Hamidreza Maleki Almani

Modern Stochastic Gaussian Models and Applications to Finance

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ACADEMIC DISSERTATION

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TIIVISTELMÄ

Tämä väitöskirja tarkastelee moderneja stokastisia prosesseja ja stokastisia malleja ja niiden sovelluksia erityisesti rahoituksessa. Tulokset on julkaistu neljässä artikkelissa. Ensimmäinen ja toinen artikkeleista käsittelevät laajennettuja gaussisia prosesseja, joilla on sekoitettu pitkän aikavälin riippuvuus (tällaisia malleja käytetään laajasti rahoituksessa). Ensimmäisessä artikkelissa esittelemme ja analysoimme monisekoitettua fraktionalista Brownin liikettä (mmfBm) ja siihen liittyvää Ornstein-Uhlenbeck-prosessia (OU), ja toisessa artikkelissa tarkastelemme parametrien estimointia OU-prosessille yleistettyjen momenttien menetelmällä (GMM). Kolmannessa artikkelissa tarkastelemme ennustamista gaussisille Volterra-prosesseille, joilla on hyppyjä. Lopuksi neljännessä artikkelissa tarkastelemme ehdollista keskisuojausta Black-Scholes-mallissa jossa on hyppyjä ja transaktiokuluja.

Avainsanoja: Yhdistetyt Poisson-hypyt, fraktionaalinen Brownin liike, fraktionaalinen malli, gaussinen Volterra-prosessi, suojaus, hyppymalli, pitkän riippuvuuden malli, matemaattinen rahoitusteoria, ennustuslait, stokastinen mallintaminen.

ABSTRACT

This thesis is an attempt to develop the modern stochastic processes and their associated models as well as their applications, particularly to finance. The results of this thesis are published in four articles. The first and the second articles deal with extended Gaussian processes, with mixed long-memory behaviour (which is widely used in finance). In the first article, we introduced and analyzed the multi-mixed fractional Brownian motion (mmfBm) and its associated Ornstein–Uhlenbeck (OU) process, and in the second one, we investigated a parameter estimation for the OU process based on the generalized method of moments (GMM). In the third article, we considered the prediction of Gaussian Volterra processes with jumps. Finally, by using these results in our fourth article, we investigated the conditional mean hedging and the conditional least square strategies in a Black-Scholes model with jumps and with transaction costs.

Keywords: Compound Poisson jumps, Fractional Brownian motion, Fractional model, Gaussian Volterra process, Hedging, Jump models, Long-memory models, Mathematical finance, Prediction laws, Stochastic modeling

To My Greatest Honor,

Long Lasting Forever,

IRAN

To My Family,

Ali, Pouran, Fatemeh, and Susanna

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First of all, I would point to the close relationship in our family that I always appreciate the most. Unfortunately, during these five years of my life in Finland, I lost both my lovely grandfathers, Mahmoud and Jacob, who were always supportive of us and they will be alive in my heart forever. I only have seen my parents once in this period, and I strongly thank my parents, Ali and Pouran, and my lovely sister Fatemeh (Leila) for their persistent patience, support, and love they have given me, even now, far away from home.

Living with a mathematician is always challenging. My career is to spend the majority of my time analyzing or machine learning with my computer. It has not been interesting to my spouse yet. I understand that has been difficult for her to be patient with it. However, we are at the beginning of our relationship and I thank her for our love and hope we compromise more in the future.

From the beginning of my career, there have been several critical times I managed to pass because of his strong support and help. I greatly thank my supervisor Prof. Tommi Sottinen, not only for his official or scientific patronage but also for his fraternal friendship.

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During my career here, I worked as a researcher and also taught closely to our faculty members of the Department of Mathematics and Statistics. I wish to thank them all for the great atmosphere of teamwork and assistance I have received from them and wish them all the best for their careers. Dear Prof. Pynnönen, Christina Gustafsson, Dr. Pape, Matti Laaksonen, and Dr. Wietsma you are the best colleagues.

Vaasa, Finland
Hamidreza Maleki Almani

CONTENTS

LIST OF PUBLICATIONS

The dissertation is based on the following four articles:

(I) Maleki Almani, H., & Sottinen, T. (2023). Multi-mixed fractional Brownian motions and Ornstein–Uhlenbeck processes. *Modern Stochastics: Theory and Applications*, 10(4), 343-366. doi:10.15559/23-VMSTA229

(II) Almani, H. M., & Sottinen, T. (2024). Parameter Estimation for multi-mixed Fractional Ornstein–Uhlenbeck Processes by Generalized Method of Moments. arXiv:2401.05114.

(III) Almani, H. M., Shokrollahi, F., & Sottinen, T. (2024). Prediction of Gaussian Volterra processes with compound Poisson jumps. *Statistics & Probability Letters*, 110054. doi:10.1016/j.spl.2024.110054

(IV) Almani, H. M., Shokrollahi, F., & Sottinen, T. (2024). Hedging in Jump Diffusion Model with Transaction Costs. arXiv: 2408.10785

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AUTHOR'S CONTRIBUTION

Publication I: "Multi-mixed fractional Brownian motions and Ornstein–Uhlenbeck processes"

The idea for this article was due to Hamidreza Maleki Almani. This article is the outcome of a joint research and all the results were proven and then simulated by Hamidreza Maleki Almani and revised with supervisor Tommi Sottinen.

Publication II: "Parameter Estimation for multi-mixed Fractional Ornstein–Uhlenbeck Processes by Generalized Method of Moments"

The idea for this article was due to Tommi Sottinen. This article is the outcome of a joint research and all the results were proven and then simulated by Hamidreza Maleki Almani and approved by supervisor Tommi Sottinen.

Publication III: "Prediction of Gaussian Volterra processes with compound Poisson jumps"

The idea for this article was due to Tommi Sottinen. This article is the outcome of a joint research. The results were proven in a collaboration by Hamidreza Maleki Almani, Foad Shokrollahi, and revised by supervisor Tommi Sottinen. The simulation was done by Hamidreza Maleki Almani.

Publication IV: "Hedging in Jump Diffusion Model with Transaction Costs"

The idea for this article was due to Tommi Sottinen that was methodologically modified by the idea due to Hamidreza Maleki Almani. This article is the outcome of a joint research and all the results were proven and then simulated by Hamidreza Maleki Almani and approved by supervisors Tommi Sottinen, and Foad Shokrollahi.

1 INTRODUCTION

The initiation of applying random processes to model the financial time series was the doctoral thesis of Bachelier (1900). Considering the Central Limit Theory, he modeled the risky asset prices by the linear Brownian motion (Bm) of Einstein (1905). Decades later, Samuelson (1965) showed that the geometric Brownian motion (gBm) models the asset prices essentially better than Bachelier's model. Applying Samuelson's gBm model for the underlying risky asset, Black and Scholes (1973) (BS model) evaluated the European option prices for the perfect hedging by the self-financing strategy in complete markets. Later on, this work was awarded the Nobel Prize in Economic Sciences in 1997 dramatically; unfortunately, Black was passed before. Leland (1985); Wilmott and Whalley (1993) modified the BS model and obtained the options and hedging strategies under the proportional transaction costs. Although the BS model was a marvelous inception that is still applicable in *business time* (see Geman, Madan, and Yor (2001a, 2001b)), some criticisms arose later.

Mandelbrot and Van Ness (1968) and related studies, showed the roughness of financial data is less than the Bm process. They observed that, the financial data has long memory (see Berg and Lyhagen (1998); Cont (2005); Dai and Singleton (2000); Hsieh (1991); Huang and Yang (1995); Lo (1991); Willinger, Taqqu, and Teverovsky (1999)). So, they suggested fractional Brownian motion (fBm), a representation of the Wiener spiral process of A. N. Kolmogorov (1940) to model prices. Many recent studies pursued this process and its applications (see Azmoodeh (2013); Shokrollahi, Kiliçman, and Magdziarz (2016); Wang (2010a, 2010b); Wang, Yan, Tang, and Zhu (2010); Wang, Zhu, Tang, and Yan (2010)). As a portfolio can include different risky assets, in practice there can be a linear combination of different fBm's to model the value of strategy. Cheridito (2001) showed a linear combination of a Bm and an independent fBm with Hurst index $H > 3/4$ is equivalent to another Bm. Y. S. Mishura and Valkeila (2002) showed this linear combination with any long-memory fBm, i.e., with Hurst index $H > 1/2$, leads to an arbitrage-free model. Almani, Hosseini, and Tahmasebi (2021); Zili (2006a) studied the finite mixed fBm's. In our first paper Almani and Sottinen (2023), we studied the Multi (infinite) mixed fBm's and its properties.

The geometric fBm model was studied in Bayraktar, Poor, and Sircar (2004); Meng and Wang (2010) to model the risky assets. However, to overcome some other problems of the BS model, such as inconsistency of returns, stochastic interest rate, and stochastic volatility, Vasicek (1977) proposed to use the Uhlenbeck and Ornstein (1930) process (OU) for the short-rate model. This model was then generalized by Hull and White (1987). OU process also was applied for the impied volatility by Heston (1993) model which using the Itô (1951a) formula transforms to the Cox, Ingersoll Jr, and Ross (1985) model (CIR). To associate such studies with appropriate roughness, Cheridito, Kawaguchi, and Maejima (2003) introduced the fractional

OU process (fOU). As mentioned above, in a portfolio of several risky assets, a linear combination of fOU processes makes sense. So, in our first paper Almani and Sottinen (2023), we also studied the Multi (infinite) mixed fOU processes and their properties. However, to apply this model one has to estimate many parameters. In our second paper Almani and Sottinen (2024), we studied the generalized method of moments (GMM) estimator for this matter.

In the BS model the underlying asset price is supposed to be continuous. However, historically it has been observed that many asset prices' time series show some random discontinuity called jumps. Stochastic jump processes have a long history of studies in potential theory by Hunt (1957); Kunita and Watanabe (1967); Lévy (1934, 1965); Meyer (1962); Watanabe (1964). In finance, Merton (1976a) applied the jump diffusion model (JD) for prices. His work was then pursued by Alòs, León, and Vives (2007); Broadie and Kaya (2006); Cont and Tankov (2002); Cont and Voltchkova (2005); Duffie, Pan, and Singleton (2000); Feng and Linetsky (2008); S. G. Kou (2002). For the JD model with compound Poisson jump of independent normal processes, Merton (1976a) evaluated the hedging strategy and option prices. Das and Foresi (1996) also calculated a close form of hedging and pricing when the interest rate has the JD model.

However, in general, as the market is incomplete under the jump-diffusion models, evaluating the perfect hedging and option pricing are impossible. To overcome this matter, Föllmer and Schweizer (1991) initiated hedging associated with the *risk measure* and Schweizer (1992, 1994, 1995) developed this idea. Recently, Sottinen and Viitasaari (2018) introduced the *conditional mean hedging* (CMH) method by employing the conditional laws from their prior paper Sottinen and Viitasaari (2016). This hedging is on the conditional average of the new model's value of the portfolio, conserning the conditional average of the BS portfolio's value. Shokrollahi and Sottinen (2017) studied this method for the *fractional* Black-Scholes model.

In our third article Almani, Shokrollahi, and Sottinen (2024b), we studied the required conditional laws for the Volterra-jump models which JD is a special case of. In our fourth paper, the idea of supervisors, Prof. Sottinen and Dr. Shokrollahi was to find the CMH strategy when the underlying asset has the general JD model. However, considering the numerical results I faced a critical fact: " *Even if the conditional means of some stochastic processes are equal, still the minimum distance of those processes is not guaranteed!* ". So, I proposed the *conditional least-square hedging* (CLH). Surprisingly this stronger hedging strategy is possible for the JD model. Using the conditional formulas of our third article, I calculated the (CLH) strategy for the JD model under transaction costs in our fourth article Almani, Shokrollahi, and Sottinen (2024a).

2 STOCHASTIC PROCESSES

In this chapter, we illustrate the probability space and stochastic processes. The initial concepts are generally from Chung (2001); Doob (1990); Jacod and Protter (2004); A. N. Kolmogorov (1950); Øksendal (2013); Resnick (1998); Shiryaev (1996). We then consider some well-known spacial cases. For the Gaussian processes, we recall some important parts of Biagini, Hu, Øksendal, and Zhang (2008); Doob (1990); Hida and Hitsuda (1993); Y. Mishura (2008); Øksendal (2013). For the Lévy processes, we point some results from Applebaum (2009); Kyprianou (2014); Sato (1999).

2.1 Preliminaries

A *probability space* is formally the set Ω of all probable states ω, with events as subsets $A \subset \Omega$, and we are interested to evaluate the probability of them $\mathbb{P}(A)$. However, some questions come up then. First; we need to distinguish what type of subsets reflect those events, i.e., what is the collection $\mathscr{F}$ of all possible events A. Second; as the function $\mathbb{P}$ is measuring the probability of events, it is indeed a measure. So, we need to characterize the probability measure function. The developement of the measure theory and probability theory is indeed an attempt to have a rational frame of principles answering such questions. Before further constructions, we review these principles here.

Definition 2.1 (σ-Algebra)**.** For a given set Ω, a σ-algebra $\mathscr{F}$ on Ω is a family of subsets in Ω that
(i) $\emptyset \in \mathscr{F}$, (empty set)
(ii) if $A \in \mathscr{F}$ then $A^c := \Omega \backslash A \in \mathscr{F}$,
(iii) if $A_1, A_2, \ldots \in \mathscr{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathscr{F}$.
The pair $(\Omega, \mathscr{F})$ is then called a measurable space.

Definition 2.2 (Measure)**.** For a measurable space $(\Omega, \mathscr{F})$, the function $\mu : \mathscr{F} \to [0, \infty)$ is a measure if
(i) $\mu(\emptyset) = 0$
(ii) for all disjoint $A_1, A_2, \ldots \in \mathscr{F}$, i.e. $A_i \cap A_j = \varnothing : i \neq j$:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

The triple $(\Omega, \mathscr{F}, \mu)$ is called a measure space then. Also, μ is called finite measure if $\mu(\Omega) < \infty$.

Definition 2.3 (Absolute Continuity)**.** Let μ, ν be two measures on $(\Omega, \mathscr{F})$, we say ν is absolutely continuous with respect to (w.r.t.) μ and denote it by $\nu \ll \mu$ iff for all $A \in \mathscr{F}$

$$\mu(A) = 0 \implies \nu(A) = 0.$$

Theorem 2.4 (Radon (1917)–Nikodym (1930) Transform)**.** *Let ν is a finite measure which is absolutely continuous w.r.t. a σ-finite measure μ, both on $(\Omega, \mathscr{F})$. Then there exists an μ-almost everywhere unique function f that $\int_\Omega |f| \,\mathrm{d}\mu < \infty$ and for all $A \in \mathscr{F}$*

$$\nu(A) = \int_A f \,\mathrm{d}\mu.$$

Proof. See Aliprantis and Burkinshaw (1998), Chapter 7, Section 39, or Folland (1999) Section 3.2. □

Remark 2.5. In such a condition the μ- (a.e.) unique function f is called the Radon–Nikodym derivative of ν w.r.t. μ and denoted by

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = f, \quad \text{or} \quad \mathrm{d}\nu = f \,\mathrm{d}\mu.$$

Definition 2.6 (Probability Measure)**.** For a measurable space $(\Omega, \mathscr{F})$, the measure μ in Definition 2.2 is a probability measure iff $\mu(\Omega) = 1$, i.e., the function $\mathbb{P} : \mathscr{F} \to [0, 1]$ is a probability measure if
(i) $\mathbb{P}(\emptyset) = 0$, and $\mathbb{P}(\Omega) = 1$,
(ii) for all disjoint $A_1, A_2, \ldots \in \mathscr{F}$, i.e. $A_i \cap A_j = \varnothing : i \neq j$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

The triple $(\Omega, \mathscr{F}, \mathbb{P})$ is called a probability space and the elements $A \in \mathscr{F}$ are called events. The events with probability 1 are called "almost sure (a.s.)" sets, i.e. they occur almost surely.

Definition 2.7 (Complete Probability Space)**.** A probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called complete if $\mathscr{F}$ contains all subsets O of Ω with $\mathbb{P}$-outer measure zero, i.e. with

$$\mathbb{P}^*(O) := \inf_{O \subset U \in \mathscr{F}} \mathbb{P}(U) = 0.$$

Remark 2.8. Any probability space can be made complete simply by adding all sets of outer measure 0 to its σ-algebra $\mathscr{F}$, and by extending its probability measure $\mathbb{P}$ accordingly. From now on we assume our probability spaces are complete.

For understanding the connection of measure theory and probability theory; before any further formulation, one may ask, if we are evaluating the probability of something, that should have been predictable beforehand, but what does the predictability means? In the mathematical language, the answer is indeed measurability.

Definition 2.9 (Random Variable)**.** For a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, a function $X : \Omega \to \mathbb{R}^n$ is a random variable or an $\mathscr{F}$-measurable function if

$$X^{-1}(B) := \{\omega \in \Omega | X(\omega) \in B\} \in \mathscr{F},$$

for all open subsets $B \subset \mathbb{R}^n$ (or Borel subsets $B \in \mathscr{B}(\mathbb{R}^n)$). The σ-algebra

$$\mathscr{F}^X := \sigma\Big\{X^{-1}(B) : B \in \mathscr{B}(\mathbb{R}^n)\Big\} \subseteq \mathscr{F}$$

is then called the generated σ-algebra by X.

Definition 2.10 (Distribution)**.** For a random variable X on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, the function $F_X : \mathscr{B}(\mathbb{R}^n) \to [0, 1]$ that

$$F_X(B) = \mathbb{P}(X^{-1}(B)),$$

is a probability measure on $\mathbb{R}^n$, called the distribution function of X. For the absolute continuous distributions F_X w.r.t. (with respect to) the Lebesgue measure μ on $\mathbb{R}^n$, the Radon-Nikodym derivative of it

$$f_X = \frac{\mathrm{d}F_X}{\mathrm{d}\mu}$$

is called the density function of X.

Definition 2.11 (L^p-Spaces)**.** For a random variable X on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and $p \in [1, \infty)$ the L^p-norm of X is

$$\|X\|_p = \|X\|_{L^p(\Omega)} = \left(\int_\Omega |X(\omega)|^p \,\mathrm{d}\mathbb{P}(\omega)\right)^{\frac{1}{p}},$$

and for $p = \infty$

$$\|X\|_\infty = \|X\|_{L^\infty(\Omega)} = \inf\big\{M \geq 0 : |X(\omega)| < M, \text{ for } a.e.\ \omega \in \Omega\big\}.$$

For $p \in [1, \infty]$, the associated L^p-space then is

$$L^p(\Omega) = \big\{X : \Omega \to \mathbb{R}^n\,;\ \|X\|_p < \infty\big\}.$$

Definition 2.12 (Expectation)**.** For a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ the expectation is a linear operator $\mathbb{E} : L^1(\Omega) \to \mathbb{R}$ that

$$\mathbb{E}[X] := \int_\Omega X(\omega)\,\mathrm{d}\mathbb{P}(\omega) = \int_{\mathbb{R}^n} x\,\mathrm{d}F_X(x).$$

For a Borel measurable function $f : \mathbb{R}^n \to \mathbb{R}$, that $f(X) \in \mathrm{L}^1(\Omega)$ it is easy to check that

$$\mathbb{E}[g(X)] = \int_\Omega g(X(\omega))\,\mathrm{d}\mathbb{P}(\omega) = \int_{\mathbb{R}^n} g(x)\,\mathrm{d}F_X(x),$$

and for the absolute continuous F_X

$$\mathbb{E}[g(X)] = \int_\Omega g(X(\omega))\,\mathrm{d}\mathbb{P}(\omega) = \int_{\mathbb{R}^n} g(x) f_X(x)\,\mathrm{d}x.$$

Remark 2.13. For $p \in [1, \infty]$ the $L^p(\Omega)$ is a Banach space, and for $p = 2$ the $L^2(\Omega)$ is indeed a Hilbert space with inner product

$$(X, Y)_{L^2(\Omega)} = \mathbb{E}[X \cdot Y]; \quad X, Y \in \mathrm{L}^2(\Omega).$$

Definition 2.14 (Independence)**.**
(I) The events $A, B \in \mathscr{F}$ are called independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$,
(II) The random variables X, Y are independent if $\mathbb{P}_{X,Y}(A, B) = \mathbb{P}_X(A) \cdot \mathbb{P}_Y(B)$, i.e., $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A) \cdot \mathbb{P}(Y \in B)$ for all Borel sets $A, B \in \mathscr{B}(\mathbb{R}^n)$. In such a condition $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

Definition 2.15. (I) For the random variables $X, Y \in \mathrm{L}^1(\Omega)$, the covariance is a bilinear symmetric function $\mathbb{C}\mathrm{ov} : L^1(\Omega) \times L^1(\Omega) \to \mathbb{R}$ that

$$\mathbb{C}\mathrm{ov}(X, Y) = \mathbb{E}\big[(X - \mathbb{E}[X])(Y - \mathbb{E}(Y))\big],$$

(II) The variance of X is $\mathbb{V}\mathrm{ar}[X] = \mathbb{C}\mathrm{ov}(X, X)$,
(III) The charactristic function of X is $\phi_X(\xi) = \mathbb{E}[\mathrm{e}^{\mathrm{i}\xi x}]$, where $\mathrm{i} = \sqrt{-1}$.

Definition 2.16 (Stochastic Process)**.** A stochastic process is an indexed collection $X = \{X_t\}_{t \in \mathscr{T}}$ of random variables $X_t : \Omega \to \mathbb{R}^n$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. It is called a discrete/continuous-indexed process if the indexing set $\mathscr{T}$ is discrete/continuous. We note, for fixed $t \in \mathscr{T}$ the mapping $\omega \mapsto X_t(\omega)$ is a random variable, and for fixed ω the mapping $t \mapsto X_t(\omega)$ is called a path of the process X.

Definition 2.17 (p-Variation)**.** Let $T > 0$ and $\{X_t\}_{t \in [0,T]}$ be a stochastic process on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. For $p > 0$, the p-variation process of X is

$$V_t^p(X) := \lim_{|\pi_n| \to 0} \sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}|^p, \quad \text{(limit in probability)}$$

where $\pi_n : 0 = t_0 < t_1 < \cdots < t_n = t$ is a partition on $[0, t]$ and $|\pi_n| = \max_{1 \le k \le n} \Delta t_k$ where $\Delta t_k = t_k - t_{k-1}$. For $p = 1$ it is called the total variation, and for $p = 2$ it is the quadratic variation and denoted by $\langle X \rangle_t = V_t^2(X)$. Similarly, for two processes $\{X_t\}_{t \in [0,T]}$ and $\{Y_t\}_{t \in [0,T]}$, the covariation of X, Y is

$$\langle X, Y \rangle_t := \lim_{|\pi_n| \to 0} \sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}| \cdot |Y_{t_k} - Y_{t_{k-1}}|, \quad \text{(limit in probability)}.$$

Definition 2.18 (Self-Similarity)**.** A stochastic process X is called self-similar of order $H \in (0,1]$ iff for all $c > 0, t \in \mathscr{T}$: $\quad X_{ct} \overset{d}{=} c^H X_t$. (Here "$\overset{d}{=}$" is equality in distribution.)

Definition 2.19 (Stationary Process)**.** A stochastic process $X = \{X_t\}_{t\in\mathscr{T}}$ is called stationary if for all $t, s \in \mathscr{T}$

$$X_t \overset{d}{=} X_s,$$

i.e., every random variables $X_t; t \in \mathscr{T}$ has equal distributions.

Theorem 2.20 (Lamperti (1962) Transform)**.**
(i) If $\{X_t\}_{t\geq 0}$ is a stationary process, then for all $c \in \mathbb{R}$ and $H \in (0,1]$ the process $Y_t = t^H X(c \log t),\ t > 0$ with initial value $Y_0 = 0$ a.s., is self-similar with index H,
(ii) If $\{Y_t\}_{t\geq 0}$ is a self-similar process with index $H \in (0,1]$, then for all $c \in \mathbb{R}$ the process $X_t = \mathrm{e}^{-cHt} Y(\mathrm{e}^{ct}),\ t \in \mathbb{R}$ is stationary.

Definition 2.21 (Stationary Increment Process)**.** A stochastic process $X = \{X_t\}_{t\in\mathscr{T}}$ has stationary increments if for all $h \geq 0$

$$X_{t+h} - X_t \overset{d}{=} X_{s+h} - X_s, \quad t, s \in \mathscr{T}$$

i.e., every equidistant increments has equal distributions.

Definition 2.22 (Independent Increment Process)**.** A stochastic process $X = \{X_t\}_{t\in\mathscr{T}}$ has independent increments if for all $s_1 < s_2 \leq t_1 < t_2$ in $\mathscr{T}$ the increments $X_{t_2} - X_{t_1}$ and $X_{s_2} - X_{s_1}$ are independent, i.e., every disjoint increments are independent.

Definition 2.23 (Spectral Representation)**.** For a stationary process X, the autocovariance function $r(h) := \mathbb{C}\mathrm{ov}(X_{t+h}, X_t) = \mathbb{C}\mathrm{ov}(X_h, X_0)$ has a representation

$$r(h) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}h\xi}\, \mathrm{d}F(\xi)$$

called the spectral representation of ρ. Also, such F is called the spectral distribution of X, and if F is absolute continuous w.r.t. the lebesgue measure, then the Radon-Nikodym derivative of it $f(\xi) = \mathrm{d}F(\xi)/\mathrm{d}\xi$ is

$$r(h) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}h\xi}\, f(\xi)\, \mathrm{d}\xi,$$

which is indeed the (inverse) Fourier transform of the density function f in spectral theory and harmonic analysis, and that is why it is called the spectral density function of the process X.

Definition 2.24 (Long Memory)**.** A stationary process X with autocovariance r given above
(i) has long memory (called long-range dependent) iff $\sum_{n=1}^{\infty} r(n) = \infty$,
(ii) has short memory (called short-range dependent) iff $\sum_{n=1}^{\infty} r(n) < \infty$.

2.2 Conditional Probability

Definition 2.25 (Conditional Probability)**.** Let X, Y be two random variables on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, the joint distribution of them is $F_{X,Y} : \mathscr{B}(\mathbb{R}^n \times \mathbb{R}^n) \to [0,1]$ that $F_{X,Y}(A,B) = \mathbb{P}(X \in A, Y \in B)$. We can define the conditional distribution of X w.r.t. Y accordingly as a function $F_{X|Y} : \mathscr{B}(\mathbb{R}^n \times \mathbb{R}^n) \to [0,1]$ that

$$F_{X|Y}(A|B) = \frac{F_{X,Y}(A \times B)}{F_Y(B)}.$$

Definition 2.26 (Conditional Expectation)**.** Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. By the Radon-Nikodym theorem, for a σ-algebra $\mathscr{H} \subset \mathscr{F}$ and a random variable $X : \Omega \to \mathbb{R}^n$ that $\mathbb{E}[|X|] < \infty$, there exist an a.s. unique function $\mathbb{E}[X|\mathscr{H}] : \Omega \to \mathbb{R}^n$ that
(i) $\mathbb{E}[X|\mathscr{H}]$ is $\mathscr{H}$-measurable,
(ii) $\int_H \mathbb{E}[X|\mathscr{H}]\, \mathrm{d}\mathbb{P} = \int_H X\, \mathrm{d}\mathbb{P}$ for all $H \in \mathscr{H}$,
it is called the conditional expectation of X w.r.t. $\mathscr{H}$. The conditional expectation of X w.r.t. another random variable Y on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is

$$\mathbb{E}[X|Y] = \mathbb{E}[X|\mathscr{F}^Y].$$

Theorem 2.27. *For a σ-algebra $\mathscr{H} \subset \mathscr{F}$ and random variables $X, Y : \Omega \to \mathbb{R}^n$ that $X, Y \in L^1(\Omega)$, and $a, b \in \mathbb{R}$*
a) $\mathbb{E}[aX + bY|\mathscr{H}] = a\mathbb{E}[X|\mathscr{H}] + b\mathbb{E}[Y|\mathscr{H}]$*,*
b) $\mathbb{E}[\mathbb{E}[X|\mathscr{H}]] = \mathbb{E}[X]$*,*
c) $\mathbb{E}[X|\mathscr{H}] = X$ *if X is $\mathscr{H}$-measurable,*
d) $\mathbb{E}[X|\mathscr{H}] = \mathbb{E}[X]$ *if X is independent from $\mathscr{H}$,*
e) $\mathbb{E}[Y \cdot X|\mathscr{H}] = Y \cdot \mathbb{E}[X]$ *if Y is $\mathscr{H}$-measurable.*

Proof. See Appendix B, Øksendal (2013). □

Definition 2.28 (Filteration)**.** A filteration is a family of σ-algebras $\{\mathscr{F}_t\}_{t \in \mathscr{T}}$, i.e.,

$$s < t \Rightarrow \mathscr{F}_s \subset \mathscr{F}_t.$$

For a filteration $\{\mathscr{F}_t\}_{t \in \mathscr{T}}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, a stochastic process $X = \{X_t\}_{t \in \mathscr{T}}$ is called $\mathscr{F}_t$ adapted if for each $t \in \mathscr{T}$ the random variable X_t is $\mathscr{F}_t$-measurable. Any stochastic process X_t is adapted to its natural filteration

$$\mathscr{F}_t^X := \sigma(X_u : u \leq t) = \sigma\left\{X_u^{-1}(B) : B \in \mathscr{B}(\mathbb{R}^n)\right\}_{u \leq t} \subseteq \mathscr{F}.$$

Definition 2.29 (Markov Process)**.** A stochastic process X on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called markov iff for all Borel measurable functions f that $f(X) \in$

$L^1(\Omega)$, and all $s \leq t$ in $\mathscr{T}$

$$\mathbb{E}[f(X_t)|\mathscr{F}_s^X] = \mathbb{E}[f(X_t)|X_s].$$

Definition 2.30 (Martingale)**.** A stochastic process M on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called a martingale w.r.t. the filteration $\{\mathscr{F}_t\}_{t\in\mathscr{T}}$ if
(i) M_t is $\mathscr{F}_t$-measurable for all $t \in \mathscr{T}$,
(ii) $\mathbb{E}[|M_t|] < \infty$ for all $t \in \mathscr{T}$, i.e., $M_t \in L^1(\Omega)$,
(iii) $\mathbb{E}[M_t|M_s] = M_s$ for all $t \geq s$.
It is called submartingale / supermartingale if in (iii) we have $\leq$ / $\geq$ instead of $=$.

Definition 2.31 (Stopping Time)**.** A random variable $\tau : \Omega \to \mathscr{T}$ on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, is called a stopping time with respect to a filtration $\{\mathscr{F}_t\}_{t\in\mathscr{T}}$ iff

$$\{\omega : \tau(\omega) \leq t\} \in \mathscr{F}_t \quad \text{for all} \quad t \in \mathscr{T}.$$

Definition 2.32 (Local Martingale)**.** Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a filteration $\{\mathscr{F}_t\}_{t\in\mathscr{T}}$. The $\mathscr{F}_t$-adapted stochastic process X is called a local martingale iff there exists a sequence $\{\tau_k\}_{k\geq 1}$ of the $\mathscr{F}_t$-stopping times that (i) the $\{\tau_k\}_{k\geq 1}$ is a.s. increasing, i.e., $\tau_k \geq \tau_{k+1}\ a.s.$,
(ii) the $\{\tau_k\}_{k\geq 1}$ diverges a.s., that is $\lim_{k\to+\infty} \tau_k = +\infty$ a.s.
(iii) the stopped process $X_t^{\tau_k} := X_{t\wedge\tau_k}$ is an $\mathscr{F}_t$-martingale for every $k \geq 1$.

Definition 2.33 (Càdlàg[1])**.** A function $f : D \subseteq \mathbb{R} \to M$ (metric space) is called càdlàg iff in every points of the domain it is right continuous, and it has a finite left-limit. It is called càglàd[2] iff in every points of the domain it is left continuous, and it has a finite right-limit.

Definition 2.34 (Semimartingale)**.** A real valued process X defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called a semimartingale iff it has the decomposition

$$X_t = M_t + Y_t,$$

where M is a local martingale and the process Y is cádlág, $\mathscr{F}_t$-adapted, with bounded total variation.

2.3 Continuity

Definition 2.35 (a.s. Versions)**.** On a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, the stochastic process $\{X_t\}_{t\in\mathscr{T}}$ is called a version (or a modification) of the process $\{Y_t\}_{t\in\mathscr{T}}$ iff

[1] continue à droite, limite à gauche
[2] continue à gauche, limite à droite

$X_t \overset{a.s.}{=} Y_t$ for all $t \in \mathscr{T}$, that is

$$\mathbb{P}(\{\omega : X_t(\omega) = Y_t(\omega)\}) = 1, \quad \text{for all} \quad t \in \mathscr{T}.$$

Definition 2.36 (Indistinguishable)**.** On a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, the stochastic process $\{X_t\}_{t\in\mathscr{T}}$ is called indistinguishable of the process $\{Y_t\}_{t\in\mathscr{T}}$ iff

$$\mathbb{P}(\{\omega : X_t(\omega) = Y_t(\omega) \quad \text{for all} \quad t \in \mathscr{T}\}) = 1.$$

Indistinguishability is stronger than a.s. equality (a.s. version).

Definition 2.37 (Hölder Continuity)**.** Let $\alpha > 0$, a stochastic process X_t is called α-Hölder continuous iff there exist some $C : \Omega \to \mathbb{R}$ that $C(\omega) > 0; \omega \in \Omega$, and for all $s, t \in \mathscr{T}$

$$|X_t - X_s| \leq C|t - s|^{\alpha} \quad a.s.$$

For $\alpha = 1$ the process is called Lipschits continuous.

Theorem 2.38 (A. Kolmogorov (1931) Continuity)**.** *Suppose that the stochastic process $X = \{X_t\}_{t\geq 0}$ satisfies the following condition: For all $T > 0$ there exist positive constants α, β, K that*

$$\mathbb{E}[|X_t - X_s|^{\alpha}] \leq K|t - s|^{1+\beta}; \quad 0 \leq s, t \leq T.$$

Then there exists a continuous version of X.

Proof. For proof, see Stroock and Varadhan (1997), Chapter 2. □

2.4 Convergence of Random Variables

In this section we consider $\{X_n\}_{n\geq 1}$, a discrete sequence of random variables on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and we define its limits with different analytical senses.

Definition 2.39 (Covergence in Distribution)**.** If $\{X_n\}_{n\geq 1}$ and X respectively has the probability distributions $\{F_n\}_{n\geq 1}$ and F, then we say $\{X_n\}_{n\geq 1}$ is convergent in distribution to X if

$$\lim_{n\to\infty} F_n(B) = F(B); \quad \text{for all} \quad B \in \mathscr{B}(\mathbb{R}^n) \quad \text{such that} \quad F(\partial B) = 0,$$

and we show this by "$X_n \xrightarrow{d} X$".

Definition 2.40 (Covergence in Probability)**.** We say $\{X_n\}_{n\geq 1}$ is convergent in probability to X if

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0; \quad \text{for all} \quad \varepsilon > 0,$$

and we show this by "$X_n \xrightarrow{\mathbb{P}} X$".

Definition 2.41 (Covergence in L^p)**.** For $p \geq 1$, we say $\{X_n\}_{n\geq 1}$ is convergent in L^p to X if

$$\lim_{n\to\infty} \mathbb{E}[\,|X_n - X|^p\,] = 0,$$

and we show this by "$X_n \xrightarrow{L^p} X$".

Definition 2.42 (Covergence Almost Surely (a.s.))**.** We say $\{X_n\}_{n\geq 1}$ is convergent almost surely to X if

$$\mathbb{P}\left(\omega \in \Omega \,:\, \lim_{n\to\infty} X_n(\omega) = X(\omega)\right) = 1,$$

and we show this by "$X_n \xrightarrow{a.s.} X$".

Theorem 2.43. *In a complete probability space*
a) If $X_n \to X$ and $X_n \to Y$ both in $\mathbb{P}, L^p$ or a.s., then $X \overset{a.s.}{=} Y$,
b) If $X_n \to X$ and $Y_n \to Y$ in $\mathbb{P}, L^p$ or a.s., then for $a, b \in \mathbb{R}\,(or\ \mathbb{C})$ we have $aX_n + bY_n \to aX + bY$ respectively in $\mathbb{P}, L^p$ or a.s.,
c) If $X_n \to X$ and $Y_n \to Y$ both in $\mathbb{P}$ or a.s., then $X_n \cdot Y_n \to X \cdot Y$ respectively in $\mathbb{P}$ or a.s.,
d) None of the above statements are valid for convergence in distribution in general,
e) For $p \geq q \geq 1$; $\xrightarrow{L^p} \;\Rightarrow\; \xrightarrow{L^q} \;\Rightarrow\; \xrightarrow{\mathbb{P}} \;\Rightarrow\; \xrightarrow{d}$,
f) In general; $\xrightarrow{a.s.} \;\Rightarrow\; \xrightarrow{\mathbb{P}} \;\Rightarrow\; \xrightarrow{d}$,
g) If $X_n \xrightarrow{\mathbb{P}} X$, then there exist a subsequence n_k such that $X_{n_k} \xrightarrow{a.s.} X$,
h) $X_n \xrightarrow{a.s.} X,\ |X_n| < Y,\ \mathbb{E}|Y| < \infty \quad \Rightarrow \quad X_n \xrightarrow{L^p} X \quad$ for all $p \geq 1$,
i) For constant $C \in \mathbb{R}^n$; $\quad X_n \xrightarrow{d} C \quad \Rightarrow \quad X_n \xrightarrow{\mathbb{P}} C$.

Proof. See Chung (2001), Chapter 4. □

Theorem 2.44 (Strong Law of Large Numbers)**.** *Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d (independent identically distributed) random variables, we have*

(i) If $\mathbb{E}(|X_1|) < \infty$, then $\quad \bar{X}_n := \frac{1}{n}\sum_{k=1}^{n} X_k \longrightarrow \mathbb{E}(X_1) \quad a.s.,$

(ii) If $\mathbb{E}(|X_1|) = \infty$, then $\quad \overline{\lim_{n\to\infty}}\, \bar{X}_n = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k = +\infty \quad a.s.$

Proof. See A. N. Kolmogorov (1950), Chapter VI, Chung (2001), Theorem 5.4.2, or Resnick (1998) Theorem 7.5.1. □

2.5 Gaussian Processes

Gaussian processes – also known as normal processes – have wide applications to natural sciences due to the central limit theorem. Here we explain some general facts of them.

