Working Papers of the University of Vaasa,
Department of Mathematics and Statistics,
11

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Preprint, December 2006

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Preprints are available at: http://www.uwasa.fi/julkaisu/sis.html

PARAMETRIZATION OF CONTRACTIVE BLOCK-OPERATOR MATRICES AND PASSIVE DISCRETE-TIME SYSTEMS

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Abstract. Passive linear systems $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ have their transfer function $\Theta_\tau(\lambda) = D + \lambda C(I - \lambda A)^{-1}B$ in the Schur class $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$. Using a parametrization of contractive block operators the transfer function $\Theta_\tau(\lambda)$ is connected to the Sz.-Nagy – Foias characteristic function $\Phi_A(\lambda)$ of the contraction A. This gives a new aspect and some explicit formulas for studying the interplay between the system τ and the functions $\Theta_\tau(\lambda)$ and $\Phi_A(\lambda)$. The method leads to some new results for linear passive discrete-time systems. Also new proofs for some known facts in the theory of these systems are obtained.

1. Introduction

A bounded linear operator T acting from a Hilbert space $\mathfrak{H}_1$ into a Hilbert space $\mathfrak{H}_2$ is said to be

(1) *contractive* if $\|T\| \le 1$;
(2) *isometric* if $\|Tf\| = \|f\|$ for all $f \in \mathfrak{H}_1 \iff T^*T = I_{\mathfrak{H}_1}$;
(3) *co-isometric* if T^* is isometric $\iff TT^* = I_{\mathfrak{H}_2}$.

A linear system $\tau = (A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N})$ with bounded linear operators A, B, C, D and separable Hilbert spaces $\mathfrak{H}$ (state space), $\mathfrak{M}$ (incoming space), and $\mathfrak{N}$ (outgoing space), of the form

$$\left\{\begin{array}{l} h_{k+1} = Ah_k + B\xi_k, \\ \sigma_k = Ch_k + D\xi_k, \end{array}\right. \qquad k \ge 0, \tag{1.1}$$

where $\{h_k\} \subset \mathfrak{H}$, $\{\xi_k\} \subset \mathfrak{M}$, $\{\sigma_k\} \subset \mathfrak{N}$, is called a *discrete-time system*. The operators A, B, C, and D are called the *main operator*, the *control operator*, the *observation operator*, and the *feedthrough operator* of τ, respectively. If the linear operator $T_\tau : \mathfrak{H} \oplus \mathfrak{M} \to \mathfrak{H} \oplus \mathfrak{N}$ defined by the block form

$$T_\tau = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix} \tag{1.2}$$

is contractive, then the corresponding discrete-time system is said to be *passive*. If the block-operator matrix T_τ is isometric (co-isometric, unitary), then the system is said to be isometric (co-isometric, conservative). Isometric and co-isometric systems were studied by L. de Branges and J. Rovnyak (see [21], [22]) and by T. Ando (see [2]), conservative systems have been investigated by B. Sz.-Nagy and C. Foias (see [33]) and M.S. Brodskiĭ (see [23]).

2000 *Mathematics Subject Classification.* Primary: 47A48, 47A56, 93B28; Secondary: 93B15, 94C05.

Key words and phrases. Passive system, transfer function, Sz.-Nagy – Foias characteristic function.

This work was supported by the Research Institute for Technology at the University of Vaasa. The first author was also supported by the Academy of Finland (projects 212146, 117617) and the Dutch Organization for Scientific Research N.W.O. (B 61–553).

Passive systems have been studied by D.Z. Arov et al (see [9], [10], [11], [12], [13], [14], [15]). The *transfer function*

$$\Theta_\tau(\lambda) := D + \lambda C(I_{\mathfrak{H}} - \lambda A)^{-1}B, \quad \lambda \in \mathbb{D}, \tag{1.3}$$

of the passive system τ in (1.1) belongs to the *Schur class* $\mathbf{S}(\mathfrak{M},\mathfrak{N})$, i.e., $\Theta_\tau(\lambda)$ is holomorphic in the unit disk $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and its values are contractive linear operators from $\mathfrak{M}$ into $\mathfrak{N}$. It is well known that a function $\Theta(\lambda)$ from the Schur class $\mathbf{S}(\mathfrak{M},\mathfrak{N})$ has almost everywhere non-tangential strong limit values $\Theta(\xi)$, $\xi \in \mathbb{T}$, where $\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$ stands for the unit circle; cf. [33]. The subspaces

$$\mathfrak{H}^c := \overline{\rm span}\,\{A^n B\mathfrak{M} :\, n \in \mathbb{N}_0\,\} \text{ and } \mathfrak{H}^o = \overline{\rm span}\,\{A^{*n}C^*\mathfrak{N} :\, n \in \mathbb{N}_0\,\} \tag{1.4}$$

are said to be the *controllable* and *observable* subspaces of the system τ, respectively. The notation $\mathbb{N}_0$ stands for the nonnegative integers; the positive integers will be denoted by $\mathbb{N}$. The system τ is said to be *controllable* (*observable*) if $\mathfrak{H}^c = \mathfrak{H}$ ($\mathfrak{H}^o = \mathfrak{H}$), and it is called *minimal* if τ is both controllable and observable. The system τ is said to be *simple* if $\mathfrak{H} = {\rm clos}\,\{\mathfrak{H}^c + \mathfrak{H}^o\}$ (the closure of the span). It follows from (1.4) that

$$(\mathfrak{H}^c)^\perp = \bigcap_{n=0}^{\infty} \ker\,(B^*A^{*n}), \quad (\mathfrak{H}^o)^\perp = \bigcap_{n=0}^{\infty} \ker\,(CA^n), \tag{1.5}$$

and therefore there are the following alternative characterizations:

(1) τ is controllable $\iff \bigcap_{n=0}^{\infty} \ker\,(B^*A^{*n}) = \{0\}$;

(2) τ is observable $\iff \bigcap_{n=0}^{\infty} \ker\,(CA^n) = \{0\}$;

(3) τ is simple $\iff \left(\bigcap_{n=0}^{\infty} \ker\,(B^*A^{*n})\right) \cap \left(\bigcap_{n=0}^{\infty} \ker\,(CA^n)\right) = \{0\}$.

It is well known that every operator-valued function $\Theta(\lambda)$ from the Schur class $\mathbf{S}(\mathfrak{M},\mathfrak{N})$ can be realized as the transfer function of some passive system, which can be chosen as controllable isometric (observable co-isometric, simple conservative, minimal passive); cf. [22], [33], [2] [9], [11], [1]. Moreover, two controllable isometric (observable co-isometric, simple conservative) systems with the same transfer function are unitarily similar: two discrete-time systems

$$\tau_1 = \{A_1, B_1, C_1, D; \mathfrak{H}_1, \mathfrak{M}, \mathfrak{N}\} \quad \text{and} \quad \tau_2 = \{A_2, B_2, C_2, D; \mathfrak{H}_2, \mathfrak{M}, \mathfrak{N}\}$$

are said to be *unitarily similar* if there exists a unitary operator U from $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ such that

$$A_1 = U^{-1}A_2U, \quad B_1 = U^{-1}B_2, \quad C_1 = C_2U;$$

cf. [21], [22], [2], [23], [1]. However, a result of D.Z. Arov [9] states that two minimal passive systems τ_1 and τ_2 with the same transfer function $\Theta(\lambda)$ are only *weakly similar*, i.e., there is a closed densely defined operator $Z : \mathfrak{H}_1 \to \mathfrak{H}_2$ such that Z is invertible, Z^{-1} is densely defined, and

$$ZA_1f = A_2Zf, \quad C_1f = C_2Zf, \quad f \in {\rm dom}\,Z, \quad \text{and} \quad ZB_1 = B_2.$$

Weak similarity preserves neither the dynamical properties of the system nor the spectral properties of its main operator A. In [13], [14] necessary and sufficient conditions have been established for minimal passive systems with the same transfer function to be similar or to be unitarily similar. These conditions involve additional operator-valued Schur functions $\varphi_\Theta(\lambda)$ and $\psi_\Theta(\lambda)$ which satisfy the inequalities

$$\varphi_\Theta^*(\xi)\varphi_\Theta(\xi) \le I_{\mathfrak{M}} - \Theta^*(\xi)\Theta(\xi), \quad \psi_\Theta(\xi)\psi_\Theta^*(\xi) \le I_{\mathfrak{N}} - \Theta(\xi)\Theta^*(\xi), \tag{1.6}$$

almost everywhere on $\mathbb{T}$, and they are uniquely (up to a constant unitary factor) determined by the following maximality property: if $\widetilde{\varphi}(\lambda)$ and $\widetilde{\psi}(\lambda)$ are operator-valued functions from the Schur class such that

$$\widetilde{\varphi}^*(\xi)\widetilde{\varphi}(\xi) \le I_{\mathfrak{M}} - \Theta^*(\xi)\Theta(\xi), \quad \widetilde{\psi}(\xi)\widetilde{\psi}^*(\xi) \le I_{\mathfrak{M}} - \Theta(\xi)\Theta^*(\xi), \tag{1.7}$$

then

$$\widetilde{\varphi}^*(\xi)\widetilde{\varphi}(\xi) \le \varphi_\Theta^*(\xi)\varphi_\Theta(\xi), \quad \widetilde{\psi}(\xi)\widetilde{\psi}^*(\xi) \le \psi_\Theta(\xi)\psi_\Theta^*(\xi), \tag{1.8}$$

almost everywhere on the unit circle $\mathbb{T}$. Here $\Theta(\xi)$, $\xi \in \mathbb{T}$, stands for the non-tangential strong limit value of $\Theta(\lambda)$ which exist almost everywhere on $\mathbb{T}$, cf. [33]. The functions $\varphi_\Theta(\lambda)$ and $\psi_\Theta(\lambda)$ are called the right and left *defect functions* (or the *spectral factors*), respectively, associated with $\Theta(\lambda)$; cf. [17], [18], [19], [20], [26].

