




Friedrichs and Kreĭn type extensions in terms of representing maps

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Received: 27 March 2024 / Accepted: 13 July 2024 / Published online: 9 August 2024
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Abstract

A semibounded operator or relation S in a Hilbert space with lower bound $\gamma \in \mathbb{R}$ has a symmetric extension $S_f = S \widehat{+} (\{0\} \times \text{mul } S^*)$, the weak Friedrichs extension of S , and a selfadjoint extension S_F , the Friedrichs extension of S , that satisfy $S \subset S_f \subset S_F$. The Friedrichs extension S_F has lower bound γ and it is the largest semibounded selfadjoint extension of S . Likewise, for each $c \leq \gamma$, the relation S has a weak Kreĭn type extension $S_{k,c} = S \widehat{+} (\ker(S^* - c) \times \{0\})$ and Kreĭn type extension $S_{K,c}$ of S , that satisfy $S \subset S_{k,c} \subset S_{K,c}$. The Kreĭn type extension $S_{K,c}$ has lower bound c and it is the smallest semibounded selfadjoint extension of S which is bounded below by c . In this paper these special extensions and, more generally, all extremal extensions of S are constructed via the semibounded sesquilinear form $t(S)$ that is associated with S ; the representing map for the form $t(S) - c$ plays an essential role here.

Keywords Sesquilinear form · Representing map · Friedrichs extension · Kreĭn type extension · Extremal extension

Mathematics Subject Classification 47A07 · 47B25 · 47A67; 47A06 · 47B65

Dedicated to Professor Fedor Sukochev.

Communicated by Fritz Gesztesy.

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1 Introduction

Let $S \in \mathbf{L}(\mathfrak{H})$ be a linear operator or relation in a Hilbert space \mathfrak{H} whose lower bound $m(S)$ will be denoted by $\gamma \in \mathbb{R}$, i.e., γ is the supremum of all $c \in \mathbb{R}$ for which

$$(\varphi', \varphi) \geq c(\varphi, \varphi), \quad \{\varphi, \varphi'\} \in S. \quad (1.1)$$

In what follows the notation $\mathbf{L}(\mathfrak{H}, \mathfrak{K})$ stands for the class of all linear relations from \mathfrak{H} to the Hilbert space \mathfrak{K} ; this is abbreviated to $\mathbf{L}(\mathfrak{H})$ when $\mathfrak{K} = \mathfrak{H}$. It is clear from (1.1) that a semibounded relation is automatically symmetric. Among all semibounded selfadjoint extensions $H \in \mathbf{L}(\mathfrak{H})$ of S there exists a largest semibounded selfadjoint extension $S_F \in \mathbf{L}(\mathfrak{H})$, the Friedrichs extension, with the same lower bound as S , such that $H \leq S_F$. Likewise, for any choice of $c \leq \gamma$, there exists among all semibounded selfadjoint extensions $H \in \mathbf{L}(\mathfrak{H})$ of S a smallest semibounded selfadjoint extension $S_{K,c} \in \mathbf{L}(\mathfrak{H})$, the so-called Kreĭn type extension, with lower bound c , such that for any selfadjoint extension $H \in \mathbf{L}(\mathfrak{H})$ of S with $H \geq c$ one has $H \geq S_{K,c}$. Thus, for any semibounded selfadjoint extension H of S one has the equivalence

$$c \leq H \Leftrightarrow S_{K,c} \leq H \leq S_F. \quad (1.2)$$

For the ordering of semibounded selfadjoint operators and relations as in (1.2), see [7]. For each $c \leq \gamma$ the Friedrichs extension $S_F \in \mathbf{L}(\mathfrak{H})$ and the Kreĭn type extension $S_{K,c} \in \mathbf{L}(\mathfrak{H})$ are connected as linear relations by the identity

$$S_{K,c} = c + (((S - c)^{-1})_F)^{-1}. \quad (1.3)$$

In addition, there are also weak versions of the Friedrichs and Kreĭn type extensions, which are connected as in (1.3), and they play an important role in applications. Note that for $\gamma \geq 0$ and $c = 0$ these results (in the context of densely defined linear operators) go back to von Neumann [25] and Kreĭn [23, 24]; and then one usually speaks of the Kreĭn-von Neumann extension instead of the Kreĭn type extension. For a recent treatment of all the above extensions in the context of linear relations, see [7], but one may also consult [14] for various other approaches.

The purpose of the present paper is to view the construction of the above semibounded selfadjoint extensions systematically from the point of view of semibounded sesquilinear forms. In what follows $\mathbf{F}(\mathfrak{H})$ stands for the set of all sesquilinear forms whose domains belong to the Hilbert space \mathfrak{H} . Let $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be any semibounded form with lower bound $\gamma \in \mathbb{R}$. For each $c \leq \gamma$ there is a linear operator $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, where \mathfrak{K} is a Hilbert space, the so-called representing map, such that the nonnegative form $\mathfrak{t} - c$ can be written as

$$(\mathfrak{t} - c)[\varphi, \psi] = (Q_c\varphi, Q_c\psi), \quad \varphi, \psi \in \text{dom } \mathfrak{t} = \text{dom } Q_c.$$

The specific choice of Q_c does not play a central role in studying the form \mathfrak{t} . The properties of representing maps for general nonnegative forms have been studied in detail in [20, 21]; see also [35]. Now return to the semibounded linear relation

$S \in \mathbf{L}(\mathfrak{H})$ and assume that it has lower bound $\gamma \in \mathbb{R}$. Then there is a natural well-defined sesquilinear form $\mathfrak{t}(S) \in \mathbf{F}(\mathfrak{H})$, defined by the left-hand side of (1.1), with $\text{dom } \mathfrak{t}(S) = \text{dom } S$ and such that $\mathfrak{t}(S)$ has the same lower bound as S . In general, this form $\mathfrak{t}(S)$ is not closed; see, for instance, in [8, 10]. By using the specific properties of the representing map for the form $\mathfrak{t}(S) - c$, $c \leq \gamma$, it follows that there exists another semibounded form $\mathfrak{s}(S) \in \mathbf{F}(\mathfrak{H})$ which is closed with lower bound c , so that

$$\mathfrak{t}(S) \subset \mathfrak{s}(S).$$

In particular, this shows that the original form $\mathfrak{t}(S)$ is closable (see Proposition 2.3 and Theorem 2.5). The present paper gives an overview of the construction of the semibounded selfadjoint extensions of the semibounded relation $S \in \mathbf{L}(\mathfrak{H})$ by means of appropriate semibounded forms. The Friedrichs extension is constructed via the form $\mathfrak{t}(S)$ (see Theorem 4.1 and Corollary 4.2). Similarly, the Kreĭn type extension is constructed via the form $\mathfrak{s}(S)$ (see Theorem 3.1, Theorem 6.1, and Corollary 6.2). Furthermore, the closely related extremal extensions are constructed via intermediate forms (see Lemma 7.2, Lemma 7.3, and Theorem 7.5). The treatment is general in the sense that the possibly multivalued operator S need not be closed or densely defined. It is emphasized that all the above constructions are based on the representing maps of the various forms, such as $\mathfrak{t}(S) - c$ and $\mathfrak{s}(S) - c$; the systematic implementation of representing maps to construct the above mentioned selfadjoint extensions of S can be seen as one of the main contributions in the present paper. The arguments are much influenced by the work of Sebestyén, Stochel, Tarscsay, and coworkers; see [26, 30–33].

The contents of the paper are now described. The forms $\mathfrak{t}(S)$ and $\mathfrak{s}(S)$, induced by the semibounded relation $S \in \mathbf{L}(\mathfrak{H})$, are constructed in Sect. 2, where also their relevant properties are discussed. In Sect. 3 there is a further discussion of the forms $\mathfrak{t}(S)$ and $\mathfrak{s}(S)$; in particular, some attention will be paid to the (non)uniqueness of their representing maps. Here also the connection with the construction of Sebestyén and Stochel is discussed; see Remark 3.4. The Friedrichs extension S_F of S and its weak version are treated in Sect. 4. A similar reasoning involving the corresponding inverse $(S - c)^{-1}$ with $c \leq \gamma$ is given in Sect. 5. This leads to analogous extensions of $(S - c)^{-1}$ in terms of appropriate representing maps and companion relations. The Kreĭn type extension $S_{K,c}$ and its weak version are introduced in Sect. 6. The identity in (1.3) is a straightforward consequence of the approach via the representing maps for the forms associated with S and $(S - c)^{-1}$; cf. Section 5. As an application of the above constructions, also the notion of extremal extensions is treated in this context, see Sect. 7. By way of illustration Sect. 8 is devoted to symmetric relations whose domain and range are orthogonal; see [13, 27]. In Sect. 9 attention is paid to the representation in terms of a closable representing map of a symmetric relation generated by a semibounded form which itself is not necessarily closed.

The authors would like to thank the referees for their careful reading and helpful suggestions.

2 Semibounded relations and representing maps

In this section representing maps are introduced for semibounded forms which are generated by semibounded relations. Also a couple of their key properties are derived for later purposes.

Definition 2.1 Let $S \in \mathbf{L}(\mathfrak{H})$ be a semibounded relation with lower bound $\gamma \in \mathbb{R}$. Then the semibounded sesquilinear form $t(S) \in \mathbf{F}(\mathfrak{H})$ is defined by

$$t(S)[\varphi, \psi] = (\varphi', \psi) = (\varphi, \psi'), \quad \{\varphi, \varphi'\}, \{\psi, \psi'\} \in S, \tag{2.1}$$

so that $\text{dom } t(S) = \text{dom } S$.

Since $\text{mul } S \subset \text{mul } S^* = (\text{dom } S)^\perp$ the form $t(S)$ is well defined and, in fact, it is semibounded with lower bound $\gamma \in \mathbb{R}$. Recall that $\text{mul } T$ stands for the multivalued part of the relevant linear relation T .

The new ingredient in the present treatment is the notion of the representing map. Let $c \leq \gamma$, then for the form $t(S) - c$ there exists a so-called *representing map* Q_c such that

$$t(S)[\varphi, \psi] = c(\varphi, \psi) + (Q_c\varphi, Q_c\psi), \quad \varphi, \psi \in \text{dom } t(S) = \text{dom } S = \text{dom } Q_c, \tag{2.2}$$

where Q_c is a linear operator in $\mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$ with a Hilbert space \mathfrak{K}_c ; see [19, Lemma 4.1]. Due to the definition of $t(S)$ in (2.1) the representing map Q_c has a so-called *companion relation* $J_c \in \mathbf{L}(\mathfrak{K}_c, \mathfrak{H})$, which is introduced by

$$J_c = \{ \{Q_c\varphi, \varphi' - c\varphi\} : \{\varphi, \varphi'\} \in S \}. \tag{2.3}$$

The operator Q_c and the relation J_c form a dual pair in the sense that

$$Q_c \subset J_c^* \quad \text{or, equivalently,} \quad J_c \subset Q_c^*, \tag{2.4}$$

as follows from a combination of the identities (2.1) and (2.2):

$$(\varphi' - c\varphi, \psi) = (\varphi, \psi' - c\psi) = (Q_c\varphi, Q_c\psi), \quad \{\varphi, \varphi'\}, \{\psi, \psi'\} \in S. \tag{2.5}$$

The identity (2.3) shows that the relation $c + J_c Q_c$ is an extension of S :

$$S \subset c + J_c Q_c. \tag{2.6}$$

Taking into account (2.4) one gets immediately two nonnegative extensions for $S - c$:

$$S - c \subset Q_c^* Q_c \quad \text{and} \quad S - c \subset J_c J_c^*.$$

The adjoint $J_c^* \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$ of the companion relation J_c is a closed relation by definition. Let π be the orthogonal projection from \mathfrak{K}_c onto

$$\text{mul } J_c^* = (\text{dom } J_c)^\perp = (\text{ran } Q_c)^\perp. \tag{2.7}$$

Then it follows from (2.4) and (2.7) that, with the orthogonal operator part $(J_c^*)_{\text{reg}}$, one also has

$$Q_c \subset (J_c^*)_{\text{reg}}, \quad \text{where } (J_c^*)_{\text{reg}} = (I - \pi) J_c^*. \tag{2.8}$$

Definition 2.2 Let $S \in \mathbf{L}(\mathfrak{H})$ be a semibounded relation with lower bound $\gamma \in \mathbb{R}$ and let $c \leq \gamma$. Let $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$ be a representing map for $t(S) - c$ with companion relation $J_c \in \mathbf{L}(\mathfrak{K}_c, \mathfrak{H})$. Then the sesquilinear form $\mathfrak{s}(S) \in \mathbf{F}(\mathfrak{H})$ is defined by

$$\mathfrak{s}(S)[\varphi, \psi] = c(\varphi, \psi) + ((J_c^*)_{\text{reg}}\varphi, (J_c^*)_{\text{reg}}\psi), \quad \varphi, \psi \in \text{dom } \mathfrak{s}(S) = \text{dom } J_c^*. \tag{2.9}$$

Consequently, the form $\mathfrak{s}(S)$ is semibounded and closed.

By definition $(J_c^*)_{\text{reg}}$ is a closed operator and it is clear from (2.8) that

$$(Q_c)^{**} \subset (J_c^*)_{\text{reg}}.$$

Hence, it follows from (2.2) and (2.9) that the closed form $\mathfrak{s}(S)$ extends the form $t(S)$. Therefore the form $t(S)$ is closable; the representation of the closure is immediate from (2.2); cf. [21]. Observe, that the form $\mathfrak{s}(S)$ is bounded below by c . In fact, $\mathfrak{s}(S)$ has lower bound c : For $c < \gamma$ this follows from

$$\ker (J_c^*)_{\text{reg}} = \ker J_c^* = (\text{ran } J_c)^\perp = (\text{ran } (S - c))^\perp = \ker (S^* - c) \neq \{0\},$$

while for $c = \gamma$ this holds by the inclusion $t(S) \subset \mathfrak{s}(S)$. These observations are summarized in the next proposition.

