Received: 16 April 2024 | Revised: 6 October 2025 | Accepted: 27 October 2025

DOI: 10.1112/jlms.70368

RESEARCH ARTICLE

Journal of the London Mathematical Society

Representing maps for semibounded forms and their Lebesgue-type decompositions

S. Hassi[1] | **H. S. V. de Snoo**[2]

[1]Department of Mathematics and Statistics, University of Vaasa, Vaasa, Finland

[2]Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, Groningen, Nederland

Correspondence
S. Hassi, Department of Mathematics and Statistics, University of Vaasa, P.O. Box 700, 65101 Vaasa, Finland.
Email: seppo.hassi@uwasa.fi

Dedicated to the memory of our friend and colleague Franciszek Hugon Szafraniec (1940-2025)

Abstract

In the Lebesgue decomposition of a lower semibounded sesquilinear form, the corresponding regular and singular parts are mutually singular. The more general Lebesgue-type decompositions studied here allow components that need not be mutually singular anymore. In the new situation, the earlier basic orthogonal space decomposition in the background is now replaced by a nonorthogonal decomposition in the sense of de Branges and Rovnyak. The relevant theory is based on Lebesgue-type decompositions for linear operators and relations via a so-called representing map. This map also makes it possible to formulate explicit analogs for representation theorems for lower semibounded forms that are not necessarily closed or closable. This new representation also appears naturally in the convergence of monotone sequences of lower semibounded forms.

MSC 2020
47A07, 47B02 (Primary) 15A63, 47A05, 47B25 (Secondary)

1 | INTRODUCTION

Let $\mathfrak{t}$ be a lower semibounded sesquilinear form on a Hilbert space $\mathfrak{H}$, which is not necessarily densely defined. Almost 50 years ago Simon [35] has constructed semibounded forms $\mathfrak{t}_{\mathrm{reg}}$ and $\mathfrak{t}_{\mathrm{sing}}$

in $\mathfrak{H}$, such that $\mathfrak{t}$ has a Lebesgue decomposition

$$\mathfrak{t} = \mathfrak{t}_{\rm reg} + \mathfrak{t}_{\rm sing}, \quad \operatorname{dom} \mathfrak{t} = \operatorname{dom} \mathfrak{t}_{\rm reg} = \operatorname{dom} \mathfrak{t}_{\rm sing}, \tag{1.1}$$

where the form $\mathfrak{t}_{\rm reg}$ is regular (closable) with the same lower bound as $\mathfrak{t}$ and the (nonnegative) form $\mathfrak{t}_{\rm sing}$ is singular (for definitions, see below), similar to the Lebesgue decomposition known from measure theory. In fact, $\mathfrak{t}_{\rm reg}$ is the largest closable form below $\mathfrak{t}$. This kind of Lebesgue decomposition appears at many places in earlier and recent literature; see, for instance, [1, 6, 22, 25, 26, 34, 37–39] and further references therein. Moreover, the regular part $\mathfrak{t}_{\rm reg}$ of a lower semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ shows up in the representation theorems for lower semibounded forms that are not necessarily closed (cf. [3]), and in the description of the limits of sequences of lower semibounded forms (cf. [27, 31, 35]); see also [28]. In the present paper, these decomposition and representation results and their generalizations will be considered from the point of view of a so-called representing map, which will now be explained.

Linear operators and relations between Hilbert spaces have Lebesgue decompositions and, more generally, Lebesgue-type decompositions based on the complementary space decomposition theory going back to de Branges and Rovnyak; see [12, 13, 16, 18]. It will be shown that such Lebesgue-type decompositions also exist for lower semibounded forms on a Hilbert space. In the setting of forms, such general complementary space decompositions appear as a nontrivial interaction between the regular and singular components. This is a new type of phenomenon not treated in earlier literature. These results rely on a reduction procedure that associates lower semibounded forms with linear operators and relations by incorporating representing maps for general forms. The following notations will be used throughout the paper: $\mathbf{F}(\mathfrak{H})$ stands for the sesquilinear forms in the Hilbert space $\mathfrak{H}$ and $\mathbf{L}(\mathfrak{H}, \mathfrak{K})$ stands for the space of all linear relations from $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$. Recall that for $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, one has $T^{-1} \in \mathbf{L}(\mathfrak{K}, \mathfrak{H})$ (note that $\{f, f'\} \in T$ if and only if $\{f', f\} \in T^{-1}$), and one uses the notations

$$\operatorname{dom} T, \quad \operatorname{ran} T, \quad \ker T, \quad \operatorname{mul} T,$$

for the domain, range, kernel, and multivalued part of T (so that $\operatorname{mul} T = \ker T^{-1}$). The notation $\mathbf{B}(\mathfrak{H}, \mathfrak{K})$ stands for the subclass of $\mathbf{L}(\mathfrak{H}, \mathfrak{K})$ consisting of the bounded linear operators T with $\operatorname{dom} T = \mathfrak{H}$. The adjoint $T^* \in \mathbf{L}(\mathfrak{K}, \mathfrak{H})$ of $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ is defined by

$$T^* = \left\{ \{f, f'\} \in \mathfrak{K} \times \mathfrak{H} : (f', h) = (f, h') \text{ for all } \{h, h'\} \in \mathrm{T} \right\},$$

so that $\operatorname{mul} T^* = (\operatorname{dom} T)^{\perp}$. It is clear that $T \subset T^{**}$, where $T^{**} := (T^*)^*$ equals the closure of T in $\mathfrak{H} \times \mathfrak{K}$. A linear relation $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ is called *regular* (or closable) when its closure T^{**} is an operator ($\operatorname{mul} T^{**} = \{0\}$ or, equivalently, $\operatorname{dom} T^*$ is dense in $\mathfrak{K}$) and called *singular* when its closure is the Cartesian product of closed linear subspaces in $\mathfrak{H}$ and $\mathfrak{K}$.

The notion of a representing map appears in [36, p. 765] for individual nonnegative forms (see also [15, 17]) and in [22, (4.6)] for a pair of nonnegative forms.

Definition 1.1. Let $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be lower semibounded and let $c \leqslant m(\mathfrak{t})$, the lower bound of $\mathfrak{t}$. A *representing map* for the nonnegative form $\mathfrak{t} - c$ is a linear operator $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$, where $\mathfrak{K}_c$ is a Hilbert space, with $\operatorname{dom} Q_c = \operatorname{dom} \mathfrak{t}$, such that

$$\mathfrak{t}[\varphi, \psi] - c(\varphi, \psi) = (Q_c\varphi, Q_c\psi)_{\mathfrak{K}_c}, \quad \varphi, \psi \in \operatorname{dom} Q_c = \operatorname{dom} \mathfrak{t}. \tag{1.2}$$

The representing map $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$ is called *minimal* when $\overline{\operatorname{ran}}\, Q_c = \mathfrak{K}_c$.

The representing map is uniquely defined up to left multiplication by a partial isometry between the ranges of the representing maps (and this partial isometry is unitary when the representing maps are minimal); see Lemma 2.1. Notice that the Hilbert space $\mathfrak{K}_c$ and the operator Q_c depend on the choice of $c \leqslant m(\mathfrak{t})$ as indicated. When no confusion is possible, the subscript c will be dropped. A characteristic example of a representing map can be found in the theory of differential operators (Dirichlet-to-Neumann maps) as explained in Example 2.6.

Let $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be lower semibounded. A sequence φ_n in $\operatorname{dom} \mathfrak{t}$ is said to converge to $\varphi \in \mathfrak{H}$ in the sense of $\mathfrak{t}$, denoted by $\varphi_n \to_{\mathfrak{t}} \varphi$, if $\varphi_n \to \varphi$ in $\mathfrak{H}$ and $\mathfrak{t}[\varphi_n - \varphi_m] \to 0$. Here, $\mathfrak{t}[\cdot]$ stands for the diagonal of $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$. Recall that this kind of convergence leads to the definition of closability, closedness, and singularity of $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$; see [27, 28]. There is a straightforward connection between these notions for lower semibounded forms and similar notions for the corresponding linear representing map in Definition 1.1. By means of this connection, one can also identify via the representing map the closure of a closable form. For details, see Lemmas 2.3 and 2.4. The main theme in this paper is the interaction between forms and their representing maps when discussing Lebesgue-type decompositions.

To proceed, first recall that for linear operators and relations in $\mathbf{L}(\mathfrak{H}, \mathfrak{K})$, there is a complete description of their Lebesgue decompositions and, more generally, of their Lebesgue-type decompositions; see [16]. In particular, this takes care of the Lebesgue-type decomposition for a pair of nonnegative bounded linear operators; see, for instance, [24, 26]. In the present paper, it will be shown how Simon's Lebesgue decomposition (1.1) of a lower semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ is obtained via the Lebesgue decomposition of the corresponding representing map $Q \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$. In fact, the construction of Simon's Lebesgue decomposition in the present approach uses an orthogonal sum decomposition of the space, and therefore, the regular and singular components of $\mathfrak{t}$ are mutually singular, analogous to the case of the Lebesgue decomposition in classical measure theory. However, the present approach also leads to the new, more general, Lebesgue-type decompositions of a lower semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$, which is based on a nonorthogonal sum decomposition of the space in the sense of de Branges and Rovnyak (see [12, 13, 16, 18]):

$$\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2, \quad \operatorname{dom} \mathfrak{t} = \operatorname{dom} \mathfrak{t}_1 = \operatorname{dom} \mathfrak{t}_2, \tag{1.3}$$

where the form $\mathfrak{t}_1 \in \mathbf{F}(\mathfrak{H})$ is semibounded and closable, while the form $\mathfrak{t}_2 \in \mathbf{F}(\mathfrak{H})$ is nonnegative and singular. Due to this new feature, the components $\mathfrak{t}_1$ and $\mathfrak{t}_2$ in the Lebesgue-type sum decomposition (1.3) may interact. The mutual singularity of the components is described in terms of a parallel sum of the components; for the parallel sum of forms, see [23].

If the lower semibounded $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ is closed, then the well-known representation theorem asserts that there is a semibounded self-adjoint relation H in $\mathfrak{H}$ with $\operatorname{dom} H \subset \operatorname{dom} \mathfrak{t}$ and having the same lower bound as $\mathfrak{t}$, such that for all elements $\{\varphi, \varphi'\} \in H$,

$$\mathfrak{t}[\varphi, \psi] = (\varphi', \psi) \quad \text{for all} \quad \psi \in \operatorname{dom} \mathfrak{t}; \tag{1.4}$$

see [7, 27, 28]. The representing map for $\mathfrak{t}$ also offers a new interpretation of this classical result for lower semibounded forms that are not necessarily closed as treated by Arendt and ter Elst in [3]. Furthermore, this interpretation will play a role when considering the convergence of monotone sequences of lower semibounded forms; cf. [27, 31, 35].

The original results of Simon [35] were preceded by a decomposition result of Ando [1] for a pair of nonnegative bounded linear operators, which are parallel to what is here called a

Lebesgue decomposition. The corresponding Lebesgue-type decompositions can be found in [19]. An extension of Simon's Lebesgue decomposition in the context of a pair of nonnegative forms was presented in [22]. The treatment of lower semibounded forms in terms of representing maps can also be used for an interpretation of the work of Sebestyén and Stochel [32] concerning the Friedrichs and Kreĭn type extensions of a semibounded operator or relation; see [17]. Moreover, it should be mentioned that many results in the present paper remain valid in the context of sectorial forms and sectorial operators or relations.

The contents of the present paper will now be described. Section 2 is centered on generalities concerning lower semibounded forms and their representing maps. Moreover, this section contains a characteristic example of a representing map that occurs in the theory of ordinary or partial differential operators. Simon's Lebesgue decomposition (1.1) for lower semibounded forms is treated in Section 3. In Section 4, the sum decomposition in (1.3) of a nonnegative form $\mathfrak{t}$ into nonnegative forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ (without any further conditions on $\mathfrak{t}_1$ and $\mathfrak{t}_2$) is parametrized by means of nonnegative contractions $K \in \mathbf{B}(\mathfrak{K})$, where $\mathfrak{K}$ is the representing Hilbert space associated with the representing map Q of $\mathfrak{t}$; see Theorem 4.1. The minimality of the parametric representation for the sum $\mathfrak{t}_1 + \mathfrak{t}_2$ is intimately connected with the interaction between the components $\mathfrak{t}_1$ and $\mathfrak{t}_2$. The mutual singularity of the forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ will be characterized by means of the parallel sum of $\mathfrak{t}_1$ and $\mathfrak{t}_2$; see [23]. This again depends on a de Branges–Rovnyak decomposition of the space; see Theorem 4.3. Section 5 is concerned with the decomposition (1.3) for lower semibounded forms (but not necessarily nonnegative) by means of a shift (using $\mathfrak{t} - c$ instead of $\mathfrak{t}$ when $c \leqslant m(\mathfrak{t})$). The nonnegative contractions $K \in \mathbf{B}(\mathfrak{K})$ are characterized for which in (1.3) the form $\mathfrak{t}_1$ is closable and the form $\mathfrak{t}_2$ is singular; cf. Theorem 5.2. Furthermore, one can characterize the case where the Lebesgue decomposition is the only Lebesgue-type decomposition; see Theorem 5.5. In Section 6, the usual representation theorem for closed lower semibounded forms as in (1.4) is being extended to lower semibounded forms that are not necessarily closed or closable; cf. Theorem 6.1. There is also a connection with the representation for the form $\mathfrak{t}_{\rm reg}$; see Theorem 6.5. The approach via representing maps will be used in Section 7 to treat the convergence of sequences of lower semibounded forms along the lines of Section 6; cf. Theorem 7.2. In the background of this section is the treatment of the convergence of nondecreasing sequences of linear operators or relations; cf. [15]. After Section 2, the terminology semibounded will be used to indicate lower semibounded.

This paper is dedicated to the memory of our friend Franek Szafraniec. More than 20 years ago, all of us, including Zoltán Sebestyén, became involved in the Lebesgue decomposition of linear operators and relations [25].

2 | REPRESENTING MAPS FOR LOWER SEMIBOUNDED SESQUILINEAR FORMS

This section contains the background of lower semibounded sesquilinear forms and their representing maps. There is also an example of a representing map in connection with differential operators.

A sesquilinear form $\mathfrak{t} = \mathfrak{t}[\cdot,\cdot]$ in a Hilbert space $\mathfrak{H}$ over $\mathbb{C}$ with an inner product $(\cdot,\cdot)$ is a mapping from $\mathfrak{D} \times \mathfrak{D}$ to $\mathbb{C}$, where $\mathfrak{D} = \operatorname{dom} \mathfrak{t}$ is a linear subspace of $\mathfrak{H}$, which is linear in the first entry and antilinear in the second entry. In what follows, sesquilinear forms over $\mathbb{C}$ will be often called just forms, and the class of forms in a Hilbert space $\mathfrak{H}$ will be denoted by $\mathbf{F}(\mathfrak{H})$. Also the diagonal

$\mathfrak{t}[\cdot]$ will be used: $\mathfrak{t}[\varphi] = \mathfrak{t}[\varphi, \varphi]$, $\varphi \in \operatorname{dom} \mathfrak{t}$. The form $\mathfrak{t}$ is called symmetric if $\mathfrak{t}[\varphi, \psi] = \overline{\mathfrak{t}[\psi, \varphi]}$ for all $\varphi, \psi \in \mathfrak{D}$. Equivalently, $\mathfrak{t}$ is symmetric if and only if $\mathfrak{t}[\varphi] \in \mathbb{R}$ for all $\varphi \in \mathfrak{D}$. Let $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathbf{F}(\mathfrak{H})$ be forms with domains $\operatorname{dom} \mathfrak{t}_1$ and $\operatorname{dom} \mathfrak{t}_2$. Then, the sum $\mathfrak{t}_1 + \mathfrak{t}_2$, defined by

$$(\mathfrak{t}_1 + \mathfrak{t}_2)[\varphi, \psi] = \mathfrak{t}_1[\varphi, \psi] + \mathfrak{t}_2[\varphi, \psi], \quad \varphi, \psi \in \operatorname{dom} \mathfrak{t}_1 \cap \operatorname{dom} \mathfrak{t}_2,$$

belongs to $\mathbf{F}(\mathfrak{H})$. In particular, $\mathfrak{t}[\varphi, \psi] + c(\varphi, \psi)$, with $\varphi, \psi \in \operatorname{dom} \mathfrak{t}$ and $c \in \mathbb{R}$, defines a form, denoted by $\mathfrak{t} + c$. The inequality $\mathfrak{t}_1 \leqslant \mathfrak{t}_2$ for symmetric forms $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathbf{F}(\mathfrak{H})$ is defined by

$$\operatorname{dom} \mathfrak{t}_2 \subset \operatorname{dom} \mathfrak{t}_1, \quad \mathfrak{t}_1[\varphi] \leqslant \mathfrak{t}_2[\varphi], \quad \varphi \in \operatorname{dom} \mathfrak{t}_2. \tag{2.1}$$

The inclusion $\mathfrak{t}_2 \subset \mathfrak{t}_1$ means that $\operatorname{dom} \mathfrak{t}_2 \subset \operatorname{dom} \mathfrak{t}_1$ and $\mathfrak{t}_1$ agree with $\mathfrak{t}_2$ on $\operatorname{dom} \mathfrak{t}_2$. In this case, $\mathfrak{t}_2$ is called a restriction of $\mathfrak{t}_1$, while $\mathfrak{t}_1$ is called an extension of $\mathfrak{t}_2$. In particular, $\mathfrak{t}_2 \subset \mathfrak{t}_1$ implies $\mathfrak{t}_1 \leqslant \mathfrak{t}_2$. All these elementary facts can be found in [7, 27, 28].

A form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ with domain $\operatorname{dom} \mathfrak{t}$ is called bounded if

$$|\mathfrak{t}[\varphi]| \leqslant a\|\varphi\|^2, \quad \varphi \in \operatorname{dom} \mathfrak{t},$$

for some $a \geqslant 0$. The form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ is called bounded from below by $c \in \mathbb{R}$ if

$$\mathfrak{t}[\varphi] \geqslant c\|\varphi\|^2, \quad \varphi \in \operatorname{dom} \mathfrak{t}, \tag{2.2}$$

and nonnegative if $\mathfrak{t}[\varphi] \geqslant 0$, $\varphi \in \operatorname{dom} \mathfrak{t}$. A lower semibounded form is automatically symmetric. The sum of forms that are lower semibounded is also lower semibounded. The lower bound $m(\mathfrak{t}) \in \mathbb{R}$ of $\mathfrak{t}$ is the largest of all numbers c in (2.2):

$$m(\mathfrak{t}) = \inf \left\{ \frac{\mathfrak{t}[\varphi, \varphi]}{(\varphi, \varphi)} \; : \; \varphi \in \operatorname{dom} \mathfrak{t} \right\}.$$

With $c \leqslant m(\mathfrak{t})$ the form

$$(\mathfrak{t} - c)[\varphi, \psi] = \mathfrak{t}[\varphi, \psi] - c(\varphi, \psi)_{\mathfrak{H}}, \quad \varphi, \psi \in \operatorname{dom} \mathfrak{t},$$

is a nonnegative and $\operatorname{dom} (\mathfrak{t} - c) = \operatorname{dom} \mathfrak{t}$. Its kernel is defined by

$$\ker\, (\mathfrak{t} - c) = \{\varphi \in \operatorname{dom} \mathfrak{t} \; : \; (\mathfrak{t} - c)[\varphi, \varphi] = 0\}.$$

By Cauchy's inequality, one sees that $\ker\, (\mathfrak{t} - c)$ is a linear subspace of $\operatorname{dom} \mathfrak{t}$.

