA realized variances cover the period from July 1963 to September 2022 (226 quarterly observations), whereas the MOM realized variance encompasses the period from November 1926 to September 2022 (382 quarterly observations). Panel A reports the estimated power-law exponents, estimated standard deviations, cutoffs and percentages of observations governed by a power law. Panel B reports the 95 % confidence intervals for point estimates derived from monthly data (Table 2) where using bootstrapped standard deviation (Table 4).

Table A7

Goodness-of-fit tests for quarterly realized factor variances.

Realized factor variance	Power law	Log-normal	Exponential
$\widehat{\sigma}^2_{MKT}$	0.95	0.00	0.00
$\widehat{\sigma}^2_{SMB}$	0.63	0.08	0.00
$\widehat{\sigma}^2_{HML}$	0.59	0.02	0.00
$\widehat{\sigma}^2_{RMW}$	0.88	0.00	0.01
$\widehat{\sigma}^2_{CMA}$	0.74	0.01	0.01
$\widehat{\sigma}^2_{MOM}$	0.24	0.07	0.00

This table reports in the first column the results from goodness-of-fit tests derived from optimal Kolmogorov Smirnoff distances to examine if our empirical data and the data-generating process coincide with the hypothesized power law distribution parametrized via x_{min} and $\widehat{\alpha}$. Also, this table reports the goodness-of-fit tests for log- normal and exponential distributions in the third and fourth columns respectively. The null hypotheses of these tests assert that the empirical data and the data-generating process coincide with the log- normal or exponential distributions, respectively. To implement the tests, we make use of 2,500 synthetic data sets for each distribution under test.

Table A8
Summary statistics for blocks bootstrapped size factor premiums.

	$\overline{RET}^{SMB}_{BOOT}$	t-statistic
Mean	0.19	1.62
Median	0.19	1.58
Maximum	0.70	7.16
Minimum	−0.25	−2.03
Std. Dev.	0.12	1.06
Skewness	0.03	0.07
Kurtosis	3.05	3.22
Jarque-Bera	1.14	13.87
Probability	0.57	0.00
Observations	5,000	5,000

We retrieve the size factor over the July 1963 to September 2022 sample from Kenneth French's website and employ blocks bootstraps as described in section 3.3. to simulate 5,000 size factor evolutions. For each synthetic size factor, we compute the sample average and the corresponding t-statistic and plot these two metrics in this Figure. The blocks bootstraps procedure uses an expected block length of 60 months. This table reports the descriptive statistics for the simulated size premium obtained via blocks bootstraps ($\overline{RET}^{SMB}_{BOOT}$) and the t-statistics.

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