Proposition 2.3 *The closed form $\mathfrak{s}(S) \in \mathbf{F}(\mathfrak{H})$ defined in (2.9) extends the form $t(S) \in \mathbf{F}(\mathfrak{H})$ in (2.1) and its lower bound is c . Consequently, the form $t(S)$ is closable and*

$$t(S) \subset \bar{t}(S) \subset \mathfrak{s}(S).$$

Furthermore, the closure $\bar{t}(s) \in \mathbf{F}(\mathfrak{H})$ is given by

$$\bar{t}(S)[\varphi, \psi] = c(\varphi, \psi) + (Q_c^{**}\varphi, Q_c^{**}\psi), \quad \varphi, \psi \in \text{dom } \bar{t}(S) = \text{dom } Q_c^{**}. \tag{2.10}$$

As a corollary of Proposition 2.3 it is shown that the formula (2.1) in Definition 2.1 holds under more general circumstances.

Corollary 2.4 *Assume that $\{\varphi, \varphi'\} \in S^*$ and that $\varphi \in \text{dom } \bar{t}(S)$. Then*

$$\bar{t}(S)[\varphi, \psi] = (\varphi', \psi) \quad \text{for all } \psi \in \text{dom } \bar{t}(S). \tag{2.11}$$

Proof Since Q_c is closable with $\text{dom } \bar{t}(S) = \text{dom } Q_c^{**}$ and $Q_c^{**} \subset J_c^*$, one has

$$(\varphi, \psi' - c\psi) = (Q_c^{**}\varphi, Q_c\psi), \quad \varphi \in \text{dom } Q_c^{**}, \quad \{\psi, \psi'\} \in S. \tag{2.12}$$

Now assume that in (2.12) one also has $\{\varphi, \varphi'\} \in S^*$. Then for all $\{\psi, \psi'\} \in S$ one sees by definition $(\varphi', \psi) = (\varphi, \psi')$. Thus (2.12), with $\text{dom } S = \text{dom } Q_c$, reads as

$$(\varphi' - c\varphi, \psi) = (Q_c^{**}\varphi, Q_c\psi), \quad \{\varphi, \varphi'\} \in S^*, \varphi \in \text{dom } Q_c^{**}, \psi \in \text{dom } Q_c. \quad (2.13)$$

Taking limits $\{\psi_n, Q_c\psi_n\} \rightarrow \{\psi, Q_c^{**}\psi\}$ in (2.13) one obtains

$$(\varphi' - c\varphi, \psi) = (Q_c^{**}\varphi, Q_c^{**}\psi), \quad \{\varphi, \varphi'\} \in S^*, \varphi, \psi \in \text{dom } Q_c^{**},$$

which leads to (2.11). □

The next result concerns the description of the (form) domain of $\bar{t}(S)$ which is easily derived from the associated representing map Q_c . This type of description goes back to Freudenthal [11]. There is also a useful description of $\text{ran } Q_c^*$ by means of S which determines the form $t(S)$; cf. [12, 17, 29]. The contents of this theorem will be interpreted in terms of Friedrichs extensions later in Sect. 4.

Theorem 2.5 *Let $S \in \mathbf{L}(\mathfrak{H})$ be semibounded with lower bound $\gamma \in \mathbb{R}$, let $c \leq \gamma$, and let $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}')$ be a representing map for $t(S) - c$. Then $\varphi \in \text{dom } \bar{t}(S)$ if and only if there is a sequence $\{\varphi_n, \varphi'_n\} \in S$ such that*

$$\varphi_n \rightarrow \varphi \quad \text{and} \quad (\varphi'_n - \varphi'_m, \varphi_n - \varphi_m) \rightarrow 0.$$

In addition, $\varphi \in \ker(\bar{t}(S) - c)$ if and only if there exists $\{\varphi_n, \varphi'_n\} \in S$ such that

$$\varphi_n \rightarrow \varphi \quad \text{and} \quad (\varphi'_n - c\varphi_n, \varphi_n) \rightarrow 0.$$

Furthermore, $\varphi \in \text{ran } Q_c^$ if and only if there exists $C_\varphi < \infty$ such that*

$$|(\psi, \varphi)|^2 \leq C_\varphi(\psi', \psi) \quad \text{for all } \{\psi, \psi'\} \in S - c. \quad (2.14)$$

Proof By Proposition 2.3 one sees that $\text{dom } \bar{t}(S) = \text{dom } Q_c^{**}$ and hence the condition that $\varphi \in \text{dom } \bar{t}(S)$ is equivalent to

$$\varphi_n \rightarrow \varphi \quad \text{and} \quad \|Q_c(\varphi_n - \varphi_m)\|^2 \rightarrow 0.$$

It remains to apply (2.5) to get the description of $\text{dom } \bar{t}(S)$ in the first statement.

As to the second statement, again by Proposition 2.3 one has similarly that $\ker(\bar{t}(S) - c) = \ker Q_c^{**}$. Hence, $\varphi \in \ker Q_c^{**}$ if and only if there exists a sequence $\{\varphi_n, \varphi'_n\} \in S$ such that $\varphi_n \rightarrow \varphi$ and $Q_c\varphi_n \rightarrow 0$. Using (2.5) this second condition can be rewritten as

$$(\varphi'_n - c\varphi_n, \varphi_n) = \|Q_c\varphi_n\|^2 \rightarrow 0.$$

Finally, to prove the last statement recall a general description of $\text{ran } T^*$ of $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, see e.g. [12, 17, 29]: $\varphi \in \text{ran } Q_c^*$ if and only if for some $c_\varphi < \infty$,

$$|(f, \varphi)| \leq c_\varphi \|Q_c f\| \quad \text{for all } f \in \text{dom } Q_c. \quad (2.15)$$

Now using (2.5) the estimate in (2.15) can be rewritten as

$$|(f, \varphi)|^2 \leq c_\varphi^2 \|Q_c f\|^2 = c_\varphi^2 (f' - cf, f) \quad \text{for all } \{f, f'\} \in S. \tag{2.16}$$

Take $C_\varphi = c_\varphi^2$ and observe that $\{f, f'\} \in S$ is equivalent to $\{f, f' - cf\} \in S - c$. It remains to replace f by ψ and $f' - cf$ by ψ' to conclude that (2.16) is equivalent to (2.14). This completes the proof. \square

The interest in the closure $\bar{t}(S)$ of the form $t(S)$ lies in the possibility to uniquely associate a semibounded selfadjoint relation which represents $\bar{t}(S)$ and extends the semibounded relation S . To be more specific, recall that for every closed semibounded form $t \in \mathbf{F}(\mathfrak{H})$ there exists a unique semibounded selfadjoint relation $H \in \mathbf{L}(\mathfrak{H})$, such that

$$t[\varphi, \psi] = (\varphi', \psi), \quad \{\varphi, \varphi'\} \in H, \quad \psi \in \text{dom } t,$$

see [7, 21]. Moreover, any semibounded selfadjoint relation $H \in \mathbf{L}(\mathfrak{H})$ is obtained in this way; cf. [21, Lemma 6.3]. Hence, there is a one-to-one correspondence between the semibounded closed forms $t \in \mathbf{F}(\mathfrak{H})$ and the semibounded selfadjoint relations $H \in \mathbf{L}(\mathfrak{H})$ which is indicated by

$$t = t_H \quad \text{or} \quad H = H_t, \tag{2.17}$$

depending on the point of view. Furthermore, it should be remembered that for any semibounded selfadjoint relation $H \in \mathbf{L}(\mathfrak{H})$ one may define the form $t(H) \in \mathbf{F}(\mathfrak{H})$ as in (2.1) by

$$t(H) = (\varphi', \psi), \quad \{\varphi, \varphi'\}, \{\psi, \psi'\} \in H.$$

Consequently, the form $t(H)$ is semibounded and closable by Proposition 2.3, so that its closure $\bar{t}(H)$ is a closed semibounded form. Since the forms $\bar{t}(H)$ and t_H coincide on $\text{dom } H$ it follows that for any semibounded selfadjoint relation $H \in \mathbf{L}(\mathfrak{H})$ one has

$$t(H) \subset \bar{t}(H) = t_H, \tag{2.18}$$

see (2.17) and [7]. As a consequence it is clear that for semibounded selfadjoint relations $H, K \in \mathbf{L}(\mathfrak{H})$ one has the equivalent statements

$$t(H) \leq t(K) \iff t_H \leq t_K \iff H \leq K, \tag{2.19}$$

see [7]. These results will be tacitly used in the rest of this paper. Note that for a semibounded relation $S \in \mathbf{L}(\mathfrak{H})$ the form $t(S) \in \mathbf{F}(\mathfrak{H})$ in (2.1) was denoted by t_S in [7]; however, this last notation will not be used in this paper (unless S is selfadjoint) as it may be confusing with the above conventions (2.17) and (2.18). At this stage it should be mentioned that for general semibounded forms one can also introduce

uniquely determined symmetric relations, see [21, Theorem 6.1] and, for special cases in the present context, Sect. 4 and Sect. 6.

Now return to the semibounded form $t(S) \in \mathbf{F}(\mathfrak{H})$ from (2.1). The closure of the form $t(S)$ in (2.10) induces a unique semibounded selfadjoint relation denoted by S_F and the closed form $s(S)$ in Definition 2.2 induces another unique semibounded selfadjoint extension denoted by $S_{K,c}$. It follows from (2.6) that S_F and $S_{K,c}$ are extensions of S . They are usually called the Friedrichs extension and the Kreĭn type extension of S , respectively. In the notation as discussed in (2.17) one may write

$$t_{S_F} = \bar{t}(S) \subset s(S) = t_{S_{K,c}} \quad \text{and} \quad t_{S_{K,c}} \leq t_{S_F}.$$

Since dealing with relations allows taking inverses, it is easy to show that J_c^{-1} is a representing map for the nonnegative form $t((S - c)^{-1})$ and that Q_c^{-1} is its companion relation. This observation leads immediately to the connecting formula (1.3). The following sections will be devoted to the details of these observations.

3 Semibounded forms induced by semibounded relations

First consider a general semibounded sesquilinear form with lower bound γ and let $c \leq \gamma$ as discussed in [21]. If $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$ and $Q'_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}'_c)$ are representing maps for $t - c$, then clearly there exists a partial isometry $V \in \mathbf{B}(\mathfrak{K}_c, \mathfrak{K}'_c)$ with initial space $\overline{\text{ran}} Q_c$ and final space $\overline{\text{ran}} Q'_c$, such that

$$Q'_c = V Q_c. \tag{3.1}$$

Here the notation $\mathbf{B}(\mathfrak{H}, \mathfrak{K})$ stands for the class of all bounded everywhere defined operators from the Hilbert space \mathfrak{H} to the Hilbert space \mathfrak{K} . Since $V \in \mathbf{B}(\mathfrak{K}_c, \mathfrak{K}'_c)$, it follows that the corresponding adjoints are connected via

$$(Q'_c)^* = (Q_c)^* V^*;$$

see e.g. [7, Proposition 1.3.9]. Therefore the representation of the form t in terms of semibounded symmetric or selfadjoint relations involving a representing map Q_c is not affected by the nonuniqueness:

$$(Q'_c)^* Q'_c = Q_c^* Q_c, \quad (Q'_c)^* (Q'_c)^{**} = Q_c^* (Q_c)^{**}, \tag{3.2}$$

for these results see [21].

In the particular case that $\text{ran } Q_c$ and $\text{ran } Q'_c$ are dense subspaces, the isometry V in (3.1) becomes a unitary operator in $\mathbf{B}(\mathfrak{K}_c, \mathfrak{K}'_c)$. This *minimality of the representing map* in [21, Definition 2.3] can be obtained by replacing the spaces \mathfrak{K}_c and \mathfrak{K}'_c by the closed subspaces $\overline{\text{ran}} Q_c$ and $\overline{\text{ran}} Q'_c$, respectively. However, in the present work this assumption is not used and the constructions are carried out without any minimality assumption. This simplifies the treatment when a representing map is to be constructed for a new form from forms whose representing maps are already given: for instance,

to define a representing map for a restriction of a given form, see e.g. Section 7 (proof of Lemma 7.2), or when constructing representing maps involving the functional calculus of given forms; cf. [21, Sections 4, 5] (sum decompositions of forms, mutual singularity of pairs of forms, etc.).

Now let $S \in \mathbf{L}(\mathfrak{H})$ be semibounded with lower bound $\gamma \in \mathbb{R}$ and let $c \leq \gamma$. The associated semibounded form $t(S)$ in (2.1) leads to a representing map Q_c via (2.5) and its companion relation J_c in (2.3) whose basic properties are now established. The following identities are straightforward consequences from the definition:

$$\begin{cases} \text{dom } J_c = \text{ran } Q_c, & \ker J_c = \{0\}, \\ \text{ran } J_c = \text{ran } (S - c), & \text{mul } J_c = \{\varphi' - c\varphi : \{\varphi, \varphi'\} \in S, Q_c\varphi = 0\}. \end{cases} \tag{3.3}$$

In the next result the identity involving $\text{mul } J_c$ will be expressed differently, independent of the representing map Q_c . In particular, it follows that J_c is an operator when S is densely defined, in which case S is an operator. The contents of the next theorem will be interpreted for the Krein type extensions later in Sect. 6.

