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Research paper

# Pricing asset-or-nothing options using Haar wavelet

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**Abstract:**
This article proposes a new numerical technique for pricing asset-or-nothing options using the Black-Scholes partial differential equation (PDE). We first use the $\theta-$weighted method to discretize the time domain, and then use Haar wavelets to approximate the functions and derivatives with respect to the asset price variable. By using some vector and matrix calculations, we reduce the PDE to a system of linear equations that can be solved at each time step for different asset prices. We perform an error analysis to show the convergence of our technique. We also provide some numerical examples to compare our technique with some existing methods and to demonstrate its efficiency and accuracy.

*Keywords:* Option pricing, Asset-or-Nothing Options, Haar Wavelets, Black-Scholes Model, Error analysis.
*Classification:* 65T60 ,91G60, 91G20.

## 1 Introduction

Haar wavelets, a class of mathematical functions, present a valuable approach for approximating solutions to partial differential equations (PDEs). Their utilization offers distinct advantages over alternative methods, characterized by traits such as simplicity, orthogonality, and sparsity. Notably, Haar wavelets exhibit versatility in handling diverse boundary conditions and nonlinearities inherent in PDEs. The subsequent dilations and translations of this core wavelet function establish a basis within the realm of square-integrable functions. Consequently, any function within this space can be represented as a linear combination of Haar wavelets. In the context of employing Haar wavelets for the resolution of partial differential equations (PDEs), a fundamental methodology involves employing the Haar wavelet expansion on the unknown function and its derivatives. Subsequently, these expansions

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are integrated into the PDE, thereby converting it into a set of algebraic equations. These equations can then be effectively addressed through a range of numerical techniques. Alternatively, one may opt to utilize the Haar wavelet operational matrix, a structured matrix that characterizes the impact of a differential operator on a Haar wavelet function. Through the multiplication of this operational matrix with the vector containing Haar wavelet coefficients, one can derive the Haar wavelet coefficients corresponding to the function's derivative. This streamlined approach significantly facilitates the PDE solution process.

Haar wavelets have been applied to solve various types of PDEs, such as elliptic, parabolic, hyperbolic, and fractional PDEs [11]. Some examples of PDEs that have been solved by Haar wavelets are the Poisson equation [10], the heat equation [3, 16], the wave equation [14], the Burgers equation [8, 13, 20], the Lane-Emden equation [11], and the Black-Scholes equation [9,18]. Haar wavelets have also been combined with other techniques, such as variational iteration [12, 21], homotopy perturbation [6], quasi-linearization [15, 19], and collocation [1, 2], to enhance the accuracy and efficiency of the solutions. These functions are utilized to deal with asset-or-nothing option pricing problems, which are considered one of the most intriguing challenges in financial mathematics.

## 2 Options and Black-Scholes Model

Options are financial instruments that grant their holders the privilege, without imposing a duty, to purchase (call option) or sell (put option) a designated underlying asset, such as stocks, commodities, or currencies, at a predefined price (strike price) during a specified timeframe (expiration date). Options are extensively utilized in financial markets and serve multiple crucial purposes.

Hedging is a key application of options, mostly used for risk management purposes. Options can be utilized by investors and corporations as a means of safeguarding their portfolios or assets against unfavorable price fluctuations.

Numerous traders employ options for speculative endeavors. If individuals anticipate an increase in the price of the underlying asset, they may elect to purchase call options. Conversely, if they anticipate a decrease in price, they may choose to purchase put options. Speculation in this context enables traders to generate profits from price fluctuations without possessing the underlying asset, hence offering the advantages of leverage and the potential for increased returns.

Utilizing options can enhance the diversification of investment portfolios. By incorporating more choices into a collection of conventional assets, investors can augment risk-adjusted returns and diminish the overall risk of their portfolio.

Options are frequently employed in the commodities market to mitigate the risk associated with price volatility. Producers and customers can utilize options to establish fixed prices for future deliveries, guaranteeing consistency in their financial planning and business activities.

## 2.1 Vanilla options

Vanilla options are a type of options that give the buyer or seller the right, but not the obligation, to buy or sell an underlying asset at a predetermined price and time. Vanilla options are simple and standard, and they do not have any special or unusual features. They are traded on exchanges, such as the Chicago Board Options Exchange1.

There are two kinds of vanilla options: call options and put options. A call option gives the buyer the right to buy the underlying asset at the strike price before or on the expiration date. A put option gives the buyer the right to sell the underlying asset at the strike price before or on the expiration date. The seller of the option, also known as the writer, has the obligation to deliver or buy the underlying asset if the buyer exercises the option.

The price of a vanilla option, also known as the premium, depends on several factors, such as the current price of the underlying asset, the strike price, the time to expiration, the volatility of the underlying asset, the interest rate, and the dividend yield. The premium is paid by the buyer to the seller when the option is purchased.

