ANDO'S LEBESGUE-TYPE DECOMPOSITIONS FOR PAIRS OF POSITIVE OPERATORS

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ABSTRACT. Let A and B be nonnegative operators in $\mathbf{B}(\mathfrak{E})$, where $\mathfrak{E}$ is a Hilbert space. It will be shown that Ando's Lebesgue-type decompositions of B with respect to A into a sum $B = B_1 + B_2$ of nonnegative operators B_1 and B_2, such that B_1 is "absolutely continuous" and B_2 is "singular" with respect to A, can be parametrized with a certain subclass of nonnegative contractions. Moreover, it will be shown that the components B_1 and B_2 in Ando's Lebesgue-type decompositions need not be mutually singular and the case of their mutual singularity will be characterized. The main idea in the paper is the use of so-called representing maps for the pair of bounded nonnegative quadratic forms associated with A and B. This approach leads also to a construction of the analog of Radon-Nikodym derivative in this setting of Ando.

1. INTRODUCTION

Almost half a century ago T. Ando introduced an abstract analog of the Lebesgue decomposition from measure theory in the setting of a pair of nonnegative operators $A, B \in \mathbf{B}(\mathfrak{E})$, where $\mathfrak{E}$ is a Hilbert space; see [2]. For an explanation of this decomposition the following linear subspaces

$$\left\{ \begin{array}{l} \mathfrak{D}(A^{\frac{1}{2}}, B^{\frac{1}{2}}) = \{\xi \in \mathfrak{E} : \, B^{\frac{1}{2}}\xi \in \operatorname{ran} A^{\frac{1}{2}}\}, \\ \mathfrak{R}(A^{\frac{1}{2}}, B^{\frac{1}{2}}) = \{\xi \in \mathfrak{E} : \, A^{\frac{1}{2}}\xi \in \operatorname{ran} B^{\frac{1}{2}}\}, \end{array} \right. \tag{1.1}$$

will be needed. The operator $B \in \mathbf{B}(\mathfrak{E})$ is said to be *Ando-regular with respect to the operator* $A \in \mathbf{B}(\mathfrak{E})$ if

$$\mathfrak{D}(A^{\frac{1}{2}}, B^{\frac{1}{2}}) \text{ is dense in } \mathfrak{E},$$

and *Ando-singular with respect to the operator* $A \in \mathbf{B}(\mathfrak{E})$ if

$$\mathfrak{D}(A^{\frac{1}{2}}, B^{\frac{1}{2}}) \subset \ker B^{\frac{1}{2}} \quad \text{or, equivalently,} \quad \mathfrak{R}(A^{\frac{1}{2}}, B^{\frac{1}{2}}) \subset \ker A^{\frac{1}{2}},$$

in fact, each of these two conditions is equivalent to

$$\operatorname{ran} A^{\frac{1}{2}} \cap \operatorname{ran} B^{\frac{1}{2}} = \{0\}.$$

The present definitions agree with Ando's original notions of absolutely continuous and singular by the characterizations in [2, Section 3, Theorem 5]; see also Corollary 3.4 below. Recall that Ando used the following terminology in [2]: B is said to be A-*absolutely continuous* if there exists a sequence (B_n), $n \in \mathbb{N}$, of nonnegative

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operators such that $B_n \nearrow B$ (strongly) and $B_n \le \varepsilon_n A$ for some $\varepsilon_n \ge 0$, while B is said to be A-singular if $0 \le D \le A$ and $0 \le D \le B$ imply $D = 0$.

The above mentioned Ando-Lebesgue-type decomposition of $B \in \mathbf{B}(\mathfrak{H})$ with respect to $A \in \mathbf{B}(\mathfrak{H})$ is concerned with the operator sum

$$B = B_1 + B_2, \tag{1.2}$$

where the nonnegative operator $B_1 \in \mathbf{B}(\mathfrak{H})$ is Ando-regular with respect to A and the nonnegative operator $B_2 \in \mathbf{B}(\mathfrak{H})$ is Ando-singular with respect to A. In fact, there always exists a so-called Ando-Lebesgue decomposition of the form (1.2), whose Ando-regular component B_1 is maximal in a particular sense. The case where the Ando-Lebesgue decomposition is the only such decomposition can be characterized by the condition that $\mathfrak{D}(A^{\frac{1}{2}}, B^{\frac{1}{2}})$ is closed. The work of Ando was at that time complemented in various papers, such as [15, 16, 19, 20], where especially the dependence of the component B_1 on B was considered, including analogs of the Radon-Nikodym derivative. More recent contributions can be found in, for instance, [1, 17, 18, 23, 25, 26].

The main aim in the present paper is to consider the above decompositions in the context of the sum decompositions for linear relations and semibounded quadratic forms as given in [12, 13]. In particular, the nonnegative operators A and B generate in a natural way nonnegative bounded forms, which can be written in terms of representing maps. This context shows the connection between the construction of Ando for pairs of nonnegative bounded operators and the slightly later construction of Simon [24] for nonnegative forms (which are not necessarily bounded); see Section 5. Following these lines the present paper offers a description of all possible sum decompositions of the form (1.2) via a class of nonnegative contractions. In addition, those decompositions will be characterized for which the summands are mutually singular, i.e., when there is no interaction between the summands; see also [13, 14]. In the case that B is Ando-regular with respect to A, one may express B in terms of A by means of a Radon-Nikodym derivative which is constructed from the Radon-Nikodym derivative introduced in [10] for the corresponding pair of representing maps associated with A and B.