Definition 2.45 (Multivariate Gaussian Variable)**.** A multivariate random variable $X : \Omega \to \mathbb{R}^n$ is called Gaussian and denoted $X \sim \mathscr{N}(\mathbf{m}, \mathbf{\Sigma})$, if its probability distribution $F_X : \mathbb{R}^n \to [0,1]$ is

$$F_X(x) = \frac{1}{\sqrt{(2\pi)^n |\mathbf{\Sigma}|}} \exp\left(-\frac{1}{2}(X-\mathbf{m})'\mathbf{\Sigma}^{-1}(X-\mathbf{m})\right),$$

where $\mathbf{m} \in \mathbb{R}^n$ is its mean and its covariance $\mathbf{\Sigma}$ is a positive-definite $n \times n$ matrix and $|\cdot|$ is the matrix determinant, and $'$ denotes the transpose. For $n = 1$ the real valued case is called normal variable, and if $\mathbf{m} = 0, \mathbf{\Sigma} = 1$ then it is the so-called standard normal, usually denoted by $\mathscr{N}(0,1)$.

Definition 2.46 (Gaussian Process)**.** A stochasic process $X = \{X_t\}_{t\in\mathscr{T}}$ is called Gaussian iff for every finite set of indices $t_1, \ldots, t_k$ in $\mathscr{T}$, the multivariate random variable

$$X_{t_1,\ldots,t_k} = (X_{t_1}, \ldots, X_{t_k}),$$

is Gaussian.

Theorem 2.47 (Central Limit Theorem)**.** *Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d random variables with $\mathbb{E}[X_k] = \mu$ and $\mathbb{V}\mathrm{ar}[X_k] = \sigma^2 < \infty$, then for $\bar{X}_n = \sum_{k=1}^n X_k/n$,*

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} Z \sim \mathscr{N}(0,1).$$

Proof. See Chung (2001), Chapter 7, and Resnick (1998), Chapter 9. □

2.5.1 Brownian Motion (Bm)

The first observation of this process takes back to 1827, when the Scottish botanist Robert Brown looked through a microscope at the plant's pollen immersed in water. The French mathematician Bachelier (1900) applied this process to model the financial data in his doctoral thesis under the supervision of Henri Poincaŕe. Five years later, Einstein (1905) modeled the motion of particles forced by water molecules with this process. Based on Einstein's model, the French physicist Perrin (1909) explained the convincing evidence for the existence of atoms and molecules. Perrin's work was awarded the Nobel Prize in Physics in 1926. After that, right on

1926 the American mathematician Wiener (1976) got interested, went to Europe, and developed some mathematical analysis of this process.

Definition 2.48. The Brownian motion (Bm) or Wiener process $W_t; t \in [0, \infty)$ beginning from $0 \in \mathbb{R}^n$ is defined as
(i) $W_0 = 0$ a.s.,
(ii) W_t is a.s. continuous,
(iii) W_t has independent increments,
(iv) W_t has stationary Gaussian increments $W_t - W_s \sim \mathcal{N}(0, t-s)$.
The Bm starting from arbitrary $x \in \mathbb{R}^n$ can be defined as $W_t^x = x + W_t$. We often denote this process also with B.

Theorem 2.49. *a) The Bm, W_t^x exists and it is unique by its multidimentional distributions*

$$\mathbb{P}(W_{t_1}^x \in A_1, \ldots, W_{t_k}^x \in A_k) = \int\limits_{A_1 \times \cdots \times A_k} p(t_1, x, x_1) \cdot p(t_2 - t_1, x_1, x_2) \cdots p(t_k - t_{k-1}, x_{k-1}, x_k) \, \mathrm{d}x_1 \cdots \mathrm{d}x_k,$$

where for $u, v \in \mathbb{R}^n$ we have $p(0, x, u) = \delta_x(u)$ and

$$p(t, u, v) = (2\pi t)^{-n/2} \cdot \exp\left(-\frac{|u - v|^2}{2t}\right); \quad t > 0,$$

b) $W_t \sim \mathcal{N}(0, t)$, $\mathbb{E}[W_t] = 0$, $\mathbb{V}\mathrm{ar}[W_t] = t$,
c) $\mathbb{C}\mathrm{ov}(W_t, W_s) = \mathbb{E}[W_t \cdot W_s] = t \wedge s \quad (= \min\{t, s\})$,
d) W_t^x is a Gaussian process, and $Z = (W_{t_1}, \ldots, W_{t_k}) \in \mathbb{R}^{nk}$ has the charactristic function

$$\phi_Z^x(\xi) = \mathbb{E}[\exp(iZ \cdot \xi)] = \exp\left(i\mathbf{m} \cdot \xi - \frac{1}{2}\xi' \mathbf{\Sigma} \xi\right); \quad \xi \in \mathbb{R}^n,$$

where "$\cdot$" is the inner product on $\mathbb{R}^n$,
e) Brownian Rescaling: For $c > 0$ the process $\widetilde{W_t} := \frac{1}{c} W_{c^2 t}$ is also a Bm,
f) Self-Similarity: Bm is self-similar of the order $H = 1/2$,
g) Quadratic Variation: $\langle W \rangle_t = t$,
h) p-Variation: For $0 < p < 2$; $V_t^p(X) = \infty$, and for $p > 2$; $V_t^p(X) = 0$,
i) Differentiability: Bm is nowhere differentiable,
j) Bm is a Markov process,
k) Bm is a continuous Martingale,
l) Bm is a semimartingale.

Proof. See Øksendal (2013), Chapter 2, Karatzas and Shreve (2014), Chapter 2. □

2.5.2 Fractional Brownian Motion (fBm)

Definition 2.50 (Fractional Brownian Motion)**.** For $H \in (0,1)$, the canonical Gaussian process $\{B_t^H\}_{t\geq 0}$ with covariance function

$$r_H(t,s) = \mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\left\{t^{2H} + s^{2H} - |t-s|^{2H}\right\}; \quad t,s \geq 0,$$

is called fractional Brownian motion (fBm) with Hurst index H. For $H = 1/2$, the fBm is the standard Brownian motion (Bm).

Theorem 2.51. *a) The fBm exists and it is unique in its multidimentional distribution sense,*
b) $B_0^H = 0$, *and* $\mathbb{E}[B_t^H] = 0$ *for all* $t \geq 0$,
c) fBm is not stationary,
d) fBm Rescaling: For $c > 0$ *the process* $\widetilde{B}_t^H = \frac{1}{c^H} B_{ct}^H$ *is also an fBm with Hurst index* H,
e) Self-Similarity: fBm is self-similar of the order H,
f) Stationary Increments: for $h,t,s \geq 0$ *we have* $B_{t+h}^H - B_t^H \overset{d}{=} B_{s+h}^H - B_s^H \overset{d}{=} B_h^H$,
g) Variance: $\mathbb{V}\mathrm{ar}[B_t^H] = \mathbb{E}[(B_t^H)^2] = t^{2H};\ t \geq 0$,
h) Gaussian Increments: $B_t^H - B_s^H \sim \mathcal{N}(0, |t-s|^{2H});\ t,s \geq 0$,
i) Hölder Continuity: For all $\epsilon > 0$, *the fBm* B^H *is* $(H-\epsilon)$*-Hölder continuous,*
j) Continuity: The fBm has a continuous version,
k) Long Memory: For $H > 1/2$ *the fBm has long-range dependent,*
l) Short Memory: For $H < 1/2$ *the fBm has short-range dependent,*
m) Quadratic Variation: For $H < 1/2, H = 1/2$, *and* $H > 1/2$ *respectively* $\langle B^H\rangle_t$ *is* ∞, t, *and* 0,
n) p-Variation: For $p < 1/H, p = 1/H$, *and* $p > 1/H$ *respectively* $V_t^p(B^H)$ *is* $\infty, t\mu_{\frac{1}{H}}$, *and* 0, *where* $\mu_{\frac{1}{H}}$ *is the* $(1/H)$*-th absolute moment of the standard normal random variable,*
o) Differentiability: fBm is nowhere differentiable,
p) For $H \neq 1/2$ *the fBm is neither a Markov process nor a semimartingale.*

Proof. See Biagini et al. (2008), Part I, Chapter 1, and Y. Mishura (2008), Section 1.2. □

Here we denote some special functions that are applied in our studies.

Definition 2.52 (Special Functions)**.** The following definitions are due to the holomorphic generalized special functions for $z \in \mathbb{C}$:

Euler Gamma Function: $\Gamma(z) = \int_0^\infty x^{z-1} \mathrm{e}^{-x} \, \mathrm{d}x; \quad z \neq -1, -2, \ldots$

Lower Gamma Function: $\gamma_a(z) = \int_0^a x^{z-1} \mathrm{e}^{-x} \, \mathrm{d}x; \quad \Re\mathrm{e}(a) > 0, \; z \neq -1, -2, \ldots$

Upper Gamma Function: $\Gamma_a(z) = \int_a^\infty x^{z-1} \mathrm{e}^{-x} \, \mathrm{d}x; \quad \Re\mathrm{e}(a) > 0, \; z \neq -1, -2, \ldots$

Euler Beta Function: $\beta(z, w) = \int_0^1 x^{z-1} (1-x)^{w-1} \, \mathrm{d}x; \quad z, w \neq -1, -2, \ldots, \; z + w \neq -1, -2, \ldots$

Gauss Hypergeometric Function: ${}_2F_1(a, b; c; z) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!};$

where $z \neq 1, \infty$, $c \neq -1, -2, \ldots$ and $(q)_n = \begin{cases} 1 & n = 0, \\ q(q+1)\cdots(q+n-1) & n > 0. \end{cases}$

The following theorem and some later ones in this chapter include some stochastic integrals, i.e., integrals w.r.t. the stochastic processes. For the definition and calculus of such integrals, the reader can see the next chapter.

Theorem 2.53 (fBm Representations)**.** *For $H, K \in (0, 1)$ the fBm's, B_t^H, B_t^K and the Bm, W_t the following representations are valid in distribution*

Mandelbrot and Van Ness (1968):

$$\begin{aligned} B_t^H &= \frac{1}{\Gamma(H + \frac{1}{2})} \int_{\mathbb{R}} \left\{ (t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right\} \mathrm{d}W_s \qquad (u_+ = \max\{u, 0\}) \\ &= \frac{1}{\Gamma(H + \frac{1}{2})} \left(\int_{-\infty}^0 \left\{ (t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right\} \mathrm{d}W_s + \int_0^t (t-s)^{H-\frac{1}{2}} \, \mathrm{d}W_s \right), \end{aligned}$$

Molchan and Golosov (1969):

$$B_t^H = \frac{1}{\Gamma(H + \frac{1}{2})} \int_0^t (t-s)^{H-\frac{1}{2}} \, {}_2F_1\left(\frac{1}{2} - H, H - \frac{1}{2}; H + \frac{1}{2}; 1 - \frac{t}{s} \right) \mathrm{d}W_s,$$

that is

- $H > 1/2:$ $\quad B_t^H = a_H \int_0^t s^{\frac{1}{2}-H} \int_s^t |u-s|^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \, \mathrm{d}u \, \mathrm{d}W_s,$

$$\text{where} \quad a_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-1/2)}},$$

- $H < 1/2:$

$$B_t^H = b_H \int_0^t \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}} \, \mathrm{d}u \right] \mathrm{d}W_s,$$

$$\text{where} \quad b_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H, H+1/2)}},$$

Samoradnitsky and Taqqu (1994):

$$B_t^H = c_H \int_R \frac{\mathrm{e}^{its}-1}{is} |s|^{1/2-H} \, \mathrm{d}W_s, \quad \text{where} \quad c_H = \sqrt{\frac{H\Gamma(2H)\sin(\pi H)}{\pi}},$$

Pipiras and Taqqu (2002):

$$\begin{aligned} B_t^H &= \tilde{c}_{H,K} \int_{\mathbb{R}} \left\{ (t-s)_+^{H-K} - (-s)_+^{H-K} \right\} \mathrm{d}B_s^K \\ &= \tilde{c}_{H,K} \left(\int_{-\infty}^0 \left\{ (t-s)^{H-K} - (-s)^{H-K} \right\} \mathrm{d}B_s^K + \int_0^t (t-s)^{H-K} \, \mathrm{d}B_s^K \right), \end{aligned}$$

$$\text{where} \quad \tilde{c}_{H,K} = \frac{1}{\Gamma(H-K+1)} \cdot \sqrt{\frac{\Gamma(2H+1)\sin(\pi H)}{\Gamma(2K+1)\sin(\pi K)}},$$

Jost (2006):

$$B_t^H = C_{H,K} \int_0^t (t-s)^{H-K} \, {}_2F_1\left(1-K-H, H-K; 1+H-K; 1-\frac{t}{s}\right) \mathrm{d}B_s^K,$$

$$\text{where} \quad C_{H,K} = \frac{1}{\Gamma(H-K+1)} \cdot \sqrt{\frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)\Gamma(2-2K)}{2K\Gamma(K+\frac{1}{2})\Gamma(\frac{3}{2}-K)\Gamma(2-2H)}}.$$

2.5.3 Mixed Fractional Brownian Motion (mfBm)

Definition 2.54. The mixed fractional Brownian motion (mfBm) is the mixture

$$M_t^{a,b} = aB_t + bB_t^H; \quad t \geq 0,$$

where $a, b \in \mathbb{R}$ and B is a Bm independent of the fBm, B^H.

Theorem 2.55. *a)* $M^{a,b}$ *is a centered Gaussian process, and* $M_0^{a,b} = 0$ *a.s.,*
b) $\mathbb{E}[M_t^{a,b}] = 0$*, and* $\mathbb{V}\mathrm{ar}[M_t^{a,b}] = \mathbb{E}[(M_t^{a,b})^2] = a^2t + b^2t^{2H}$ *for all* $t \geq 0$*,*
c) $\mathbb{C}\mathrm{ov}(M_t^{a,b}, M_s^{a,b}) = a^2(t \wedge s) + \frac{b^2}{2}\left\{t^{2H} + s^{2H} - |t-s|^{2H}\right\}; \quad t, s \geq 0,$
d) mfBm is not stationary,
e) mfBm Rescaling: For $c > 0$ *the process* $M_c^{a,b}t \overset{d}{=} M_t^{a\sqrt{c}, bc^H}$ *is also a mfBm,*
f) mfBm is not self-similar for $H \neq 1/2$*,*
g) Stationary Increments: for $h, t, s \geq 0: \ M_{t+h}^{a,b} - M_t^{a,b} \overset{d}{=} M_{s+h}^{a,b} - M_s^{a,b} \overset{d}{=} M_h^{a,b}$*,*
h) Gaussian Increments: $M_t^{a,b} - M_s^{a,b} \sim \mathcal{N}(0, a^2|t-s| + b^2|t-s|^{2H});\ t, s \geq 0,$
i) Hölder Continuity: For all $\epsilon > 0$*, the* $M^{a,b}$ *is* $(\frac{1}{2} \wedge H - \epsilon)$*-Hölder continuous,*
j) Continuity: The mfBm has a continuous version,
k) Long Memory: For $b \neq 0$ *and* $H > 1/2$ *the fBm has long-range dependent,*
l) Short Memory: For $b \neq 0$ *and* $H < 1/2$ *the fBm has short-range dependent,*
m) Quadratic Variation and p-Variation:

$$\langle M^{a,b}\rangle_t = \begin{cases} \infty & H < \frac{1}{2} \\ (a^2+b^2)t & H = \frac{1}{2} \\ a^2t & H > \frac{1}{2} \end{cases} \qquad V_t^p(M^{a,b})_t = \begin{cases} \infty & p(H \wedge \frac{1}{2}) < 1, \\ a^2t & H > \frac{1}{2}, p = 2 \\ (a^2+b^2)t & H = \frac{1}{2}, p = 2 \\ b^{\frac{1}{H}}\mu_{\frac{1}{H}}t & H < \frac{1}{2}, p = \frac{1}{H} \\ 0 & p(H \wedge \frac{1}{2}) > 1 \end{cases}$$

n) Differentiability: mfBm is nowhere differentiable,
o) For $H \neq 1/2$ *and* $b \neq 0$ *the fBm is not a Markov process,*
p) For $H \in (3/4, 1]$ *the mfBm is equivalent to a Bm in distribution.*

Proof. See Almani and Sottinen (2023); Cheridito (2001); Marinucci and Robinson (1999); van Zanten (2007); Zili (2006b). □

2.5.4 Multi-Mixed Fractional Brownian Motion (mmfBm)

The following definition is due to the work Almani and Sottinen (2023).

Definition 2.56. (mmfBm) We call an infinite-mixture process

$$M_t = \sum_{k=1}^{\infty} \sigma_k B_t^{H_k}, \quad t \geq 0 \quad \text{(if the limit exists in } L^2(\Omega \times \mathscr{T})\text{)}$$ [3]

[3]This requires $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$

as the multi-mixed fractional Brownian motion (mmfBm) with parameters σ_k, H_k, $k \in \mathbb{N}$ iff B^{H_k}'s are independent fBm's with Hurst indices $H_k \in (0, 1)$.

2.6 Jump Process

Understanding the jump processes is not only important for modelling the discontinuity points but also to understand the behaviour of the Lévy process paths. Here we consider some important particular cases.

Definition 2.57 (Oscillation)**.** The oscillation of a stochastic process $X = \{X_t\}_{t\in\mathscr{T}}$, on a time subset $U \subset \mathscr{T}$ is a random variable

$$w_X(U, \omega) = \sup_{t\in U} X_t(\omega) - \inf_{t\in U} X_t(\omega),$$

and the oscilation of X on a time-point $t \in \mathscr{T}$ is

$$w_X(t, \omega) = \lim_{\epsilon\to 0} w_X(U_\epsilon(t); \omega), \quad \text{where} \quad U_\epsilon(t) = \{s \in \mathscr{T} : |s - t| < \epsilon\},$$

Trivially, $X_t(\omega)$ is continuous at time t iff $w_X(t, \omega) = 0$.

Definition 2.58 (Jump)**.** A function $f : \mathbb{R} \to \mathbb{R}^n$ has a jump discontinuous point at $t \in \mathbb{R}$ iff
(i) The right and left limits of f both exist and are finite in t, i.e., there exist some $L^-, L^+ \neq \pm\infty$ that $\lim_{s\to t^+} f(s) = L^+$ and $\lim_{s\to t^-} f(s) = L^-$,
(ii) The right and left limits of f are not equal on t, i.e., $L^- \neq L^+$.
For a stochastic process $X = \{X_t\}_{t\geq 0}$, any path $X_t(\omega)$ has a jump in time point t iff it has a jump discontinuity on t as mentioned above, i.e., $X_{t+}(\omega) = \lim_{s\to t^-} X_s(\omega)$ and $X_{t-}(\omega) = \lim_{s\to t^+} X_s(\omega)$ both exist and are finite, but they are not equal.

Definition 2.59 (Jump Process)**.** A (pure) jump process $Y = \{Y_t\}_{t\geq 0}$ is a stochastic process that its paths moves only by jumps on random time points, i.e., for each $\omega \in \Omega$ there exist a random time set $\mathscr{T}(\omega) \subset [0, \infty)$ that is the set of all jump times of $Y_t(\omega)$, and for all $t \geq 0$

$$Y_t(\omega) = \sum_{s\in\mathscr{T}_t(\omega)} w_Y(s, \omega)$$

where $\mathscr{T}_t(\omega) = \mathscr{T}(\omega) \cap [0, t] = \{s \in \mathscr{T} : s \leq t\}$.

Definition 2.60 (Poisson Process)**.** The stochastic process $\{N_t\}_{t\geq 0}$ is called a poisson process with index $\lambda > 0$, denoted by $N \sim \mathscr{P}oisson(\lambda)$, iff
(i) $N_0 = 0 \quad$ a.s.,

(ii) N_t has independent increments,
(iii) for $0 \leq s < t < \infty$, its increments admits the discrete probability function

$$f_N(n, t-s; \lambda) = \mathbb{P}(N_t - N_s = n) = \frac{\lambda^n (t-s)^n}{n!} \mathrm{e}^{-\lambda(t-s)}; \quad n = 0, 1, \ldots$$

Definition 2.61 (Compound Poisson Process)**.** A compound Poisson process, with index $\lambda > 0$ and jump size distribution F, is a process $\{J_t\}_{t\geq 0}$ given by

$$J_t = \sum_{k=1}^{N_t} X_k,$$

where $N = (N_t)_{t\geq 0}$ is a Poisson process with intensity λ and $X_k \sim F; k \in \mathbb{N}$ are i.i.d. with common distribution F (called the child distribution of J), and they are independent of the Poisson process N. The sum is zero for $N_t = 0$.

Definition 2.62 (Compensated Generalized Poisson process)**.** A compensated generalized Poisson process is a stochastic process $\tilde{N} = \{\tilde{N}_t\}_{t\geq 0}$ that
i) $\tilde{N}$ is a martingale,
ii) The set of all jump discontinuity of $\tilde{N}$ on every time-interval I is almost surely countable, i.e.,

$$\#\left(\left\{t \in I : \tilde{N}_{t+} \neq \tilde{N}_{t-}\right\}\right) \preccurlyeq \aleph_0 \quad a.s.$$

where $\#$ is cardinal and $\aleph_0 = \#(\mathbb{N})$,
iii) Every jump discontinuity of $\tilde{N}_t$ never exceed 1 almost surely, i.e., for all $t \geq 0$,

$$w_{\tilde{N}}(t, \omega) = \left|\tilde{N}_{t+} - \tilde{N}_{t-}\right| \leq 1 \quad a.s.$$

Theorem 2.63. *a) The density, probability distribution, and charactristic functions of N are respectively*

$$f_N(n, t; \lambda) = \frac{\lambda^n t^n}{n!} \mathrm{e}^{-\lambda t}, \quad F_N(n, t; \lambda) = \mathrm{e}^{-\lambda t} \sum_{k=0}^{n} \frac{\lambda^k t^k}{k!}, \quad \phi_N(\xi, t) = \lambda t(\mathrm{e}^{i\xi} - 1),$$

b) $\mathbb{E}[N_t] = \mathbb{V}\mathrm{ar}[N_t] = \lambda t$,
c) $N_t - N_s \stackrel{d}{=} N_{t-s}; \quad 0 \leq s \leq t$,
d) The waiting time between two consecutive Poisson events, i.e., the random variable $\tau = \min\{h : N_{t+h} > N_t\}$, has the exponential distribution of the parameter λ, indeed its density function is $f_T(\tau) = \frac{1}{\lambda}\mathrm{e}^{-\lambda\tau}; \ \tau \geq 0$,
e) The probability distribution, density, and charactristic functions of the random

variable J_t are respectively

$$F_J(y,t) = \mathbb{P}(J_t \le y) = \mathrm{e}^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}\cdot \mathbb{P}(J_t \le y|N_t = n),$$

$$f_J(y,t) = \mathrm{e}^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}\cdot \frac{\partial \mathbb{P}(J_t \le y|N_t = n)}{\partial y},$$

$$\phi_J(\xi,t) = \exp\Big(\lambda t\left(\phi_X(\xi)-1\right)\Big),$$

f) $\mathbb{E}[J_t] = \lambda t\,\mathbb{E}[X], \quad \mathbb{V}\mathrm{ar}[J_t] = \lambda t\,\mathbb{E}[X^2].$

Proof. See Applebaum (2009); Daley and Vere-Jones (2007); Kingman (1992); S. M. Ross (2014). □

2.7 Lévy Process

The Lévy processes are a mixture of the Brownian motion and some jump processes. In this section we consider the fundamentals of this process.

Definition 2.64 (Lévy Process)**.** A (standard) Lévy process is a stochastic process $\{X_t\}_{t\ge 0}$ with the following properties
(i) $X_0 = 0$ a.s.,
(ii) independent increments,
(iii) stationary increments,
(iv) coninuity in probability, i.e., for all $\varepsilon > 0$ and $t \ge 0$ it holds

$$\lim_{h\to 0}\mathbb{P}(\,|X_{t+h} - X_t| > \varepsilon\,) = 0.$$

Definition 2.65 (Infinite Divisibility)**.** A probability distribution F is called infinite divisible iff for every given positive integers $n = 1, 2, \dots$ there exist i.i.d random variables $X_1^n, X_2^n, \dots X_n^n$ that the sum of them $S_n = X_1^n + X_2^n + \cdots + X_n^n$ has the probability distribution F.

Theorem 2.66. *I) Every Lévy processes are semimartingales,*
II) Every Lévy processes have infinite devisible distributions, and moreover; a distribution F is infinite divisible iff there exist a Lévy process with the probability distribution F,
III) For a Lévy process X with finite moments, the nth moment $\mu_n(t) = \mathbb{E}[X_t^n]$ is a polynomial on t, satisfying the binomial property

$$\mu_n(t+s) = \sum_{k=0}^{n}\binom{n}{k}\mu_k(t)\mu_{n-k}(s); \quad t, s \ge 0.$$

Proof. See Applebaum (2009); Kyprianou (2014); Sato (1999). □

The two following theorems both play the key roles to charactrize the Lévy processes.

Theorem 2.67 (Lévy (1934)–Khinchine (1937) Representation)**.** *The charactristic function of every Lévy processes X is*

$$\phi_X(\xi,t) = \exp\left(t\left(\alpha i\xi - \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}\setminus\{0\}} \left(\mathrm{e}^{i\xi x} - 1 - i\xi x\mathbf{1}_{\{|x|<1\}}\right)\ell(\mathrm{d}x)\right)\right),$$

where $\alpha \in \mathbb{R}, \sigma \geq 0$, and ℓ is a σ-finite measure that

$$\int_{\mathbb{R}\setminus\{0\}} \min(1,x^2)\,\ell(\mathrm{d}x) < \infty.$$

In such condition, ℓ is called a Lévy measure of X.

Theorem 2.68 (Lévy (1965)–Itô (1941) Decomposition)**.** *Let ℓ be a Lévy measure, $\mu = \ell\big|_{(-1,1)\setminus\{0\}}$, and $\nu = \dfrac{\ell\big|_{\mathbb{R}\setminus(-1,1)}}{\ell(\mathbb{R}\setminus(-1,1))}$ be the renormalized restrictions of ℓ as probability measures. Then*

$$\int_{\mathbb{R}\setminus\{0\}} \left(\mathrm{e}^{i\xi x} - 1 - i\xi x\mathbf{1}_{\{|x|<1\}}\right)\ell(\mathrm{d}x)$$
$$= \ell(\mathbb{R}\setminus(-1,1))\int_{\mathbb{R}}(\mathrm{e}^{i\xi x}-1)\,\nu(\mathrm{d}x) + \int_{\mathbb{R}}\left(\mathrm{e}^{i\xi x}-1-i\xi x\right)\,\mu(\mathrm{d}x).$$

Remark 2.69. In right side of above equality; the former is indeed the characteristic function of a compound Poisson process with intensity $\lambda = \ell(\mathbb{R}\setminus(-1,1))$ and child distribution ν. The latter is also the charactristic function of a compensated generalized Poisson process (CGP). Further, in case $\int_{\mathbb{R}}|x|\,\mu(\mathrm{d}x) < \infty$ then the (CGP) is a pure jump process (see Kyprianou (2014); Lawler (2010)). The result of Lévy–Khintchine representation and Lévy–Itô Decomposition is that; every Lévy processes ℓ_t has the following decomposition in distribution:

$$\ell_t \stackrel{d}{=} \alpha t + W_t + J_t + \tilde{N}_t,$$

where $\alpha \in \mathbb{R}$ and $W_t, J_t, \tilde{N}_t$ are respectively a Brownian motion, a generalized compound Poisson process, and a compensated generalized Poisson process.

3 STOCHASTIC CALCULUS

The work of Itô (1944) and later on Itô (1951a, 1951b) revealed that the calculus for stochastic processes (stochastic calculus) is different to deterministic calculus. In this chapter, we first consider the stochastic calculus w.r.t. semimartingales from Chung and Williams (1990); Revuz and Yor (2013); Rogers and Williams (2000), and in particular the Brownian motion and Lévy processes mainly from Karatzas and Shreve (2014); Øksendal (2013), then we consider some basics of the stochastic calculus w.r.t. the fBm and related processes from Biagini et al. (2008); Y. Mishura (2008).

3.1 Itô Calculus

The main difficulty to identify the framework of the stochastic calculus was indeed first; to develop a "*stochastic integral*" matching by the deterministic case, the classic Riemann (1868)–Stieltjes (1894) integral, and its generalization, the Lebesgue (1928) integral, and second; to understand the meaning of the "*stochastic differential equation*", and study the existance and uniquness of its solution, which it is widely demanded for stochastic modeling. In this section, we review them both w.r.t. (continuous) semimartingales.

3.1.1 Stochastic Integral

Definition 3.1 (Stochastic Integrals)**.** Let $\{X_t\}_{t\in[a,b]}$ be a stochastic process on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and $f : \Omega \times [a,b] \to \mathbb{R}$. The Itô integral of f w.r.t. X on time-interval $[a,b]$ is

$$\int_a^b f_t \, \mathrm{d}X_t := \lim_{|\pi_n|\to 0} \sum_{k=1}^{n} f_{t_{k-1}} (X_{t_k} - X_{t_{k-1}}),$$

iff this limit exists in probability for every partitions $\pi_n : a = t_0 < t_1 < \cdots < t_n = b$ on $[a,b]$, where $|\pi_n| = \max_{t_{k-1}, t_k \in \pi_n} |t_k - t_{k-1}|$. The Stratonovich integral of f w.r.t. X on time-interval $[a,b]$ is

$$\int_a^b f_t \circ \mathrm{d}X_t := \lim_{|\pi_n|\to 0} \sum_{k=1}^{n} \left(\frac{f_{t_{k-1}} + f_{t_k}}{2} \right) (X_{t_k} - X_{t_{k-1}}),$$

iff this limit exists in probability for every partitions $\pi_n : a = t_0 < t_1 < \cdots < t_n = b$ on $[a, b]$, where $|\pi_n| = \max_{t_{k-1}, t_k \in \pi_n} |t_k - t_{k-1}|$.

Remark 3.2. The Itô integral has had considerably wider applications in stochastic theory and modeling. So, here we provide the analysis related to this type of stochastic integral.

Definition 3.3. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[a,b]}, \mathbb{P})$ be a filtered probability space and M_t be a square-integrable continuous semimartingale on it w.r.t. $\{\mathscr{F}_t\}_{t\in[a,b]}$. Define $\mathfrak{L}^2_M(a, b)$ as the space of all functions $f : [a, b] \times \Omega \to \mathbb{R}$ that
(i) f is $(\mathscr{B} \times \mathscr{F})$-measurable,
(ii) $f_t(\omega)$ is $\mathscr{F}_t$-adapted for every $\omega \in \Omega$ and $t \in [a, b]$,
(iii) $\int_a^b \mathbb{E}[f_t^2]\, \mathrm{d}\langle M\rangle_t < \infty$.

Theorem 3.4 (Itô (1944) Isometry)**.** *In the conditions of Definition 3.3, for a semimartingale M and every $f \in \mathfrak{L}^2_M(a, b)$ the Itô integral of f w.r.t. M exists and converges in $L^2(\Omega \times [a, b])$ and*

$$\mathbb{E}\left[\int_a^b f_t\, \mathrm{d}M_t\right] = 0, \quad \textit{and} \quad \mathbb{E}\left[\left(\int_a^b f_t\, \mathrm{d}M_t\right)^2\right] = \int_a^b \mathbb{E}[f_t^2]\, \mathrm{d}\langle M\rangle_t.$$

Moreover, for every $f, g \in \mathfrak{L}^2_M(a, b)$

$$\mathbb{E}\left[\left(\int_a^b f_t\, \mathrm{d}M_t\right) \cdot \left(\int_a^b g_t\, \mathrm{d}M_t\right)\right] = \int_a^b \mathbb{E}[f_t g_t]\, \mathrm{d}\langle M\rangle_t.$$

Proposition 3.5. *By the same assumptions of Theorem 3.4;*
a) For all $t_0 \in [a, b]$; $\int_{t_0}^{t_0} f_t\, \mathrm{d}M_t = 0, \quad a.s.,$
b) For all $c \in [a, b]$; $\int_a^b f_t\, \mathrm{d}M_t = \int_a^c f_t\, \mathrm{d}M_t + \int_c^b f_t\, \mathrm{d}M_t \quad a.s.,$
c) The Itô integral process $Y_t := \int_a^t f_s\, \mathrm{d}M_s;\ t \in [a, b]$ is a $L^2(\Omega)$-martingale that has a cádlág version,
d) If M has continuous paths, then there is a version of Y with continuous paths.

Corollary 3.6. *As Brownian motion and Lévy processes are semimartingales, by Theorem 3.4 Itô integrals exists w.r.t. them, and also for the Bm case as $\langle B\rangle_t = t$, for all $f, g \in \mathfrak{L}^2_B(a, b)$*

$$\mathbb{E}\left[\int_a^b f_t\, \mathrm{d}B_t\right] = 0, \quad \mathbb{E}\left[\left(\int_a^b f_t\, \mathrm{d}B_t\right)^2\right] = \int_a^b \mathbb{E}[f_t^2]\, \mathrm{d}t, \quad \textit{and}$$
$$\mathbb{E}\left[\left(\int_a^b f_t\, \mathrm{d}B_t\right) \cdot \left(\int_a^b g_t\, \mathrm{d}B_t\right)\right] = \int_a^b \mathbb{E}[f_t g_t]\, \mathrm{d}t.$$

Definition 3.7 (Itô Process)**.** Providing the existence and well-definition of the Itô integral of $\beta_t(\omega)$ w.r.t. M_t on a time interval $t \in [t_0, T]$, the process

$$Y_t = \xi + \int_{t_0}^{t} \alpha_s(\omega)\,\mathrm{d}s + \int_{t_0}^{t} \beta_s(\omega)\,\mathrm{d}M_s, \quad a.s.$$

is called an Itô process, where ξ is a random variable independent of $M_t : t \in [t_0, T]$. When $\Delta t = t - t_0 \to 0$ the above integral formula becomes the following differential formula

$$\mathrm{d}Y_t = \alpha_t(\omega)\,\mathrm{d}t + \beta_t(\omega)\,\mathrm{d}M_t, \quad Y_{t_0} = \xi \quad a.s.$$

Theorem 3.8 (Itô (1951a) Formula)**.** *Let M be a continuous semimartingale and $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous and its partial derivative $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial t \partial x}$ exist and are continuous on all (t, x) then for all $t \in \mathbb{R}$ we have almost surely (a.s.)*

$$\begin{aligned} f(t, M_t) - f(t_0, M_{t_0}) = &\int_{t_0}^{t} \frac{\partial f}{\partial s}(s, M_s)\,\mathrm{d}s \\ &+ \int_{t_0}^{t} \frac{\partial f}{\partial x}(s, M_s)\,\mathrm{d}M_s + \frac{1}{2}\int_{t_0}^{t} \frac{\partial^2 f}{\partial s \partial x}(s, M_s)\,\mathrm{d}\langle M\rangle_s, \end{aligned}$$

where the second integral in right side is an Itô integral.

Remark 3.9. The Itô formula plays the key rule in stochastic differential equasions (SDE) theory. Indeed, for $\Delta t = t - t_0 \to 0$ the above integral formula becomes

$$\mathrm{d}f(t, M_t) = \frac{\partial f}{\partial t}(t, M_t)\,\mathrm{d}t + \frac{\partial f}{\partial x}(t, M_t)\,\mathrm{d}M_t + \frac{1}{2}\frac{\partial^2 f}{\partial t \partial x}(t, M_t)\,\mathrm{d}\langle M\rangle_t,$$

which is known as the differential form of Itô formula and makes it possible to tranform many SDE's and derive the solution of them.

Corollary 3.10. *As Brownian motion and Lévy processes are semimartingales, the Itô formula is valid on them, and for the Bm case*

$$\begin{aligned} f(t, B_t) - f(t_0, B_{t_0}) &= \int_{t_0}^{t} \left\{\frac{\partial f}{\partial s}(s, B_s) + \frac{1}{2}\frac{\partial^2 f}{\partial s \partial x}(s, B_s)\right\}\mathrm{d}s + \int_{t_0}^{t} \frac{\partial f}{\partial x}(s, B_s)\,\mathrm{d}B_s, \\ \mathrm{d}f(t, B_t) &= \left\{\frac{\partial f}{\partial t}(t, B_t) + \frac{1}{2}\frac{\partial^2 f}{\partial t \partial x}(t, B_t)\right\}\mathrm{d}t + \frac{\partial f}{\partial x}(t, B_t)\,\mathrm{d}B_t. \end{aligned}$$

Theorem 3.11 (Girsanov (1960) Transform)**.** *Suppose $\mathbb{P}, \mathbb{Q}$ are probability measures on a measurable filtered space $(\Omega, \{\mathscr{F}_t\}_{t\in[t_0,T]}, \mathscr{F})$, with $\mathscr{F} = \mathscr{F}_T$ and $\mathbb{Q} \ll \mathbb{P}$. Also, denote the Radon-Nykodim measures derivative $z = \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$, and suppose the derivative martingale $z_t := \mathbb{E}^{\mathbb{P}}[z|\mathscr{F}_t]$ has a continuous version Z_t. Suppose $M = \{M_t\}_{t\in[t_0,T]}$ is a continuous local martingale w.r.t. $\{\mathscr{F}_t\}_{t\in[t_0,T]}$ and $\mathbb{P}$.*

Let

$$Y_t := \int_{t_0}^{t} Z_s^{-1}\,\mathrm{d}\langle Z, M\rangle_s,$$

wherever the right side is well defined and finite for all $t \in [t_0, T]$ *(i.e., it is well-defined* $\mathbb{P}$*-(a.s.)). In these conditions; If* Y *is identically zero on the remaining* $\mathbb{P}$*-null set, then* Y *is locally of bounded variation (under either* $\mathbb{P}$ *or* $\mathbb{Q}$*), and the process* $\tilde{M} := M - Y$ *is a continuous local martingale under* $\mathbb{Q}$*, with the same* $\mathbb{Q}$*-quadratic variation (process) as the* $\mathbb{P}$*-quadratic variation of* M*, i.e.,*

$$\langle \tilde{M}\rangle^{\mathbb{Q}} = \langle M\rangle^{\mathbb{P}} \quad a.s.$$

Remark 3.12. The Girsanov transform changes probability measure from $\mathbb{P}$ to $\mathbb{Q}$ on a measurable space while changing a continuous local martingale M under $\mathbb{P}$ to a continuous semimartingale $\tilde{M}$ under $\mathbb{Q}$.