The profit or loss of a vanilla option depends on whether the option is in the money, at the money, or out of the money at expiration. An option is in the money if the exercise of the option results in a positive payoff. For example, a call option is in the money if the underlying asset price is higher than the strike price. An option is at the money if the exercise of the option results in a zero payoff. For example, a call option is at the money if the underlying asset price is equal to the strike price. An option is out of the money if the exercise of the option results in a negative payoff. For example, a call option is out of the money if the underlying asset price is lower than the strike price.

Vanilla options are used for various purposes, such as hedging, speculation, and arbitrage. Hedging is the use of options to reduce the risk of adverse price movements in the underlying asset. Speculation is the use of options to profit from the expected price movements in the underlying asset. Arbitrage is the use of options to exploit the price differences between the option and the underlying asset or other related instruments.

## 2.2 Black-Scholes Model

The Black-Scholes model [5] is a mathematical equation that estimates the theoretical value of options contracts based on current stock prices, expected dividends, strike price, risk-free rate, and volatility. Scholes and Merton won the 1997 Nobel Memorial Prize in Economic Sciences for discovering "a new method to determine the value of derivatives." Black died two years previously, so he could not receive a Nobel Prize, but the committee recognized his work in the Black-Scholes model.

The Black-Scholes model requires five input variables: the strike price of an option, the current stock price, the time to expiration, the risk-free rate, and the volatility. The model then calculates the price of a call option or a put option using a formula that involves these variables and some mathematical constants. The formula is known as the Black-Scholes formula.

In deriving this formula for the value of an option in terms of the price of the stock, they assumed ideal conditions in the market for the stock and for the option [5]:

(a) The short-term interest rate is known and is constant through time.

(b) The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is lognormal. The variance rate of the return on the stock is constant.

(c) The stock pays no dividends or other distributions.

(d) The option is European, that is, it can only be exercised at maturity.

(e) There are no transaction costs in buying or selling the stock or the option.

(f) It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate.

(g) There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date.

In the subsequent subsections, we assume that the stock price adheres to a geometric Brownian motion with a consistent drift ($\mu$) and volatility ($\sigma$).

$$dS(t) = \mu S(t)d\tau + \sigma dW(t), \tag{1}$$

which $W(t)$ is a Brownian motion. Let $V(t, S)$ denote the value of the option at time $t$, if the stock price at that time is $S(t) = S$. The following equation can be deduced using Itö lemma and the Doeblin formula, as well as the application of the delta hedging rule [17]

$$V_t(t, S) + rSV_S(t, S) + \frac{1}{2}\sigma^2 S^2 V_{SS}(t, S) - rV(t, S) = 0, \ t \in [0, T], \ S \geq 0, \tag{2}$$

with the following terminal condition

$$\begin{aligned} V(T, S) &= f(S), & (3)\\ V(t, 0) &= g(t), t \in [0, T], & (4)\\ \lim_{S \to \infty} V(t, S) &= h(t), & (5) \end{aligned}$$

which is a partial differential equation of the type called backward parabolic.

## 2.3 Asset-or-Nothing Option

Asset-or-nothing options are a type of binary or digital options that pay a fixed amount of the underlying asset or nothing at all, depending on whether the asset price is above or below a certain strike price at expiration. They are different from regular options, which pay the difference between the asset price and the strike price, and from cash-or-nothing options, which pay a fixed amount of cash or nothing at all.

Asset-or-nothing options can be used as a simplified way of hedging or speculating on the price movements of the underlying asset. For example, an investor who is bullish on a stock can buy an asset-or-nothing call option to profit from a rise in the stock price, while an investor who is bearish can buy an asset-or-nothing put option to profit from a fall in the stock price. Asset-or-nothing options can also be used to create synthetic positions that mimic the payoff of other financial instruments, such as futures, forwards, or swaps.These kind of options are often traded on unregulated platforms and may carry a higher risk of fraud or manipulation. They may also be subject to different regulations and taxation than standard options. Therefore, investors who wish to trade asset-or-nothing options should use platforms that are regulated by the Securities and Exchange Commission (SEC), the Commodity Futures Trading Commission (CFTC), or other regulators. They should also be aware of the potential advantages and disadvantages of asset-or-nothing options compared to other types of options.

**Black-Scholes model for Asset-or-Nothing Options**

Let $C(S,t)$ and $P(S,t)$ denote the values of the asset-or-nothing call and put options, respectively, for asset price $S$ and time $t$. Using the hedging argument [7] we can get the following equations

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0. \tag{6}$$

The payoff of a asset-or-nothing call option at the expiry date is

$$f_{\text{call}}(S(T)) = C(S(T),T) = \begin{cases} S(T), & S(T) > K, \\ 0, & S(T) < K, \end{cases} \tag{7}$$

in case of asset-or-nothing put option, we have

$$f_{\text{put}}(S(T)) = P(S(T),T) = \begin{cases} 0, & S(T) > K, \\ S(T), & S(T) < K, \end{cases} \tag{8}$$

where in both of above equations $K$ denotes the strike price. When $S = 0$, the asset remain at zero for all times hence the payoff is zero. This gives the boundary condition

$$g_{\text{call}}(t) = C(0,t) = 0, \qquad \text{for all } \ 0 \le t \le T. \tag{9}$$

When $S$ is very large, the option is almost certain to pay off the amount $S$. So, after discounting for interest, we find that

$$h_{\text{call}}(t) = C(S,t) \approx Se^{-r(T-t)}, \qquad \text{for large} \quad S. \tag{10}$$

We can consider a portfolio consisting of a asset-or-nothing put and a asset-or-nothing call with the same strike prices and expiry dates and drive the following relation which is called asset-or-nothing put-call parity

$$C(S,t) + P(S,t) = Se^{-r(T-t)}. \tag{11}$$

Using the put-call parity(11), the put option value can be derived from the call option value. The payoff functions for asset-or-nothing call and put options are plotted in figure (1).