The contents of the paper are now briefly described. In Section 2 there is a general sum decomposition of nonnegative bounded operators into the sum of nonnegative bounded operators. The methods to be used in the present paper are built on the notion of representing maps for the bounded nonnegative forms associated with the pair of nonnegative operators A and B; an approach being developed in a more general setting in [13] to study general, not necessarily closed or closable, semibounded forms. There is also a description of the case where the components are mutually singular. The concepts of Ando-regular and Ando-singular are studied in Section 3, where these concepts are connected to the corresponding notions for linear operators or relations which are generated by pairs of representing maps for the forms associated with A and B. The notion of Radon-Nikodym derivative of B in terms of A is considered in Section 4. To distinguish this notion for the present setting of nonnegative operators the term *Ando-Radon-Nikodym derivative of B with respect to A* will be used. The particular Ando-Lebesgue decomposition from [2] is revisited in Section 5: first this decomposition is constructed via the present approach and then it is shown how this leads to Ando's construction in terms of parallel sums and differences. The general Ando-Lebesgue-type decompositions are parametrized in Section 6; the uniqueness result to guarantee that the Ando-Lebesgue decomposition is the only Ando-Lebesgue-type decomposition is proved by means of this parametrization. For the convenience of the reader some of the more technical details have been collected in Section 7.

2. Sum decompositions of bounded nonnegative operators

In this section the general decomposition of a nonnegative operator $B \in \mathbf{B}(\mathfrak{E})$ into the sum of two such operators is considered. This also includes a discussion of the interaction between the summands of the decomposition. The relevant results from [13] are collected and presented in the context of the present paper. In later sections these decompositions of $B \in \mathbf{B}(\mathfrak{E})$ will be considered in the presence of another nonnegative operator $A \in \mathbf{B}(\mathfrak{E})$.

Let $\mathfrak{E}$ be a Hilbert space and let $B \in \mathbf{B}(\mathfrak{E})$. It is clear that $B \in \mathbf{B}(\mathfrak{E})$ is nonnegative if and only if

$$B = \Psi^*\Psi, \tag{2.1}$$

for some $\Psi \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$, where $\mathfrak{K}$ is a Hilbert space. Note that if B and Ψ are connected via (2.1), then

$$(B\eta, \zeta) = (\Psi\eta, \Psi\zeta), \quad \eta, \zeta \in \mathfrak{E}. \tag{2.2}$$

Thus Ψ may be interpreted as a representing map for the bounded nonnegative form in the left-hand side of (2.2), generated by B; see [13]. Of course, one can always choose Ψ minimal, i.e., $\overline{\operatorname{ran}}\, \Psi = \mathfrak{K}$. As a possible choice for Ψ one may take $\mathfrak{K} = \overline{\operatorname{ran}}\, B^{\frac{1}{2}}$ and $\Psi = B^{1/2}$. It follows from (2.1) that there exist a unique partial isometry $V \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ which connects the representing maps

$$\Psi = V\, B^{\frac{1}{2}}. \tag{2.3}$$

The partial isometry V has initial space $\overline{\operatorname{ran}}\, B^{\frac{1}{2}}$ and final space $\overline{\operatorname{ran}}\, \Psi$; therefore, $V^*V \in \mathbf{B}(\mathfrak{E})$ is an orthogonal projection onto $\overline{\operatorname{ran}}\, B^{\frac{1}{2}}$, while $VV^* \in \mathbf{B}(\mathfrak{K})$ is an orthogonal projection onto $\overline{\operatorname{ran}}\, \Psi$.

The following parametrization result goes back to [13, Theorem 4.1] in the setting of general nonnegative quadratic forms. For the present case a simple direct proof is provided.

Proposition 2.1. *Let $B \in \mathbf{B}(\mathfrak{E})$ be a nonnegative operator with a representing map $\Psi \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ as in* (2.1). *Then B has the sum decomposition*

$$B = B_1 + B_2 \tag{2.4}$$

with nonnegative operators $B_1, B_2 \in \mathbf{B}(\mathfrak{E})$ if and only if for some nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$

$$B_1 = \Psi^*(I - K)\Psi \quad \textit{and} \quad B_2 = \Psi^* K \Psi. \tag{2.5}$$

Proof. Assume that $B = B_1 + B_2$ is a sum decomposition of B as in (2.4), so that B_1 and B_2 are nonnegative. Hence, it follows from $0 \le B_2 \le B$ and $B = \Psi^*\Psi$ in (2.1), that $B_2^{\frac{1}{2}} = Z\Psi$ with some contraction $Z \in \mathbf{B}(\mathfrak{K}, \mathfrak{E})$. Therefore, one sees that

$$B_2 = \Psi^* K \Psi,$$

where $K = Z^*Z \in \mathbf{B}(\mathfrak{K})$ a nonnegative contraction. Hence,

$$B_1 = B - B_2 = \Psi^*(I-K)\Psi,$$

and (2.5) has been shown. Conversely, if (2.5) holds for some nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$, then B_1 and B_2 are nonnegative operators in $\mathbf{B}(\mathfrak{E})$ and it is clear that the sum decomposition in (2.4) is satisfied. □

The parametrization in Proposition 2.1 is given in terms of the representing map $\Psi = VB^{\frac{1}{2}}$ in (2.3). If the representation is minimal, i.e., $\overline{\mathrm{ran}}\,\Psi = \mathfrak{K}$, then the nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$ is uniquely determined. However, if the representation is not minimal then only the compression of K to the closed linear subspace $\overline{\mathrm{ran}}\,\Psi \subset \mathfrak{K}$ plays a role. For instance, if instead of Ψ one chooses the representing map $B^{\frac{1}{2}}$, then it is clear that any nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$ in (2.5) will result in

$$B_1 = B^{\frac{1}{2}}(I-K')B^{\frac{1}{2}}, \quad B_2 = B^{\frac{1}{2}}K'B^{\frac{1}{2}}, \tag{2.6}$$

where $K' = V^*KV \in \mathbf{B}(\mathfrak{H})$ is a nonnegative contraction whose compression from $\mathfrak{H}$ to the closed linear subspaces $\mathrm{ran}\,B^{\frac{1}{2}}$ transforms likewise. Moreover, any nonnegative contraction in $\mathbf{B}(\mathfrak{E})$, leaving $\overline{\mathrm{ran}}\,\Psi$ invariant, can be seen as the result of such a transform, since $K = VK'V^*$. When K in Proposition 2.1 is selected such that $\mathrm{ran}\,K \subset \overline{\mathrm{ran}}\,\Psi$, then it is uniquely determined.