The Girsanov transform for the case $M = B$ (Brownian morion) has the following results:

Corollary 3.13. *Let* Y *be an Itô process w.r.t. the Brownian motion* B *on the filtered probability space* $(\Omega, \{\mathscr{F}_t\}_{t\in[t_0,T]}, \mathscr{F}, \mathbb{P})$*, i.e.*

$$\mathrm{d}Y_t = \alpha_t(\omega)\,\mathrm{d}t + \beta_t(\omega)\,\mathrm{d}B_t, \quad t \le T.$$

If for and itô integrable process $u_t(\omega)$ *w.r.t.* B *and* $\mathbb{P}$

$$\alpha_t(\omega) = \theta_t(\omega) - \beta_t(\omega)\cdot u_t(\omega), \quad t \le T$$

and the exponential process

$$M_t = \exp\left(\int_{t_0}^{t} u_s\,\mathrm{d}B_s - \frac{1}{2}\int_{t_0}^{t} u_s^2\,\mathrm{d}s\right), \quad t \le T$$

is a $\mathbb{P}$*-martingale w.r.t. the filter* $\{\mathscr{F}_t\}_{t\in[t_0,T]}$*, and*

$$\mathrm{d}\mathbb{Q} = M_T\,\mathrm{d}\mathbb{P}, \quad on \quad \mathscr{F}_T,$$

then $\mathbb{Q}$ *is a probability measure on* $\mathscr{F}_T$*, and*

$$\tilde{B}_t = \int_{t_0}^{t} u_s\,\mathrm{d}s + B_t, \quad t \le T$$

is a Brownian motion w.r.t. $\mathbb{Q}$*, and also*

$$\mathrm{d}Y_t = \theta_t(\omega)\,\mathrm{d}t + \beta_t(\omega)\,\mathrm{d}\tilde{B}_t, \quad t \le T.$$

Remark 3.14 (Novikov (1972) Condition)**.** By the same notation of the Corollary

3.13, the process M is a martingale if

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_{t_0}^{T} u_t^2 \,\mathrm{d}t\right)\right] < \infty.$$

3.1.2 Stochastic Differential Equation (SDE)

For a stochastic process $M = \{M_t\}_{t\geq t_0}$ and a random variable ξ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and functions $b, \sigma : [t_0, T] \times \mathbb{R} \to \mathbb{R}$, a solution X to the following integral equation is widely used in finance, physics, biology and many other scientific branches

$$X_t = \xi + \int_{t_0}^{t} b(s, X_s)\,\mathrm{d}s + \int_{t_0}^{t} \sigma(s, X_s)\,\mathrm{d}M_s \quad a.s.$$

Providing the existance of the stochastic integral in the second part of the right side, this equation can be written in differential form

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}M_t, \quad X_{t_0} = \xi \quad a.s. \tag{3.1}$$

This is called a stochastic differential equation (SDE), and the following theorem indeed identifies the existance and uniquness of the solution of it, when the process M is a semimartingale and the stochastic integration is taken as an Itô integral.

Theorem 3.15 (Existence & Uniqueness)**.** *Let M be a continuous semimartingale, and $b(\cdot, x), \sigma(\cdot, x)$ are local Lipschits on x, i.e., on a compact set $K \subset \mathbb{R}$, for all $T \geq 0$ there exist some $C_{T,K} > 0$ that*

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq C_{T,K}|x - y|; \quad x, y \in \mathrm{K},\ t \in [t_0, T],$$

Then the equation (3.1) has an almost surely (a.s.) unique t-continuous solution X adapted to the generated filter by the initial variable ξ and the semimartingale M. Moreover, this solution is square integrable, i.e.

$$\int_{t_0}^{T} \mathbb{E}\,|X_t|^2\,\mathrm{d}\langle M\rangle_t < \infty.$$

Proof. See He, Wang, and Yan (2019), Chapter IX, Section 7. □

3.2 Fractional Calculus

In this section, we first investigate the stochastic integral w.r.t. the fBm. Then we develop the stochastic differential equations w.r.t. the fBm.

3.2.1 Fractional Stochastic Integral

We extend the pathwise stochastic integral (called also "$\omega\, by\, \omega$") w.r.t. the fBm, B^H, i.e., to generalize the Lebesgue–Stieltjes (L-S) integral

$$\int_a^b f_t \,\mathrm{d}B_t^H := \lim_{|\pi_n|\to 0} \sum_{k=1}^n f_{t_{k-1}}(\omega) \left[B_{t_k}^H(\omega) - B_{t_{k-1}}^H(\omega)\right], \tag{3.2}$$

for each $\omega \in \Omega$, if the right limit exists uniquely for every partitions $\pi_n : a = t_0 < t_1 < \cdots < t_n = b$ on $[a, b]$, where $|\pi_n| = \max_{t_{k-1},\, t_k \in \pi_n} |t_k - t_{k-1}|$. As the fBm is not a semimartingale for $H \neq 1/2$, the approach of Itô integral and the Itô isometry are not feasible w.r.t. the fBm (see Nualart (2003)). However, to overcome this matter, it is sufficient to apply the L-S integrals for Hölder continuous functions, called Young (1936) integral (see also Dudley and Norvaisa (1998); Kondurar (1937)).

Definition 3.16. For $0 < \gamma < 1$, we denote the space of all γ-Hölder functions $f : [a, b] \to \mathbb{R}$ by $C^\gamma([a, b])$ equipped with the norm

$$\|f\|_\gamma = \sup_{t\in[a,b]} |f(t)| + \sup_{a\le s<t\le b} \frac{|f(t) - f(s)|}{(t-s)^\gamma} < \infty.$$

Theorem 3.17. *Let $f \in C^\beta([a, b])$, $g \in C^\gamma([a, b])$ and $\beta + \gamma > 1$, then the L-S integral $\int_a^t f_s \,\mathrm{d}g_s$ exists for all $t \in [a, b]$.*

Proof. See Ruzmaikina (2000). □

Corollary 3.18. *For all $f \in C^\beta([a, b])$ that $\beta + H > 1$ the pathwise integral $\int_a^t f_s \,\mathrm{d}B_s^H$ exists almust surely (a.s.) for all $t \in [a, b]$.*

Proof. It is sufficient to consider there exists a real number γ that $1 - \beta < \gamma < H$. Now, since $B^H \in C^\gamma([a, b])$ almost surely (a.s.), one can apply Theorem 3.17. □

To identify the (probabilistic) properties of the stochastic integral (3.2), we need the connections of it with some generalizations from functional analysis called "*symmetric*" and "*forward*" integrals.

Definition 3.19. Let $\{f_t\}_{t\in[a,b]}$ be a process with integrable trajectories. The symmetric integral of f w.r.t. B^H on $[a,b]$ is

$$\int_a^b f_t \circ \mathrm{d}B_t^H := \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_a^b f_t\,(B_{t+\epsilon}^H - B_{t-\epsilon}^H)\,\mathrm{d}t,$$

and similarly the forward integral of f w.r.t. B^H on $[a,b]$ is

$$\int_a^b f_t\,\mathrm{d}^- B_t^H := \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_a^b f_t\,(B_{t+\epsilon}^H - B_t^H)\,\mathrm{d}t, \tag{3.3}$$

provided that the limits exist in probability. The reason of the above notation is that indeed for the case Bm where $H = 1/2$ the symmetric and forward integrals respectively are the Stratonovich and Itô integral.

This definition of forward integral (3.3) coinsides with the Lebesgue–Stieltjes limit (3.2) for $H \geq 1/2$ if both exists in probability. However, for the case $H < 1/2$ they don't match in some cases (see Nualart (2003)). So, the following generalization for $H \in (0,1)$ by Crauel, Gundlach, and Zähle (1999) are applicable for such cases.

Definition 3.20. Let $\{f_t\}_{t\in[a,b]}$ be a process with integrable trajectories, the extended forward integral of f w.r.t. B^H on $[a,b]$ is

$$\int_a^t f_s\,\mathrm{d}^- B_s^H := \lim_{\epsilon\to 0} \frac{1}{\Gamma(\epsilon)} \int_a^b u^{\epsilon-1} \int_a^t f_s \frac{B_{s+u}^H - B_s^H}{s}\,\mathrm{d}s\,\mathrm{d}u, \tag{3.4}$$

provided that the limits exist uniformly on compact sets in probability (ucp) as a function of $t \in [a,b]$.

The following theorem from Biagini and Øksendal (2008) shows that the generalization (3.4) indeed matches to (3.2) and (3.3) if all exist in ucp.

Theorem 3.21. *Let $f_t = f(t,\omega)$ be a càglàd measurable process. If f is forward integrable w.r.t. B^H, then it is pathwise Lebesgue–Stieltjes integrable w.r.t. B^H, and*

$$\int_a^b f_t\,\mathrm{d}B_t^H = \int_a^b f_t\,\mathrm{d}^- B_t^H.$$

3.2.2 Fractional Stochastic Differential Equation

For the lack of Itô calculus w.r.t. fBm, the existance and uniqueness of the SDE w.r.t. fBm

$$\mathrm{d}X_t = b(t,X_t)\,\mathrm{d}t + \sigma(t,X_t)\,\mathrm{d}B_t^H, \quad X_{t_0} = \xi, \tag{3.5}$$

must be reconsidered. Here we mention two important results, the first one is a straight result of fractional calculus, the second one is a stochastic generalization of the classic Picard (1893) successive approximation approach.

Theorem 3.22 (Ruzmaikina (2000)). *Let $1/2 < H < 1$ and $b, \sigma, \frac{\partial\sigma}{\partial t}, \frac{\partial\sigma}{\partial x}$ exist and are globally Lipschitz on both t and x in $[t_0, T] \times \mathbb{R}$. Then for all $1 - H < \beta < H$, and the initial (random) variable ξ, the SDE*

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}^- B_t^H, \quad X_{t_0} = \xi$$

has an almust sure (a.s.) unique solution in $C^\beta([t_0, T])$.

Theorem 3.23 (Nualart and Ouknine (2002)). *Let $\sigma(t, x) \equiv \sigma$ be constant. In each of the following conditions:*
(i) $H \leq 1/2$, and $b(t, x)$ be a bounded Borel function with linear growth on x, i.e. for some $c_T > 0 : \quad |b(t, x)| \leq c_T(1 + |x|); \quad x \in \mathbb{R},\ t \in [t_0, T]$,
(ii) $H > 1/2$, and for some $\alpha > H - 1/2$, $\beta > 1 - 1/2H$ the $b(t, x)$ is α-Hölder continuous on t and β-Hölder continuous on x,
then for all initial (random) variable ξ, the SDE

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}^- B_t^H, \quad X_{t_0} = \xi$$

has an almust sure (a.s.) continuous unique solution.

3.3 Geometric Brownian Motion (gBm)

The gBm is the solution of the SDE

$$\mathrm{d}S_t = \mu\, S_t\,\mathrm{d}t + \sigma\, S_t\,\mathrm{d}B_t, \quad S_0 = \xi,$$

on $t \in [0, T]$, where μ, σ are constants and B is a Bm. Indeed, applying the Itô formula for $f(t, S_t) = \ln(S_t/\xi)$, this SDE admits the solution

$$S_t = \xi\, \exp\left((\mu - \frac{\sigma^2}{2})\, t \,+\, \sigma\, B_t \right).$$

3.4 Ornstein-Uhlenbeck Process (OU)

The OU process is the solution of the SDE

$$\mathrm{d}U_t = \theta\,(\mu - U_t)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t, \quad U_0 = \xi,$$

on $t \in [0,T]$, where μ, σ, θ are constants and B is a Bm. Indeed, applying the Itô formula for $f(t, U_t) = \mathrm{e}^{\theta t} U_t$, this SDE admits the solution

$$U_t = \xi\, \mathrm{e}^{-\theta t} + \mu\, (1 - \mathrm{e}^{-\theta t}) + \sigma \int_0^t \mathrm{e}^{-\theta(t-s)}\, \mathrm{d}B_s.$$

3.4.1 Fractional Ornstein–Uhlenbeck Process (fOU)

Cheridito et al. (2003) showed the OU equation w.r.t. fBm

$$\mathrm{d}U_t = \theta\, (\mu - U_t)\, \mathrm{d}t + \sigma\, \mathrm{d}B_t^H, \quad U_0 = \xi,$$

on $t \in [0,T]$, where μ, σ, θ are constants and B^H is a fBm, admits the solution

$$U_t^H = \xi\, \mathrm{e}^{-\theta t} + \mu\, (1 - \mathrm{e}^{-\theta t}) + \sigma \int_0^t \mathrm{e}^{-\theta(t-s)}\, \mathrm{d}B_s^H,$$

called fractional Ornstein–Uhlenbeck Process (fOU).

3.5 Jump Diffusion Process (JDP)

Consider the SDE

$$\frac{\mathrm{d}S_t}{S_{t-}} = \mu\, \mathrm{d}t + \sigma\, \mathrm{d}B_t + \mathrm{d}J_t, \tag{3.6}$$

where B is the standard Brownian motion and J is an independent compound Poisson jump process with intensity λ and jump distribution F. In other words

$$J_t = \sum_{k=1}^{N_t} \xi_k,$$

where $N = (N_t)_{t\in[0,T]}$ is a Poisson process with intensity λ and the jumps ξ_k, $k \in \mathbb{N}$, are i.i.d. with common distribution F, and they are independent of the Poisson process N and the Brownian motion B. As "$\sigma B + J$" is a semimartingale (indeed it is a Lévy process), one can apply the Itô formula with $f(t, S_t) = \ln(S_t/S_0)$, and find the path-wise solution of the SDE (3.6) is

$$S_t = S_0\, \mathrm{e}^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t} \prod_{k=1}^{N_t} (1 + \xi_k), \tag{3.7}$$

called the jump diffusion process (JDP), see e.g. Lamberton and Lapeyre (2011).

3.6 Gaussian Volterra Process (GV)

Let $G = (G_t)_{t\in[0,T]}$ be a centered Gaussian process with $G_0 = 0$ and covariance function R: $[0, T]^2 \to \mathbb{R}$ defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. A function $K : [0, T]^2 \to \mathbb{R}$ is called a Volterra kernel if $K(t, s) = 0$ whenever $t < s$. For a Volterra kernel K we define its associated operator K as

$$\mathrm{K}[f](t) = \int_0^t f(s)K(t, s)\,\mathrm{d}s.$$

Denote

$$\mathbf{1}_t(s) \quad = \quad \mathbf{1}_{[0,t)}(s) = \begin{cases} 1, & \text{if } s \in [0, t), \\ 0, & \text{otherwise} \end{cases} .$$

The adjoint associated operator K^* of the Volterra kernel K is given by extending linearly the relation

$$\mathrm{K}^*[\mathbf{1}_t](s) = K(t, s).$$

It turns out that K^* for a Gaussian Volterra process with covariance

$$R(t, s) = \int_0^{t\wedge s} K(t, u)K(s, u)\,\mathrm{d}v(u)$$

extends to an isometry from Λ to $L^2([0, T], \mathrm{d}v)$ where v is given in Definition 3.25 (i) and Λ, the space of Wiener integrands, is the closure of the indicator functions $\mathbf{1}_t$, $t \in [0, T]$, in the inner product

$$\langle \mathbf{1}_t, \mathbf{1}_s \rangle_\Lambda = R(t, s).$$

Remark 3.24. By Alòs, Mazet, and Nualart (2001), if K is of bounded variation in its first argument, we can write for any simple function f

$$\mathrm{K}^*[f](t) = f(t)K(T, t) + \int_t^T [f(u) - f(t)]\, K(\mathrm{d}u, t).$$

Moreover, as in Alòs et al. (2001) Lemma 1, for simple functions f and g we have

$$\int_0^T \mathrm{K}^*[f](t)g(t)\,\mathrm{d}t = \int_0^T f(t)\mathrm{K}[g](\mathrm{d}t)$$

justifying the name “adjoint” associated operator.

For Gaussian Volterra representations we recall what is the co-called abstract Wiener integral (for more information on abstact Wiener integrals and their relation to conditioning we refer to Sottinen and Yazigi (2014)). The linear space $\mathscr{L}$ is the closure of the random variables G_t, $t \in [0,T]$, in $L^2(\Omega, \mathscr{F}, \mathbb{P})$. The spaces Λ and $\mathscr{L}$ are isometric. Indeed, the mapping

$$\mathbf{1}_t \mapsto G_t$$

extends to an isometry. This isometry is called the abstract Wiener integral and we denote it

$$\int_0^T f(t)\,\mathrm{d}G_t$$

for a $f \in \Lambda$.

Definition 3.25 (Gaussian Volterra process)**.** Let $G = (G_t)_{t\in[0,T]}$ be a centered Gaussian process with covariance function $R\colon [0,\infty)^2 \to \mathbb{R}$. Assume that
(i) there exists an increasing function $v\colon [0,T] \to \mathbb{R}$ and a Volterra kernel $K\colon [0,T]^2 \to \mathbb{R}$ such that $\int_0^t K(t,s)^2\,\mathrm{d}v(s) < \infty$ for all $t \in [0,T]$ and

$$R(t,s) = \int_0^{t\wedge s} K(t,u)K(s,u)\,\mathrm{d}v(u),$$

(ii) for each $t \in [0,T]$ the equation

$$\mathrm{K}^*[K^{-1}(t,\cdot)](s) = \mathbf{1}_t(s)$$

admits a solution $K^{-1}(t,\cdot)$.

Note that by Definition 3.25 (ii) the operator K^* is invertible and we have

$$(\mathrm{K}^*)^{-1}[\mathbf{1}_t](s) = K^{-1}(t,s).$$

We note that due to Definition 3.25 (i) for a Gaussian Volterra process the space Λ is isometric to $L^2([0,T],\mathrm{d}v)$. Indeed, we have

$$\langle f,g\rangle_\Lambda = \langle \mathrm{K}^*[f], \mathrm{K}^*[g]\rangle_{L^2([0,T],\mathrm{d}v)}.$$

In particular, this means that the mapping K^* in Definition 3.25 (ii) is an isometry between Λ and $L^2([0,t],\mathrm{d}v)$. The following representation proposition is a direct consequence of Definition 3.25. Indeed, Proposition 3.26 could have been taken as the definition of Gaussian Volterra process.

Proposition 3.26 (Volterra representation)**.** *Let G be a Gaussian Volterra process. Let K^{-1} be the kernel in Definition 3.25 (ii). Then the process*

$$M_t = \int_0^t K^{-1}(t,s)\,\mathrm{d}G_s$$

is a Gaussian martingale with bracket $\langle M\rangle_t = v(t)$. Moreover,

$$G_t = \int_0^t K(t,s)\,\mathrm{d}M_s,$$

where v and K are as in Definition 3.25(i).

Note that form Proposition 3.26 we immediately see that the filtrations $\mathbb{F}^G$ and $\mathbb{F}^M$ coincide. The Volterra representations of Proposition 3.26 extend immediately to the following transfer principle for Wiener integrals.

Proposition 3.27 (Transfer principle)**.** *Let $f \in \Lambda$ and $g \in L^2([0,T], \mathrm{d}v)$. Then*

$$\begin{aligned} \int_0^T f(t)\,\mathrm{d}G_t &= \int_0^T \mathrm{K}^*[f](t)\,\mathrm{d}M_t, \\ \int_0^T g(t)\,\mathrm{d}M_t &= \int_0^T (\mathrm{K}^*)^{-1}[g](t)\,\mathrm{d}G_t. \end{aligned}$$

4 STOCHASTIC FINANCE

The previous chapters provided the mathematics required for the advanced technical analysis to finance, called mathematical finance. Here, we first review some fundamentals of finance associated to our works, from Björk (2009); Clark and Ghosh (2004); Föllmer and Schied (2016); Hull (2012); S. Ross (2003); Shiryaev (1999) and the online financial reference investopedia.com. Next, we consider the well-known Black and Scholes (1973) (BS) model, and then explain the motivation of the new technique of conditional mean hedging. Finally, we link the models we employed in our studies.

4.1 Principles of Finance

4.1. Contract: In finance, a contract is a deal between two parties to trade (buy or sell) an asset, comodity, security, or wealth.

4.2. Forward Contract: A customized deal between two parties to buy or sell an asset at a specified price on a future date. A forward is made over the counter (OTC) and settles just once, at the end of the contract.

4.3. Future Contract: A legal agreement to buy or sell a particular commodity asset, or security at a predetermined price at a specified time in the future. Futures contracts, meanwhile, are standardized to trade on stock exchanges. As such, they are settled daily.

4.4. Numeraire (Unit): A tradable economic entity which other tradables' prices are expressed relative to its price. It usually can be money or bond.

4.5. Spot Price: The current price in the marketplace at which a given asset can be bought or sold for immediate delivery, called exercise price also. It is usually denoted by S in finance.

4.6. Strike Price: The dealed price at which the contract can be exercised by the holders of the contract. It is usually denoted by K in finance.

4.7. Maturity Date: The date of the final day of the contract's validation time. It is also called the expiration date, and usually in finance is denoted by T.

4.8. Positions: Commonly, in a financial contract to trade an asset, the buyer is called the "*long position*". Inversely, the seller is called the "*short position*".

4.9. Traders: They are the parties - individuals, groups, companies, instidutions, or gvernments - who take some positions in a market of assets. Generally based on their objections, traders belong to one of the following categories:

- *Hedger*: Who takes a new position to offset or reduce the price risk of her/his other existing position(s),

- *Arbitrageur*: Who attempts to profit from market inefficiencies, seeking to profit from the same asset being priced differently in separate markets by simultaneously buying the asset at a lower price and selling it at a higher price,

- *Speculator*: Who conducts finantial trades that have substantial risk of losing value but also holds the expectation of a significant gain or other major value (in future).

4.10. Assets: An asset is a resource with economic value (price) - such as a security, commodity, or currency - that an individual, corporation, or country owns or controls that it can provide a future benefit or cost. In every contract, the parties deal on a type of buy or sell of one or multi assets under some circumstances.

4.11. Risk-free Asset: An asset whose price is deterministic. Indeed, the price of such an asset follows the growth or decay dynamic

$$\mathrm{d}A_t = rA_t\mathrm{d}t,$$

here $r \in \mathbb{R}$ is called the (risk free) interest rate (the rate of return).

4.12. Risky Asset: An asset whose price changes randomly (stochastically or uncertainly). Modeling the dynamics of the price of the risky asset has been a fundamental and historical question in the heart of finance. The most populer model is

$$\mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}n_t,$$

here $\mu \in \mathbb{R}$, $\sigma > 0$, and n_t is a random noise (stochastic process).

4.13. Derivatives: The term derivative refers to a type of financial contract whose value is dependent on some underlying securities or assets. They can be traded as both future or forward contracts.

4.14. Options: An option is a derivative (contract) that allows the holder the right to buy or sell an underlying asset or financial instrument at a specified strike price on or before a specified date, depending on the form of the option. Options are classified by the following categories in finance:

- *Call Option*: An option that allows the right to buy the underlying asset(s),
- *Put Option*: An option that allows the right to sell the underlying asset(s),
- *European Option*: An option that can be exercised only at the maturity date,
- *American Option*: An option that can be exercised in any time till the end of the maturity date.

Remark 4.15. Denoting the spot and strike prices by S, K, the maturity by T; the European call option pays off the value $(S_T - K)_+$ to the holder of it, and the European put option pays off the value $(K - S_T)_+$ to the holder of it, where

$x_+ = \max\{x, 0\}$. Consequently, this values are considered as the price of call and put options by the maturity T.

4.16. Strategy: The vector of assets' volume that a trader holds in his portfolio. That is the vector $\boldsymbol{\pi}_t = (\pi_t^1, \ldots, \pi_t^n)'$ if the trader holds respectively the volumes $\pi_t^1, \ldots, \pi_t^n$ of the assets $S_t^1, \ldots, S_t^n$ at time $t \in [0, T]$. Consequently, the value of the strategy $\boldsymbol{\pi}$ at time t is

$$V_t^{\boldsymbol{\pi}} = \boldsymbol{\pi}_t \cdot \mathbf{S}_t = \sum_{i=1}^{n} \pi_t^i S_t^i,$$

where $\mathbf{S}_t = (S_t^1, \ldots, S_t^n)'$.

4.17. Transaction Costs: Expenses incurred when trading an asset, outside the cost of that asset itself. They can be considered as the fees that a bank, broker, underwriter or other financial intermediary charges, or the difference between what a seller and buyer paid for a security. They are usually considered as a portion $\kappa \in (0, 1)$ of the trading assets, called the "*proportional transaction cost*".

4.18. Self-Financing Strategy: The strategy $\boldsymbol{\pi}$ that

$$\begin{aligned} V_t^{\boldsymbol{\pi}} &= V_0^{\boldsymbol{\pi}} + \int_0^t \boldsymbol{\pi}_u \cdot \mathrm{d}\mathbf{S}_u - \int_0^t \mathbf{S}_u' \,\boldsymbol{\kappa}\, |\mathrm{d}\boldsymbol{\pi}_u| \\ &= V_0^{\boldsymbol{\pi}} + \sum_{i=1}^{n} \int_0^t \pi_u^i \mathrm{d}S_u^i - \sum_{i=1}^{n} \int_0^t \kappa^i S_u^i \,|\mathrm{d}\pi_u^i|, \end{aligned} \tag{4.1}$$

where $\boldsymbol{\kappa} = \mathrm{diag}(\kappa^1, \ldots, \kappa^n) \in (0, 1)^{n \times n}$ is the diagonal $n \times n$ matrix of the proportional transaction costs for the assets $S_t^1, \ldots, S_t^n$.

4.19. Arbitrage Opportunity: A situation with possibilities to make a profit without incurring any loss with no capital. In other word, it is a self-financing strategy with

(I) $V_0^{\boldsymbol{\pi}} \leq 0 \quad a.s.,$ (II) $V_T^{\boldsymbol{\pi}} \geq 0 \quad a.s.,$ (III) $\mathbb{P}[V_T^{\boldsymbol{\pi}} > 0] > 0.$

A market with no arbitrage opportunity is called an **Arbitrage-free** market.

4.20. Hedging: The hedging strategy describes a variety of techniques used by traders to reduce risk exposure in a strategy. Hedging can be done by investing on some assets to minimize the negative impact of the adverse price swings in other assets.

4.21. Risk-Neutral Measure: An equivalent probability measure under whichS each share price is exactly equal to the discounted expectation of that share price. In other words, it is an equivalent measure $\mathbb{Q}$ to the market probability $\mathbb{P}$, such that all discounted asset prices are martingales w.r.t. $\mathbb{Q}$. This measure has centeral role in the derivatives pricing theory in finance.

4.2 Complete and Incomplete Markets

In financial market, one can consider the activities as a "*game*" of tradings between the traders. Even as a game, the fundamentalists believe these activities must be adjusted in a "*fair framework*", associated with the "*equilibrium economy*". However, the modern economists orient to the unnecessity of faireness in the market game, reflecting more to the nature of the "*competetive economy*". Among this debate, a crucial concept of mathematical finance arises; "*completeness of a market*".

Definition 4.22. A market in which every derivative can be hedged is called a **Complete Market**. Otherwise that market is an **Incomplete Market**.

The following theorem reveals the relationship between arbitrage-free and complete markets.

Theorem 4.23. *In a discrete (i.e. finite state) market, the following hold:*

(I) The First Fundamental Theorem of Asset Pricing: *A discrete market on a discrete probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ *is arbitrage-free iff there exists at least one risk-neutral probability measure* $\mathbb{Q}$ *equivalent to* $\mathbb{P}$*,*

(II) The Second Fundamental Theorem of Asset Pricing: *An arbitrage-free market* (S, b) *consisting of a collection of stocks* S *and a risk-free bond* b *is complete iff there exists a unique risk-neutral probability measure* $\mathbb{Q}$ *equivalent to* $\mathbb{P}$ *and has numeraire(unit)* b.

Proof. See Björk (2009); Pascucci (2011). □

4.3 Black-Scholes Model

The decade 1970-80 was undoubtably the initiation of the modern finance. The main core of several novel theories of that time was the option pricing theory. Pioneer to all of them, Black and Scholes (1973) (BS model) applied the geometric Brownian motion (gBm) of Samuelson (1965) adjusted by the theoretical framework of finance, the *no-arbitrage* theory. This continuous hedging-pricing model later was employed successfully for pricing of rational options by Merton (1973), and tested for transaction costs and taxes by Ingersoll Jr (1976). We illustrate it briefly here. Consider a trader holding a portfolio including: a risk-free asset A with interest rate r, and a risky asset S that

$$\text{(geometric model)} \qquad \mathrm{d}A_t = rA_t\mathrm{d}t, \tag{4.2}$$

$$\text{(gBm model)} \qquad \mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t, \tag{4.3}$$

on trading period of $t \in [0, T]$, where W is the Wiener process (Brownian motion), and μ, σ are some constants. So, if the market is complete (transaction costs are negligable), and he aims to hedge his investment with a *self-financing* trading strategy (η, π) then the value of the strategy at time $t \geq 0$ satisfies

$$\mathrm{d}V_t = \eta\,\mathrm{d}A_t + \pi\,\mathrm{d}S_t. \tag{4.4}$$

Applying Itô (1951a) calculus, Black–Scholes (BS) established the following fundamental theorem

Theorem 4.24 (Black and Scholes (1973))**.** *Under the follownig circumstances*
(i) The investment strategy is self-financing,
(ii) The interest rate r of the risk-free asset A is constant,
(iii) The risky asset S has the gBm dynamics (4.3)*,*
(iv) For some $g \in \mathscr{C}^{1,2}([0,T] \times \mathbb{R}_+)$ we have $V_t = g(t, S_t)$, particularely

$$V_T = g(T, S_T),$$

then for the function g we have

$$\text{(BS Equation)} \qquad \frac{\partial g}{\partial t} + rx\frac{\partial g}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 g}{\partial x^2} - rg = 0, \tag{4.5}$$

and from (iv), on $[0, T]$

$$\pi = \frac{\partial g}{\partial x}(t, S_t). \tag{4.6}$$

Remark 4.25. By the no-arbitrage theory, value of the portfolio V must be equal to the value of the option of the risky asset that the investor holds. So, (iv) is equivalent to have the option price only dependent on time t and the asset price S_t. This is valid for the Europian *vanilla* options, and for such option's boundary conditions, the equation (4.5) has a unique solution.

Theorem 4.26 (Black and Scholes (1973))**.** *Under the conditions (i)-(iv) of the Theorem 4.24, for the European call option price $C(t, S_t)$ and European put option price $P(t, S_t)$, if*

$$\begin{aligned} &C(t, 0) = 0, \quad t \in [0, T], \\ &C(t, x) \to x - K, \quad \text{as } x \to \infty, \\ &C(T, x) = (x - K)_+, \end{aligned}$$

the BS equation (4.5) returns the following solutions

$$
\begin{aligned}
C(t,S_t) &= \Phi(d_+)S_t - \Phi(d_-)K\mathrm{e}^{-r(T-t)},\\
P(t,S_t) &= K\mathrm{e}^{-r(T-t)} - S_t + C(t,S_t),\\
d_+ &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)(T-t)\right],\\
d_- &= d_+ - \sigma\sqrt{T-t}.
\end{aligned}
$$

Theorem 4.27 (Kac (1949), Feynman–Kac formula)**.** *Under the conditions (i)-(iv) of the Theorem 4.24, for a complete market, there is an equivalent probability measure $\mathbb{Q}$ to the market probability measure $\mathbb{P}$, that the BS equation (4.5) admites the following solution*

$$
V(t,S_t) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)}V(T,S_T)\middle|\mathscr{F}_t\right],
$$

and there exists a Wiener process $\widetilde{W}$ (Brownian motion) w.r.t. the measure $\mathbb{Q}$ that

$$
\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}\widetilde{W}_t.
$$

4.4 Conditional Mean Hedging

As explained above, the BS model makes it possible to perfectly hedge all derivatives. Moreover, the BS model is free of arbitrage. Although, this was a phenomenal inception that is still applicable on *business time* (see Geman et al. (2001a, 2001b)), some critiques arrised later. In fact, the main criticism was even published some years before Black–Scholes paper, the Mandelbrot and Van Ness (1968). Working on stock price data, they had concluded that the noise of price is far different from the Wiener process. This rules out the gBm model (4.3), and also returns the possiblity of arbitrage.

For models that the BS hedging method is not applicable, economists investigated other hedging methods. Föllmer and Schweizer (1991) initiated hedging associated to the *risk measure* and this idea got was developed in Schweizer (1992, 1994, 1995). Sottinen and Viitasaari (2018) introduced the *conditional mean hedging* method by employing the conditional laws from their prior paper Sottinen and Viitasaari (2017). This hedging is on the conditional average of the new model's value of portfolio, with respect to the conditional average of the BS portfolio's value. Shokrollahi and Sottinen (2017) studied this method for the *fractional* Black-Scholes model.

Assume the trading only takes place at given fixed time points $0 = t_0 < t_1 < \ldots <$

$t_N = T$. So, the trader by the self-financing strategy (4.1) would hedge at least once on each time period $[t_i, t_{i+1})$. In other word, the simplest strategy he can take is the discrete trading amount

$$\pi_t^N = \sum_{i=0}^{N-1} \pi_{t_i}^N \mathbf{1}_{[t_i, t_{i+1})}(t).$$

of the risky asset S. Then, the value of the strategy π^N is the integral of (4.4) minus the associated transaction cost of it. That is

$$V_t^{\pi^N,\kappa} = V_0^{\pi^N,\kappa} + \int_0^t \pi_u^N \, \mathrm{d}S_u - \int_0^t \kappa S_u |\mathrm{d}\pi_u^N|,$$

where $\kappa \in (0, 1)$ is the proportional transaction cost. Since under the transactional costs the perfect hedging is not possible, it is natural to try hedging on average in the sense of the following definition:

Definition 4.28. Let $f(t, S_t)$ be a European vanilla option with convex or concave payoff f, and π be its Black–Scholes strategy. We call the discrete-time strategy π^N is a *conditional mean hedging* (CMH) strategy, if for all trading times t_i,

$$\mathbb{E}\left[V_{t_{i+1}}^{\pi^N,\kappa} \,|\, \mathscr{F}_{t_i} \right] = \mathbb{E}\left[V_{t_{i+1}}^{\pi} \,|\, \mathscr{F}_{t_i} \right],$$

where $\mathscr{F}_{t_i}$ is the information filter, generated by the asset price S upto time t_i.

In our fourth paper, we studied the CMH strategy for the markets including jumps under transaction costs. We showed the existence of such strategy requires many restrictions for those markets. So, I introduced the following method.

Definition 4.29. Let $f(S_T)$ be a financial derivative with convex or concave payoff f. Let π be its Black–Scholes strategy. We call the discrete-time strategy ${}^*\pi^N$ is a *conditional least square hedging* (CLH) strategy, if for all trading times t_i,

$${}^*\pi_{t_{i+1}}^N \in \underset{\pi_{t_{i+1}}^N \in \mathbb{R}}{\operatorname{argmin}} \, \mathbb{E}\left[\left(V_{t_{i+1}}^{\pi^N,\kappa} - V_{t_{i+1}}^{\pi} \right)^2 \,\Big|\, \mathscr{F}_{t_i} \right].$$

Here $\mathscr{F}_{t_i}$ is the information filter, generated by the asset price S upto time t_i.

4.5 Fractional Models

In modern finance, the perfect hedging is not possible for incomplete markets as explained before; however, to develope the technical analysis for competetive finance, mathematical economists has applied (or some introduced) other stochastic processes in (4.3) instead of Wiener process. Some considered the *stable* processes (see Itô (2006); Mandelbrot (1997); Samoradnitsky and Taqqu (1994)), and some tried the processes with *long memory* to model the financial time-series with long range dependence (see Azmoodeh (2013); Shokrollahi et al. (2016); Wang (2010a, 2010b); Wang, Yan, et al. (2010); Wang, Zhu, et al. (2010)). Thereby, they employed the following fractional SDE for the risky asset(s) in their works

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t^H,$$

where B^H is an fBm with Hurst index $H \in (1/2, 1)$. In particular, Shokrollahi and Sottinen (2017) studied the conditional mean hedging for this model with transaction costs.

4.6 Jump Diffusion Model

In many recent studies, the real financial price data suggested the presence of jumps (see Carr, Geman, Madan, and Yor (2002); Geman (2002); S. Kou and Wang (2001); S. G. Kou (2002); Madan (2001); Merton (1976a)). The simplest model with jump is the so called *jump diffusion (JD)* model

$$\frac{\mathrm{d}S_t}{S_{t^-}} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + \mathrm{d}J_t. \tag{4.7}$$

This model nicely callibrates prices data, although the perfect hedging is impossible (see Lamberton and Lapeyre (2011), Chapter 7).

5 CONCLUSIONS

5.1 Summaries of Articles

Article I: "Multi-mixed fractional Brownian motions and Ornstein–Uhlenbeck processes"

This article initiates a study on the multi-mixed fractional Brownian motions (mmfBm)

$$M = \sum_{k=1}^{\infty} \sigma_k B^{H_k},$$

and the associated multi-mixed fractional Ornstein–Ulhenbeck (mmfOU) processes of the type

$$\begin{aligned} \mathrm{d}U_t &= -\lambda U_t \mathrm{d}t + \mathrm{d}M_t, \quad U_0 = \xi, \\ U_t &= \mathrm{e}^{-\lambda t}\xi + \int_0^t \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}M_s, \end{aligned}$$

Where $H_k \neq H_l$ for $k \neq l$, $H_{\inf} = \inf_{k\in\mathbb{N}} H_k > 0$, $H_{\sup} = \sup_{k\in\mathbb{N}} H_k < 1$, $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$, and $\lambda > 0$. These processes are constructed by mixing by superimposing (infinitely many) independent fractional Brownian motions (fBm) and fractional Ornstein–Uhlenbeck processes (fOU), respectively. We prove their existence as $L^2(\Omega \times [0, T])$ processes and study their path properties, viz. long-range and short-range dependence, Hölder continuity, p-variation, and conditional full support.

Article II: "Parameter Estimation for multi-mixed Fractional Ornstein–Uhlenbeck Processes by Generalized Method of Moments"

This article is an attempt to develop the generalized method of moments (GMM) estimation for the parameters of the finitely mixed multi-mixed fractional Ornstein–Uhlenbeck (mmfOU) processes:

$$\lambda > 0, \qquad H_k \in [H_{\inf}, H_{\sup}] \subset (0,1), \qquad \sigma_k > 0; \qquad k = 1, 2, \ldots n,$$

i.e. the following vector will be estimated

$$\theta = (\lambda; \mathbf{H}; \boldsymbol{\sigma})' \in \mathbb{R}^{2n+1} \quad \text{where} \quad \mathbf{H} = (H_1, \ldots, H_n), \quad \boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n).$$

and analyze the consistency and assymptotic normality of this estimator. We also include some simulations and provide numerical observations considering different statistical errors.