In this paper passive discrete-time systems $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ of the form (1.1) are considered. Some new proofs and new formulas concerning these systems and their transfer functions $\Theta_\tau(\lambda)$ in (1.3) are presented. Also some new facts concerning the realization of operator-valued Schur functions $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ as transfer functions of passive systems τ are established. One of the main consequences of the approach used and developed in this paper can be formulated as follows:

Theorem. *Let $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ and assume that $\Theta(\lambda)$ is not a constant function.*

(i) *Suppose that $\varphi_\Theta(\lambda) = 0$, $\psi_\Theta(\lambda) = 0$, and that $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ is a simple passive system with transfer function $\Theta(\lambda)$. Then τ is conservative and minimal. Furthermore, if $\Theta(\lambda)$ is bi-inner, then in addition $A \in C_{00}$.*

(ii) *Suppose that $\varphi_\Theta(\lambda) = 0$ ($\psi_\Theta(\lambda) = 0$) and that $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ is a controllable (observable) passive system with transfer function $\Theta(\lambda)$. Then τ is isometric (co-isometric) and minimal. Furthermore, if $\Theta(\lambda)$ is inner (co-inner), then in addition $A \in C_{0\cdot}$ ($C_{\cdot 0}$).*

The classes $C_{0\cdot}$, $C_{\cdot 0}$, and C_{00} are introduced in [33]; see also Section 6. The above theorem is very close to the following result established by D.Z. Arov, which was proved by means of the so-called *optimal* and $*$-*optimal* realizations of Schur class functions (see [10], [11], [14]):

Theorem. ([10]) *Let $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$. Then:*

(i) *if $\Theta(\lambda)$ is bi-inner and τ is a simple passive system with transfer function $\Theta(\lambda)$ then τ is conservative;*

(ii) *if $\varphi_\Theta(\lambda) = 0$ or $\psi_\Theta(\lambda) = 0$ then all passive minimal systems with the same transfer function $\Theta(\lambda)$ are unitarily equivalent and if $\varphi_\Theta(\lambda) = 0$ and $\psi_\Theta(\lambda) = 0$ then they are in addition conservative.*

The arguments in the present paper use a parametrization of contractive block-operator matrices of the form

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{K} \\ \mathfrak{N} \end{pmatrix},$$

established in the papers [16], [24], and [32]; a new proof of the parametrization is presented. This parametrization leads to some explicit formulas for realizing operator-valued Schur functions as transfer functions of passive systems. In particular, the transfer function of a passive system is expressed in terms of the characteristic function of the main operator A of the system; cf. B. Sz.-Nagy and C. Foias [33]. The connection is used to study passive systems and their transfer functions via the Sz.-Nagy – Foias characteristic function. For instance, an exact form of the inner (co-inner, bi-inner) dilations for a passive system with a strongly stable (co-stable, bi-stable) main operator is established.

In what follows the class of all continuous linear operators defined on a complex Hilbert space $\mathfrak{H}_1$ and taking values in a complex Hilbert space $\mathfrak{H}_2$ is denoted by $\mathbf{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ and $\mathbf{L}(\mathfrak{H}) := \mathbf{L}(\mathfrak{H},\mathfrak{H})$. The domain, the range, and the null-space of a linear operator T are denoted by $\operatorname{dom} T$, $\operatorname{ran} T$, and $\ker T$, respectively. The set of all regular points of a closed operator T is denoted by $\rho(T)$.

2. The model of Sz-Nagy and Foias

For a contraction $A \in \mathbf{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ the nonnegative square root $D_A = (I - A^*A)^{1/2}$ is said to be the *defect operator* of S and $\mathfrak{D}_A$ stands for the closure of the range $\operatorname{ran} D_A$. It is well known that the defect operators satisfy the following commutation relation:

$$AD_A = D_{A^*}A, \tag{2.1}$$

and that the block operator

$$\begin{pmatrix} A^* & D_A \\ D_{A^*} & -A \end{pmatrix} : \begin{pmatrix} \mathfrak{H}_2 \\ \mathfrak{D}_A \end{pmatrix} \to \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{D}_{A^*} \end{pmatrix} \tag{2.2}$$

is unitary, cf. [33]. If $\mathfrak{H}_1 = \mathfrak{H}_2 = \mathfrak{H}$ then the transfer function of the conservative system

$$\{A^*, D_A, D_{A^*}, -A; \mathfrak{H}, \mathfrak{D}_A, \mathfrak{D}_{A^*}\}.$$

is given by

$$\Phi_A(\lambda) := \left(-A + \lambda D_{A^*}(I_{\mathfrak{H}} - \lambda A^*)^{-1} D_A\right) \upharpoonright \mathfrak{D}_A, \quad \lambda \in \mathbb{D}. \tag{2.3}$$

The function $\Phi_A(\lambda)$ is the Sz.-Nagy – Foias characteristic function of the contraction A and it belongs to the Schur class $\mathbf{S}(\mathfrak{D}_A, \mathfrak{D}_{A^*})$; cf. [33]. For the adjoint operator A^* the characteristic function takes the form

$$\Phi_{A^*}(\lambda) := \left(-A^* + \lambda D_A(I_{\mathfrak{H}} - \lambda A)^{-1} D_{A^*}\right) \upharpoonright \mathfrak{D}_{A^*} = \Phi_A(\bar{\lambda})^*. \tag{2.4}$$

Observe that $\Phi_{A^*}(\lambda)$ is the transfer function of the conservative system

$$\Sigma = \{A, D_{A^*}, D_A, -A^*; \mathfrak{H}, \mathfrak{D}_{A^*}, \mathfrak{D}_A\}. \tag{2.5}$$

The controllable and observable subspaces of the system Σ take the form

$$\mathfrak{H}^c_\Sigma = \overline{\operatorname{span}}\,\{A^n D_{A^*}\mathfrak{D}_{A^*} : n \in \mathbb{N}_0, \}, \quad \mathfrak{H}^o_\Sigma = \overline{\operatorname{span}}\,\{A^{*n} D_A \mathfrak{D}_A : n \in \mathbb{N}_0 \}. \tag{2.6}$$

It follows that

$$(\mathfrak{H}^c_\Sigma)^\perp = \bigcap_{n=0}^{\infty} \ker\,(D_{A^*}A^{*n}) = \bigcap_{n=1}^{\infty} \ker\, D_{A^{*n}}, \tag{2.7}$$

and that

$$(\mathfrak{H}^o_\Sigma)^\perp = \bigcap_{n=0}^{\infty} \ker\,(D_A A^n) = \bigcap_{n=1}^{\infty} \ker\, D_{A^n}. \tag{2.8}$$

The subspace $(\mathfrak{H}^c_\Sigma)^\perp$ ($(\mathfrak{H}^o_\Sigma)^\perp$) is invariant under A^* (A, respectively) and the operator $A^* \upharpoonright (\mathfrak{H}^c_\Sigma)^\perp$ ($A\upharpoonright (\mathfrak{H}^o_\Sigma)^\perp$, respectively) is isometric. Clearly,

$$(\mathfrak{H}^c_\Sigma)^\perp \cap (\mathfrak{H}^o_\Sigma)^\perp = \{\, f \in \mathfrak{H} : \, \|f\| = \|A^n f\| = \|A^{*n} f\|,\ n \in \mathbb{N}\,\}\,.$$

This yields some basic facts, which are formulated in the next remark.

Remark 2.1. The conservative system Σ in (2.4) admits the following properties:

(i) Σ is simple if and only if A is completely non-unitary; cf. [33, Theorem 3.2];
(ii) if Σ is simple and $A^*\,(\mathfrak{H}^c_\Sigma)^\perp = (\mathfrak{H}^c_\Sigma)^\perp$, then $(\mathfrak{H}^c_\Sigma)^\perp = \{0\}$, i.e., Σ is controllable;
(iii) if Σ is simple and $A\,(\mathfrak{H}^o_\Sigma)^\perp = (\mathfrak{H}^o_\Sigma)^\perp$, then $(\mathfrak{H}^o_\Sigma)^\perp = \{0\}$, i.e., Σ is observable.

3. An identity for contractions

An identity is derived for a class of contractions. It is useful for the parametrization of contractions in block form and for the representation of transfer functions of passive systems.

Lemma 3.1. *Let $\mathfrak{H}$, $\mathfrak{K}$, $\mathfrak{M}$, and $\mathfrak{N}$ be Hilbert spaces, and let the operator $F \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be a contraction, let the operators $M \in \mathbf{L}(\mathfrak{M}, \mathfrak{D}_{F^*})$ and $K \in \mathbf{L}(\mathfrak{D}_F, \mathfrak{N})$ be contractions, and let the operator $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$ be a contraction. Then the operator G defined by*

$$G = KFM + D_{K^*} X D_M \in \mathbf{L}(\mathfrak{M}, \mathfrak{N}) \tag{3.1}$$

satisfies the identity

$$\|h\|^2 - \|Gh\|^2 = \|D_F M h\|^2 + \|D_X D_M h\|^2 + \|(D_K F M - K^* X D_M)\, h\|^2\,, \tag{3.2}$$

for all $h \in \mathfrak{M}$. In particular, G is a contraction.