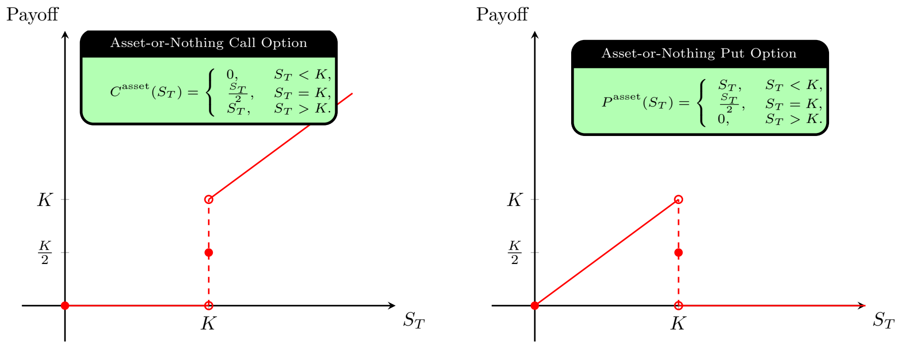


Figure 1: **Left:** Payoff function for asset-or-nothing call option. **Right:** Payoff function for asset-or-nothing put option.

## 3 Implementation of Haar wavelets

Haar wavelets are the rectangular-shaped wave-forms consist of piecewise constant functions. Haar wavelet transform does not permit the overlapping of the window while approximating a function. Haar wavelets are a discrete type of wave-forms that are generated by operating translation and dilation on a single prototype function. The Haar wavelet family of orthogonal functions for $x \in [0,1]$ is defined as follows:

$$\varphi(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & elsewhere, \end{cases} \tag{12}$$

and using this function, the Haar wavelet will define as follows

$$\psi(x) = \varphi(2x) - \varphi(2x-1) = \begin{cases} 1, & 0 \leq x < \frac{1}{2} \\ -1, & \frac{1}{2} \leq x < 1. \end{cases} \tag{13}$$

Let

$$j = 0, 1, 2, \cdots, m = 2^j, k = 0, 1, 2, \cdots, m-1, i = m+k+1, \tag{14}$$

put $h_1(x) = \varphi(x)$ and

$$h_i(x) = \begin{cases} 1, & \alpha_i \leq x < \beta_i, \\ -1, & \beta_i \leq x < \gamma_i, \\ 0, & elsewhere, \end{cases} \tag{15}$$

where

$$\alpha_i = \frac{k}{m}, \quad \beta_i = \frac{2k+1}{2m}, \quad \gamma_i = \frac{k+1}{m}. \tag{16}$$

## 3.1 Function approximation

Assuming that $f$ is an square integrable function on the interval [0.1), this function can be written as a linear combination of the Haar wavelet family as follows

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x), \tag{17}$$

which $a_i, i = 1, 2, \cdots$ are constants. Considering $J$ as the maximum value for $j$ and placing $M = 2^J$, we can get an approximation for the square integrable function $f$ on the interval $[0, 1)$ as follows.

$$f(x) \simeq \sum_{i=1}^{2M} a_i h_i(x). \tag{18}$$

We use the following notations to simplify the calculations in the following sections.

$$p_{i,1}(x) = \int_0^x h_i(t) d\tau, \tag{19}$$

$$p_{i,n+1} = \int_0^x p_{i,n}(t) d\tau, n = 1, 2, \cdots, \tag{20}$$

$$C_{i,n} = \int_0^1 p_{i,n}(x) dx, n = 1, 2, \cdots. \tag{21}$$

The following relationships can be easily obtained by the definition of the Haar wavelet and performing some preliminary calculations [4].