In what follows, it is important to observe that for every lower semibounded form, there exists a representing map as introduced in Definition 1.1. For the convenience of the reader, a proof of this result is included; see [15, Lemma 4.1].

Lemma 2.1. *Let* $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ *be lower semibounded. Then, for each* $c \leqslant m(\mathfrak{t})$*, there exists a representing map* $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$ *for the nonnegative form* $\mathfrak{t} - c$*. Moreover, this representing map is uniquely defined in the following sense: if* $Q_c' \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c')$ *is an another representing map for* $\mathfrak{t} - c$ *with* $\operatorname{dom} Q_c' = \operatorname{dom} \mathfrak{t}$*, then there exists a partial isometry* $V \in \mathbf{B}(\mathfrak{K}_c, \mathfrak{K}_c')$ *with initial space* $\overline{\operatorname{ran}}\, Q_c$ *and final space* $\overline{\operatorname{ran}}\, Q_c'$*, such that* $Q_c' = VQ_c$*. When both* Q_c *and* Q_c' *are minimal, then* V *is unitary.*

Proof. First, the existence of a representing map will be proved. Since $\ker(\mathfrak{t}-c)$ is a linear subspace of $\operatorname{dom}\mathfrak{t}$, the quotient space $\operatorname{dom}\mathfrak{t}/\ker(\mathfrak{t}-c)$ is well defined and consists of the equivalence classes $\varphi+\ker(\mathfrak{t}-c)$, $\varphi\in\operatorname{dom}\mathfrak{t}$. The equivalence classes $\varphi+\ker(\mathfrak{t}-c)$ and $\psi+\ker(\mathfrak{t}-c)$ are equal if and only $\varphi-\psi\in\ker(\mathfrak{t}-c)$. With the usual addition, it is clear that the quotient space $\operatorname{dom}\mathfrak{t}/\ker(\mathfrak{t}-c)$ is a linear space. The nonnegative form $\mathfrak{t}-c$ induces a natural inner product $(\cdot,\cdot)_{\mathfrak{t}-c}$ on $\operatorname{dom}\mathfrak{t}/\ker(\mathfrak{t}-c)$ by

$$(\varphi+\ker(\mathfrak{t}-c),\psi+\ker(\mathfrak{t}-c))_{\mathfrak{t}-c}=(\mathfrak{t}-c)[\varphi,\psi],\quad \varphi,\psi\in\operatorname{dom}\mathfrak{t}. \tag{2.3}$$

The Hilbert space completion of $\operatorname{dom}\mathfrak{t}/\ker(\mathfrak{t}-c)$ with the inner product (2.3) is denoted by $\mathfrak{K}_c$ with inner product $(\cdot,\cdot)_{\mathfrak{t}-c}$. Denote by Q_c the linear relation from $\mathfrak{H}$ to $\mathfrak{K}_c$, defined by

$$Q_c=\big\{\{\varphi,\varphi+\ker(\mathfrak{t}-c)\}:\ \varphi\in\operatorname{dom}\mathfrak{t}\big\}, \tag{2.4}$$

which is easily seen to be a representing map for $\mathfrak{t}-c$,

$$Q_c\varphi=\varphi+\ker(\mathfrak{t}-c),\quad \varphi\in\operatorname{dom}\mathfrak{t}.$$

Hence, the existence of a representing map for $\mathfrak{t}-c$ has been established.

Next, the uniqueness property of representing maps will be shown. Let Q_c' be an another representing map for $\mathfrak{t}-c$ from $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}_c'$ with the properties $\operatorname{dom}Q_c'=\operatorname{dom}\mathfrak{t}$ and

$$(\mathfrak{t}-c)[\varphi,\psi]=(Q_c'\varphi,Q_c'\psi),\quad \varphi,\psi\in\operatorname{dom}Q_c'. \tag{2.5}$$

Then, the linear relation V from $\mathfrak{K}_c$ to $\mathfrak{K}_c'$, defined by

$$\big\{\{Q_c\varphi,Q_c'\varphi\}:\ \varphi\in\operatorname{dom}\mathfrak{t}\big\},$$

is an isometric operator from $\operatorname{ran}Q_c$ onto $\operatorname{ran}Q_c'$, which can be extended as an isometric operator from $\overline{\operatorname{ran}}\,Q_c$ onto $\overline{\operatorname{ran}}\,Q_c'$, such that $Q_c'f=VQ_cf$ holds for all $f\in\operatorname{dom}\mathfrak{t}$. To get the desired partial isometry V, it remains to continue the isometric map to $\mathfrak{K}_c\ominus\overline{\operatorname{ran}}\,Q_c$ as a zero mapping. This proves the stated uniqueness property of the representing map for $\mathfrak{t}-c$. □

It should be stressed that the representing map Q_c in (1.2) depends on the choice of the bound $c\leqslant m(\mathfrak{t})$, and that

$$\ker Q_c=\ker(\mathfrak{t}-c).$$

Thus, for instance, $\ker Q_c=\{0\}$ for all $c<m(\mathfrak{t})$, while, in general, $\ker Q_c\neq\{0\}$ when $c=m(\mathfrak{t})$. However, observe that, by definition,

$$\operatorname{dom}Q_c=\operatorname{dom}Q_{m(\mathfrak{t})}=\operatorname{dom}\mathfrak{t},\quad c\leqslant m(\mathfrak{t}).$$

Furthermore, it follows from (1.2) that

$$\inf\left\{\frac{\|Q_c\varphi\|^2}{\|\varphi\|^2}:\ \varphi\in\operatorname{dom}\mathfrak{t}\right\}=m(\mathfrak{t})-c. \tag{2.6}$$

Of course, one could always take $c = m(\mathfrak{t})$, but the more general choice $c \leqslant m(\mathfrak{t})$ allows some useful flexibility.

Recall from [27] that a sequence φ_n in $\operatorname{dom} \mathfrak{t}$ is said to converge to $\varphi \in \mathfrak{H}$ in the sense of $\mathfrak{t}$, denoted by $\varphi_n \to_{\mathfrak{t}} \varphi$, if

$$\varphi_n \to \varphi \quad \text{in } \mathfrak{H}, \quad \text{and} \quad \mathfrak{t}[\varphi_n - \varphi_m] \to 0.$$

Here, $\mathfrak{t}[\cdot]$ stands for the diagonal of $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$. The following definitions are standard for a lower semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$; see, for example, [27, 28].

Definition 2.2. Let $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be lower semibounded. Then, $\mathfrak{t}$ is said to be

(a) *closable* if for any sequence $\varphi_n \in \operatorname{dom} \mathfrak{t}$:

$$\varphi_n \to_{\mathfrak{t}} 0 \quad \Rightarrow \quad \mathfrak{t}[\varphi_n] \to 0.$$

(b) *closed* if for any sequence $\varphi_n \in \operatorname{dom} \mathfrak{t}$:

$$\varphi_n \to_{\mathfrak{t}} \varphi \quad \Rightarrow \quad \varphi \in \operatorname{dom} \mathfrak{t} \quad \text{and} \quad \mathfrak{t}[\varphi_n - \varphi] \to 0.$$

(c) *singular* if $\mathfrak{t}$ is nonnegative and if for every $\varphi \in \operatorname{dom} \mathfrak{t}$, there is a sequence $\varphi_n \in \operatorname{dom} \mathfrak{t}$ such that

$$\varphi_n \to \varphi, \quad \mathfrak{t}[\varphi_n] \to 0.$$

Note that a nonnegative form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$, which is singular, automatically has lower bound $m(\mathfrak{t}) = 0$. An equivalent definition for a nonnegative form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ to be singular is that for every $\varphi \in \operatorname{dom} \mathfrak{t}$, there is a sequence $\varphi_n \in \operatorname{dom} \mathfrak{t}$ such that

$$\varphi_n \to 0, \quad \mathfrak{t}[\varphi_n - \varphi] \to 0. \tag{2.7}$$

The connection between these notions for lower semibounded forms and for the corresponding linear representing maps is now easily established; cf. [27, Ch. VI, Examples 1.13, 1.23] for items (a) and (b).

Lemma 2.3. *Let $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be lower semibounded and let $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$ be a representing map for $\mathfrak{t}$ as in (1.2) with $c \leqslant m(\mathfrak{t})$. Then, the following statements hold:*

(a) *$\mathfrak{t}$ is closable if and only if Q_c is closable;*
(b) *$\mathfrak{t}$ is closed if and only if Q_c is closed.*

Moreover, if $\mathfrak{t}$ is nonnegative, then

(c) *$\mathfrak{t}$ is singular if and only if Q_0 is singular,*

and in this case $m(\mathfrak{t}) = 0$.

Proof. (a) and (b) It suffices to observe that $\varphi_n \to_{\mathfrak{t}} \varphi$ is equivalent to $\varphi_n \to \varphi$, while $Q_c\varphi_n$ converges in $\mathfrak{K}_c$.

(c) Since $\mathfrak{t}$ is nonnegative, it follows that $m(\mathfrak{t}) \geqslant 0$. Thus, by taking $c = 0$ in (1.2), one sees that

$$\mathfrak{t}[\varphi, \psi] = (Q_0\varphi, Q_0\psi)_{\mathfrak{K}_0}, \quad \varphi, \psi \in \operatorname{dom} Q_0, \tag{2.8}$$

holds for some representing map Q_0 from $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}_0$. Furthermore, $m(\mathfrak{t}) = 0$ is a consequence of (2.6). By definition, $\mathfrak{t}$ in (2.8) is singular if and only if for every $\varphi \in \operatorname{dom} \mathfrak{t} = \operatorname{dom} Q_0$, there is a sequence $\varphi_n \in \operatorname{dom} \mathfrak{t}$ such that

$$\varphi_n \to \varphi, \quad \mathfrak{t}[\varphi_n] = \|Q_0\varphi_n\|^2 \to 0. \tag{2.9}$$

This condition means that $\operatorname{dom} Q_0 \subset \ker Q_0^{**}$, that is, Q_0 is a singular operator. Conversely, if Q_0 is singular, it follows from (2.8) that the form $\mathfrak{t}$ is nonnegative and singular. ☐

Assume that the lower semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ has a lower semibounded form extension $\mathfrak{t}' \in \mathbf{F}(\mathfrak{H})$, that is, $\mathfrak{t} \subset \mathfrak{t}'$. Let Q_c and Q_c' be the representing maps for $\mathfrak{t}$ and $\mathfrak{t}'$ for $c \leqslant m(\mathfrak{t}')$. If $\mathfrak{t}'$ is closed, then Q_c' is closed by Lemma 2.3, which implies that Q_c is closable; thus, $\mathfrak{t}$ is closable by Lemma 2.3. Therefore, if the form $\mathfrak{t}$ has a closed extension, then $\mathfrak{t}$ is closable. A converse to this statement is contained in Lemma 2.4; its proof is now given in terms of a representing map.

Lemma 2.4. *Let* $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ *be lower semibounded and assume that* $\mathfrak{t}$ *is closable. Then,* $\mathfrak{t}$ *has a closure* $\bar{\mathfrak{t}} \in \mathbf{F}(\mathfrak{H})$ *(the smallest closed extension) defined as follows: the domain* $\operatorname{dom} \bar{\mathfrak{t}}$ *is given by all* $\varphi \in \mathfrak{H}$ *for which there exists a sequence* $\varphi_n \in \operatorname{dom} \mathfrak{t}$ *with* $\varphi_n \to_{\mathfrak{t}} \varphi$*; and in this case,*

$$\bar{\mathfrak{t}}[\varphi, \psi] = \lim_{n\to\infty} \mathfrak{t}[\varphi_n, \psi_n] \quad \textit{for any} \quad \varphi_n \to_{\mathfrak{t}} \varphi, \quad \psi_n \to_{\mathfrak{t}} \psi. \tag{2.10}$$

The forms $\mathfrak{t}$ *and* $\bar{\mathfrak{t}}$ *have the same lower bound. In fact, if* $c \leqslant m(\mathfrak{t})$ *and* $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$ *is a closable representing map for* $\mathfrak{t} - c$*, then* $\bar{\mathfrak{t}}$ *is given by*

$$\bar{\mathfrak{t}}[\varphi, \psi] - c(\varphi, \psi) = (Q_c^{**}\varphi, Q_c^{**}\psi)_{\mathfrak{K}_c}, \quad \varphi, \psi \in \operatorname{dom} \bar{\mathfrak{t}} = \operatorname{dom} Q_c^{**}, \tag{2.11}$$

where Q_c^{**} *denotes the closure of* Q_c *in* $\mathfrak{H} \times \mathfrak{K}_c$.

Proof. Since $\mathfrak{t}$ is closable, also Q_c is closable by Lemma 2.3, that is, Q_c^{**} is a closed operator. Hence, the form $\bar{\mathfrak{t}}$ defined by (2.11) is closed and it clearly extends $\mathfrak{t}$. Since Q_c^{**} is the smallest closed linear operator extension of Q_c, it follows that $\bar{\mathfrak{t}}$ is the smallest closed form-extension of $\mathfrak{t}$. By construction, one has $\operatorname{dom} \bar{\mathfrak{t}} = \operatorname{dom} Q_c^{**}$, so that $\operatorname{dom} \bar{\mathfrak{t}}$ is given by all $\varphi \in \mathfrak{H}$ for which there exists a sequence $\varphi_n \in \operatorname{dom} \mathfrak{t}$ with $\varphi_n \to_{\mathfrak{t}} \varphi$ and (2.10). ☐

Closability and closedness of a lower semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ are *translation invariant*, that is, for all $a \in \mathbb{R}$, the forms $\mathfrak{t}$ and $\mathfrak{t} - a$ are simultaneously closable or closed, respectively. Moreover, in this case, one has

$$\overline{\mathfrak{t} - a} = \bar{\mathfrak{t}} - a, \quad a \in \mathbb{R};$$

cf. Lemma 2.4. It should be stressed that the closure process in Lemma 2.4 depends neither on the choices of the sequences $\varphi_n \to_{\mathfrak{t}} \varphi$ and $\psi_n \to_{\mathfrak{t}} \psi$ in (2.14), (2.10), nor on the choices of $c \leqslant m(\mathfrak{t})$ and of Q_c in the representation (1.2). The translation invariance of the closability of lower semibounded forms $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ is quite natural. The situation is different in the case of nonnegative singular forms. For instance, it follows from the definition that if $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ is nonnegative and singular, then $\mathfrak{t} + c$ is not singular for any $c > 0$.

The notion of a core is closely connected to Lemma 2.4. Let the form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be lower semibounded and closed. Then, a *core for* $\mathfrak{t}$ is a subset $\mathfrak{D}$ of $\operatorname{dom} \mathfrak{t}$ such that the (form) closure of the restriction $\mathfrak{t} \restriction \mathfrak{D}$ is equal to $\mathfrak{t}$; cf. [27, p. 317]. Likewise, let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be a closed linear operator. Then, a *core for* T is a subset $\mathfrak{D}$ of $\operatorname{dom} T$ such that the (graph) closure of $\{\{\varphi, T\varphi\} : \varphi \in \mathfrak{D}\}$ is equal to T; cf. [27, p. 166]. Therefore, if the closed semibounded form $\mathfrak{t}$ with lower bound $\gamma \in \mathbb{R}$ and the closed linear operator Q_c with $c \leqslant m(\mathfrak{t})$ are connected by

$$\mathfrak{t}[\varphi, \psi] - c(\varphi, \psi) = (Q_c\varphi, Q_c\psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{t} = \operatorname{dom} Q_c,$$

and $\mathfrak{D}$ is a subset of $\operatorname{dom} \mathfrak{t} = \operatorname{dom} Q_c$, then $\mathfrak{D}$ is a core for $\mathfrak{t}$ if and only if $\mathfrak{D}$ is a core for Q_c; cf. Lemma 2.4.

It is clear that a null form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$, that is, $\operatorname{dom} \mathfrak{t} = \ker \mathfrak{t}$, is nonnegative and, simultaneously, closable and singular. Conversely, if $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ is a nonnegative form that is both regular and singular, then it follows from Lemma 2.3 that $\mathfrak{t} = 0$.

Recall that for linear operators $Q_1 \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_1)$ and $Q_2 \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_2)$ one says that Q_1 is *contractively dominated* by Q_2, in notation $Q_1 \prec_c Q_2$, if

$$\operatorname{dom} Q_2 \subset \operatorname{dom} Q_1 \quad \text{and} \quad \|Q_1\varphi\| \leqslant \|Q_2\varphi\|, \quad \varphi \in \operatorname{dom} Q_2, \tag{2.12}$$

see [24, Definition 8.1, Lemma 8.2]. If for $i = 1, 2$ one has

$$\mathfrak{t}_i[\varphi, \psi] = c(\varphi, \psi) + (Q_i\varphi, Q_i\psi)_{\mathfrak{K}_i}, \quad \varphi, \psi \in \operatorname{dom} \bar{\mathfrak{t}}_i = \operatorname{dom} Q_i,$$

then clearly

$$\mathfrak{t}_1 \leqslant \mathfrak{t}_2 \quad \Leftrightarrow \quad Q_1 \prec_c Q_2. \tag{2.13}$$

This equivalence will be tacitly used elsewhere in the paper. In particular, one sees that if the lower semibounded forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are closable, then

$$\mathfrak{t}_1 \leqslant \mathfrak{t}_2 \quad \Rightarrow \quad \bar{\mathfrak{t}}_1 \leqslant \bar{\mathfrak{t}}_2,$$

as $Q_1 \prec_c Q_2$ implies that $(Q_1)^{**} \prec_c (Q_2)^{**}$; cf. [24] and [15, (2.4)].

Finally some remarks will be made concerning the definitions of closable, closed, and singular forms in a Hilbert space.

Remark 2.5. Assume that $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ is lower semibounded and let $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be a representing map, with $c \leqslant \gamma$. If $\varphi_n \to_{\mathfrak{t}} \varphi$ and $\psi_n \to_{\mathfrak{t}} \psi$, then $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in $\mathfrak{H}$, while $Q_c\varphi_n$ and $Q_c\psi_n$ converge in $\mathfrak{K}_c$ to, say, h and k in $\operatorname{ran} Q_c^{**}$. This implies that

$$\lim_{n\to\infty} \mathfrak{t}[\varphi_n, \psi_n] = c(\varphi, \psi) + (h, k). \tag{2.14}$$

Assume that $\operatorname{mul} Q_c^{**}$ is not trivial. Let $\omega \in \operatorname{mul} Q_c^{**}$ with $\omega \neq 0$. Then, there is a sequence $\chi_n \in \operatorname{dom} Q_c$ such that $\chi_n \to 0$ and $Q_c\chi_n \to \omega$. Therefore, $\varphi_n + a\chi_n \to_{\mathfrak{t}} \varphi$ for all $a \in \mathbb{C}$, and therefore,

$$\lim_{n\to\infty} \mathfrak{t}[\varphi_n + a\chi_n, \psi_n] = c(\varphi,\psi) + (h,k) + a(\omega,k).$$

Thus, it follows that for any $\omega \in \operatorname{mul} Q_c^{**}$ that is not orthogonal to k the limit in (2.14) can take every possible value in $\mathbb{C}$, as can be seen from

$$\operatorname{ran} Q_c^{**} = (\operatorname{ran} Q_c^{**} \ominus \operatorname{mul} Q_c^{**}) \oplus \operatorname{mul} Q_c^{**}.$$

Consequently, the limit in (2.14) depends on the choice of the sequences φ_n and ψ_n. Hence, in general, there does not exist a well-defined extension for the form $\mathfrak{t}$.