Proof. ¿From the definition of G in (3.1) one obtains

$$\begin{aligned} \|h\|^2 - \|Gh\|^2 &= \|h\|^2 - \|(KFM + D_{K^*} X D_M)\, h\|^2 \\ &= \|h\|^2 - \|KFMh\|^2 - \|D_{K^*} X D_M h\|^2 - 2\mathrm{Re}\,(KFMh, D_{K^*} X D_M h)\,. \end{aligned} \tag{3.3}$$

Taking adjoints in (2.1) gives $K^* D_{K^*} = D_K K^*$, and hence

$$(KFMh, D_{K^*} X D_M h) = (D_K FMh, K^* X D_M h)\,.$$

The definition of D_{K^*} shows that

$$-\|KFMh\|^2 = \|D_K FMh\|^2 - \|FMh\|^2,$$

and, likewise,

$$-\|D_{K^*} X D_M h\|^2 = -\|X D_M h\|^2 + \|K^* X D_M h\|^2.$$

Now the righthand side of (3.3) becomes

$$\begin{aligned} &\|h\|^2 - \|FMh\|^2 - \|X D_M h\|^2 \\ &\qquad + \|D_K FMh\|^2 + \|K^* X D_M h\|^2 - 2\mathrm{Re}\,(D_K FMh, K^* X D_M h) \\ &= \|h\|^2 - \|FMh\|^2 - \|X D_M h\|^2 + \|(D_K FM - K^* X D_M)\, h\|^2\,. \end{aligned}$$

Finally, observe that

$$\|D_F Mh\|^2 = \|Mh\|^2 - \|FMh\|^2, \quad \|D_X D_M h\|^2 = \|h\|^2 - \|Mh\|^2 - \|XD_M h\|^2.$$

Hence the proof of (3.2) is complete. □

4. Contractive block operators

The following theorem goes back to [16], [24], [32]; other proofs of the theorem can be found in [30], [31], [6], and an equivalent parametrization is given in [28]. The present proof is based on an approximation procedure and is along the lines of the proof in [8] for the parametrization of all quasi-selfadjoint extensions of a symmetric contraction.

Theorem 4.1. *Let $A \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, $B \in \mathbf{L}(\mathfrak{M}, \mathfrak{K})$, $C \in \mathbf{L}(\mathfrak{H}, \mathfrak{N})$, and $D \in \mathbf{L}(\mathfrak{M}, \mathfrak{N})$. The operator matrix*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{K} \\ \mathfrak{N} \end{pmatrix} \tag{4.1}$$

is a contraction if and only if T is of the form

$$T = \begin{pmatrix} A & D_{A^*} M \\ KD_A & -KA^*M + D_{K^*} X D_M \end{pmatrix}, \tag{4.2}$$

where $A \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, $M \in \mathbf{L}(\mathfrak{M}, \mathfrak{D}_{A^*})$, $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$, and $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$ are contractions, all uniquely determined by T. Furthermore, the following equality holds for all $f \in \mathfrak{H}$, $h \in \mathfrak{M}$:

$$\begin{aligned} &\left\| \begin{pmatrix} f \\ h \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} A & D_{A^*} M \\ KD_A & -KA^*M + D_{K^*} X D_M \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix} \right\|^2 \\ &\qquad = \|D_K(D_A f - A^* Mh) - K^* X D_M h\|^2 + \|D_X D_M h\|^2 \geq 0. \end{aligned} \tag{4.3}$$

Proof. Assume that T is of the form (4.2), where $A \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, $M \in \mathbf{L}(\mathfrak{M}, \mathfrak{D}_{A^*})$, $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$, and $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$ are contractions. Then T can be written in the form (4.6). By applying Lemma 3.1 to (4.6) one obtains (4.3) from (3.2). Thus, T is a contraction.

Conversely, assume that $T \in \mathbf{L}(\mathfrak{H} \oplus \mathfrak{M}, \mathfrak{K} \oplus \mathfrak{N})$ in (4.1) is a contraction. Denote by $P_{\mathfrak{K}}$ and $P_{\mathfrak{N}}$ the orthogonal projections in the Hilbert space $\mathfrak{K} \oplus \mathfrak{N}$ onto $\mathfrak{K}$ and $\mathfrak{N}$, respectively, so that $A = P_{\mathfrak{K}} T \upharpoonright \mathfrak{H}$, $B = P_{\mathfrak{K}} T \upharpoonright \mathfrak{M}$, $C = P_{\mathfrak{N}} T \upharpoonright \mathfrak{H}$, and $D = P_{\mathfrak{N}} T \upharpoonright \mathfrak{M}$. Since $T \upharpoonright \mathfrak{H}$ is a contraction, one has

$$\|Cf\|^2 \leq \|f\|^2 - \|Af\|^2 \quad \text{for all } f \in \mathfrak{H}.$$

It follows that $C = KD_A$, where $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$ is a contraction, which is uniquely determined by A and C. The operators T^* and $T^* \upharpoonright \mathfrak{K}$ are also contractions. Therefore, one concludes that $B^* = ND_{A^*}$, where $N \in \mathbf{L}(\mathfrak{D}_{A^*}, \mathfrak{M})$ is a contraction, uniquely determined by A and B. Let $M := N^* \in \mathbf{L}(\mathfrak{M}, \mathfrak{D}_{A^*})$. Contractivity of T and the relation (2.1) imply

$$\begin{aligned} 0 &\leq \|f\|^2 + \|h\|^2 - \|Af + D_{A^*} Mh\|^2 - \|KD_A f + Dh\|^2 \\ &= \|f\|^2 + \|h\|^2 - \|Af\|^2 - \|D_{A^*} Mh\|^2 - 2\mathrm{Re}\,(Af, D_{A^*} Mh) - \|KD_A f + Dh\|^2 \\ &= \|D_A f\|^2 + \|A^* Mh\|^2 - 2\mathrm{Re}\,(D_A f, A^* Mh) + \|D_M h\|^2 - \|KD_A f + Dh\|^2 \\ &= \|D_A f - A^* Mh\|^2 + \|D_M h\|^2 - \|KD_A f + Dh\|^2, \end{aligned} \tag{4.4}$$

for all $f \in \mathfrak{H}$ and $h \in \mathfrak{M}$. Since $\operatorname{ran} M \subset \mathfrak{D}_{A^*}$ and $A^*\mathfrak{D}_{A^*} \subset \mathfrak{D}_A$, there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathfrak{D}_A$ such that for a given vector $h \in \mathfrak{M}$ the equality

$$\lim_{n\to\infty} D_A f_n = A^*Mh$$

is satisfied. Hence (4.4) implies that

$$\|KA^*Mh + Dh\|^2 \le \|D_M h\|^2, \quad h \in \mathfrak{M}.$$

Similarly taking into account that T^* is a contraction one gets

$$\|M^*AK^*g + D^*g\|^2 \le \|D_{K^*} g\|^2, \quad g \in \mathfrak{N}.$$

The last inequality yields that there exists a contraction $Z \in \mathbf{L}(\mathfrak{D}_{K^*}, \mathfrak{M})$ such that

$$M^*AK^* + D^* = ZD_{K^*},$$

and Z is uniquely determined by M, A, K, and D; thus, in particular, by T. Substituting $D = -KA^*M + D_{K^*}Z^*$ into (4.4) shows that for all $f \in \mathfrak{H}$, $h \in \mathfrak{M}$,

$$\begin{aligned}
&\|D_A f - A^*Mh\|^2 + \|D_M h\|^2 - \|KD_A f - KA^*Mh + D_{K^*}Z^*h\|^2 \\
&= \|D_A f - A^*Mh\|^2 + \|D_M h\|^2 - \|K(D_A f - A^*Mh)\|^2 + \|K^*Z^*h\|^2 - \|Z^*h\|^2 \\
&\quad - 2\mathrm{Re}\,(D_K(D_A f - A^*Mh), K^*Z^*h) \\
&= \|D_K(D_A f - A^*Mh) - K^*Z^*h\|^2 + \|D_M h\|^2 - \|Z^*h\|^2 \ge 0.
\end{aligned}$$

Finally, choose a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathfrak{D}_A$ such that for a given vector $h \in \mathfrak{M}$ the equality

$$\lim_{n\to\infty} D_K D_A f_n = D_K A^*Mh + K^*Z^*h$$

is satisfied. This yields $\|Z^*h\| \le \|D_M h\|$ for all $h \in \mathfrak{M}$. Therefore there exists a contraction $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$, uniquely determined by Z and M, such that $Z^* = XD_M$. Thus

$$D = -KA^*M + D_{K^*}XD_M \tag{4.5}$$

and here all the contractions are uniquely determined by T. This completes the proof. □

Observe that if T is given by (4.2), then it can be rewritten in the form

$$T = \begin{pmatrix} I_{\mathfrak{K}} & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} A & D_{A^*} \\ D_A & -A^* \end{pmatrix} \begin{pmatrix} I_{\mathfrak{H}} & 0 \\ 0 & M \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & D_{K^*}XD_M \end{pmatrix},$$

where the operators

$$\mathcal{K} = \begin{pmatrix} I_{\mathfrak{K}} & 0 \\ 0 & K \end{pmatrix} : \begin{pmatrix} \mathfrak{K} \\ \mathfrak{D}_A \end{pmatrix} \to \begin{pmatrix} \mathfrak{K} \\ \mathfrak{N} \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} I_{\mathfrak{H}} & 0 \\ 0 & M \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_{A^*} \end{pmatrix}$$

are contractions and the operator

$$\mathcal{U} = \begin{pmatrix} A & D_{A^*} \\ D_A & -A^* \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_{A^*} \end{pmatrix} \to \begin{pmatrix} \mathfrak{K} \\ \mathfrak{D}_A \end{pmatrix}$$

is unitary. Introduce the contraction $\mathcal{X}$ by

$$\mathcal{X} = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_M \end{pmatrix} \to \begin{pmatrix} \mathfrak{K} \\ \mathfrak{D}_{K^*} \end{pmatrix}.$$

Since $D_{\mathcal{K}^*} = 0 \oplus D_{K^*} \in \mathbf{L}(\mathfrak{K} \oplus \mathfrak{N})$ and $D_{\mathcal{M}} = 0 \oplus D_M \in \mathbf{L}(\mathfrak{H} \oplus \mathfrak{M})$ one can write

$$T = \mathcal{K}\,\mathcal{U}\,\mathcal{M} + D_{\mathcal{K}^*}\,\mathcal{X}\,D_{\mathcal{M}}. \tag{4.6}$$