$$p_{i,n}(x) = \begin{cases} 0, & 0 \leq x < \alpha_i, \\ \frac{1}{n!}(x-\alpha_i)^n, & \alpha_i \leq x < \beta_i, \\ \frac{1}{n!}\left[(x-\alpha_i)^n - 2(x-\beta_i)^n\right], & \beta_i \leq x < \gamma_i, \\ \frac{1}{n!}\left[(x-\alpha_i)^n - 2(x-\beta)^n + (x-\gamma_i)^n\right], & \gamma_i \leq x < 1, \end{cases} \tag{22}$$

and

$$C_{1,n} = \frac{1}{(n+1)!}\left[(x-\alpha_i)^{n+1} - 2(x-\beta_i)^{n+1} + (x-\gamma_i)^{n+1}\right], \tag{23}$$

where $n = 1, 2, \cdots$ and $i = 2, 3, \cdots$. For the case $i = 1$ we have

$$p_{1,n} = \frac{x^n}{n!}, n = 1, 2, \cdots \tag{24}$$

and

$$C_{1,n} = \frac{1}{(n+1)!}, n = 1, 2, \cdots . \tag{25}$$

## 3.2 Implementation of Haar wavelets

**Change of variables**

Equations (6) are defined on the spatial domain $[0, \infty)$, which represents the range of the stock price. The application of Haar wavelets to this problem requires the truncation of the semi-finite interval $[0, \infty)$ to a finite interval $[0, S_{\max}]$ as the initial step. A careful selection of $S_{\max}$ is required to achieve a satisfactory level of approximation for the interval $[0, S_{\max}]$. Some research papers adopt $S_{\max} = 4E$ as a common choice. Moreover, we introduce the variable $x = \frac{S}{S_{\max}}$ to transform the spatial domain from $[0, S_{\max}]$ to $[0, 1]$, which facilitates the application of Haar wavelets to the problem. We also use the variable $\tau = T - t$ to change the problem from a backward to a forward time domain. Applying the above-mentioned change of variables to equation (6), we obtain the following equations for the asset-or-nothing call option:

$$-\frac{\partial C}{\partial \tau} + \frac{1}{2}\sigma^2 x \frac{\partial^2 C}{\partial x^2} + rx\frac{\partial C}{\partial x} - rC^{\text{cash}} = 0,\ \tau \in [0, T],\ x \in [0, 1], \tag{26}$$

**$\theta$-weighted approach**

Applying $\theta$-weighted $(0 \leq \theta \leq 1)$ scheme to spatial part and forward difference to temporal part of equation (26) yields

$$\begin{aligned} C^{k+1}(x) - \theta d\tau \left[\frac{\sigma^2}{2}x^2 C^{k+1}_{xx}(x) + rxC^{k+1}_{x}(x) - rC^{k+1}(x)\right] &= \\ C^{k}(x) + (1-\theta) d\tau \left[\frac{\sigma^2}{2}x^2 C^{k}_{xx}(x) + rxC^{k}_{x}(x) - rC^{k}(x)\right], \end{aligned} \tag{27}$$

where $C^k(x) = C(x, \tau_k)$, $\tau_{k+1} = \tau_k + d\tau$ and $d\tau$ is time step. Now approximate mixed order derivative by Haar wavelets as follows:

$$C^{k+1}_{xx}(x) = \sum_{i=1}^{2M} \alpha_i h_i(x), \tag{28}$$

where $\alpha_j$ are wavelets coefficients to be determined and $h_i(x)$ are wavelets defined in equation (15). Integrating equation (28) from 0 to $x$, we obtain

$$C_x^{k+1}(x) = \sum_{i=1}^{2M} \alpha_i p_{i,1}(x) + C_x^{k+1}(0). \tag{29}$$

The unknown term $C_x^{k+1}(0)$ in equation (29) can be computed by integration of equation (29) w.r.t $x$ from 0 to 1. By doing so we get following,

$$C_x^{k+1}(0) = C^{k+1}(1) - C^{k+1}(0) - \sum_{i=1}^{2M} \alpha_i p_{i,2}(1), \tag{30}$$

substituting equation (30) in equation (29) we have

$$C_x^{k+1}(x) = \sum_{i=1}^{2M} \alpha_i (p_{i,1}(x) - p_{i,2}(1)) + C^{k+1}(1) - C^{k+1}(0). \tag{31}$$

Substituting $C_x^{k+1}(0)$ from equation (30) in equation (29) and integrating from 0 to $x$ leads to

$$C^{k+1}(x) = \sum_{i=1}^{2M} \alpha_i \left(p_{i,2}(x) - x p_{i,2}(1)\right) + x\left(C^{k+1}(1) - C^{k+1}(0)\right) + C^{k+1}(0). \tag{32}$$

For the simplifying, we use the following notations

$$\begin{aligned} C_{xx}^k(x) &= {}^kH(x), \\ C_x^k(x) &= {}^kP(x) + c^k, \\ C^k(x) &= {}^kQ(x) + xc^k + d^k, \end{aligned} \tag{33}$$

where

$$\begin{aligned} x &= (x_1, x_2, \cdots, x_{2M}), \\ {}^k &= [\alpha_1 \;\; \alpha_2 \;\; \cdots \;\; \alpha_{2M}], \\ H &= [h_1(x) \; h_2(x) \;\; \cdots \;\; h_{2M}(x)]^T, \\ P &= [p_{1,1}(x) - p_{1,2}(1) \;\; p_{2,1}(x) - p_{2,2}(1) \;\; \cdots \;\; p_{2M,1}(x) - p_{2M,2}(1)]^T, \\ Q &= [p_{1,2}(x) - xp_{1,2}(1) \;\; p_{2,2}(x) - xp_{2,2}(1) \;\; \cdots \;\; p_{2M,2}(x) - xp_{2M,2}(1)]^T, \\ c^k &= C^k(1) - C^k(0), \\ d^k &= C^k(0). \end{aligned} \tag{34}$$