Finally, notice that the alternative characterization of singularity in (2.7), that is, for every $\chi \in \operatorname{dom} \mathfrak{t}$, there is a sequence $\chi_n \in \operatorname{dom} \mathfrak{t}$, such that

$$\chi_n \to 0, \quad \text{and} \quad \mathfrak{t}[\chi_n - \chi] \to 0,$$

is equivalent to saying that $\operatorname{ran} Q \subset \operatorname{mul} Q^{**}$.

This section is finished with the following example that involves λ-depending representing maps in the connection of differential operators.

Example 2.6. It is explained how representing maps that are closable naturally appear in the setting of Dirichlet-to-Neumann maps for elliptic partial differential equations and local point interactions for some classical differential operators in mathematical physics. For this purpose, first recall the notion of Nevanlinna families of linear relations $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, in a Hilbert space $\mathcal{H}$, which are characterized by the following properties:

(a) for $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$, the relation $M(\lambda)$ is maximal dissipative (maximal accumulative, respectively);
(b) $M(\lambda)^* = M(\bar{\lambda})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
(c) for some, and hence for all, $\mu \in \mathbb{C}_+(\mathbb{C}_-)$ the operator family $(M(\lambda) + \mu)^{-1}$ in $\mathbf{B}(\mathcal{H})$ is holomorphic for all $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$.

By the condition (a) or (b), the values $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, are necessarily closed. A Nevanlinna family is said to be strict if $M(\lambda) \cap M(\lambda)^* = \{0\}$ for some (and hence for all) $\lambda \in \mathbb{C} \setminus \mathbb{R}$, or equivalently, for some (equivalently for every) $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $M(\lambda)$ is a densely defined operator and satisfies the following strictness condition:

$$\operatorname{Im}(M(\lambda)u, u) = 0, \quad u \in \operatorname{dom} M(\lambda) \quad \Longrightarrow \quad u = 0. \tag{2.15}$$

This subclass of Nevanlinna functions is sufficient for most typical applications in partial differential equations, where the Dirichlet-to-Neumann map $\Lambda(\cdot)$, up to a sign change, can be connected to a strict Nevanlinna function. To show the connection to the present notion of representing maps, associate with each $M(\cdot)$ a family of nonnegative quadratic forms $\mathfrak{t}_{M(\lambda)}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, in $\mathcal{H}$:

$$\mathfrak{t}_{M(\lambda)}[u,v] := \frac{(M(\lambda)u,v) - (u, M(\lambda)v)}{\lambda - \bar{\lambda}}, \quad u,v \in \operatorname{dom} \mathfrak{t}_{M(\lambda)} = \operatorname{dom} M(\lambda).$$

By the conditions (a) and (b), the form $\mathfrak{t}_{M(\lambda)}$ is nonnegative on its domain $\operatorname{dom}\mathfrak{t}_{M(\lambda)}$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$. According to Lemma 2.1 for each $\lambda \in \mathbb{C}\setminus\mathbb{R}$, there exists a minimal representing map $g(\lambda)$ from $\operatorname{dom}(M(\lambda))$ to a Hilbert space $\mathfrak{K}_\lambda$ such that

$$\frac{(M(\lambda)u, v) - (u, M(\lambda)v)}{\lambda - \bar{\lambda}} = (g(\lambda)u, g(\lambda)v)_{\mathfrak{K}_\lambda}, \quad u, v \in \operatorname{dom}\mathfrak{t}_{M(\lambda)}.$$

Due to the condition (2.15), the space $\mathfrak{K}_\lambda$ can be obtained as a completion of the natural form inner product $(\cdot,\cdot)_{\mathfrak{t}}$ on $\operatorname{dom}\mathfrak{t}$. The mapping $g(\lambda)$ can be constructed more explicitly by means of the notion of a *unitary boundary triplet* for a symmetric operator S in a Hilbert space $\mathfrak{H}$ involving a pair of boundary mappings $\{\Gamma_0, \Gamma_1\}$ acting from a dense subspace T of (the graph of) S^* to another Hilbert space $\mathcal{H}$ (often called a boundary space). This mapping is typically called the γ-field and denoted by $\gamma(\lambda)$. It maps $\operatorname{dom} M(\lambda)$ onto a dense subspace $\mathfrak{N}_\lambda(T)$ of $\mathfrak{N}_\lambda(S^*)$ (the eigenspaces of T and S^*, respectively); see [11, Lemma 2.14 & eq. (4.5)]. Therefore, by Lemma 2.1, the space $\mathfrak{K}_\lambda$ and the mapping $g(\lambda)$ can be identified (via a unitary mapping) with the closed subspace $\mathfrak{N}_\lambda(S^*)$ of the underlying subspace $\mathfrak{H}$ and the γ-field $\gamma(\lambda)$, respectively. Recall also that $\{\Gamma_0, \Gamma_1\} : \mathfrak{H}^2 \to \mathcal{H}^2$ is called *minimal*, if

$$\mathfrak{H} = \mathfrak{H}_{min} := \overline{\operatorname{span}}\{\, \mathfrak{N}_\lambda(T) : \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \,\},$$

in which case also $\{\Gamma_0, \Gamma_1\}$ is (up to unitary equivalence) uniquely determined by the function $M(\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$; see [11, Theorem 3.9].

In general, the mapping $\gamma(\lambda)$ is not closable and can even be singular; for an example, see [11, Example 6.7] and [9, Proposition 5.24]. In [9], a unitary boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for S^* is said to be *essentially self-adjoint* if $A_0 := \ker \Gamma_0$ is essentially self-adjoint in $\mathfrak{H}$. The main result in [9] shows that this simple property concerning A_0 is necessary and sufficient for the form $\mathfrak{t}_{M(\lambda)}$ to be closable for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and that for this, it is even sufficient that the form $\mathfrak{t}_{M(\lambda)}$ is closable at one point in the upper and the lower half-plane, for example, at the points $\lambda = \pm i$. Moreover, in this case, the form domain of its closure does not depend on $\lambda \in \mathbb{C}\setminus\mathbb{R}$; see [9, Theorem 5.24]. For various applications of form domain invariant Nevanlinna functions, see [10, Section 3] for Laplace operators, and [10, Section 4] for sequences of local point interactions for momentum, Schrödinger, and Dirac operators.

3 | LEBESGUE DECOMPOSITIONS OF SEMIBOUNDED FORMS

It will be shown that for a semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$, there exists a sum decomposition $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$, $\operatorname{dom}\mathfrak{t} = \operatorname{dom}\mathfrak{t}_1 = \operatorname{dom}\mathfrak{t}_2$, where $\mathfrak{t}_1 \in \mathbf{F}(\mathfrak{H})$ is closable and $\mathfrak{t}_2 \in \mathbf{F}(\mathfrak{H})$ is singular. Moreover, it will be shown that this decomposition is unique by the property that $\mathfrak{t}_1$ is the largest of all closable forms below $\mathfrak{t}$. This so-called Lebesgue decomposition and its uniqueness go back to Simon [35]; the present proofs are based on the Lebesgue decomposition of the representing map of the form $\mathfrak{t}$.

Let the form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be semibounded, let $c \leqslant m(\mathfrak{t})$, and assume that $\mathfrak{t}$ has the representation

$$\mathfrak{t}[\varphi, \psi] = c(\varphi, \psi) + (Q\varphi, Q\psi), \quad \varphi, \psi \in \operatorname{dom}\mathfrak{t} = \operatorname{dom} Q, \tag{3.1}$$

where $Q \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ is a representing map for $\mathfrak{t} - c$. The closure of Q is a closed linear relation $Q^{**} \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ and let P_0 be the orthogonal projection from $\mathfrak{K}$ onto $\operatorname{mul} Q^{**}$. Then, the sum decomposition

$$Q = Q_{\rm reg} + Q_{\rm sing}, \quad Q_{\rm reg} = (I - P_0)Q, \quad Q_{\rm sing} = P_0 Q, \tag{3.2}$$

where $\operatorname{dom} Q_{\rm reg} = \operatorname{dom} Q_{\rm sing} = \operatorname{dom} Q$ is the so-called Lebesgue decomposition of the representing Q; cf. [16, 24]. Here, $Q_{\rm reg}$ is a closable operator and $Q_{\rm sing}$ is a singular operator. Recall that $Q^{**} \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ has a similar decomposition

$$Q^{**} = (Q^{**})_{\rm reg} + (Q^{**})_{\rm sing}, \quad (Q^{**})_{\rm reg} = (I - P_0)Q^{**}, \quad (Q^{**})_{\rm sing} = P_0 Q^{**},$$

where $\operatorname{dom}(Q^{**})_{\rm reg} = \operatorname{dom}(Q^{**})_{\rm sing} = \operatorname{dom} Q^{**}$. Here, $(Q^{**})_{\rm reg}$ is the usual orthogonal operator part of the closed linear relation Q^{**}. It is important to observe that

$$(Q_{\rm reg})^{**} = (Q^{**})_{\rm reg}, \quad \operatorname{dom}(Q_{\rm reg})^{**} = \operatorname{dom} Q^{**}.$$

For more details, see [15, 16, 24].

Theorem 3.1. *Let the form* $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ *be semibounded. Then,* $\mathfrak{t}$ *has the sum decomposition*

$$\mathfrak{t} = \mathfrak{t}_{\rm reg} + \mathfrak{t}_{\rm sing}, \quad \operatorname{dom} \mathfrak{t} = \operatorname{dom} \mathfrak{t}_{\rm reg} = \operatorname{dom} \mathfrak{t}_{\rm sing}, \tag{3.3}$$

with a regular (closable) semibounded form $\mathfrak{t}_{\rm reg} \in \mathbf{F}(\mathfrak{H})$ *and a singular nonnegative form* $\mathfrak{t}_{\rm sing} \in \mathbf{F}(\mathfrak{H})$*. In fact, let* $c \leqslant m(\mathfrak{t})$ *and assume that* $\mathfrak{t}$ *has the representation (3.1). Then*

$$\mathfrak{t}_{\rm reg}[\varphi, \psi] = c(\varphi, \psi) + (Q_{\rm reg}\varphi, Q_{\rm reg}\psi), \quad \mathfrak{t}_{\rm sing}[\varphi, \psi] = (Q_{\rm sing}\varphi, Q_{\rm sing}\psi), \tag{3.4}$$

with $\varphi, \psi \in \operatorname{dom} \mathfrak{t} = \operatorname{dom} Q$*. Moreover, the closure* $\overline{\mathfrak{t}_{\rm reg}}$ *of the closable part* $\mathfrak{t}_{\rm reg}$ *is given by*

$$\overline{\mathfrak{t}_{\rm reg}}\,[\varphi, \psi] = c(\varphi, \psi) + ((Q_{\rm reg})^{**}\varphi, (Q_{\rm reg})^{**}\psi), \quad \varphi, \psi \in \operatorname{dom} \overline{\mathfrak{t}_{\rm reg}} = \operatorname{dom} Q^{**}. \tag{3.5}$$

Proof. Since P_0 in (3.2) is an orthogonal projection, it is clear from (3.4) that $\mathfrak{t} = \mathfrak{t}_{\rm reg} + \mathfrak{t}_{\rm sing}$; cf. (3.1). Since $Q_{\rm reg}$ is a closable operator and $Q_{\rm sing}$ is a singular operator, it also follows from (3.4) that the forms $\mathfrak{t}_{\rm reg}$ and $\mathfrak{t}_{\rm sing}$ are regular and singular, respectively; cf. Lemma 2.3. Finally, (3.5) is a direct consequence of the definition of $\mathfrak{t}_{\rm reg}$; cf. Lemma 2.4. □

The decomposition of the semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ in (3.3) into the regular part $\mathfrak{t}_{\rm reg}$ and the singular part $\mathfrak{t}_{\rm sing}$ in (3.4) is called the *Lebesgue decomposition* of $\mathfrak{t}$. It will be shown in the following theorem that $\mathfrak{t}_{\rm reg}$ in (3.4) does not depend on the choice of c and Q in (3.1), and neither does $\mathfrak{t}_{\rm sing} = \mathfrak{t} - \mathfrak{t}_{\rm reg}$.

Theorem 3.2. *Let the form* $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ *be semibounded. Then,* $\mathfrak{t}_{\rm reg}$ *as defined in Theorem 3.1 satisfies* $\mathfrak{t}_{\rm reg} \leqslant \mathfrak{t}$*. Let* $\mathfrak{u} \in \mathbf{F}(\mathfrak{H})$ *be a semibounded form and assume that* $\mathfrak{u} \leqslant \mathfrak{t}$*, then*

$$\mathfrak{u}_{\rm reg} \leqslant \mathfrak{t}_{\rm reg}. \tag{3.6}$$

In particular, if the form $\mathfrak{u} \in \mathbf{F}(\mathfrak{H})$ *is closable and* $\mathfrak{u} \leqslant \mathfrak{t}$*, then*

$$\mathfrak{u} \leqslant \mathfrak{t}_{\rm reg}.$$

Consequently, $\mathfrak{t}_{\mathrm{reg}}$ *is the largest of all closable semibounded forms below* $\mathfrak{t}$.

Proof. By assumption, $\mathfrak{u} \in \mathbf{F}(\mathfrak{H})$ is a semibounded form with $\mathfrak{u} \leqslant \mathfrak{t}$, that is,

$$\operatorname{dom} \mathfrak{t} \subset \operatorname{dom} \mathfrak{u} \quad \text{and} \quad \mathfrak{u}[\varphi] \leqslant \mathfrak{t}[\varphi], \quad \varphi \in \operatorname{dom} \mathfrak{t}.$$

Let the lower bound of $\mathfrak{u}$ be $c \in \mathbb{R}$. Thanks to the inequality $\mathfrak{u} \leqslant \mathfrak{t}$, it follows that $c \leqslant m(\mathfrak{t})$. By Lemma 2.1, the form $\mathfrak{u}$ (having lower bound c) has a representation of the form

$$\mathfrak{u}[\varphi, \psi] = c(\varphi, \psi) + (R_c\varphi, R_c\psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{u} = \operatorname{dom} R_c,$$

with a representing map R_c from $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}_c'$. Since $c \leqslant m(\mathfrak{t})$, it follows from the same lemma that the form $\mathfrak{t}$ (with lower bound γ) has a representation of the form

$$\mathfrak{t}[\varphi, \psi] = c(\varphi, \psi) + (Q_c\varphi, Q_c\psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{t} = \operatorname{dom} Q_c,$$

where Q_c is a representing map from $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}_c$. As in Theorem 3.1, one sees that

$$\mathfrak{u}_{\mathrm{reg}}[\varphi, \psi] = c(\varphi, \psi) + (R_{c,\mathrm{reg}}\varphi, R_{c,\mathrm{reg}}\psi), \quad \varphi, \psi \in \operatorname{dom} R_c,$$

and, likewise, it follows from Theorem 3.1 that

$$\mathfrak{t}_{\mathrm{reg}}[\varphi, \psi] = c(\varphi, \psi) + (Q_{c,\mathrm{reg}}\varphi, Q_{c,\mathrm{reg}}\psi), \quad \varphi, \psi \in \operatorname{dom} Q_c.$$

Using the representations of $\mathfrak{u}$ and $\mathfrak{t}$, one sees that the inequality $\mathfrak{u} \leqslant \mathfrak{t}$ is equivalent to

$$\operatorname{dom} Q_c \subset \operatorname{dom} R_c \quad \text{and} \quad \|R_c\varphi\| \leqslant \|Q_c\varphi\|, \quad \varphi \in \operatorname{dom} Q_c.$$

As a consequence from [24, Theorem 8.3], one finds that

$$R_{c,\mathrm{reg}} \prec Q_{c,\mathrm{reg}}, \tag{3.7}$$

see (2.12). Now, using the above representations of $\mathfrak{u}_{\mathrm{reg}}$ and $\mathfrak{t}_{\mathrm{reg}}$, it follows that the inequality (3.6) holds; cf. (2.13). Since $\mathfrak{t}_{\mathrm{reg}} \leqslant \mathfrak{t}$ and $\mathfrak{t}_{\mathrm{reg}}$ is closable, it is clear that $\mathfrak{t}_{\mathrm{reg}}$ has the stated maximality property. ☐

Remark 3.3. The following useful observation goes back to [35]. Let the form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be semibounded, let the form $\mathfrak{b} \in \mathbf{F}(\mathfrak{H})$ be symmetric with $\operatorname{dom} \mathfrak{t} \subset \operatorname{dom} \mathfrak{b}$, and assume that for some $\alpha \geqslant 0$,

$$|\mathfrak{b}[\varphi]| \leqslant \alpha\|\varphi\|^2, \quad \varphi \in \operatorname{dom} \mathfrak{t}.$$

Then, the sum $\mathfrak{t} + \mathfrak{b} \in \mathbf{F}(\mathfrak{H})$ is semibounded (with $\operatorname{dom}(\mathfrak{t} + \mathfrak{b}) = \operatorname{dom} \mathfrak{t}$). Moreover, if $\mathfrak{t}$ is closable, then $\mathfrak{t} + \mathfrak{b}$ is closable, cf. [27, p. 320] and [7, Theorem 5.1.16], and in this case,

$$(\mathfrak{t} + \mathfrak{b})_{\mathrm{reg}} = \mathfrak{t}_{\mathrm{reg}} + \mathfrak{b} \quad \text{and} \quad (\mathfrak{t} + \mathfrak{b})_{\mathrm{sing}} = \mathfrak{t}_{\mathrm{sing}}. \tag{3.8}$$

To see this, note that $\mathfrak{t}_{\rm reg} \leqslant \mathfrak{t}$ implies $\mathfrak{t}_{\rm reg} + \mathfrak{b} \leqslant \mathfrak{t} + \mathfrak{b}$, and, since the left-hand side is closable, one obtains $\mathfrak{t}_{\rm reg} + \mathfrak{b} \leqslant (\mathfrak{t} + \mathfrak{b})_{\rm reg}$ by Theorem 3.2. Likewise $(\mathfrak{t} + \mathfrak{b})_{\rm reg} - \mathfrak{b}$ is closable and $(\mathfrak{t} + \mathfrak{b})_{\rm reg} - \mathfrak{b} \leqslant \mathfrak{t}$, so that $(\mathfrak{t} + \mathfrak{b})_{\rm reg} - \mathfrak{b} \leqslant \mathfrak{t}_{\rm reg}$ by Theorem 3.2. Combining these inequalities gives the first identity in (3.8) and the second identity in (3.8) is clear from the first one.