Theorem 3.1 *Let $S \in \mathbf{L}(\mathfrak{H})$ be semibounded with lower bound $\gamma \in \mathbb{R}$ and let $c \leq \gamma$. Then the multivalued part of J_c in (2.3) is given by*

$$\text{mul } J_c = \text{ran } (S - c) \cap \text{mul } S^*. \tag{3.4}$$

The element $\psi' \in \text{mul } J_c^{**}$ if and only if there exists $\{\varphi_n, \varphi'_n\} \in S$ such that

$$\varphi'_n - c\varphi_n \rightarrow \psi' \quad \text{and} \quad (\varphi'_n - c\varphi_n, \varphi_n) \rightarrow 0. \tag{3.5}$$

Furthermore, $\psi' \in \text{dom } J_c^*$ if and only if there exists $C_{\psi'} < \infty$ such that

$$|(\varphi', \psi')|^2 \leq C_{\psi'}(\varphi', \varphi) \quad \text{for all } \{\varphi, \varphi'\} \in S - c. \tag{3.6}$$

Proof The following equivalence is a consequence of (2.5):

$$(\varphi' - c\varphi, \varphi) = 0 \quad \Leftrightarrow \quad Q_c\varphi = 0$$

for all $\{\varphi, \varphi'\} \in S$. In other words,

$$\text{mul } J_c = \{\varphi' - c\varphi \in \mathfrak{H} : (\varphi' - c\varphi, \varphi) = 0 \text{ for some } \{\varphi, \varphi'\} \in S\}. \tag{3.7}$$

Now (3.4) can be easily proved; cf. [15, Lemma 3.1].

Assume that $\varphi' - c\varphi \in \text{ran } (S - c) \cap \text{mul } S^*$ for some $\{\varphi, \varphi'\} \in S$. This implies that $(\varphi' - c\varphi, \varphi) = 0$ and therefore, by (3.7), $\varphi' - c\varphi \in \text{mul } J_c$.

Conversely, assume that $\varphi' - c\varphi \in \text{mul } J_c$. Therefore, by (3.7), one sees that $\{\varphi, \varphi'\} \in S$ and $(\varphi' - c\varphi, \varphi) = 0$. Clearly with $\{\psi, \psi'\} \in S$ the Cauchy-Schwarz inequality for the nonnegative form $t(S) - c$ gives

$$|(\varphi' - c\varphi, \psi)|^2 \leq (\varphi' - c\varphi, \varphi)(\psi' - c\psi, \psi).$$

Hence $(\varphi' - c\varphi, \psi) = 0$ for all $\psi \in \text{dom } S$, so that $\varphi' - c\varphi \in (\text{dom } S)^\perp = \text{mul } S^*$. Therefore, it follows that $\varphi' - c\varphi \in \text{ran } (S - c) \cap \text{mul } S^*$.

To prove (3.5) let $\psi' \in \text{mul } J_c^{**}$. Using (2.3) this means that there exists a sequence $\{\varphi_n, \varphi'_n\} \in S$ such that $Q_c \varphi_n \rightarrow 0$ and $\varphi'_n - c\varphi_n \rightarrow \psi'$. Now apply (2.5) to see that

$$(\varphi'_n - c\varphi_n, \varphi_n) = \|Q_c \varphi_n\|^2 \rightarrow 0.$$

The converse statement is shown similarly.

To prove (3.6) observe that by (3.3) J_c^{-1} is an operator and that $\text{dom } J_c^* = \text{ran } (J_c^{-1})^*$. Now proceeding as in the proof of Theorem 2.5 one concludes that $\psi' \in \text{dom } J_c^*$ if and only if there exists $c_{\psi'} < \infty$ such that

$$|(f' - cf, \psi')| \leq c_{\psi'} \|J_c^{-1}(f' - cf)\| = c_{\psi'} \|Q_c f\| \text{ for all } \{f, f'\} \in S.$$

Again squaring this, using (2.5), and then replacing f by φ and $f' - cf$ by φ' yields

$$|(\varphi', \psi')|^2 \leq c_{\psi'}^2 (\varphi, \varphi') \text{ for all } \{\varphi, \varphi'\} \in S - c.$$

Thus with $C_{\psi'} = c_{\psi'}^2$, one gets (3.6). This completes the proof. □

Since the representing map $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$ in (2.2) is only determined up to a partial isometry, its choice will influence the companion relation $J_c \in \mathbf{L}(\mathfrak{K}_c, \mathfrak{H})$ in (2.3). Thus it will have to be checked whether the choice of Q_c influences the relation $c + J_c Q_c$ in (2.6) and the form $\mathfrak{s}(S)$ in (2.9).

Lemma 3.2 *Let $S \in \mathbf{L}(\mathfrak{H})$ be semibounded with lower bound $\gamma \in \mathbb{R}$ and let $c \leq \gamma$. Assume that $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$ and $Q'_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}'_c)$ are representing maps for $\mathfrak{t}(S) - c$ which are connected via $Q'_c = V Q_c$ by a partial isometry $V \in \mathbf{B}(\mathfrak{K}_c, \mathfrak{K}'_c)$ as in (3.1). Let $J_c \in \mathbf{L}(\mathfrak{K}_c, \mathfrak{H})$ and $J'_c \in \mathbf{L}(\mathfrak{K}'_c, \mathfrak{H})$ be the companion relations of Q_c and Q'_c , respectively. Then J_c and J'_c are connected by*

$$J'_c (V \upharpoonright \text{ran } Q_c) = J_c. \tag{3.8}$$

In particular, one has

$$S \subset c + J'_c Q'_c = c + J_c Q_c; \tag{3.9}$$

here the inclusion is an equality if and only if $\text{ran } (S - c) \cap \text{mul } S^ = \text{mul } S$. Moreover, the adjoints are connected via*

$$((J'_c)^*)_{\text{reg}} = V (J_c^*)_{\text{reg}}. \tag{3.10}$$

Proof Let V be the indicated partial isometry. Then the identity (3.8) is obtained from (3.1) by means of the formula (2.3).

The identity in (3.9) is a straightforward consequence of (3.1) and (3.8). The inclusion in (3.9) was already stated in (2.6). Next observe that $\text{dom } J_c Q_c = \text{dom } S$. Hence, the inclusion in (3.9) is an equality if and only if $\text{mul } S = \text{mul } J_c Q_c = \text{mul } J_c$. Now the assertion follows from (3.4) in Theorem 3.1.

In order to prove the identity (3.10) note that

$$(\text{mul } (J'_c)^*)^\perp = \overline{\text{ran } Q'_c} = V \overline{\text{ran } Q_c} = V (\text{mul } J_c^*)^\perp. \tag{3.11}$$

Next assume that $\{f, f'\} \in V (J_c^*)_{\text{reg}}$. Then $\{f, h\} \in (J_c^*)_{\text{reg}}$ and $f' = Vh$. Since J_c^* is closed $\{f, h\} \in (J_c^*)_{\text{reg}}$ implies that $\{f, h\} \in J_c^*$ and $h \in \overline{\text{ran } Q_c}$. Hence, $f' \in \overline{\text{ran } Q'_c}$, and using (2.3) for all $\{\varphi, \varphi'\} \in S$ one gets

$$\begin{aligned} 0 &= (h, Q_c \varphi) - (f, \varphi' - c\varphi) \\ &= (Vh, V Q_c \varphi) - (f, \varphi' - c\varphi) \\ &= (f', Q'_c \varphi) - (f, \varphi' - c\varphi), \end{aligned}$$

which gives $\{f, f'\} \in (J'_c)^*$. Hence the inclusion $V (J_c^*)_{\text{reg}} \subset ((J'_c)^*)_{\text{reg}}$ has been shown; cf. (3.11). For the reverse inclusion, let $\{f, f'\} \in ((J'_c)^*)_{\text{reg}}$. In other words, $\{f, f'\} \in (J'_c)^*$ and $f' = Vh$ for some $h \in (\text{mul } (J_c^*)^\perp$ by (3.11). Therefore, retracing the above steps shows $\{f, h\} \in (J_c^*)_{\text{reg}}$, and hence $\{f, f'\} \in V (J_c^*)_{\text{reg}}$. \square

Corollary 3.3 *Under the circumstances of Lemma 3.2 one has*

$$\begin{cases} J'_c (J'_c)^* = J'_c ((J'_c)^*)_{\text{reg}} = J_c (J_c^*)_{\text{reg}} = J_c J_c^*, \\ (J'_c)^{**} (J'_c)^* = (((J'_c)^*)_{\text{reg}})^* ((J'_c)^*)_{\text{reg}} = ((J_c^*)_{\text{reg}})^* (J_c^*)_{\text{reg}} = J_c^{**} J_c^*. \end{cases}$$

Proof Recall that for any linear relation $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ one has

$$TT^* = T(T^*)_{\text{reg}} \quad \text{and} \quad T^{**}T^* = ((T^*)_{\text{reg}})^*(T^*)_{\text{reg}}, \tag{3.12}$$

see [19] for similar results. Therefore, due to the identities in (3.12) it suffices to verify that

$$J'_c ((J'_c)^*)_{\text{reg}} = J_c (J_c^*)_{\text{reg}} \quad \text{and} \quad (((J'_c)^*)_{\text{reg}})^* ((J'_c)^*)_{\text{reg}} = ((J_c^*)_{\text{reg}})^* (J_c^*)_{\text{reg}}.$$

However, these identities follow immediately from (3.8) and (3.10). \square

Remark 3.4 Let $S \in \mathbf{L}(\mathfrak{H})$ be a semibounded relation in a Hilbert space \mathfrak{H} with lower bound $\gamma \in \mathbb{R}$ and $c \leq \gamma$. Then $S - c \geq 0$ and this implies that

$$(\varphi' - c\varphi, \psi' - c\psi)_{S-c} := (\varphi' - c\varphi, \psi) = (\varphi, \psi' - c\psi), \quad \{\varphi, \varphi'\}, \{\psi, \psi'\} \in S, \tag{3.13}$$

defines a semi-inner product on the linear subspace $\text{ran } (S - c)$. It is immediately clear that the linear subspace of null elements of $\text{ran } (S - c)$, provided with this semi-inner product, is given by $\text{ran } (S - c) \cap \text{mul } S^*$; see (3.7) and Theorem 3.1. The Hilbert space \mathfrak{H}_{S-c} is defined as the completion of

$$\text{ran } (S - c) / (\text{ran } (S - c) \cap \text{mul } S^*)$$

with the inner product

$$([\varphi' - c\varphi], [\psi' - c\psi])_{S-c} := (\varphi' - c\varphi, \psi' - c\psi)_{S-c}, \quad \{\varphi, \varphi'\}, \{\psi, \psi'\} \in S,$$

where $[\varphi' - c\varphi]$ and $[\psi' - c\psi]$ denote the relevant equivalence classes. Then, clearly, the operator $q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{H}_{S-c})$ given by

$$q_c\varphi = [\varphi' - c\varphi], \quad \{\varphi, \varphi'\} \in S,$$

is well defined and, automatically, one has that $\text{ran } q_c$ is dense in \mathfrak{H}_{S-c} . It follows from the above identity (3.13) that

$$(q_c\varphi, q_c\psi)_{S-c} = (\mathfrak{t}(S) - c)[\varphi, \psi], \quad \{\varphi, \varphi'\}, \{\psi, \psi'\} \in S.$$

Thus the operator $q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{H}_{S-c})$ acts as a minimal representing map for the form $\mathfrak{t}(S) - c$. This construction has been used for nonnegative relations in [15], see also [16, 28] for the sectorial case. In the case when S is densely defined one has $\text{mul } S^* = \{0\}$, i.e., S^* and, therefore, also J_c is an operator. When S is a closed densely defined nonnegative operator, this construction reduces to that of Sebestyén and Stochel appearing in [30]: in other words, their construction produced a minimal representing map for the closable form $\mathfrak{t}(S) \geq 0$.

4 A characterization of the Friedrichs extensions

Let $S \in \mathbf{L}(\mathfrak{H})$ be a semibounded relation with lower bound $\gamma \in \mathbb{R}$. Let $\mathfrak{t}(S)$ be the corresponding closable form with representing map Q_c in (2.2) and $\bar{\mathfrak{t}}(S)$ be its closure with representing map Q_c^{**} in (2.10). Then the semibounded selfadjoint relation $S_F \in \mathbf{L}(\mathfrak{H})$ which is uniquely determined by the semibounded closed form $\bar{\mathfrak{t}}(S) \in \mathbf{F}(\mathfrak{H})$, is given by

$$S_F = c + Q_c^* Q_c^{**}, \tag{4.1}$$

while the symmetric semibounded relation $S_f \in \mathbf{L}(\mathfrak{H})$ which is uniquely determined by the closable form $\mathfrak{t}(S) \in \mathbf{F}(\mathfrak{H})$, is given by

$$S_f = c + Q_c^* Q_c, \tag{4.2}$$

see [21, Theorem 6.1 and Corollary 6.4] for a characterization and details. Moreover, the relations S_F and S_f do not depend on the particular choice of the representing map Q_c ; see (3.2). It is a direct consequence of (2.6) (using $J_c \subset Q_c^*$) and Lemma 3.2 that S_f and S_F are extensions of S , such that

$$S \subset S_f = c + Q_c^* Q_c \subset S_F = c + Q_c^* Q_c^{**}. \tag{4.3}$$

The multivalued parts $\text{mul } S_f = \text{mul } S_F = \text{mul } Q_c^* = (\text{dom } Q_c)^\perp$ satisfy

$$\text{mul } S_f = \text{mul } S_F = \text{mul } S^*. \tag{4.4}$$

Moreover, it follows from [21, Corollary 6.4] that the lower bounds of S_f and S_F are given by

$$m(S_f) = m(S_F) = \gamma.$$

The semibounded selfadjoint extension S_F in (4.1) is the well-known *Friedrichs extension*, while the semibounded symmetric extension S_f in (4.2) will be called the *weak Friedrichs extension*. It is clear that for any linear extension T of S with $S \subset T \subset S_F$ it follows that also the inclusions $S \subset \bar{T} \subset S_F$ hold, where \bar{T} stands for the closure of T . Hence it follows that $\mathfrak{t}(S) \subset \mathfrak{t}(\bar{T}) \subset \mathfrak{t}(S_F) \subset \mathfrak{t}_{S_F} = \bar{\mathfrak{t}}(S)$, which gives

$$\bar{\mathfrak{t}}(\bar{T}) = \bar{\mathfrak{t}}(S) \quad \text{and} \quad (\bar{T})_F = S_F, \tag{4.5}$$

so that the Friedrichs extension of such a semibounded extension T of S remains the same. In particular, one has $(\bar{S})_F = S_F$. From the definition the following translation invariance property of S_F is immediate

$$(S - c)_F = S_F - c.$$

The extensions S_f and S_F are intrinsically characterized in the following theorem.