Substituting the right hands of equations (28), (31) and (32) into equation (27), using the collocation points $x_j = \frac{j+0.5}{2M}$ and the notations introduced in (33), we will obtain the following system of algebraic equations,

$${}^{k+1}G =^k N + R_1 - R_2, \tag{35}$$

where

$$
\begin{aligned}
G &= (1+\theta r d\tau)Q - \theta r d\tau x P - \frac{1}{2}\theta\sigma^2 x^2 H, \\
N &= \left[1-(1-\theta) r d\tau\right] Q + (1-\theta) r d\tau x P + \frac{1}{2}(1-\theta)\sigma^2 x^2 H, \\
R_1 &= c^{k+1}x + r\theta d\tau d^{k+1} + d^{k+1}, \\
R_2 &= c^k x - (1-\theta) r d\tau d^k + d^k,
\end{aligned} \tag{36}
$$

and $^1$ could be obtained easily from the below equation

$$C(0,x) = ^1 Q + x(C(0,1) - C(0,0)) + C(0,0). \tag{37}$$

## 3.3 Error analysis

Considering the notations introduced in (14), we have the following lemma for the relations between $m$, $k$ and $i$.

**Lemma 3.1.** *According to the above notions and suppose the value of $i$ is given then, we have*

$$m = 2^{\lfloor \log_2 i \rfloor} \tag{38}$$

$$k = i - m \tag{39}$$

*where $\lfloor . \rfloor$ is the floor function.*

**Theorem 3.2.** *If $C(x)$ satisfies a Lipschitz condition on $[0,1]$,that is, there exists a positive $L$ such that for all $x_1, x_2 \in [0,1]$ we have $|C(x_1) - C(x_2)| \leq L|x_1 - x_2|$. Then the error bound for $\|e_{2M}(x)\|$ is obtained as*

$$\|e_{2M}(x)\|_2 \leq \frac{L}{2\sqrt{15}M^2}, \tag{40}$$

*where $e_{2M}(x) = C(x,t) - C_{2M}(x,t)$ in which $C_{2M}(x,t)$ is obtained by Haar wavelet method. Also Haar wavelet method will converge in the sense that $e_{2M}(x)$ goes to zero as $M$ goes to infinity.*

*Proof.* Let $C_{2M}(x,t)$ which is obtained by Haar wavelet method, denotes the approximation of $C(x,t)$, then error at the $J$th level ($M = 2^J$) of resolution is defined as

$$
\begin{aligned}
e_{2M}(x) &= C(x,t) - C_{2M}(x,t) \\
&= \sum_{i=2M+1}^{\infty} a_i h_i(x),
\end{aligned} \tag{41}
$$

where $C_{2M}(x,t) = \sum_{i=1}^{2M} a_i h_i(x)$ and the coefficients $a_i$ are determined by

$$a_i = \int_0^1 C(x,t) h_i(x) dx = \langle h_i(x), C(x,t) \rangle,$$

here $\langle ., . \rangle$ shows the inner product. Now, the orthogonality of the sequence $h_i(x)$ on $[0, 1]$ implies that

$$\begin{aligned} \|e_{2M}(x)\|_2^2 &= \int_0^1 \left( \sum_{i=2M+1}^{\infty} a_i h_i(x) \right)^2 dx = \sum_{i=2M+1}^{\infty} a_i^2 \int_0^1 h_i^2(x) dx \\ &= \sum_{i=2M+1}^{\infty} \frac{a_i^2}{m_i}. \end{aligned} \tag{42}$$

Equation (15) implies that

$$a_i = \langle h_i(x), C(x,t) \rangle = \int_0^1 h_i(x) C(x,t) dx = \int_{\frac{k_i}{m_i}}^{\frac{k_i+0.5}{m_i}} C(x,t)\, dx - \int_{\frac{k_i+0.5}{m_i}}^{\frac{k_i+1}{m_i}} C(x,t)\, dx. \tag{43}$$

By using the mean value theorem for integrals, we obtain some $x_1 \in \left[\frac{k_i}{m_i} - \frac{k_i+0.5}{m_i}\right]$ and some $x_2 \in \left[\frac{k_i+0.5}{m_i} - \frac{k_i+1}{m_i}\right]$ such that

$$\begin{aligned} a_i &= \langle h_i(x), C(x,t) \rangle \\ &= \left[\frac{k_i + 0.5}{m_i} - \frac{k_i}{m_i}\right] C(x_1, t) - \left[\frac{k_i + 1}{m_i} - \frac{k_i + 0.5}{m_i}\right] C(x_2, t) \\ &= \frac{1}{2m_i} \left( C(x_1, t) - C_2(x_2, t) \right). \end{aligned} \tag{44}$$