Corollary 4.2. *Let $A \in \mathbf{L}(\mathfrak{H},\mathfrak{K})$ be a contraction. Assume that $K \in \mathbf{L}(\mathfrak{D}_A,\mathfrak{N})$, $M \in \mathbf{L}(\mathfrak{M},\mathfrak{D}_{A^*})$, and $X \in \mathbf{L}(\mathfrak{D}_M,\mathfrak{D}_{K^*})$ are contractions. Then the operator T in (4.2) is:*

(i) *isometric if and only if $D_X D_M = 0$ and $D_K D_A = 0$;*
(ii) *co-isometric if and only if $D_{X^*} D_{K^*} = 0$ and $D_{M^*} D_{A^*} = 0$.*

Proof. By symmetry it suffices to prove statement (i). Suppose that $D_X D_M = 0$. If, in addition, A is isometric, i.e., if $D_A = 0$, then $\mathfrak{D}_A = \{0\}$, $A^*{\upharpoonright}\mathfrak{D}_{A^*} = 0$, and $\operatorname{dom} K = \operatorname{dom} D_K = \{0\}$, so that $K^* = 0 \in \mathbf{L}(\mathfrak{N},\{0\})$. Now the identity (4.3) in Theorem 4.1 shows that T is isometric. On the other hand, if A is not isometric but $D_K D_A = 0$, then $D_K = 0$, i.e., K is isometric, since $\operatorname{dom} D_K = \mathfrak{D}_A$. In this case $K^* D_{K^*} = D_K K^* = 0$ and since $\operatorname{ran} X \subset \mathfrak{D}_{K^*}$, one has also $K^* X = 0$. Thus, again (4.3) shows that T is isometric.

Conversely, assume that T is isometric. Then from (4.3) it is clear that $D_X D_M = 0$. Moreover, taking $h = 0$ in (4.3) one obtains $D_K D_A = 0$. □

As the proof shows the equality $D_K D_A = 0$ means that there are two cases:

(1) $D_A = 0$, i.e. $\mathfrak{D}_A = \{0\}$;
(2) $\mathfrak{D}_A \neq \{0\}$ and $D_K = 0$.

In the case (1) A is isometric. In the case (2) the operator K is isometric. Likewise, one can interpret the equality $D_X D_M = 0$: either M is isometric, or M is not isometric, in which case X is isometric.

5. Transfer functions of passive systems

Let $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ be a passive linear system with the corresponding block representation

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}. \tag{5.1}$$

The next theorem gives an expression of the transfer function $\Theta_\tau(\lambda)$ of τ by means of the characteristic function of the main operator A and the parameters of the block representation of the operator T in (4.2). For this purpose, define the following operator-valued holomorphic functions

$$\varphi(\lambda) := \begin{pmatrix} D_K \Phi_{A^*}(\lambda) M - K^* X D_M \\ -D_X D_M \end{pmatrix} : \mathfrak{M} \to \begin{pmatrix} \mathfrak{D}_K \\ \mathfrak{D}_M \end{pmatrix}, \quad \lambda \in \mathbb{D}, \tag{5.2}$$

and

$$\psi(\lambda) := \begin{pmatrix} K\Phi_{A^*}(\lambda) D_{M^*} - D_{K^*} X M^* & D_{K^*} D_{X^*} \end{pmatrix} : \begin{pmatrix} \mathfrak{D}_{M^*} \\ \mathfrak{D}_{K^*} \end{pmatrix} \to \mathfrak{N}, \quad \lambda \in \mathbb{D}. \tag{5.3}$$

Theorem 5.1. *Let $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ be a passive linear system and let (4.2) be the representation of the block operator T in (5.1). Then the transfer function $\Theta_\tau(\lambda)$ of τ and the characteristic function $\Phi_{A^*}(\lambda)$ of A^* in (2.4) are connected via*

$$\Theta_\tau(\lambda) = K\Phi_{A^*}(\lambda) M + D_{K^*} X D_M, \quad \lambda \in \mathbb{D}; \tag{5.4}$$

in particular, $\Theta_\tau(\lambda) \in \mathbf{S}(\mathfrak{M},\mathfrak{N})$. In addition, the identities

$$\left\| D_{\Theta_\tau(\lambda)} h \right\|^2 = \left\| D_{\Phi_{A^*}(\lambda)} M h \right\|^2 + \|\varphi(\lambda) h\|^2, \quad h \in \mathfrak{M}, \tag{5.5}$$

$$\left\|D_{\Theta_\tau^*(\lambda)}g\right\|^2=\left\|D_{\Phi_A(\bar\lambda)}K^*g\right\|^2+\|\psi(\lambda)^*g\|^2,\quad g\in\mathfrak{N}, \tag{5.6}$$

hold and the functions $\varphi(\lambda)$ and $\psi(\lambda)$ in (5.2) *and* (5.3) *are Schur functions.*

Proof. Using (4.2) the equalities (1.3) and (2.4) yield (5.4). It is clear that $\Phi_{A^*}(\lambda)$ is a Schur function. Hence, by Lemma 3.1 $\Theta_\tau(\lambda)$ is a Schur function, too. The relations

$$\begin{aligned}\left\|D_{\Theta_\tau(\lambda)}h\right\|^2=&\left\|D_{\Phi_{A^*}(\lambda)}Mh\right\|^2+\|D_XD_Mh\|^2\\&+\left\|(D_K\Phi_{A^*}(\lambda)M-K^*XD_M)\,h\right\|^2,\quad h\in\mathfrak{M},\end{aligned} \tag{5.7}$$

$$\begin{aligned}\left\|D_{\Theta_\tau^*(\lambda)}g\right\|^2=&\left\|D_{\Phi_A(\bar\lambda)}K^*g\right\|^2+\|D_{X^*}D_{K^*}g\|^2\\&+\left\|\left(D_{M^*}\Phi_A(\bar\lambda)K^*-MX^*D_{K^*}\right)g\right\|^2,\quad g\in\mathfrak{N},\end{aligned} \tag{5.8}$$

follow from (3.2) and (2.4). Furthermore, the definitions (5.2) and (5.3) show that

$$\|\varphi(\lambda)h\|^2=\|D_XD_Mh\|^2+\left\|(D_K\Phi_{A^*}(\lambda)M-K^*XD_M)\,h\right\|^2,\quad h\in\mathfrak{M}, \tag{5.9}$$

and

$$\|\psi^*(\lambda)g\|^2=\|D_{X^*}D_{K^*}g\|^2+\left\|\left(D_{M^*}\Phi_A(\bar\lambda)K^*-MX^*D_{K^*}\right)g\right\|^2,\quad g\in\mathfrak{N}. \tag{5.10}$$

Now (5.7) and (5.8), together with (5.9) and (5.10) yield (5.5) and (5.6). It clear from these identities that the values of $\varphi(\lambda)$ and $\psi(\lambda)$, $\lambda\in\mathbb{D}$, are contractive operators and, hence, they are Schur functions. □

Proposition 5.2. *Let $\tau=\{A,B,C,D;\mathfrak{H},\mathfrak{M},\mathfrak{N}\}$ be a passive linear system and let Σ be the conservative system in* (2.5) *induced by the contraction A. Then the controllable and observable subspaces of the systems τ and Σ satisfy the inclusions*

$$\mathfrak{H}^c\subset\mathfrak{H}^c_\Sigma\quad\text{and}\quad\mathfrak{H}^o\subset\mathfrak{H}^o_\Sigma. \tag{5.11}$$

In particular, if the system τ is controllable (observable, minimal, simple), then so is the system Σ. Moreover, if τ is isometric (co-isometric), then the equality $\mathfrak{H}^o=\mathfrak{H}^o_\Sigma$ ($\mathfrak{H}^c=\mathfrak{H}^c_\Sigma$) holds.

Proof. The block representation (4.2) in Theorem 4.1 shows that $B=D_{A^*}M$ and $C=KD_A$. Hence the controllable and observable subspaces (1.4) for τ can be rewritten as

$$\mathfrak{H}^c=\overline{\operatorname{span}}\,\{\,A^nD_{A^*}M\mathfrak{M}:\,n\in\mathbb{N}_0\,\},\quad\mathfrak{H}^o=\overline{\operatorname{span}}\,\{\,A^{*n}D_AK^*\mathfrak{N}:\,n\in\mathbb{N}_0\,\}. \tag{5.12}$$

Since $\operatorname{ran}M\subset\mathfrak{D}_{A^*}$ and $\operatorname{ran}K^*\subset\mathfrak{D}_A$ the inclusions (5.11) follow directly from the representations of $\mathfrak{H}^c_\Sigma$ and $\mathfrak{H}^o_\Sigma$ in (2.6).

If τ is isometric then $D_KD_A=0$ by Corollary 4.2. Here either $D_A=0$, or $D_A\neq0$ in which case $D_K=0$. If $D_K=0$, i.e. K is isometric, then from (1.5) and (2.8) one obtains

$$(\mathfrak{H}^o)^\perp=\bigcap_{n=0}^\infty\ker\,(KD_AA^n)=\bigcap_{n=0}^\infty\ker\,(D_AA^n)=(\mathfrak{H}^o_\Sigma)^\perp.$$

If $D_A=0$ then clearly $\mathfrak{H}^o=\mathfrak{H}^o_\Sigma=\{0\}$. Thus in both cases the equality $\mathfrak{H}^o=\mathfrak{H}^o_\Sigma$ holds.