Article III: "Prediction of Gaussian Volterra processes with compound Poisson jumps"

This article establishes the technical framework required for the next article on conditional mean hedging. To do this, we study the conditionsl probability, expectation, and covariance of the processes

$$X = G + J,$$

where the continuous part G is a Gaussian Volterra process, i.e.,

$$G_t = \int_0^t K(t,s)\,\mathrm{d}M_s,$$

where K and M are as in Definition 3.25 and Proposition 3.26. Further, J is an independent compound Poisson process with intensity and jump distribution F. In other words

$$J_t = \sum_{k=1}^{N_t} \xi_k,$$

where $N = (N_t)_{t\in[0,T]}$ is a Poisson process and the jumps ξ_k, $k \in \mathbb{N}$, are i.i.d. with common distribution, independent of both N and G.

Article IV: "Hedging in Jump Diffusion Model with Transaction Costs"

Using the technical results of the previous article, this work studies the conditional prediction laws for the (forward) jump diffusion model

$$\frac{\mathrm{d}S_t}{S_{t^-}} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + \mathrm{d}J_t,$$

that are required for our further buildings. Next, we develop the conditional mean hedging for the jump diffusion model with respect to the Black-Scholes (BS) model's strategy and characterize the explicit strategies with interpretations related to the investor priorities and the transaction costs. We consider this result for the particular case of the European call option under the transaction costs and formulate the consequential hedging strategies for that. Finally, we present a decision tree, table of values, and figures to support our results.

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Multi-mixed fractional Brownian motions and Ornstein–Uhlenbeck processes

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Abstract The so-called multi-mixed fractional Brownian motions (mmfBm) and multi-mixed fractional Ornstein–Uhlenbeck (mmfOU) processes are studied. These processes are constructed by mixing by superimposing or mixing (infinitely many) independent fractional Brownian motions (fBm) and fractional Ornstein–Uhlenbeck processes (fOU), respectively. Their existence as L^2 processes is proved, and their path properties, viz. long-range and short-range dependence, Hölder continuity, p-variation, and conditional full support, are studied.

Keywords Fractional Brownian motion, Gaussian processes, long-range dependence, multi-mixed fractional Brownian motion, multi-mixed fractional Ornstein–Uhlenbeck process, short-range dependence, stationary-increment processes, stationary processes
2010 MSC 60G10, 60G15, 60G22

1 Introduction and preliminaries

The first attempt to formulate the long-term memory of time series was in hydrology when Hurst (1951) and his colleagues were studying the fluctuation of the reservoir of the Nile river over a long period of time (see [16]). Later on, after the works of Mandelbrot (1968) in [24], it was clarified that this behavior of time series is because of including long-range depended noises called fractional Brownian motion

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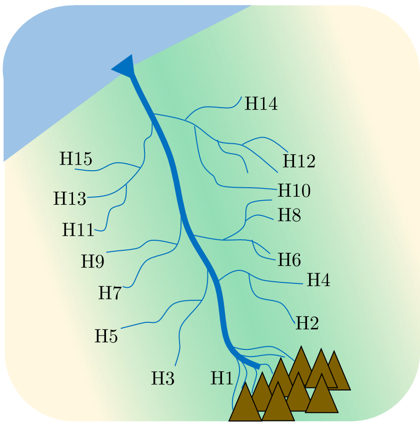
(fBm) B^H. Regarding sources of these noises, in hydrology, they are accumulated from factors such as waterfalls, glaciers melting, riverbed shape and material, slope and direction, width and depth, local temperature, etc. Moreover, we know a river (especially a large one like the Nile) is a combination of many sub-rivers, and each sub-river (or even a large river) is also a combination of many sources like streams, mountain glaciers, underground water reservoirs, and rainfall in general. Now, one may ask

> *How many sources of such noises are there for a river reservoir in reality?*

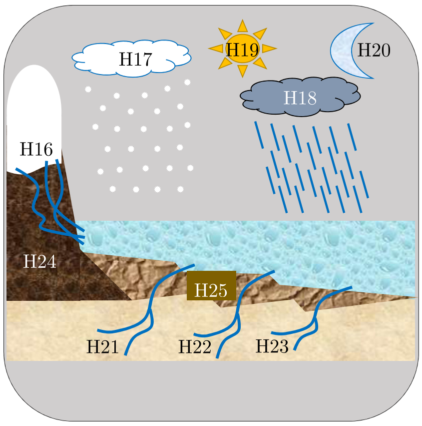
The answer of nature then, in practice, is infinity!

Table 1. Multi-mixed fBm arising from different Hurst exponents imposed to a river reservoir

River path



Riverbed



So, let us consider the source i has the noise B^{H_i} with the weight of effect σ_i to the river reservoir, then the general noise of the river can be written as a linear combination

$$M_t = \sum_{i=1}^{\infty} \sigma_i B_t^{H_i}. \tag{1}$$

On the other hand, about the dynamic of a particle in a liquid, Langevin (1908) in [21] modeled the particle's velocity U with an equation which was wisely revised later on by Doob (1942) [9] as

$$\mathrm{d}U_t = -\lambda U_t \,\mathrm{d}t + \mathrm{d}M_t,$$

where $\lambda > 0$ is the mean reversion parameter and M is a noise, caused by a fluctuating force imposed by an impact of the molecules of the surrounding medium. If $U_0 = \xi$ then the unique solution of this equation is

$$U_t = \mathrm{e}^{-\lambda t}\xi + \int_0^t \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}M_s.$$

First, this solution was given for all cases where M is semimartingale, then Cheridito et al. (2003) in [8] confirmed this solution for the case the noise process is an fBm

$M = B^H$. Now, let us think the liquid is not purely homogeneous, so the local surrounding molecules (particles) can have different imposing forces according to their different sizes, weights, density, or dynamic patterns. Hence, if molecule (particle) i

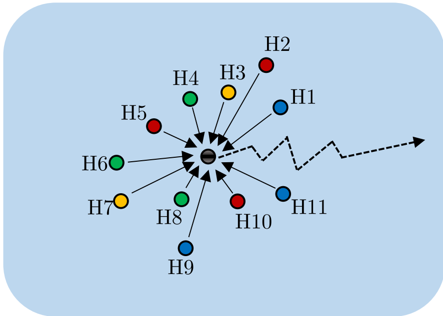


Fig. 1. Free particle movement in a liquid bombarded by multiple molecules (particles) imposing multi-mixed fBm noises

imposes the noise force B^{H_i} with the weight of effect σ_i to the free particle, then the Langevin equation takes the form

$$U_t = \mathrm{e}^{-\lambda t}\xi + \int_0^t \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}\left(\sum_{i=1}^{\infty} \sigma_i B_t^{H_i}\right). \tag{2}$$

In this article, our aim is to develop the analysis and some properties of the stochastic processes in equations (1) and (2). To do this, first, we review some mathematical concepts. The fractional Brownian motion (fBm) B^H, with parameter $H \in (0, 1)$ called the Hurst index, is the unique (up to a multiplicative constant) centered H-self-similar stationary-increment Gaussian process. The fBm was first studied in [19]. The name fractional Brownian motion comes from the influential article [24]. For further information of the fBm, see the monographs [6, 25]. The covariance of the fBm with the Hurst index H is given by

$$r_H(t,s) = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right].$$

For $H = 1/2$ this process is well known as the Brownian motion (BM) or the Wiener process: $B^{\frac{1}{2}} = W$. As a stationary-increment process, the fBm has the spectral representation

$$r_H(t,s) = \int_{\mathbb{R}} \frac{(\mathrm{e}^{\mathrm{i}sx}-1)(\mathrm{e}^{\mathrm{i}tx}-1)}{x^2} f_H(x)\,\mathrm{d}x,$$

where

$$f_H(x) = \frac{\sin(\pi H)\Gamma(1+2H)}{2\pi}|x|^{1-2H}. \tag{3}$$

Here Γ is the complete gamma function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t,$$

see [28].

Let

$$\varrho_H(\delta;t) = \mathbb{E}\left[(B^H_\delta - B^H_0)(B^H_{t+\delta} - B^H_t)\right]$$

be the incremental autocovariance (with lag δ) of the fBm. For $t \to \infty$ we have the power decay

$$\varrho_H(\delta;t) \sim H(2H-1)\delta^2 t^{2H-2}.$$

This means that the increments of the fBm, called the fractional Gaussian noise (fGn), for $H > \frac{1}{2}$, are positively correlated and long-range dependent. However, for $H < \frac{1}{2}$ they are negatively correlated and short-range dependent.

In the Bm case $B^{\frac{1}{2}} = W$ we have independent increments, i.e. no dependence:

$$\varrho_{\frac{1}{2}}(\delta;t) = 0.$$

The fBm has almost surely Hölder continuous paths with any order $H - \varepsilon$ for any $\varepsilon > 0$. This follows, e.g., from Theorem 1 of [2].

In addition to Hölder continuity, we have the p-variation as a measure of the path regularity. For a process X and $p \in [1,\infty)$ for the partitions $\pi_n := \{t_k = \frac{k}{n}T : k = 0,1,\ldots,n\}$, if

$$V^p_T(X) := \lim_{n\to\infty} \sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}|^p < \infty \quad \text{(limit in probability)},$$

then it is said X has equidistant p-variation on $[0,T]$, and its p-variation on $[0,T]$ is $V^p_T(X)$. For the fBm B^H then the p-variation is

$$V^p_T(B^H) = \begin{cases} \infty & ; \quad pH < 1 \\ T\mu_p & ; \quad pH = 1 \\ 0 & ; \quad pH > 1 \end{cases}$$

where μ_p is the pth moment of a standard Gaussian process, see [10, 11].

While the fBm has been proposed as a model for financial time series, modeling with it makes arbitrage possible, see [4]. To eliminate this problem, a generalization called mixed fractional Brownian motion (mfBm) was introduced in [7]. This is the mixture model

$$M^{a,b} = aB + bB^H,$$

where $a, b \in \mathbb{R}$ and B is a standard Brownian motion (Bm) independent of the fBm B^H. If $H > 1/2$, the mfBm has the path roughness governed by the Bm part and the long-range dependence governed by the fBm part. Hence, e.g., in pricing of financial derivatives the corresponding mixed Black–Scholes model yields the same option prices as the standard Brownian model, see [5].

A natural generalization of the mfBm is to consider two (or n) independent fBm mixtures, see [23]. In this paper, we study an independent infinite-mixture generalization that we call the multi-mixed fractional Brownian motion (mmfBm) with parameters σ_k, H_k, $k \in \mathbb{N}$:

$$M = \sum_{k=1}^{\infty} \sigma_k B^{H_k},$$

where B^{H_k}'s are independent fBm's with Hurst indices $H_k \in (0,1)$, and σ_k's are positive volatility constants satisfying $\;_{k=1}^{\infty} \sigma_k^2 < \infty$. This study extends the work of [30].

For other kinds of generalizations of the fBm, see, e.g., [14, 22, 26, 27].

The fractional Ornstein–Uhlenbeck process (fOU) $U^{\lambda,H}$, with parameters $\lambda > 0$ and $H \in (0,1)$, is the stationary solution of the Langevin equation

$$\mathrm{d}U_t^{\lambda,H} = -\lambda U_t^{\lambda,H}\,\mathrm{d}t + \mathrm{d}B_t^H,$$

which is given by

$$U_t^{\lambda,H} = \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}B_s^H,$$

where $(B_s^H)_{s\leq 0}$ is an independent copy of the fBm $(B_s^H)_{s\geq 0}$, see [8]. Note that the Langevin equation and its solution can be understood via integration by parts. As a stationary process, the fOU admits the spectral density

$$f_{\lambda,H}(x) = \frac{f_H(x)}{x^2+\lambda^2}, \tag{4}$$

where f_H is the spectral density of the driving fBm (3), see [3]. Denote, for $\alpha \in (-1,0) \cup (0,1)$,

$$\gamma_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^x s^{\alpha-1}\mathrm{e}^s\,\mathrm{d}s, \tag{5}$$

$$\Gamma_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty s^{\alpha-1}\mathrm{e}^{-s}\,\mathrm{d}s, \tag{6}$$

and $\gamma_0(x) = 1$, $\Gamma_0(x) = 0$. The functions γ_α and Γ_α are related to the incomplete Gamma functions and they can be calculated, e.g., by approximating the integrals with sums. The autocovariance function of the fOU process can be written as

$$\rho_{\lambda,H}(t) = \frac{\Gamma(1+2H)}{4}\frac{\mathrm{e}^{-\lambda t}}{\lambda^{2H}}\;\; 1+\gamma_{2H-1}(\lambda t) + \mathrm{e}^{2\lambda t}\Gamma_{2H-1}(\lambda t)\;\; . \tag{7}$$

See Proposition 4.

A stationary process X with the autocovariance function satisfying

$$\rho(t) \sim c|t|^{-\alpha} \quad \text{as} \quad t \to \infty$$

where $0 \neq c \in \mathbb{R}$, and "$\sim$" means the ratio of left and right sides tends to 1, is called long-range dependent (having long memory) if $0 < \alpha \leq 1$, and short-range dependent (having short memory) if $\alpha > 1$, see [18].

For $H = \frac{1}{2}$ we recover the well-known Bm case

$$\rho_{\lambda,\frac{1}{2}}(t) = \frac{\mathrm{e}^{-\lambda t}}{2\lambda}.$$

For $t \to \infty$ we have the power decay

$$\rho_{\lambda,H}(t) = \frac{1}{2}\sum_{n=1}^{N} \lambda^{-2n} \left(\prod_{j=0}^{2n-1}(2H-j)\right) t^{2H-2n} + O(t^{2H-2N-2}),$$

for $N = 1, 2, \dots$, i.e., the fOU process with $H > \frac{1}{2}$ is long-range dependent, and for $H \le \frac{1}{2}$ it is short-range dependent, see [8].

The Hölder continuity and p-variation of fOU is the same as for the mmfBm.

In this paper we study the multi-mixed fractional Ornstein–Uhlenbeck process (mmfOU) with parameters $\lambda > 0$ and σ_k, H_k, $k \in \mathbb{N}$, that is defined naturally as the stationary solution of Langevin equation with mmfBm as the driving noise:

$$\mathrm{d}U_t = -\lambda U_t \,\mathrm{d}t + \mathrm{d}M_t,$$

with

$$U_0 = \int_{-\infty}^{0} \mathrm{e}^{\lambda s} \,\mathrm{d}M_s,$$

where $(M_s)_{s \le 0}$ is an independent copy of the mmfBm. This study develops the work of [17].

The rest of the paper is organized as follows. In Section 2 we define the multi-mixed fractional Brownian motions (mmfBm) and the associated multi-mixed fractional Ornstein–Uhlenbeck (mmfOU) processes, prove their existence in $L^2(\Omega \times [0, T])$, and provide their basic properties. The long-range dependence of these processes are studied in Section 3. In Section 4 we analyze the Hölder continuity and p-variation of mmfBm's and mmfOU processes. The p-variations of these processes are calculated in Section 5. In Section 6 we show that the mmfBm's and mmfOU processes have the conditional full support property. Finally, In Section 7 some simulated paths of these processes are given.

2 Definitions and basic properties

Definition 1. Let σ_k, $k \in \mathbb{N}$, satisfy

$$\sum_{k=1}^{\infty} \sigma_k^2 < \infty, \tag{8}$$

and let H_k, $k \in \mathbb{N}$, satisfy

$$\begin{aligned} H_k &\neq H_l \quad \text{for } k \neq l, \\ H_{\inf} &= \inf_{k \in \mathbb{N}} H_k > 0 \\ H_{\sup} &= \sup_{k \in \mathbb{N}} H_k < 1. \end{aligned} \tag{9}$$

The multi-mixed fractional Brownian motion (mmfBm) is

$$M = \sum_{k=1}^{\infty} \sigma_k B^{H_k},$$

where B^{H_k}, $k \in \mathbb{N}$, are independent fBm's.

The following proposition shows the existence of the mmfBm.

Proposition 1. *The mmfBm M exists as a random function taking values in $L^2(\Omega \times [0, T])$ for all $T > 0$.*

Proof. Let $M^n = \sum_{k=1}^{n} \sigma_k B^{H_k}$. Clearly M^n takes values in $L^2(\Omega \times [0, T])$. Let $n, m \in \mathbb{N}$ with $n > m$. Then

$$
\begin{aligned}
\|M^n - M^m\|^2_{L^2(\Omega\times[0,T])} &= \int_0^T \mathbb{E}\left[(M_t^n - M_t^m)^2\right] \mathrm{d}t \\
&= \int_0^T \mathbb{E}\left[\left(\sum_{k=m+1}^{n} \sigma_k B_t^{H_k}\right)^2\right] \mathrm{d}t \\
&= \sum_{k=m+1}^{n} \int_0^T \sigma_k^2 \mathbb{E}\left[(B_t^{H_k})^2\right] \mathrm{d}t \\
&= \sum_{k=m+1}^{n} \int_0^T \sigma_k^2 t^{2H_k} \,\mathrm{d}t \\
&= \sum_{k=m+1}^{n} \sigma_k^2 \frac{T^{1+2H_k}}{1+2H_k} \\
&\leq \sum_{k=m+1}^{n} \sigma_k^2 \max\left\{1, T^3\right\},
\end{aligned}
$$

which shows that $(M^n)_{n\in\mathbb{N}}$ is the Cauchy sequence. Thus $M^n \to M$ in $L^2(\Omega\times[0, T])$ showing the existence. □

In the same way we see that the mmfBm $(M_t)_{t\geq 0}$ exists in the sense that $M_t^n \to M_t$ in $L^2(\Omega)$ for all $t \geq 0$.

The following is now obvious.

Proposition 2. *The mmfBm has stationary increments, its covariance function is*

$$
r(t, s) = \sum_{k=1}^{\infty} \sigma_k^2 r_{H_k}(s, t) = \frac{1}{2} \sum_{k=1}^{\infty} \sigma_k^2 \left[|t|^{2H_k} + |s|^{2H_k} - |t - s|^{2H_k}\right], \tag{10}
$$

and it admits the spectral density

$$
f(x) = \sum_{k=1}^{\infty} \sigma_k^2 f_{H_k}(x) = \sum_{k=1}^{\infty} \frac{\sin(\pi H_k)\Gamma(1 + 2H_k)}{2\pi} \sigma_k^2 |x|^{1-2H_k}. \tag{11}
$$

Definition 2. The multi-mixed fractional Ornstein–Uhlenbeck process (mmfOU) U with parameter $\lambda > 0$ is the stationary solution of the Langevin equation

$$
\mathrm{d}U_t = -\lambda U_t \mathrm{d}t + \mathrm{d}M_t, \tag{12}
$$

where the equation is understood in the integration by parts sense.

Proposition 3. *On $L^2(\Omega \times [0, T])$, the mmfOU can be represented as the integral*

$$U_t = \mathrm{e}^{-\lambda t}\xi + \int_0^t \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}M_s,$$

where the integral is understood in the integration by parts sense, and

$$\xi = \int_{-\infty}^0 \mathrm{e}^{\lambda s}\,\mathrm{d}M_s,$$

where $(M_s)_{s\le 0}$ is an independent copy of the mmfBm $(M_s)_{s\ge 0}$.

Proof. Let $M^n = {}_{k=1}^{n} \sigma_k B^{H_k}$. Then, the stationary solution of the Langevin equation

$$\mathrm{d}U_t^n = -\lambda U_t^n \mathrm{d}t + \mathrm{d}M_t^n$$

is given by

$$U_t^n = \mathrm{e}^{-\lambda t}\xi_n + \int_0^t \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}M_s^n,$$

where

$$\xi_n = \int_{-\infty}^0 \mathrm{e}^{\lambda s}\,\mathrm{d}M_s^n.$$

Then, with integration by parts

$$\begin{aligned}\int_0^t \mathrm{e}^{\lambda s}\,\mathrm{d}M_s^n &= \mathrm{e}^{\lambda t}M_t^n - \lambda\int_0^t \mathrm{e}^{\lambda s}M_s^n\,\mathrm{d}s\\ &\to \mathrm{e}^{\lambda t}M_t - \lambda\int_0^t \mathrm{e}^{\lambda s}M_s\,\mathrm{d}s = \int_0^t \mathrm{e}^{\lambda s}\,\mathrm{d}M_s,\end{aligned}$$

because $M^n \to M$ in $L^2(\Omega \times [0, T])$. With the same arguments $\xi_n \to \xi$ in $L^2(\Omega)$. This yields $U^n \to U$ in $L^2(\Omega \times [0, T])$. □

Lemma 1. *For $0 \ne p \in (-1, 1)$, $\lambda > 0$, $t > 0$,*

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\frac{|x|^p}{\lambda^2 + x^2}\,\mathrm{d}x = \frac{\pi\,\mathrm{e}^{-\lambda t}}{2\cos(\frac{p\pi}{2})\lambda^{1-p}}\left\{1 + \gamma_{-p}(\lambda t) + \mathrm{e}^{2\lambda t}\Gamma_{-p}(\lambda t)\right\}, \tag{13}$$

where γ_{-p} and Γ_{-p} are given by (5) and (6).

Proof. Recall that for the Fourier transform

$$\mathscr{F}(f)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx} f(t)\,\mathrm{d}t$$

we have the convolution theorem

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\mathscr{F}(f)(x)\mathscr{F}(g)(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(t-\xi)g(\xi)\,\mathrm{d}\xi. \tag{14}$$

Moreover, we have

$$\mathscr{F}\left(e^{-\lambda|t|}\right) = \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\lambda^2 + x^2}, \tag{15}$$

$$\mathscr{F}\left(|t|^{\alpha}\right) = \sqrt{\frac{2}{\pi}} \cdot \Gamma(\alpha+1)\cos\left(\frac{(\alpha+1)\pi}{2}\right)|x|^{-(\alpha+1)}. \tag{16}$$

The first formula (15) is valid for $\lambda > 0$. The second formula (16) is valid for $-1 < \alpha < 0$. For $-2 < \alpha < -1$, because of the function $|t|^{\alpha}$, some singular terms arise at the origin. Nevertheless, it admits a unique meromorphic extension as a tempered distribution, also denoted $|t|^{\alpha}$ as a homogeneous distribution on all real line $\mathbb{R}$ including the origin (see [13]). So, we use that extension and formula (16) will be valid for all $-1 \neq \alpha \in (-2, 0)$. So, using $f(t) = e^{-\lambda|t|}$ and $g(t) = |t|^{\alpha}$ in (14) we obtain

$$\begin{aligned}
&\frac{2}{\pi} \cdot \Gamma(\alpha+1)\cos\left(\frac{(\alpha+1)\pi}{2}\right)\lambda \int_{-\infty}^{\infty} \mathrm{e}^{itx}\frac{|x|^{-(\alpha+1)}}{\lambda^2+x^2}\,\mathrm{d}x \\
&\quad = \int_{-\infty}^{\infty} |\xi|^{\alpha}\mathrm{e}^{-\lambda|t-\xi|}\,\mathrm{d}\xi \\
&\quad = \int_{-\infty}^{0} (-\xi)^{\alpha}\mathrm{e}^{-\lambda(t-\xi)}\,\mathrm{d}\xi \\
&\qquad + \int_{0}^{t} \xi^{\alpha}\mathrm{e}^{-\lambda(t-\xi)}\,\mathrm{d}\xi \\
&\qquad + \int_{t}^{\infty} \xi^{\alpha}\mathrm{e}^{-\lambda(\xi-t)}\,\mathrm{d}\xi \\
&\quad = \frac{\mathrm{e}^{-\lambda t}}{\lambda^{(\alpha+1)}}\int_{0}^{\infty} u^{\alpha}\mathrm{e}^{-u}\,\mathrm{d}u \\
&\qquad + \frac{\mathrm{e}^{-\lambda t}}{\lambda^{(\alpha+1)}}\int_{0}^{\lambda t} u^{\alpha}\mathrm{e}^{u}\,\mathrm{d}u \\
&\qquad + \frac{\mathrm{e}^{\lambda t}}{\lambda^{(\alpha+1)}}\int_{\lambda t}^{\infty} u^{\alpha}\mathrm{e}^{-u}\,\mathrm{d}u \\
&\quad = \frac{\mathrm{e}^{-\lambda t}\Gamma(\alpha+1)}{\lambda^{(\alpha+1)}}\left\{1 + \gamma_{(\alpha+1)}(\lambda t) + \mathrm{e}^{2\lambda t}\Gamma_{(\alpha+1)}(\lambda t)\right\}.
\end{aligned}$$

Now, choosing $p = -(\alpha+1)$ proves (13). □

Proposition 4 follows from Lemma 1 (see also [20]).

Proposition 4. *The covariance function of the fOU is*

$$\begin{aligned}
\rho_{\lambda,H}(t) &= \mathbb{E}[U_s^{\lambda,H}U_{s+t}^{\lambda,H}] \\
&= \frac{\Gamma(1+2H)}{4}\frac{\mathrm{e}^{-\lambda t}}{\lambda^{2H}}\quad 1 + \gamma_{2H-1}(\lambda t) + \mathrm{e}^{2\lambda t}\Gamma_{2H-1}(\lambda t)\quad .
\end{aligned} \tag{17}$$

Proposition 5. *The covariance function of the mmfOU is*

$$\begin{aligned}\rho_\lambda(t) &= \mathbb{E}[U_s U_{s+t}] \\ &= \sum_{k=1}^{\infty} \sigma_k^2 \frac{\Gamma(1+2H_k)\mathrm{e}^{-\lambda t}}{4\lambda^{2H_k}} \quad 1 + \gamma_{2H_k-1}(\lambda t) + \mathrm{e}^{2\lambda t}\Gamma_{2H_k-1}(\lambda t) \quad , \qquad (18)\end{aligned}$$

and it admits the spectral density

$$f_\lambda(x) = \sum_{k=1}^{\infty} \sigma_k^2 \frac{\sin(\pi H_k)\Gamma(1+2H_k)}{2\pi} \frac{|x|^{1-2H_k}}{x^2+\lambda^2}. \qquad (19)$$

Proof. Let U^n be like in the proof of Proposition 3, then

$$f_{\lambda,n}(x) = \sum_{k=1}^{n} \sigma_k^2 \frac{\sin(\pi H_k)\Gamma(1+2H_k)}{2\pi} \frac{|x|^{1-2H_k}}{x^2+\lambda^2},$$

and $f_{\lambda,n}(x) \to f_\lambda(x)$ because $U^n \to U$ in $L^2(\Omega \times [0,T])$. This proves (19). Similarly, (18) follows by Proposition 4. □

Remark 1. Proposition 4 represents the covariance function $\rho_{\lambda,H}(t)$ in a form involving special functions. However, these special complex functions are usually not suitable for numerical computations. For example, in [3], Lemma B.1, the following representation was used for $H > \frac{1}{2}$:

$$\begin{aligned}\rho_{\lambda,H}(t) &= H\Gamma(2H)\frac{e^{-\lambda t}}{\lambda^{2H}} \quad \frac{1+e^{2\lambda t}}{2} - \frac{\lambda}{\Gamma(2H-1)} I_{\lambda,H}(t) \quad , \\ I_{\lambda,H}(t) &= \int_0^t \int_0^{\lambda v} e^{2\lambda v} e^{-s} s^{2H-2}\, \mathrm{d}s \mathrm{d}v.\end{aligned}$$

The double integral above seems reasonable enough, but yields slow numerical calculation in practice. This can be improved by calculating the inner integral as follows:

$$\begin{aligned}I_{\lambda,H}(t) &= \int_0^{\lambda t} \int_{s/\lambda}^{t} e^{2\lambda v} e^{-s} s^{2H-2}\, \mathrm{d}s \mathrm{d}v \\ &= \frac{1}{2\lambda} \int_0^{\lambda t} s^{2H-2}(e^{2\lambda t - s} - e^s)\, \mathrm{d}s \\ &= \frac{e^{\lambda t}}{\lambda} \int_0^{\lambda t} s^{2H-2} \sinh(\lambda t - s)\, \mathrm{d}s.\end{aligned}$$

Consequently,

$$\rho_{\lambda,H}(t) = \frac{\Gamma(2H+1)}{2\lambda^{2H}} \quad \cosh(\lambda t) - \frac{1}{\Gamma(2H-1)} \int_0^{\lambda t} s^{2H-2} \sinh(\lambda t - s)\, ds \quad . \qquad (20)$$

For the case $H < 1/2$ we use the following developed version of Lemma 5.1 in [15] for $\alpha > -1$. The proof is similar.

Lemma 2. *For $\alpha > -1$*

$$\int_0^\infty \int_0^\infty e^{-(x+y)}|x-y|^\alpha \, dx dy = \Gamma(\alpha+1).$$

Theorem 1. *For the fOU process $U^{\lambda,H}$, we have*

$$\rho_{\lambda,H}(t) = \frac{\Gamma(2H+1)}{2\lambda^{2H}} \quad \cosh(\lambda t) - \frac{1}{\Gamma(2H)} \int_0^{\lambda t} s^{2H-1} \cosh(\lambda t - s) \, ds \quad , \qquad (21)$$

and so for mmfOU process we have

$$\rho_\lambda(t) = \sum_{k=0}^\infty \sigma_k^2 \frac{\Gamma(2H_k+1)}{2\lambda^{2H_k}} \quad \cosh(\lambda t) - \frac{1}{\Gamma(2H_k)} \int_0^{\lambda t} s^{2H_k-1} \cosh(\lambda t - s) \, ds \quad .$$

Proof. For $H = 1/2$, the right-hand side of (21) is $e^{-\lambda t}/2\lambda$, equal to the autocovariance of the classical Ornstein–Uhlenbeck process with respect to the standard Brownian motion. For $H > 1/2$, we obtain (21) from (20) via integration by parts. To prove it for $H < 1/2$, we will apply the same approach as in the proof of *Lemma B.1* in [3]

$$\begin{aligned}
\rho_{\lambda,H}(t) &= \mathbb{E}[U_t^{\lambda,H} U_0^{\lambda,H}] \\
&= \mathbb{E} \quad \int_{-\infty}^0 e^{\lambda u} \, dB_u^H \int_{-\infty}^t e^{-\lambda(t-v)} \, dB_v^H \Big] \\
&= e^{-\lambda t} \quad \mathbb{V}ar(U_0^{\lambda,H}) + \mathbb{E} \quad \int_{-\infty}^0 e^{\lambda u} \, dB_u^H \int_0^t e^{\lambda v} \, dB_v^H \Big]\Big\}.
\end{aligned}$$

To obtain the term $\mathbb{V}ar(U_0^{\lambda,H})$ in a closed form, [3] referred to *Lemma 5.2* in [15]; however, such form was only obtained for $H \geq 1/2$, and so we need to extend their result for $H < 1/2$.

Since

$$U_0^{\lambda,H} = \int_{-\infty}^0 e^{\lambda u} \, dB_u^H = -\lambda \int_{-\infty}^0 e^{\lambda u} B_u^H \, du,$$

we have

$$\begin{aligned}
\mathbb{V}ar(U_0^{\lambda,H}) &= \mathbb{V}ar \quad -\lambda \int_{-\infty}^0 e^{\lambda u} B_u^H \, du \Big] \\
&= \lambda^2 \, \mathbb{V}ar \quad \int_0^\infty e^{-\lambda u} B_u^H \, du \Big] \\
&= \lambda^2 \mathbb{E}\left[\left(\int_0^\infty e^{-\lambda u} B_u^H \, du\right)^2\right] \\
&= \lambda^2 \mathbb{E} \quad \int_0^\infty \int_0^\infty e^{-\lambda(u+v)} B_u^H B_v^H \, du dv \Big]
\end{aligned}$$

$$= \frac{\lambda^2}{2} \int_0^\infty \int_0^\infty e^{-\lambda(u+v)} \cdot \left\{u^{2H} + v^{2H} - |u-v|^{2H}\right\} \mathrm{d}u\mathrm{d}v$$
$$= \frac{\lambda^2}{2} \; 2\left(\int_0^\infty e^{-\lambda u}\,\mathrm{d}u\right)\left(\int_0^\infty e^{-\lambda v} v^{2H}\,\mathrm{d}v\right)$$
$$- \int_0^\infty \int_0^\infty e^{-\lambda(u+v)}|u-v|^{2H}\,\mathrm{d}u\mathrm{d}v \quad .$$

Now choosing $x = \lambda u$, $y = \lambda v$ by Lemma 2 we have

$$\mathbb{V}ar(U_0^{\lambda,H}) = \frac{\lambda^{-2H}}{2} \; 2\int_0^\infty e^{-y} y^{2H}\,\mathrm{d}y$$
$$- \int_0^\infty \int_0^\infty e^{-(x+y)}|x-y|^{2H}\,\mathrm{d}x\mathrm{d}y$$
$$= \frac{\lambda^{-2H}}{2}\Big[2\Gamma(2H+1) - \Gamma(2H+1)\Big]$$
$$= \lambda^{-2H} H\Gamma(2H). \tag{22}$$

On the other hand, as in Lemma 2.1 in [8] and the proof of Lemma B.1 in [3], using formula

$$\gamma_\ell(z,x) = \frac{\gamma_\ell(z+1,x) + x^z e^{-x}}{z},$$

where γ_ℓ is the well-known lower Gamma function, for $H < 1/2$ we have

$$\mathbb{E}\left[\int_{-\infty}^0 e^{\lambda u}\,\mathrm{d}B_u^H \int_0^t e^{\lambda v}\,\mathrm{d}B_v^H\right]$$
$$= H(2H-1)\int_{-\infty}^0 \int_0^t e^{-\lambda(u+v)}|u-v|^{2H-2}\,\mathrm{d}u\mathrm{d}v$$
$$= \mathbb{V}ar(U_0^{\lambda,H}) \; \frac{e^{2\lambda t}-1}{2}$$
$$- \frac{\lambda}{\Gamma(2H-1)} \int_0^t e^{2\lambda v} \int_0^{\lambda v} e^{-s} s^{2H-2}\,\mathrm{d}s\mathrm{d}v$$
$$= \mathbb{V}ar(U_0^{\lambda,H}) \; \frac{e^{2\lambda t}-1}{2}$$
$$- \frac{\lambda}{\Gamma(2H-1)} \int_0^t e^{2\lambda v} \gamma_\ell(2H-1,\lambda v)\,\mathrm{d}v$$
$$= \mathbb{V}ar(U_0^{\lambda,H}) \; \frac{e^{2\lambda t}-1}{2}$$
$$- \frac{\lambda}{\Gamma(2H)} \int_0^t e^{2\lambda v} \gamma_\ell(2H,\lambda v)\,\mathrm{d}v$$
$$- \frac{\lambda^{2H}}{\Gamma(2H)} \int_0^t e^{\lambda v} v^{2H-1}\,\mathrm{d}v$$
$$= \mathbb{V}ar(U_0^{\lambda,H}) \; \frac{e^{2\lambda t}-1}{2}$$

$$- \frac{\lambda}{\Gamma(2H)} \int_0^t e^{2\lambda v} \int_0^{\lambda v} e^{-s} s^{2H-1}\, ds dv$$
$$- \frac{\lambda^{2H}}{\Gamma(2H)} \int_0^t e^{\lambda v} v^{2H-1}\, dv \quad . \tag{23}$$

Using (22) and (23), with similar arguments as we did for (20), we obtain (21). □

3 Long-range dependence

The increments of fBm are a well-known stationary process, that is long-range dependent (LRD) if $H > 1/2$, see [18]. Motivated by this, we consider the LRD for the increments of the mmfBm

$$\Delta_\delta M_t = \sum_{k=1}^{\infty} \sigma_k \Delta_\delta B_t^{H_k},$$

with covariance function

$$\varrho(\delta; t) = \mathbb{E}\big[\Delta_\delta M_{s+t} \Delta_\delta M_s\big],$$

where $\delta > 0$ is the lag and $\Delta_\delta x_t = x_{t+\delta} - x_t$ for a process x.

Theorem 2. *For $t \to \infty$,*

$$\varrho(\delta; t) \sim \delta^2 \sum_{k=1}^{\infty} \sigma_k^2 H_k(2H_k - 1)t^{2H_k-2} = O(t^{2H_{\sup}-2}). \tag{24}$$

So the mmfBm increment process $\Delta_\delta M_t$ is LRD if and only if $H_k > 1/2$ for some $k \geq 0$.

Proof. By using the generalized binomial theorem,

$$\begin{aligned}
\varrho(\delta; t) &= \frac{1}{2} \sum_{k=1}^{\infty} \sigma_k^2 \Big\{(t+\delta)^{2H_k} + (t-\delta)^{2H_k} - 2t^{2H_k}\Big\} \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \sigma_k^2 t^{2H_k} \quad \left(1 + \frac{\delta}{t}\right)^{2H_k} + \left(1 - \frac{\delta}{t}\right)^{2H_k} - 2 \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \sigma_k^2 t^{2H_k} \quad \sum_{r=0}^{\infty} \binom{2H_k}{r} \left(\frac{\delta}{t}\right)^r + \sum_{r=0}^{\infty} \binom{2H_k}{r} (-1)^r \left(\frac{\delta}{t}\right)^r - 2 \\
&\sim \delta^2 \sum_{k=1}^{\infty} \sigma_k^2 H_k(2H_k - 1)t^{2H_k-2}.
\end{aligned} \tag{25}$$

Since

$$\sigma_k^2 H_k(2H_k - 1)t^{2H_k-2} \leq \sigma_k^2,$$

the series (25) is uniformly convergent. So we have

$$\lim_{t\to\infty}\sum_{k=1}^{\infty}\sigma_k^2 H_k(2H_k-1)t^{2H_k-2}=\sum_{k=1}^{\infty}\lim_{t\to\infty}\sigma_k^2 H_k(2H_k-1)t^{2H_k-2}.$$

This yields (24). □

To investigate LRD for the mmfOU process, we first need some lemmas.

The following theorem shows that similar to the mmfBm increment process, the long-range dependence of the mmfOU is governed by the long-range dependence of the largest Hurst index in the driving mmfBm.