Theorem 4.1 *Let $S \in \mathbf{L}(\mathfrak{H})$ be a semibounded linear relation with lower bound $\gamma \in \mathbb{R}$ and let $\mathfrak{t}(S)$ be the corresponding semibounded form in (2.1). Then the following statements hold:*

(a) *The Friedrichs extension $S_F \in \mathbf{L}(\mathfrak{H})$ of S in (4.1) is given by*

$$S_F = \{ \{ \varphi, \varphi' \} \in S^* : \varphi \in \text{dom } \bar{\mathfrak{t}}(S) \}. \tag{4.6}$$

If $H \in \mathbf{L}(\mathfrak{H})$ is a selfadjoint extension of S with

$$\text{dom } H \subset \text{dom } \bar{\mathfrak{t}}(S), \tag{4.7}$$

then $H = S_F$.

(b) *The weak Friedrichs extension $S_f \in \mathbf{L}(\mathfrak{H})$ of S in (4.2) is given by*

$$S_f = S \widehat{+} (\{0\} \times \text{mul } S^*). \tag{4.8}$$

If $H \in \mathbf{L}(\mathfrak{H})$ is a symmetric extension of S with

$$\text{dom } H = \text{dom } S, \tag{4.9}$$

then $H \subset S_f$.

(c) *The equality $S_f = S_F$ holds if and only if*

$$\text{dom } S = \overline{\text{dom } S} \cap \text{dom } S^*. \tag{4.10}$$

In particular, (4.10) holds if $\text{dom } S$ is closed.

Proof (a) It is clear from $S \subset S_F$ that $S_F \subset S^*$. Moreover, it follows from (4.1) that $\text{dom } S_F \subset \text{dom } Q_c^{**} = \text{dom } \bar{t}(S)$. Thus S_F belongs to the right-hand side of (4.6). However, observe that

$$(\varphi', \varphi) = \bar{t}(S)[\varphi, \varphi] \in \mathbb{R}, \quad \{\varphi, \varphi'\} \in S^*, \quad \varphi \in \text{dom } \bar{t}(S),$$

cf. Corollary 2.4, so that the right-hand side of (4.6) is a symmetric relation. Therefore, (4.6) has been shown.

Let $H \in \mathbf{L}(\mathfrak{H})$ be a selfadjoint extension of S with (4.7). Then it follows from (4.6) that $H \subset S_F$. Since both relations are selfadjoint, one concludes that $H = S_F$.

(b) It is clear from $S \subset S_f$ that $S_f \subset S^*$. Moreover, it follows from (4.2) that $\text{dom } S_f \subset \text{dom } Q_c = \text{dom } S$. Hence one obtains

$$S_f \subset \{ \{\varphi, \varphi'\} \in S^* : \varphi \in \text{dom } S \}. \tag{4.11}$$

Now observe that

$$\{ \{\varphi, \varphi'\} \in S^* : \varphi \in \text{dom } S \} = S \hat{+} (\{0\} \times \text{mul } S^*) \subset S_f,$$

so that (4.11) gives (4.8).

Let $H \in \mathbf{L}(\mathfrak{H})$ be a symmetric extension of S with (4.9). Then it follows from (4.11) that $H \subset S_f$.

(c) Thanks to (4.3), it is clear that $S_f = S_F$ if and only if S_f is selfadjoint, which is equivalent to $\text{dom } S = \overline{\text{dom } S} \cap \text{dom } S^*$; cf. [7, Lemma 1.5.7]. \square

The linear subspace $\text{ran } (S_F - c)^{\frac{1}{2}}$ can be characterized in terms of the representing map Q_c for $t(S) \in \mathbf{F}(\mathfrak{H})$. By (4.1) one has $S_F - c = Q_c^* Q_c^{**}$ and therefore

$$\text{ran } (S_F - c)^{\frac{1}{2}} = \text{ran } Q_c^*.$$

Thus the following result is clear from Theorem 2.5; cf. [12, 17, 26, 31].

Corollary 4.2 *Let $S \in \mathbf{L}(\mathfrak{H})$ be as in Theorem 4.1 and let S_F be its Friedrichs extension. Then $\varphi \in \text{ran } (S_F - c)^{\frac{1}{2}}$ if and only if there exists $C_\varphi < \infty$ such that*

$$|(\psi, \varphi)|^2 \leq C_\varphi (\psi', \psi) \text{ for all } \{\psi, \psi'\} \in S - c.$$

Similarly, one has in terms of the representing map Q_c

$$\text{dom } (S_F - c)^{\frac{1}{2}} = \text{dom } \bar{t}(S) = \text{dom } Q_c^{**},$$

and this domain was described already in Theorem 2.5; cf. [14, Remark 4.6], [12, eq. (7)], [7, Corollary 5.3.4]. Additionally, one has

$$\ker (S_F - c) = \ker (\bar{t}(S) - c) = \ker Q_c^{**},$$

which was also described in Theorem 2.5. In particular, if S is nonnegative, i.e. $\gamma \geq 0$, then one can take $c = 0$; cf. [14, Proposition 4.10].

The Friedrichs extension $S_F \in \mathbf{L}(\mathfrak{H})$ guarantees the existence of at least one semi-bounded selfadjoint extension of a semibounded relation $S \in \mathbf{L}(\mathfrak{H})$ and, in fact, S_F has the same lower bound as S . It is a consequence of (4.4) that S_F is an operator if and only if S is densely defined. In this case S is automatically an operator and all selfadjoint extensions of S are also operators. In the general nondensely defined case, S_F is the largest semibounded selfadjoint extension of S . In particular, one should note that if $S \in \mathbf{L}(\mathfrak{H})$ is semibounded and selfadjoint then $S_F = S$.

Corollary 4.3 *Let $S \in \mathbf{L}(\mathfrak{H})$ be a semibounded linear relation with lower bound $\gamma \in \mathbb{R}$. Let $H \in \mathbf{L}(\mathfrak{H})$ be any semibounded selfadjoint extension of S , then*

$$H \leq S_F. \tag{4.12}$$

Proof First recall that for any semibounded selfadjoint extension H the corresponding form $\mathfrak{t}(H)$ is semibounded and that $\mathfrak{t}(S) \subset \mathfrak{t}(H)$. Since H is selfadjoint $\mathfrak{t}(H)$ is closable and its closure is denoted by \mathfrak{t}_H ; see (2.17). Thus it is clear that

$$\mathfrak{t}_{S_F} = \bar{\mathfrak{t}}(S) \subset \bar{\mathfrak{t}}(H) = \mathfrak{t}_H \quad \text{and} \quad \mathfrak{t}_H \leq \mathfrak{t}_{S_F},$$

which gives the inequality (4.12); see [7]. □

5 Forms and representing maps induced by inverses of semibounded relations

In Sect. 4 the explicit formula (4.1) for the Friedrichs extension of a semibounded relations S and its extremality property in Corollary 4.3 was established. This was done by means of the representing map Q_c of the form $\mathfrak{t}(S) - c$ using the fact that \mathfrak{t}_{S_F} is the closure of the form $\mathfrak{t}(S)$ and that $\mathfrak{t}_{S_F} \subset \mathfrak{t}_H$ for every semibounded selfadjoint extension H of S . Among all semibounded selfadjoint extensions H of S the lower bound $m(H)$ can be any number below γ . After fixing a real number $c \leq \gamma$ a natural question is whether there is a minimal semibounded selfadjoint extension among all semibounded selfadjoint extensions of S satisfying $m(H) \geq c$. The most interesting case here is the case where $c = \gamma$. This problem has a long history in the case of nonnegative operators S and an affirmative answer to the problem was given by M.G. Krein in his famous papers [23, 24], where all nonnegative selfadjoint extensions $H \geq 0$ in the densely defined case were characterized via the operator interval

$$S_K \leq H \leq S_F.$$

The minimal solution S_K is nowadays called the Krein extension or the Krein-von Neumann extension of S . These results were extended to the case of nonnegative linear relations in [9] using the key connection

$$S_K = ((S^{-1})_F)^{-1}. \tag{5.1}$$

The general semibounded situation has been treated in [7] using the form $t(S)$ and its closure to define S_F and the analog (1.3) of the formula (5.1) for the nonnegative relation $S - c$; see [7, Sections 5.3 & 5.4]. The new ingredient in the present approach is to use representing maps for the nonnegative forms induced by $S - c$ and its inverse $(S - c)^{-1}$ with a fixed $c \leq \gamma$. For the form $t((S - c)^{-1})$ this leads to an explicit formula for the corresponding minimal extension $S_{K,c}$ of S in (1.2) as will be seen in Sect. 6.

Now let $S \in \mathbf{L}(\mathfrak{H})$ be a semibounded linear relation with lower bound $\gamma \in \mathbb{R}$ and let $c \leq \gamma$ be fixed. The relation $S - c$ is nonnegative and so is its inverse

$$(S - c)^{-1} = \{ \{ \varphi' - c\varphi, \varphi \} : \{ \varphi, \varphi' \} \in S \}, \quad c \leq \gamma. \tag{5.2}$$

The inverse $(S - c)^{-1}$ induces a nonnegative sesquilinear form $t((S - c)^{-1})$, whose domain is given by

$$\text{dom } t((S - c)^{-1}) = \text{ran } (S - c),$$

and according to (2.1), (2.5), and (5.2) one has for all $\{ \varphi, \varphi' \}, \{ \psi, \psi' \} \in S$,

$$t((S - c)^{-1})[\varphi' - c\varphi, \psi' - c\psi] = (\varphi, \psi' - c\psi) = (t(S) - c)[\varphi, \psi] = (Q_c\varphi, Q_c\psi). \tag{5.3}$$

The next proposition identifies a representing map and a corresponding companion relation for the nonnegative form $t((S - c)^{-1})$ which, in fact, can be expressed in terms of the corresponding data for the semibounded form $t(S)$ introduced already in Sect. 2.

Proposition 5.1 *Let $S \in \mathbf{L}(\mathfrak{H})$ be a semibounded relation with lower bound $\gamma \in \mathbb{R}$ and let $c \leq \gamma$. Let $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$ be a representing map for $t(S) - c$ with the companion relation $J_c \in \mathbf{L}(\mathfrak{K}_c, \mathfrak{H})$ in (2.3). Then the inverse $J_c^{-1} \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$ of J_c is an operator and it is a representing map for the form $t((S - c)^{-1})$,*

$$t((S - c)^{-1})[\eta, \zeta] = (J_c^{-1}\eta, J_c^{-1}\zeta), \quad \eta, \zeta \in \text{ran } (S - c). \tag{5.4}$$

The companion relation of the representing map J_c^{-1} is given by $Q_c^{-1} \in \mathbf{L}(\mathfrak{K}_c, \mathfrak{H})$. Moreover, with the orthogonal operator part of the adjoint of Q_c^{-1} ,

$$((Q_c^{-1})^*)_{\text{reg}} = (I - \pi)(Q_c^{-1})^*, \tag{5.5}$$

where π is the orthogonal projection from \mathfrak{K}_c onto $\text{mul } J_c^*$ as in (2.7), the sesquilinear form $\mathfrak{s}((S - c)^{-1}) \in \mathbf{F}(\mathfrak{H})$ (cf. Definition 2.8) is given by

$$\mathfrak{s}((S - c)^{-1})[\eta, \zeta] = ((Q_c^{-1})^*)_{\text{reg}} \eta, ((Q_c^{-1})^*)_{\text{reg}} \zeta), \quad \eta, \zeta \in \text{dom } ((Q_c^{-1})^*)_{\text{reg}}. \tag{5.6}$$

It is well-defined, nonnegative, and closed.

Proof According to (2.3) the inverse $(J_c)^{-1}$ has the form

$$(J_c)^{-1} = \{ \{ \varphi' - c\varphi, Q_c\varphi \} : \{ \varphi, \varphi' \} \in S \} \tag{5.7}$$

and as noted in (3.3) J_c^{-1} is an operator. The formula (5.4) is now clear from (5.3).

It follows from (5.7) that

$$Q_c^{-1} = \{ \{ J_c^{-1}(\psi' - c\psi), \psi \} : \{ \psi, \psi' \} \in S \},$$

so that Q_c^{-1} is the companion relation (with 0 as base point) of the representing map J_c^{-1} . The adjoint $(Q_c^{-1})^*$ of the companion relation Q_c^{-1} is closed by definition and due to the identity

$$\text{mul } (Q_c^{-1})^* = (\text{ran } Q_c)^\perp = \text{mul } J_c^*,$$

its orthogonal operator part is given by (5.5). Therefore, Definition 2.2 can be applied to the form $\mathfrak{s}((S-c)^{-1})$ and this yields a well-defined nonnegative closed form defined on the linear space $\text{dom } (Q_c^{-1})^*$ given by the formula (5.6). □

It follows from Proposition 5.1 that the operator J_c^{-1} and the relation Q_c^{-1} form a dual pair

$$J_c^{-1} \subset (Q_c^{-1})^* \quad \text{or, equivalently,} \quad Q_c^{-1} \subset (J_c^{-1})^*; \tag{5.8}$$

cf. (2.4). Moreover, the relation $Q_c^{-1}J_c^{-1}$ is an extension of $(S-c)^{-1}$:

$$(S-c)^{-1} \subset Q_c^{-1}J_c^{-1}, \tag{5.9}$$

which is symmetric as follows from (5.8); cf. (2.6). The analog of the pairing stated in (2.8), when applied to the operator J_c^{-1} and the relation $((Q_c)^{-1})^*$ in (5.8), reads as

$$J_c^{-1} \subset ((Q_c^{-1})^*)_{\text{reg}}.$$

Thus the form $\mathfrak{t}((S-c)^{-1})$, as a restriction of the closed form $\mathfrak{s}((S-c)^{-1})$, is closable and its closure is given by

$$\bar{\mathfrak{t}}((S-c)^{-1})[\eta, \zeta] = ((J_c^{-1})^{**}\eta, (J_c^{-1})^{**}\zeta), \tag{5.10}$$

for all $\eta, \zeta \in \text{dom } \bar{\mathfrak{t}}((S-c)^{-1}) = \text{dom } (J_c^{-1})^{**}$.