From the Lipschitz condition, it follows that

$$\begin{aligned} |a_i| &= \frac{1}{2m_i} |C(x_1) - C(x_2)| \leq \frac{1}{2m_i} L |x_1 - x_2| \\ &\leq \frac{L}{2m_i} \left( \frac{k_i + 1}{m_i} - \frac{k_i}{m_i} \right) = \frac{L}{2m_i^2} \end{aligned} \tag{45}$$

Based on equations (42) and (45) we can express that

$$\begin{aligned}
\|e_{2M}(x)\|_2^2 \leq \sum_{i=2M+1}^{\infty} \frac{L^2}{4m_i^5} &= \sum_{i=2M+1}^{\infty} \frac{L^2}{4\left(2^{\lfloor \log_2 i \rfloor}\right)^5} \\
&= \frac{L^2}{4}\left[\underbrace{\left(\frac{1}{2^{5J+5}} + \cdots + \frac{1}{2^{5J+5}}\right)}_{2^{J+1}} + \underbrace{\left(\frac{1}{2^{5J+10}} + \cdots + \frac{1}{2^{5J+10}}\right)}_{2^{J+2}} + \cdots\right] \\
&= \frac{L^2}{4}\left(\frac{1}{2^{4J+4}} + \frac{1}{2^{4J+8}} + \frac{1}{2^{4J+12}} + \dots\right) \\
&= \frac{L^2}{4}\frac{1}{2^{4J}}\left(\frac{1}{2^4} + \frac{1}{2^8} + \frac{1}{2^{12}} + \dots\right) \\
&= \frac{L^2}{64}\frac{1}{2^{4J}}\left[1 + \left(\frac{1}{16}\right) + \left(\frac{1}{16}\right)^2 + \dots\right] = \frac{L^2}{64M^4}\sum_{n=0}^{\infty}\left(\frac{1}{16}\right)^n \\
&= \frac{L^2}{64M^4}\frac{16}{15},
\end{aligned} \tag{46}$$

Taking the square root of the final equation yields

$$\|e_{2M}(x)\|_2 \leq \frac{L}{2\sqrt{15}M^2}. \tag{47}$$

□

# 4 Numerical results

To validate the efficacy of the proposed approach, a series of test issues are solved. We seek to obtain a numerical solution for both the asset-or-nothing call option and the asset-or-nothing put option, which will serve as two test cases. For comparison, each numerical result has been juxtaposed with the precise answer for each scenario, which is displayed below. The findings demonstrate that the approach outlined in the research is a remarkably precise and effective one. Given the significance of option prices when initiating a trade, we have placed particular emphasis on error-checking during this stage. We performed all computations and simulations in this paper using Python 3.11.

## 4.1 Pricing asset-or-nothing call option

The asset-or-nothing call option pricing problem is described by a partial differential equation, which is accompanied by its corresponding boundary conditions. The subsequent equations are derived from the variable transformations covered in the preceding sections.

$$-C_{\tau}^{\text{Cash}}+\frac{1}{2}\sigma^2 x C_{xx}^{\text{Cash}}+rxC_x^{\text{Cash}}-rC^{\text{cash}}=0,\ \tau\in[0,T],\ x\in[0,1], \tag{48}$$

$$C^{\text{Cash}}(x,T)=\begin{cases} A, & x>\frac{E}{S_{\max}},\\ \frac{A}{2}, & x=\frac{E}{S_{\max}},\\ 0, & x<\frac{E}{S_{\max}},\end{cases} \tag{49}$$

$$C^{\text{Cash}}(0,t) = 0, \tag{50}$$
$$C^{\text{Cash}}(1,t) = Ae^{-r(T-t)}. \tag{51}$$

Given the equations (48-51) and the values $r=0.03$, $T=1$, $K=A=100$, and $d\tau=0.01$, the numerical solutions are depicted in the figures. Figure (2) shows the numerical solution of this problem based on haar wavelets method.