Analogous to the interpretation of Ψ as a representing map of the nonnegative form in (2.2), the sum decomposition (2.4) will be written as

$$(B\eta,\zeta) = \left(\begin{pmatrix} (I-K)^{\frac{1}{2}} \\ K^{\frac{1}{2}} \end{pmatrix} \Psi\eta, \begin{pmatrix} (I-K)^{\frac{1}{2}} \\ K^{\frac{1}{2}} \end{pmatrix} \Psi\zeta \right), \quad \eta,\zeta \in \mathfrak{E}. \tag{2.7}$$

Therefore, when $\overline{\mathrm{ran}}\,(I-K)$ and $\overline{\mathrm{ran}}\,K$ are interpreted as Hilbert spaces with the inner product inherited from $\mathfrak{K}$, the following operator

$$\begin{pmatrix} (I-K)^{\frac{1}{2}} \\ K^{\frac{1}{2}} \end{pmatrix} \Psi : \mathfrak{E} \to \begin{pmatrix} \overline{\mathrm{ran}}\,(I-K) \\ \overline{\mathrm{ran}}\,K \end{pmatrix}, \tag{2.8}$$

serves as a representing map for B; cf. [12, 13]. The interest is now in the minimality of this new representing map. The next result is proved for a general nonnegative form in [13, Theorem 4.3].

Proposition 2.2. *Let $B \in \mathbf{B}(\mathfrak{E})$ be a nonnegative operator with a representing map $\Psi \in \mathbf{B}(\mathfrak{E},\mathfrak{K})$ as in* (2.1)*, and assume that $\overline{\mathrm{ran}}\,\Psi = \mathfrak{K}$. Let B have the sum decomposition* (2.5)*, so that* (2.7) *holds with the nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$. Then the following statements are equivalent:*

(i) *the representing map* (2.8) *is minimal;*
(ii) *K is an orthogonal projection.*

The summands B_1 and B_2 in (2.4) are said to be *mutually singular* if their parallel sum satisfies $B_1 : B_2 = 0$; for the definition of $B_1 : B_2$, see e.g. [6, 14, 22] and Subsection 7.4. Recall that when B_1 and B_2 are given as in (2.5), then

$$\big((B_1 : B_2)\,\eta,\zeta\big) = \big(((I-K):K)\Psi\eta,\Psi\zeta\big), \quad \eta,\zeta \in \mathrm{dom}\,\mathfrak{K}. \tag{2.9}$$

For the argument behind this identity, see [13, Proposition 4.4]. It is easy to check that $(I-K):K = (I-K)K$, see e.g. [6, p. 277] or [14, Section 8.3]. In particular, $(I-K):K = 0$ precisely when K is an orthogonal projection. The next result is now clear from (2.9).

Corollary 2.3. *Let the representing map* Ψ *in* (2.1) *be minimal, i.e.,* $\overline{\operatorname{ran}}\,\Psi = \mathfrak{K}$. *Then* B_1 *and* B_2 *in* (2.5) *are mutually singular if and only if* $K \in \mathbf{B}(\mathfrak{K})$ *is an orthogonal projection.*

The above sum decompositions form the main ingredient of the paper. Later on they will be considered in the context of another nonnegative operator $A \in \mathbf{B}(\mathfrak{E})$ such that

$$A = \Phi^*\Phi \tag{2.10}$$

for some $\Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$, where $\mathfrak{H}$ is a Hilbert space. As above, Φ may be seen as a representing map for the bounded nonnegative form in the left-hand side of (2.2), generated by A:

$$(A\eta, \zeta) = (\Phi\eta, \Phi\zeta), \quad \eta, \zeta \in \mathfrak{E}. \tag{2.11}$$

As a possible choice for Φ one may take $\mathfrak{H} = \mathfrak{E}$ and $\Phi = A^{1/2}$. It follows from (2.1) that there exists a unique partial isometry $U \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ which connects the representing maps

$$\Phi = U\,A^{\frac{1}{2}}.$$

The partial isometry U has initial space $\overline{\operatorname{ran}}\,A^{\frac{1}{2}}$ and final space $\overline{\operatorname{ran}}\,\Phi$; therefore, $U^*U \in \mathbf{B}(\mathfrak{E})$ is an orthogonal projection onto $\overline{\operatorname{ran}}\,A^{\frac{1}{2}}$, while $UU^* \in \mathbf{B}(\mathfrak{H})$ is an orthogonal projection onto $\overline{\operatorname{ran}}\,\Phi$.

3. Ando-regularity and Ando-singularity

The Ando-regularity and Ando-singularity of a pair of nonnegative operators $A, B \in \mathbf{B}(\mathfrak{E})$ will now be explained in terms of the corresponding representing maps $\Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $\Psi \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ in (2.1) and (2.10), respectively. The main tool here is the corresponding operator range relation $L(\Phi, \Psi) \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, defined for any pair $\Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $\Psi \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$, by

$$L(\Phi, \Psi) = \big\{ \{\Phi\eta, \Psi\eta\} : \eta \in \mathfrak{E} \big\}. \tag{3.1}$$

Recall that an arbitrary linear relation $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ is called *regular* when it is closable, i.e., when its graph-closure $T^{**} \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ is an operator or $\operatorname{mul} T^{**} = \{0\}$; it is called *singular* when its graph-closure $T^{**} = \mathfrak{X} \times \mathfrak{Y}$, where $\mathfrak{X} \subset \mathfrak{H}$ and $\mathfrak{Y} \subset \mathfrak{K}$ are closed linear subspaces; for more, see Subsection 7.1. It will be explained how these notions can interpreted in the context of pairs of bounded linear operators, and in the context of pairs of nonnegative bounded linear operators.