If τ is co-isometric then $D_{M^*}D_{A^*}=0$ by Corollary 4.2. If here $D_{M^*}=0$ then (1.5) and (2.7) imply

$$(\mathfrak{H}^c)^\perp=\bigcap_{n=0}^\infty\ker\,(M^*D_{A^*}A^{*n})=\bigcap_{n=0}^\infty\ker\,(D_{A^*}A^{*n})=(\mathfrak{H}^c_\Sigma)^\perp.$$

In the case that $D_{A^*}=0$ one has $\mathfrak{H}^c=\mathfrak{H}^c_\Sigma=\{0\}$. Therefore $\mathfrak{H}^c=\mathfrak{H}^c_\Sigma$. □

Corollary 5.3. *Let $\tau=\{A,B,C,D;\mathfrak{H},\mathfrak{M},\mathfrak{N}\}$ be a passive linear system and let (4.2) be the representation of the block operator T in (5.1). Then:*

(i) *If τ is isometric, then $\varphi(\lambda)=0$. In this case*

$$\left\|D_{\Theta_\tau(\lambda)}h\right\|=\left\|D_{\Phi_{A^*}(\lambda)}Mh\right\|,\quad h\in\mathfrak{M}.$$

Conversely, if $\varphi(\lambda)=0$ and τ is controllable, then τ is isometric.

(ii) *If τ is co-isometric, then $\psi(\lambda)=0$. In this case*

$$\left\|D_{\Theta^*_\tau(\lambda)}g\right\|=\left\|D_{\Phi_A(\bar\lambda)}K^*g\right\|,\quad g\in\mathfrak{N}.$$

Conversely, if $\psi(\lambda)=0$ and τ is observable, then τ is co-isometric.

Proof. (i) Assume that τ is isometric. According to Corollary 4.2 $D_XD_M=0$ and $D_KD_A=0$. Here either $D_A=0$, so that $\operatorname{dom}K=\operatorname{dom}D_K=\{0\}$ and $K^*=0$, or $D_K=0$ and then $K^*X=0$. In each case the definition (5.2) shows that $\varphi(\lambda)=0$.

Conversely, assume that $\varphi(\lambda)=0$ and that τ is controllable. In view of (5.2) the condition $\varphi(\lambda)=0$ means that

$$D_XD_M=0,\quad D_K\Phi_{A^*}(\lambda)M=K^*XD_M,\quad \lambda\in\mathbb{D}. \tag{5.13}$$

The definition (2.4) of $\Phi_{A^*}(\lambda)$ implies the power series representation

$$\Phi_{A^*}(\lambda)=-A^*+\sum_{n=0}^\infty\lambda^{n+1}D_AA^nD_{A^*}, \tag{5.14}$$

which together with the second identity in (5.13) gives

$$-D_KA^*M=K^*XD_M \tag{5.15}$$

and

$$D_KD_AA^nD_{A^*}M=0,\quad n\in\mathbb{N}_0. \tag{5.16}$$

Since τ is controllable, (5.16) combined with (5.12) yields $D_KD_A=0$. By Corollary 4.2 τ is isometric and (i) is proved.

The proof of (ii) is similar. For later use we only mention that $\psi(\lambda)=0$ is equivalent to

$$D_{X^*}D_{K^*}=0,\quad D_{M^*}\Phi_A(\lambda)K^*=MX^*D_{K^*},\quad \lambda\in\mathbb{D}, \tag{5.17}$$

where $\Phi_A(\lambda)=\Phi_{A^*}(\bar\lambda)^*$ is the characteristic function of the contraction A; see (2.3). □

6. Isometric, co-isometric, and conservative systems

A function $\Theta(\lambda)\in\mathbf{S}(\mathfrak{M},\mathfrak{N})$ is said to be *inner* if $\Theta^*(\xi)\Theta(\xi)=I_{\mathfrak{M}}$ for almost all $\xi\in\mathbb{T}$, and is said to be *co-inner* if $\Theta(\xi)\Theta^*(\xi)=I_{\mathfrak{N}}$ for almost all $\xi\in\mathbb{T}$. A function $\Theta(\lambda)\in\mathbf{S}(\mathfrak{M},\mathfrak{N})$ is said to be *bi-inner* if it is both inner and co-inner. A contraction A in a Hilbert space $\mathfrak{H}$ belongs to the classes $C_{0\cdot}$ or $C_{\cdot 0}$ if

$$s-\lim_{n\to\infty}A^n=0\quad\text{or}\quad s-\lim_{n\to\infty}A^{*n}=0,$$

respectively. By definition, $C_{00}:=C_{0\cdot}\cap C_{\cdot 0}$. The completely non-unitary part of a contraction A belongs to the class $C_{\cdot 0}$, $C_{0\cdot}$, or C_{00} if and only if its characteristic function $\Phi_A(\lambda)$ in (2.3) is inner, co-inner, or bi-inner, respectively; cf. [33].

Lemma 6.1. *Let $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ be a passive system with transfer function $\Theta_\tau(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$. If $\Theta_\tau(\lambda)$ is inner, then the restriction $A{\upharpoonright}\mathfrak{H}^c$ belongs to the class $C_{0\cdot}$. If $\Theta_\tau(\lambda)$ is co-inner, then the restriction $A^*{\upharpoonright}\mathfrak{H}^o$ belongs to the class $C_{0\cdot}$.*

Proof. If $\Theta_\tau(\lambda)$ is inner, then (5.5) in Theorem 5.1 implies that $D_{\Phi_{A^*}}(\xi)M = 0$ for almost all $\xi \in \mathbb{T}$, i.e.,

$$||\Phi_{A^*}(\xi)Mh||^2 = ||Mh||^2 \quad \text{for almost all} \quad \xi \in \mathbb{T}, \quad h \in \mathfrak{M}.$$

Therefore, the norm of the vector-function $\Phi_{A^*}(\xi)Mh$ in the Hardy space $H^2(\mathfrak{D}_A)$ equals $||Mh||$; cf. [33]. From (2.4) one obtains

$$||Mh||^2 = ||\Phi_{A^*}(\xi)Mh||^2_{H^2(\mathfrak{D}_A)} = \sum_{n=0}^{\infty} ||D_A A^n D_{A^*} Mh||^2 + ||A^*Mh||^2, \quad h \in \mathfrak{M}.$$

This implies that

$$||D_{A^*}Mh||^2 - \lim_{m\to\infty} ||A^m D_{A^*}Mh||^2 = ||D_{A^*}Mh||^2, \quad h \in \mathfrak{M},$$

and, consequently,

$$\lim_{m\to\infty} A^m D_{A^*}Mh = 0, \quad h \in \mathfrak{M}.$$

Now for every $n \in \mathbb{N}_0$

$$\lim_{m\to\infty} A^m \left(A^n D_{A^*}Mh\right) = A^n \left(\lim_{m\to\infty} A^m D_{A^*}Mh\right) = 0, \quad h \in \mathfrak{M}. \tag{6.1}$$

Since $\mathfrak{H}^c = \overline{\mathrm{span}}\,\{ A^n D_{A^*} M\mathfrak{M} \,:\, n \in \mathbb{N}_0 \}$ and A is contractive, the identity (6.1) implies that $\lim_{m\to\infty} A^m k = 0$ for all $k \in \mathfrak{H}^c$, i.e., the restriction $A{\upharpoonright}\mathfrak{H}^c$ belongs to the class $C_{0\cdot}$.

Similarly one can prove the other statement. □

The following result from [33] is needed in the sequel.

Theorem 6.2. ([33]) *Let $\mathfrak{M}$ be a separable Hilbert space and let $N(\xi)$, $\xi \in \mathbb{T}$, be an $\mathbf{L}(\mathfrak{M})$-valued measurable function such that $0 \le N(\xi) \le I_{\mathfrak{M}}$. Then there exist a Hilbert space $\mathfrak{K}$ and an outer function $\varphi(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{K})$ satisfying the following conditions:*

(i) *$\varphi^*(\xi)\varphi(\xi) \le N^2(\xi)$ almost everywhere on $\mathbb{T}$;*
(ii) *if $\widetilde{\mathfrak{K}}$ is a Hilbert space and $\widetilde{\varphi}(\lambda) \in \mathbf{S}(\mathfrak{M}, \widetilde{\mathfrak{K}})$ is such that $\widetilde{\varphi}^*(\xi)\widetilde{\varphi}(\xi) \le N^2(\xi)$ almost everywhere on $\mathbb{T}$, then $\widetilde{\varphi}^*(\xi)\widetilde{\varphi}(\xi) \le \varphi^*(\xi)\varphi(\xi)$ almost everywhere on $\mathbb{T}$.*

Moreover, the function $\varphi(\lambda)$ is uniquely defined up to a left constant unitary factor.

Assume that $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ and denote by $\varphi_\Theta(\xi)$ and $\psi_\Theta(\xi)$, $\xi \in \mathbb{T}$, the functions which are described in (1.6), (1.7), and (1.8). Their existence is guaranteed by Theorem 6.2 with $N^2(\xi) = I_{\mathfrak{M}} - \Theta^*(\xi)\Theta(\xi)$ and $N^2(\bar{\xi}) = I_{\mathfrak{N}} - \Theta(\bar{\xi})\Theta^*(\bar{\xi})$, respectively. Clearly, if $\Theta(\lambda)$ is inner or co-inner, then $\varphi_\Theta = 0$ or $\psi_\Theta = 0$, respectively. In the case that the system τ is simple and conservative the following result has been established in [10], [11], [14], [18], [19], [20].