Theorem 3. *For $t\to\infty$ and each $N=1,2,\dots,$*

$$\rho_\lambda(t)=\frac{1}{2}\sum_{k=1}^{\infty}\sum_{n=1}^{N}\sigma_k^2\lambda^{-2n}\left(\prod_{j=0}^{2n-1}(2H_k-j)\right)t^{2H_k-2n}+O(t^{2H_{\sup}-2N-2}). \tag{26}$$

So the mmfOU process U is LRD if and only if $H_k>1/2$ for some $k\geq 0$.

Proof. By the proof of Lemma 2.2 and Theorem 2.3 in [8]

$$\begin{aligned}
\rho_\lambda(t)&=\mathbb{E}\left[\int_{-\infty}^{0}\mathrm{e}^{\lambda u}\,\mathrm{d}M_u\int_{-\infty}^{t}\mathrm{e}^{-\lambda(t-v)}\,\mathrm{d}M_v\right]\\
&=\mathrm{e}^{-\lambda t}\mathbb{E}\left[\int_{-\infty}^{0}\mathrm{e}^{\lambda u}\,\mathrm{d}M_u\int_{-\infty}^{1/\lambda}\mathrm{e}^{\lambda v}\,\mathrm{d}M_v\right]\\
&\quad+\mathrm{e}^{-\lambda t}\sum_{i=1}^{\infty}\sigma_i^2 H_i(2H_i-1)\\
&\quad\times\int_{-\infty}^{0}\mathrm{e}^{\lambda u}\left(\int_{1/\lambda}^{t}\mathrm{e}^{\lambda v}(v-u)^{2H_i-2}\,\mathrm{d}v\right)\mathrm{d}u\\
&=O(\mathrm{e}^{-\lambda t})\\
&\quad+\frac{1}{2}\sum_{i=1}^{\infty}\sigma_i^2\frac{H_i(2H_i-1)}{\lambda^{2H_i}}\quad\mathrm{e}^{-\lambda t}\int_{1}^{\lambda t}\mathrm{e}^{y}y^{2H_i-2}\,\mathrm{d}y\\
&\quad+\mathrm{e}^{\lambda t}\int_{\lambda t}^{\infty}\mathrm{e}^{-y}y^{2H_i-2}\,\mathrm{d}y\\
&\leq O(\mathrm{e}^{-\lambda t})\\
&\quad+\frac{1}{2}\sum_{k=1}^{\infty}\sum_{n=1}^{N}\sigma_k^2\lambda^{-2n}\left(\prod_{j=0}^{2n-1}(2H_k-j)\right)t^{2H_k-2n}\\
&\quad+\frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2\frac{\left|H_k(2H_k-1)\cdots(2H_k-2-2N)\right|}{\lambda^{2H_k}}\\
&\quad\times\left[\mathrm{e}^{-\frac{\lambda t}{2}}+(1+2^{2H_k-2N-3})(\lambda t)^{2H_k-2N-3}\right].
\end{aligned}\tag{27}$$

Now, for $t \in [1, \infty)$,

$$\frac{\left|H_k(2H_k-1)\cdots(2H_k-2-2N)\right|}{\lambda^{2H_k}} \mathrm{e}^{-\frac{\lambda t}{2}} < \Lambda_N$$

$$\frac{\left|H_k(2H_k-1)\cdots(2H_k-2-2N)\right|}{\lambda^{2H_k}} (1+2^{2H_k-2N-3})(\lambda t)^{2H_k-2N-3} < \Pi_N,$$

where

$$\Lambda_N = H_{\sup} \frac{\max\left(|2H_{\inf}-1|, |2H_{\sup}-1|\right)}{\max\left(\lambda^{2H_{\inf}}, \lambda^{2H_{\sup}}\right)} \left|(2H_{\inf}-2)\cdots(2H_{\inf}-2-2N)\right|,$$

$$\Pi_N = H_{\sup} \frac{\max\left(|2H_{\inf}-1|, |2H_{\sup}-1|\right)}{\lambda^{2N+3}} \left|(2H_{\inf}-2)\cdots(2H_{\inf}-2-2N)\right| \\ \times (1+2^{2H_{\sup}-2N-3}).$$

So, as $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$, the series in the right-hand side of the inequality (27) is uniformly convergent on $t \in [1, \infty)$. Hence

$$\lim_{t\to\infty} \sum_{k=1}^{\infty} \sigma_k^2 \frac{\left|H_k(2H_k-1)\cdots(2H_k-2-2N)\right|}{\lambda^{2H_k}} \\ \times \left[\mathrm{e}^{-\frac{\lambda t}{2}} + (1+2^{2H_k-2N-3})(\lambda t)^{2H_k-2N-3}\right] \\ = \sum_{k=1}^{\infty} \sigma_k^2 \frac{\left|H_k(2H_k-1)\cdots(2H_k-2-2N)\right|}{\lambda^{2H_k}} \\ \times \lim_{t\to\infty} \left[\mathrm{e}^{-\frac{\lambda t}{2}} + (1+2^{2H_k-2N-3})(\lambda t)^{2H_k-2N-3}\right].$$

This proves (26). □

4 Continuity

Definition 3. Let $X = (X_t)$ be a continuous stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If

$$H := \sup\left\{\nu > 0 : \sup_{s,t} \frac{X_t - X_s}{|t-s|^\nu} < \infty\right\} < \infty,$$

the process X is called Hölder continuous with index H, and H is its Hölder index.

Theorem 4. *Both mmfBm and mmfOU have Hölder index* $H_{\inf}$.

Proof. For $\epsilon > 0$ and $|t-s| < 1$, the mmfBm satisfies

$$\mathbb{E}\left[(M_t - M_s)^2\right] = \sum_{k=1}^{\infty} \sigma_k^2 |t-s|^{2H_k} \leq \left(\sum_{k=1}^{\infty} \sigma_k^2\right) |t-s|^{2H_{\inf}-\epsilon} = C_0 |t-s|^{2H_{\inf}-\epsilon},$$

where $C_0 := \sum_{k=1}^{\infty} \sigma_k^2 > 0$. Thus, the claim follows from Theorem 1 of [2]. On the other hand, for some $j \geq 1$ we have $H_{\inf} \leq H_j < H_{\inf} + \epsilon$ and so the fBm B^{H_j} is not $(H_{\inf} + \epsilon)$-Hölder continuous. Hence the process $M = \sigma_j B^{H_j} + \sum_{k \neq j} \sigma_k B^{H_k}$ is not $(H_{\inf} + \epsilon)$-Hölder continuous. This proves the claim for mmfBm.

For the mmfOU, we apply Corollary 2 of [2]. That states the process U_t is Hölder-continuous of any order $0 < a < H_{\inf}$ if and only if for each $0 < \epsilon < 2H_{\inf}$, there is some $0 < \delta < 1$ such that

$$\int_0^\infty (1 - \cos(sx)) f_\lambda(x) dx < C_\epsilon s^{2H_{\inf} - \epsilon}, \quad s \in (0, \delta). \tag{28}$$

This is equivalent to

$$\int_0^\infty \frac{(1 - \cos(sx))}{s^{2H_{\inf} - \epsilon}} f_\lambda(x) dx < C_\epsilon, \quad s \in (0, \delta).$$

To show this, here for $s < 1$ we have

$$\begin{aligned}
&\int_0^\infty \frac{(1 - \cos(sx))}{s^{2H_k - \epsilon}} f_{\lambda, H_k}(x) dx \\
&= s^\epsilon c_{H_k} \int_0^\infty (1 - \cos(sx)) \frac{x \cdot (sx)^{-2H_k}}{\lambda^2 + x^2} dx \\
&= s^\epsilon c_{H_k} \int_0^\infty (1 - \cos u) \frac{u^{1-2H_k}}{s^2\lambda^2 + u^2} du \quad (u = sx) \\
&\leq c_{H_k} \int_0^\infty (1 - \cos u) \frac{u^{1-2H_k}}{s^2\lambda^2 + u^2} du \quad (0 < s < 1) \\
&\leq c_{H_k} \left\{ \int_0^\epsilon (1 - \cos u) \frac{u^{1-2H_k}}{u^2} du + \int_\epsilon^\infty \frac{u^{1-2H_k}}{u^2} du \right\} \\
&= c_{H_k} \left\{ \int_0^\epsilon \frac{2\sin^2(\frac{u}{2})}{u^2} u^{1-2H_k} du + \int_\epsilon^\infty u^{-1-2H_k} du \right\} \\
&\leq c_{H_k} \left\{ \int_0^\epsilon \frac{1}{2} u^{1-2H_k} du + \int_\epsilon^\infty u^{-1-2H_k} du \right\} \\
&= c_{H_k} \left\{ \frac{\epsilon^{2-2H_k}}{4(1 - H_k)} + \frac{\epsilon^{-2H_k}}{2H_k} \right\} =: C_{\epsilon, H_k} < \infty.
\end{aligned}$$

Therefore,

$$\int_0^\infty (1 - \cos(sx)) f_{\lambda, H_k}(x) dx \leq C_{\epsilon, H_k} s^{2H_k - \epsilon} \leq C_{\epsilon, H_k} s^{2H_{\inf} - \epsilon}. \tag{29}$$

Also, we have

$$\begin{aligned}
\sum_{k=1}^\infty \sigma_k^2 C_{\epsilon, H_k} &= \sum_{k=1}^\infty \sigma_k^2 \frac{\sin(\pi H_k)\Gamma(1 + 2H_k)}{2\pi} \left\{ \frac{\epsilon^{2-2H_k}}{4(1 - H_k)} + \frac{\epsilon^{-2H_k}}{2H_k} \right\} \\
&\leq \frac{\Gamma(3)}{2\pi} \left\{ \frac{\epsilon^{2-2H_{\sup}}}{4(1 - H_{\sup})} + \frac{\epsilon^{-2H_{\inf}}}{2H_{\inf}} \right\} \left(\sum_{k=1}^\infty \sigma_k^2 \right) =: C_\epsilon < \infty,
\end{aligned} \tag{30}$$

if and only if $0 < H_{\inf} \leq H_{\sup} < 1$ and $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$. Now, (29) and (30) yield (28). Moreover, for some $j \geq 1$ we have $H_{\inf} \leq H_j < H_{\inf} + \epsilon$ and so the fOU U^{H_j} is not $(H_{\inf} + \epsilon)$-Hölder continuous. Hence the process $U = \sigma_j U^{H_j} + \sum_{k \neq j} \sigma_k U^{H_k}$ is not $(H_{\inf} + \epsilon)$-Hölder continuous. This proves the claim for mmfOU. □

5 Variation

Recall that the p-variation of fBm with $H \in (0, 1)$ on the time-interval $[0, T]$ is given in Definition 3.4 of [29] as

$$V_T^p(B^H) = \lim_{|\pi_n| \to 0} \sum_{t_k \in \pi_n} |\Delta B_{t_k}^H|^p = \begin{cases} \infty & ; \quad pH < 1 \\ T\mu_p & ; \quad pH = 1 \\ 0 & ; \quad pH > 1 \end{cases}$$

where $\pi_n = \{t_k = \frac{k}{n}\}_{k=0}^n$ is a partition of $[0, T]$, and μ_p is the pth absolute moment of a standard Gaussian process, and the limit is taken in probability. With the same argument, it is easy to check that for the mixed fractional Brownian motion (mfBm) $Y = aB + bB^H$ the p-variation is

$$V_T^p(Y) = \lim_{|\pi_n| \to 0} \sum_{t_k \in \pi_n} |\Delta Y_{t_k}|^p = \begin{cases} \infty & ; \quad p \min(1/2, H) < 1 \\ Ta^p \mu_p & ; \quad H > 1/2, \ p/2 = 1 \\ T(a^2 + b^2)^{p/2} \mu_p & ; \quad H = 1/2, \ p/2 = 1 \\ Tb^p \mu_p & ; \quad H < 1/2, \ pH = 1 \\ 0 & ; \quad p \min(1/2, H) > 1 \end{cases}$$

where $a, b > 0$, and B is the standard Brownian motion, and B^H is a standard fBm independent from B. Now, for the p-variation of the mmfBm we have the next theorem.

Theorem 5. *For $p > 0$, the p-variations of the mmfBm M and the mmfOU U on the time-interval $[0, T]$ are equal and*

$$V_T^p(M) = V_T^p(U) = \begin{cases} \infty & ; \quad pH_{\inf} < 1 \\ T\left(\sum_{H_i = H_{\inf}} \sigma_i^2 \right)^{p/2} \mu_p & ; \quad pH_{\inf} = 1 \\ 0 & ; \quad pH_{\inf} > 1 \end{cases} \tag{31}$$

Proof. For the mmfBm M, we have

$$\begin{aligned} v_{\pi_n}^p(M) &:= \sum_{t_k \in \pi_n} |\Delta M_{t_k}|^p \\ &= \sum_{t_k \in \pi_n} \left| \sum_{i=1}^{\infty} \sigma_i^2 (\Delta t_k)^{2H_i} \right|^{p/2} \left| \frac{\Delta M_{t_k}}{[\sum_{i=1}^{\infty} \sigma_i^2 (\Delta t_k)^{2H_i}]^{1/2}} \right|^p \\ &\stackrel{d}{=} \left(\sum_{i=1}^{\infty} \sigma_i^2 T^{2H_i} n^{2/p - 2H_i} \right)^{p/2} \cdot \frac{1}{n} \sum_{k=1}^{n} |Z_k|^p \end{aligned}$$

as $|\pi_n| \to 0$, or equivalently $n \to \infty$. Here Z_k is a standard Gaussian process and so by the proof of Lemma 3.7 in [29]

$$\frac{1}{n}\sum_{k=1}^{n}|Z_k|^p \to \mu_p,$$

as $n \to \infty$, where μ_p is the pth absolute moment of the standard Gaussian process. Now, if $pH_{\inf} < 1$ then $H_{\inf} < 1/p$, and so there exists some $j \geq 1$ that $H_j < 1/p$, and so $2/p - 2H_j > 0$. Therefore

$$v^p_{\pi_n}(M) \geq \left(\sigma_j^2 T^{2H_j} n^{2/p-2H_j}\right)^{p/2} \cdot \frac{1}{n}\sum_{k=1}^{n}|Z_k|^p \to \infty.$$

On the other hand, if $pH_{\inf} \geq 1$, for $x \in (1, \infty)$

$$\sigma_i^2 T^{2H_i} x^{2/p-2H_i} \leq \sigma_i^2 T^2,$$

and because $\sum_{i=1}^{\infty}\sigma_i^2 < \infty$, the $\sum_{i=1}^{\infty}\sigma_i^2 T^{2H_i} x^{2/p-2H_i}$ is uniformly convergent on $x \in [1, \infty)$. So for $pH_{\inf} \geq 1$,

$$\lim_{n\to\infty}\sum_{i=1}^{\infty}\sigma_i^2 T^{2H_i} n^{2/p-2H_i} = \sum_{i=1}^{\infty}\lim_{n\to\infty}\sigma_i^2 T^{2H_i} n^{2/p-2H_i}.$$

This yields the values mentioned in (31) are correct for the p-variation of M. For the mmfOU U, as it is stationary, we have

$$\begin{aligned} v^p_{\pi_n}(U) &:= \sum_{t_k\in\pi_n}|\Delta U_{t_k}|^p \\ &\stackrel{d}{=} \sum_{k=1}^{n}\left(\mathbb{V}ar[\Delta U_{t_1}]\right)^{p/2}|Z_k|^p \\ &= n\left(\mathbb{V}ar[U_{\frac{T}{n}} - U_0]\right)^{p/2} \cdot \frac{1}{n}\sum_{k=1}^{n}|Z_k|^p. \end{aligned}$$

As $\frac{1}{n}\sum_{k=1}^{n}|Z_k|^p \to \mu_p$ for $n \to \infty$, the problem reduces to the limit

$$\lim_{n\to\infty} n\left(\mathbb{V}ar[U_{\frac{T}{n}} - U_0]\right)^{p/2}.$$

To find it, again because U is stationary, and using the proof of Theorem 1 we have

$$\begin{aligned} \mathbb{V}ar[U_{\frac{T}{n}} - U_0] &= \mathbb{V}ar\, U_{\frac{T}{n}} + \mathbb{V}ar\, U_0 - 2\,\mathbb{C}ov\left(U_{\frac{T}{n}}, U_0\right) \\ &= 2\,\mathbb{V}ar\, U_0 - 2\,\mathbb{C}ov\left(U_{\frac{T}{n}}, U_0\right) \\ &= 2\sum_{i=1}^{\infty}\sigma_i^2\lambda^{-2H_i} H_i\Gamma(2H_i) \end{aligned}$$

$$
\begin{aligned}
&- 2\sum_{i=1}^{\infty}\sigma_i^2\frac{\Gamma(2H_i+1)}{2\lambda^{2H_i}} \quad \cosh\left(\frac{\lambda T}{n}\right) \\
&- \frac{1}{\Gamma(2H_i)}\int_0^{\frac{\lambda T}{n}} s^{2H_i-1}\cosh\left(\frac{\lambda T}{n}-s\right)\mathrm{d}s \\
= &\sum_{i=1}^{\infty}\sigma_i^2\frac{\Gamma(2H_i+1)}{\lambda^{2H_i}} \quad 1-\cosh\left(\frac{\lambda T}{n}\right) \\
&+ \frac{1}{\Gamma(2H_i)}\int_0^{\frac{\lambda T}{n}} s^{2H_i-1}\cosh\left(\frac{\lambda T}{n}-s\right)\mathrm{d}s \quad .
\end{aligned}
$$

For the large values of n, the final series in the right-hand side above is uniformly convergent. So, the $\lim_{n\to\infty}$ and $^{\infty}_{i=1}$ could change places. This yields

$$
\begin{aligned}
&\lim_{n\to\infty} n\Big(\mathbb{V}ar[U_{\frac{T}{n}}-U_0]\Big)^{p/2} \\
&= \lim_{n\to\infty}\Big(n^{2/p}\,\mathbb{V}ar[U_{\frac{T}{n}}-U_0]\Big)^{p/2} \\
&= \Bigg(\sum_{i=1}^{\infty}\sigma_i^2\frac{\Gamma(2H_i+1)}{\lambda^{2H_i}}\cdot\lim_{n\to\infty} n^{2/p} \quad 1-\cosh\left(\frac{\lambda T}{n}\right) \\
&\quad + \frac{1}{\Gamma(2H_i)}\int_0^{\frac{\lambda T}{n}} s^{2H_i-1}\cosh\left(\frac{\lambda T}{n}-s\right)\mathrm{d}s\Big\}\Bigg)^{p/2}.
\end{aligned}
$$

Now for $t\to 0$, by the Taylor expansion

$$
1-\cosh t = -\sum_{r=1}^{\infty}\frac{t^{2r}}{(2r)!},
$$

and via integration by parts

$$
\int_0^t s^{2H_i-1}\cosh(t-s)\,\mathrm{d}s = \frac{t^{2H_i}}{2H_i}+\frac{1}{2H_i}\int_0^t s^{2H_i}\sinh(t-s)\,\mathrm{d}s.
$$

Again for $t\to 0$, by the Taylor expansion,

$$
\int_0^t s^{2H_i}\sinh(t-s)\,\mathrm{d}s \le \int_0^t t^{2H_i}\sinh t\,\mathrm{d}s = t^{2H_i+1}\sinh t = \sum_{r=1}^{\infty}\frac{t^{2r+2H_i}}{(2r-1)!}.
$$

These yield for $t\to 0$

$$
1-\cosh t+\frac{1}{\Gamma(2H_i)}\int_0^t s^{2H_i-1}\cosh(t-s)\,\mathrm{d}s \sim \frac{t^{2H_i}}{2H_i+1}.
$$

Therefore

$$
\lim_{n\to\infty} n\Big(\mathbb{V}ar[U_{\frac{T}{n}}-U_0]\Big)^{p/2} = \left(\sum_{i=1}^{\infty}\sigma_i^2 T^{2H_i}\lim_{n\to\infty} n^{2/p-2H_i}\right)^{p/2},
$$

this proves (31). □

6 Conditional full support

As explained in [5], in mathematical finance models one of the must require features is the so-called Conditional Full Support (CFS) to avoid simple kind of arbitrage. This means that, given the information up to any time $\tau \in [0, T]$, the process is inherently free enough to go anywhere after time τ with positive probability. This motivates us to study the CFS property of the mmfBm and mmfOU processes but first we restate the precise definition of CFS from [12].

Definition 4. Let $X = (X_t)_{0 \le t \le T}$ be a continuous stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(\mathcal{F}_t)$ be its natural filtration. The process X is said to have CFS if, for all $t \in [0, T]$, the conditional law of $(X_u)_{t \le u \le T}$ given $(\mathcal{F}_t)$, almost surely has support $C_{X_t}[t, T]$, where $C_x[t, T]$ is the space of continuous functions f on $[t, T]$ satisfying $f(t) = x$. Equivalently, this means that, for all $t \in [0, T]$, $f \in C_0[t, T]$, and $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{t \le u \le T} |X_u - X_t - f(u)| < \varepsilon \,\middle|\, \mathcal{F}_t \right) > 0,$$

almost surely.

Theorem 6. *Both the mmfBm and the mmfOU have conditional full support.*

Proof. It is easy to check that

$$f(x) = \sum_{k=1}^{\infty} \sigma_k^2 f_{H_k}(x) \ge \begin{cases} \frac{\varepsilon_H \Gamma(1)}{2\pi} \left(\sum_{k=1}^{\infty} \sigma_k^2 \right) |x|^{1-2H_{\inf}} & : \quad |x| \le 1 \\ \frac{\varepsilon_H \Gamma(1)}{2\pi} \left(\sum_{k=1}^{\infty} \sigma_k^2 \right) |x|^{1-2H_{\sup}} & : \quad |x| \ge 1 \end{cases} =: h(x),$$

where

$$\varepsilon_H := \inf \left\{ \sin(\pi H_k) \right\}_{k \ge 1} = \inf \left\{ \sin(\pi H_{\inf}), \sin(\pi H_{\sup}) \right\}.$$

Since $0 < H_{\inf} \le H_{\sup} < 1$, $\varepsilon_H > 0$. Thus $h(x) > 0$ for $x \ne 0$. Therefore, for any $x_0 > 1$ we have

$$\begin{aligned} \int_{x_0}^{\infty} \frac{\log f(x)}{x^2} \, \mathrm{d}x &\ge \int_{x_0}^{\infty} \frac{\log h(x)}{x^2} \, \mathrm{d}x \\ &= \log \left\{ \frac{\varepsilon_H}{2\pi} \left(\sum_{k=1}^{\infty} \sigma_k^2 \right) \right\} \int_{x_0}^{\infty} \frac{\mathrm{d}x}{x^2} \\ &\quad + (1 - 2H_{\sup}) \int_{x_0}^{\infty} \frac{\log x}{x^2} \, \mathrm{d}x > -\infty, \end{aligned}$$

and by Theorem 2.1 of [12] this proves that M has conditional full support.

For mmfOU it is easy to check that

$$f_\lambda(x) = \sum_{k=1}^{\infty} \sigma_k^2 f_{\lambda,H_k}(x) \geq \begin{cases} \frac{\varepsilon_H \Gamma(1)}{2\pi}\left(\sum_{k=1}^{\infty}\sigma_k^2\right)\frac{|x|^{1-2H_{\mathrm{inf}}}}{\lambda^2+x^2} & : \quad |x| \leq 1 \\ \frac{\varepsilon_H \Gamma(1)}{2\pi}\left(\sum_{k=1}^{\infty}\sigma_k^2\right)\frac{|x|^{1-2H_{\mathrm{sup}}}}{\lambda^2+x^2} & : \quad |x| \geq 1 \end{cases} =: h(x),$$

where

$$\varepsilon_H := \inf\left\{\sin(\pi H_k)\right\}_{k\geq 1} = \inf\left\{\sin(\pi H_{\mathrm{inf}}), \sin(\pi H_{\mathrm{sup}})\right\}.$$

Since $0 < H_{\mathrm{inf}} \leq H_{\mathrm{sup}} < 1$, we have $\varepsilon_H > 0$. Consequently, $h(x) > 0$ for $x \neq 0$. Therefore, for any $x_0 > 1$ we have that

$$\begin{aligned} \int_{x_0}^{\infty} \frac{\log f_\lambda(x)}{x^2}\,\mathrm{d}x &\geq \int_{x_0}^{\infty} \frac{\log h(x)}{x^2}\,\mathrm{d}x \\ &= \log\left\{\frac{\varepsilon_H}{2\pi}\left(\sum_{k=1}^{\infty}\sigma_k^2\right)\right\}\int_{x_0}^{\infty}\frac{\mathrm{d}x}{x^2} \\ &\quad + (1-2H_{\mathrm{sup}})\int_{x_0}^{\infty}\frac{\log x}{x^2}\,\mathrm{d}x \\ &\quad - \int_{x_0}^{\infty}\frac{\log(\lambda^2+x^2)}{x^2}\,\mathrm{d}x > -\infty. \end{aligned}$$

The claim follows now from Theorem 2.1 of [12]. □

7 Sample paths

Here we aim to present some replications of the mmfOU and its related mmfBm with different limitations for its Hurst exponents. Obviously the limitations of the Hurst exponents characterize the roughness of the sample paths. In each of these replications, the mmfOU is given on $N = 1000$ equidistant points $t_k = k/(N-1)$ of the time interval [0, 1], with $n = 10$ equidistant Hurst exponents $H_i = H_{\mathrm{inf}} + (i-1)(H_{\mathrm{sup}} - H_{\mathrm{inf}})/(n-1)$ on the Hurst interval $[H_{\mathrm{inf}}, H_{\mathrm{sup}}]$. Also, the coefficients $\sigma_i = i^{-1}, i!^{-1}, \mathrm{e}^{-i}$ are used and indicated in each figure. In all paths here $\lambda = 1$.

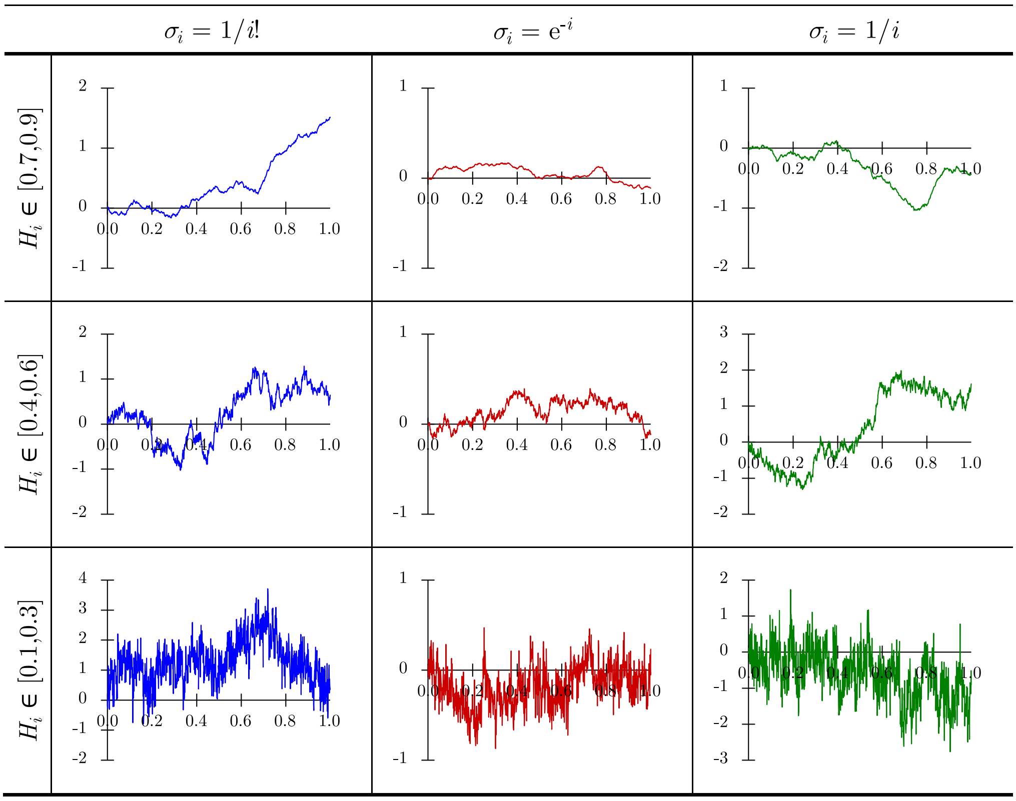


Fig. 2. Sample paths of mmfBm with equidistant time points and equidistant Hurst parameters

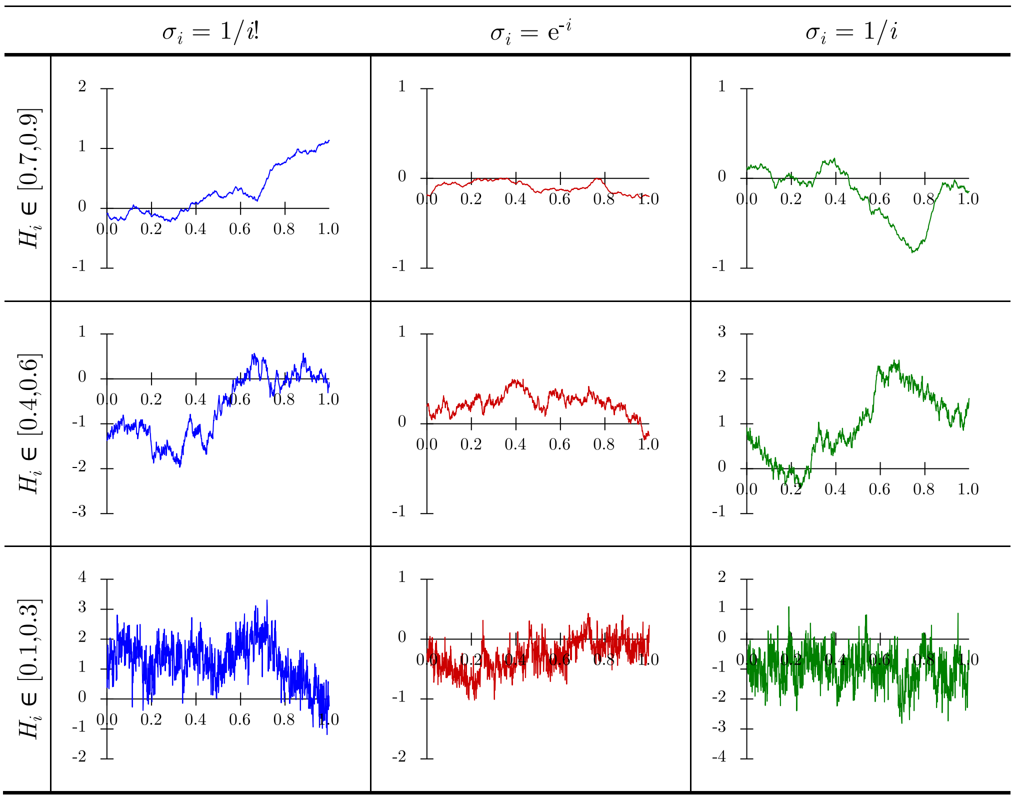


Fig. 3. Sample paths of mmfOU with equidistant time points and equidistant Hurst parameters

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PARAMETER ESTIMATION FOR MULTI-MIXED FRACTIONAL ORNSTEIN–UHLENBECK PROCESSES BY GENERALIZED METHOD OF MOMENTS

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ABSTRACT. We develope the generalized method of moments (GMM) estimation for the parameters of the finitely mixed multi-mixed fractional Ornstein–Uhlenbeck (mmfOU) processes, and analyze the consistency and assymptotic normality of this estimator. We also include some simulations and provide numerical observations considering different statistical errors.

1. INTRODUCTION AND PRELIMINARIES

Langevin [13] postulated the model

$$\frac{\mathrm{d}U_t}{\mathrm{d}t} = -\frac{f}{m}U_t + \frac{F_t}{m}, \tag{1}$$

to the velocity U of a free particle with mass $m > 0$, moving in a liquid by the alternating force F of surrounding molecules, and a constant friction coefficient f. By imposing probabilistic hypotheses to the force F, Ornstein and Uhlenbeck [18] derived that the solution of (1) has a Gaussian distribution with exponential mean. Later on, considering the initial variable U_0 to be central Gaussian and independent from F, Doob [5] showed that the solution to (1) is stationary and for some constant $c > 0$, the time-changed scaled process $t^{1/2}\,U(c\log t)$ is in fact the well-known Einstein's [6] Brownian motion for $t \geq 0$. As the Brownian motion in nowhere differentiable, Doob understood (1) as

$$\mathrm{d}U_t = -\lambda U_t \mathrm{d}t + \mathrm{d}X_t, \tag{2}$$

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where $U_0 = \xi$ is the initial variable and X is a Lévy process. He derived the solution

$$U_t = \mathrm{e}^{-\lambda t}\xi + \int_0^t \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}X_s, \tag{3}$$

which is called the Ornstein–Uhlenbeck (OU) process. This solution was later extent for all cases that X is a semimartingale (see e.g. [17]). Lamperti [12] proved the processes $V_t, t \geq 0$ is H-selfsimilar if and only if $U_t = e^{-\lambda t}V(c\exp(\lambda t/H))$ is stationary for all $\lambda, c > 0$. Cheridito and et al. [4] applied Lamperti transform and verified the solution (3) is valid for $X = B^H$, where B^H is the fractional Brownian mtoion (fBm) introduced by Mandelbrot and Van Ness [14].

In [2] the Langevin equation (2) with $X = \sum_{i=1}^{\infty} \sigma_i B^{H_i}$, the multi-mixed fractional Brownian motion (mmfBm), is the dynamic of particle's velocity when the liquid is not purely homogeneous, i.e. the local surrounding molecules can have imposing forces of different roughnesses. That is

$$\mathrm{d}U_t = -\lambda U_t \mathrm{d}t + \mathrm{d}\left(\sum_{i=1}^{\infty} \sigma_i B_t^{H_i}\right). \tag{4}$$

They confirmed the solution (3) for this model is possible if for all $i \geq 1$, $H_i \in [H_{\inf}, H_{\sup}] \subset (0, 1)$ and $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$. The solution is

$$U_t = \mathrm{e}^{-\lambda t}\xi + \int_0^t \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}\left(\sum_{i=1}^{\infty} \sigma_i B_t^{H_i}\right). \tag{5}$$

When $U_0 = \xi = \sum_i \sigma_i \xi_i$ is the initial condition, this solution can also be written as

$$U_t = \sum_{i=1}^{\infty} \sigma_i U_t^{\lambda, H_i}, \tag{6}$$

where

$$\mathrm{d}U_t^{\lambda,H_i} = -\lambda U_t^{\lambda,H_i}\mathrm{d}t + \mathrm{d}B_t^{H_i}, \tag{7}$$

and $U_0^{\lambda,H_i} = \xi_i$. To approximate U, one can naturally consider

$$U_t^n = \sum_{i=1}^{n} \sigma_i U_t^{\lambda,H_i}, \tag{8}$$

that is convergent to U in $L^2(\Omega \times [0, T])$, for any $T \geq 0$ (see [2]). To apply the process (8) for modelling the real data, we need $2n+1$ real-valued parameters

$$\lambda;\ H_1, \ldots, H_n;\ \sigma_1, \ldots, \sigma_n$$

to be statistically estimated. To do this, we use the generalized method of moments (GMM) introduced in Barbiza and Viens [3].

The outline of this paper is as follows. In Section 2 we develop the GMM estimator for the process (8) analytically. Next, in Section 3, we investigate the consistency of the introduced estimator. In Section 4 we prove the asymptotic normality of our estimator. Finally, in Section 5, we provide some simulations.

2. Generalized Methdod of Moments Estimation

The parameters we aim to estimate are

$$\lambda > 0, \qquad H_k \in [H_{\text{inf}}, H_{\text{sup}}] \subset (0,1), \qquad \sigma_k > 0; \qquad k = 1,2,\dots n,$$

i.e. the following vector will be estimated

$$\theta = (\lambda; \mathbf{H}; \boldsymbol{\sigma})' \in \mathbb{R}^{2n+1}$$

where

$$\mathbf{H} = (H_1, \dots, H_n),$$

$$\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n).$$

To develope GMM estimator, the autocovariance function of U^n is required. It is given in [2] by

$$\begin{aligned}
(9) \qquad \rho_\theta(t) &= \rho_{\lambda,n}(t) = \mathbb{E}[U^n_s U^n_{s+t}] \\
&= \sum_{k=1}^{n} \sigma_k^2 \frac{\Gamma(1+2H_k)}{4} \frac{\mathrm{e}^{-\lambda t}}{\lambda^{2H_k}} \left\{ 1 + \gamma_{2H_k-1}(\lambda t) + \mathrm{e}^{2\lambda t}\Gamma_{2H_k-1}(\lambda t) \right\} \\
&= \sum_{k=1}^{n} \sigma_k^2 \frac{\Gamma(2H_k+1)}{2\lambda^{2H_k}} \left\{ \cosh(\lambda t) - \frac{1}{\Gamma(2H_k)} \int_0^{\lambda t} s^{2H_k-1} \cosh(\lambda t - s)\, ds \right\},
\end{aligned}$$

where for $\alpha \in (-1,0) \cup (0,1)$

$$\begin{aligned}
\gamma_\alpha(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x s^{\alpha-1} \mathrm{e}^{s}\, \mathrm{d}s, \\
\Gamma_\alpha(x) &= \frac{1}{\Gamma(\alpha)} \int_x^\infty s^{\alpha-1} \mathrm{e}^{-s}\, \mathrm{d}s,
\end{aligned}$$

and $\gamma_0(x) = 1$, $\Gamma_0(x) = 0$. Here γ_α, Γ_α are the normalized incomplete gamma functions (see [1]). The formula (9) is continuous in t, and so U^n is ergodic (see [8], page 257). By Lemma 5.2 in [11] for sufficiently small $t \searrow 0$ we have

$$(10) \qquad \rho_\theta(t) \approx \sum_{k=1}^{n} \sigma_k^2 \left(\frac{H\Gamma(2H)}{\lambda^{2H}} - \frac{t^{2H}}{2} \right).$$

Further, for our estimation, we need some filters associated to the parameter θ and a positive integers $L \geq 2n+1$. We define them as following.

Definition 1. A filter of length $L+1$ and order l is a sequence of real numbers $\mathbf{a} = (a_k)_{k=0}^{L} = (a_0, \dots, a_L)$, that for $l > 0$

$$\sum_{q=0}^{L} a_q q^r = 0; \quad 0 \leq r \leq l-1, \; r \in \mathbb{Z},$$

$$\sum_{q=0}^{L} a_q q^l \neq 0.$$

When $l = 0$, we put $a_0 = 1$ and $a_q = 0$ for $0 < q \leq L$.