The treatment of the operator range relation $L(\Phi, \Psi) \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ in (3.1) involves additionally the linear subspaces $\mathfrak{D}(\Phi, \Psi) \subset \mathfrak{K}$ and $\mathfrak{R}(\Phi, \Psi) \subset \mathfrak{H}$ defined by

$$\mathfrak{D}(\Phi, \Psi) = \{\xi \in \mathfrak{K} : \Psi^*\xi \in \operatorname{ran} \Phi^*\}, \quad \mathfrak{R}(\Phi, \Psi) = \{\nu \in \mathfrak{H} : \Phi^*\nu \in \operatorname{ran} \Psi^*\}, \tag{3.2}$$

and note the following identities

$$(\mathfrak{D}(\Phi, \Psi))^\perp = \operatorname{mul} L(\Phi, \Psi)^{**}, \quad (\mathfrak{R}(\Phi, \Psi))^\perp = \ker\, L(\Phi, \Psi)^{**},$$

see Subsection 7.2. The next definition is inspired by similar concepts for the linear relation $L(\Phi, \Psi)$ in (3.1); see Subsection 7.2. Note, in particular, that the definition of singularity is symmetric in Φ and Ψ.

Definition 3.1. Let $\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{H})$ and $\Psi \in \mathbf{B}(\mathfrak{E},\mathfrak{K})$. Then the operator Ψ is called *regular with respect to* Φ if $\mathfrak{D}(\Phi,\Psi)$ is dense in $\mathfrak{K}$, which is the case if and only if the relation $L(\Phi,\Psi)$ in (3.1) is regular. Likewise, the operator Ψ is called *singular with respect to* Φ if one, and hence all, of the following equivalent conditions hold:

(i) $\mathfrak{D}(\Phi,\Psi) \subset \ker \Psi^*$;
(ii) $\mathfrak{R}(\Phi,\Psi) \subset \ker \Phi^*$;
(iii) $\operatorname{ran} \Phi^* \cap \operatorname{ran} \Psi^* = \{0\}$,

which is the case if and only if the relation $L(\Phi,\Psi)$ in (3.1) is singular.

It is also possible to characterize the above notions for pairs of operators Φ and Ψ in a way suggested by measure theory. The operator $\Psi \in \mathbf{B}(\mathfrak{E},\mathfrak{K})$ is said to be *dominated by* Φ, denoted by $\Psi \prec \Phi$, if $\|\Psi\eta\| \le \varepsilon\|\Phi\eta\|$, $\eta \in \mathfrak{E}$, for some $\varepsilon \ge 0$. Likewise, Ψ is *contractively dominated by* Φ, denoted by $\Psi \prec_c \Phi$, if $\|\Psi\eta\| \le \|\Phi\eta\|$, $\eta \in \mathfrak{E}$. Moreover, the operator Ψ is said to be *almost dominated by* $\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{H})$ if there exists a sequence $\varepsilon_n \ge 0$ and a sequence $\Psi_n \in \mathbf{B}(\mathfrak{E},\mathfrak{K}_n)$, such that for all $\eta \in \mathfrak{E}$:

(a) $\|\Psi_n\eta\| \le \varepsilon_n\|\Phi\eta\|$;
(b) $\|\Psi_n\eta\| \le \|\Psi_{n+1}\eta\|$;
(c) $\|\Psi_n\eta\| \nearrow \|\Psi\eta\|$.

Of course (a) means that each approximating Ψ_n is dominated by Φ. All these notions are discussed in [10, 11, 12].

Lemma 3.2. *Let $\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{H})$ and $\Psi \in \mathbf{B}(\mathfrak{E},\mathfrak{K})$. Then the following statements are equivalent:*

(i) *Ψ is regular with respect to Φ;*
(ii) *Ψ is almost dominated by Φ.*

Likewise, the following statements are equivalent:

(iii) *Ψ is singular with respect to Φ (or Φ is singular with respect to Ψ);*
(iv) *for any $\Xi \in \mathbf{B}(\mathfrak{E},\mathfrak{L})$, where $\mathfrak{L}$ is a Hilbert space, one has*

$$\Xi \prec \Phi \quad \text{and} \quad \Xi \prec \Psi \quad \Rightarrow \quad \Xi = 0.$$

Proof. (i) $\Leftrightarrow$ (ii) First observe that Ψ is regular with respect to Φ if and only if $L(\Phi,\Psi)$ is (the graph of) a closable operator. In [10, Definition 6.3 and Theorem 6.4] and [11] it is shown that a regular relation can be defined by a suitable approximation property, from which the assertion follows.

(iii) $\Rightarrow$ (iv) Assume that Ψ is singular with respect to Φ or, equivalently, that $\operatorname{ran} \Phi^* \cap \operatorname{ran} \Psi^* = \{0\}$. Now let $\Xi \in \mathbf{B}(\mathfrak{E},\mathfrak{L})$ and assume that for all $\eta \in \mathfrak{E}$

$$\|\Xi\eta\| \le \|\Phi\eta\| \quad \text{and} \quad \|\Xi\eta\| \le \|\Psi\|.$$

Then by the Douglas lemma [5] $\Xi = H\Phi$ and $\Xi = K\Psi$ for some contractions $H \in \mathbf{B}(\mathfrak{H},\mathfrak{L})$ and $K \in \mathbf{B}(\mathfrak{K},\mathfrak{L})$, which are uniquely determined when $\operatorname{ran} H^* \subset (\ker \Phi^*)^\perp$ and $\operatorname{ran} K^* \subset (\ker \Phi^*)^\perp$; cf. (7.15), (7.16). Since $\Phi^*H^* = \Psi^*K^*$ one concludes that $H = 0$ and $K = 0$. Thus, $\Xi = 0$ and this proves (iv).