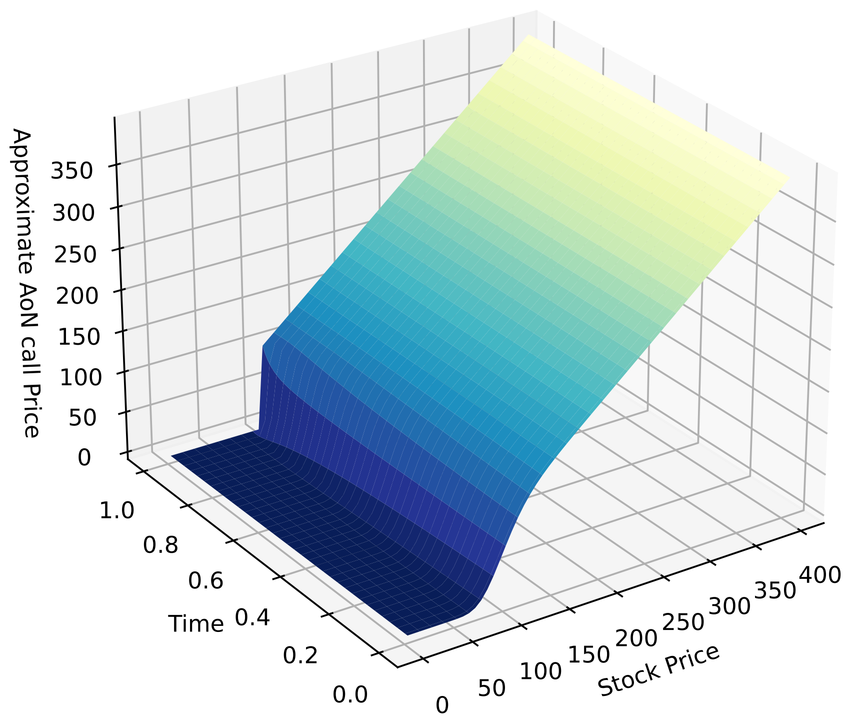


Figure 2: Plot of the approximate solution for the asset-or-nothing call option problem.

As stated earlier, the option value at $t = 0$ is of great significance. Hence, we will analyze the error in the option value at this particular time. Figure (3) displays the error plots for various values of $T$, $\sigma$, $K$, and $J$. The top-left plot corresponds to different values of $T$, the bottom-left plot to different values of $\sigma$, the top-right plot to different values of $K$, and the bottom-right plot to different values of $J$. The convergence of the method can be seen in this figure.

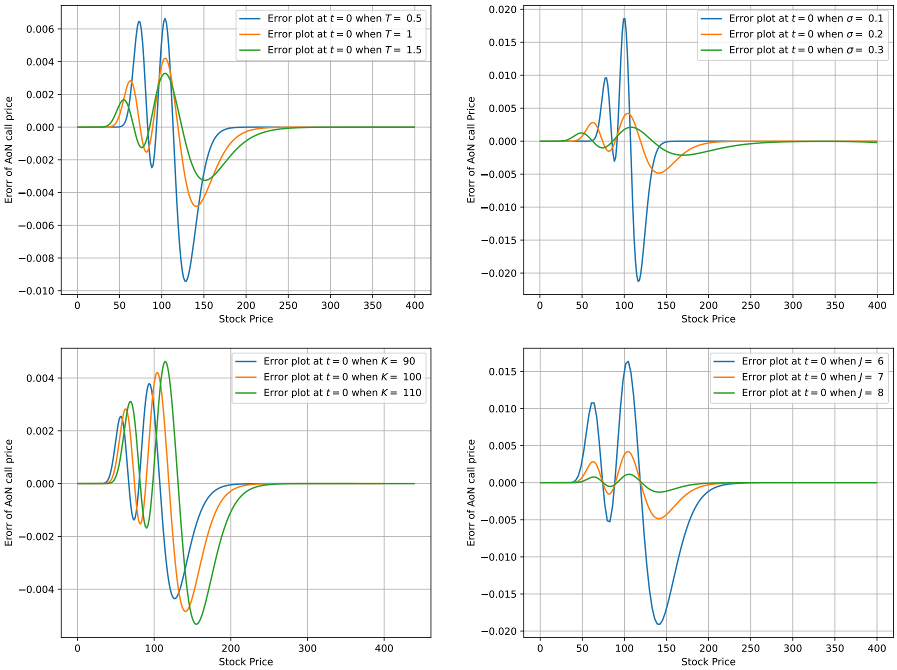


Figure 3: Error plot at time $t = 0$ with different values of $\sigma$ (Top-Right), $T$ (Top-Left), $J$ (Bottom-Right) and $K$ (Bottom-Left).

## 4.2 Pricing asset-or-nothing put option

For an asset-or-nothing put option, the following holds:

$$-P_{\tau}^{\text{Cash}} + \frac{1}{2}\sigma^2 x P_{xx}^{\text{Cash}} + rxP_{x}^{\text{Cash}} - rP^{\text{cash}} = 0, \;\; \tau \in [0, T], \;\; x \in [0, 1], \tag{52}$$

$$P^{\text{Cash}}(x, T) = \begin{cases} 0, & x > \frac{E}{S_{\max}}, \\ \frac{A}{2}, & x = \frac{E}{S_{\max}}, \\ A, & x < \frac{E}{S_{\max}}, \end{cases} \tag{53}$$

$$P^{\text{Cash}}(0,t) = Ae^{-r(T-t)}, \quad (54)$$
$$P^{\text{Cash}}(1,t) = 0. \quad (55)$$

The numerical solution of this problem using the Haar wavelet method is presented in Figure (4). Again, as mentioned before, the value of the option at $t = 0$ is very important. Hence, we will analyze the error in the option value at this particular time. Figure (5) displays the error plots for various values of $T$, $\sigma$, $K$, and $J$. The top-left plot corresponds to different values of $T$, the bottom-left plot to different values of $\sigma$, the top-right plot to different values of $K$, and the bottom-right plot to different values of $J$. This figure illustrates the convergence of the method.