The GMM estimation requires a family of filters by the length L and orders $l = 0, 1, \dots, L$. For each filter $\mathbf{a}$ with order l define

$$\mathbf{b} = (b_0, \dots, b_L)',$$

where

$$b_0 := \sum_{q=0}^{L} a_q^2,$$

$$b_k := 2\sum_{q=0}^{L-k} a_{q+k}a_q\,; \quad k = 1, \dots, L.$$

For different orders $l_i \neq l_j$, the corresponding vectors $\mathbf{b}_i$ and $\mathbf{b}_j$ are linearly independent. So, we can take L filters $\mathbf{a}_1, \dots, \mathbf{a}_L$ with respective orders $l_1, \dots, l_L$ such that the $L \times (L+1)$ matrix $B = [\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_L]'$ satisfies $2n+1 \leq \mathrm{rank}(B) \leq L$. Then, for a fixed step size $\alpha > 0$, and $\theta \in \Theta$ (a closed subset of $\mathbb{R}^{2n+1}$), we can define the function

$$V(\theta) := \sum_{k=0}^{L} b_k \rho_\theta(\alpha k) = \sum_{k=0}^{L} b_k \rho_{\lambda,n}(\alpha k),$$

and the filtered process of the step size $\alpha > 0$ for $t \geq 0$

$$\varphi(t) := \sum_{q=0}^{L} a_q U^n_{t-q\alpha}.$$

Next, let θ_0 be the exact value of the parameter. Due to the stationarity of U^n, the random variable $\varphi(t)$ is centered Gaussian with variance $\mathbb{E}[\varphi(t)^2] = V(\theta_0)$. Following the notation of [15], for any $\theta \in \Theta$ we have the vector of differences

$$\mathbf{g}(t,\theta) := (g_1(t,\theta), \dots, g_L(t,\theta))'\,,$$

where

$$g_\ell(t,\theta) = \varphi_\ell(t)^2 - V_\ell(\theta), \quad 1 \leq \ell \leq L,$$

and the subscript ℓ means that we use the filter $\mathbf{a}_\ell$ in the filtered process and variance. Moreover, $\mathbf{g}(t,\theta)$ satisfies a *population moment condition* (see [9] and [15])

$$\mathbb{E}[\mathbf{g}(t,\theta_0)] = 0, \quad t \geq 0.$$

Also, the vector $\mathbf{g}$ is in the second Wiener chaos (see [16]).

Further, assume that we have observed the process U^n at discrete equidistant times $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ with fixed time step $\alpha = t_i - t_{i-1}$. Let A be an $L \times L$ symmetric positive-definite matrix (a suitably chosen A ensures that the GMM estimate is efficient, see [3] Section 2.3). Now, for $\theta \in \Theta$ and time $t \geq 0$ denote

$\mathbf{g}_0(\theta) := \mathbb{E}[\mathbf{g}(t,\theta)]$ (vector of expected differences),
$\hat{\mathbf{g}}_N(\theta) := \frac{1}{N-L+1}\sum_{i=L}^{N} \mathbf{g}(t_i,\theta)$ (vector of sampled differences),
$Q_0(\theta) := \mathbf{g}_0(\theta)' A \mathbf{g}_0(\theta)$ (squared distance of expected differences),
$\hat{Q}_N(\theta) := \hat{\mathbf{g}}_N(\theta)' A \hat{\mathbf{g}}_N(\theta)$ (sampled version of the above distance).

Note that $\mathbf{g}_0(\theta)$ does not depend on time t, due to the stationarity of U^n. Finally, we define the GMM estimator of $\theta_0 = (\lambda_0; \mathbf{H}_0; \boldsymbol{\sigma}_0^2)$ as

$$\hat{\theta}_N = (\hat{\lambda}_N; \hat{\mathbf{H}}_N; \hat{\boldsymbol{\sigma}}_N)' = \arg\min_{\theta\in\Theta} \hat{Q}_N(\theta). \tag{11}$$

In fact, if the function $Q_0(\theta)$ attains a unique zero at θ_0, then we can find a value $\hat{\theta}_N$ such that the approximation of Q_0, that is $\hat{Q}_N$, should be the smallest possible (see [9, 10]).

Moreover, all the remaining results are still valid if we substitute the fixed matrix A with a sequence of random matrices (perhaps depending on the data) with a deterministically bounded eigenstructure. In particular, we can choose such sequence in order to attain convergence in probability to the efficient alternative of A (see [3], Section 2.3).

In the following we will show the consistency and asymptotic normality of the GMM estimator (11).

3. Consistency

As mentioned in [3], for the consistency, it is sufficient to find conditions under which the mapping $\theta \mapsto \boldsymbol{\rho}_{\theta,2n+1}(\alpha)$ is smooth and invertible at least in a closed subset $\theta \in \Theta \subseteq \mathbb{R}^{2n+1}$, where

$$\boldsymbol{\rho}_{\theta,2n+1}(\alpha) = (\rho_\theta(0\cdot\alpha), \ldots, \rho_\theta((2n)\cdot\alpha))',$$

and $\alpha > 0$ is the step size. This is equivalent for $\nabla_\theta \boldsymbol{\rho}_{\theta,2n+1}(\alpha)$ to be non-singular in a closed subset $\theta \in \Theta \subseteq \mathbb{R}^{2n+1}$. To show this for the mmfOU processes, with $\theta = (\lambda; \mathbf{H}; \boldsymbol{\sigma})' \in \mathbb{R}^{2n+1}$, we have the next lemma and theorem.

Lemma 1. *In the closed rectangle*

$$\Upsilon_3 = \left\{ \theta = (\lambda, H, \sigma)' \;\middle|\; \begin{array}{l} \lambda, \sigma > 0,\ H \in (0,1), \\ \lambda < \exp\big(\Psi(2H+1)\big) \end{array} \right\} \subset \Theta \subseteq \mathbb{R}^3, \tag{12}$$

where $\Psi(\cdot)$ *is the so called digamma function*

$$\Psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

for $\rho_\theta(t) = \sigma^2\rho_{\lambda,H}(t)$, *the matrix*

$$\nabla_\theta \boldsymbol{\rho}_{\theta,3}(\alpha) = \begin{bmatrix} \frac{\partial\rho_\theta}{\partial\lambda}(0) & \frac{\partial\rho_\theta}{\partial H}(0) & \frac{\partial\rho_\theta}{\partial\sigma}(0) \\ \frac{\partial\rho_\theta}{\partial\lambda}(\alpha) & \frac{\partial\rho_\theta}{\partial H}(\alpha) & \frac{\partial\rho_\theta}{\partial\sigma}(\alpha) \\ \frac{\partial\rho_\theta}{\partial\lambda}(2\alpha) & \frac{\partial\rho_\theta}{\partial H}(2\alpha) & \frac{\partial\rho_\theta}{\partial\sigma}(2\alpha) \end{bmatrix}$$

is a P-matrix for sufficiently small α.

Proof. Proof is similar to Lemma B.2 in [3]. □

Theorem 1. *In the closed rectangle*

$$\Upsilon_{2n+1} = \left\{ \theta = (\lambda; \mathbf{H}; \boldsymbol{\sigma})' \,\middle|\, \begin{array}{l} \lambda, \sigma_k > 0,\ H_k \in (0,1), \\ \lambda < \exp\big(\Psi(2H_{\inf}+1)\big) \end{array} \right\} \subset \Theta \subseteq \mathbb{R}^{2n+1}, \tag{13}$$

the mapping

$$\boldsymbol{\rho}_{\theta,2n+1}(\alpha) = (\rho_\theta(0\cdot\alpha), \ldots, \rho_\theta((2n)\cdot\alpha))', \tag{14}$$

is injective for sufficiently small α.

Proof. Let $\theta \in \Upsilon_{2n+1}$, and $x = (x_1, \ldots, x_{2n+1})' > 0$ is a positive vector in $\mathbb{R}^{2n+1}$. Then, the kth component of the vector $\nabla_\theta \boldsymbol{\rho}_{\theta,2n+1}(\alpha)\, x$ is

$$\begin{aligned}
\Big[\nabla_\theta \boldsymbol{\rho}_{\theta,2n+1}(\alpha)\, x\Big]_k &= \sum_{j=1}^{2n+1} \frac{\partial \rho_{\lambda,n}}{\partial \theta_j}((k-1)\alpha) x_j \\
&= \frac{\partial \rho_{\lambda,n}}{\partial \lambda}((k-1)\alpha)\, x_1 + \sum_{j=1}^{n} \frac{\partial \rho_{\lambda,n}}{\partial H_j}((k-1)\alpha)\, x_{j+1} \\
&+ \sum_{j=1}^{n} \frac{\partial \rho_{\lambda,n}}{\partial \sigma_j}((k-1)\alpha)\, x_{n+j+1} \\
&= \sum_{j=1}^{n} \sigma_j^2 \frac{\partial \rho_{\lambda,H_j}}{\partial \lambda}((k-1)\alpha)\, x_1 + \sum_{j=1}^{n} \sigma_j^2 \frac{\partial \rho_{\lambda,H_j}}{\partial H_j}((k-1)\alpha)\, x_{j+1} \\
&+ \sum_{j=1}^{n} 2\sigma_j \frac{\partial \rho_{\lambda,H_j}}{\partial \sigma_j}((k-1)\alpha)\, x_{n+j+1} \\
&= \sum_{j=1}^{n} \Bigg[\sigma_j^2 \frac{\partial \rho_{\lambda,H_j}}{\partial \lambda}((k-1)\alpha)\, x_1 + \sigma_j^2 \frac{\partial \rho_{\lambda,H_j}}{\partial H_j}((k-1)\alpha)\, x_{j+1} \\
&+ 2\sigma_j \frac{\partial \rho_{\lambda,H_j}}{\partial \sigma_j}((k-1)\alpha)\, x_{n+j+1} \Bigg].
\end{aligned} \tag{15}$$

Now, for all $j = 1, \ldots, 2n+1$, let $\tilde{\rho}_j(t) = \sigma_j^2 \rho_{\lambda,H_j}(t)$ and

$$\tilde{\alpha} = \begin{cases} (k-1)\alpha & ; \quad 1 \le k \le n, \\ (k-1)\alpha/2 & ; \quad n+1 \le k \le 2n+1. \end{cases}$$

Then, since $\exp(\Psi(\cdot))$ is an strictly increasing function,

$$\lambda < \exp\big(\Psi(2H_{\inf}+1)\big) = \min_{1\le i\le 2n+1} \exp\big(\Psi(2H_i+1)\big) \le \exp\big(\Psi(2H_j+1)\big),$$

and so by Lemma 1 the matrix

$$\nabla_{\lambda,H_j,\sigma_j} \tilde{\rho}_j(\tilde{\alpha}) = \begin{bmatrix} \dfrac{\partial \tilde{\rho}_j}{\partial \lambda}(0) & \dfrac{\partial \tilde{\rho}_j}{\partial H_j}(0) & \dfrac{\partial \tilde{\rho}_j}{\partial \sigma_j}(0) \\[2ex] \dfrac{\partial \tilde{\rho}_j}{\partial \lambda}(\tilde{\alpha}) & \dfrac{\partial \tilde{\rho}_j}{\partial H_j}(\tilde{\alpha}) & \dfrac{\partial \tilde{\rho}_j}{\partial \sigma_j}(\tilde{\alpha}) \\[2ex] \dfrac{\partial \tilde{\rho}_j}{\partial \lambda}(2\tilde{\alpha}) & \dfrac{\partial \tilde{\rho}_j}{\partial H_j}(2\tilde{\alpha}) & \dfrac{\partial \tilde{\rho}_j}{\partial \sigma_j}(2\tilde{\alpha}) \end{bmatrix} \tag{16}$$

is a P-matrix for all $\tilde{\alpha} \leq \tilde{\alpha}_{kj}$ sufficiently small associated to k and $\tilde{\rho}_j$. So, (16) is also P-matrix for

$$\tilde{\alpha} \leq \tilde{\alpha}^*_{2n+1} := \min_{\substack{1\leq k\leq 2n+1\\ 1\leq j\leq 2n+1}} \tilde{\alpha}_{kj},$$

for all $j = 1, \ldots, 2n+1$.
Hence, by the Theorem 2 of [7], for the subvector $\tilde{x}_j = (x_1, x_{j+1}, x_{n+j+1})'$ we have

$$\nabla_{\lambda,H_j,\sigma_j} \tilde{\rho}_j(\tilde{\alpha}) \cdot \tilde{x}_j > 0,$$

and so for $\theta \in \Upsilon_{2n+1}$

$$\sigma_j^2 \frac{\partial \rho_{\lambda,H_j}}{\partial \lambda}((k-1)\alpha)\, x_1 + \sigma_j^2 \frac{\partial \rho_{\lambda,H_j}}{\partial H_j}((k-1)\alpha)\, x_{j+1} + 2\sigma_j \frac{\partial \rho_{\lambda,H_j}}{\partial \sigma_j}((k-1)\alpha)\, x_{n+j+1} > 0.$$

Consequently, for

$$\alpha \leq \frac{\tilde{\alpha}^*_{2n+1}}{2n+1}, \tag{17}$$

(15) is positive for all $k = 1, \ldots, 2n+1$, and so the vector $\nabla_\theta \boldsymbol{\rho}_{\theta,2n+1}(\alpha)\, x$ is positive. Hence, by the Theorem 2 of [7], the gradiant $\nabla_\theta \boldsymbol{\rho}_{\theta,2n+1}(\alpha)$ is a P-matrix, and so the mapping (14) is injective on Υ_{2n+1}. □

Now, we can obtain the strong consistency for the GMM estimators for the mmfOU processes.

Theorem 2. *For the GMM parameter estimator* (11) *for the mmfOU process* U^n*, we have*

$$\hat{\theta}_N = (\hat{\lambda}_N; \hat{\mathbf{H}}_N; \hat{\boldsymbol{\sigma}}_N)' \xrightarrow{a.s.} \theta_0 = (\lambda_0; \mathbf{H}_0; \boldsymbol{\sigma}_0).$$

Proof. As an straightforward result of Theorem 1, Assumption 2.1 of [3] is valid and we can apply Theorem 2.1 in [3]. □

Remark 1. First, we note that for $\lambda \in [0, \mathrm{e}^{\Psi(1)}]$ the consistency that we studied above is valid. Second, the proof of Theorem 1 returns an important result as the nature of numerical estimation. By (17), the larger the number of parameters we aim to estimate, the smaller the sufficient step $\alpha > 0$ we need. However, in any case, there exists some sufficient α for which the consistency is garanteed.

4. Asymptotic Normality

For $\theta = (\lambda; \mathbf{H}; \boldsymbol{\sigma})' \in \mathbb{R}^{2n+1}$, the spectral density function (SDF) of U^n is

$$f_\theta(x) = f(\theta; x) = \sum_{i=1}^n \sigma_i^2 f_{\lambda,H_i}(x), \tag{18}$$

where

$$f_{\lambda,H_i}(x) = \frac{\sin(\pi H_i)\Gamma(1+2H_i)}{2\pi} \frac{|x|^{1-2H_i}}{x^2+\lambda^2}, \tag{19}$$

is the SDF of the fOU process U^{λ,H_i} (see [2]). Denote

$$\bar{f}_\theta(x) := \sum_{p\in\mathbb{Z}} f_\theta\left(x + \frac{2p\pi}{\alpha}\right); \quad x \in \left[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right],$$

and for $l_i, l_j \in \{1, \ldots, 2n+1\}$

$$R_\theta(x|i,j) := \min(1, |x|^{l_i+l_j})\bar{f}_\theta(x).$$

Then, as explained in [3], the GMM estimator has asymptotic normality if $R_{\theta_0}(\cdot|i,j) \in L^2(-\pi,\pi)$. For the process U^n we have

$$\begin{aligned} R_\theta(x|i,j) &= \min(1, |x|^{l_i+l_j})\overline{f}_\theta(x) \\ &= \sum_{k=1}^n \sigma_k^2 R_{\lambda,H_k}(x|i,j), \end{aligned}$$

where

$$\begin{aligned} R_{\lambda,H_k}(x|i,j) &:= \min(1, |x|^{l_i+l_j})\overline{f}_{\lambda,H_k}(x), \\ \bar{f}_{\lambda,H_k}(x) &:= \sum_{p\in\mathbb{Z}} f_{\lambda,H_k}\left(x + \frac{2p\pi}{\alpha}\right); \quad x \in \left[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right]. \end{aligned}$$

So, $R_\theta(\cdot|i,j) \in L^2(-\pi,\pi)$ if and only if $R_{\lambda,H_k}(\cdot|i,j) \in L^2(-\pi,\pi)$ for all $k = 1, \ldots, n$. The following Lemma is a generalization of Lemma 3.1 in [3].

Lemma 2. *For the U^n process $R_{\theta_0}(\cdot|i,j) \in L^2(-\pi,\pi)$ for*

(i) $l_i + l_j \geq 1$, *if all* $H_k \in (0,1)$,
(ii) $l_i + l_j = 0$, *if all* $H_k \in (0, \frac{3}{4})$, *i.e.* $H_{\sup} \leq 3/4$,

where $n \geq 1$ ($L \geq 3$).

Now, for $\theta \in \Theta$ we define

$$\begin{aligned} \hat{G}_N(\theta) &:= \nabla_\theta \hat{\mathbf{g}}_N(\theta), \\ G(\theta) &:= \mathbb{E}[\nabla_\theta \mathbf{g}(\cdot,\theta)], \end{aligned}$$

and note that $G(\theta)$ does not depend on time t because of the stationarity of U^n.

Finally, we have the following theorem for asymptotic normality of the GMM estimator of U^n.

Theorem 3. *The GMM estimator* (11) *for the mmfOU processes U^n satisfies*

(i) $\mathbb{E}[\|\hat{\theta}_N - \theta_0\|^2] = O(N^{-1})$,
(ii) $N^c\|\hat{\theta}_N - \theta_0\| \xrightarrow{a.s.} 0$ *for* $c < \frac{1}{2}$,
(iii) $\sqrt{N}(\hat{\theta}_N - \theta_0) \xrightarrow{d} \mathcal{N}(0, C(\theta_0)\Lambda C(\theta_0)')$,

where $C(\theta_0) = [G(\theta_0)' A G(\theta_0)]^{-1} G(\theta_0)' A$, *and* Λ *is a* $L \times L$ *matrix (for* $L \geq 2n+1$*) such that*

$$\Lambda_{ij} = 2 \sum_{p \in \mathbb{Z}} \left[\sum_{k=0}^{L} b_k^{i,j} \rho_{\theta_0} (\alpha \cdot (k+p)) \right]^2 ,$$

$$b_k^{i,j} = \sum_{q=0}^{L-k} a_q^i a_{q+k}^j + \sum_{q=0}^{L-k} a_q^j a_{q+k}^i .$$

Proof. Theorem 1 and Lemma 2 yield that in fact the Assumptions 1.2-1.3 of [3] are valid. So, we can apply Lemma 2.2 and Theorem 2.3 of [3]. □

5. Simulation

In this section, we perform a simulation study to test our GMM estimator for the parameters vector $\theta = (\lambda; \mathbf{H}; \boldsymbol{\sigma})$ for $n = 3$. The statistical error metrics we use are

$$\widehat{MSE} := \frac{1}{m} \sum_{i=1}^{m} \|\hat{\theta}_{N,i} - \theta\|^2,$$

$$e(\widehat{Var}) := \text{maximum eigenvalue of } \widehat{Var}(\theta_N),$$

$$\widehat{Bias}^2 := \left\| \frac{1}{m} \sum_{i=1}^{m} \hat{\theta}_{N,i} - \theta \right\|^2 ,$$

where $\widehat{Var}(\theta_N)$ is the empirical covariance matrix of the estimated parameters based on the m replications. For statistically high accuracy (cooking), we setup $m = 500$ replications in our estimations.

To have the consistency, as mentioned by Remark 1, we take $\lambda = \exp(\Psi(1))$. For (17) to hold we take $N = 200, 600$, and 1000 in time interval $[0, 1]$. Also, Theorem 1 is valid when $\boldsymbol{\sigma}$ has a fixed sign. So, we choose the positive-sign values $\boldsymbol{\sigma} = (0.5, 1, 1.5)'$.

For asymptotic normality, we are restricted by the Lemma 2. So, we take filters of positive orders $l = 1, 2, \ldots, 2n+1$. On the other hand, we are interested to test our estimator for varying Hurst values. Specially, for applications sense, we are interested to include the standard Brownian motion (Bm), the case $H_j = 1/2$ for some j. Hence, we take $\mathbf{H} = (0.3, 0.5, 0.7)'$.

In each estimation, we first simulate replications for the U^n process, and then estimate its parameters $\theta = (\lambda; \mathbf{H}; \boldsymbol{\sigma})$. To produce U^n, we use iterations of the Langevin equation

$$\mathrm{d}U_t^n = -\lambda U_t^n \mathrm{d}t + \mathrm{d} \left(\sum_{i=1}^{n} \sigma_i B_t^{H_i} \right), \tag{20}$$

at N points of time interval $[0, 1]$ with uniform step-sizes. In each replication, we use the autocovariance function $\rho_{\lambda,n}$ given in (10) to estimate the parameters $\hat{\theta}_N = (\hat{\lambda}_N; \hat{\mathbf{H}}_N; \hat{\boldsymbol{\sigma}}_N)$. To do this, we employ finite difference filters of orders $L = 2n + 1$ (number of parameters) with $A = I_{2n+1}$ the identity matrix to minimize the calculation cost (see [3]).

The resulting values of the statistical error metrics for our simulations are given in Table 5. As one can check, the method produces reasonably accurate estimations and the errors vanishe while inreasing the time step accuracy by increasing N. This fact is nicely observable in Figure 5, confirming our analytical formula (17) in action.

N	200	600	1000
$\widehat{MSE}$	0.006867	0.001149	0.000558
$e(\widehat{Var})$	0.004257	0.000564	0.000242
$\widehat{Bias}^2$	0.00082	0.000294	0.000184

TABLE 1. Statistical errors for estimations by $m = 500$ replications.

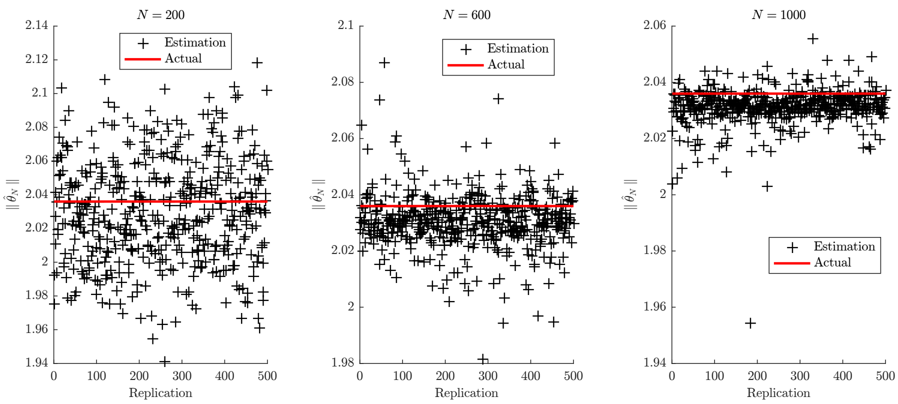

FIGURE 1. Estimations of $\theta = (\lambda; \mathbf{H}; \boldsymbol{\sigma})$ for $m = 500$ replications.

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Prediction of Gaussian Volterra processes with compound Poisson jumps

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ABSTRACT

We consider a Gaussian Volterra process with compound Poisson jumps and derive its prediction law.

1. Introduction

We are interested in a mixed process $X = G + J$, where the continuous part G is a so-called Gaussian Volterra process. In older terminology these are processes that admit canonical representation of multiplicity one. A typical Gaussian Volterra process is the fractional Brownian motion as shown in Norros et al. (1999). See Section 4 for the definition of fractional Brownian motion and for other examples. Intuitively, a Gaussian process G is a Gaussian Volterra process if one can construct a martingale M from by using a non-anticipative linear transformation and then represent the original process G in a non-anticipative way as a linear transformation of the martingale M. The motivation to use Gaussian Volterra processes is that for them one can calculate their prediction law in terms of the kernels that transfer the Gaussian Volterra process into its driving martingale M and vice versa. In the mixture $X = G+J$ the jump part J will be an independent compound Poisson process with square-integrable jump distribution.

One motivation to study processes of the type $X = G + J$ comes from mathematical finance. Indeed, it is well-known that the returns of financial assets do not follow Gaussian distribution (Blattberg and Gonedes, 2010; Fama, 1965; Mandelbrot and Mandelbrot, 1997) and the returns also exhibit jumps, or shocks (Akgiray and Booth, 1986, 1987; Ball and Torous, 1985; Jarrow and Rosenfeld, 1984; Press, 1967). Also, there is evidence of long-range dependence in the returns also explain the presence of long-memory (Baillie, 1996; Chan and Hameed, 2006; Harvey, 1995; Kim and Wu, 2008; Rajan and Zingales, 2003). Thus models where the returns are Gaussian with jumps seem more reasonable: The Gaussian part could take care of the long-range dependence with fractional Brownian motion (fBm) as the Gaussian Volterra process, and the shocks would come from the compound Poisson part. Then one can use the result of this paper to calculate imperfect hedges in the mixed model in the similar way as done in Shokrollahi and Sottinen (2017) and Sottinen and Viitasaari (2018). Indeed, this is work in progress by the authors.

In this paper we derive the prediction law of the mixed process $X = G + J$.

The rest of the paper is organized as follows. In Section 2 we define Gaussian Volterra processes and derive their prediction laws. Section 3 is the main section of the paper where we introduce the Gaussian Volterra processes with compound Poisson jumps and derive their prediction laws. Finally, in Section 4 we provide examples of Gaussian Volterra processes.

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2. Gaussian Volterra processes

A Gaussian Volterra process is a Gaussian process that has a canonical representation of multiplicity one with respect to a Gaussian martingale. The Gaussian Volterra process is defined in Definition 2.1 below in terms of covariance functions. The Gaussian Volterra representations follow from Definition 2.1 and are stated in Proposition 2.1.

For convenience we consider processes over the compact time-interval $[0,T]$ with an arbitrary but fixed time horizon $T>0$.

Let $G=(G_t)_{t\in[0,T]}$ be a centered Gaussian process with $G_0=0$ and covariance function $R:[0,T]^2\to\mathbb{R}$ defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$.

A kernel $K:[0,T]^2\to\mathbb{R}$ is a Volterra kernel if $K(t,s)=0$ whenever $t<s$. For a Volterra kernel K we define its associated operator K as.

$$\mathrm{K}[f](t)=\int_0^t f(s)K(t,s)\,\mathrm{d}s.$$

Denote

$$\mathbf{1}_t(s)=\mathbf{1}_{[0,t)}(s)=\begin{cases}1, & \text{if } s\in[0,t),\\ 0, & \text{otherwise}\end{cases}.$$

The adjoint associated operator K^* of the Volterra kernel K is given by extending linearly the relation

$$\mathrm{K}^*[\mathbf{1}_t](s)=K(t,s).$$

It turns out that K^* for a Gaussian Volterra process with covariance

$$R(t,s)=\int_0^{t\wedge s}K(t,u)K(s,u)\,\mathrm{d}v(u)$$

extends to an isometry from Λ to $L^2([0,T],\mathrm{d}v)$ where v is given in Definition 2.1(i) and Λ, the space of Wiener integrands, is the closure of the indicator functions $\mathbf{1}_t$, $t\in[0,T]$, in the inner product

$$\langle\mathbf{1}_t,\mathbf{1}_s\rangle_\Lambda=R(t,s).$$

Remark 2.1. By Alòs et al. (2001), if K is of bounded variation in its first argument, we can write for any simple function f

$$\mathrm{K}^*[f](t)=f(t)K(T,t)+\int_t^T[f(u)-f(t)]\,K(\mathrm{d}u,t).$$

Moreover, as in Alòs et al. (2001) Lemma 1, we have for simple functions f and g that

$$\int_0^T\mathrm{K}^*[f](t)g(t)\,\mathrm{d}t=\int_0^T f(t)\mathrm{K}[g](\mathrm{d}t)$$

justifying the name "adjoint" associated operator.

For Gaussian Volterra representations we recall what is the co-called abstract Wiener integral (for more information on abstract Wiener integrals and their relation to conditioning we refer to Sottinen and Yazigi (2014)). The linear space $\mathcal{L}$ is the closure of the random variables G_t, $t\in[0,T]$, in $L^2(\Omega,\mathcal{F},\mathbb{P})$. The spaces Λ and $\mathcal{L}$ are isometric. Indeed, the mapping

$$\mathbf{1}_t\mapsto G_t$$

extends to an isometry. This isometry is called the abstract Wiener integral and we denote it

$$\int_0^T f(t)\,\mathrm{d}G_t$$

for a $f\in\Lambda$.

Definition 2.1 (*Gaussian Volterra Process*)**.** Let $G=(G_t)_{t\in[0,T]}$ be a centered Gaussian process with covariance function $R:[0,\infty)^2\to\mathbb{R}$. Assume that

(1) there exists an increasing function $v:[0,T]\to\mathbb{R}$ and a Volterra kernel $K:[0,T]^2\to\mathbb{R}$ such that $\int_0^t K(t,s)^2\,\mathrm{d}v(s)<\infty$ for all $t\in[0,T]$ and

$$R(t,s)=\int_0^{t\wedge s}K(t,u)K(s,u)\,\mathrm{d}v(u),$$

(2) for each $t\in[0,T]$ the equation

$$\mathrm{K}^*[K^{-1}(t,\cdot)](s)=\mathbf{1}_t(s)$$

admits a solution $K^{-1}(t,\cdot)$.

Note that by Definition 2.1(ii) the operator K^* is invertible and we have

$$(\mathrm{K}^*)^{-1}[\mathbf{1}_t](s) = K^{-1}(t,s).$$

We note that due to Definition 2.1(i) for a Gaussian Volterra process the space Λ is isometric to $L^2([0,T],\mathrm{d}v)$. Indeed, we have

$$\langle f, g\rangle_\Lambda = \langle \mathrm{K}^*[f], \mathrm{K}^*[g]\rangle_{L^2([0,T],\mathrm{d}v)}.$$

In particular, this means that the mapping K^* in Definition 2.1(ii) is an isometry between Λ and $L^2([0,t],\mathrm{d}v)$.

The following representation proposition is a direct consequence of Definition 2.1. Indeed, Proposition 2.1 could have been taken as the definition of Gaussian Volterra process.

Proposition 2.1 (*Volterra Representation*). *Let G be a Gaussian Volterra process. Let K^{-1} be the kernel in Definition 2.1(ii). Then the process*

$$M_t = \int_0^t K^{-1}(t,s)\,\mathrm{d}G_s$$

is a Gaussian martingale with bracket $\langle M\rangle_t = v(t)$. Moreover,

$$G_t = \int_0^t K(t,s)\,\mathrm{d}M_s,$$

where v and K are as in Definition 2.1(i).

Note that form Proposition 2.1 we immediately see that the filtrations $\mathbb{F}^G$ and $\mathbb{F}^M$ coincide.

The Volterra representations of Proposition 2.1 extend immediately to the following transfer principle for Wiener integrals.

Proposition 2.2 (*Transfer Principle*). *Let $f \in \Lambda$ and $g \in L^2([0,T],\mathrm{d}v)$. Then*

$$\begin{aligned}
\int_0^T f(t)\,\mathrm{d}G_t &= \int_0^T \mathrm{K}^*[f](t)\,\mathrm{d}M_t,\\
\int_0^T g(t)\,\mathrm{d}M_t &= \int_0^T (\mathrm{K}^*)^{-1}[g](t)\,\mathrm{d}G_t.
\end{aligned}$$

In what follows we will use the following notation for the conditional mean, the conditional covariance and the conditional law of a stochastic process Y:

$$\begin{aligned}
\hat{m}_t^Y(u) &= \mathbb{E}\left[Y_t \middle| \mathcal{F}_u^Y\right],\\
\hat{R}_Y(t,s|u) &= \mathbb{C}\mathrm{ov}[Y_t, Y_s \big| \mathcal{F}_u^Y],\\
\hat{P}_t^Y(\mathrm{d}y|u) &= \mathbb{P}\left[Y_t \in \mathrm{d}y \middle| \mathcal{F}_u^Y\right]
\end{aligned}$$

We end this section by stating the prediction formula for Gaussian Volterra processes. The formula and its proof is similar to that given in Sottinen and Viitasaari (2017). We give here the proof in detail for the convenience of the readers.

Proposition 2.3 (*Volterra Prediction*). *Let G be a Gaussian Volterra process as in Definition 2.1. Let $u \le s \le t \le T$. Denote*

$$\Psi(t,s|u) = (\mathrm{K}^*)^{-1}[K(t,\cdot) - K(u,\cdot)](s)$$

and

$$\Phi(\mathrm{d}x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}\,\mathrm{d}x.$$

Then

$$\begin{aligned}
\hat{m}_t^G(u) &= G_u - \int_0^u \Psi(t,s|u)\,\mathrm{d}G_u,\\
\hat{R}_G(t,s|u) &= R(t,s) - \int_0^u K(t,x)K(s,x)\,\mathrm{d}v(x),\\
\hat{P}_t^G(\mathrm{d}x|u) &= \Phi(\mathrm{d}x;\hat{m}_t^G(u),\hat{R}(t,t|u)).
\end{aligned}$$

Proof. It is well-known that conditional Gaussian processes are Gaussian. Therefore it is enough to identify the conditional mean and conditional covariance.

We consider first the conditional mean. Now

$$\begin{aligned}
\hat{m}_t^G(u) &= \mathbb{E}\left[G_t \middle| \mathcal{F}_u^G\right]\\
&= \mathbb{E}\left[\int_0^t K(t,s)\,\mathrm{d}M_s \middle| \mathcal{F}_u^M\right]
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^u K(t,s)\,\mathrm{d}M_s \\
&= \int_0^u K(u,s)\,\mathrm{d}M_s + \int_0^u [K(t,s)-K(u,s)]\,\mathrm{d}M_s \\
&= G_u - \int_0^u (\mathrm{K}^*)^{-1}\,[K(t,\cdot)-K(u,\cdot)]\,(s)\,\mathrm{d}G_s.
\end{aligned}
$$

Let us then consider the conditional covariance. Now

$$
\begin{aligned}
\hat{R}_G(t,s|u) &= \mathbb{E}\left[\left(G_t-\hat{m}_t^G(u)\right)\left(G_s-\hat{m}_s^G(u)\right)\Big|\mathcal{F}_u^G\right] \\
&= \mathbb{E}\left[\left(\int_0^t K(t,x)\,\mathrm{d}M_x-\hat{m}_t^G(u)\right)\left(\int_0^s K(s,x)\,\mathrm{d}M_x-\hat{m}_s^G(u)\right)\Big|\mathcal{F}_u^M\right] \\
&= \mathbb{E}\left[\int_u^t K(t,x)\,\mathrm{d}M_x\int_u^s K(s,x)\,\mathrm{d}M_x\Big|\mathcal{F}_u^M\right] \\
&= \mathbb{E}\left[\int_u^t K(t,x)\,\mathrm{d}M_x\int_u^s K(s,x)\,\mathrm{d}M_x\right] \\
&= \int_u^{t\wedge s} K(t,x)K(s,x)\,\mathrm{d}v(x) \\
&= \int_0^{t\wedge s} K(t,x)K(s,x)\,\mathrm{d}v(x)-\int_0^u K(t,x)K(s,x)\,\mathrm{d}v(x) \\
&= R(t,s)-\int_0^u K(t,x)K(s,x)\,\mathrm{d}v(x). \quad \square
\end{aligned}
$$

3. Gaussian Volterra process with jumps

In this section we prove our main result, the prediction formula for Gaussian Volterra processes with compound Poisson jumps. Indeed, we consider the process $X=(X_t)_{t\in[0,T]}$ given by

$$
X = G + J, \tag{3.1}
$$

where G is a continuous Gaussian Volterra process and J is an independent compound Poisson process with intensity λ and jump distribution F. In other words

$$
J_t = \sum_{k=1}^{N_t} \xi_k,
$$

where $N=(N_t)_{t\in[0,T]}$ is a Poisson process with intensity λ and the jumps ξ_k, $k\in\mathbb{N}$, are i.i.d. with common distribution F, and they are independent of the Poisson process N and the Gaussian Volterra process G. We denote

$$
\begin{aligned}
\mu_1 &= \mathbb{E}[\xi_k], \\
\mu_2 &= \mathbb{E}[\xi_k^2].
\end{aligned}
$$

We note that it is crucial that the Gaussian Volterra part is continuous since it implies that $\mathbb{F}^X=\mathbb{F}^{G,J}$, i.e. the signals G and J can be separated from the signal X. We refer to Azmoodeh et al. (2014) and references therein on the continuity of Gaussian processes.

Our main theorem is the following.

Theorem 3.1 (*Mixed Prediction*). *Let X be given by (3.1). Then*

$$
\begin{aligned}
\hat{m}_t^X(u) &= X_u - \int_0^u \Psi(t,s|u)\,\mathrm{d}G_s + \lambda(t-u)\mu_1, \\
\hat{R}_X(t,s|u) &= R(t,s) - \int_0^u K(t,x)K(s,x)\,\mathrm{d}v(x) + \lambda(t\wedge s-u)\mu_2, \\
\hat{P}_t^X(\mathrm{d}x|u) &= \int_{y\in\mathbb{R}} \Phi\left(\mathrm{d}x-y;\hat{m}_t^G(u),\hat{R}_G(t,t|u)\right)\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda(t-u)}(\lambda(t-u))^n}{n!}F^{*n}\left(\mathrm{d}y-J_u\right).
\end{aligned}
$$

Here F^{*n} is the n-fold convolution of the distribution F:

$$
\begin{aligned}
F^{*1}(\mathrm{d}x) &= F(\mathrm{d}x), \\
F^{*n}(\mathrm{d}x) &= \int_{y\in\mathbb{R}} F(\mathrm{d}x-y)F^{*(n-1)}(\mathrm{d}y).
\end{aligned}
$$

Proof. Let us begin with the mean $\hat{m}_t^X(u)$. The conditional mean of G is already known. As for the conditional mean of J, we have, by independence, that

$$
\hat{m}_t^J(u) = \mathbb{E}\left[J_t\Big|\mathcal{F}_u^J\right]
$$

$$
\begin{aligned}
&= J_u + \mathbb{E}\left[J_t - J_u \middle| \mathcal{F}_u^J\right] \\
&= J_u + \mathbb{E}\left[J_{t-u}\right] \\
&= J_u + \lambda(t-u)\mu_1.
\end{aligned}
$$

The formula for the conditional mean follows from this.