(iv) $\Rightarrow$ (iii) Assume that (iv) holds. Suppose that $\operatorname{ran} \Phi^* \cap \operatorname{ran} \Psi^* \ne \{0\}$. Then there exists a proper orthogonal projection $\Xi \in \mathbf{B}(\mathfrak{E})$ with $\operatorname{ran} \Xi \subset \operatorname{ran} \Phi^* \cap \operatorname{ran} \Psi^*$ or, since Ξ is selfadjoint,

$$\operatorname{ran} \Xi^* \subset \operatorname{ran} \Phi^* \quad \text{and} \quad \operatorname{ran} \Xi^* \subset \operatorname{ran} \Psi^*.$$

By the Douglas lemma this means that $\Xi \prec \Phi$ and $\Xi \prec \Psi$, which leads to the contradiction $\Xi = 0$. $\square$

The notions Ando-regular and Ando-singular, as given in the introduction for a pair of nonnegative operators $A, B \in \mathbf{B}(\mathfrak{E})$, are related to the concepts of regularity and singularity for a pair of representing maps $\Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $\Psi \in \mathbf{B}(\mathfrak{H}, \mathfrak{K})$ for A and B in (2.1) and (2.10), respectively. It is a consequence of the next lemma that the definitions of Ando-regular and Ando-singular are independent of the choice of the representing maps; cf. Subsection 7.1.

Lemma 3.3. *Let $A, B \in \mathbf{B}(\mathfrak{E})$ be a pair of nonnegative operators and assume that their representing maps Φ, Ψ are given by* (2.1) *and* (2.10). *Then*

(a) *$B \in \mathbf{B}(\mathfrak{E})$ is Ando-regular with respect to $A \in \mathbf{B}(\mathfrak{E})$ if and only if Ψ is regular with respect to Φ;*
(b) *$B \in \mathbf{B}(\mathfrak{E})$ is Ando-singular with respect to $A \in \mathbf{B}(\mathfrak{E})$ if and only if Ψ is singular with respect to Φ.*

Proof. From (1.1) and Definition 3.1 it is clear that $B \in \mathbf{B}(\mathfrak{E})$ is Ando-regular with respect to $A \in \mathbf{B}(\mathfrak{E})$ if and only if $B^{\frac{1}{2}}$ is regular with respect to $A^{\frac{1}{2}}$. Therefore the assertion in (a) is clear from Corollary 7.2.

Likewise, it is clear that $B \in \mathbf{B}(\mathfrak{E})$ is Ando-singular with respect to $A \in \mathbf{B}(\mathfrak{E})$ if and only if $B^{\frac{1}{2}}$ is singular with respect to $A^{\frac{1}{2}}$. Therefore the assertion in (b) is clear from Corollary 7.2. $\square$

Now let $A, B \in \mathbf{B}(\mathfrak{E})$ be a general pair of nonnegative operators. The following notions, going back to Ando, have been inspired by measure theory; cf. [2]. The operator $B \in \mathbf{B}(\mathfrak{E})$ is said to be *Ando-dominated by A* if $(B\eta, \eta) \le c(A\eta, \eta)$, $\eta \in \mathfrak{E}$, for some $c \ge 0$. If $c = 1$ one speaks of *contractively Ando-dominated*. The operator B is said to be *almost Ando-dominated by A* if there exists a sequence $c_n \ge 0$ and a sequence of nonnegative operators $B_n \in \mathbf{B}(\mathfrak{E})$, such that for all $\eta \in \mathfrak{E}$:

(a) $(B_n\eta, \eta) \le c_n(A\eta, \eta)$;
(b) $(B_n\eta, \eta) \le (B_{n+1}\eta, \eta)$;
(c) $(B_n\eta, \eta) \nearrow (B\eta, \eta)$.

Here (a) means that every approximating B_n is Ando-dominated by A. Note that (c) implies that $(B_n\eta, \eta) \le \|B\|\|\eta\|^2$, $\eta \in \mathfrak{E}$. With this last bound the condition (b) implies that B_n converges strongly to an element in $\mathbf{B}(\mathfrak{E})$; cf. [4, p. 79]. Hence condition (c) may also be replaced by: B_n converges strongly to B; cf. [2]. The above measure-theoretic notions are shown to be equivalent to the operator-theoretic notions in the corollary below.

Corollary 3.4. *Let $A, B \in \mathbf{B}(\mathfrak{E})$ be nonnegative operators. Then the following statements are equivalent:*

(i) *$B \in \mathbf{B}(\mathfrak{E})$ is Ando-regular with respect to $A \in \mathbf{B}(\mathfrak{E})$;*
(ii) *B is almost Ando-dominated by A.*

Likewise, the following statements are equivalent:

(iii) *$B \in \mathbf{B}(\mathfrak{E})$ is Ando-singular with respect to $A \in \mathbf{B}(\mathfrak{E})$;*
(iv) *for every nonnegative $D \in \mathbf{B}(\mathfrak{E})$ such that*

$$D \le A \quad \textit{and} \quad D \le B \quad \Rightarrow \quad D = 0.$$

Proof. Let $A, B \in \mathbf{B}(\mathfrak{E})$ be a pair of nonnegative operators and assume that their representing maps Φ, Ψ are given by (2.1) and (2.10).