It follows from (3.2) that $\operatorname{ran} Q_{\rm reg} \subset \operatorname{ran}(I - P_0)$ and $\operatorname{ran} Q_{\rm sing} \subset \operatorname{ran} P_0$. If the representing map Q is minimal: $\overline{\operatorname{ran}}\, Q = \mathfrak{K}$, then $\overline{\operatorname{ran}}\, Q_{\rm reg} = \operatorname{ran}(I - P_0)$ and $\overline{\operatorname{ran}}\, Q_{\rm sing} = \operatorname{ran} P_0$. Clearly, the regular and singular components in the Lebesgue decomposition of Q have only a trivial intersection of the ranges. The Lebesgue decomposition is an example of a more general decomposition of the form $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$ where $\mathfrak{t}_1$ is closable and $\mathfrak{t}_2$ is singular. In fact, by considering orthogonal Lebesgue-type decompositions of the representing map Q, one may easily construct such decompositions in the same way as the Lebesgue decomposition. Moreover, also in these cases, the ranges of the components of Q have only a trivial intersection; cf. Corollary 5.4. One then says that the corresponding forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are mutually singular. However, not every decomposition $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$ where $\mathfrak{t}_1$ is closable and $\mathfrak{t}_2$ is singular, can be obtained via an orthogonal Lebesgue-type decomposition of the representing map Q. The required class of decompositions is much wider, and only in this more general situation, one meets components $\mathfrak{t}_1$ and $\mathfrak{t}_2$ that are not mutually singular; cf. Theorem 5.2. In order to characterize all such decompositions, it is helpful to first investigate the sum decompositions of nonnegative forms into nonnegative forms in Section 4. These will then be used to study Lebesgue-type decompositions of semibounded forms in Section 5.

4 | SUM DECOMPOSITIONS OF NONNEGATIVE FORMS

In this section, the interest will be in the sum decomposition $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$ of a nonnegative form $\mathfrak{h} \in \mathbf{F}(\mathfrak{H})$ with nonnegative forms $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathbf{F}(\mathfrak{H})$ with the additional property that $\operatorname{dom} \mathfrak{h} = \operatorname{dom} \mathfrak{h}_1 = \operatorname{dom} \mathfrak{h}_2$. The parametrization of such decompositions will be given by means of the representing map of the sum. The minimality of the representation of such a sum will be characterized. Furthermore, the interaction of the components $\mathfrak{h}_1$ and $\mathfrak{h}_2$ in the sum decomposition will be described in terms of the parallel sum of $\mathfrak{h}_1 : \mathfrak{h}_2$.

First, a characterization will be presented of all possible sum decompositions of a nonnegative form. The construction in Theorem 4.1 is based on the fact that associated with a sum decomposition of nonnegative forms are natural inequalities that arise from this decomposition; cf. [30].

Theorem 4.1. *Let* $\mathfrak{h} \in \mathbf{F}(\mathfrak{H})$ *be a nonnegative form and assume that* $\mathfrak{h}$ *has the representation*

$$\mathfrak{h}[\varphi, \psi] = (Q\varphi, Q\psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{h} = \operatorname{dom} Q, \tag{4.1}$$

where $Q \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ *is a representing map.*

Let $K \in \mathbf{B}(\mathfrak{K})$ *be a nonnegative contraction and define the forms* $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathbf{F}(\mathfrak{H})$ *for all* $\varphi, \psi \in \operatorname{dom} \mathfrak{h} = \operatorname{dom} Q$ *by*

$$\mathfrak{h}_1[\varphi, \psi] = ((I - K)^{\frac{1}{2}} Q\varphi, (I - K)^{\frac{1}{2}} Q\psi), \quad \mathfrak{h}_2[\varphi, \psi] = (K^{\frac{1}{2}} Q\varphi, K^{\frac{1}{2}} Q\psi). \tag{4.2}$$

Then, the form $\mathfrak{h}$ *has the sum decomposition*

$$\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2, \quad \operatorname{dom} \mathfrak{h} = \operatorname{dom} \mathfrak{h}_1 = \operatorname{dom} \mathfrak{h}_2, \tag{4.3}$$

where $\mathfrak{h}_1$ *and* $\mathfrak{h}_2$ *are nonnegative.*

Conversely, let the nonnegative form $\mathfrak{h} \in \mathbf{F}(\mathfrak{H})$ *in (4.1) have a sum decomposition (4.3), where* $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathbf{F}(\mathfrak{H})$ *are nonnegative. Then, there exists a nonnegative contraction* $K \in \mathbf{B}(\mathfrak{K})$ *such that (4.2) holds.*

Proof. Assume that $K \in \mathbf{B}(\mathfrak{K})$ is a nonnegative contraction and let $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathbf{F}(\mathfrak{H})$ be defined by (4.2). Due to the identity

$$((I-K)^{\frac{1}{2}})^*(I-K)^{\frac{1}{2}} + (K^{\frac{1}{2}})^*K^{\frac{1}{2}} = K + (I-K) = I_{\mathfrak{K}},$$

it is clear from (4.2) that $\mathfrak{h}_1 + \mathfrak{h}_2 = \mathfrak{h}$. Hence, K generates a sum decomposition of the form $\mathfrak{h}$ in (4.1).

Conversely, let the form $\mathfrak{h}$ have the decomposition as in (4.3). According to Lemma 2.1, there exist representing maps Q_1 and Q_2 from $\mathfrak{H}$ to Hilbert spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively, with domains equal to dom $\mathfrak{h}$, such that

$$\mathfrak{h}_1[\varphi,\psi] = (Q_1\varphi, Q_1\psi)_{\mathfrak{K}_1}, \quad \mathfrak{h}_2[\varphi,\psi] = (Q_2\varphi, Q_2\psi)_{\mathfrak{K}_2}, \tag{4.4}$$

for all $\varphi, \psi \in \operatorname{dom} \mathfrak{h}$. It follows from (4.3) and (4.4) that

$$\begin{aligned} (Q\varphi, Q\psi)_{\mathfrak{K}} &= \mathfrak{h}[\varphi,\psi] = \mathfrak{h}_1[\varphi,\psi] + \mathfrak{h}_2[\varphi,\psi] \\ &= (Q_1\varphi, Q_1\psi)_{\mathfrak{K}_1} + (Q_2\varphi, Q_2\psi)_{\mathfrak{K}_2}, \end{aligned} \tag{4.5}$$

for all $\varphi, \psi \in \operatorname{dom} \mathfrak{h}$. The identity (4.5) shows that there are natural inequalities: for all $\varphi \in \operatorname{dom} \mathfrak{h}$, one has

$$(Q_1\varphi, Q_1\varphi)_{\mathfrak{K}_1} \leqslant (Q\varphi, Q\varphi)_{\mathfrak{K}}, \quad (Q_2\varphi, Q_2\varphi)_{\mathfrak{K}_2} \leqslant (Q\varphi, Q\varphi)_{\mathfrak{K}}.$$

Hence, the linear relations C_1 from $\mathfrak{K}$ to $\mathfrak{K}_1$ and C_2 from $\mathfrak{K}$ to $\mathfrak{K}_2$, defined by

$$C_1 = \left\{\{Q\varphi, Q_1\varphi\} : \varphi \in \operatorname{dom} \mathfrak{h}\right\}, \quad C_2 = \left\{\{Q\varphi, Q_2\varphi\} : \varphi \in \operatorname{dom} \mathfrak{h}\right\},$$

are actually contractive operators from $\operatorname{ran} Q$ onto $\operatorname{ran} Q_1$ and $\operatorname{ran} Q_2$, respectively. Thus, they can be uniquely extended to all of $\overline{\operatorname{ran}}\, Q$ and these extensions are denoted again by C_1 and C_2, respectively. Finally, extend these operators trivially to all of $\mathfrak{K}$ and give these extensions the same notation. Now define the operators K_1 and K_2 by

$$K_1 = C_1^* C_1^{**} \quad \text{and} \quad K_2 = C_2^* C_2^{**}.$$

Then, $K_1, K_2 \in \mathbf{B}(\mathfrak{K})$ are nonnegative contractions and they satisfy

$$\mathfrak{h}_1[\varphi,\psi] = (C_1Q\varphi, C_1Q\psi)_{\mathfrak{K}_1} = (K_1Q\varphi, Q\psi)_{\mathfrak{K}} \tag{4.6}$$

and

$$\mathfrak{h}_2[\varphi,\psi] = (C_2Q\varphi, C_2Q\psi)_{\mathfrak{K}_2} = (K_2Q\varphi, Q\psi)_{\mathfrak{K}} \tag{4.7}$$

for all $\varphi, \psi \in \operatorname{dom} \mathfrak{h}$. By combining (4.6) and (4.7) with (4.5), it follows that

$$K_1 + K_2 = P_{\overline{\operatorname{ran}}\, Q},$$

where the right-hand side stands for the orthogonal projection from $\mathfrak{K}$ onto $\overline{\operatorname{ran}}\, Q$. Now let $K = K_2$, then the second identity in (4.2) follows from (4.7). Note that

$$I_{\mathfrak{K}} - K = P_{\overline{\operatorname{ran}}\, Q} - K_2 + (I - P_{\overline{\operatorname{ran}}\, Q}) = K_1 + (I - P_{\overline{\operatorname{ran}}\, Q}),$$

which leads to the identity

$$((I_{\mathfrak{K}} - K)Q\varphi, Q\psi) = (K_1 Q\varphi, Q\psi)_{\mathfrak{K}}, \quad \varphi, \psi \in \operatorname{dom} \mathfrak{h}.$$

Thus, the first identity in (4.2) follows from (4.6). □

Remark 4.2. If the representing map $Q \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ in (4.1) is minimal, then the contraction $K \in \mathbf{B}(\mathfrak{K})$ in (4.2) is unique (see for a similar feature [16, Remark 3.7]). To see this directly, assume that K and $\widetilde{K}$ are nonnegative contractions in $\mathbf{B}(\mathfrak{K})$ that satisfy (4.2). Then, it follows that,

$$\mathfrak{h}_2[\varphi, \psi] = (K^{\frac{1}{2}} Q\varphi, K^{\frac{1}{2}} Q\psi) = (\widetilde{K}^{\frac{1}{2}} Q\varphi, \widetilde{K}^{\frac{1}{2}} Q\psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{h}.$$

Hence, $((K - \widetilde{K})Q\varphi, Q\psi) = 0$ for all $\varphi, \psi \in \operatorname{dom} \mathfrak{h}$. By the assumption $\overline{\operatorname{ran}}\, Q = \mathfrak{K}$, one concludes that $K - \widetilde{K} = 0$.

Let $Q_1 \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_1)$ and $Q_2 \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_2)$ be linear operators; here $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are Hilbert spaces. The corresponding *column operator* $\operatorname{col}(Q_1, Q_2)$ from $\mathfrak{H}$ to the orthogonal sum $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ is defined by

$$\operatorname{col}(Q_1, Q_2) = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} : \operatorname{dom} Q_1 \cap \operatorname{dom} Q_2 \to \mathfrak{K} = \begin{pmatrix} \mathfrak{K}_1 \\ \mathfrak{K}_2 \end{pmatrix}.$$

The column operator is used to describe the representing map for a sum of nonnegative forms. Let the forms $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathbf{F}(\mathfrak{H})$ be nonnegative with $\operatorname{dom} \mathfrak{h}_1 = \operatorname{dom} \mathfrak{h}_2$. For $i = 1, 2$, let $Q_i \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_i)$ be the representing maps for the forms $\mathfrak{h}_i$ with $\operatorname{dom} \mathfrak{h} = \operatorname{dom} Q_1 = \operatorname{dom} Q_2$, such that

$$\mathfrak{h}_1[\varphi, \psi] = (Q_1\varphi, Q_1\psi), \quad \mathfrak{h}_2[\varphi, \psi] = (Q_2\varphi, Q_2\psi), \tag{4.8}$$

where $\varphi, \psi \in \operatorname{dom} \mathfrak{h}_1 = \operatorname{dom} \mathfrak{h}_2 = \operatorname{dom} Q_1 = \operatorname{dom} Q_2$. Then, $\operatorname{col}(Q_1, Q_2)$ is a representing map for the form $\mathfrak{h}_1 + \mathfrak{h}_2$, as follows with $\varphi, \psi \in \operatorname{dom} \mathfrak{h}_1 = \operatorname{dom} \mathfrak{h}_2$ from

$$\begin{aligned} (\mathfrak{h}_1 + \mathfrak{h}_2)[\varphi, \psi] &= (Q_1\varphi, Q_1\psi) + (Q_2\varphi, Q_2\psi) \\ &= (\operatorname{col}(Q_1, Q_2)\varphi, \operatorname{col}(Q_1, Q_2)\psi). \end{aligned} \tag{4.9}$$

If in (4.8), both Q_1 and Q_2 are minimal in $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively, it does not necessarily follow that $\operatorname{col}(Q_1, Q_2)$ is minimal in $\mathfrak{K}_1 \oplus \mathfrak{K}_2$. A characterization of minimality in the context of Theorem 4.1 will be given in Theorem 4.3. For this purpose, let $K \in \mathbf{B}(\mathfrak{K})$ be a nonnegative contraction and define the Hilbert spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$ by

$$\mathfrak{K}_1 = \overline{\operatorname{ran}}\,(I-K) \quad \text{and} \quad \mathfrak{K}_2 = \overline{\operatorname{ran}}\,K,$$

each provided with the inner product of $\mathfrak{K}$. They satisfy $\mathfrak{K} = \mathfrak{K}_1 + \mathfrak{K}_2$ with overlapping space $\mathfrak{K}_1 \cap \mathfrak{K}_2 = \overline{\operatorname{ran}}\,(I-K)K$; for a treatment of the corresponding de Branges–Rovnyak decompositions, see [16]. With the representations in (4.2), one defines $Q_1 = (I-K)^{\frac{1}{2}}Q$ and $Q_2 = K^{\frac{1}{2}}Q$. Then, $\operatorname{col}(Q_1, Q_2)$ is a representing map in $\mathbf{L}(\mathfrak{H}, \mathfrak{K}_1 \times \mathfrak{K}_2)$, so that for all $\varphi, \psi \in \operatorname{dom}\mathfrak{h} = \operatorname{dom} Q$

$$\mathfrak{h}[\varphi,\psi] = \left(\begin{pmatrix} (I-K)^{\frac{1}{2}} \\ K^{\frac{1}{2}} \end{pmatrix} Q\varphi, \begin{pmatrix} (I-K)^{\frac{1}{2}} \\ K^{\frac{1}{2}} \end{pmatrix} Q\psi \right). \tag{4.10}$$

Theorem 4.3. *Let the form $\mathfrak{h} \in \mathbf{F}(\mathfrak{H})$ be nonnegative with the representation (4.1), where $Q \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ is a minimal representing map, that is, $\overline{\operatorname{ran}}\,Q = \mathfrak{K}$. Let $K \in \mathbf{B}(\mathfrak{K})$ be a nonnegative contraction and assume that with respect to K, the form $\mathfrak{h}$ has the representation (4.10). Then, $(I-K)^{\frac{1}{2}}Q$ is an operator from $\mathfrak{H}$ into $\mathfrak{K}_1$ with dense range and $K^{\frac{1}{2}}Q$ is an operator from $\mathfrak{H}$ into $\mathfrak{K}_2$ with dense range. Moreover, the following statements are equivalent:*

(i) *the representation (4.10) is minimal;*
(ii) *K is an orthogonal projection.*

Proof. The representation (4.10) for the form $\mathfrak{h} \in \mathbf{F}(\mathfrak{H})$ follows from (4.2) and (4.9). By assumption, Q has dense range in $\mathfrak{K}$, which implies that $(I-K)^{\frac{1}{2}}Q$ has dense range in $\overline{\operatorname{ran}}\,(I-K)^{\frac{1}{2}} = \mathfrak{K}_1$, and similarly, $K^{\frac{1}{2}}Q$ has dense range in $\overline{\operatorname{ran}}\,K^{\frac{1}{2}} = \mathfrak{K}_2$.

Recall from [16] that $W = \operatorname{col}((I-K)^{\frac{1}{2}}, K^{\frac{1}{2}})$ is an isometry from $\mathfrak{K}$ into the Hilbert space $\mathfrak{K}_1 \oplus \mathfrak{K}_2$. Moreover, the isometry W is surjective if and only if K is an orthogonal projection; cf. [16, Proposition 2.8]. Finally, observe that due to $\overline{\operatorname{ran}}\,Q = \mathfrak{K}$, the mapping W is surjective if and only if the representing map $\operatorname{col}((I-K)^{\frac{1}{2}}Q, K^{\frac{1}{2}}Q)$ is minimal. ☐

Next, the interaction between the components in a sum decomposition will be considered. This interaction is measured in terms of parallel sums. To explain this, assume that $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathbf{F}(\mathfrak{H})$ are nonnegative with $\operatorname{dom}\mathfrak{h}_1 = \operatorname{dom}\mathfrak{h}_2$. The *parallel sum* of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ is a nonnegative form defined for $\varphi \in \operatorname{dom}\mathfrak{h}_1 = \operatorname{dom}\mathfrak{h}_2$ by

$$(\mathfrak{h}_1 : \mathfrak{h}_2)[\varphi] = \inf\{\mathfrak{h}_1[h+\varphi] + \mathfrak{h}_2[h] : h \in \operatorname{dom}\mathfrak{h}\}, \tag{4.11}$$

and polarization. The definition in (4.11) and a proof that it is actually a nonnegative form can be found in [22, Proposition 2.2]. Likewise, for nonnegative operators $A, B \in \mathbf{B}(\mathfrak{K})$, one defines the *parallel sum* $A : B \in \mathbf{B}(\mathfrak{K})$ for $\varphi \in \mathfrak{K}$ by

$$((A : B)\varphi, \varphi) = \inf\{(A(h+\varphi), h+\varphi) + (Bh, h) : h \in \mathfrak{K}\}, \tag{4.12}$$

and polarization. Recall that if $\operatorname{ran}(A+B)$ is closed, then it follows that

$$(A : B) = A(A+B)^{(-1)}B, \tag{4.13}$$

where $(A+B)^{(-1)}$ denotes the Moore–Penrose inverse of $A+B$. For further details, see [14, 18].

Proposition 4.4. *Let the forms* $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathbf{F}(\mathfrak{H})$ *be nonnegative with sum* $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$. *Let* $Q \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ *be a minimal representing map for the sum* $\mathfrak{h}$:

$$\mathfrak{h}[\varphi, \psi] = (Q\varphi, Q\psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{h}.$$

Let $K \in \mathbf{B}(\mathfrak{K})$ *be the nonnegative contraction for which:*

$$\mathfrak{h}_1[\varphi, \psi] = ((I-K)^{\frac{1}{2}} Q\varphi, (I-K)^{\frac{1}{2}} Q\psi), \quad \mathfrak{h}_2[\varphi, \psi] = (K^{\frac{1}{2}} Q\varphi, K^{\frac{1}{2}} Q\psi),$$

where $\varphi, \psi \in \operatorname{dom} \mathfrak{h} = \operatorname{dom} Q$. *Then, the parallel sum* $\mathfrak{h}_1 : \mathfrak{h}_2 \in \mathbf{F}(\mathfrak{H})$ *has the representation*

$$(\mathfrak{h}_1 : \mathfrak{h}_2)[\varphi, \psi] = (((I-K) : K)Q\varphi, Q\psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{h} = \operatorname{dom} Q. \tag{4.14}$$

Proof. Let $\varphi \in \operatorname{dom} Q$ and consider the quadratic form defined by (4.11). It follows from the definition in (4.11) and the representations (4.2) in Theorem 4.1 that

$$\begin{aligned}
(\mathfrak{h}_1 : \mathfrak{h}_2)[\varphi] &= \inf \left\{ \|(I-K)^{\frac{1}{2}} Q(h+\varphi)\|^2 + \|K^{\frac{1}{2}} Qh\|^2 : h \in \operatorname{dom} Q \right\} \\
&\geqslant \inf \left\{ \|(I-K)^{\frac{1}{2}} (Q\varphi - g)\|^2 + \|K^{\frac{1}{2}} g\|^2 : g \in \mathfrak{K} \right\} \\
&= \inf \{ ((I-K)f, f) + (Kg, g) : f + g = Q\varphi : f, g \in \mathfrak{K} \} \\
&= (((I-K) : K)Q\varphi, Q\varphi).
\end{aligned} \tag{4.15}$$

The last equality follows from the definition in (4.12).