In the terminology of Sect. 4, applied to the closable form $\mathfrak{t}((S-c)^{-1})$, it follows from (5.4) that the weak Friedrichs extension $((S-c)^{-1})_f$ of $(S-c)^{-1}$ is given by

$$((S-c)^{-1})_f = (J_c^{-1})^*(J_c^{-1}), \tag{5.11}$$

while, similarly, it follows from (5.10) that the Friedrichs extension $((S-c)^{-1})_F$ of $(S-c)^{-1}$ is given by

$$((S-c)^{-1})_F = (J_c^{-1})^*(J_c^{-1})^{**}. \tag{5.12}$$

According to (5.9) these two nonnegative relations are extensions of $(S - c)^{-1}$, they satisfy

$$(S - c)^{-1} \subset ((S - c)^{-1})_f \subset ((S - c)^{-1})_F,$$

and they all have the same lower bound.

6 A characterization of the Kreĭn type extensions

Let $S \in \mathbf{L}(\mathcal{H})$ be a semibounded relation with lower bound $\gamma \in \mathbb{R}$ and let $\mathfrak{s}(S) \in \mathbf{F}(\mathcal{H})$ be the corresponding closed form with representing map $(J_c^*)_{\text{reg}}$ as in (2.9). Then the semibounded selfadjoint relation $S_{K,c}$, determined by the closed form $\mathfrak{s}(S)$, is clearly given by

$$S_{K,c} = c + ((J_c^*)_{\text{reg}})^*(J_c^*)_{\text{reg}} = c + J_c^{**}J_c^*, \tag{6.1}$$

where the last identity follows from Corollary 3.3. By Corollary 3.3 the relation $S_{K,c}$ does not depend on the particular choice of representing map Q_c ; cf. [21]. It is a direct consequence of (2.6) (using $Q_c \subset J_c^*$ in (2.6)) that $S_{K,c}$ is an extension of S , i.e.,

$$S \subset S_{K,c}.$$

The multivalued part $\text{mul } S_{K,c}$ satisfies

$$\text{mul } S_{K,c} = \text{mul } J_c^{**},$$

and the formula (3.5) in Theorem 3.1 offers a description for this. By Proposition 2.3 the lower bound of $\mathfrak{s}(S)$ is c and consequently c is also the lower bound of $S_{K,c}$; cf. [21]. The semibounded selfadjoint extension $S_{K,c}$ is the so-called *Kreĭn type extension* relative to the point $c \leq \gamma$. By Corollary 4.3 one sees immediately that $S_{K,c} \leq S_F$. From the right-hand side of (6.1) it follows that

$$S_{K,c} - c = J_c^{**}J_c^* = \left((J_c^{-1})^*(J_c^{-1})^{**} \right)^{-1} = \left(((S - c)^{-1})_F \right)^{-1},$$

where for the last identity (5.12) was used. Therefore, one concludes that the Kreĭn type extension $S_{K,c}$ can be expressed in terms of the Friedrichs extension induced by the closable form $t((S - c)^{-1})$:

$$S_{K,c} = c + (((S - c)^{-1})_F)^{-1}. \tag{6.2}$$

Let T be any linear extension T of S with $S \subset T \subset S_{K,c}$. In conjunction with (4.5) it becomes clear that

$$\bar{t}((\bar{T} - c)^{-1}) = \bar{t}((S - c)^{-1}) \quad \text{and} \quad (\bar{T})_{K,c} = S_{K,c},$$

since $\text{clos}(T - c)^{-1} = (\overline{T} - c)^{-1}$. In particular, one has $(\overline{S})_{K,c} = S_{K,c}$. There is also a weak version of the Krein type extension, but it has to be approached in an indirect way (since the form $\mathfrak{s}(S)$ is closed). Inspired by the definition in (6.1) the semibounded symmetric relation $S_{k,c}$ will be introduced as follows

$$S_{k,c} := c + J_c(J_c^*)_{\text{reg}} = c + J_c J_c^*, \tag{6.3}$$

where the last identity follows from Corollary 3.3. By Corollary 3.3 the relation $S_{k,c}$ does not depend on the particular choice of representing map Q_c . It is a direct consequence of (2.6) or, equivalently, of (5.9) (using $Q_c \subset J_c^*$) that $S_{k,c}$ is an extension of S and that, in fact,

$$S \subset S_{k,c} = c + J_c J_c^* \subset S_{K,c} = c + J_c^{**} J_c^*.$$

Moreover, (3.3) together with Theorem 3.1 implies that the multivalued parts satisfy

$$\text{mul } S_{k,c} = \text{ran}(S - c) \cap \text{mul } S^* \subset \text{mul } S_{K,c} = \text{mul } J_c^{**}.$$

Since $m(S_{K,c}) = c$, it follows from (6.1) and (6.3) that $m(S_{k,c}) = c$. The semibounded extension $S_{k,c}$ is called the *weak Krein type extension* relative to the point $c \leq \gamma$. From the right-hand side of (6.3) it follows that

$$S_{k,c} - c = J_c J_c^* = \left((J_c^{-1})^* J_c^{-1} \right)^{-1} = \left(((S - c)^{-1})_f \right)^{-1},$$

where for the last identity (5.11) was used. Therefore one concludes

$$S_{k,c} = c + (((S - c)^{-1})_f)^{-1}. \tag{6.4}$$

Although $S_{k,c}$ has been defined directly in (6.3), it is now clear from

$$(S_{k,c} - c)^{-1} = ((S - c)^{-1})_f,$$

that $(S_{k,c} - c)^{-1}$ can actually be expressed in terms of the weak Friedrichs extension induced by the closable form $\mathfrak{t}((S - c)^{-1})$.

With (6.1) and (6.3) in mind, it is now possible to translate Theorem 4.1 to the present context. Thus the extensions $S_{k,c}$ and $S_{K,c}$ are intrinsically characterized in the following theorem. Remember for (a) and (b) that

$$\text{ran}(S - c) = \text{dom } \mathfrak{t}((S - c)^{-1}) \subset \text{dom } \overline{\mathfrak{t}}((S - c)^{-1})$$

Theorem 6.1 *Let $S \in \mathbf{L}(\mathfrak{S})$ be a semibounded linear relation with lower bound $\gamma \in \mathbb{R}$ and let $\mathfrak{t}(S)$ be the corresponding semibounded form in (2.1). Let $c \leq \gamma$. Then the following statements hold:*

(a) The Kreĭn type extension $S_{K,c} \in \mathbf{L}(\mathfrak{H})$ of S in (6.1) is given by

$$S_{K,c} = \{ \{ \varphi, \varphi' \} \in S^* : \varphi' - c\varphi \in \text{dom } \bar{t}((S - c)^{-1}) \}. \tag{6.5}$$

If $H \in \mathbf{L}(\mathfrak{H})$ is a selfadjoint extension of S with

$$\text{ran } (H - c) \subset \text{dom } \bar{t}((S - c)^{-1}),$$

then $H = S_{K,c}$.

(b) The weak Kreĭn type extension $S_{k,c} \in \mathbf{L}(\mathfrak{H})$ of S in (6.3) is given by

$$S_{k,c} = S \hat{+} \widehat{\mathfrak{N}}_c(S^*). \tag{6.6}$$

If $H \in \mathbf{L}(\mathfrak{H})$ is a symmetric extension of S with

$$\text{ran } (H - c) \subset \text{dom } t((S - c)^{-1}) = \text{ran } (S - c),$$

then $H \subset S_{k,c}$.

(c) For all $c < \gamma$

$$S_{K,c} = \bar{S} \hat{+} \widehat{\mathfrak{N}}_c(S^*),$$

and the identity $S_{k,\gamma} = S_{K,\gamma}$ holds if and only if

$$\text{ran } (S - \gamma) = \overline{\text{ran}}(S - \gamma) \cap \text{ran } (S^* - \gamma). \tag{6.7}$$

In particular the identity $S_{k,\gamma} = S_{K,\gamma}$ holds if $\text{ran } (S - \gamma)$ is closed.

Proof By Theorem 4.1 one sees that

$$((S - c)^{-1})_F = \{ \{ \varphi, \varphi' \} \in ((S - c)^{-1})^* : \varphi \in \text{dom } \bar{t}(S - c)^{-1} \}$$

and, likewise,

$$((S - c)^{-1})_f = (S - c)^{-1} \hat{+} (\{0\} \times \ker(S^* - c)).$$

It follows from (6.4) and (6.2) that

$$(S_{k,c} - c)^{-1} = ((S - c)^{-1})_f \quad \text{and} \quad (S_{K,c} - c)^{-1} = ((S - c)^{-1})_F,$$

and then, by Theorem 4.1, (6.6) and (6.5) follow. Also by Theorem 4.1 one has for $c \leq \gamma$ that $S_{k,c} = S_{K,c}$ if and only if (6.7) holds.

$$\text{ran } (S - c) = \overline{\text{ran}}(S - c) \cap \text{ran } (S^* - c)$$

For every $c \leq \gamma$ the inclusion

$$\overline{S} \widehat{+} \widehat{\mathfrak{N}}_c(S^*) \subset S_{K,c}$$

is clear, and for $c < \gamma$ there is actually equality as both sides are selfadjoint; cf. [7, Lemma 5.4.1]. □

The linear subspace $\text{dom}(S_{K,c} - c)^{\frac{1}{2}}$ can be characterized in terms of the representing map $(J_c^*)_{\text{reg}}$ for $\mathfrak{s}(S) \in \mathbf{F}(\mathfrak{H})$. By (6.1) one has $S_{K,c} - c = J_c^{**} J_c^*$ and therefore

$$\text{dom}(S_{K,c} - c)^{\frac{1}{2}} = \text{dom } J_c^*.$$

Thus the following result is clear from Theorem 3.1, cf. [1, 5, 12, 14, 17].

Corollary 6.2 *Let $S \in \mathbf{L}(\mathfrak{H})$ be as in Theorem 6.1 and let $S_{K,c}$ be its Krein type extension. Then $\psi' \in \text{dom}(S_{K,c} - c)^{\frac{1}{2}}$ if and only if there exists $C_{\psi'} < \infty$ such that*

$$|(\varphi', \psi')|^2 \leq C_{\psi'}(\varphi', \varphi) \quad \text{for all } \{\varphi, \varphi'\} \in S - c. \tag{6.8}$$

The description of $\text{dom}(S_{K,c} - c)^{\frac{1}{2}}$ can also be obtained from Corollary 4.2 by means of the identity (6.2). Moreover, it follows from (6.2) that

$$\text{ran}(S_{K,c} - c)^{\frac{1}{2}} = \text{dom}(((S - c)^{-1})_{\mathbf{F}})^{\frac{1}{2}} = \text{dom} \bar{\mathfrak{t}}((S - c)^{-1}),$$

where the right-hand side can be described by means of Theorem 2.5.

Let $S \in \mathbf{L}(\mathfrak{H})$ be semibounded with lower bound $\gamma \in \mathbb{R}$ and let $c \leq \gamma$. The Krein type extension $S_{K,c}$ of S has lower bound c and, in fact, it is the smallest semibounded selfadjoint extension of S which is bounded below by c .

Corollary 6.3 *Let $S \in \mathbf{L}(\mathfrak{H})$ be a semibounded linear relation with lower bound $\gamma \in \mathbb{R}$ and $c \leq \gamma$. Let $H \in \mathbf{L}(\mathfrak{H})$ be a semibounded selfadjoint extension of S , then there is the equivalence*

$$c \leq H \iff \mathfrak{t}_{S_{K,c}} \leq \mathfrak{t}_H.$$

Proof (\Rightarrow) Assume that $c \leq H$. Since $S \subset H$, it follows that $(S - c)^{-1} \subset (H - c)^{-1}$. Thus also $(H - c)^{-1} \leq ((S - c)^{-1})_{\mathbf{F}}$ (see (4.12)) and, by antitonicity, one obtains

$$(((S - c)^{-1})_{\mathbf{F}})^{-1} \leq ((H - c)^{-1})^{-1} = H - c,$$

see [7], or by (6.2) $S_{K,c} \leq H$. Consequently, $\mathfrak{t}_{S_{K,c}} \leq \mathfrak{t}_H$.

(\Leftarrow) This implication is straightforward. □

For a semibounded relation $S \in \mathbf{L}(\mathfrak{H})$ it is clear that $S_F \in \mathbf{L}(\mathfrak{H})$ is an operator if and only if S is densely defined; and in this case all selfadjoint extensions of S are operators. The question when $S_{K,c} \in \mathbf{L}(\mathfrak{H})$ is an operator will be addressed now; note that then S is automatically an operator. First observe, that if S is a closed (or closable) operator, then by Theorem 6.1

$$\text{dom } S_{K,c} = \text{dom } \overline{S} + \ker (S^* - c), \quad c < \gamma,$$

is dense in \mathfrak{H} , and the selfadjoint relation $S_{K,c}$ with $c < \gamma$ is an operator. The next proposition deals with the case that S is an operator which is not necessarily closed or closable; see also [14, Proposition 4.9], [7, Corollary 5.4.8], [32, Theorem 4.1], and [33, Theorem 2.1]. The limit criterion (6.9) below (with $c = 0$) was introduced in [1] for a closed nonnegative operator S , which was then called *positively closable*.