(i) $\Rightarrow$ (ii) Assume that B is Ando-regular with respect to A or, equivalently, that Ψ is regular with respect to Φ; cf. Lemma 3.3. Thus there exists a sequence $\varepsilon_n \geq 0$ and a sequence $\Psi_n \in \mathbf{B}(\mathfrak{E}, \mathfrak{K}_n)$, where $\mathfrak{K}_n$ are Hilbert spaces, such that for all $\eta \in \mathfrak{E}$:

$$\|\Psi_n\eta\| \leq \varepsilon_n \|\Phi\eta\|, \quad \|\Psi_n\eta\| \leq \|\Psi_{n+1}\eta\|, \quad \|\Psi_n\eta\| \nearrow \|\Psi\eta\|, \tag{3.3}$$

see (a)–(c) preceding Lemma 3.2. Since $A = \Phi^*\Phi$ and $B = \Psi^*\Psi$, it follows from (3.3) that the sequence of nonnegative operators $B_n = \Psi_n^*\Psi_n \in \mathbf{B}(\mathfrak{E})$ satisfies

$$(B_n\eta, \eta) \leq c_n(A\eta, \eta), \quad (B_n\eta, \eta) \leq (B_{n+1}\eta, \eta), \quad (B_n\eta, \eta) \nearrow (B\eta, \eta), \tag{3.4}$$

where $c_n = \varepsilon_n^2$. Hence B is almost Ando-dominated by A.

(ii) $\Rightarrow$ (i) Assume that B is almost Ando-dominated by A. Then there exists a sequence of bounded nonnegative operators $B_n \in \mathbf{B}(\mathfrak{E})$ as in (3.4). This implies for the sequence $B_n^{\frac{1}{2}} \in \mathbf{B}(\mathfrak{E})$ that one has for all $\eta \in \mathfrak{E}$:

$$\|B_n^{\frac{1}{2}}\eta\| \leq \sqrt{c_n}\,\|\Phi\eta\|, \quad \|B_n^{\frac{1}{2}}\eta\| \leq \|B_{n+1}^{\frac{1}{2}}\eta\|, \quad \|B_n^{\frac{1}{2}}\eta\| \nearrow \|\Psi\eta\|.$$

Thus it is clear that Ψ is almost dominated by Φ.

(iii) $\Rightarrow$ (iv) Assume that B is Ando-singular with respect to A or, equivalently, that Ψ is singular with respect to Φ; cf. Lemma 3.3. Then by Lemma 3.2 for every $\varXi \in \mathbf{B}(\mathfrak{E}, \mathfrak{L})$ one has

$$\|\varXi\eta\| \leq \|\Phi\eta\| \quad \text{and} \quad \|\varXi\eta\| \leq \|\Psi\eta\| \quad \text{for all} \quad \eta \in \mathfrak{E} \quad \Rightarrow \quad \varXi = 0. \tag{3.5}$$

Now let $D \in \mathbf{B}(\mathfrak{E})$ be a nonnegative operator, for which

$$D \leq A \quad \text{and} \quad D \leq B.$$

Write $D = \varXi^*\varXi$ with $\varXi \in \mathbf{B}(\mathfrak{E}, \mathfrak{L})$, then it follows from (3.5) that $D = 0$.

(iv) $\Rightarrow$ (iii) Assume that A and B satisfy the assertion in (iv). It suffices to show that Ψ is singular with respect to Φ; cf. Lemma 3.3. Therefore let $\varXi \in \mathbf{B}(\mathfrak{E}, \mathfrak{L})$ satisfy

$$\varXi \prec \Phi \quad \text{and} \quad \varXi \prec \Psi.$$

Then $D = \varXi^*\varXi$ satisfies $D \in \mathbf{B}(\mathfrak{E})$, $D \leq A$, and $D \leq B$, so that $D = 0$. Therefore one concludes $\varXi = 0$. □

4. Radon-Nikodym derivatives

Let $A, B \in \mathbf{B}(\mathfrak{E})$ be a pair of nonnegative operators. If B is Ando-regular with respect to A, then it is possible to express B in terms of A by means of an analog of the Radon-Nikodym derivative. This section is concerned with a construction of an appropriate version of such a derivative.

Let $A, B \in \mathbf{B}(\mathfrak{E})$ be a pair of nonnegative operators and let the corresponding representing maps $\Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{E})$ and $\Psi \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ be given by (2.1) and (2.10). If B is Ando-regular with respect to A, then Ψ is regular with respect to Φ or, equivalently, the relation $L(\Phi, \Psi)$ is closable; see Lemma 3.3. In this case the Radon-Nikodym derivative of Ψ with respect to Φ defined by

$$R(\Phi, \Psi) = L(\Phi, \Psi)^{**} \tag{4.1}$$

is a closed operator with the property that $R(\Phi, \Psi)\Phi = \Psi$; see Subsection 7.3. Recall that Φ and Ψ can be seen as representing maps for the forms (2.2) and

(2.11) generated by A and B, respectively. In particular, the Radon-Nikodym derivative $R(\Phi,\Psi)$ in (4.1) combined with the formulas (2.1) and (2.10) yields

$$A=\Phi^*\Phi, \quad B=(R(\Phi,\Psi)\Phi)^*R(\Phi,\Psi)\Phi. \tag{4.2}$$

To make the connection between A and B more precise, some explicit expressions for $R(\Phi,\Psi)$ established in the recent paper [10] will be used; for completeness the key facts to be used here have been summarized in Subsection 7.3. Indeed, an application of Douglas lemma [5] shows that there exists a unique pair of contractions $C_\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{H})$ and $C_\Psi \in \mathbf{B}(\mathfrak{E},\mathfrak{K})$, such that

$$\Phi=C_\Phi\,(\Phi^*\Phi+\Psi^*\Psi)^{\frac{1}{2}}, \quad \Psi=C_\Psi\,(\Phi^*\Phi+\Psi^*\Psi)^{\frac{1}{2}} \tag{4.3}$$

with

$$\ker\,(\Phi^*\Phi+\Psi^*\Psi)\subset \ker\,C_\Phi, \quad \ker\,(\Phi^*\Phi+\Psi^*\Psi)\subset \ker\,C_\Psi,$$

cf. (7.15), (7.16). This leads to the following explicit formulas for the Radon-Nikodym derivative in (4.1):