Theorem 6.3. *Let $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ be a simple conservative system with transfer function $\Theta_\tau(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$. Then:*

(i) *the subspace $(\mathfrak{H}^o)^\perp$ $((\mathfrak{H}^c)^\perp)$ is invariant under A (A^*) and the restriction $A{\upharpoonright}(\mathfrak{H}^o)^\perp$ $(A^*{\upharpoonright}(\mathfrak{H}^c)^\perp)$ is a unilateral shift;*

(ii) *the functions* $\varphi_\Theta(\lambda)$ *and* $\psi_\Theta(\lambda)$ *take the form*

$$\varphi_\Theta(\lambda) = P_\Omega(I_{\mathfrak{H}} - \lambda A)^{-1}B, \quad \psi_\Theta(\lambda) = C(I_{\mathfrak{H}} - \lambda A)^{-1}\upharpoonright \Omega_*, \tag{6.2}$$

where

$$\Omega = (\mathfrak{H}^o)^\perp \ominus A(\mathfrak{H}^o)^\perp, \quad \Omega_* = (\mathfrak{H}^c)^\perp \ominus A^*(\mathfrak{H}^c)^\perp, \tag{6.3}$$

and P_Ω *is the orthogonal projector in* $\mathfrak{H}$ *onto* Ω.

By Theorem 5.1 the functions $\varphi(\lambda)$ and $\psi(\lambda)$ defined by (5.2) and (5.3) satisfy

$$\varphi^*(\lambda)\varphi(\lambda) \leq I_{\mathfrak{M}} - \Theta_\tau^*(\lambda)\Theta_\tau(\lambda), \quad \psi(\lambda)\psi^*(\lambda) \leq I_{\mathfrak{N}} - \Theta_\tau(\lambda)\Theta_\tau^*(\lambda), \tag{6.4}$$

for all $\lambda \in \mathbb{D}$; see (5.5) and (5.6). Since all the functions involved in these inequalities have limiting values almost everywhere on $\mathbb{T}$, it follows from (6.4) that

$$\varphi^*(\xi)\varphi(\xi) \leq I_{\mathfrak{M}} - \Theta_\tau^*(\xi)\Theta_\tau(\xi), \quad \psi(\lambda)\psi^*(\lambda) \leq I_{\mathfrak{N}} - \Theta_\tau(\lambda)\Theta_\tau^*(\lambda), \tag{6.5}$$

for almost all $\xi \in \mathbb{T}$. Hence, by Theorem 6.2, the functions $\varphi(\lambda)$ and $\psi(\lambda)$ satisfy the inequalities

$$\varphi(\xi)^*\varphi(\xi) \leq \varphi_\Theta^*(\xi)\varphi_\Theta(\xi), \quad \psi(\xi)\psi^*(\xi) \leq \psi_\Theta(\xi)\psi_\Theta^*(\xi), \tag{6.6}$$

for almost all $\xi \in \mathbb{T}$. In particular, (6.6) shows that if $\varphi_\Theta(\xi) = 0$, then $\varphi(\xi) = 0$ and if $\psi_\Theta(\xi) = 0$, then $\psi(\xi) = 0$.

For a proof of Theorem 6.3 see [10], [11], [14]; the proof is based on the notions of *optimal* and $*$-*optimal* passive systems. In the sequel the representations of the functions $\varphi_\Theta(\lambda)$ and $\psi_\Theta(\lambda)$ as given in Theorem 6.3 are needed. Furthermore, the connections between the system τ and the system Σ in (2.5) will be used; cf. Theorem 5.1.

Corollary 6.4. *If the system* $\tau = \{A, B, C, D, \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ *is simple and conservative then* $\varphi_\Theta(\lambda) = 0$ *(*$\psi_\Theta(\lambda) = 0$*) if and only if the system* τ *is observable (controllable).*

Proof. Let $\varphi_\Theta(\lambda) = 0$ for all $\lambda \in \mathbb{D}$. In view of (6.2) this means that $P_\Omega(I_{\mathfrak{H}} - \lambda A)^{-1}B = 0$ for all $\lambda \in \mathbb{D}$. Therefore, $P_\Omega A^n Bf = 0$ for all $f \in \mathfrak{M}$ and $n = 0, 1, \ldots$. This is equivalent to the equality $P_\Omega \mathfrak{H}^c = 0$, i.e., $\Omega \subset (\mathfrak{H}^c)^\perp$. On the other hand, (6.3) shows that $\Omega \subset (\mathfrak{H}^o)^\perp$. Thus, $\Omega \subset (\mathfrak{H}^c)^\perp \bigcap (\mathfrak{H}^o)^\perp$ and, because the system τ is simple, this gives $\Omega = \{0\}$, i.e., $A(\mathfrak{H}^o)^\perp = (\mathfrak{H}^o)^\perp$. Since τ is isometric, the equality $\mathfrak{H}^o = \mathfrak{H}^o_\Sigma$ holds by Proposition 5.2 and hence by Remark 2.1 $(\mathfrak{H}^o_\Sigma)^\perp = \{0\}$, i.e., the systems Σ and τ are observable.

Conversely, if τ is observable then $(\mathfrak{H}^o_\Sigma)^\perp = (\mathfrak{H}^o_\tau)^\perp = \{0\}$, so that $\Omega = \{0\}$ and $\varphi_\Theta(\lambda) = 0$.

Similarly it is seen that $\psi_\Theta(\lambda) = 0$ if and only if τ is controllable. □

Theorem 6.5. *Let* $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ *be a passive system with transfer function* $\Theta_\tau(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$. *Assume that* $\Theta_\tau(\lambda)$ *is not constant. Then:*

(i) *If* τ *is controllable and* $\varphi_\Theta(\lambda) = 0$, *then* τ *is isometric and minimal. Moreover, if* $\Theta_\tau(\lambda)$ *is inner, then* $A \in C_{0\,\cdot}$.

(ii) *If* τ *is observable and* $\psi_\Theta(\lambda) = 0$, *then* τ *is co-isometric and minimal. Moreover, if* $\Theta_\tau(\lambda)$ *is co-inner, then* $A \in C_{\cdot\,0}$.

(iii) *If* τ *is simple,* $\varphi_\Theta(\lambda) = 0$, *and* $\psi_\Theta(\lambda) = 0$, *then* τ *is conservative and minimal. Moreover, if* $\Theta_\tau(\lambda)$ *is bi-inner, then* $A \in C_{00}$.

Proof. (i) & (ii) It suffices to prove (i), as the proof of (ii) is completely similar. Therefore, assume that τ is controllable and that $\varphi_\Theta(\lambda)=0$. Then (6.6) implies that $\varphi(\lambda)=0$ and hence τ is isometric by Corollary 5.3. By Corollary 4.2 this means that $D_X D_M=0$ and $D_K D_A=0$. If $D_A=0$, i.e., A is isometric, then $\mathfrak{D}_A=\{0\}$, $D_{K^*}=I_{\mathfrak{N}}$, and (5.4) in Theorem 5.1 shows that $\Theta_\tau(\lambda)=XD_M$ for all $\lambda\in\mathbb{D}$, which is impossible as $\Theta_\tau(\lambda)$ is not constant. Therefore, $D_A\neq 0$ and then $D_K=0$, i.e., K is isometric. Next it is shown that for the conservative system Σ in (2.5) one has $\varphi_\Sigma(\lambda)=0$. Since τ is controllable, also Σ is controllable and in particular simple; see Proposition 5.2. By (6.2) in Theorem 6.3

$$\varphi_\Sigma(\lambda)=P_\Omega(I_{\mathfrak{H}}-\lambda A)^{-1}D_{A^*}=P_\Omega\left(\sum_{n=0}^{\infty}\lambda^n A^n D_{A^*}\right),\quad \lambda\in\mathbb{D},$$

where P_Ω is the orthogonal projection from $\mathfrak{H}$ onto $\Omega:=(\mathfrak{H}^o_\Sigma)^\perp\ominus A(\mathfrak{H}^o_\Sigma)^\perp$, see (2.8). From the definition of the function $\varphi_\Sigma(\lambda)$ and (5.5) one obtains

$$\|\varphi_\Sigma(\lambda)Mh\|^2\le\|D_{\Phi_{A^*}(\lambda)}Mh\|^2\le\|D_{\Theta_\tau(\lambda)}h\|^2,\quad h\in\mathfrak{M}.$$

Now the assumption $\varphi_\Theta(\lambda)=0$ and Theorem 6.2 imply that $\varphi_\Sigma(\lambda)M=0$, $\lambda\in\mathbb{D}$. Hence

$$P_\Omega A^n D_{A^*}M=0,\quad n\in\mathbb{N}_0,$$

and thus $P_\Omega\mathfrak{H}^c=\{0\}$; see (5.12). Since τ is controllable, one has $P_\Omega=0$. This shows that $\varphi_\Sigma(\lambda)=0$ and hence by Corollary 6.4 Σ is also observable, i.e., $\mathfrak{H}^o_\Sigma=\mathfrak{H}$. Since τ is isometric, Proposition 5.2 shows that also τ is observable. Thus, τ is minimal.

Now assume that $\Theta_\tau(\lambda)$ is inner. Since τ is controllable one has $\mathfrak{H}^c=\mathfrak{H}$ and thus $A\in C_{0\,\cdot}$ by Lemma 6.1.