Proposition 6.4 *Let $S \in \mathbf{L}(\mathfrak{H})$ be a semibounded operator with lower bound $\gamma \in \mathbb{R}$ and let $c \leq \gamma$. Then the following conditions are equivalent:*

- (i) *the Krein type extension $S_{K,c}$ is an operator or, equivalently, the form $\mathfrak{s}(S)$ is densely defined;*
- (ii) *there exists a Hilbert space \mathfrak{K} and a closable operator $B \in \mathbf{L}(\mathfrak{K}, \mathfrak{H})$ such that $S = c + BB^* \upharpoonright \text{dom } S$;*
- (iii) *S admits a selfadjoint operator extension H with lower bound $m(H) = c$;*
- (iv) *every sequence $\{\varphi_n\} \in \text{dom } S$ satisfies the following property:*

$$((S - c)\varphi_n, \varphi_n) \rightarrow 0 \quad \text{and} \quad (S - c)\varphi_n \rightarrow \omega \quad \text{imply} \quad \omega = 0. \quad (6.9)$$

Proof According to the general representation theorem one has that

$$\text{mul } S_{K,c} = (\text{dom } \mathfrak{s}(S))^\perp,$$

so that $S_{K,c}$ is an operator if and only if the form $\mathfrak{s}(S)$ is densely defined.

(i) \Rightarrow (ii) Let $S_{K,c}$ be an operator. By (6.1) it follows that $\text{mul } J_c^{**} = \text{mul } S_{K,c}$, so that $J_c \in \mathbf{L}(\mathfrak{K}_c, \mathfrak{H})$ is a closable operator. Then by (3.4) and Lemma 3.2 one has $S = c + J_c Q_c$, where $Q_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$ is the representing map (2.2) for the form $\mathfrak{t}(S) - c$. Now recall that $Q_c \subset (J_c^*)_{\text{reg}} = (I - \pi)J_c^*$, where $I - \pi$ is the orthogonal projection onto $(\text{mul } J_c^*)^\perp = \overline{\text{ran}} Q_c$ (see (2.7)) and that $(J_c^*)_{\text{reg}}$ is a closed operator which extends Q_c . Therefore,

$$S = c + J_c Q_c = c + J_c ((J_c^*)_{\text{reg}}) \upharpoonright \text{dom } S = c + J_c J_c^* \upharpoonright \text{dom } S,$$

see Corollary 3.3. Thus one can take $\mathfrak{K} = \mathfrak{K}_c$ and $B = J_c$.

(ii) \Rightarrow (iii) Since B is closable the representation for B implies that

$$S \subset c + BB^* \subset c + B^{**}B^*,$$

where by assumption $B^{**}B^*$ is a nonnegative selfadjoint operator. Thus S admits a selfadjoint operator extension $H = c + B^{**}B^*$ with lower bound $m(H) \geq c$.

(iii) \Rightarrow (i) By the extremality property of Kreĭn type extensions stated in Corollary 6.3 one has $t_{S_{K,c}} \leq t_H$ and, equivalently, $S_{K,c} \leq H$, see (2.19). Since H is an operator the same is true for $S_{K,c}$.

(i) \Leftrightarrow (iv) This is a direct consequence of the explicit formula (6.1) and the description of $\text{mul } J_c^{**}$ in Theorem 3.1, since according to the condition (6.9) one has that $\text{mul } J_c^{**} = \{0\}$. □

By the extremality property of Kreĭn type extensions it is also clear that if $S_{K,\gamma}$ is an operator then the same is true for every selfadjoint extension $S_{K,c}$ with $c < \gamma$. Similarly, if $S_{K,\gamma} \in \mathbf{B}(\mathfrak{H})$ then also $S_{K,c} \in \mathbf{B}(\mathfrak{H})$ for all $c < \gamma$.

7 Extremal semibounded extensions

Let $S \in \mathbf{L}(\mathfrak{H})$ be a semibounded relation with lower bound $\gamma \in \mathbb{R}$ and let $c \leq \gamma$. Then S has semibounded selfadjoint extensions S_F and $S_{K,c}$, which satisfy

$$t(S) \subset \bar{t}(S) = t_{S_F} \subset t_{S_{K,c}}. \tag{7.1}$$

Moreover, for any $c \leq \gamma$ a semibounded selfadjoint extension $H \in \mathbf{L}(\mathfrak{H})$ of S satisfies the following equivalence

$$c \leq H \Leftrightarrow t_{S_{K,c}} \leq t_H \leq t_{S_F}, \tag{7.2}$$

see Corollaries 4.3, 6.3. In fact, the form inequalities in (7.2) imply for a semibounded selfadjoint relation $H \in \mathbf{L}(\mathfrak{H})$ that it is automatically an extension of S ; see [7].

Theorem 6.1 shows that for $c < \gamma$ one has $m(S_{K,c}) = c < m(S) = m(S_F)$. In the case that $c = \gamma$ one still has $S_{K,\gamma} \leq S_F$. It is possible that here equality holds and the next result contains some criteria for this situation; cf. [14, Theorem 4.14], [12, Proposition 12], see also [7, Corollary 5.3.10].

Proposition 7.1 *Let $S \in \mathbf{L}(\mathfrak{H})$ be semibounded with lower bound $\gamma \in \mathbb{R}$ and let $c \leq \gamma$. Then the following statements are equivalent:*

- (i) *the equality $S_{K,\gamma} = S_F$ holds;*
- (ii) *$\ker(S^* - c) \cap \text{dom}(S_{K,\gamma} - \gamma)^{\frac{1}{2}} = \{0\}$ for some (equivalently for all) $c < \gamma$;*
- (iii) *for some (equivalently for all) $c < \gamma$ one has*

$$\sup_{\{\varphi, \varphi'\} \in S - \gamma} \frac{|\langle \varphi', h \rangle|^2}{|\langle \varphi', \varphi \rangle|} = \infty \quad \text{for every } h \in \ker(S^* - \gamma). \tag{7.3}$$

Proof For the equivalence of (i) and (ii) see [7, Corollary 5.3.10].

To see the equivalence of (ii) and (iii) let $c < \gamma$ and let $h \in \ker(S^* - c)$. Then (6.8) shows that $h \in \text{dom}(S_{K,\gamma} - \gamma)^{\frac{1}{2}}$ if and only if there exists $C_h < \infty$ such that

$$|\langle \varphi', h \rangle|^2 \leq C_h \langle \varphi', \varphi \rangle \quad \text{for all } \{\varphi, \varphi'\} \in S - c.$$

Thus, $\ker (S^* - c) \cap \text{dom} (S_{K,\gamma} - \gamma)^{\frac{1}{2}} = \{0\}$ if and only if (7.3) holds. \square

In the rest of this section it is assumed that $S_{K,\gamma} \neq S_F$, in which case there are also other semibounded selfadjoint extensions H of S between these two extreme extensions. In particular, in this case $\mathfrak{t}_{S_F} \subsetneq \mathfrak{t}_{S_{K,c}}$ and the form inclusions in (7.1) with the inequalities in (7.2) suggest the class of extremal extensions that will be introduced and discussed in this section; see Definition 7.4 below. The definition of this class will be motivated by the following lemma.

Lemma 7.2 *Let $S \in \mathbf{L}(\mathfrak{H})$ be semibounded with lower bound $\gamma \in \mathbb{R}$ and let $c \leq \gamma$. Then the following statements are equivalent:*

(i) $H \in \mathbf{L}(\mathfrak{H})$ is a selfadjoint extension of S which is bounded below by c and which satisfies

$$\mathfrak{t}_{S_F} \subset \mathfrak{t}_H \subset \mathfrak{t}_{S_{K,c}}. \tag{7.4}$$

(ii) $H = R_c^* R_c^{**} + c$ for some $R_c \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$ which satisfies

$$Q_c \subset R_c \subset (J_c^*)_{\text{reg}}. \tag{7.5}$$

Proof (i) \Rightarrow (ii) Assume that H is a semibounded selfadjoint extension of S , which is bounded below by c , and which satisfies $\mathfrak{t}_H \subset \mathfrak{t}_{S_{K,c}}$. The inclusion $\mathfrak{t}_H \subset \mathfrak{t}_{S_{K,c}}$ means that

$$\mathfrak{t}_H[\varphi, \psi] = c(\varphi, \psi) + ((J_c^*)_{\text{reg}}\varphi, (J_c^*)_{\text{reg}}\psi), \quad \varphi, \psi \in \text{dom } \mathfrak{t}_H \subset \text{dom } J_c^*,$$

see (6.1). Then the operator R_c defined by $R_c = (J_c^*)_{\text{reg}}|_{\text{dom } \mathfrak{t}_H}$ is a representing map for the closed form \mathfrak{t}_H which extends the representing map Q_c in (2.2). In particular, R_c is a closed operator and $H = c + R_c^* R_c$. Thus (7.5) has been shown.

(ii) \Rightarrow (i) It follows from (2.6) and (7.5) that

$$S - c \subset J_c Q_c \subset R_c^* R_c \subset R_c^* R_c^{**},$$

and, hence, $H = c + R_c^* R_c^{**}$ is a semibounded selfadjoint extension of S . It is clear from the construction that $R_c^{**} \subset (J_c^*)_{\text{reg}}$, so that $\mathfrak{t}(H) \subset \mathfrak{t}(S_{K,c})$. Since H is a semibounded selfadjoint extension of S it follows automatically that $\mathfrak{t}(S_F) \subset \mathfrak{t}(H)$. Therefore, one sees (7.4). \square

The next lemma is helpful in the discussion of extremal extensions in Definition 7.4. Recall from (2.18) that for a semibounded selfadjoint relation $H \in \mathbf{L}(\mathfrak{H})$ one has $\mathfrak{t}(H) \subset \mathfrak{t}(H) = \mathfrak{t}_H$.

Lemma 7.3 *Let $S \in \mathbf{L}(\mathfrak{H})$ be semibounded with lower bound $\gamma \in \mathbb{R}$. Assume that $c \leq \gamma$ and let $H \in \mathbf{L}(\mathfrak{H})$ be a semibounded selfadjoint extension of S which is bounded below by c . Then there are the following statements.*

(a) For $\{h, h'\} \in S$, $\{f, f'\} \in H$, and $f \in \text{dom } J_c^*$ one has

$$\begin{aligned} (\mathfrak{t}(H) - c)[f - h, f - h] - \|(J_c^*)_{\text{reg}}f - Q_c h\|^2 \\ = (\mathfrak{t}(H) - c)[f, f] - \|(J_c^*)_{\text{reg}}f\|^2. \end{aligned} \tag{7.6}$$

(b) For $\{h, h'\} \in S$ and $\{f, f'\} \in H$ one has

$$(\mathfrak{t}(H) - c)[f - h, f - h] \geq \|(J_c^*)_{\text{reg}} f - Q_c h\|^2. \tag{7.7}$$

(c) If $\mathfrak{t}_H \subset \mathfrak{t}_{S_{K,c}}$, then for $\{h, h'\} \in S$ and $\{f, f'\} \in H$:

$$(\mathfrak{t}(H) - c)[f - h, f - h] = \|(J_c^*)_{\text{reg}} f - Q_c h\|^2. \tag{7.8}$$

Proof (a) Since $\{h, h'\} \in S$ there is the identity

$$(h' - ch, h) = (Q_c h, Q_c h). \tag{7.9}$$

The assumption $f \in \text{dom } J_c^*$ implies $\{f, (J_c^*)_{\text{reg}} f\} \in J_c^*$. Recall that for $\{h, h'\} \in S$ one has $\{Q_c h, h' - ch\} \in J_c$. Hence, one obtains

$$(h' - ch, f) = (Q_c h, (J_c^*)_{\text{reg}} f). \tag{7.10}$$

Since $\{f, f'\} \in H \subset S^*$ and $\{h, h'\} \in S$, one has $(f', h) = (f, h')$, so that by (7.10),

$$(f' - cf, h) = (f, h' - ch) = ((J_c^*)_{\text{reg}} f, Q_c h). \tag{7.11}$$

Thus, it follows from (7.9), (7.10), and (7.11) that

$$\begin{aligned} & (\mathfrak{t}(H) - c)[f - h, f - h] \\ &= (f' - cf - (h' - ch), f - h) \\ &= (f' - cf, f) - (f' - cf, h) - (h' - ch, f) + (h' - ch, h) \\ &= (f' - cf, f) - ((J_c^*)_{\text{reg}} f, Q_c h) - (Q_c h, (J_c^*)_{\text{reg}} f) + (Q_c h, Q_c h) \\ &= (f' - cf, f) - ((J_c^*)_{\text{reg}} f, (J_c^*)_{\text{reg}} f) + \|(J_c^*)_{\text{reg}} f - Q_c h\|^2, \end{aligned}$$

which gives the identity (7.6), since $(\mathfrak{t}(H) - c)[f, f] = (f' - cf, f)$.

(b) and (c) Since $c \leq m(H)$, Corollary 6.3 shows that $\mathfrak{t}_{S_{K,c}} \leq \mathfrak{t}_H$. Then one sees that $\{f, f'\} \in H$ implies that $f \in \text{dom } S_{K,c} = \text{dom } J_c^*$ and

$$((J_c^*)_{\text{reg}} f, (J_c^*)_{\text{reg}} f) = (\mathfrak{t}(S_{K,c}) - c)[f, f] \leq (\mathfrak{t}(H) - c)[f, f],$$

with equality if $\mathfrak{t}_H \subset \mathfrak{t}_{S_{K,c}}$. Thus (7.7) and (7.8) follow from (7.6). □

The identity (7.8) in Lemma 7.3 is the basis for the following definition.