$$R(\Phi,\Psi)=L(C_\Phi,C_\Psi)=C_\Psi(C_\Phi)^{-1}=C_\Psi\,(I-\mathcal{P})(C_\Phi)^{-1}=C_\Psi\,((C_\Phi)^{-1})_{\mathrm{reg}}, \tag{4.4}$$

where $\mathcal{P}$ is the orthogonal projection from $\mathfrak{E}$ onto $\ker C_\Phi$; cf. (7.18). If B is Ando-regular with respect to A then $C_\Psi(C_\Phi)^{-1}$ is a closed, in general, unbounded operator and one has $\ker C_\Phi \subset \ker C_\Psi$. Observe that $\ker \Phi^*=\{0\}$ precisely when Φ as a representing map for A in (2.10) is chosen minimal. If this is not the case, it is convenient to extend $R(\Phi,\Psi)$ as a zero operator from $\mathrm{ran}\, C_\Phi$ to $\ker\,(C_\Phi)^*=\ker\,\Phi^*$. This extension can be rewritten as $R(\Phi,\Psi)\mathcal{Q}$, where $\mathcal{Q}$ is the orthogonal projection from $\mathfrak{H}$ onto $\overline{\mathrm{ran}}\, C_\Phi=\overline{\mathrm{ran}}\,\Phi$. It is a closed densely defined operator from $\mathfrak{H}$ to $\mathfrak{K}$ and satisfies the key identity $(R(\Phi,\Psi)\mathcal{Q})\Phi=\Psi$. Moreover, the operator $\mathcal{R}(\Phi,\Psi)=\mathcal{Q}R(\Phi,\Psi)^*R(\Phi,\Psi)\mathcal{Q}\in \mathbf{L}(\mathfrak{H})$ has the block form

$$\mathcal{R}(\Phi,\Psi)=\begin{pmatrix}\mathcal{R}_{\mathrm{op}}(\Phi,\Psi) & 0\\ 0 & 0\end{pmatrix}:\begin{pmatrix}\overline{\mathrm{ran}}\,\Phi\\ \ker\,\Phi^*\end{pmatrix}\to\begin{pmatrix}\overline{\mathrm{ran}}\,\Phi\\ \ker\,\Phi^*\end{pmatrix} \tag{4.5}$$

and it is nonnegative and selfadjoint. Here $\mathcal{R}_{\mathrm{op}}(\Phi,\Psi)$ stands for the orthogonal operator part of $R(\Phi,\Psi)^*R(\Phi,\Psi)$ and considered as a nonnegative selfadjoint operator in $\overline{\mathrm{ran}}\,\Phi$; see (7.1), (7.3).

Now let B be Ando-regular with respect to A. Then the selfadjoint nonnegative operator $\mathcal{R}(\Phi,\Psi)$ and the operator part $\mathcal{R}_{\mathrm{op}}(\Phi,\Psi)$ in (4.5), respectively, will be called the *Ando-Radon-Nikodym derivative* and the *minimal Ando-Radon-Nikodym derivative* of B with respect to A corresponding to the representations of A and B in (2.1) and (2.10). In particular, if Φ as a representing map for A in (2.10) is chosen minimal, then one has $\mathcal{R}(\Phi,\Psi)=\mathcal{R}_{\mathrm{op}}(\Phi,\Psi)$. This terminology makes sense and in fact only the operator part $\mathcal{R}_{\mathrm{op}}(\Phi,\Psi)$ in (4.5) will play a role. Namely, the combination $\mathcal{R}(\Phi,\Psi)\Phi$ can be rewritten as

$$\mathcal{R}(\Phi,\Psi)\Phi=\begin{pmatrix}\mathcal{R}(\Phi,\Psi)_{\mathrm{op}} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\Phi\\ 0\end{pmatrix}=\begin{pmatrix}\mathcal{R}(\Phi,\Psi)_{\mathrm{op}}\Phi\\ 0\end{pmatrix},$$

so that with the right interpretation one has $\Phi^*\mathcal{R}(\Phi,\Psi)\Phi=\Phi^*\mathcal{R}(\Phi,\Psi)_{\mathrm{op}}\Phi$. Likewise one may write $\mathcal{R}(\Phi,\Psi)^{\frac{1}{2}}\Phi=(\mathcal{R}(\Phi,\Psi)_{\mathrm{op}})^{\frac{1}{2}}\Phi$. If B is Ando-dominated by A or, equivalently, Ψ is dominated by Φ, then $\mathcal{R}(\Phi,\Psi)\in \mathbf{B}(\mathfrak{H})$ and $\mathrm{dom}\,\mathcal{R}_{\mathrm{op}}(\Phi,\Psi)=\overline{\mathrm{ran}}\,\Phi$.

Theorem 4.1. *Let $A, B \in \mathbf{B}(\mathfrak{E})$ be a pair of nonnegative operators and let the representing maps Φ, Ψ be given by* (2.1) *and* (2.10). *Assume that B is Ando-regular with respect to A and let $\mathcal{R}(\Phi,\Psi) \in \mathbf{L}(\mathfrak{E})$ be the Ando-Radon-Nikodym derivative of B with respect to A in* (4.5) *corresponding to* (2.1) *and* (2.10). *Then there is the inclusion*

$$\Phi^*\mathcal{R}(\Phi,\Psi)\Phi \subset B, \tag{4.6}$$

with equality if B is Ando-dominated by A. Moreover,

$$\mathcal{R}(\Phi,\Psi)^{\frac{1}{2}}\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{H}) \quad \textit{and} \quad \operatorname{clos}(\Phi^*\mathcal{R}(\Phi,\Psi)^{\frac{1}{2}}) \in \mathbf{B}(\mathfrak{H},\mathfrak{E}), \tag{4.7}$$

and, consequently, there is the identity

$$\operatorname{clos}(\Phi^*\mathcal{R}(\Phi,\Psi)^{\frac{1}{2}})\,\mathcal{R}(\Phi,\Psi)^{\frac{1}{2}}\Phi = B. \tag{4.8}$$