The reverse inequality in (4.15) follows from the denseness of $\operatorname{ran} Q$ in $\mathfrak{K}$. To see this, let $\varepsilon > 0$ and select $g \in \mathfrak{K}$ such that

$$\|(I-K)^{\frac{1}{2}} (Q\varphi - g)\|^2 + \|K^{\frac{1}{2}} g\|^2 < (((I-K) : K)Q\varphi, Q\varphi) + \varepsilon.$$

Since $\operatorname{ran} Q$ is dense in $\mathfrak{K}$, one has $\lim_{n\to\infty} \|g - Qh_n\| = 0$ for some sequence $h_n \in \operatorname{dom} Q$. It follows that for every $\varepsilon > 0$, there exists $h \in \operatorname{dom} Q$ such that

$$\|(I-K)^{\frac{1}{2}} (Q(\varphi - h)\|^2 + \|K^{\frac{1}{2}} Qh\|^2 < (((I-K) : K)Q\varphi, Q\varphi) + 2\varepsilon.$$

Taking the infimum over all $h \in \operatorname{dom} Q$, this leads to

$$\begin{aligned}
&\inf \left\{ \|(I-K)^{\frac{1}{2}} Q(\varphi - h)\|^2 + \|K^{\frac{1}{2}} Qh\|^2 : h \in \operatorname{dom} Q \right\} \\
&\qquad\qquad\qquad \leqslant (((I-K) : K)Q\varphi, Q\varphi).
\end{aligned}$$

Therefore, equality prevails in (4.15) and this implies the statement by the polarization formula for forms. ☐

Thus, the representing map for the form $\mathfrak{h}_1 : \mathfrak{h}_2$ is given by $((I-K) : K)^{\frac{1}{2}} Q$, which involves the parallel sum of the operators $I - K$ and K. This result is a special case of the functional calculus developed in [30]. The parallel sum $\mathfrak{h}_1 : \mathfrak{h}_2$ in (4.14) measures the interaction of the forms $\mathfrak{h}_1$ and

$\mathfrak{h}_2$. The forms $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are called *mutually singular* if $\mathfrak{h}_1 : \mathfrak{h}_2 = 0$; cf. [22, Proposition 2.10], in which case there is no interaction between the forms $\mathfrak{h}_1$ and $\mathfrak{h}_2$.

Corollary 4.5. *Under the conditions of Proposition 4.4, the following statements are equivalent:*

(i) *the forms $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are mutually singular;*
(ii) *K is an orthogonal projection.*

Proof. It follows from (4.15) that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are mutually singular if and only if $((I-K) : K)^{\frac{1}{2}} Q = 0$. Since the representing map Q in Proposition 4.4 is assumed to be minimal, this is equivalent to $((I-K) : K)^{\frac{1}{2}} = 0$. Observe that

$$(I-K) : K = (I-K)K,$$

cf. (4.13), from which the assertion follows. □

Now return to the context of Theorem 4.1 and Theorem 4.3. The sum decompositions (4.3) of a form $\mathfrak{h}$ can be classified into two different categories: the forms $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are mutually singular (precisely when K is an orthogonal projection) or they are not mutually singular (precisely when K is not an orthogonal projection); see the remark following Theorem 2.5 in [35].

5 | LEBESGUE-TYPE DECOMPOSITIONS OF SEMIBOUNDED FORMS

The Lebesgue decomposition of a semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ in Section 3 provides a sum decomposition of $\mathfrak{t}$ into a closable part $\mathfrak{t}_{\rm reg}$ and a singular part $\mathfrak{t}_{\rm sing}$. Recall that $\mathfrak{t}_{\rm reg}$ is the largest of all closable semibounded forms below $\mathfrak{t}$. By means of the sum decompositions in Section 4, it is now possible to describe the general situation.

Based on the Lebesgue decomposition in Section 3, the following definition is quite natural.

Definition 5.1. Let the form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be semibounded. Then a sum decomposition of $\mathfrak{t}$, given by

$$\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2, \quad \operatorname{dom} \mathfrak{t} = \operatorname{dom} \mathfrak{t}_1 = \operatorname{dom} \mathfrak{t}_2, \tag{5.1}$$

is called a Lebesgue-type sum decomposition of $\mathfrak{t}$ if $\mathfrak{t}_1 \in \mathbf{F}(\mathfrak{H})$ is semibounded and closable, and $\mathfrak{t}_2 \in \mathbf{F}(\mathfrak{H})$ is nonnegative and singular.

The following characterization of all Lebesgue-type decompositions of a semibounded form is an immediate consequence of Theorem 4.1 and Lemma 2.3, by a reduction to nonnegative forms.

Theorem 5.2. *Let the form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be semibounded, let $c \leqslant m(\mathfrak{t})$, and assume that $\mathfrak{t}$ has the representation*

$$\mathfrak{t}[\varphi, \psi] = c(\varphi, \psi) + (Q\varphi, Q\psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{t} = \operatorname{dom} Q, \tag{5.2}$$

where $Q \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ is a representing map for $\mathfrak{t} - c$.

Let $K \in \mathbf{B}(\mathfrak{K})$ be a nonnegative contraction that satisfies the conditions

$$\operatorname{clos}\{k \in \overline{\operatorname{ran}}\,(I-K) : (I-K)^{\frac{1}{2}}k \in \operatorname{dom} Q^*\} = \overline{\operatorname{ran}}\,(I-K), \tag{5.3}$$

$$\operatorname{ran} K^{\frac{1}{2}} \cap \operatorname{dom} Q^* \subset \ker Q^*, \tag{5.4}$$

and define the forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ by

$$\begin{aligned} \mathfrak{t}_1[\varphi,\psi] &= c(\varphi,\psi) + ((I-K)^{\frac{1}{2}}Q\varphi,(I-K)^{\frac{1}{2}}Q\psi), \\ \mathfrak{t}_2[\varphi,\psi] &= (K^{\frac{1}{2}}Q\varphi,K^{\frac{1}{2}}Q\psi). \end{aligned} \tag{5.5}$$

Then, the sum $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$ in (5.1) is a Lebesgue-type decomposition of $\mathfrak{t}$ in the sense of Definition 5.1.

Conversely, let the sum $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$ in (5.1) be a Lebesgue-type decomposition of $\mathfrak{t}$ in the sense of Definition 5.1. Then, there exists a nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$ such that (5.3), (5.4), and (5.5) are satisfied.

Proof. According to (5.2) the form $\mathfrak{h} = \mathfrak{t} - c \in \mathbf{F}(\mathfrak{H})$ is nonnegative with representing map Q. Assume $K \in \mathbf{B}(\mathfrak{K})$ is a nonnegative contraction that satisfies (5.3) and (5.4). Define the nonnegative forms $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathbf{F}(\mathfrak{H})$ by

$$\mathfrak{h}_1[\varphi,\psi] = ((I-K)^{\frac{1}{2}}Q\varphi,(I-K)^{\frac{1}{2}}Q\psi), \quad \mathfrak{h}_2[\varphi,\psi] = (K^{\frac{1}{2}}Q\varphi,K^{\frac{1}{2}}Q\psi).$$

Then, by Theorem 4.1, one has $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$. Conditions (5.3) and (5.4) guarantee that $(I-K)^{\frac{1}{2}}Q$ is regular and $K^{\frac{1}{2}}Q$ is singular; cf. [16, Lemma 4.1, Lemma 4.3]. Hence, by Lemma 2.3, it follows that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are regular and singular, respectively. Setting $\mathfrak{t}_1 = \mathfrak{h}_1 + c$ and $\mathfrak{t}_2 = \mathfrak{h}_2$, then $\mathfrak{t}_1$ is semibounded and $\mathfrak{t}_2$ is nonnegative. It is now clear that $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$ is a Lebesgue-type decomposition of $\mathfrak{t}$.

For the converse, let $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be a semibounded form and let it have a Lebesgue-type decomposition (5.1). Since the form $\mathfrak{t}$ is assumed to have the representation (5.2), one notes that

$$\mathfrak{t} - c = \mathfrak{h}_1 + \mathfrak{h}_2, \quad \mathfrak{h}_1 = \mathfrak{t}_1 - c \geqslant 0, \quad \mathfrak{h}_2 = \mathfrak{t}_2 \geqslant 0,$$

is a sum decomposition of the nonnegative form $\mathfrak{h} = \mathfrak{t} - c$ into the nonnegative forms $\mathfrak{h}_1$ and $\mathfrak{h}_2$. By Theorem 4.1, there exists a nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$ such that (4.2) holds. Hence, it follows that (5.5) is then satisfied. Since $\mathfrak{t}_1$ is regular and $\mathfrak{t}_2$ is singular, it is clear that $(I-K)^{\frac{1}{2}}Q$ is regular and $K^{\frac{1}{2}}Q$ is singular. Thus, by [16, Lemma 4.1, Lemma 4.3], the conditions (5.3) and (5.4) are satisfied. □

It is a consequence of Proposition 4.4 and Corollary 4.5 that the interaction between the components in a Lebesgue-type decomposition (5.1) can be specified in the following sense.

Corollary 5.3. *Let the conditions of Theorem 5.2 be satisfied. Then the parallel sum $((\mathfrak{t}_1 - c) : \mathfrak{t}_2) \in \mathbf{F}(\mathfrak{H})$ has the representation*

$$((\mathfrak{t}_1 - c) : \mathfrak{t}_2)[\varphi,\psi] = (((I-K) : K)Q\varphi, Q\psi), \quad \varphi,\psi \in \operatorname{dom} \mathfrak{h} = \operatorname{dom} Q.$$

In particular, the following statements are equivalent:

(i) *the forms* $\mathfrak{t}_1 - c$ *and* $\mathfrak{t}_2$ *are mutually singular;*
(ii) K *is an orthogonal projection.*

For the convenience of the reader, the case of orthogonal projections in Theorem 5.2 will be considered separately. According to Corollary 5.3, in this case, there is only a trivial interaction between the components $\mathfrak{t}_1 - c$ and $\mathfrak{t}_2$,

Corollary 5.4. *Let the form* $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ *be semibounded, let* $c \leqslant m(\mathfrak{t})$*, and let* $\mathfrak{t}$ *have the representation (5.2) with representing map* $Q \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$*. Let* $P \in \mathbf{B}(\mathfrak{K})$ *be an orthogonal projection that satisfies the conditions*

$$\operatorname{clos}(\ker P \cap \operatorname{dom} Q^*) = \ker P, \tag{5.6}$$

$$\operatorname{ran} P \cap \operatorname{dom} Q^* \subset \ker Q^*, \tag{5.7}$$

and define the forms $\mathfrak{t}_1$ *and* $\mathfrak{t}_2$ *by*

$$\begin{aligned} \mathfrak{t}_1[\varphi, \psi] &= c(\varphi, \psi) + ((I - P)Q\varphi, (I - P)Q\psi), \\ \mathfrak{t}_2[\varphi, \psi] &= (PQ\varphi, PQ\psi). \end{aligned} \tag{5.8}$$

Then, the sum $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$ *in (5.1) is a Lebesgue-type decomposition of* $\mathfrak{t}$*. In particular, if* $P = P_0$ *is the orthogonal projection onto* $\operatorname{mul} Q^{**}$*, then (5.6) and (5.7) are satisfied and (5.5) leads to the Lebesgue decomposition.*

Proof. If $K \in \mathbf{B}(\mathfrak{K})$ is an orthogonal projection, then the conditions (5.3) and (5.4) are equivalent with the conditions (5.6) and (5.7); see [16, Lemma 4.1, Lemma 4.2]. If P_0 is the orthogonal projection onto $\operatorname{mul} Q^{**}$, then

$$\operatorname{ran} P_0 = \ker (I - P_0) = \operatorname{mul} Q^{**} \quad \text{and} \quad \operatorname{ran}(I - P_0) = \ker P_0 = \overline{\operatorname{dom}} \, Q^*.$$

With $P = P_0$, the conditions (5.6) and (5.7) are satisfied. Comparing with (3.2), one sees that this choice corresponds to the Lebesgue decomposition. □

Recall that the representing map Q of a semibounded form $\mathfrak{t}$ may always be taken minimal. If this is the case, then the conditions (5.4) and (5.7) read as

$$\operatorname{ran} K^{\frac{1}{2}} \cap \operatorname{dom} Q^* = \{0\} \quad \text{and} \quad \operatorname{ran} P \cap \operatorname{dom} Q^* = \{0\},$$

respectively. The following theorem was originally established in the context of pairs of nonnegative forms in [22].

Theorem 5.5. *Let the form* $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ *be semibounded. Then, the following statements are equivalent:*

(i) *the Lebesgue decomposition of* $\mathfrak{t}$ *is the only Lebesgue-type decomposition of the form* $\mathfrak{t}$*;*
(ii) *the form* $\mathfrak{t}_{\mathrm{reg}}$ *is bounded.*

Proof. (i) ⇒ (ii) Assume that any Lebesgue-type decomposition $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$ of $\mathfrak{t}$ is equal to the Lebesgue decomposition of $\mathfrak{t}$. Then, in particular, any Lebesgue-type decomposition $Q = (I -$

$P)Q + PQ$, with P an orthogonal projection satisfying (5.3) and (5.4), gives the Lebesgue decomposition of Q. By [24, Theorem 6.1], this implies that Q_{reg} is bounded, so that also the form $\mathfrak{t}_{\mathrm{reg}}$ is bounded.

(ii) ⇒ (i) Assume that $\mathfrak{t}_{\mathrm{reg}}$ is bounded. Let the form $\mathfrak{t}$ have the representation

$$\mathfrak{t}[\varphi, \psi] = c(\varphi, \psi) + (Q\varphi, Q\psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{t} = \operatorname{dom} Q,$$

where $c \leqslant m(\mathfrak{t})$, Q is a representing map, and assume that $\overline{\operatorname{ran}}\, Q = \mathfrak{K}$. Then, the regular part Q_{reg} is bounded or, equivalently, $\operatorname{dom} Q^*$ is closed; cf. [24, Theorem 6.1] or [16, Theorem 5.7]. Now let $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$ be a Lebesgue-type decomposition of $\mathfrak{t}$ of the form in Theorem 5.2 with a nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$. It follows from $\operatorname{mul} Q^{**} \subset \ker\,(I-K)^{\frac{1}{2}}$ or, equivalently, $\operatorname{ran}\,(I-K)^{\frac{1}{2}} \subset \operatorname{dom} Q^*$, that

$$\operatorname{ran}\,(I-K)^{\frac{1}{2}} \cap \operatorname{ran} K^{\frac{1}{2}} \subset \operatorname{dom} Q^* \cap \operatorname{ran} K^{\frac{1}{2}} = \{0\},$$

which shows that K is an orthogonal projection. This gives an orthogonal Lebesgue-type decomposition $Q = (I-K)Q + KQ$ for Q. Since Q_{reg} is bounded, one sees that $(I-K)Q = Q_{\mathrm{reg}}$ by [24]. Hence, the Lebesgue-type decomposition of $\mathfrak{t}$ is equal to the Lebesgue decomposition. □

At the end of this section, there is a brief discussion concerning the connection between the Lebesgue-type decompositions of a form $\mathfrak{t}$ and the Lebesgue-type decompositions of its representing map Q (see [16]). Let $K \in \mathbf{B}(\mathfrak{K})$ be any nonnegative contraction, then one can write Q as

$$Q = Q_1 + Q_2, \quad Q_1 = (I-K)Q, \quad Q_2 = KQ. \tag{5.9}$$

Recall that the decomposition (5.9) is a Lebesgue-type decomposition of Q if and only if K satisfies

$$(I-K)Q \quad \text{is closable and} \quad KQ \text{ is singular.} \tag{5.10}$$

However, the nonnegative contraction K generates a Lebesgue-type decomposition $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$ via (5.8) if and only if

$$(I-K)^{\frac{1}{2}}Q \text{ regular} \quad \text{and} \quad K^{\frac{1}{2}}Q \text{ singular;} \tag{5.11}$$

cf. (5.3) and (5.4). The conditions in (5.10) and (5.11) are equivalent when K is an orthogonal projection. In the general case when K is a nonnegative contraction, there is, for closability, only the implication

$$(I-K)Q \text{ is closable} \quad \Rightarrow \quad (I-K)^{\frac{1}{2}}Q \text{ is closable} \tag{5.12}$$

(use [16, Corollary 4.2]), while both statements are equivalent with the closability of T when $\|K\| < 1$. Likewise, there is, for singularity, only the implication

$$K^{\frac{1}{2}}Q \text{ is singular} \quad \Rightarrow \quad KQ \text{ is singular.} \tag{5.13}$$

Furthermore, it should be mentioned that the converse statements in (5.12) and (5.13) do not hold in general. In fact, there is an example that shows this simultaneously; see [20].

An extension of the Lebesgue-type decompositions for a single semibounded form to the case of a pair of nonnegative bounded operators or forms (see [1, 35] and [22]) has connections with a recent approach of Lebesgue-type decompositions for a pair of measures via reproducing kernel Hilbert spaces in [6].

6 | REPRESENTATION THEOREMS FOR SEMIBOUNDED FORMS

Let $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be a semibounded form. If $\mathfrak{t}$ is closed then the well-known representation theorem (see [7, 27]) asserts that there is a semibounded self-adjoint relation H in $\mathfrak{H}$ with the same lower bound as $\mathfrak{t}$, such that for all elements $\{\varphi, \varphi'\} \in H$

$$\mathfrak{t}[\varphi, \psi] = (\varphi', \psi) \quad \text{for all} \quad \psi \in \operatorname{dom} \mathfrak{t}.$$

In this section, it will be shown that for an arbitrary semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$, a similar observation can be made when the above identity is interpreted in a limiting sense. Moreover, the regular part $\mathfrak{t}_{\mathrm{reg}}$ of $\mathfrak{t}$ is represented by the same semibounded self-adjoint relation. This version of the representation of semibounded forms goes back to Arendt and ter Elst (in the setting of sectorial forms); see [3, Theorem 3.2, Proposition 3.10]. The representing map for $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ allows a simple and direct exposition of the arguments.