Definition 7.4 Let $S \in \mathbf{L}(\mathfrak{H})$ be semibounded with lower bound $\gamma \in \mathbb{R}$ and let $c \leq \gamma$. A selfadjoint extension $H \in \mathbf{L}(\mathfrak{H})$ of S is called *extremal* with respect to c if for all $\{f, f'\} \in H$ one has

$$\inf \{ (f' - h' - c(f - h), f - h) : \{h, h'\} \in S \} = 0. \tag{7.12}$$

Let H be a selfadjoint extension of S which is extremal with respect to c . Then it is clear from the definition that for all $\{f, f'\} \in H$ and $\{h, h'\} \in S$ one has

$$(f' - h' - c(f - h), f - h) \geq 0.$$

By taking $\{h, h'\} = \{0, 0\}$, it follows that H is automatically bounded below by c . Thus one may write (7.12) in the equivalent form

$$\inf \{ (t(H) - c)[f - h, f - h] : \{h, h'\} \in S \} = 0.$$

Theorem 7.5 *Let $S \in \mathbf{L}(\mathfrak{H})$ be a semibounded relation with lower bound $\gamma \in \mathbb{R}$ and let $c \leq \gamma$. Then the following statements are equivalent:*

- (i) $H \in \mathbf{L}(\mathfrak{H})$ is a semibounded selfadjoint extension of S which is bounded below by c and which satisfies (7.4);
- (ii) $H \in \mathbf{L}(\mathfrak{H})$ is a semibounded selfadjoint extension of S which is extremal with respect to c .

Proof (i) \Rightarrow (ii) This follows from (7.8) in Lemma 7.3 by observing that $\text{ran } Q_c$ is dense in $(\text{mul } J_c^*)^\perp$.

(ii) \Rightarrow (i) Let $H \in \mathbf{L}(\mathfrak{H})$ be a selfadjoint extension of S which is extremal with respect to c . Then H is bounded below by c and by Corollary 6.3 it follows that $t_{S_{K,c}} \leq t_H$. Therefore one may apply (7.7) in Lemma 7.3 with $\{f, f'\} \in H$ and $\{h, h'\} \in S$:

$$(f' - h' - c(f - h), f - h) \geq \|(J_c^*)_{\text{reg}} f - Q_c h\|^2.$$

Therefore, both sides of the identity (7.6) are nonnegative for all $\{f, f'\} \in H$ and $\{h, h'\} \in S$. Note that the element $\{h, h'\} \in S$ appears only in the left-hand side of (7.6).

Now let $\{f, f'\} \in H$ and choose $\epsilon > 0$. Since H is extremal, it follows from (7.12) that there exists $\{h, h'\} \in S$ such that

$$(f' - h' - c(f - h), f - h) < \epsilon. \tag{7.13}$$

From (7.13) one concludes for the left-hand side of (7.6)

$$\begin{aligned} 0 &\leq (t(H) - c)[f - h, f - h] - \|(J_c^*)_{\text{reg}} f - Q_c h\|^2 \\ &\leq (t(H) - c)[f - h, f - h] < \epsilon. \end{aligned}$$

Therefore, it follows from the right-hand side of (7.6) that for any $\{f, f'\} \in H$ and any $\epsilon > 0$:

$$0 \leq (t(H) - c)[f, f] - (t(S_{K,c}) - c)[f, f] < \epsilon.$$

Hence $t(H)[f, f] = t(S_{K,c})[f, f]$, $\{f, f'\} \in H$, and by polarization one obtains

$$t(H)[f, g] = t(S_{K,c})[f, g], \quad \{f, f'\}, \{g, g'\} \in H.$$

Since $\text{dom } t(H) \subset \text{dom } S_{K,c}$, this implies that $t(H) \subset t(S_{K,c})$ and, since H is a semibounded selfadjoint extension of S one also has $t(S_F) \subset t(H)$. Thus (7.4) is satisfied. \square

Definition 7.4 goes back to Arlinskiĭ and Tsekanovskiĭ [6], see also [3, 4]. The results in this section are based on [5, 15], where also an earlier form of Theorem 7.5 can be found.

8 Symmetric relations whose domain and range are orthogonal

Let $S \in \mathbf{L}(\mathfrak{H})$ be a linear relation. Then the *numerical range* $\mathcal{W}(S)$ of S is defined by

$$\mathcal{W}(S) = \{(\varphi', \varphi) : \{\varphi, \varphi'\} \in S : \|\varphi\| = 1\} \subset \mathbb{C},$$

when $\text{dom } S \neq \{0\}$, and by $\{0\} \subset \mathbb{C}$ if $\text{dom } S = \{0\}$, i.e., if S is purely multivalued. For linear relations $S \in \mathbf{L}(\mathfrak{H})$ the numerical range $\mathcal{W}(S)$ is a convex set; see [22, Proposition 2.18]. Moreover, it is clear that all eigenvalues in \mathbb{C} of S belong to $\mathcal{W}(S)$. Clearly, the numerical range of the inverse of S is given by

$$\mathcal{W}(S^{-1}) = \{\lambda \in \mathbb{C} : \bar{\lambda} \in \mathcal{W}(S)\}.$$

The interest in this section is in linear relations $S \in \mathbf{L}(\mathfrak{H})$ for which $\mathcal{W}(S) = \{0\}$; this class was discussed earlier in [8]. Here is a characterization of it; see [13] and also [27].

Lemma 8.1 *For every linear relation $S \in \mathbf{L}(\mathfrak{H})$ the following statements are equivalent:*

- (i) $\mathcal{W}(S) = \{0\}$;
- (ii) $\text{dom } S \perp \text{ran } S$.

Now assume that $\mathcal{W}(S) = \{0\}$. Then the linear relation S is nonnegative and the form $t(S) \in \mathbf{F}(\mathfrak{H})$, determined by S in (2.1), is trivial by Lemma 8.1:

$$t(S)[\varphi, \psi] = (\varphi', \psi) = 0, \quad \{\varphi, \varphi'\}, \{\psi, \psi'\} \in S.$$

Thus $t(S) \in \mathbf{F}(\mathfrak{H})$ is the null form defined on $\text{dom } S$:

$$t(S) = 0, \quad \text{dom } t(S) = \text{dom } S.$$

This shows that $\gamma = 0$ and, therefore, in (2.2) one may choose $c = 0$ and a representing map $Q \in \mathbf{L}(\mathfrak{H})$ by

$$Q = \text{dom } S \times \{0\}. \tag{8.1}$$

Clearly, the corresponding companion relation $J \in \mathbf{L}(\mathfrak{H})$, given by (2.3), reduces to

$$J = \{ \{0, \varphi'\} : \{\varphi, \varphi'\} \in S \} = \{0\} \times \text{ran } S,$$

so that

$$J^* = (\text{ran } S)^\perp \times \mathfrak{H} = \ker S^* \times \mathfrak{H}.$$

Since $\text{mul } J^* = \mathfrak{H}$, it follows that

$$(J^*)_{\text{reg}} = \ker S^* \times \{0\}.$$

Hence the form $\mathfrak{s}(S)$ defined in (2.9) is given by

$$\mathfrak{s}(S)[\varphi, \psi] = 0, \quad \varphi, \psi \in \text{dom } \mathfrak{s}(S) = \ker S^*.$$

Note that (2.4) implies $\text{dom } S \subset \ker S^*$ and $\text{ran } S \subset \text{mul } S^*$, which can be seen immediately from the assumption $\mathcal{W}(S) = \{0\}$. It is clear from these observations that $\mathfrak{t}(S) \subset \mathfrak{s}(S)$.

Lemma 8.2 *Let $S \in \mathbf{L}(\mathfrak{H})$ be a linear relation such that $\mathcal{W}(S) = \{0\}$. Then the Friedrichs extension of S is given by*

$$S_F = \overline{\text{dom } S} \times \text{mul } S^* = \overline{\ker } S \times \text{mul } S^*, \tag{8.2}$$

and the Kreĭn type extension $S_K = S_{K,0}$ is given by

$$S_K = \ker S^* \times \overline{\text{ran } S} = \ker S^* \times \overline{\text{mul } S}. \tag{8.3}$$

Proof It follows from the representation (8.1) that

$$Q^{**} = \overline{\text{dom } S} \times \{0\} \quad \text{and} \quad Q^* = \mathfrak{H} \times \text{mul } S^*.$$

By (4.1) one has $S_F = Q^* Q^{**}$, which gives (8.2). Likewise, one has

$$((J_c^*)_{\text{reg}})^* = \mathfrak{H} \times \overline{\text{ran } S}.$$

By (6.1) one has $S_K = ((J_c^*)_{\text{reg}})^* (J_c^*)_{\text{reg}}$, which gives (8.3). □

Thus one sees that both nonnegative selfadjoint extensions S_F and S_K are singular relations. It follows from Lemma 8.2 that $S_F = S_K$ if and only if

$$\ker S^* = \overline{\text{dom } S} \quad \text{and} \quad \text{mul } S^* = \overline{\text{ran } S}.$$

Furthermore,

$$S_F \cap S_K = \overline{\ker } S \times \overline{\text{ran } S},$$

which implies that $S_F \cap S_K = \overline{S}$. In addition, observe that the description in Corollary 4.2, when applied to S with $\mathcal{W}(S) = \{0\}$, shows that

$$\text{ran } S_F^{\frac{1}{2}} = (\text{dom } S)^\perp = \text{mul } S^* = \text{mul } S_F,$$

which agrees with (8.2). Similarly Corollary 6.2 yields

$$\text{dom } S_K^{\frac{1}{2}} = (\text{ran } S)^\perp = \text{ker } S^* = \text{ker } S_K,$$

and this agrees with (8.3).

Notice that if $\mathcal{W}(S) = \{0\}$, then by Lemma 8.2 it follows that $\mathcal{W}(S_F) = \{0\}$ and, likewise, $\mathcal{W}(S_K) = \{0\}$. This observation gives a connection to the nonnegative extremal extensions of S (i.e., extremal with respect to 0).

Proposition 8.3 *Let $S \in \mathbf{L}(\mathfrak{H})$ be nonnegative relation such that $\mathcal{W}(S) = \{0\}$. Let $H \in \mathbf{L}(\mathfrak{H})$ be a selfadjoint extension of S . Then the following conditions are equivalent:*

- (i) $\mathcal{W}(H) = \{0\}$;
- (ii) H is a nonnegative extremal extension of S .

Proof The assumption $\mathcal{W}(S) = \{0\}$ implies that $\mathcal{W}(S_K) = \{0\}$. In other words, $\text{dom } S_K \perp \text{ran } S_K$. Hence the form $\mathfrak{t}(S_K)$ corresponding to S_K is the zero form on $\text{dom } S_K$.

(i) \Rightarrow (ii) Assume that $\mathcal{W}(H) = \{0\}$. Then clearly $\mathfrak{t}(H)$ is the zero form and $\mathfrak{t}_H = 0$. Since the assumption $\mathcal{W}(H) = \{0\}$ implies that $\mathcal{W}(S) = \{0\}$, it follows from Lemma 8.2 that $\mathcal{W}(S_F) = 0$ and $\mathcal{W}(S_K) = 0$. Thanks to H being a nonnegative selfadjoint extension of S , Corollary 6.3 implies that $\mathfrak{t}_{S_K} \leq \mathfrak{t}_H$, which gives $\text{dom } \mathfrak{t}_H \subset \text{dom } \mathfrak{t}_{S_K}$. Recall that $\mathfrak{t}_{S_F} \subset \mathfrak{t}_H$. All these forms are null forms and one obtains $\mathfrak{t}_{S_F} \subset \mathfrak{t}_H \subset \mathfrak{t}_{S_K}$. Now Theorem 7.4 shows that H is extremal.

(ii) \Rightarrow (i) Let H be an extremal extension of S . Then by Theorem 7.4 one has $\mathfrak{t}(H) \subset \mathfrak{t}(S_K)$. Hence $\mathfrak{t}(H)$ is the zero form on $\text{dom } \mathfrak{t}(H)$. In particular, it follows that $\mathcal{W}(H) = 0$. □

9 Symmetric relations defined by semibounded quadratic forms

In this section attention is paid to the representation of a general semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ as given in [21]. It is known that such a form induces a semibounded relation $S_{\mathfrak{t}} \in \mathbf{L}(\mathfrak{H})$ and a semibounded selfadjoint relation $\tilde{A}_{\mathfrak{t}} \in \mathbf{L}(\mathfrak{H})$ which extends $S_{\mathfrak{t}}$. The relation $\tilde{A}_{\mathfrak{t}}$ is the Friedrichs extension of an, in general, nontrivial extension of $S_{\mathfrak{t}}$. The semibounded relation $S_{\mathfrak{t}}$ itself gives rise to a semibounded form $\mathfrak{t}(S_{\mathfrak{t}})$, which induces a Friedrichs extension of $S_{\mathfrak{t}}$.

The general semibounded form $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ is given by

$$\mathfrak{t}[\varphi, \psi] = c(\varphi, \psi) + (Q\varphi, Q\psi), \quad \varphi, \psi \in \text{dom } Q, \tag{9.1}$$

for some $c \in \mathbb{R}$ and a representing map $Q \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$. Note that the form t is closable (closed) or singular if and only if the operator Q is closable (closed) or singular, respectively. As mentioned above, the form t induces a semibounded selfadjoint relation $\tilde{A}_t \in \mathbf{L}(\mathfrak{H})$ and a semibounded relation $S_t \in \mathbf{L}(\mathfrak{H})$, and they are given by

$$\tilde{A}_t = c + Q^*Q^{**} \quad \text{and} \quad S_t = c + Q^*Q, \tag{9.2}$$

which are characterized in [21, Theorem 6.1], see also [2, Theorem 3.2]. It follows from (9.2) that

$$S_t \subset \tilde{A}_t.$$

Denote the regular part of Q by Q_{reg} and the regular part of t by τ , so that

$$\tau[\varphi, \psi] = c(\varphi, \psi) + (Q_{\text{reg}}\varphi, Q_{\text{reg}}\psi), \quad \varphi, \psi \in \text{dom } Q, \tag{9.3}$$

is a regular (closable) form in $\mathbf{F}(\mathfrak{H})$; its closure is given by

$$\bar{\tau}[\varphi, \psi] = c(\varphi, \psi) + (Q_{\text{reg}}^{**}\varphi, Q_{\text{reg}}^{**}\psi), \quad \varphi, \psi \in \text{dom } Q^{**}.$$

Then, likewise, there are a semibounded selfadjoint relation $\tilde{A}_\tau \in \mathbf{L}(\mathfrak{H})$ and a semibounded relation $S_\tau \in \mathbf{L}(\mathfrak{H})$, given by

$$S_\tau = c + (Q_{c,\text{reg}})^*Q_{c,\text{reg}} \quad \text{and} \quad \tilde{A}_\tau = c + (Q_{c,\text{reg}})^*(Q_{c,\text{reg}})^{**}, \tag{9.4}$$

and they satisfy

$$S_t \subset S_\tau \quad \text{and} \quad \tilde{A}_t = \tilde{A}_\tau = A_{\bar{\tau}};$$

where $A_{\bar{\tau}} \in \mathbf{L}(\mathfrak{H})$ is the semibounded selfadjoint relation corresponding to the closed form $\bar{\tau} \in \mathbf{F}(\mathfrak{H})$; see [21, Theorem 6.5]. It is clear from (9.4) that $\tilde{A}_t = \tilde{A}_\tau = A_{\bar{\tau}}$ is the Friedrichs extension of the semibounded relation $S_\tau \in \mathbf{L}(\mathfrak{H})$; see Sect. 4.