(iii) Let τ be simple and assume that $\varphi_\Theta(\lambda)=0$ and $\psi_\Theta(\lambda)=0$. Then (6.6) implies that $\varphi(\lambda)=0$ and $\psi(\lambda)=0$, and hence the inequalities in (5.13) and (5.17) hold. Therefore, see (5.15) and (5.16), one obtains

$$-D_K A^*M=K^*XD_M,\quad -D_{M^*}AK^*=MX^*D_{K^*},\tag{6.7}$$

and

$$D_K D_A A^n D_{A^*}M=0,\quad D_{M^*}D_{A^*}A^{*n}D_A K^*=0,\quad n\in\mathbb{N}_0.\tag{6.8}$$

Let $f\in\mathfrak{M}$. The equality $-D_K A^*Mf=K^*XD_Mf$ and

$$\operatorname{ran}D_K\cap\operatorname{ran}K^*=\operatorname{ran}D_K K^*=\operatorname{ran}K^*D_{K^*}$$

(cf. [7]), imply that $K^*(XD_Mf-D_{K^*}v)=0$ for some $v\in\mathfrak{N}$. Since $\ker K^*\subset\operatorname{ran}D^2_{K^*}$, one has $XD_Mf=D_{K^*}h_1$ for some $h_1\in\mathfrak{N}$. Then $D_K(A^*Mf+K^*h_1)=0$ so that $g_0:=A^*Mf+K^*h_1\in\ker D_K=\ker D^2_K$, i.e., $g_0=K^*Kg_0$, and here $h_0:=Kg_0\in\ker D_{K^*}$. Hence, $A^*Mf=-K^*h_1+K^*h_0=-K^*h$ with $h=h_1-h_0$ and, moreover, $D_{K^*}h=D_{K^*}h_1=XD_Mf$. Now $-D_{M^*}AK^*h=MX^*D_{K^*}h$ gives

$$D_{M^*}AA^*Mf=MX^*XD_Mf.$$

Taking into account the equality $D_XD_M=0$ one obtains

$$D_{M^*}AA^*Mf=MX^*XD_Mf=MD_Mf=D_{M^*}Mf$$

for every $f\in\mathfrak{M}$. Hence $D_{M^*}D^2_{A^*}M=0$. It follows that

$$0=D_{M^*}D^2_{A^*}MD_M=(D_{A^*}D_{M^*})^*\,(D_{A^*}D_{M^*})\,M$$

and hence $D_{A^*}D_{M^*}M = 0$. Since $M^* \in \mathbf{L}(\mathfrak{D}_{A^*}, \mathfrak{M})$, one has $\mathfrak{D}_{M^*} \subset \mathfrak{D}_{A^*}$. Therefore

$$D_{M^*}M = MD_M = 0.$$

This means that the operator M^* is a partial isometry. Similarly it can be proved that the operator K is a partial isometry. It follows that

$$\mathfrak{D}_A = \operatorname{ran} K^* \oplus \ker K, \quad D_K \upharpoonright \operatorname{ran} K^* = 0, \quad D_K \upharpoonright \ker K = I_{\ker K},$$
$$\mathfrak{D}_{A^*} = \operatorname{ran} M \oplus \ker M^*, \quad D_{M^*} \upharpoonright \operatorname{ran} M = 0, \quad D_{M^*} \upharpoonright \ker M^* = I_{\ker M^*}.$$

Since $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$, (6.7) gives $D_K A^* M = 0$ and $D_{M^*}AK^* = 0$. This means that

$$A^* : \operatorname{ran} M \to \operatorname{ran} K^*, \quad A : \operatorname{ran} K^* \to \operatorname{ran} M,$$

and consequently

$$A : \ker K \to \ker M^*, \quad A^* : \ker M^* \to \ker K.$$

Therefore

$$D_{A^*}^2 : \ker M^* \to \ker M^*, \quad D_A^2 : \ker K \to \ker K,$$

so that

$$M^* D_{A^*}^2 \varphi = 0 \text{ for all } \varphi \in \ker M^*, \quad KD_A^2\psi = 0 \text{ for all } \psi \in \ker K.$$

The equalities (6.8) yield

$$M^* D_{A^*} A^{*n} D_A D_K = 0, \quad KD_A A^n D_{A^*} D_{M^*} = 0, \quad n \in \mathbb{N}_0.$$

Now, let $\psi \in \ker K$. Then $\varphi = A\psi \in \ker M^*$ and $D_{M^*}A\psi = A\psi$, so that

$$0 = KD_A A^n D_{A^*} A\psi = KD_A A^{n+1} D_A \psi \quad \text{for all } n \in \mathbb{N}_0.$$

Since $KD_A^2\psi = 0$, one has in fact

$$KD_A A^m D_A \psi = 0, \quad m \in \mathbb{N}_0.$$

Similarly, $M^* D_{A^*} A^{*m} D_A \psi = 0$, $m \in \mathbb{N}_0$. This means that the vector $D_A\psi$ belongs to $(\mathfrak{H}^c)^\perp \cap (\mathfrak{H}^o)^\perp$. Since τ is simple, it follows that $D_A\psi = 0$ and thus $\psi = 0$, i.e., $\ker K = \{0\}$. Similarly $\ker M^* = \{0\}$. Thus, the operators K and M^* are isometries. In addition $D_X D_M = 0$ and $D_{X^*}D_{K^*} = 0$; see (5.13), (5.17). Hence, by Corollary 4.2 the operator T in (4.2) is unitary, i.e., τ is conservative. Furthermore, minimality of τ follows from Corollary 6.4.

The last assertion is now obtained directly from (i) and (ii). Also, if $\Theta_\tau(\lambda)$ is bi-inner then $D_{\Phi^*(\xi)}M = 0$ and $D_{\Phi(\xi)}K^* = 0$ almost everywhere on $\mathbb{T}$. Since $\operatorname{ran} M = \mathfrak{D}_{A^*}$ and $\operatorname{ran} K^* = \mathfrak{D}_A$, the characteristic function $\Phi_{A^*}(\lambda)$ is bi-inner by Corollary 5.3. Since τ and hence also Σ is simple, the operator A is completely non-unitary; see Remark 2.1. Therefore, A belongs to the class C_{00}. □

Since every two controllable isometric (observable co-isometric) realizations of an operator-valued function from the Schur class are unitarily similar (see [2], [1]), the following theorem is a corollary of Theorem 6.5; cf. [10], [11], [14].

Theorem 6.6. *Let $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$. Then:*

(i) *if $\Theta(\lambda)$ is bi-inner and τ is a simple passive system with transfer function $\Theta(\lambda)$, then τ is conservative;*

(ii) *if $\varphi_\Theta(\lambda) = 0$ or $\psi_\Theta(\lambda) = 0$, then all passive minimal systems with transfer function $\Theta(\lambda)$ are unitarily equivalent, and if $\varphi_\Theta(\lambda) = 0$ and $\psi_\Theta(\lambda) = 0$, then they are in addition conservative.*

7. Bi-stable passive systems and bi-inner dilations of their transfer functions

Let $\Theta(\lambda)$ be a function from the Schur class $\mathbf{S}(\mathfrak{M},\mathfrak{N})$. Following [15] the function $\Theta(\lambda)$ is said to have an *inner dilation* if there exists a function $\Theta_r(\lambda)$ such that

$$\mathbf{\Theta}(\lambda)=\begin{pmatrix}\Theta(\lambda)\\ \Theta_r(\lambda)\end{pmatrix}\in \mathbf{S}(\mathfrak{M},\mathfrak{N}\oplus\mathfrak{L})$$

is inner. The function $\Theta(\lambda)$ is said to have a *co-inner dilation* if there exists a function $\Theta_l(\lambda)$ such that

$$\mathbf{\Theta}(\lambda)=\begin{pmatrix}\Theta_l(\lambda) & \Theta(\lambda)\end{pmatrix}\in \mathbf{S}(\mathfrak{K}\oplus\mathfrak{M},\mathfrak{N})$$

is co-inner. The function $\Theta(\lambda)$ is said to have a *bi-inner dilation* if there exist functions $\Theta_{11}(\lambda)$, $\Theta_{22}(\lambda)$, and $\Theta_{21}(\lambda)$ such that

$$\mathbf{\Theta}(\lambda)=\begin{pmatrix}\Theta_{11}(\lambda) & \Theta(\lambda)\\ \Theta_{21}(\lambda) & \Theta_{22}(\lambda)\end{pmatrix}\in \mathbf{S}(\mathfrak{K}\oplus\mathfrak{M},\mathfrak{N}\oplus\mathfrak{L})$$

is bi-inner.

Recall that a system $\tau=\{A,B,C,D,\mathfrak{H},\mathfrak{M},\mathfrak{N}\}$ is said to be *strongly stable* (*strongly co-stable*) if the operator A belongs to the class $C_{0\cdot}$ ($C_{\cdot 0}$); cf. [10], [15]. The following result is well known; cf. [10]. The present proof is based on the parametrization in Theorem 4.1 and the relations between the transfer function $\Theta_\tau(\lambda)$ and the characteristic function $\Phi_{A^*}(\lambda)$ established in Theorem 5.1.

Proposition 7.1. (cf. [10]) *Let $\tau=\{A,B,C,D;\mathfrak{H},\mathfrak{M},\mathfrak{N}\}$ be a passive system with transfer function $\Theta_\tau(\lambda)$. Then:*

(i) *if τ is strongly stable then $\Theta_\tau(\lambda)$ has an inner dilation;*
(ii) *if τ is strongly co-stable then $\Theta_\tau(\lambda)$ has a co-inner dilation;*
(iii) *if τ is strongly stable and strongly co-stable then $\Theta_\tau(\lambda)$ has a bi-inner dilation.*

Proof. (i) Let τ be strongly stable. Then the characteristic function Φ_{A^*} is an inner function, i.e. $\Phi_{A^*}(\xi)^*\Phi_{A^*}(\xi)=I_{\mathfrak{D}_{A^*}}$ for almost all $\xi\in\mathbb{T}$. It follows from (5.5) that

$$I_{\mathfrak{N}}-\Theta_\tau(\xi)^*\Theta_\tau(\xi)=\varphi(\xi)^*\varphi(\xi),$$

for almost all $\xi\in\mathbb{T}$. In other words, the function

$$\mathbf{\Theta}(\lambda):=\begin{pmatrix}\Theta_\tau(\lambda)\\ \varphi(\lambda)\end{pmatrix},\quad \lambda\in\mathbb{D},$$

is an inner dilation of Θ_τ.