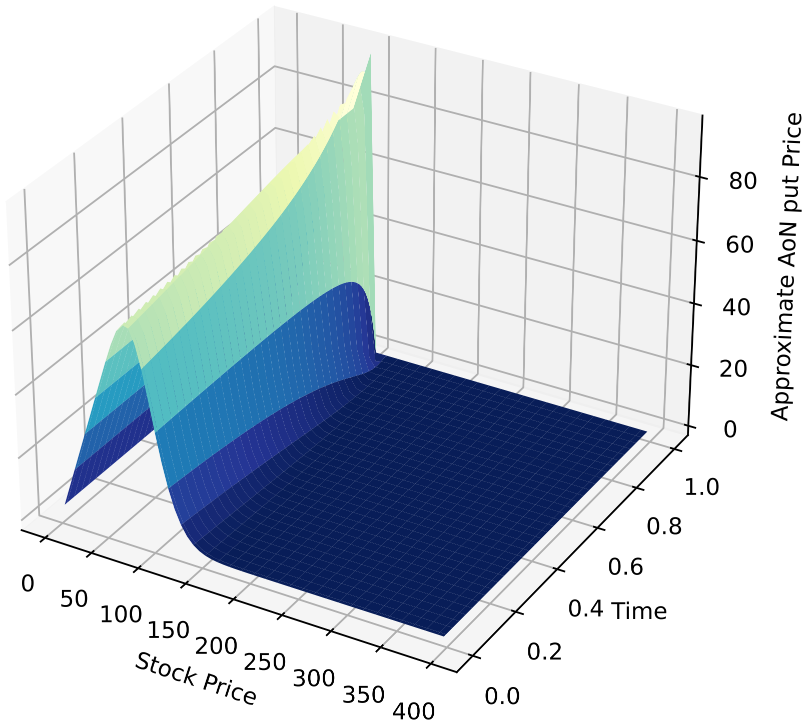


Figure 4: Plot of the approximate solution for the asset-or-nothing put option problem.

## 5 Conclusion

In this article, we have explored the application of the Haar wavelet method for pricing asset-or-nothing options, which are a type of binary or digital option that

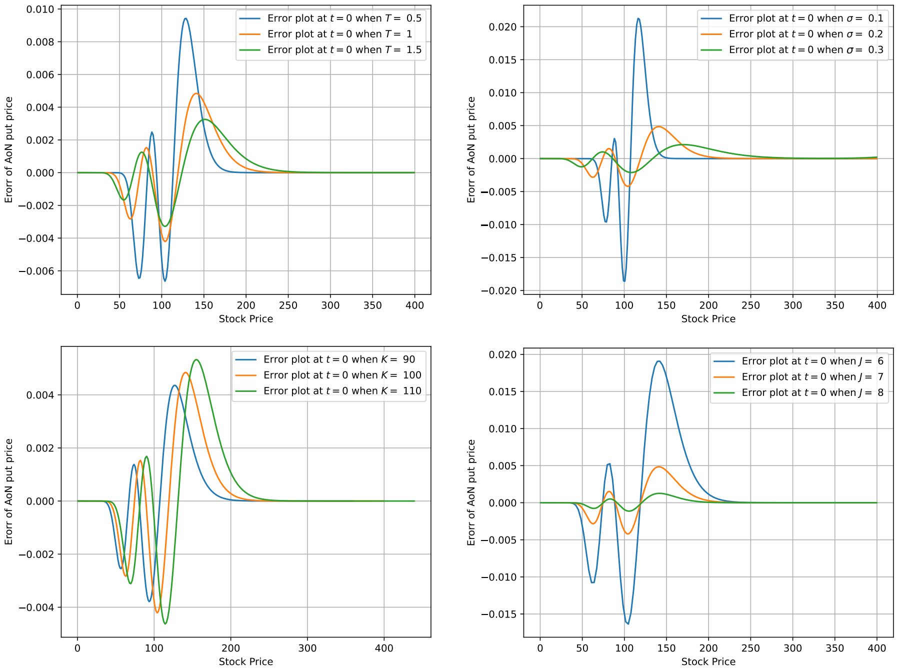


Figure 5: Error plot at time $t = 0$ with different values of $\sigma$ (Top-Right), $T$ (Top-Left), $J$ (Bottom-Right) and $K$ (Bottom-Left).

pays a fixed amount if the underlying asset is above or below a certain level at maturity. The Haar wavelet method can approximate the functions involved in the Black-Scholes partial differential equation, which is the standard model for option pricing. We have also provided some examples of asset-or-nothing options and solved them by using the Haar wavelet method. The results have demonstrated the accuracy and efficiency of the Haar wavelet method in valuing asset-or-nothing options. The Haar wavelet method is a promising technique for solving option pricing problems, as it offers several advantages over other numerical methods, such as simplicity, stability, and scalability. Other types of options and financial derivatives, such as barrier options, and exotic options, can also benefit from the application of the Haar wavelet method. We hope that this article will inspire further research and development of the Haar wavelet method and its applications in finance.

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