Theorem 6.1. *Let the form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be semibounded. Then the following statements hold:*

(A) *There exists a semibounded relation $S_\mathfrak{t} \in \mathbf{L}(\mathfrak{H})$ that is bounded from below by $m(\mathfrak{t})$, such that $\operatorname{dom} S_\mathfrak{t} \subset \operatorname{dom} \mathfrak{t}$ and for each $\{\varphi, \varphi'\} \in S_\mathfrak{t}$*

$$\mathfrak{t}[\varphi, \psi] = (\varphi', \psi) \quad \textit{for all} \quad \psi \in \operatorname{dom} \mathfrak{t}. \tag{6.1}$$

The relation $S_\mathfrak{t}$ admits the following maximality property: if $H \in \mathbf{L}(\mathfrak{H})$ is any symmetric relation, such that $\operatorname{dom} H \subset \operatorname{dom} \mathfrak{t}$ and for each $\{\varphi, \varphi'\} \in H$, the statement (6.1) holds, then $H \subset S_\mathfrak{t}$.

(B) *There exists a semibounded self-adjoint extension $\widetilde{A}_\mathfrak{t} \in \mathbf{L}(\mathfrak{H})$ of $S_\mathfrak{t}$ that is bounded from below by $m(\mathfrak{t})$, such that for each $\{\varphi, \varphi'\} \in \widetilde{A}_\mathfrak{t}$, there exists a sequence $\varphi_n \in \operatorname{dom} \mathfrak{t}$ for which*

$$\varphi_n \to_\mathfrak{t} \varphi \quad \textit{and} \quad \mathfrak{t}[\varphi_n, \psi] \to (\varphi', \psi) \quad \textit{for all} \quad \psi \in \operatorname{dom} \mathfrak{t}. \tag{6.2}$$

The self-adjoint relation $\widetilde{A}_\mathfrak{t}$ is uniquely determined: if $H \in \mathbf{L}(\mathfrak{H})$ is any self-adjoint relation, such that for each $\{\varphi, \varphi'\} \in H$, there exists a sequence $\varphi_n \in \operatorname{dom} \mathfrak{t}$ for which the statement (6.2) holds, then $H = \widetilde{A}_\mathfrak{t}$.

(C) *For all $c \leqslant m(\mathfrak{t})$, the symmetric relation $S_\mathfrak{t}$ and the self-adjoint relation $\widetilde{A}_\mathfrak{t}$ admit the representations*

$$S_\mathfrak{t} = Q_c^* Q_c + c \quad \textit{and} \quad \widetilde{A}_\mathfrak{t} = Q_c^* Q_c^{**} + c, \tag{6.3}$$

where $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ is a representing map for $\mathfrak{t} - c$. In particular, the relations $S_\mathfrak{t}$ and $\widetilde{A}_\mathfrak{t}$ do not depend on the choice of $c \leqslant m(\mathfrak{t})$ and the representing map Q_c.

Proof. (A) Fix $c \leqslant m(\mathfrak{t})$ and let $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be some representing map for $\mathfrak{t} - c$:

$$\mathfrak{t}[\varphi, \psi] = (Q_c\varphi, Q_c\psi) + c(\varphi, \psi), \quad \varphi, \psi \in \operatorname{dom} Q_c = \operatorname{dom} \mathfrak{t}. \tag{6.4}$$

Define $S_c = Q_c^* Q_c + c$. Then, clearly, $S_c \geqslant c$ and $\operatorname{dom} S_c \subset \operatorname{dom} \mathfrak{t}$.

Now let $\{\varphi, \varphi'\} \in Q_c^* Q_c$. Then, $\{Q_c\varphi, \varphi'\} \in Q_c^*$ and if $\{\psi, Q_c\psi\} \in Q_c$, then $(Q_c\varphi, Q_c\psi) = (\varphi', \psi)$. From (6.4), one gets

$$\mathfrak{t}[\varphi, \psi] = (\varphi', \psi) + c(\varphi, \psi) = (\varphi' + c\varphi, \psi).$$

Therefore, $S_c = Q_c^* Q_c + c$ satisfies (6.1).

Next let H with $\operatorname{dom} H \subset \operatorname{dom} \mathfrak{t}$ satisfy (6.1). If $\{\varphi, \varphi'\} \in H$ and $\psi \in \operatorname{dom} \mathfrak{t}$ then using (6.4), one obtains

$$(\varphi', \psi) = (Q_c\varphi, Q_c\psi) + c(\varphi, \psi) \quad \text{or} \quad (\varphi' - c\varphi, \psi) = (Q_c\varphi, Q_c\psi).$$

Since this holds for all $\psi \in \operatorname{dom} Q_c$, one concludes that

$$\{Q_c\varphi, \varphi' - c\varphi\} \in Q_c^* \quad \text{or} \quad \{\varphi, \varphi' - c\varphi\} \in Q_c^* Q_c.$$

Therefore, $H - c \subset Q_c^* Q_c \Leftrightarrow H \subset S_c$.

(B) Define $A_c = Q_c^* Q_c^{**} + c$ and let $\{\varphi, \varphi'\} \in A_c$, so that $\{\varphi, \varphi' - c\varphi\} \in Q_c^* Q_c^{**}$. Then $\{\varphi, g\} \in Q_c^{**}$ and $\{g, \varphi' - c\varphi\} \in Q_c^*$ for some $g \in \mathfrak{K}_c$. Hence, there exists a sequence $\varphi_n \in \operatorname{dom} \mathfrak{t}$ such that $\varphi_n \to \varphi$ in $\mathfrak{H}$ and $Q_c\varphi_n \to g$ in $\mathfrak{K}_c$. Then, $(Q_c\varphi_n)$ is a Cauchy sequence and

$$\mathfrak{t}[\varphi_n - \varphi_m] = \|Q_c(\varphi_n - \varphi_m)\|_{\mathfrak{K}_c}^2 + c\|\varphi_n - \varphi_m\|_{\mathfrak{H}}^2 \to 0, \quad n, m \to \infty.$$

Thus, $\varphi_n \to_{\mathfrak{t}} \varphi$. On the other hand, for all $\psi \in \operatorname{dom} \mathfrak{t}$, one has

$$\lim_{n\to\infty} (\mathfrak{t} - c)[\varphi_n, \psi] = \lim_{n\to\infty} (Q_c\varphi_n, Q_c\psi)_{\mathfrak{K}_c} = (g, Q_c\psi)_{\mathfrak{K}_c}. \tag{6.5}$$

It follows from $\{g, \varphi' - c\varphi\} \in Q_c^*$ and $\psi \in \operatorname{dom} Q_c$ that

$$(g, Q_c\psi)_{\mathfrak{K}_c} = (\varphi' - c\varphi, \psi)_{\mathfrak{H}}. \tag{6.6}$$

By combining (6.5) and (6.6), one obtains for all $\psi \in \operatorname{dom} \mathfrak{t}$,

$$\lim_{n\to\infty} \mathfrak{t}[\varphi_n, \psi] = (\varphi' - c\varphi, \psi)_{\mathfrak{H}} + c \lim_{n\to\infty} (\varphi_n, \psi)_{\mathfrak{H}} = (\varphi', \psi)_{\mathfrak{H}}.$$

Thus, $A_c = Q_c^* Q_c^{**} + c$ satisfies (6.2).

Next, let $H = H^*$ be such that for each $\{\varphi, \varphi'\} \in H$, there exist $\varphi_n \in \operatorname{dom} \mathfrak{t}$ satisfying (6.2). Then $\varphi_n \to_{\mathfrak{t}} \varphi$ implies that

$$\varphi_n \to \varphi \quad \text{and} \quad \|Q_c(\varphi_n - \varphi_m)\|_{\mathfrak{K}_c} \to 0.$$

Thus, $(Q_c\varphi_n)$ is a Cauchy sequence in $\mathfrak{K}_c$ and $Q_c\varphi_n \to g$ for some $g \in \mathfrak{K}_c$. Hence, $\{\varphi, g\} = \lim_{n\to\infty}\{\varphi_n, Q_c\varphi_n\} \in Q_c^{**}$. Now from the second condition in (6.2), one gets for all $\psi \in \operatorname{dom} \mathfrak{t}$

$$\lim_{n\to\infty} (\mathfrak{t} - c)[\varphi_n, \psi] = (\varphi', \psi)_{\mathfrak{H}} - c \lim_{n\to\infty} (\varphi_n, \psi)_{\mathfrak{H}} = (\varphi' - c\varphi, \psi)_{\mathfrak{H}}.$$

Thus, for all $\psi \in \operatorname{dom} \mathfrak{t} = \operatorname{dom} Q_c$,

$$(g, Q_c\psi)_{\mathfrak{K}_c} = \lim_{n\to\infty} (Q_c\varphi_n, Q_c\psi)_{\mathfrak{K}_c} = \lim_{n\to\infty} (\mathfrak{t} - c)[\varphi_n, \psi] = (\varphi' - c\varphi, \psi)_{\mathfrak{H}}.$$

Hence, $\{g, \varphi' - c\varphi\} \in Q_c^*$ and, since $\{\varphi, g\} \in Q_c^{**}$, one has

$$\{\varphi, \varphi' - c\varphi\} \in Q_c^* Q_c^{**}.$$

Thus, $H - c \subset Q_c^* Q_c^{**}$ or, equivalently, $H \subset A_c$. Since both relations are self-adjoint, this gives $H = A_c$.

(C) The proofs of (A) and (B) show that $S_c = Q_c^* Q_c + c$ and $A_c = Q_c^* Q_c^{**} + c$ satisfy the desired conditions with a choice of $c \leqslant m(\mathfrak{t})$ and a representing map Q_c for $\mathfrak{t} - c$.

On the other hand, if $Q_c' = VQ_c$ with a unitary map V from $\overline{\operatorname{ran}}\, Q_c$ onto $\overline{\operatorname{ran}}\, Q_c'$, then

$$Q_c^* Q_c + c = (Q_c')^* Q_c' + c; \quad Q_c^* Q_c^{**} + c = (Q_c')^* (Q_c')^{**} + c.$$

Thus, S_c and A_c do not depend on the choice of Q_c.

Finally, if $c' \leqslant \gamma$ and $Q_{c'}$ is some representing map for $\mathfrak{t} - c'$, then the proof of (A) shows that

$$S_{c'} = Q_{c'}^* Q_{c'} + c'$$

also satisfies the conditions in (A), and thus, $S_{c'} \subset S_c$. By symmetry, one also has $S_c \subset S_{c'}$. Therefore, $S_{c'} = S_c$. This proves that $S_{\mathfrak{t}}$ in (6.3) does not depend on $c \leqslant m(\mathfrak{t})$. In particular, $S_{\mathfrak{t}} = S_c \geqslant c$ for each $c \leqslant m(\mathfrak{t})$, so that $S_{\mathfrak{t}} \geqslant m(\mathfrak{t})$. A similar reasoning applies to A_c: $\widetilde{A}_{\mathfrak{t}} = Q_c^* Q_c^{**} + c$ for all $c \leqslant m(\mathfrak{t})$ and thus $\widetilde{A}_{\mathfrak{t}} \geqslant m(\mathfrak{t})$.

This completes the proof. □

According to Theorem 6.1 any semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ generates a semibounded (symmetric) relation $S_{\mathfrak{t}}$ and a semibounded self-adjoint relation $\widetilde{A}_{\mathfrak{t}}$ in $\mathbf{L}(\mathfrak{H})$. Moreover, it follows from (6.3) that they satisfy the following identities:

$$\operatorname{mul} S_{\mathfrak{t}} = (\operatorname{dom} \mathfrak{t})^{\perp}, \quad \operatorname{mul} \widetilde{A}_{\mathfrak{t}} = (\operatorname{dom} \mathfrak{t})^{\perp}. \tag{6.7}$$

The semibounded relation $\widetilde{A}_{\mathfrak{t}}$ in Theorem 6.1 is self-adjoint with $\overline{\operatorname{dom}}\, \widetilde{A}_{\mathfrak{t}} = \overline{\operatorname{dom}}\, \mathfrak{t}$; its orthogonal operator part $(\widetilde{A}_{\mathfrak{t}})_{\mathrm{reg}}$ is given by $(\widetilde{A}_{\mathfrak{t}})_{\mathrm{reg}} = (I - P_{\mathrm{reg}})\widetilde{A}_{\mathfrak{t}}$, where P_{reg} is the orthogonal projection from $\mathfrak{H}$ onto $\operatorname{mul} \widetilde{A}_{\mathfrak{t}}$. Therefore, in (6.2) one may replace (φ', ψ) by $((\widetilde{A}_{\mathfrak{t}})_{\mathrm{reg}}\varphi, \psi)$, so that (6.2) reads

$$\varphi_n \to_{\mathfrak{t}} \varphi \quad \text{and} \quad \mathfrak{t}[\varphi_n, \psi] \to ((\widetilde{A}_{\mathfrak{t}})_{\mathrm{reg}}\varphi, \psi) \quad \text{for all} \quad \psi \in \operatorname{dom} \mathfrak{t},$$

where $(\widetilde{A}_{\mathfrak{t}})_{\mathrm{reg}}$ is a semibounded self-adjoint operator in $\mathfrak{H} \ominus \operatorname{mul} \widetilde{A}_{\mathfrak{t}}$. In general, the inclusion $S_{\mathfrak{t}} \subset \widetilde{A}_{\mathfrak{t}}$ is not an equality: for instance, this is the case when the form $\mathfrak{t}$ is defined by a semibounded relation. In the context of closed and closable forms, the main result leads to more special results that will be discussed now. A further discussion of the main theorem and its connections to the case of closed or closable forms can be found in Theorem 6.5 below.

When the form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ in Theorem 6.1 is closed, then the statements simplify: the relations $S_{\mathfrak{t}}$ and $\widetilde{A}_{\mathfrak{t}}$ coincide and will be denoted by $A_{\mathfrak{t}}$. Thus, Theorem 6.1 leads to the so-called first representation theorem as a straightforward consequence; cf. [7, 27].

Corollary 6.2. *Let the form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be semibounded and closed. Then there exists a semibounded self-adjoint relation $A_{\mathfrak{t}} \in \mathbf{L}(\mathfrak{H})$ with lower bound $m(\mathfrak{t})$, such that* $\operatorname{dom} A_{\mathfrak{t}} \subset \operatorname{dom} \mathfrak{t}$ *and for each $\{\varphi, \varphi'\} \in A_{\mathfrak{t}}$*

$$\mathfrak{t}[\varphi, \psi] = (\varphi', \psi) \quad \textit{for all} \quad \psi \in \operatorname{dom} \mathfrak{t}. \tag{6.8}$$

If $H \in \mathbf{L}(\mathfrak{H})$ is a self-adjoint relation, such that $\operatorname{dom} H \subset \operatorname{dom} \mathfrak{t}$ *and for each $\{\varphi, \varphi'\} \in H$, the statement (6.8) holds, then $H = A_{\mathfrak{t}}$. Moreover, for all $c \leqslant m(\mathfrak{t})$, the self-adjoint relation $A_{\mathfrak{t}}$ is given by*

$$A_{\mathfrak{t}} = Q_c^* Q_c + c, \tag{6.9}$$

where the representing map $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ for $\mathfrak{t} - c$ is closed.

Note that it follows from (6.7) or (6.9) that the following identities

$$\operatorname{mul} A_{\mathfrak{t}} = (\operatorname{dom} \mathfrak{t})^{\perp} \quad \text{and} \quad \overline{\operatorname{dom}} A_{\mathfrak{t}} = \overline{\operatorname{dom}}\, \mathfrak{t}$$

hold. Recall that the identity (6.8) can be written for each $\varphi \in \operatorname{dom} A_{\mathfrak{t}}$ as

$$\mathfrak{t}[\varphi, \psi] = ((A_{\mathfrak{t}})_{\mathrm{reg}} \varphi, \psi) \quad \text{for all} \quad \psi \in \operatorname{dom} \mathfrak{t},$$

where the orthogonal operator part $(A_{\mathfrak{t}})_{\mathrm{reg}}$ is a semibounded self-adjoint operator in $\mathfrak{H} \ominus \operatorname{mul} A_{\mathfrak{t}}$. It is now simple to see that any semibounded self-adjoint relation in a Hilbert space $\mathfrak{H}$ is connected to a closed semibounded form; cf. [7].

Lemma 6.3. *Let $A \in \mathbf{L}(\mathfrak{H})$ be a self-adjoint relation with lower bound $\gamma \in \mathbb{R}$ and let $c \leqslant \gamma$. Then, the form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$, defined by*

$$\operatorname{dom} \mathfrak{t} = \operatorname{dom} (A_{\mathrm{reg}} - c)^{\frac{1}{2}}$$

and

$$\mathfrak{t}[\varphi, \psi] = ((A_{\mathrm{reg}} - c)^{\frac{1}{2}} \varphi, (A_{\mathrm{reg}} - c)^{\frac{1}{2}} \psi) + c(\varphi, \psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{t}_A,$$

is closed, independent of $c \leqslant \gamma$, and has lower bound $m(\mathfrak{t}) = \gamma$. Moreover, the unique semibounded self-adjoint relation in $\mathbf{L}(\mathfrak{H})$ associated with $\mathfrak{t}$ is A.

Proof. Note that the mapping, defined by

$$\varphi \mapsto (A_{\mathrm{reg}} - c)^{\frac{1}{2}} \varphi, \quad \varphi \in \operatorname{dom} (A_{\mathrm{reg}} - c)^{\frac{1}{2}},$$

takes $\operatorname{dom} (A_{\mathrm{reg}} - c)^{\frac{1}{2}}$ into the Hilbert space $\mathfrak{H} \ominus \operatorname{mul} A$ with dense range. When $\varphi \in \operatorname{dom} A$, it follows that $\mathfrak{t}[\varphi, \psi] = (A_{\mathrm{reg}} \varphi, \psi)$. ☐

In Corollary 6.2, it has been shown that for every closed semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$, there is a unique semibounded self-adjoint relation $H = A_{\mathfrak{t}}$ in $\mathbf{L}(\mathfrak{H})$ such that (6.8) holds. Furthermore, according to Lemma 6.3, for each semibounded self-adjoint relation $H \in \mathbf{L}(\mathfrak{H})$, there is a closed semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ such that $H = A_{\mathfrak{t}}$ holds. In the following, the notation $\mathfrak{t}(H)$ is used to denote this one-to-one correspondence.