Next consider the connection of the above facts with the constructions in the present paper. Let the semibounded form $t \in \mathbf{F}(\mathfrak{H})$ be given by (9.1) and let the corresponding semibounded relation S_t be as in (9.2). Then this relation S_t induces a semibounded form $t(S_t) \in \mathbf{F}(\mathfrak{H})$ as in Definition 2.1. First observe that the domain of S_t in (9.2) satisfies $\text{dom } S_t = \mathcal{R}$, when the linear subspace $\mathcal{R} \subset \text{dom } Q$ is defined by

$$\mathcal{R} = \{\varphi \in \text{dom } Q : Q\varphi \in \text{dom } Q^*\},$$

so that in general one has

$$\{0\} \subset \mathcal{R} \subset \text{dom } Q.$$

Define the operator $\dot{Q} \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ as the restriction of Q to the linear subspace $\mathcal{R} \subset \text{dom } Q$:

$$\dot{Q} = Q|_{\mathcal{R}}, \quad \text{where } \text{dom } \dot{Q} = \mathcal{R}.$$

Then it follows directly from the definition that with $\varphi \in \mathcal{R}$ one has

$$Q\varphi \in \text{dom } Q^* \subset (\text{mul } Q^{**})^\perp,$$

which shows $Q\varphi = (I - P)Q\varphi$, when P is the orthogonal projection onto $\text{mul } Q^{**}$. Hence, one sees the inclusion

$$\dot{Q} \subset Q_{\text{reg}}, \tag{9.5}$$

which shows that $\dot{Q} \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ is closable. Observe that $\dot{Q} = Q_{\text{reg}}$ if and only if $\mathcal{R} = \text{dom } Q$. Now let $\{\varphi, \varphi'\}, \{\psi, \psi'\} \in S_{\mathfrak{t}}$, so that in particular $\{Q\varphi, \varphi'\} \in Q^*$ and $\{\psi, Q\psi\} \in Q$. As a consequence, it follows by Definition 2.2 that the form $\mathfrak{t}(S_{\mathfrak{t}}) \in \mathbf{F}(\mathfrak{H})$ is given by

$$\mathfrak{t}(S_{\mathfrak{t}})[\varphi, \psi] = c(\varphi, \psi) + (\dot{Q}\varphi, \dot{Q}\psi), \quad \varphi, \psi \in \text{dom } \dot{Q} = \text{dom } S_{\mathfrak{t}} = \mathcal{R}, \tag{9.6}$$

which confirms that $\mathfrak{t}(S_{\mathfrak{t}})$ is closable. It follows from the inclusion (9.5) that

$$\mathfrak{t}(S_{\mathfrak{t}}) \subset \mathfrak{t},$$

and, since both forms are closable, one obtains $\bar{\mathfrak{t}}(S_{\mathfrak{t}}) \subset \bar{\mathfrak{t}}_{\mathfrak{t}}$. This last inclusion implies the inequality $\bar{\mathfrak{t}}_{\mathfrak{t}} \leq \bar{\mathfrak{t}}(S_{\mathfrak{t}})$ or, equivalently,

$$c + Q_{\text{reg}}^* Q_{\text{reg}}^{**} = (S_{\mathfrak{t}})_{\mathbf{F}} \leq (S_{\mathfrak{t}})_{\mathbf{F}} = c + \dot{Q}^* \dot{Q}^{**},$$

see [21]. This shows the ordering between the Friedrichs extension of $S_{\mathfrak{t}}$ and the Friedrichs extension of $S_{\mathfrak{t}}$.

The above discussion will be illustrated by the following extreme example involving a nonnegative and singular form.

Proposition 9.1 *Let $\mathfrak{t} \in \mathbf{F}(\mathfrak{H})$ be a nonnegative singular form and let $Q \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be a singular representing map for \mathfrak{t} , so that*

$$\mathfrak{t}[\varphi, \psi] = (Q\varphi, Q\psi), \quad \varphi, \psi \in \text{dom } \mathfrak{t} = \text{dom } Q,$$

in which case $Q_{\text{reg}} = 0$ and $\mathfrak{r} = \mathfrak{t}_{\text{reg}} = 0$. Then the following relations, corresponding to the forms \mathfrak{t} and \mathfrak{r} in (9.1) and (9.3), are singular:

$$\begin{cases} S_{\mathfrak{r}} = (Q_{\text{reg}})^* Q_{\text{reg}} = \text{dom } Q \times (\text{dom } Q)^\perp, \\ \tilde{A}_{\mathfrak{t}} = (Q_{\text{reg}})^* (Q_{\text{reg}})^{**} = \tilde{A}_{\mathfrak{t}} = \overline{\text{dom } Q} \times (\text{dom } Q)^\perp, \end{cases} \tag{9.7}$$

and $S_{\mathfrak{r}}$ is essentially selfadjoint. Furthermore, the following relations corresponding to the form $\mathfrak{t}(S_{\mathfrak{t}})$ in (9.6), are singular,

$$\begin{cases} S_{\mathfrak{t}} = Q^* Q = \ker Q \times (\text{dom } Q)^\perp, \\ (S_{\mathfrak{t}})_{\mathbf{F}} = Q^* Q^{**} = \overline{\ker Q} \times (\ker Q)^\perp, \end{cases} \tag{9.8}$$

and S_t is essentially selfadjoint if and only if $\overline{\text{dom } Q} = \overline{\text{ker } Q}$.

Proof Since t is singular one has $\tau = t_{\text{reg}} = 0$. Thus, equivalently, $Q = Q_{\text{sing}}$ or $Q_{\text{reg}} = 0$. Therefore it is clear that

$$Q_{\text{reg}} = \text{dom } t \times \{0\}, \quad (Q_{\text{reg}})^* = \mathfrak{K} \times (\text{dom } t)^\perp, \quad (Q_{\text{reg}})^{**} = \overline{\text{dom } t} \times \{0\},$$

from which the assertions in (9.7) follow.

Since Q is singular, recall that $\text{dom } Q^* = \text{ker } Q^*$. Let $\{f, f'\} \in Q^*Q$, so that $\{f, \varphi\} \in Q$ and $\{\varphi, f'\} \in Q^*$ for some $\varphi \in \mathfrak{K}$. In fact, $\varphi \in \text{ran } Q \cap \text{ker } Q^* = \{0\}$. Thus $f \in \text{ker } Q$ and $f' \in \text{mul } Q^*$, which gives $\{f, f'\} \in \text{ker } Q \times \text{mul } Q^*$. Therefore $Q^*Q \subset \text{ker } Q \times \text{mul } Q^*$. Since the reverse inclusion is obvious, one obtains the identity $Q^*Q = \text{ker } Q \times \text{mul } Q^*$. Thus (9.8) follows. \square

As an application a Lebesgue decomposition of a form involving nonzero regular and singular parts is given in the following example.

Example 9.2 Let $t \in \mathbf{F}(\mathfrak{H})$ be the singular form from Proposition 9.1 and assume without loss of generality that the singular representing map $Q \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ for t is minimal; see [21]). Since t is singular its lower bound satisfies $\gamma = 0$, see [21]. Observe that t_{reg} has representing map $Q_{\text{reg}} = \text{dom } t \times \{0\}$, while t_{sing} has Q as a representing map. Now consider the shifted form $t - c \in \mathbf{F}(\mathfrak{H})$ with $c < \gamma = 0$. Observe that

$$(t - c)_{\text{reg}} = t_{\text{reg}} - c = -c \quad \text{and} \quad (t - c)_{\text{sing}} = t_{\text{sing}} = t,$$

see [34, Corollary 2.3] or [21]. This leads to the Lebesgue decomposition of the form $t - c$:

$$t - c = (t - c)_{\text{reg}} + (t - c)_{\text{sing}} = -c + t_{\text{sing}} = -c + t, \tag{9.9}$$

see [18, 20, 21]. In order to determine a representing map for $t - c$, first define for $c < \gamma$

$$q_c : \mathfrak{H} \rightarrow \mathfrak{H} = \overline{\text{dom } t} \oplus (\text{dom } t)^\perp, \quad \varphi \rightarrow \begin{pmatrix} \sqrt{|c|}\varphi \\ 0 \end{pmatrix}, \quad \varphi \in \text{dom } t, \tag{9.10}$$

so that q_c is a representing map for the nonnegative form $-c(\varphi, \psi)$, $\varphi, \psi \in \text{dom } t$. It is clear that the column operator Q_c defined by

$$Q_c = \begin{pmatrix} q_c \\ Q \end{pmatrix} : \text{dom } t \rightarrow \mathfrak{H} \times \mathfrak{K}, \tag{9.11}$$

is a representing map for $t - c$, which in general is not minimal. Consider the Lebesgue decomposition of Q : $Q = Q_{\text{reg}} + Q_{\text{sing}}$. Since Q is singular and q_c is a bounded operator on $\text{dom } t$ one sees that

$$\text{mul } Q_c^{**} = \{0\} \times \text{mul } Q^{**} = \{0\} \times \mathfrak{K} \subset \mathfrak{H} \times \mathfrak{K},$$

where the second identity holds by minimality of Q . Hence, with $P_{\mathfrak{K}}$ the orthogonal projection from $\mathfrak{H} \times \mathfrak{K}$ onto \mathfrak{K} , it follows that

$$Q_{c,\text{reg}} = (I_{\mathfrak{H} \times \mathfrak{K}} - P_{\mathfrak{K}})Q_c = q_c \quad \text{and} \quad Q_{c,\text{sing}} = P_{\mathfrak{K}}Q_c = Q \tag{9.12}$$

are representing maps for $(t - c)_{\text{reg}}$ and $(t - c)_{\text{sing}}$.

Next the interest is in the semibounded relations generated by the form $t - c$, given in terms of the corresponding representing map Q_c . It follows from the definition in (9.10) that

$$(Q_{c,\text{reg}})^* = q_c^* = \{ \{ \psi, \psi' \} \in \mathfrak{H} \times \mathfrak{H} : \psi' - \sqrt{|c|} \psi \in (\text{dom } t)^\perp \}. \tag{9.13}$$

Therefore it is clear from (9.13) that

$$(Q_{c,\text{reg}})^* Q_{c,\text{reg}} = q_c^* q_c = \{ \{ \varphi, -c\varphi + \chi \} : \varphi \in \text{dom } t, \chi \in (\text{dom } t)^\perp \}, \tag{9.14}$$

and

$$(Q_{c,\text{reg}})^* (Q_{c,\text{reg}})^{**} = q_c^* q_c^{**} = \{ \{ \varphi, -c\varphi + \chi \} : \varphi \in \overline{\text{dom } t}, \chi \in (\text{dom } t)^\perp \}. \tag{9.15}$$

On the other hand, using (9.11) and Proposition 9.1 one concludes that

$$Q_c^* Q_c = Q^* Q + q_c^* q_c = \{ \{ \varphi, -c\varphi + \chi \} : \varphi \in \ker t, \chi \in (\text{dom } t)^\perp \},$$

and

$$Q_c^* Q_c^{**} = Q^* Q^{**} + q_c^* q_c^{**} = \{ \{ \varphi, -c\varphi + \chi \} : \varphi \in \overline{\ker t}, \chi \in (\text{dom } t)^\perp \}.$$

Thus one has

$$Q_c^* Q_c = Q^* Q - c \quad \text{and} \quad Q_c^* Q_c^{**} = Q^* Q^{**} - c$$

Therefore the relations S_{t-c} and \tilde{A}_{t-c} associated with the shifted form $t - c$ in (9.2) are given by

$$S_{t-c} = Q_c^* Q_c = Q^* Q - c, \quad \tilde{A}_{t-c} = Q_c^* Q_c^{**} = Q^* Q^{**} - c,$$

with the explicit formulas for $S_t = Q^* Q$ and $\tilde{A}_t = Q^* Q^{**}$ as given in Proposition 9.1. In particular, $(S_{t-c})^{**} \subset \tilde{A}_{t-c}$, but $(S_{t-c})^{**} \neq \tilde{A}_{t-c}$ whenever the form t is not a zero form on $\text{dom } t$. Finally, from (9.7), (9.14) and (9.15), it is directly seen that

$$(Q_{c,\text{reg}})^* (Q_{c,\text{reg}})^{**} = q_c^* q_c^{**} = Q^* Q^{**} - c = \tilde{A}_{t-c} = (S_{t-c})^{**} = (S_t)^{**} - c.$$

The connection between Lebesgue decompositions of linear relations in (9.12) and Lebesgue decompositions of semibounded forms in (9.9) is a general fact, which relies on the existence of representing maps for forms; for details see [19–21].

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