Let us then consider the conditional variance $\hat{R}_X(t,s|u)$. By independence we have

$$
\hat{R}_X(t,s|u) = \hat{R}_G(t,s|u) + \hat{R}_J(t,s|u).
$$

Now $\hat{R}_G(t,s|u)$ is known and for $\hat{R}_J(t,s|u)$ we have

$$
\begin{aligned}
\hat{R}_J(t,s|u) &= \mathbb{C}\mathrm{ov}\left[J_t, J_s \middle| \mathcal{F}_u^J\right] \\
&= \mathbb{C}\mathrm{ov}\left[J_t - J_u, J_s - J_u \middle| \mathcal{F}_u^J\right] \\
&= \mathbb{C}\mathrm{ov}\left[J_{t-u}, J_{s-u}\right] \\
&= \lambda(t \wedge s - u)\mu_2.
\end{aligned}
$$

Finally, let us consider the conditional law $\hat{P}_t^X(\mathrm{d}x|u)$. By the law of total probability and independence we have

$$
\begin{aligned}
\hat{P}_t(\mathrm{d}x|u) &= \int_{y\in\mathbb{R}} \mathbb{P}\left[G_t \in \mathrm{d}x - y \middle| J_t = y, \mathcal{F}_u^X\right] \mathbb{P}\left[J_t \in \mathrm{d}y \middle| \mathcal{F}_u^X\right] \\
&= \int_{y\in\mathbb{R}} \mathbb{P}\left[G_t \in \mathrm{d}x - y \middle| \mathcal{F}_u^G\right] \mathbb{P}\left[J_t \in \mathrm{d}y \middle| \mathcal{F}_u^J\right] \\
&= \int_{y\in\mathbb{R}} \hat{P}_t^G(\mathrm{d}x - y|u)\mathbb{P}\left[J_t - J_u \in \mathrm{d}y - J_u \middle| J_u\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{P}\left[J_t - J_u \in \mathrm{d}y - J_u \middle| J_u\right] \\
&= \sum_{n=0}^{\infty} \mathbb{P}\left[J_t - J_u \in \mathrm{d}y - J_u \middle| N_t - N_u = n, J_u\right] \mathbb{P}\left[N_t - N_u = n \middle| J_u\right] \\
&= \sum_{n=0}^{\infty} \mathbb{P}\left[\sum_{k=1}^{n} \xi_k \in \mathrm{d}y - J_u \middle| J_u\right] \mathbb{P}[N_{t-u} = n] \\
&= \sum_{n=0}^{\infty} F^{*n}\left(\mathrm{d}y - J_u\right) \mathbb{P}[N_{t-u} = n]
\end{aligned}
$$

The formula for the conditional law follows from this by plugging in $\hat{P}_t^G(\mathrm{d}x|u)$ and the Poisson probabilities. □

Remark 3.1. It is interesting to note that just like in the Gaussian case, also in the mixed case, the conditional covariance is deterministic.

4. Examples

In this section we give examples for different Gaussian Volterra processes G for the prediction formula of Theorem 3.1. This means that we give the kernel K, the function v, and the kernel Ψ.

Example 4.1 (*fBm*)**.** The fractional Brownian motion B^H with Hurst index $H \in (0,1)$ is a centered Gaussian process with covariance

$$
R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).
$$

By Norros et al. (1999) (see also (Sottinen and Viitasaari, 2017)) B^H is a Gaussian Volterra process with $v(t) = t$ and the Volterra kernel

$$
K_H(t,s) = c_H\left\{\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}\,\mathrm{d}u\right\}, \tag{4.1}
$$

with a normalizing constant

$$
c_H = \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(\frac{1}{2}+H)\Gamma(2-2H)}},
$$

and we have

$$
\Psi_H(t,s|u) = \frac{\sin(\pi(H-\frac{1}{2}))}{\pi}s^{\frac{1}{2}-H}(u-s)^{\frac{1}{2}-H}\int_u^t \frac{z^{H-\frac{1}{2}}(z-u)^{H-\frac{1}{2}}}{z-s}\,\mathrm{d}z. \tag{4.2}
$$

Example 4.2 (*ccmfBm*)**.** The long-range dependent completely correlated mixed fractional Brownian motion (ccmfBm) was introduced in Dufitinema et al. (2021). It is a Gaussian process

$$M = aW + bB^H$$

where B^H is a fractional Brownian motion with $H > 1/2$, and it is constructed from the Brownian motion W by using the Volterra representation

$$B_t^H = \int_0^t K_H(t,s)\,\mathrm{d}W_s$$

(see Example 4.1 above). The ccfBm, M is also a Gaussian Volterra process with $v(t) = t$ and the Volterra kernel

$$K_{a,b,H}(t,s) = a\,\mathbf{1}_t(s) + b\,K_H(t,s).$$

In other word, it has the following Volterra representation

$$M_t = \int_0^t K_{a,b,H}(t,s)\,\mathrm{d}W_s,$$

and so

$$\Psi_{a,b,H}(t,s|u) = (\mathrm{K}^*_{a,b,H})^{-1}\left[K_{a,b,H}(t,\cdot) - K_{a,b,H}(u,\cdot)\right](s).$$

Here

$$\mathrm{K}^*_{a,b,H}[f](t) = af(t) + \frac{b\,c(H)(H-\frac{1}{2})}{t^{H-\frac{1}{2}}}\int_t^T f(u)\frac{u^{H-\frac{1}{2}}}{(u-t)^{\frac{3}{2}-H}}\,\mathrm{d}u,$$

for all $f \in \Lambda_{a,b,H}$, the space of all integrands from M, which is simply $L^2[0,T]$ in this case. The $(\mathrm{K}^*_{a,b,H})^{-1}$ is the inverse operator of $\mathrm{K}^*_{a,b,H}$, where from Dufitinema et al. (2021)

$$(\mathrm{K}^*_{a,b,H})^{-1}[f](t) = f(T)K^{-1}_{a,b,H}(t,T) - \int_t^T f(s)K^{-1}_{a,b,H}(\mathrm{d}s,t),$$

$$K^{-1}_{a,b,H}(t,s) = \frac{1}{a}\mathbf{1}_t(s) + \frac{1}{a}\sum_{k=1}^{\infty}(-1)^k\left(\frac{b}{a}\right)^k\gamma_k(t,s),$$

$$\gamma_k(t,s) = \frac{c(H)^k\Gamma(H+\frac{1}{2})^k}{\Gamma(k(H-\frac{1}{2}))}\frac{1}{s^{H-\frac{1}{2}}}\int_s^t u^{H-\frac{1}{2}}(u-s)^{k(H-\frac{1}{2})-1}\mathrm{d}u.$$

Example 4.3 (*mfBm*)**.** The mixed fractional Brownian motion (mfBm) is the process

$$\tilde{M} = W + B^H$$

where the Brownian motion W and the fractional Brownian motion B^H are independent. The mfBm was introduced by Cheridito (2001). In Cai et al. (2016) it was shown that the mfBm is a Gaussian Volterra process with $v(t) = t$ and a certain kernel $\tilde{K}_H$.

Indeed, let $H > 1/2$ and let $L(t,s)$ be the solution of the equation

$$L(t,s) + H(2H-1)\int_0^t L(t,x)|s-x|^{2H-2}\,\mathrm{d}x = -H(2H-1)|t-s|^{2H-2}, \quad 0 \le s,t$$

Then by Theorem 2.2 of Cai et al. (2016) we have the following: denote

$$\phi(t) = 1 - \int_0^t L(t,x)\,\mathrm{d}x.$$

Then

$$\tilde{W}_t = \mathbb{E}\left[\int_0^t \phi(s)\,\mathrm{d}W_s \,\middle|\, \mathcal{F}_t^{\tilde{M}}\right] = \int_0^t q(t,s)\,\mathrm{d}\tilde{M}_s,$$

is a Brownian motion, where $q(t,s)$ is the unique solution of the Wiener–Hopf equation:

$$q(t,s) + H(2H-1)\int_0^t q(t,x)|s-x|^{2H-2}\,\mathrm{d}x = \phi(s), \quad 0 \le s,t$$

and

$$\tilde{M}_t = \int_0^t \tilde{K}_H(t,s)\,\mathrm{d}\tilde{W}_s,$$

where

$$\tilde{K}_H(t,s) = -\frac{\partial}{\partial s}\int_s^t q(t,x)\mathrm{d}x.$$

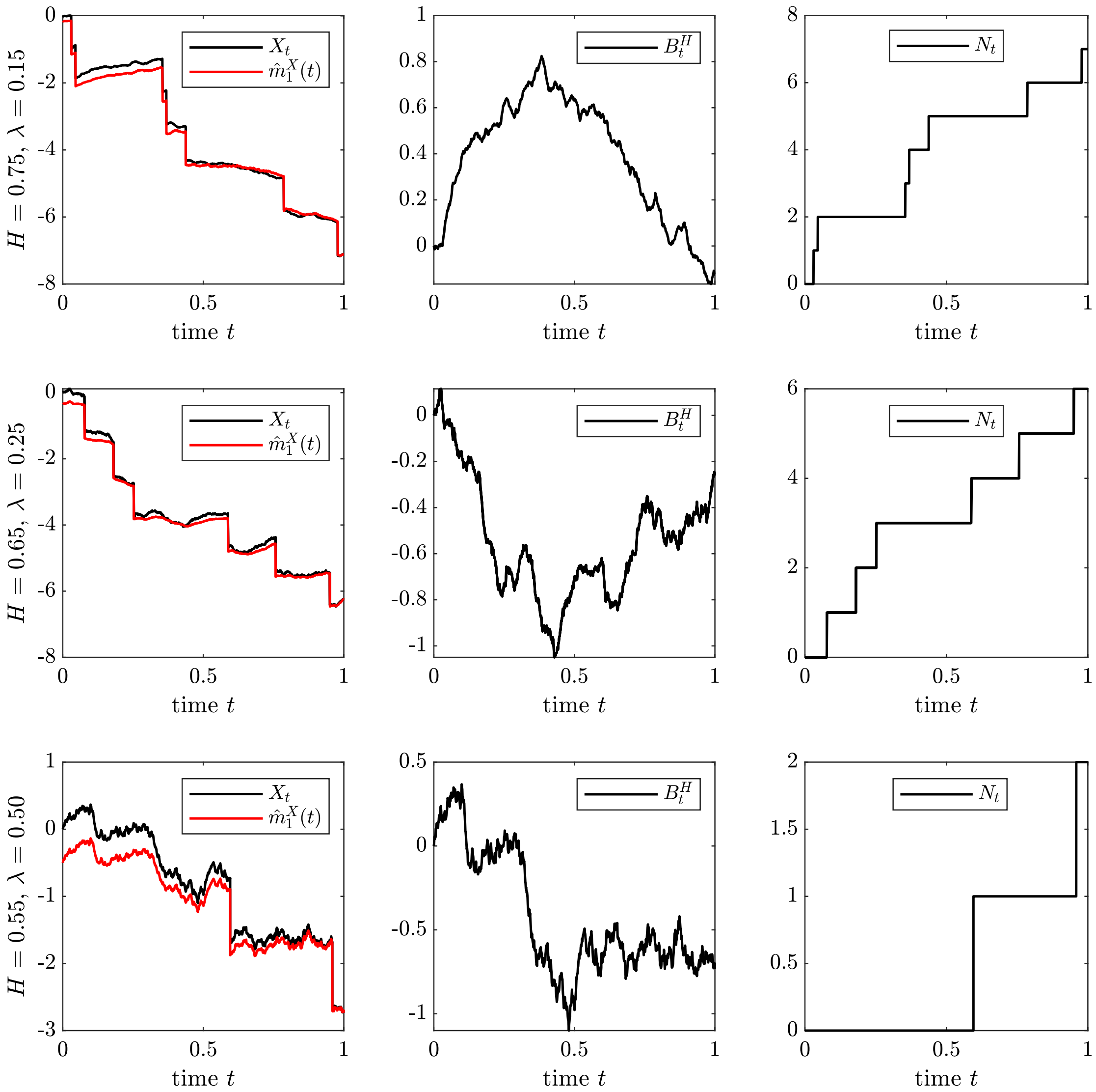

Fig. 1. Real value path, conditional mean, continuous noise path, and the jump part of the process $X = B^H - N$.

and here we have

$$\tilde{\Psi}(t,s|u) = (\widetilde{\mathrm{K}}_H^*)^{-1}\left[\tilde{K}_H(t,\cdot) - \tilde{K}_H(u,\cdot)\right](s),$$

where

$$\widetilde{\mathrm{K}}_H^*[f](t) = f(t)\tilde{K}_H(T,t) + \int_t^T [f(u) - f(t)]\,\tilde{K}_H(\mathrm{d}u,t),$$

for all $f \in \tilde{\Lambda}$, the space of all integrands from $\tilde{M}$, and $(\widetilde{\mathrm{K}}_H^*)^{-1}$ is the inverse operator of $\widetilde{\mathrm{K}}_H^*$.

5. Simulation

We consider the case $X = B^H - N$ where B^H is a fractional Brownian motion (fBm) with Hurst index H and N is a Poisson process with parameter λ. The paths of this process X_t and its conditional mean $\hat{m}_1^X(t)$ for time t, are given in Fig. 1 below. Plots of Fig. 1 for different λ and H are simulated for $N = 1000$ time points in a period of $[0,1]$. As one can see, the closer to the maturity time $t = 1$, the more accurate the conditional mean is.

Data availability

No data was used for the research described in the article.

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HEDGING IN JUMP DIFFUSION MODEL WITH TRANSACTION COSTS

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Abstract. We consider the jump-diffusion risky asset model and study its conditional prediction laws. Next, we explain the conditional least square hedging strategy and calculate its closed form for the jump-diffusion model, considering the Black-Scholes framework with interpretations related to investor priorities and transaction costs. We investigate the explicit form of this result for the particular case of the European call option under transaction costs and formulate recursive hedging strategies. Finally, we present a decision tree, table of values, and figures to support our results.

1. Introduction

The decade of 1970-80 was undoubtedly the initiation of modern finance. During that period, the main issue economists were eager to solve was to evaluate a closed form for the option price. Pioneer among them, Black–Scholes (BS) (1973) [3] applied the geometric Brownian motion (gBm) of Samuelson (1965) [27] adjusted by the theoretical framework of finance, the *no-arbitrage* theory. This continuous hedging-pricing model was later employed successfully for pricing of rational options by Merton (1976) [23], and extened for transaction costs and taxes by Ingersoll (1976) [11].

The gBm is the stochastic differential equation (SDE):

$$\frac{\mathrm{d}S_t}{S_t} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \tag{1.1}$$

where S is the asset price, W is the Wiener process, and μ, σ are constants. In a *self-financing* trading strategy, the investor holds a portfolio η, π of some risk-free and risky assets respectively with prices A, S, specially the case that A is the

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value of the cash account. So, the value V of the associated portfolio at time $t \geq 0$ changes by

$$\mathrm{d}V_t = \eta_t \,\mathrm{d}A_t + \pi_t \,\mathrm{d}S_t. \tag{1.2}$$

Applying Itô's (1951) [12, 13] calculus, Black–Scholes showed that if

(i) The investment strategy is self-financing,
(ii) The interest rate r of the risk-free asset A is constant,
(iii) The risky asset S has the gBm dynamic (1.1),
(iv) For some $g \in \mathscr{C}^{1,2}([0,T] \times \mathbb{R}_+)$ we have $V_t = g(t, S_t)$,

then

$$\frac{\partial g}{\partial t} + rx\frac{\partial g}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 g}{\partial x^2} - rg = 0, \tag{1.3}$$

and from (iv), on $[0,T]$

$$\pi_t = \frac{\partial g}{\partial x}(t, S_t). \tag{1.4}$$

By the no-arbitrage theory, value of the portfolio V must match to the value of the option of the risky asset that the investor holds. So, (iv) is equivalent to having the option price depend only on time t and the asset price S_t. This is valid for the European *vanilla* options, and for such option's boundary conditions, the equation (1.3) has a unique solution. Later on, Leland (1985) [19] developed this approach under transaction costs. For the mathematical reconstruction, see related references [6, 15].

As explained above, the BS model makes it possible to identify a *perfect hedging* strategy. In other word, if in a market we have all (i)-(iv) valid, postulating of the no-arbitrage theory (which is interesting in equilibrium finance) is possible. Although, this was a marvelous inception that is still applicable on *business time* (see [9, 10]), some criticisms arose later. In fact, the main criticism was even published some years before Black–Scholes work, that is the Mandelbrot & Van Ness (1968) [22]. Working on stock price data, they had concluded the noise of price is far different with a Wiener process. This rules out the gBm model (1.1) and also returns the possibility of arbitrage.

In modern finance, to overcome such issues, mathematical economists have applied (or some introduced) other stochastic processes in (1.1) place of Wiener process. Some considered the *stable* processes (see [14, 21, 26]), some tried processes with *long memory* (see [2, 31, 35–38]), and most recently they have studied the processes including *jumps* (see [4, 8, 16, 17, 20, 24]). The simplest model with these jump processes (1.1) is the *jump diffusion (JD)* model

$$\frac{\mathrm{d}S_t}{S_{t-}} = \mu \,\mathrm{d}t + \sigma \,\mathrm{d}W_t + \mathrm{d}J_t. \tag{1.5}$$

For the JD model with compound Poisson jump of independent normal processes, [24] evaluated the hedging strategy and option prices. [5] also calculated a closed forms of hedging and pricing when the interest rate has the JD model. Although the general JD model nicely calibrates the price data, it allows arbitrage opportunities. So, the perfect hedging is impossible for the general JD model (see [18] chapter 7).

For models where the BS hedging method is not applicable, economists investigated other hedging methods. Föllmer and Schweizer (1991) [7] initiated hedging associated to the *risk measure* and Schweizer (1992-95) developed this idea in [28–30]. However, as the BS is still valid on *business time*, a financial investor naturally may ask the following question:

*"In an incomplete market, what is the discrete strategy with the **closest price** to the BS strategy of the associated background complete market?"*

Motivated to answer this, Sottinen & Viitasaari (2018) [34] introduced the *conditional mean hedging* (CMH) method by employing the conditional laws from their prior paper [33]. This hedging is based on the coalitional average of the new model's value of portfolio, with respect to the coalitional average of the BS portfolio's value. Shokrollahi & Sottinen (2017) [32] studied this method for the *fractional* Black-Scholes model.

In our recent article [1], we studied the required conditional laws for the Volterra-jump models, which JD is a special case of. We aimed to find the CMH strategy when the underlying asset has the general JD model. However, considering the numerical results revealed a critical fact:

*"Even if the conditional means of some stochastic processes are equal, still the **minimum distance** of those processes is not guaranteed!"*

Therefore, we considered a stronger approach. The idea is to investigate a discrete strategy that has the minimum conditional difference with the BS strategy. This is the *conditional least-square hedging* (CLH). Surprisingly this stronger hedging strategy is possible for the JD model. Using the conditional formulas of our previous article, we calculate the (CLH) strategy for the JD model under transaction costs here.

The rest of this paper is as follows. In Section 2 we study the conditional prediction laws for the jump diffusion model. In Section 3 we consider the CMH strategies for this model under proportional transaction costs with a decision tree for the investor. In Section 4 we explain and investigate the CLH strategies for it. In Section 5 we investigate the explicit form of CLH strategies for the European call options. In Section 6 we visualize our finding by some simulations and figures.

2. Preliminaries

We consider the discounted pricing model where the riskless interest rate r is zero $(\mathrm{d}A = 0)$ and the risky asset S is given by the following jump–diffusion stochastic differential equation (SDE)

$$\frac{\mathrm{d}S_t}{S_{t^-}} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + \mathrm{d}J_t, \tag{2.1}$$

where W is the standard Brownian motion and J is an independent compound Poisson jump process with intensity λ and jump distribution F. In other words

$$J_t = \sum_{k=1}^{N_t} \xi_k,$$

where $N = (N_t)_{t\in[0,T]}$ is a Poisson process with intensity λ and the jumps ξ_k, $k \in \mathbb{N}$, are i.i.d. with common distribution F, and they are independent of the Poisson process N and the Brownian motion W. We denote

$$\begin{aligned} \epsilon_1 &= \mathbb{E}[\xi_k], \\ \epsilon_2 &= \mathbb{E}[\xi_k^2]. \end{aligned}$$

The path-wise solution of the SDE (2.1) is

$$S_t = S_0 \, \mathrm{e}^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t} \prod_{k=1}^{N_t} (1+\xi_k), \tag{2.2}$$

see e.g. [18]. Note that as the asset is positive $S_t > 0, \forall t \geq 0$, we must have $\xi_k > -1, \forall k \geq 1$.

Now, as in real world, continuous trading is impossible, we assume the trading only takes place at some given fixed time points $0 = t_0 < t_1 < \ldots < t_N = T$. The trader by the self-financing strategy would hedge at least once on each time period $[t_i, t_{i+1})$. So, the simplest strategy he can take is to hold the discrete amount

$$\pi_t^N = \sum_{i=0}^{N-1} \pi_{t_i}^N \mathbb{1}_{[t_i,t_{i+1})}(t), \tag{2.3}$$

of the risky asset S at time $t \geq 0$. Then, the value of the strategy π^N is the integral of (1.2) minus the proportional transaction cost of it. That is

$$V_t^{\pi^N,\kappa} = V_0^{\pi^N,\kappa} + \int_0^t \pi_u^N \, \mathrm{d}S_u - \int_0^t \kappa S_u |\mathrm{d}\pi_u^N|,$$

where $\kappa \in (0,1)$ is the proportion of transaction cost.

Definition 2.1. Consider a European option of the type $f(S_T)$, with convex or concave payoff f. Let π be its Black–Scholes strategy. We call the discrete-time strategy π^N is a *conditional mean hedging* (CMH) strategy, if for all trading times t_i,

$$\mathbb{E}\left[V_{t_{i+1}}^{\pi^N,\kappa} \,|\, \mathscr{F}_{t_i}\right] = \mathbb{E}\left[V_{t_{i+1}}^{\pi} \,|\, \mathscr{F}_{t_i}\right], \tag{2.4}$$

where $\mathscr{F}_{t_i}$ is the information filter, generated by the asset price process S up to time t_i.

To extract the strategy π^N for such a definition, we need to study those conditional expectations. The following lemmas provide the required conditional laws. To distinguish the gBm asset model, implied to the BS strategy in the background complete market, from the JD model of the incomplete market, we denote it by S^{BS}, and that is

$$\frac{\mathrm{d}S_t^{\mathrm{BS}}}{S_t^{\mathrm{BS}}} = \mu \, \mathrm{d}t + \sigma \, \mathrm{d}W_t, \tag{2.5}$$

with the solution

$$S_t^{\mathrm{BS}} = S_0^{\mathrm{BS}}\,\mathrm{e}^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t}, \tag{2.6}$$

and we note for all $t \geq 0$

$$S_t = S_t^{\mathrm{BS}} \prod_{j=1}^{N_t}(1+\xi_j). \tag{2.7}$$

Lemma 2.1 (Conditional Moments)**.** *For $t \geq u$, and $k \geq 1$ the conditional JD and gBm processes, i.e., $S_t(u) := S_t|\mathscr{F}_u$ and $S_t^{\mathrm{BS}}(u) := S_t^{\mathrm{BS}}|\mathscr{F}_u$ has the moments*

$$\widehat{S_t^k}(u) = \mathbb{E}\left[S_t^k\middle|\mathscr{F}_u\right] = S_u^k \exp\left[\left(k\mu + k(k-1)\frac{\sigma^2}{2} + \lambda\sum_{j=1}^{k}\binom{k}{j}\epsilon_j\right)(t-u)\right], \tag{2.8}$$

$$\widehat{(S_t^{\mathrm{BS}})^k}(u) = \mathbb{E}\left[(S_t^{\mathrm{BS}})^k\middle|\mathscr{F}_u\right] = (S_u^{\mathrm{BS}})^k \exp\left[\left(k\mu + k(k-1)\frac{\sigma^2}{2}\right)(t-u)\right], \tag{2.9}$$

where $\epsilon_j = \mathbb{E}[\xi^j]$. In particular for $k=1,2$

$$\hat{S}_t(u) = \mathbb{E}\left[S_t\middle|\mathscr{F}_u\right] = S_u \exp\Big[(\mu+\lambda\epsilon_1)(t-u)\Big], \tag{2.10}$$

$$\widehat{S_t^2}(u) = \mathbb{E}\left[S_t^2\middle|\mathscr{F}_u\right] = S_u^2 \exp\left[\Big(2\mu+\sigma^2+2\lambda\epsilon_1+\lambda\epsilon_2\Big)(t-u)\right]. \tag{2.11}$$

Proof.

$$\begin{aligned}
\widehat{S_t^k}(u) &= \mathbb{E}\left[S_t^k\middle|\mathscr{F}_u\right] \\
&= S_0^k\,\mathrm{e}^{k(\mu-\frac{\sigma^2}{2})t}\,\mathbb{E}\Big[\mathrm{e}^{k\sigma W_t}\Big|\mathscr{F}_u^W\Big]\,\mathbb{E}\Big[\prod_{j=1}^{N_t}(1+\xi_i)^k\Big|\mathscr{F}_u^{\xi,N}\Big] \\
&= S_0^k\,\mathrm{e}^{k(\mu-\frac{\sigma^2}{2})t}\,\mathrm{e}^{k\sigma W_u}\,\mathbb{E}\Big[\mathrm{e}^{k\sigma(W_t-W_u)}\Big|\mathscr{F}_u^W\Big] \\
&\times \prod_{j=1}^{N_u}(1+\xi_j)^k\,\mathbb{E}\Big[\prod_{j=N_u+1}^{N_t}(1+\xi_j)^k\Big|\mathscr{F}_u^{\xi,N}\Big] \\
&= S_u^k\,\mathrm{e}^{k(\mu-\frac{\sigma^2}{2})(t-u)}\,\mathbb{E}\Big[\mathrm{e}^{k\sigma W_{t-u}}\Big]\,\mathbb{E}\Big[\prod_{j=1}^{N_{t-u}}(1+\xi_j)^k\Big] \\
&= S_u^k\,\mathrm{e}^{(k\mu+k(k-1)\frac{\sigma^2}{2})(t-u)}\sum_{n=0}^{\infty}\mathbb{E}\Big[\prod_{j=1}^{n}(1+\xi_j)^k\Big]\,\mathbb{P}\big[N_{t-u}=n\big] \\
&= S_u\,\mathrm{e}^{(k\mu+k(k-1)\frac{\sigma^2}{2})(t-u)}\sum_{n=0}^{\infty}\frac{\lambda^n(t-u)^n}{n!}\,\mathrm{e}^{-\lambda(t-u)}\Big(\mathbb{E}[(1+\xi)^k]\Big)^n \\
&= S_u^k\,\mathrm{e}^{(k\mu+k(k-1)\frac{\sigma^2}{2}-\lambda)(t-u)}\sum_{n=0}^{\infty}\frac{\lambda^n(t-u)^n}{n!}\left(\sum_{j=0}^{k}\binom{k}{j}\mathbb{E}[\xi^j]\right)^n \\
&= S_u^k\,\mathrm{e}^{\left(k\mu+k(k-1)\frac{\sigma^2}{2}+\lambda\sum_{j=1}^{k}\binom{k}{j}\epsilon_j\right)(t-u)}.
\end{aligned}$$

□

Remark 2.1. Formula (2.10) is a special case of [1, Theorem 3.1], for $X = \sigma W + J$. However, the properties of Wiener process increments cause a better closed form for the conditional laws here.

Lemma 2.2 (Conditional Expectation)**.** *For $t \geq u$, the conditional processes $f_t^{\mathrm{BS}}(u) := f(t, S_t^{\mathrm{BS}})|\mathscr{F}_u$ and $S.f_t^{\mathrm{BS}}(u) := S_t \cdot f(t, S_t^{\mathrm{BS}})|\mathscr{F}_u$, we have the following rules*

$$\widehat{f_t^{\mathrm{BS}}}(u) = \mathbb{E}\left[f(t, S_t^{\mathrm{BS}})\middle|\mathscr{F}_u\right] = \int_{-\infty}^{\infty} f\left(t, S_u^{\mathrm{BS}} \mathrm{e}^{(\mu-\frac{\sigma^2}{2})(t-u)+\sigma z}\right)\phi(z;0;t-u)\mathrm{d}z, \tag{2.12}$$

$$\widehat{S.f_t^{\mathrm{BS}}}(u) = \mathbb{E}\left[S_t \cdot f(t, S_t^{\mathrm{BS}})\middle|\mathscr{F}_u\right] = S_u \mathrm{e}^{(\mu+\lambda\epsilon_1-\frac{\sigma^2}{2})(t-u)} \int_{-\infty}^{\infty} \mathrm{e}^{\sigma z} f\left(t, S_u^{\mathrm{BS}} \mathrm{e}^{(\mu-\frac{\sigma^2}{2})(t-u)+\sigma z}\right)\phi(z;0;t-u)\mathrm{d}z, \tag{2.13}$$

where $\phi(z;a;b^2)$ is the normal density function of mean a and variance b^2, and G^{*n} is the nth fold convolution of the distribution $\zeta_k = \log(1+\xi_k) \sim G$.

Proof. We have

$$\begin{aligned}
\widehat{f_t^{\mathrm{BS}}}(u) &= \mathbb{E}\left[f(t, S_t^{\mathrm{BS}})\middle|\mathscr{F}_u\right] \\
&= \mathbb{E}\left[f\left(t, S_u^{\mathrm{BS}}\, \mathrm{e}^{(\mu-\frac{\sigma^2}{2})(t-u)\,+\,\sigma(W_t-W_u)}\right)\middle|\mathscr{F}_u^{W,J}\right] \\
&= \mathbb{E}\left[f\left(t, S_u^{\mathrm{BS}}\, \mathrm{e}^{(\mu-\frac{\sigma^2}{2})(t-u)\,+\,\sigma(W_{t-u})}\right)\middle|\mathscr{F}_u^{W}\right]. \qquad (2.14)
\end{aligned}$$

Then we note S_u^{BS} and $W_{t-u} \overset{d}{=} W_t - W_u$ are respectively measurable and independent of $\mathscr{F}_u^W$, the expectation is indeed just with respect to the variable W_{t-u}. So, using the density function of W_{t-u} which is $\phi(z;0;t-u)$, we can continue (2.14) as

$$= \int_{-\infty}^{\infty} f\left(t, S_u^{\mathrm{BS}} \mathrm{e}^{(\mu-\frac{\sigma^2}{2})(t-u)+\sigma z}\right)\phi(z;0;t-u)\mathrm{d}z.$$

Next, we note W, N, ξ_k are all independent, $W_{t-u} \overset{d}{=} W_t - W_u$ and $N_{t-u} \overset{d}{=} N_t - N_u$ are both independent from $\mathscr{F}_u = \mathscr{F}_u^{W,N,\xi}$, and also S_u and S_u^{BS} both are $\mathscr{F}_u$-measurable. So,

$$\begin{aligned}
\widehat{S.f_t^{\mathrm{BS}}}(u) &= \mathbb{E}\left[S_t \cdot f(t, S_t^{\mathrm{BS}})\middle|\mathscr{F}_u\right] \\
&= \mathbb{E}\Bigg[S_u \mathrm{e}^{(\mu-\frac{\sigma^2}{2})(t-u)+\sigma(W_t-W_u)} \prod_{j=N_u}^{N_t}(1+\xi_j) \\
&\quad \times f\left(t, S_u^{\mathrm{BS}} \mathrm{e}^{(\mu-\frac{\sigma^2}{2})(t-u)+\sigma(W_t-W_u)}\right)\Bigg|\mathscr{F}_u^{W,N,\xi}\Bigg]
\end{aligned}$$

$$= S_u e^{(\mu - \frac{\sigma^2}{2})(t-u)} \mathbb{E}\left[\prod_{j=1}^{N_{t-u}} (1+\xi_j) \Big| \mathscr{F}_u^{N,\xi}\right]$$

$$\times \mathbb{E}\left[e^{\sigma W_{t-u}} f\Big(t, S_u^{\mathrm{BS}} e^{(\mu-\frac{\sigma^2}{2})(t-u)+\sigma W_{t-u}}\Big) \Big| \mathscr{F}_u^W\right]$$

$$= S_u e^{(\mu+\lambda\epsilon_1-\frac{\sigma^2}{2})(t-u)} \mathbb{E}\left[e^{\sigma W_{t-u}} f\Big(t, S_u^{\mathrm{BS}} e^{(\mu-\frac{\sigma^2}{2})(t-u)+\sigma W_{t-u}}\Big) \Big| \mathscr{F}_u^W\right],$$

and again the expectation is just with respect to the variable W_{t-u}. This proves (2.13). □

3. Conditional Mean Strategies

From now on, we consider Δ as the backward difference operator i.e. $\Delta t_{i+1} = t_{i+1} - t_i$ and for function f it is $\Delta f_{t_{i+1}} = f_{t_{i+1}} - f_{t_i}$.

Lemma 3.1 (Conditional Gains)**.** *Let S be left continuous at $\{t_i\}_{i\geq 0}$, i.e. no jump happens right before the transaction payment time points, then*

$$\hat{S}_{t_{i+1}}(t_i) = S_{t_i} \exp\Big[(\mu + \lambda\epsilon_1)\Delta t_{i+1}\Big], \tag{3.1}$$

$$\widehat{S^2}_{t_{i+1}}(t_i) = S_{t_i}^2 \exp\Big[\Big(2\mu + \sigma^2 + 2\lambda\epsilon_1 + \lambda\epsilon_2\Big)\Delta t_{i+1}\Big], \tag{3.2}$$

$$\begin{aligned} \widehat{V}^{\pi}_{t_{i+1}}(t_i) &= \mathbb{E}\left[g(t_{i+1}, S^{\mathrm{BS}}_{t_{i+1}}) \Big| \mathscr{F}_{t_i}\right] \\ &= \int_{-\infty}^{\infty} g\Big(t_{i+1}, S^{\mathrm{BS}}_{t_i} e^{(\mu-\frac{\sigma^2}{2})\Delta t_{i+1}+\sigma z}\Big)\phi(z;0;\Delta t_{i+1})\mathrm{d}z \end{aligned} \tag{3.3}$$

$$\begin{aligned} \widehat{S.V^{\pi}}_{t_{i+1}}(t_i) &= \mathbb{E}\left[S_{t_{i+1}} g(t_{i+1}, S^{\mathrm{BS}}_{t_{i+1}}) \Big| \mathscr{F}_{t_i}\right] \\ &= S_{t_i} e^{(\mu+\lambda\epsilon_1-\frac{\sigma^2}{2})\Delta t_{i+1}} \int_{-\infty}^{\infty} e^{\sigma z} g\Big(t_{i+1}, S^{\mathrm{BS}}_{t_i} e^{(\mu-\frac{\sigma^2}{2})\Delta t_{i+1}+\sigma z}\Big)\phi(z;0;\Delta t_{i+1})\mathrm{d}z \end{aligned} \tag{3.4}$$

$$V^{\pi^N,\kappa}_{t_{i+1}} = V^{\pi^N,\kappa}_{t_i} + \pi^N_{t_i}\Delta S_{t_{i+1}} - \kappa S_{t_{i+1}}|\Delta\pi^N_{t_{i+1}}|, \tag{3.5}$$

$$\hat{V}^{\pi^N,\kappa}_{t_{i+1}}(t_i) = V^{\pi^N,\kappa}_{t_i} + \pi^N_{t_i}\Delta\hat{S}_{t_{i+1}}(t_i) - \kappa\hat{S}_{t_{i+1}}(t_i)|\Delta\pi^N_{t_{i+1}}|, \tag{3.6}$$

where $\Delta\hat{S}_{t_{i+1}}(t_i) = \hat{S}_{t_{i+1}}(t_i) - S_{t_i}$.

Proof. (3.1) and (3.2) are straight results of Lemma 2.1, and formulas (3.3) and (3.4) are the results of Lemma 2.2. To prove (3.5), we note that from (2.4)

$$\begin{aligned} V^{\pi^N,\kappa}_{t_{i+1}} &= V^{\pi^N,\kappa}_{t_i} + \int_{t_i}^{t_{i+1}} \pi^N_u \,\mathrm{d}S_u - \int_{t_i}^{t_{i+1}} \kappa S_u |\mathrm{d}\pi^N_u| \\ &= V^{\pi^N,\kappa}_{t_i} + \pi^N_{t_i}(S_{t_{i+1}} - S_{t_i}) - \kappa \lim_{|\Lambda_n|\to 0} \sum_{j=0}^{n-1} S_{u_j^*}|\Delta\pi^N_{u_{j+1}}|, \end{aligned} \tag{3.7}$$

where $\Lambda_n : t_i = u_0, u_1, \ldots, u_{n-1}, u_n = t_{i+1}$ is a partition of $n+1$ time points on $[t_i, t_{i+1}]$, that $u_j^* \in [u_j, u_{j+1}]$, and $|\Lambda_n| = \max_{0\leq j\leq n-1}|\Delta u_j|$. Now, as the asset price S is left continuous at $\{t_i\}_{i\geq 0}$, we can continue (3.7) as folloing

$$= V^{\pi^N,\kappa}_{t_i} + \pi^N_{t_i}(S_{t_{i+1}} - S_{t_i}) - \kappa S_{t_{i+1}}|\Delta\pi^N_{t_{i+1}}|.$$

Taking the conditional expectation $\mathbb{E}[\,\cdot\,|\mathscr{F}_{t_i}]$ to the right side of the final equality above, we have (3.6). □

Theorem 3.1 (Conditional Mean Hedging)**.** *If the asset price S is left continuous, π^N is the CMH strategy for the European option of type $f(S_T)$ with convex or concave positive payoff function f, and transaction costs proportion κ, if and only if*

$$|\pi^N_{t_{i+1}} - \pi^N_{t_i}| = \frac{V^{\pi^N,\kappa}_{t_i} - \hat{V}^{\pi}_{t_{i+1}}(t_i) + \pi^N_{t_i}\Delta\hat{S}_{t_{i+1}}(t_i)}{\kappa\hat{S}_{t_{i+1}}(t_i)}, \tag{3.8}$$

for all $i = 0, \ldots, n-1$.