Let $\mathfrak{t}_1 \in \mathbf{F}(\mathfrak{H})$ and $\mathfrak{t}_2 \in \mathbf{F}(\mathfrak{H})$ be semibounded forms with representations

$$\begin{cases} \mathfrak{t}_1[\varphi, \psi] = c(\varphi, \psi) + (Q_1 \varphi, Q_1 \psi), & \varphi, \psi \in \operatorname{dom} \mathfrak{t}_1 = \operatorname{dom} Q_1, \\ \mathfrak{t}_2[\varphi, \psi] = c(\varphi, \psi) + (Q_2 \varphi, Q_2 \psi), & \varphi, \psi \in \operatorname{dom} \mathfrak{t}_2 = \operatorname{dom} Q_2, \end{cases}$$

with representing maps $Q_1 \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_1)$ and $Q_2 \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_2)$. Recall that $\mathfrak{t}_1 \leqslant \mathfrak{t}_2$ if and only if $Q_1 \prec_c Q_2$; see (2.1), (2.12), and (2.13). Now assume, in addition, that $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are closed forms or, equivalently, that Q_1 and Q_2 are closed operators, so that the corresponding semibounded self-adjoint relations H_1 and H_2 in $\mathbf{L}(\mathfrak{H})$ are given by

$$H_1 = Q_1^* Q_1 + c, \quad H_2 = Q_2^* Q_2 + c.$$

Recall from [7, Definition 5.2.8] that two nonnegative relations H_1 and H_2 in $\mathbf{L}(\mathfrak{H})$ satisfy $H_1 \leqslant H_2$ when

$$\operatorname{dom} H_2^{\frac{1}{2}} \subset \operatorname{dom} H_1^{\frac{1}{2}} \quad \text{and} \quad \|(H_{1,\mathrm{reg}})^{\frac{1}{2}} f\| \leqslant \|(H_{2,\mathrm{reg}})^{\frac{1}{2}}\|, \ f \in \operatorname{dom} H_2^{\frac{1}{2}}.$$

According to (2.1) and [15, Theorem 2.2], one has the equivalent statements

$$\mathfrak{t}_1 \leqslant \mathfrak{t}_2 \quad \Leftrightarrow \quad Q_1 \prec_c Q_2 \quad \Leftrightarrow \quad H_1 \leqslant H_2.$$

In particular, one has for semibounded self-adjoint relations H and K in $\mathbf{L}(\mathfrak{H})$ that $H \leqslant K$ is equivalent to $\mathfrak{t}(H) \leqslant \mathfrak{t}(K)$ (here the above notation has been used).

When the form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ in Theorem 6.1 is closable, the results simplify in the sense that the description in (B) of that theorem can be stated directly. The following result is straightforward; the main tool is Lemma 2.4 in conjunction with the representations (6.3) in Theorem 6.1. Although Q_c is assumed to be closable, the difference between $Q_c^* Q_c$ and $Q_c^* Q_c^{**}$ is in general very large as will be shown in [20].

Corollary 6.4. *Let the form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be semibounded and closable. Let $\bar{\mathfrak{t}} \in \mathbf{F}(\mathfrak{H})$ be the closure of $\mathfrak{t}$. Then, the semibounded self-adjoint relation $A_{\bar{\mathfrak{t}}} \in \mathbf{L}(\mathfrak{H})$ is an extension of $S_{\mathfrak{t}}$ (introduced in Theorem 6.1) and it has the same lower bound $m(\mathfrak{t})$. For all $c \leqslant m(\mathfrak{t})$, the semibounded relation $S_{\mathfrak{t}}$ and the self-adjoint relation $A_{\bar{\mathfrak{t}}}$ admit the representations*

$$S_{\mathfrak{t}} = Q_c^* Q_c + c \quad \text{and} \quad A_{\bar{\mathfrak{t}}} = Q_c^* Q_c^{**} + c,$$

where $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ is a representing map for $\mathfrak{t} - c$. In particular, $A_{\bar{\mathfrak{t}}}$ coincides with the self-adjoint relation $\widetilde{A}_{\mathfrak{t}}$ in Theorem 6.1.

Now return to the general context of Theorem 6.1, where $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$. Fix $c \leqslant m(\mathfrak{t})$ and let $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be some representing map for $\mathfrak{t} - c$, so that

$$\mathfrak{t}[\varphi, \psi] = c(\varphi, \psi) + (Q_c \varphi, Q_c \psi), \quad \varphi, \psi \in \operatorname{dom} Q_c = \operatorname{dom} \mathfrak{t}. \tag{6.10}$$

In the general situation of Theorem 6.1, the representing map Q_c need not be closable. However, the product $Q_c^* Q_c^{**}$ makes sense and is a nonnegative self-adjoint relation in $\mathbf{L}(\mathfrak{H})$. To proceed, one turns to the Lebesgue decomposition of Q_c:

$$Q_c = Q_{c,\mathrm{reg}} + Q_{c,\mathrm{sing}}.$$

The closable component $Q_{c,\mathrm{reg}}$ defines a closable form $\mathfrak{t}_{\mathrm{reg}} \in \mathbf{F}(\mathfrak{H})$ that in the following will be denoted by

$$\mathfrak{r} = \mathfrak{t}_{\mathrm{reg}}.$$

It is clear that the closable form $\mathfrak{r}$ and its closure $\bar{\mathfrak{r}}$ have the representations

$$\mathfrak{r}[\varphi, \psi] = c(\varphi, \psi) + (Q_{c,\mathrm{reg}}\varphi, Q_{c,\mathrm{reg}}\psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{r} = \operatorname{dom} \mathfrak{t}, \tag{6.11}$$

and

$$\bar{\mathfrak{r}}[\varphi, \psi] = c(\varphi, \psi) + ((Q_{c,\mathrm{reg}})^{**}\varphi, (Q_{c,\mathrm{reg}})^{**}\psi), \quad \varphi, \psi \in \operatorname{dom} \bar{\mathfrak{r}}. \tag{6.12}$$

Observe that one may also apply Theorem 6.1 to the closable form $\mathfrak{r} \in \mathbf{F}(\mathfrak{H})$; it turns out that the semibounded self-adjoint relations generated by $\mathfrak{t}$ and $\mathfrak{r}$ coincide.

Theorem 6.5. *Let the form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be semibounded, let $c \leqslant m(\mathfrak{t})$, and let $\mathfrak{t} - c$ have a representing map $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, such that (6.10) holds. Let $\mathfrak{r} = \mathfrak{t}_{\mathrm{reg}}$ so that (6.11) and (6.12) are satisfied. Then the relations $S_{\mathfrak{r}}$ and $\widetilde{A}_{\mathfrak{r}}$ (corresponding to $S_{\mathfrak{t}}$ and $\widetilde{A}_{\mathfrak{t}}$ introduced for the form $\mathfrak{t}$ in Theorem 6.1) are given by*

$$S_{\mathfrak{r}} = (Q_{c,\mathrm{reg}})^* Q_{c,\mathrm{reg}} + c \quad \textit{and} \quad \widetilde{A}_{\mathfrak{r}} = (Q_{c,\mathrm{reg}})^*(Q_{c,\mathrm{reg}})^{**} + c, \tag{6.13}$$

and they satisfy

$$S_{\mathfrak{t}} \subset S_{\mathfrak{r}} \quad \textit{and} \quad \widetilde{A}_{\mathfrak{t}} = \widetilde{A}_{\mathfrak{r}} = A_{\bar{\mathfrak{r}}}. \tag{6.14}$$

In particular, the relations $S_{\mathfrak{r}}$ and $A_{\bar{\mathfrak{r}}}$ in (6.13) and (6.14) do not depend on the choice of $c \leqslant m(\mathfrak{t})$ and the representing map Q_c.

Proof. The inclusion $S_{\mathfrak{t}} \subset S_{\mathfrak{r}}$ in (6.14) is equivalent to $Q_c^* Q_c \subset (Q_{c,\mathrm{reg}})^* Q_{c,\mathrm{reg}}$ and the equality $\widetilde{A}_{\mathfrak{t}} = \widetilde{A}_{\mathfrak{r}}$ is equivalent to $Q_c^* Q_c^{**} = (Q_{c,\mathrm{reg}})^*(Q_{c,\mathrm{reg}})^{**}$. This last inclusion and equality are both easily established by means of the formula $Q_{c,\mathrm{reg}} = (I - P)Q_c$, where P is the orthogonal projection onto $\operatorname{mul} Q_c^{**} = (\operatorname{dom} Q_c^*)^{\perp}$; for details, see [15, Appendix]. The equality $\widetilde{A}_{\mathfrak{r}} = A_{\bar{\mathfrak{r}}}$ holds by Corollary 6.4. The remaining statements follow from Theorem 6.1. ☐

Closely connected to the topics in this section are the semibounded forms induced by semibounded operators or relations. Their representing maps can be used to define the extremal extensions, including the Friedrichs extension and the Kreĭn type extension; cf. [2, 4, 5, 7, 8, 21, 40]. This connection with the work of Sebestyén and Stochel and their coworkers (see, for instance, [29, 32, 33]) can be found in [17]. In particular, one can see there that in (6.14) of Theorem 6.5 above the relation $\widetilde{A}_{\mathfrak{t}} = \widetilde{A}_{\mathfrak{r}} = A_{\bar{\mathfrak{r}}}$ is the Friedrichs extension of the semibounded relation $S_{\mathfrak{r}}$.

7 | MONOTONE SEQUENCES OF SEMIBOUNDED FORMS

Nondecreasing sequences of semibounded forms have a limit and, likewise, nonincreasing sequences of semibounded forms with a common lower bound have a limit; see, for instance, [27, 31, 35]. In this section, the convergence of monotone sequences of semibounded forms will be considered in connection with the convergence of the corresponding representing maps as in Section 6.

First, some general facts are established. Let $\mathfrak{t}_n \in \mathbf{F}(\mathfrak{H})$ be a sequence of lower semibounded forms whose lower bounds are uniformly bounded:

$$\gamma \leqslant m(\mathfrak{t}_n) \quad \text{for some} \quad \gamma \in \mathbb{R},$$

and let $c \leqslant \gamma$. Then, there exists a sequence of representing maps $Q_n \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$, where $\mathfrak{K}_n$ are Hilbert spaces, such that

$$\mathfrak{t}_n[\varphi, \psi] = c + (Q_n\varphi, Q_n\psi), \quad \varphi, \psi \in \operatorname{dom}\mathfrak{t}_n = \operatorname{dom} Q_n. \tag{7.1}$$

Moreover, each sequence of linear operators $Q_n \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$ defines via (7.1) a sequence of semibounded forms $\mathfrak{t}_n \in \mathbf{F}(\mathfrak{H})$ such that $m(\mathfrak{t}_n) \geqslant c$. Now assume that the sequence of semibounded forms $\mathfrak{t}_n \in \mathbf{F}(\mathfrak{H})$ satisfies

$$\mathfrak{t}_m \leqslant \mathfrak{t}_n, \quad m \leqslant n. \tag{7.2}$$

In this case, one can take $\gamma = m(\mathfrak{t}_1)$ and the representing maps $Q_n \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$ in (7.1) satisfy

$$Q_m \prec_c Q_n, \quad m \leqslant n. \tag{7.3}$$

Conversely, if $Q_n \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$ satisfies (7.3), then the semibounded forms $\mathfrak{t}_n \in \mathbf{F}(\mathfrak{H})$ in (7.1) satisfy (7.2). It is clear from (7.1) that

$$\bigcap_{n=1}^{\infty} \operatorname{dom}\mathfrak{t}_n = \bigcap_{n=1}^{\infty} \operatorname{dom} Q_n,$$

and, moreover, for an element φ in this set, one has

$$\sup_{n\in\mathbb{N}} \mathfrak{t}_n[\varphi] < \infty \quad \Leftrightarrow \quad \sup_{n\in\mathbb{N}} \|Q_n\varphi\| < \infty.$$

Recall that if (7.3) is satisfied, then there exists a linear operator $Q \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, where $\mathfrak{K}$ is a Hilbert space, which satisfies

$$Q_n \prec_c Q \quad \text{and} \quad \|Q_n\varphi\| \nearrow \|Q\varphi\|, \quad \varphi \in \operatorname{dom} Q, \tag{7.4}$$

where $\operatorname{dom} Q$ is given by

$$\operatorname{dom} Q = \left\{ \varphi \in \bigcap_{n\in\mathbb{N}} \operatorname{dom} Q_n \; : \; \sup_{n\in\mathbb{N}} \|Q_n\varphi\| < \infty \right\},$$

see [15, Theorem 5.1]. The linear operator $Q \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ serves as a representing map for the semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ defined by

$$\mathfrak{t}[\varphi, \psi] = c + (Q\varphi, Q\psi), \quad \varphi, \psi \in \operatorname{dom}\mathfrak{t} = \operatorname{dom} Q. \tag{7.5}$$

Hence, the following lemma, going back to Simon (see [31, 35]), is now straightforward.

Lemma 7.1. *Let $\mathfrak{t}_n \in \mathbf{F}(\mathfrak{H})$ be a sequence of semibounded forms, represented as in (7.1). Assume that the sequence satisfies*

$$\mathfrak{t}_m \leqslant \mathfrak{t}_n, \quad m \leqslant n. \tag{7.6}$$

Then there exists a unique semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$*, represented in (7.5), such that*

$$\operatorname{dom} \mathfrak{t} = \left\{ \varphi \in \bigcap_{n \in \mathbb{N}} \operatorname{dom} \mathfrak{t}_n \;:\; \sup_{n \in \mathbb{N}} \mathfrak{t}_n[\varphi] < \infty \right\}$$

and which satisfies

$$\mathfrak{t}_n \leqslant \mathfrak{t} \quad \text{and} \quad \mathfrak{t}_n[\varphi] \nearrow \mathfrak{t}[\varphi], \quad \varphi \in \operatorname{dom} \mathfrak{t}.$$

Furthermore, let $\mathfrak{u} \in \mathbf{F}(\mathfrak{H})$ *be a semibounded form. Then, there is the implication*

$$\mathfrak{t}_n \leqslant \mathfrak{u}, \quad n \in \mathbb{N} \quad \Rightarrow \quad \mathfrak{t} \leqslant \mathfrak{u}.$$

The following statements hold:

(a) *if* $\mathfrak{t}_n$ *is closable for all* $n \in \mathbb{N}$*, then* $\mathfrak{t}$ *is closable;*
(b) *if* $\mathfrak{t}_n$ *is closed for all* $n \in \mathbb{N}$*, then* $\mathfrak{t}$ *is closed.*

If $\mathfrak{t}_n$ *is a bounded everywhere defined form for all* $n \in \mathbb{N}$*, then* $\mathfrak{t}$ *is a bounded everywhere defined form.*

In the limit procedure of Lemma 7.1, the properties of being closable and closed are preserved, respectively. This observation will be used in the following discussion of the regular parts of $\mathfrak{t}_n$ and $\mathfrak{t}$ in $\mathbf{F}(\mathfrak{H})$. It follows from (7.1) and (7.5) that $\mathfrak{r}_n = \mathfrak{t}_{n,\mathrm{reg}}$ and $\mathfrak{r} = \mathfrak{t}_{\mathrm{reg}}$ have the representations

$$\mathfrak{r}_n[\varphi, \psi] = c + (Q_{n,\mathrm{reg}}\varphi, Q_{n,\mathrm{reg}}\psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{r}_n = \operatorname{dom} \mathfrak{t}_n = \operatorname{dom} Q_n, \tag{7.7}$$

and

$$\mathfrak{r}[\varphi, \psi] = c + (Q_{\mathrm{reg}}\varphi, Q_{\mathrm{reg}}\psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{r} = \operatorname{dom} \mathfrak{t} = \operatorname{dom} Q. \tag{7.8}$$

The assumption $Q_m \prec_c Q_n$ in (7.3) implies that

$$Q_{m,\mathrm{reg}} \prec_c Q_{n,\mathrm{reg}}, \quad m \leqslant n, \tag{7.9}$$

and, likewise, the inequality $Q_n \prec_c Q$ in (7.4) gives

$$Q_{n,\mathrm{reg}} \prec_c Q_{\mathrm{reg}}; \tag{7.10}$$

cf. (3.7). In particular, Q_{reg} is an upper bound for $Q_{n,\mathrm{reg}}$. By (7.9) and Lemma 7.1, it follows from the closability of the operators $Q_{n,\mathrm{reg}}$ that there exists a closable operator $Q_{\mathrm{r}} \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}')$ such that its domain is given by

$$\operatorname{dom} Q_{\mathrm{r}} = \left\{ \varphi \in \bigcap_{n \in \mathbb{N}} \operatorname{dom} Q_n \;:\; \sup_{n \in \mathbb{N}} \|Q_{n,\mathrm{reg}}\varphi\| < \infty \right\} \tag{7.11}$$

and which satisfies

$$\begin{cases} Q_{n,\mathrm{reg}} \prec_c Q_{\mathrm{r}} \prec Q_{\mathrm{reg}}, \\ \|Q_{n,\mathrm{reg}}\varphi\| \nearrow \|Q_{\mathrm{r}}\varphi\|, \quad \varphi \in \operatorname{dom} Q_{\mathrm{r}}. \end{cases} \tag{7.12}$$

The closable operator Q_{r} serves as a representing map for the closable semibounded form $\mathfrak{t}_{\mathrm{r}} \in \mathbf{F}(\mathfrak{H})$ defined by

$$\mathfrak{t}_{\mathrm{r}}[\varphi,\psi] = c(\varphi,\psi) + (Q_{\mathrm{r}}\varphi, Q_{\mathrm{r}}\psi), \quad \varphi,\psi \in \operatorname{dom} Q_{\mathrm{r}}. \tag{7.13}$$

Since the regular parts $Q_{n,\mathrm{reg}}$ and Q_{r} are closable, the semibounded forms $\mathfrak{r}_n$ and $\mathfrak{t}_{\mathrm{r}}$ are closable, and one obtains from (7.7) that

$$\bar{\mathfrak{r}}_n[\varphi,\psi] = c + ((Q_{n,\mathrm{reg}})^{**}\varphi, (Q_{n,\mathrm{reg}})^{**}\psi) \tag{7.14}$$

for all $\varphi,\psi \in \operatorname{dom} \bar{\mathfrak{r}}_n = \operatorname{dom}(Q_{n,\mathrm{reg}})^{**}$ and from (7.13) that

$$(\operatorname{clos} \mathfrak{t}_{\mathrm{r}})[\varphi,\psi] = c + ((Q_{\mathrm{r}})^{**}\varphi, (Q_{\mathrm{r}})^{**}\psi) \tag{7.15}$$

for all $\varphi,\psi \in \operatorname{dom}(\operatorname{clos} \mathfrak{t}_{\mathrm{r}}) = \operatorname{dom}(Q_{\mathrm{r}})^{**}$. It follows from the inequalities (7.9) that

$$(Q_{m,\mathrm{reg}})^{**} \prec_c (Q_{n,\mathrm{reg}})^{**}, \quad m \leqslant n, \tag{7.16}$$

and from (7.10) that

$$(Q_{n,\mathrm{reg}})^{**} \prec_c (Q_{\mathrm{r}})^{**}. \tag{7.17}$$

By (7.16) and Lemma 7.1, it follows that there exists a closed operator $S_{\mathrm{r}} \in \mathbf{L}(\mathfrak{H},\mathfrak{K}'')$ such that its domain is given by

$$\operatorname{dom} S_{\mathrm{r}} = \left\{ \varphi \in \bigcap_{n\in\mathbb{N}} \operatorname{dom}(Q_{n,\mathrm{reg}})^{**} \,:\, \sup_{n\in\mathbb{N}} \|(Q_{n,\mathrm{reg}})^{**}\varphi\| < \infty \right\}, \tag{7.18}$$

and which satisfies

$$\begin{cases} (Q_{n,\mathrm{reg}})^{**} \prec_c S_{\mathrm{r}} \prec_c (Q_{\mathrm{reg}})^{**}, \\ \|(Q_{n,\mathrm{reg}})^{**}\varphi\| \nearrow \|S_{\mathrm{r}}\varphi\|, \quad \varphi \in \operatorname{dom} S_{\mathrm{r}}. \end{cases} \tag{7.19}$$

The closed operator S_{r} serves as a representing map for the closed semibounded form $\mathfrak{s}_{\mathrm{r}} \in \mathbf{F}(\mathfrak{H})$ defined by

$$\mathfrak{s}_{\mathrm{r}}[\varphi,\psi] = c + (S_{\mathrm{r}}\varphi, S_{\mathrm{r}}\psi), \quad \varphi,\psi \in \operatorname{dom} \mathfrak{s}_{\mathrm{r}} = \operatorname{dom} S_{\mathrm{r}}. \tag{7.20}$$

It is clear from the above that $S_{\mathrm{r}} \prec (Q_{\mathrm{r}})^{**}$.