(ii) Let τ be strongly co-stable. Then the characteristic function $\Phi_A(\lambda)=\Phi_{A^*}(\overline{\lambda})^*$ is an inner function, i.e., $\Phi_A(\xi)^*\Phi_A(\xi)=I_{\mathfrak{D}_A}$ for almost all $\xi\in\mathbb{T}$. Now it follows from (5.6) that

$$I_{\mathfrak{N}}-\Theta_\tau(\xi)\Theta_\tau(\xi)^*=\psi(\xi)\psi(\xi)^*,$$

for almost all $\xi\in\mathbb{T}$. In other words, the function

$$\mathbf{\Theta}(\lambda):=\begin{pmatrix}\psi(\lambda) & \Theta_\tau(\lambda)\end{pmatrix},\quad \lambda\in\mathbb{D},$$

is a co-inner dilation of $\Theta(\lambda)$.

(iii) Let τ be strongly stable and strongly co-stable. Define

$$\Theta_{21}(\lambda) = \begin{pmatrix} K^*XM^* + D_K\Phi_{A^*}(\lambda)D_{M^*} & -K^*D_{X^*} \\ D_XM^* & X^* \end{pmatrix}, \quad \lambda \in \mathbb{D}.$$

Using the formulas in Theorem 5.1 it can be checked with a straightforward calculation that the function

$$\mathbf{\Theta}(\lambda) := \begin{pmatrix} \psi(\lambda) & \Theta_\tau(\lambda) \\ \Theta_{21}(\lambda) & \varphi(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{D},$$

satisfies the following two identities:

$$I - \mathbf{\Theta}(\lambda)^*\mathbf{\Theta}(\lambda) = M_1^* D^2_{\Phi_{A^*}(\lambda)} M_1, \quad I - \mathbf{\Theta}(\lambda)\mathbf{\Theta}(\lambda)^* = K_1^* D^2_{\Phi_A(\bar{\lambda})} K_1, \tag{7.1}$$

where $M_1 = \begin{pmatrix} D_{M^*} & 0 & M \end{pmatrix}$ and $K_1 = \begin{pmatrix} K^* & D_K & 0 \end{pmatrix}$. Since the characteristic function $\Phi_{A^*}(\lambda)$ is bi-inner, (7.1) shows that the function $\mathbf{\Theta}(\lambda)$ is a bi-inner dilation of $\Theta_\tau(\lambda)$. □

8. Operators of the class $C(\alpha)$ and corresponding passive systems

A bounded operator A on a Hilbert space $\mathfrak{H}$ is said to belong to the class $C(\alpha)$, $\alpha \in (0, \pi/2)$, if

$$\|A \sin\alpha \pm i\cos\alpha\, I\| \le 1, \tag{8.1}$$

cf. [4]. Let $A_R = (A + A^*)/2$ and $A_I = (A - A^*)/2i$ be the real and imaginary parts of A. Then the condition (8.1) is equivalent to

$$|(A_I f, f)| \le \frac{\tan\alpha}{2}\|D_A f\|^2 \quad \text{for all } f \in \mathfrak{H}, \tag{8.2}$$

cf. [5]. In particular (8.2) shows that the operators in $C(\alpha)$ are contractive. The inequality (8.2) also implies that it is natural to define the class $C(0)$ as the set of all selfadjoint contractions. Let

$$\widetilde{C} = \bigcup \{\, C(\alpha) : \ \alpha \in [0, \pi/2) \,\}.$$

The class $\widetilde{C}$ was studied in [4], [5]. In particular, it was proved in [4] that if $A \in \widetilde{C}$, then

(i) $\operatorname{ran} D_{A^n} = \operatorname{ran} D_{A^{*n}} = \operatorname{ran} D_{A_R}$ for all $n \in \mathbb{N}$;

(ii) the subspace $\mathfrak{D}_A = \mathfrak{D}_{A^*}$ reduces the operator A and, moreover, $A{\upharpoonright}\mathfrak{D}_A$ is a completely non-unitary contraction of the class C_{00}, while $A{\upharpoonright}\ker D_A$ is selfadjoint and unitary.

Let A belong to the class $\widetilde{C}$ and let $\Phi_A(\lambda)$ in (2.3) be its characteristic function. Then $\Phi_A(\lambda)$ is bi-inner (see [33]) and there exist unitary non-tangential strong limit values

$$\Phi_A(\pm 1) = s - \lim_{\lambda \to \pm 1} \Phi_A(\lambda);$$

cf. [4]. Observe that if A is a selfadjoint contraction (i.e. belongs to the class $C(0)$) then

$$\Phi_A(\pm 1) = \pm I_{\mathfrak{D}_A}.$$

Define the sets

$$P_+(\alpha) := \{\lambda : |\lambda\sin\alpha + i\cos\alpha| < 1\}, \quad P_-(\alpha) := \{\lambda : |\lambda\sin\alpha - i\cos\alpha| < 1\}.$$

Theorem 8.1. ([5]) *Let $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{N}, \mathfrak{N}\}$ be a passive linear system. If the operator T in* (5.1) *belongs to the class $C(\alpha)$ for $\alpha \in [0, \pi/2)$, then the transfer function $\Theta_\tau(\lambda)$ has the following properties:*

(i) $\Theta_\tau(\lambda)$ *is holomorphic on the domain*

$$P(\alpha) = P_+(\alpha) \cup P_-(\alpha);$$

(ii) *the following implications hold for all* $\beta \in [\alpha, \pi/2]$*:*

$$\lambda \in P_+(\beta) \Rightarrow \|\Theta_\tau(\lambda) \sin\beta + i\cos\beta I\|_{\mathfrak{N}} \leq 1,$$

and

$$\lambda \in P_-(\beta) \Rightarrow \|\Theta_\tau(\lambda) \sin\beta - i\cos\beta I\|_{\mathfrak{N}} \leq 1;$$

(iii) *the non-tangential limit values* $\Theta_\tau(\pm 1)$ *exist and they belong to the class* $C(\alpha)$ *in the Hilbert space* $\mathfrak{N}$*;*

(iv) *the coefficients* $\{G_n\}$ *of the Taylor expansion*

$$\Theta_\tau(\lambda) = \sum_{n=0}^{\infty} \lambda^n G_n, \quad |\lambda| < 1,$$

belong to the class $C(\alpha)$ *in the Hilbert space* $\mathfrak{N}$.

Observe, that $\sigma(T) \subset \overline{P_+(\alpha)} \cap \overline{P_-(\alpha)}$, where $\overline{P_\pm(\alpha)} := \{\lambda : |\lambda \sin\alpha \pm i\cos\alpha| \leq 1\}$. It follows from (ii) that for $\beta \in [\alpha, \pi/2)$ the values $\Theta_\tau(\lambda)$ with $\lambda \in \overline{P_+(\beta)} \cap \overline{P_-(\beta)}$ belong to the class $C(\beta)$ in the Hilbert space $\mathfrak{N}$.

The next proposition, when combined with Proposition 7.1, shows that if the operator T in (5.1) corresponding to the passive system $\tau = \{A, B, C, D, \mathfrak{H}, \mathfrak{N}, \mathfrak{N}\}$ belongs to the class $\widetilde{C}$, then the transfer function of τ admits a bi-inner dilation.

Proposition 8.2. *Let* $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{N}, \mathfrak{N}\}$ *be a passive linear system and let* T *in* (5.1) *belong to* $\widetilde{C}$*. If* τ *is controllable (observable), then* τ *is strongly stable and strongly co-stable.*

Proof. According to Theorem 4.1 the operator T in (5.1) takes the form (4.2), where $A \in \mathbf{L}(\mathfrak{H})$, $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$, $M \in \mathbf{L}(\mathfrak{M}, \mathfrak{D}_{A^*})$, and $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$ are contractions. Suppose that T belongs to $C(\alpha)$ for some $\alpha \in [0, \pi/2)$, i.e,

$$\|T \sin\alpha \pm i\cos\alpha I\| \leq 1.$$

Then

$$\|A \sin\alpha \pm i\cos\alpha I\| = \|(P_{\mathfrak{H}} T \upharpoonright \mathfrak{H}) \sin\alpha \pm i\cos\alpha I_{\mathfrak{H}}\| \leq 1.$$

This means that the operator A belongs to the class $C(\alpha)$ in the subspace $\mathfrak{H}$. It follows that $\operatorname{ran} D_A = \operatorname{ran} D_{A^*}$. As a consequence of Douglas theorem [25] it is seen that there exists a bounded and boundedly invertible operator L in the subspace $\mathfrak{H}$ such that

$$D_A = D_{A^*} L.$$

It follows by induction from the equalities $AD_A = D_{A^*}A$ and $A^*D_{A^*} = D_A A^*$ that

$$A^n D_{A^*} = D_{A^*}(AL^{-1})^n, \quad A^{*n} D_A = D_A (A^* L)^n, \quad n \in \mathbb{N}. \tag{8.3}$$

Suppose that the system τ is controllable, so that

$$\mathfrak{H}^c = \overline{\operatorname{span}}\{\operatorname{ran} A^n D_{A^*} M : n \in \mathbb{N}_0\} = \mathfrak{H}.$$

Then the first identities in (8.3) imply $\mathfrak{H}^c \subset \mathfrak{D}_{A^*}$ and hence $\mathfrak{D}_{A^*} = \mathfrak{H}$. Because $A \in C(\alpha)$, one has $\mathfrak{D}_A = \mathfrak{D}_{A^*}$ and therefore A belongs to the class C_{00}, i.e., the system τ is strongly stable and strongly co-stable.

Suppose that the system τ is observable. Then

$$\mathfrak{H}^o = \overline{\operatorname{span}}\,\{\, \operatorname{ran} A^{*n} D_A K^* : n \in \mathbb{N}_0 \,\} = \mathfrak{H}$$

and from the second identities in (8.3) one obtains $\mathfrak{H}^o \subset \mathfrak{D}_A$, so that $\mathfrak{D}_A = \mathfrak{H}$. Since $A \in C(\alpha)$, one obtains once again that A belongs to the class C_{00}, i.e. the system τ is strongly stable and strongly co-stable. □

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