Proof. The definition 2.1 and equation (2.4), are equivalent to have (3.3) and (3.6) equal. This returns the equation (3.8). □

Remark 3.1. Taking a long position (buying) of S for the period $[t_i, t_{i+1})$ means $\Delta\pi^N_{t_i} > 0$, and so from (3.8)

$$\pi^N_{t_{i+1}} = \pi^N_{t_i} + \frac{V^{\pi^N,\kappa}_{t_i} - \hat{V}^{\pi}_{t_{i+1}}(t_i) + \pi^N_{t_i}\Delta\hat{S}_{t_{i+1}}(t_i)}{\kappa\hat{S}_{t_{i+1}}(t_i)}. \tag{3.9}$$

On the other hand, a short position (selling) of S for the period $[t_i, t_{i+1})$ means $\Delta\pi^N_{t_i} < 0$, hence from (3.8)

$$\pi^N_{t_{i+1}} = \pi^N_{t_i} - \frac{V^{\pi^N,\kappa}_{t_i} - \hat{V}^{\pi}_{t_{i+1}}(t_i) + \pi^N_{t_i}\Delta\hat{S}_{t_{i+1}}(t_i)}{\kappa\hat{S}_{t_{i+1}}(t_i)}. \tag{3.10}$$

These cause a binary decision tree of strategies that a trader can take on transaction time intervals as follows.

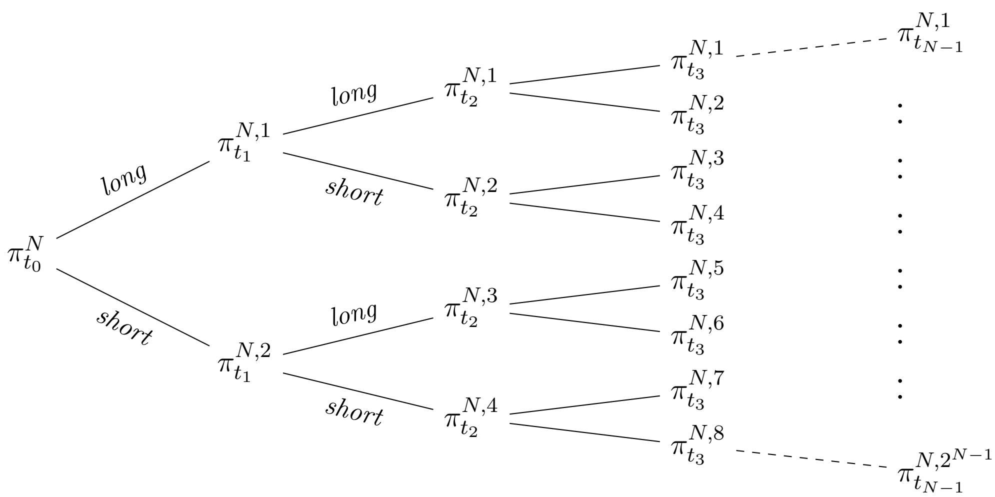

FIGURE 1. The decision tree of CMH method.

4. Conditional Least-Square Strategies

Remark 4.1 (Interpretations)**.** Although Theorem 3.1 results in an explicit form for the CMH strategy, there are some considerable points in it:

1. Denote

$$\text{(Empirical)} \quad \theta^N_{t_i} = V^{\pi^N,\kappa}_{t_i} + \pi^N_{t_i}\Delta\hat{S}_{t_{i+1}}(t_i), \tag{4.1}$$

$$\text{(Theoretical)} \quad \theta^{\pi}_{t_i} = \hat{V}^{\pi}_{t_{i+1}}(t_i). \tag{4.2}$$

Theorem 3.1 shows the CMH strategy is plausible only if $\theta^N_{t_i} - \theta^\pi_{t_i} \geq 0$, i.e., the empirical strategy's values always exceed the theoretical values for $i = 0, \dots, n-1$ all $(\theta^N_{t_i} \geq \theta^\pi_{t_i})$. However, this theorem vanishes if just for some $0 \leq j \leq n-1$ we happen to $\theta^N_{t_j} < \theta^\pi_{t_j}$.

2. On the other hand, even if we successfully extract the CMH strategy, that is matching the conditional means of the variables $V^{\pi^N,\kappa}_{t_{i+1}}$ and $V^\pi_{t_{i+1}}$. However, this strategy does not guarantee the minimum conditional distance of these variables necessarily.

3. Since under the transactional costs the perfect hedging is not possible. However, we note that applying the JD model (1.5) for the risky asset price, is indeed a generalization of the gBm model (1.1) by adding a jump to it. So, for a trader in JD incomplete market, it is natural to investigate the closest discrete strategy to the perfect hedging strategy in the background gBm complete market, i.e., the BS strategy. Now, if one consider the difference (distance) as the conditional least-square norm, then the mathematical formulation of this methodology for hedging forms the following definition as a modified and alternative framework instead of the CMH.

Definition 4.1 (Conditional Least Square Hedging)**.** Let $f(S_T)$ be a financial derivative with convex or concave payoff f. Let π be its Black–Scholes strategy. We call the discrete-time strategy ${}^*\pi^N$ is a *conditional least square hedging* (CLH) strategy, if for all trading times t_i,

$$(4.3) \qquad {}^*\pi^N_{t_{i+1}} \in \underset{\pi^N_{t_{i+1}} \in \mathbb{R}}{\operatorname{argmin}}\, \mathbb{E}\left[\left(V^{\pi^N,\kappa}_{t_{i+1}} - V^\pi_{t_{i+1}}\right)^2 \,\Big|\, \mathscr{F}_{t_i}\right].$$

Here $\mathscr{F}_{t_i}$ is the information filter, generated by the asset price S upto time t_i.

Lemma 4.1. *Consider the problem*

$$(4.4) \qquad \min_{x \in \mathbb{R}} f(x) = a(x - x_0)^2 + b|x - x_0| + c,$$

where $a > 0$, $b, x_0 \in \mathbb{R}$, *and* $c \geq 0$.
i) If $b \geq 0$, *it has the unique solution* $x^* = x_0$ *with* $f(x^*) = c$,
ii) If $b < 0$, *it has the two solutions* $x^* = x_0 \pm b/2a$ *with* $f(x^*) = 0$.

Proof. First, if $b \geq 0$ then f is a convex function which has its unique minimum on $x^* = x_0$ with $f(x^*) = c$. Second, if $b < 0$, we have

$$(4.5) \qquad f(x) = \begin{cases} a(x-x_0)^2 - b(x-x_0) + c & ; \quad x < x_0 \\ c & ; \quad x = x_0 \\ a(x-x_0)^2 + b(x-x_0) + c & ; \quad x > x_0 \end{cases}$$

and so

$$(4.6) \qquad f'(x) = \begin{cases} 2a(x-x_0) - b & ; \quad x < x_0 \\ 2a(x-x_0) + b & ; \quad x > x_0 \end{cases}$$

and $f''(x) \equiv 2a > 0$ on all $x \in \mathbb{R}$. So, f has minimums around the roots of its derivative f', which are $x^* = x_0 \pm b/2a$ with values $f(x^*) = 0$. □

Theorem 4.1 (CLH Strategy)**.** *Denote*

$$(4.7) \qquad U_{t_i} = \left(V^{\pi^N,\kappa}_{t_i} - \pi^N_{t_i} S_{t_i}\right) \hat{S}_{t_{i+1}}(t_i) - \widehat{S.V^\pi}_{t_{i+1}}(t_i) + \pi^N_{t_i} \widehat{S^2}_{t_{i+1}}(t_i).$$

If the asset price S is left continuous, for the European option of the type $f(S_T)$ with convex or concave positive payoff function f, and proportion of transaction costs κ the CLH strategy admits the following recursive equations:
(I) If $U_{t_i} \leq 0$, then ${}^*\pi^N_{t_{i+1}} = \pi^N_{t_i}$,
(II) If $U_{t_i} > 0$, then for a long position on t_{i+1}

$$(4.8)\quad {}^*\pi^N_{t_{i+1}}(long) = \pi^N_{t_i} + \frac{U_{t_i}}{\kappa \widehat{S^2}_{t_{i+1}}(t_i)} = \pi^N_{t_i} + \frac{1}{\kappa}\left[\pi^N_{t_i} - \frac{\widehat{S.V^\pi}_{t_{i+1}}(t_i) - (V^{\pi^N,\kappa}_{t_i} - \pi^N_{t_i} S_{t_i})\hat{S}_{t_{i+1}}(t_i)}{\widehat{S^2}_{t_{i+1}}(t_i)}\right],$$

and for a short position on t_{i+1}

$$(4.9)\quad {}^*\pi^N_{t_{i+1}}(short) = \pi^N_{t_i} - \frac{U_{t_i}}{\kappa \widehat{S^2}_{t_{i+1}}(t_i)} = \pi^N_{t_i} - \frac{1}{\kappa}\left[\pi^N_{t_i} - \frac{\widehat{S.V^\pi}_{t_{i+1}}(t_i) - (V^{\pi^N,\kappa}_{t_i} - \pi^N_{t_i} S_{t_i})\hat{S}_{t_{i+1}}(t_i)}{\widehat{S^2}_{t_{i+1}}(t_i)}\right].$$

Proof. By Lemma 3.1

$$V^{\pi^N,\kappa}_{t_{i+1}} - V^\pi_{t_{i+1}} = V^{\pi^N,\kappa}_{t_i} - V^\pi_{t_{i+1}} + \pi^N_{t_i}(S_{t_{i+1}} - S_{t_i}) - \kappa S_{t_{i+1}}\left|\pi^N_{t_{i+1}} - \pi^N_{t_i}\right|,$$

and so

$$\begin{aligned}\left(V^{\pi^N,\kappa}_{t_{i+1}} - V^\pi_{t_{i+1}}\right)^2 &= \kappa^2 S^2_{t_{i+1}}\left(\pi^N_{t_{i+1}} - \pi^N_{t_i}\right)^2 \\ &\quad - 2\kappa S_{t_{i+1}}\left\{V^{\pi^N,\kappa}_{t_i} - V^\pi_{t_{i+1}} + \pi^N_{t_i}(S_{t_{i+1}} - S_{t_i})\right\}\left|\pi^N_{t_{i+1}} - \pi^N_{t_i}\right| \\ &\quad + \left(V^{\pi^N,\kappa}_{t_i} - V^\pi_{t_{i+1}} + \pi^N_{t_i}(S_{t_{i+1}} - S_{t_i})\right)^2,\end{aligned}$$

and taking the conditional expectation we have

$$\begin{aligned}&\mathbb{E}\left[\left(V^{\pi^N,\kappa}_{t_{i+1}} - V^\pi_{t_{i+1}}\right)^2 \middle| \mathscr{F}_{t_i}\right] = \kappa^2 \widehat{S^2}_{t_{i+1}}\left(\pi^N_{t_{i+1}} - \pi^N_{t_i}\right)^2 \\ &- 2\kappa\left\{\left(V^{\pi^N,\kappa} - \pi^N_{t_i} S_{t_i}\right)\hat{S}_{t_{i+1}}(t_i) - \widehat{S.V^\pi}_{t_{i+1}}(t_i) + \pi^N_{t_i}\widehat{S^2}_{t_{i+1}}(t_i)\right\}\left|\pi^N_{t_{i+1}} - \pi^N_{t_i}\right| \\ &+ \mathbb{E}\left[\left(V^{\pi^N,\kappa}_{t_i} - V^\pi_{t_{i+1}} + \pi^N_{t_i}(S_{t_{i+1}} - S_{t_i})\right)^2 \middle| \mathscr{F}_{t_i}\right] \\ &= a_i\left(\pi^N_{t_{i+1}} - \pi^N_{t_i}\right)^2 + b_i\left|\pi^N_{t_{i+1}} - \pi^N_{t_i}\right| + c_i,\end{aligned}$$

and this is a function of the form (4.4). So, by the Lemma 4.1, if $b_i \geq 0$ $(U_i \leq 0)$ then it has a unique minimum at ${}^*\pi^N_{t_{i+1}} = \pi^N_{t_i}$. If $b_i < 0$ $(U_i > 0)$ then it has twople minimums at ${}^*\pi^N_{t_{i+1}} = \pi^N_{t_i} \pm b_i/2a_i = \pi^N_{t_i} \mp U_i/\kappa\widehat{S^2}_{t_{i+1}}(t_i)$. □

Remark 4.2. Similar to the CMH method, here also we will face a decision tree of the optimal strategies of CLH method. However, the decision tree of CLH method is different. In some branches that strategy does not change, the branch continues straight but not in a binary shape. In other words, we face a decision tree similar to Figure 2.

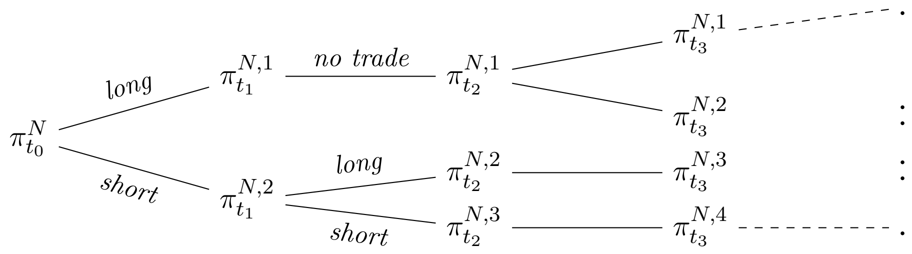

FIGURE 2. The decision tree of CLH method.

Remark 4.3. Denote

$$\text{(Empirical)} \quad \ell_{t_i}^{N} = \left(V_{t_i}^{\pi^N,\kappa} - \pi_{t_i}^{N} S_{t_i}\right) \hat{S}_{t_{i+1}}(t_i) + \pi_{t_i}^{N} \widehat{S^2}_{t_{i+1}}(t_i), \tag{4.10}$$

$$\text{(Theoretical)} \quad \ell_{t_i}^{\pi} = \widehat{S.V^{\pi}}_{t_{i+1}}(t_i). \tag{4.11}$$

As $U_{t_i} = \ell_{t_i}^{N} - \ell_{t_i}^{\pi}$, further than the explicit strategies, Theorem 4.1 reveals a test. In other words, it indicates that if the empirical value remains less or equal to the theoretical value ($\ell_{t_i}^{N} \leq \ell_{t_i}^{\pi}$), the trader does not need to change the volume of the strategy to stay close to the BS strategy's value. However, if the empirical value exceeds the theoretical value ($\ell_{t_i}^{N} > \ell_{t_i}^{\pi}$), in order to stay close to BS strategy's value, the trader should update the volume of the strategy according to the formulas (4.8) and (4.9).

5. European Vanilla Call Option

Next, we are interested in calculating the European Call option case. To do this, first we need the following lemma. From now on, we denote the density, cumulative probability, and the expectation functions of the normal standard distribution respectively by φ, Φ, $\mathbb{E}_z$, and the expectation function of the normal distribution $\mathcal{N}(\mu, \sigma^2)$ by $\mathbb{E}_z^{\mu,\sigma^2}$, i.e., for all $f : \mathbb{R} \to \mathbb{R}$

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{z^2}{2}}, \qquad \Phi(z) = \int_{-\infty}^{z} \varphi(u)\,\mathrm{d}u = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{e}^{-\frac{u^2}{2}}\,\mathrm{d}u,$$
$$\mathbb{E}_z\big[f(z)\big] = \int_{-\infty}^{\infty} f(u)\,\varphi(u)\,\mathrm{d}u = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\,\mathrm{e}^{-\frac{u^2}{2}}\,\mathrm{d}u,$$
$$\mathbb{E}_z^{\mu,\sigma^2}\big[f(z)\big] = \int_{-\infty}^{\infty} f(u)\,\phi(u;\mu;\sigma^2)\,\mathrm{d}u = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} f(u)\,\mathrm{e}^{-\frac{(u-\mu)^2}{2\sigma^2}}\,\mathrm{d}u,$$

providing the right side integrals exist.

Lemma 5.1. *For all given real values* $a, b, \alpha, \mu \in \mathbb{R}$ *and* $\sigma > 0$

$$\int_{-\infty}^{\infty} \mathrm{e}^{\alpha z}\,\Phi(az+b)\,\phi(z;\mu;\sigma^2)\,\mathrm{d}z \;=\; \mathrm{e}^{\alpha\mu + \frac{\alpha^2\sigma^2}{2}}\,\Phi\left(\frac{a(\mu+\alpha\sigma^2)+b}{\sqrt{1+a^2\sigma^2}}\right), \tag{5.1}$$

in other word

$$\mathbb{E}_z^{\mu,\sigma^2}\Big[\mathrm{e}^{\alpha z}\,\Phi(az+b)\Big] = \mathrm{e}^{\alpha\mu + \frac{\alpha^2\sigma^2}{2}}\,\Phi\left(\frac{a(\mu+\alpha\sigma^2)+b}{\sqrt{1+a^2\sigma^2}}\right). \tag{5.2}$$

In particular

$$\mathbb{E}_z\Big[\mathrm{e}^{\alpha z}\,\Phi(az+b)\Big] = \int_{-\infty}^{\infty}\mathrm{e}^{\alpha z}\,\Phi(az+b)\,\varphi(z)\,\mathrm{d}z \;=\; \mathrm{e}^{\frac{\alpha^2}{2}}\,\Phi\left(\frac{a\alpha+b}{\sqrt{1+a^2}}\right), \tag{5.3}$$

$$\mathbb{E}_z^{\mu,\sigma^2}\Big[\Phi(az+b)\Big] = \Phi\left(\frac{a\mu+b}{\sqrt{1+a^2\sigma^2}}\right). \tag{5.4}$$

Proof. By changing the variable $u=\frac{z-\mu}{\sigma}$ we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\mathrm{e}^{\alpha z}\,\Phi(az+b)\,\phi(z;\mu;\sigma^2)\,\mathrm{d}z\\
&=\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\mathrm{e}^{\alpha z}\,\Phi(az+b)\,\mathrm{e}^{-\frac{(z-\mu)^2}{2\sigma^2}}\,\mathrm{d}z\\
&=\frac{\mathrm{e}^{\alpha\mu}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(a\sigma u+a\mu+b)\,\mathrm{e}^{\alpha\sigma u-\frac{u^2}{2}}\,\mathrm{d}u\\
&=\frac{\mathrm{e}^{\alpha\mu+\frac{\alpha^2\sigma^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(a\sigma u+a\mu+b)\,\mathrm{e}^{-\frac{(u-\alpha\sigma)^2}{2}}\,\mathrm{d}u\\
&=\mathrm{e}^{\alpha\mu+\frac{\alpha^2\sigma^2}{2}}\int_{-\infty}^{\infty}\Phi(a\sigma u+a\mu+b)\,\phi(u;\alpha\sigma;1)\,\mathrm{d}u.
\end{aligned}\tag{5.5}$$

Next, we not $\Phi(z)=\frac{1}{2}\left\{1+\mathrm{erf}\left(\frac{z}{\sqrt{2}}\right)\right\}$ for $\mathrm{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z\mathrm{e}^{-u^2}\,\mathrm{d}u$. So, one can continue (5.5) following

$$\begin{aligned}
&=\mathrm{e}^{\alpha\mu+\frac{\alpha^2\sigma^2}{2}}\int_{-\infty}^{\infty}\frac{1}{2}\left\{1+\mathrm{erf}\left(\frac{a\sigma u+a\mu+b}{\sqrt{2}}\right)\right\}\,\phi(u;\alpha\sigma;1)\,\mathrm{d}u\\
&=\frac{\mathrm{e}^{\alpha\mu+\frac{\alpha^2\sigma^2}{2}}}{2}\left\{1+\int_{-\infty}^{\infty}\mathrm{erf}\left(\frac{a\sigma u+a\mu+b}{\sqrt{2}}\right)\,\phi(u;\alpha\sigma;1)\,\mathrm{d}u\right\}.
\end{aligned}\tag{5.6}$$

Now, from [25, Section 4.3, Formula 13], for arbitrary $A,B,m\in\mathbb{R}$ and $\nu>0$

$$\int_{-\infty}^{\infty}\mathrm{erf}(Au+B)\,\phi(u;m;\nu^2)\,\mathrm{d}u=\mathrm{erf}\left(\frac{Am+B}{\sqrt{1+2A^2\nu^2}}\right).$$

So, we can continue (5.6) as following

$$\begin{aligned}
&=\frac{\mathrm{e}^{\alpha\mu+\frac{\alpha^2\sigma^2}{2}}}{2}\left\{1+\mathrm{erf}\left(\frac{a\alpha\sigma^2+a\mu+b}{\sqrt{1+a^2\sigma^2}}\Big/\sqrt{2}\right)\right\}\\
&=\mathrm{e}^{\alpha\mu+\frac{\alpha^2\sigma^2}{2}}\,\Phi\left(\frac{a(\mu+\alpha\sigma^2)+b}{\sqrt{1+a^2\sigma^2}}\right).
\end{aligned}$$

□

Corollary 5.1 (European Call Option)**.** *For the European call option with left continuous underlying asset price S and strike-price K, denote*

$$a_i=\frac{\sigma\Delta t_{i+1}}{\sqrt{T-t_{i+1}}},\quad c_i=\sqrt{\frac{\Delta t_{i+1}}{T-t_{i+1}}},\quad b_i^-=b_i^+-\sigma\sqrt{T-t_{i+1}},$$

$$b_i^+=\frac{\ln\frac{S_{ti}^{\mathrm{BS}}}{K}+(\mu-\frac{\sigma^2}{2})\Delta t_{i+1}+\frac{\sigma^2}{2}(T-t_{i+1})}{\sigma\sqrt{T-t_{i+1}}},$$

then

$$\ell_{t_i}^{\text{Call}} = \widehat{S.V}_{t_{i+1}}^{\text{Call}}(t_i) \tag{5.7}$$

$$= S_{t_i} e^{(\mu+\lambda\epsilon_1)\Delta t_{i+1}} \left[S_{t_i}^{\text{BS}} e^{(\mu+\sigma^2)\Delta t_{i+1}} \Phi\left(\frac{2a_i + b_i^+}{\sqrt{1+c_i^2}}\right) - K\Phi\left(\frac{a_i + b_i^-}{\sqrt{1+c_i^2}}\right)\right],$$

and if $\ell_{t_i}^N \le \ell_{t_i}^{\text{Call}}$, *the CLH strategy admits* ${}^*\pi_{t_{i+1}}^N = \pi_{t_i}^N$. *If* $\ell_{t_i}^N > \ell_{t_i}^{\text{Call}}$ *then*

$$ {}^*\pi_{t_{i+1}}^N(long) = \pi_{t_i}^N + \frac{\ell_{t_i}^N - \ell_{t_i}^{\text{Call}}}{\kappa \widehat{S^2}_{t_{i+1}}(t_i)}, \quad {}^*\pi_{t_{i+1}}^N(short) = \pi_{t_i}^N - \frac{\ell_{t_i}^N - \ell_{t_i}^{\text{Call}}}{\kappa \widehat{S^2}_{t_{i+1}}(t_i)}. \tag{5.8}$$

Proof. For the European call option with zero rate riskless asset we have the Black–Scholes solution to the equation (1.3) is

$$g_{\text{Call}}(t, S_t^{\text{BS}}) = S_t^{\text{BS}}\Phi(d_t^+) - K\Phi(d_t^-),$$
$$d_t^+ = \frac{\ln\frac{S_t^{\text{BS}}}{K} + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}},$$
$$d_t^- = d_t^+ - \sigma\sqrt{T-t}.$$

So by Lemma 3.1

$$\ell_{t_i}^{\text{Call}} = \widehat{S.V}_{t_{i+1}}^{\text{Call}}(t_i) = \mathbb{E}\left[S_{t_{i+1}} g_{\text{Call}}(t_{i+1}, S_{t_{i+1}}^{\text{BS}}) \middle| \mathscr{F}_{t_i}\right] \tag{5.9}$$
$$= S_{t_i} e^{(\mu+\lambda\epsilon_1-\frac{\sigma^2}{2})\Delta t_{i+1}}$$
$$\times \int_{-\infty}^{\infty} e^{\sigma z} g_{\text{Call}}(t_{i+1}, S_{t_{i+1}}^{\text{BS}})\phi(z;0;\Delta t_{i+1})\mathrm{d}z$$
$$= S_{t_i} e^{(\mu+\lambda\epsilon_1-\frac{\sigma^2}{2})\Delta t_{i+1}}$$
$$\times \int_{-\infty}^{\infty} e^{\sigma z}\Big(S_{t_{i+1}}\Phi(d_{t_{i+1}}^+) - K\Phi(d_{t_{i+1}}^-)\Big)\phi(z;0;\Delta t_{i+1})\mathrm{d}z. \tag{5.10}$$

Here the inner integral is

$$\mathbb{E}_z^{0,\Delta t_{i+1}}\left[e^{\sigma z}\Big(S_{t_{i+1}}^{\text{BS}}\Phi(d_{t_{i+1}}^+) - K\Phi(d_{t_{i+1}}^-)\Big)\right]$$
$$= S_{t_i}^{\text{BS}} e^{(\mu-\frac{\sigma^2}{2})\Delta t_{i+1}}\ \mathbb{E}_z^{0,\Delta t_{i+1}}\left[e^{2\sigma z}\Phi\left(\frac{\ln\frac{S_{t_i}^{\text{BS}} e^{(\mu-\frac{\sigma^2}{2})\Delta t_{i+1}+\sigma z}}{K} + \frac{\sigma^2}{2}(T-t_{i+1})}{\sigma\sqrt{T-t_{i+1}}}\right)\right]$$
$$- K\ \mathbb{E}_z^{0,\Delta t_{i+1}}\left[e^{\sigma z}\Phi\left(\frac{\ln\frac{S_{t_i}^{\text{BS}} e^{(\mu-\frac{\sigma^2}{2})\Delta t_{i+1}+\sigma z}}{K} - \frac{\sigma^2}{2}(T-t_{i+1})}{\sigma\sqrt{T-t_{i+1}}}\right)\right]$$
$$= S_{t_i}^{\text{BS}} e^{(\mu-\frac{\sigma^2}{2})\Delta t_{i+1}}\ \mathbb{E}_z^{0,\Delta t_{i+1}}\left[e^{2\sigma z}\Phi\left(\frac{\ln\frac{S_{t_i}^{\text{BS}}}{K} + (\mu-\frac{\sigma^2}{2})\Delta t_{i+1} + \sigma z + \frac{\sigma^2}{2}(T-t_{i+1})}{\sigma\sqrt{T-t_{i+1}}}\right)\right]$$
$$- K\ \mathbb{E}_z^{0,\Delta t_{i+1}}\left[e^{\sigma z}\Phi\left(\frac{\ln\frac{S_{t_i}^{\text{BS}}}{K} + (\mu-\frac{\sigma^2}{2})\Delta t_{i+1} + \sigma z - \frac{\sigma^2}{2}(T-t_{i+1})}{\sigma\sqrt{T-t_{i+1}}}\right)\right],$$

applying Lemma 5.1 we can continue

$$= S^{\mathrm{BS}}_{t_i} \mathrm{e}^{(\mu+\frac{3}{2}\sigma^2)\Delta t_{i+1}+y} \Phi\left(\frac{2a_i + b_i^+}{\sqrt{1+c_i^2}}\right) - K\mathrm{e}^{\frac{\sigma^2}{2}\Delta t_{i+1}} \Phi\left(\frac{a_i + b_i^-}{\sqrt{1+c_i^2}}\right).$$

Now, by substituting this to (5.10) proves (5.7), and applying this result to the Theorem 4.1 and Remark 4.3 proves the rest of this corollary. □

6. Simulations

In this section, we simulate the result of Corollary 5.1. First we consider the case $\xi \equiv -0.5$, i.e., pure negative Poisson jumps in Figure 3 and its decision tree in Figure 4. Then, we consider the case $\xi \equiv 0.5$, the pure positive Poisson jumps and its results are given in Figures 5 and 6. The simulations are done for $T = 12$ months (1 year), including 5 transaction payment times, with $\kappa = 0.1$. In both simulations we consider $\mu = 0.15, \sigma = 0.25$ and $\lambda = 0.3$ per month.

Remark 6.1. Considering the simulations and figures, one can see in branches that the optimal strategy ${}^*\pi^N_{t_{i+1}}$ changes, it is diverging from $\pi^N_{t_i}$ rapidly. To explain the reason of this, we must note to the equations (3.5) and (4.3). Indeed, we aim to minimize the difference of

$$V^{\pi^N,\kappa}_{t_{i+1}} = V^{\pi^N,\kappa}_{t_i} + \pi^N_{t_i}(S_{t_{i+1}} - S_{t_i}) - \kappa S_{t_{i+1}} |\pi^N_{t_{i+1}} - \pi^N_{t_i}|,$$

and $V^\pi_{t_{i+1}}$, with respect to the overall information up to time t_i, i.e., $\mathscr{F}_{t_i}$. While V^π changes slightly (with no jump) from t_i to t_{i+1}, the JD asset price S changes roughly high in that time period if some jumps happen. In other word, the part $\pi^N_{t_i}(S_{t_{i+1}} - S_{t_i})$ is roughly high when some jumps happen on $[t_i, t_{i+1})$. So, the methodology of CLH tries to compensate this by the part $-\kappa S_{t_{i+1}}|\pi^N_{t_{i+1}} - \pi^N_{t_i}|$. To do this, it takes a value for ${}^*\pi^N_{t_{i+1}}$ extremely different from $\pi^N_{t_i}$, to overcome the reducing effect of proportion κ as well as the roughness of the part $\pi^N_{t_i}(S_{t_{i+1}} - S_{t_i})$. In other word, the jumps in optimal strategy values are the *"cost"* of overcoming to the jumps in the underlying asset price.

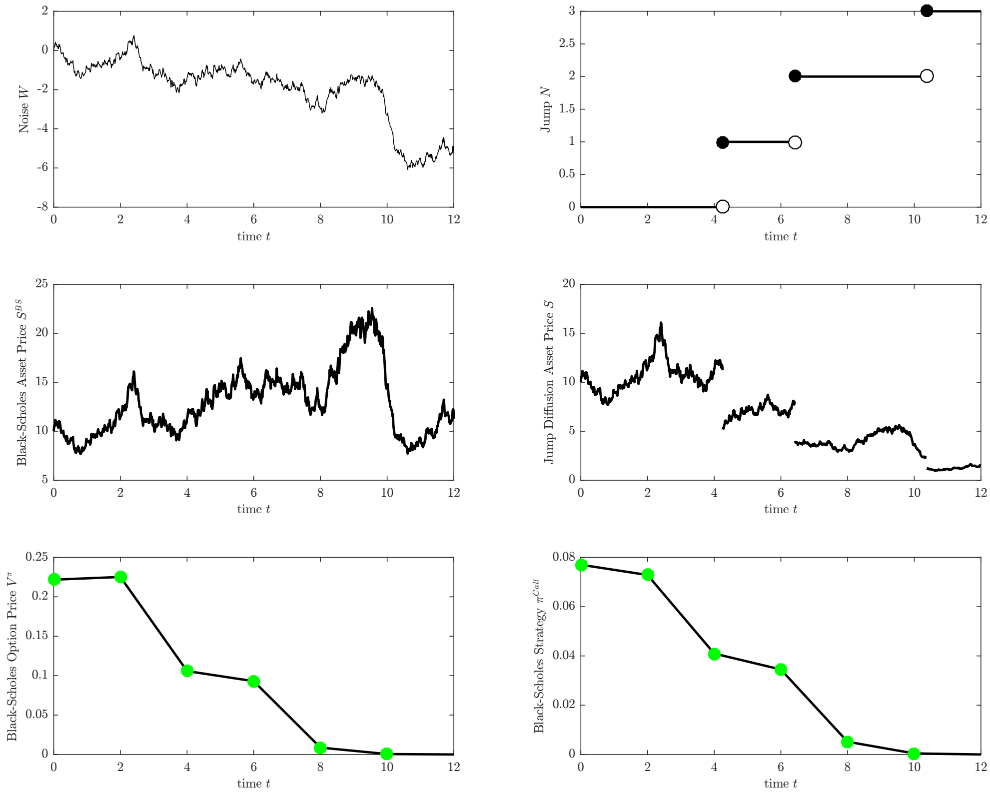

FIGURE 3. The underlying asset price, European call option price and strategy, the noise and jumps for constants $\mu = 0.15, \sigma = 0.25, \xi \equiv -0.5, T = 12, \kappa = 0.1$ and $\lambda = 0.3$.

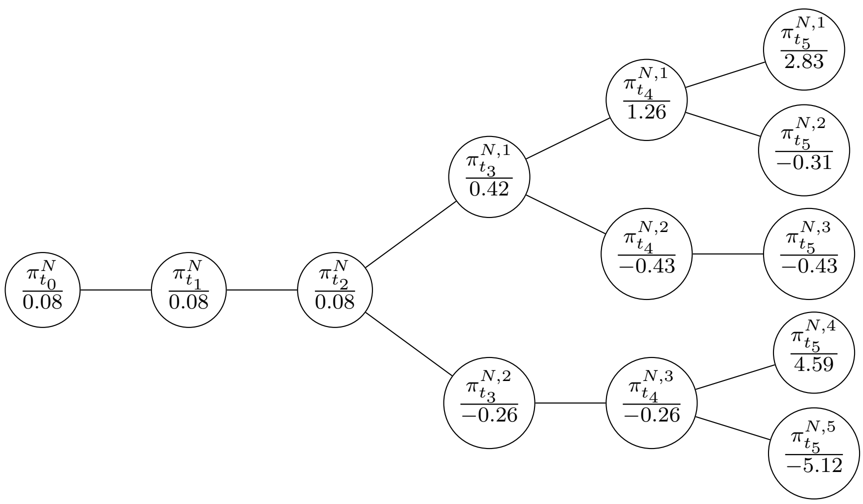

FIGURE 4. The decision tree of optimal strategies associated to the model in 3.

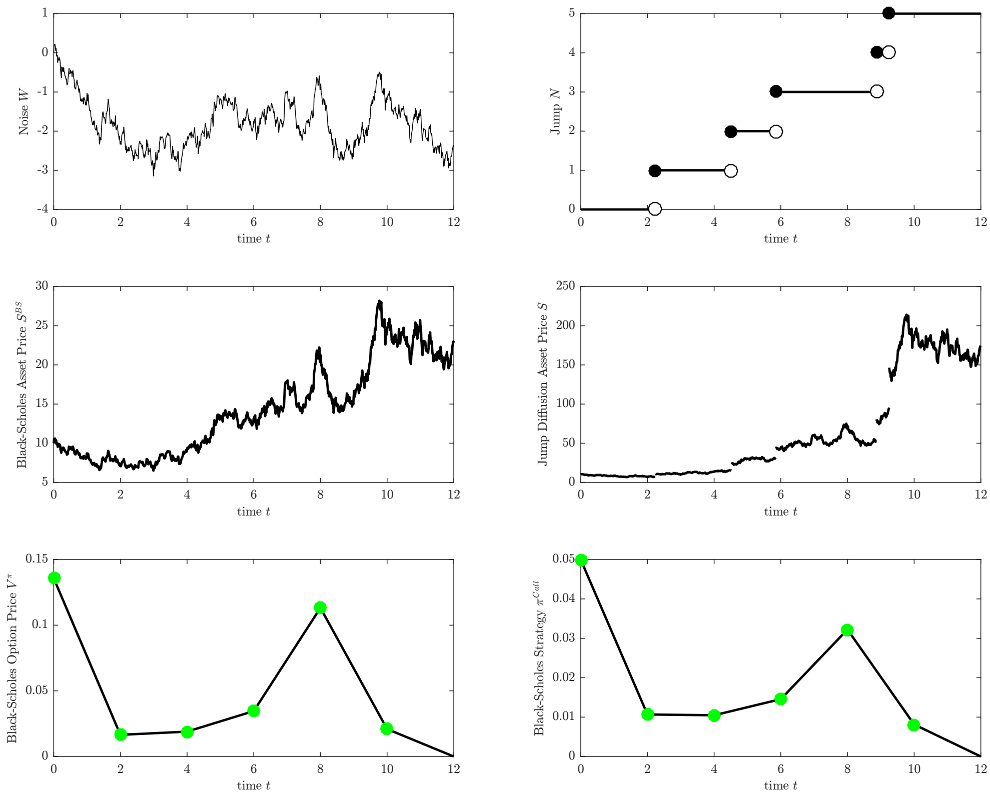

FIGURE 5. The underlying asset price, European call option price and strategy, the noise and jumps for constants $\mu = 0.15, \sigma = 0.25, \xi \equiv 0.5, T = 12, \kappa = 0.1$ and $\lambda = 0.3$.

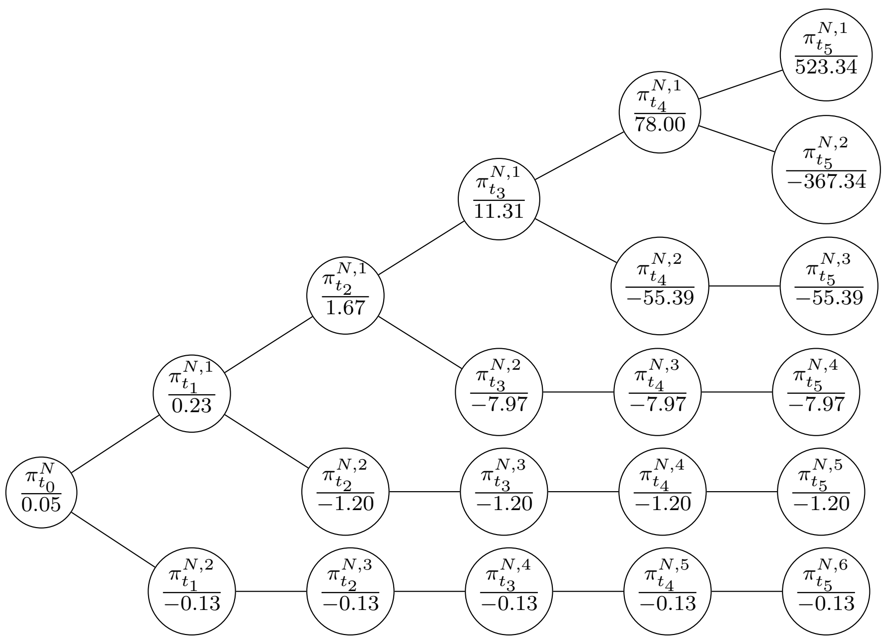

FIGURE 6. The decision tree of optimal strategies associated to the model in 5.

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