The above facts together with Lemma 7.1 now lead to the following theorem, which ultimately describes the limit behavior of the semibounded self-adjoint relations corresponding to $\mathfrak{t}_n$ as described in Theorems 6.1 and 6.5.

Theorem 7.2. *Let $\mathfrak{t}_n \in \mathbf{F}(\mathfrak{H})$ be a sequence of semibounded forms, represented in (7.1), which satisfies (7.6). Let the semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$, represented in (7.5), be the limit of $\mathfrak{t}_n$. Then the following statements hold:*

(A) *The regular parts $\mathfrak{r}_n \in \mathbf{F}(\mathfrak{H})$, represented in (7.7), satisfy*

$$\mathfrak{r}_m \leqslant \mathfrak{r}_n, \quad m \leqslant n, \quad \textit{and} \quad \mathfrak{r}_n \leqslant \mathfrak{t}_{\mathrm{reg}}, \tag{7.21}$$

where $\mathfrak{t}_{\mathrm{reg}}$ *is represented in (7.8). Moreover, there is a closable semibounded form* $\mathfrak{t}_{\mathrm{r}} \in \mathbf{F}(\mathfrak{H})$*, represented in (7.13), such that*

$$\operatorname{dom}\mathfrak{t}_{\mathrm{r}} = \left\{ \varphi \in \bigcap_{n\in\mathbb{N}} \operatorname{dom}\mathfrak{t}_n \,:\, \sup_{n\in\mathbb{N}} \mathfrak{r}_n[\varphi] < \infty \right\}, \tag{7.22}$$

and which satisfies

$$\mathfrak{r}_n \leqslant \mathfrak{t}_{\mathrm{r}} \leqslant \mathfrak{t}_{\mathrm{reg}} \quad \textit{and} \quad \mathfrak{r}_n[\varphi] \nearrow \mathfrak{t}_{\mathrm{r}}[\varphi], \quad \varphi \in \operatorname{dom}\mathfrak{t}_{\mathrm{r}}. \tag{7.23}$$

(B) *The closures* $\bar{\mathfrak{r}}_n \in \mathbf{F}(\mathfrak{H})$ *of the regular parts* $\mathfrak{r}_n$*, represented in (7.14), satisfy*

$$\bar{\mathfrak{r}}_m \leqslant \bar{\mathfrak{r}}_n \quad m \leqslant n, \quad \textit{and} \quad \bar{\mathfrak{r}}_n \leqslant \operatorname{clos}\,\mathfrak{t}_{\mathrm{r}}, \tag{7.24}$$

where $\operatorname{clos}\,\mathfrak{t}_{\mathrm{r}}$ *is represented in (7.15). Moreover, there exists a closed semibounded form* $\mathfrak{s}_{\mathrm{r}} \in \mathbf{F}(\mathfrak{H})$*, represented in (7.20), such that*

$$\operatorname{dom}\mathfrak{s}_{\mathrm{r}} = \left\{ \varphi \in \bigcap_{n\in\mathbb{N}} \operatorname{dom}\bar{\mathfrak{r}}_n \,:\, \sup_{n\in\mathbb{N}} \bar{\mathfrak{r}}_n[\varphi] < \infty \right\}, \tag{7.25}$$

and which satisfies

$$\bar{\mathfrak{r}}_n \leqslant \mathfrak{s}_{\mathrm{r}} \leqslant \operatorname{clos}\,\mathfrak{t}_{\mathrm{r}} \quad \textit{and} \quad \bar{\mathfrak{r}}_n[\varphi] \nearrow \mathfrak{s}_{\mathrm{r}}[\varphi], \quad \varphi \in \operatorname{dom}\mathfrak{s}_{\mathrm{r}}. \tag{7.26}$$

In fact, $\operatorname{dom}(\operatorname{clos}\mathfrak{t}_{\mathrm{r}}) \subset \operatorname{dom}\mathfrak{s}_{\mathrm{r}}$ *and*

$$(\operatorname{clos}\mathfrak{t}_{\mathrm{r}})\,[\varphi,\psi] = \mathfrak{s}_{\mathrm{r}}[\varphi,\psi], \quad \varphi,\psi \in \operatorname{dom}(\operatorname{clos}\,\mathfrak{t}_{\mathrm{r}}). \tag{7.27}$$

(C) *The semibounded self-adjoint relations* $\widetilde{A}_{\mathfrak{t}_n} \in \mathbf{L}(\mathfrak{H})$ *corresponding to the semibounded forms* $\mathfrak{t}_n \in \mathbf{F}(\mathfrak{H})$ *converge to the semibounded self-adjoint relation* $A_{\mathfrak{s}} \in \mathbf{L}(\mathfrak{H})$ *corresponding to the closed semibounded form* $\mathfrak{s} \in \mathbf{F}(\mathfrak{H})$*:*

$$\widetilde{A}_{\mathfrak{t}_n} \to A_{\mathfrak{s}} \tag{7.28}$$

in the strong resolvent sense or, equivalently, the strong graph sense in $\mathfrak{H}$*.*

Proof. Due to the assumption (7.6), it follows for the corresponding representing maps that $Q_m \prec_c Q_n$, $m \leqslant n$, and that $Q_n \prec_c Q$.

(A) The inequalities in (7.21) follow from (7.9) and (7.10). The statements about $\mathfrak{t}_{\mathrm{r}}$ in (7.22) and (7.23) follow from (7.11) and (7.12).

(B) The inequalities in (7.24) follow from (7.16) and (7.17). The statements about $\mathfrak{s}_{\mathrm{r}}$ in (7.25) and (7.26) follow from (7.18) and (7.19). The equality (7.27) holds by polarization, after observing that $\mathfrak{t}_{\mathrm{r}}[\varphi]$ is the limit of $\mathfrak{r}_n[\varphi]$ for $\varphi \in \operatorname{dom}\mathfrak{t}_{\mathrm{r}}$, while $\mathfrak{s}_{\mathrm{r}}[\varphi]$ is the limit of $\bar{\mathfrak{r}}_n[\varphi]$ for $\varphi \in \operatorname{dom}\mathfrak{s}_{\mathrm{r}}$, see (7.23) and (7.26). This equality is then preserved also for the closure $\operatorname{clos}\mathfrak{t}_{\mathrm{r}}$, since $\mathfrak{t}_{\mathrm{r}} \subset \mathfrak{s}_{\mathrm{r}}$ and $\mathfrak{s}_{\mathrm{r}}$ is closed.

(C) The semibounded self-adjoint relation $\widetilde{A}_n \in \mathbf{L}(\mathfrak{H})$ corresponding to $\mathfrak{t}_n \in \mathbf{F}(\mathfrak{H})$ via Theorem 6.1 is given by

$$\widetilde{A}_n = c + (Q_n)^*(Q_n)^{**} = c + (Q_{n,\mathrm{reg}})^*(Q_{n,\mathrm{reg}})^{**},$$

where the second equality follows again from $(Q_n)^*(Q_n)^{**} = (Q_{n,\mathrm{reg}})^*(Q_{n,\mathrm{reg}})^{**}$, cf. [15, Appendix]. Hence, by Corollary 6.2, $\widetilde{A}_n$ is the unique semibounded self-adjoint relation corresponding to the closed form $\bar{\mathfrak{r}}_n$. The sequence $(Q_{n,\mathrm{reg}})^{**}$ satisfies (7.19), while (7.26) shows that the sequence $\bar{\mathfrak{r}}_n$ has the closed limit form $\mathfrak{s}_{\mathrm{r}}$ with the closed representing operator S_{r} in (7.20). One concludes that $A_{\mathfrak{s}} = (S_{\mathrm{r}})^* S_{\mathrm{r}}$ and now the strong resolvent or, equivalently, the strong graph convergence in (7.28) follows from the standard monotonicity principle for closed forms; cf. for example [7, Theorem 5.2.15]. □

Finally, let $\mathfrak{t}_n \in \mathbf{F}(\mathfrak{H})$, $n \in \mathbb{N}$, be a sequence of (lower) semibounded forms that is nonincreasing:

$$c < \mathfrak{t}_n \leqslant \mathfrak{t}_m, \quad m \leqslant n, \tag{7.29}$$

in the sense of (2.1). The assumption of the common lower bound $c \in \mathbb{R}$ guarantees the existence of a limit. Due to (7.29), the lower bounds satisfy

$$c \leqslant m(\mathfrak{t}_n) \leqslant m(\mathfrak{t}_m), \quad m \leqslant n.$$

The following result is straightforward, see [15, Theorem 10.3] and [31, 35].

Lemma 7.3. *Let $\mathfrak{t}_n \in \mathbf{F}(\mathfrak{H})$ be a sequence of (lower) semibounded forms that satisfies (7.29). Then there exists a (lower) semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ such that*

$$\operatorname{dom} \mathfrak{t} = \bigcup_{n \in \mathbb{N}} \operatorname{dom} \mathfrak{t}_n, \tag{7.30}$$

and which satisfies

$$c \leqslant \mathfrak{t} \leqslant \mathfrak{t}_n \quad \text{and} \quad \mathfrak{t}_n[\varphi] \searrow \mathfrak{t}[\varphi], \quad \varphi \in \operatorname{dom} \mathfrak{t}. \tag{7.31}$$

In the case of nonincreasing sequences, the notions of closability or closedness are in general not preserved; see [15, Example 10.5], [31]. There is a useful result for nonincreasing sequences of closed forms which goes back to [35]; see also [31]. The following result is included for completeness: it is the analog of [15, Theorem 10.4], when adapted to the setting of nonincreasing sequences of forms.

Theorem 7.4. *Let $\mathfrak{t}_n \in \mathbf{F}(\mathfrak{H})$, represented as in (7.1), be a sequence of closed (lower) semibounded forms such that (7.29) holds. Let the semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$, represented as in (7.5), be the limit of $\mathfrak{t}_n$ as in (7.30) and (7.31). Then, the semibounded self-adjoint relations $A_{\mathfrak{t}_n} \in \mathbf{L}(\mathfrak{H})$ corresponding to $\mathfrak{t}_n \in \mathbf{F}(\mathfrak{H})$ converge to a semibounded self-adjoint relation $A_\infty \in \mathbf{L}(\mathfrak{H})$:*

$$A_{\mathfrak{t}_n} \to A_\infty,$$

in the strong resolvent sense or, equivalently, in the strong graph sense in $\mathfrak{H}$. Let $\mathfrak{t}_\infty \in \mathbf{F}(\mathfrak{H})$ be the closed semibounded form corresponding to $A_\infty \in \mathbf{L}(\mathfrak{H})$. Then, the semibounded forms $\mathfrak{t}$ and $\mathfrak{t}_\infty$ are connected by

$$\operatorname{clos} \mathfrak{t}_{\mathrm{reg}} = \mathfrak{t}_\infty.$$

Moreover, for the semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$, *one has*

(a) $\mathfrak{t}$ *is closable if and only if* $\mathfrak{t} \subset \mathfrak{t}_\infty$;
(b) $\mathfrak{t}$ *is closed if and only if* $\mathfrak{t} = \mathfrak{t}_\infty$;
(c) $\mathfrak{t}$ *is singular if and only if* $A_\infty - c$ *is singular.*

ACKNOWLEDGMENTS

A first version of the present paper was made available on arXiv in December 2023. The authors thank the referees for their constructive comments that have resulted in the improvement of the presentation of this paper.

Open access publishing facilitated by Vaasan yliopisto, as part of the Wiley - FinELib agreement.

JOURNAL INFORMATION

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID

S. Hassi https://orcid.org/0000-0002-0102-1087

REFERENCES

1. T. Ando, *Lebesgue-type decomposition of positive operators*, Acta Sci. Math. (Szeged) **38** (1976), 253–260.
2. T. Ando and K. Nishio, *Positive selfadjoint extensions of positive symmetric operators*, Tohóku Math. J. **22** (1970), 65–75.
3. W. Arendt and T. ter Elst, *Sectorial forms and degenerate differential operators*, J. Operator Theory **67** (2012), 33–72.
4. Yu. M. Arlinskiĭ, *Positive spaces of boundary values and sectorial extensions of nonnegative symmetric operators*, Ukrainian Math. J. **40** (1988), 8–15.
5. Y. M. Arlinskiĭ, S. Hassi, H. S. V. de Snoo, and Z. Sebestyén, *On the class of extremal extensions of a nonnegative operator*, Oper. Theory: Adv. Appl. (B. Sz.-Nagy memorial volume) **127** (2001), 41–81.
6. J. Bal, R. T. W. Martin, and F. Naderi, *A reproducing kernel approach to Lebesgue decomposition*, Canad. J. Math. **77** (2025), 1570–1610.
7. J. Behrndt, S. Hassi, and H. S. V. de Snoo, *Boundary value problems, Weyl functions, and differential operators*, Monographs in Mathematics, vol. 108, Birkhäuser, Cham, 2020.
8. E. A. Coddington and H. S. V. de Snoo, *Positive selfadjoint extensions of positive symmetric subspaces*, Math. Z. **159** (1978), 203–214.
9. V. A. Derkach, S. Hassi, and M. M. Malamud, *Generalized boundary triples, I. Some classes of isometric and unitary boundary pairs and realization problems for subclasses of Nevanlinna functions*, Math. Nachr. **293** (2020), 1278–1327.
10. V. Derkach, S. Hassi, and M. Malamud, *Generalized boundary triples, II. Some applications of generalized boundary triples and form domain invariant Nevanlinna functions*, Math. Nachr. **295** (2022), 1113–1162.
11. V. A. Derkach, S. Hassi, M. M. Malamud, and H. S. V. de Snoo, *Boundary relations and their Weyl families*, Trans. Amer. Math. Soc. **358** (2006), 5351–5400.
12. M. Dritschel and J. Rovnyak, *Extension theorems for contraction operators on Kreĭn spaces*, Oper. Theory Adv. Appl. **47** (1990), 221–305.
13. M. Dritschel and J. Rovnyak, *Julia operators and complementation in Kreĭn spaces*, Indiana Univ. Math. J. **40** (1991), 885–901.
14. P. A. Fillmore and J. P. Williams, *On operator ranges*, Adv. Math. **7** (1971), 254–281.

15. S. Hassi and H. S. V. de Snoo, *Sequences of operators, monotone in the sense of contractive domination*, Complex Anal. Oper. Theory, **18** (2024), 81, 10.1007/s11785-024-01507-3.
16. S. Hassi and H. S. V. de Snoo, *Complementation and Lebesgue-type decompositions of linear operators and relations*, J. London Math. Soc. (2) **109** (2024), DOI 10.1112/jlms.12900.
17. S. Hassi and H. S. V. de Snoo, *Friedrichs and Kreĭn type extensions in terms of representing maps*, Ann. Funct. Anal. **15** (2024), no. 78, DOI 10.1007/s43034-024-00380-7.
18. S. Hassi and H. S. V. de Snoo, *Nonorthogonal Hilbert space decompositions with operator range spaces and reproducing kernel Hilbert spaces*, D. Alpay et al. (eds.), Operator theory, Springer, Basel, 2025, DOI 10.1007/978-3-0348-0692-3.
19. S. Hassi and H. S. V. de Snoo, *Ando's Lebesgue-type decompositions for pairs of positive operators*, Oper. Theory Adv. Appl. **307** (2025), 299–326.
20. S. Hassi and H. S. V. de Snoo, *Non-standard behaviour of semibounded forms and their representations*, in preparation.
21. S. Hassi, M. M. Malamud, and H. S. V. de Snoo, *On Kreĭn's extension theory of nonnegative operators*, Math. Nachr. **274/275** (2004), 40–73.
22. S. Hassi, Z. Sebestyén, and H. S. V. de Snoo, *Lebesgue type decompositions for nonnegative forms*, J. Funct. Anal. **257** (2009), 3858–3894.
23. S. Hassi, Z. Sebestyén, and H. S. V. de Snoo, *Domain and range descriptions for adjoint relations, and parallel sums and differences of forms*, Oper. Theory Adv. Appl. **198** (2010), 211–227.
24. S. Hassi, Z. Sebestyén, and H. S. V. de Snoo, *Lebesgue type decompositions for linear relations and Ando's uniqueness criterion*, Acta Sci. Math. (Szeged) **84** (2018), 465–507.
25. S. Hassi, Z. Sebestyén, H. S. V. de Snoo, and F. H. Szafraniec, *A canonical decomposition for linear operators and linear relations*, Acta Math. Hungar. **115** (2007), 281–307.
26. S. Izumino, *Decomposition of quotients of bounded operators with respect to closability and Lebesgue-type decomposition of positive operators*, Hokkaido Math. J. **18** (1989), 199–209.
27. T. Kato, *Perturbation theory for linear operators*, Springer, Berlin, 1980.
28. V. D. Koshmanenko, *Singular quadratic forms in perturbation theory*, Mathematics and Its Applications, vol. 474, Kluwer Academic Publishers, Dordrecht/Boston/London, 1999.
29. V. Prokaj and Z. Sebestyén, *On Friedrichs extensions of operators*, Acta Sci. Math. (Szeged) **62** (1996), 243–246.
30. W. Pusz and S. L. Woronowicz, *Functional calculus for sesquilinear forms and the purification map*, Rep. Math. Phys. **5** (1975), 159–170.
31. M. Reed and B. Simon, *Methods of modern physics. I.*, Academic Press, New York, 1980.
32. Z. Sebestyén and J. Stochel, *Restrictions of positive self-adjoint operators*, Acta Sci. Math. (Szeged) **55** (1991), 149–154.
33. Z. Sebestyén and Z. Tarcsay, *Extensions of positive symmetric operators and Krein's uniqueness criteria*, Linear Multilinear Algebra **72** (2024), 1663–1688.
34. Z. Sebestyén, Z. Tarcsay, and T. Titkos, *Lebesgue decomposition theorems*, Acta Sci. Math. (Szeged) **79** (2013), 219–233.
35. B. Simon, *A canonical decomposition for quadratic forms with applications to monotone convergence theorems*, J. Funct. Anal. **28** (1978), 377–385.
36. W. Szymański, *Positive forms and dilations*, Trans. Amer. Math. Soc. **301** (1987), 761–780.
37. Z. Szücs and B. Takács, *Finite dimensional irreducible representations and the uniqueness of the Lebesgue decomposition of positive functionals*, J. Operator Theory **91** (2024), 55–95.
38. Z. Tarcsay, *Lebesgue decomposition for representable functionals on $*$-algebras*, Glasg. Math. J. **58** (2016), 491–501.
39. Z. Tarcsay, *Operators on anti-dual pairs: Lebesgue decomposition of positive operators*, J. Math. Anal. Appl. **484** (2020), 123753, 28 pp.
40. Z. Tarcsay and Z. Sebestyén, *Basic representation theorems of forms*, Complex Anal. Oper. Theory **19** (2025), no. 178, DOI 10.1007/s11785-025-01806-3.