Proof. The Radon-Nikodym derivative $R(\Phi,\Psi) \in \mathbf{L}(\mathfrak{H},\mathfrak{K})$ satisfies $R(\Phi,\Psi)\Phi = \Psi$, so that

$$R(\Phi,\Psi)\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{K}). \tag{4.9}$$

Since Ψ is regular with respect to Φ, the adjoint relation $R(\Phi,\Psi)^* \in \mathbf{L}(\mathfrak{K},\mathfrak{H})$ is densely defined. Furthermore, one sees from (4.4) that $\operatorname{mul} R(\Phi,\Psi)^* = \ker\,(C_\Phi)^* = \ker \Phi^*$, so that $\Phi^*R(\Phi,\Psi)^* \in \mathbf{L}(\mathfrak{K},\mathfrak{E})$ is an operator. Thus $\Phi^*R(\Phi,\Psi)^* \in \mathbf{L}(\mathfrak{K},\mathfrak{E})$ is a densely defined operator which obviously satisfies

$$\Phi^*R(\Phi,\Psi)^* \subset (R(\Phi,\Psi)\Phi)^*. \tag{4.10}$$

Therefore, multiplying this inclusion from the right by $R(\Phi,\Psi)\Phi$ leads to the inclusion (4.6). If B is Ando-dominated by A, recall that $\mathcal{R}(\Phi,\Psi) \in \mathbf{B}(\mathfrak{H})$ and equality in (4.10) is a consequence.

In order to proceed, first observe at this stage that in addition to (4.9) one has

$$\operatorname{clos}(\Phi^*R(\Phi,\Psi)^*) = (R(\Phi,\Psi)\Phi)^*,$$

which follows from (4.10) since the right-hand side of (4.10) belongs to $\mathbf{B}(\mathfrak{K},\mathfrak{E})$. Next observe that it follows from the definition in (4.5) that there exists a partial isometry W such that

$$\mathcal{R}(\Phi,\Psi)^{\frac{1}{2}} = WR(\Phi,\Psi) \quad \text{and, consequently,} \quad \mathcal{R}(\Phi,\Psi)^{\frac{1}{2}} = R(\Phi,\Psi)^*W^*;$$

cf. (2.3). Therefore, one sees that

$$\mathcal{R}(\Phi,\Psi)^{\frac{1}{2}}\Phi = WR(\Phi,\Psi)\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{H}),$$

and, likewise,

$$\begin{cases} \Phi^*\mathcal{R}(\Phi,\Psi)^{\frac{1}{2}} = \Phi^*R(\Phi,\Psi)^*W^*, \\ \operatorname{clos}(\Phi^*\mathcal{R}(\Phi,\Psi)^{\frac{1}{2}}) = \operatorname{clos}(\Phi^*R(\Phi,\Psi)^*)W^*. \end{cases}$$

Thus the assertions in (4.7) follow. □

Observe that in Theorem 4.1 one can replace $\mathcal{R}(\Phi,\Psi)$ by its operator part $\mathcal{R}_{\rm op}(\Phi,\Psi)$ without changing otherwise the stated formulas. The formulation of Theorem 4.1 can be made more explicit by incorporating the pair of contractions $C_\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{H})$ and $C_\Psi \in \mathbf{B}(\mathfrak{E},\mathfrak{K})$ appearing in (4.3). Indeed, the formulas (4.3) and (4.4) imply that

$$(I-\mathcal{P})C_\Phi^{-1}\Phi = (I-\mathcal{P})C_\Phi^{-1}C_\Phi(\Phi^*\Phi+\Psi^*\Psi)^{\frac{1}{2}} = (\Phi^*\Phi+\Psi^*\Psi)^{\frac{1}{2}}$$

is a bounded nonnegative operator in $\mathbf{B}(\mathfrak{E})$. On the other hand,

$$((I-\mathcal{P})C_\Phi^{-1}\Phi)^* \supset \Phi^*(C_\Phi^*)^{-1}(I-\mathcal{P}) = (\Phi^*\Phi+\Psi^*\Psi)^{\frac{1}{2}}C_\Phi^*(C_\Phi^*)^{-1}(I-\mathcal{P})$$

and here $C_\Phi^*(C_\Phi^*)^{-1} = I \upharpoonright \operatorname{ran} C_\Phi^*$. Thus, clearly

$$\operatorname{clos}(\Phi^*(C_\Phi^*)^{-1})(I-\mathcal{P}) = ((I-\mathcal{P})C_\Phi^{-1}\Phi)^* = (\Phi^*\Phi+\Psi^*\Psi)^{\frac{1}{2}}$$

and consequently the adjoint in (4.2) is given by

$$(R(\Phi,\Psi)\Phi)^* = (C_\Psi(I-\mathcal{P})C_\Phi^{-1}\Phi)^* = \operatorname{clos}(\Phi^*(C_\Phi^*)^{-1})(I-\mathcal{P})C_\Psi^* = \Psi^*.$$

Now the formula (4.8) in Theorem 4.1 can be also rewritten as

$$B = \operatorname{clos}(\Phi^*(C_\Phi^*)^{-1})(I-\mathcal{P})C_\Psi^*C_\Psi(I-\mathcal{P})C_\Phi^{-1}\Phi = \operatorname{clos}(\Phi^*(C_\Phi^*)^{-1})C_\Psi^*C_\Psi C_\Phi^{-1}\Phi,$$

since $\ker C_\Phi \subset \ker C_\Psi$, so that $\operatorname{ran} C_\Psi^* \subset \overline{\operatorname{ran}}\, C_\Phi^* = \operatorname{ran}(I-\mathcal{P})$. Hence one has the following explicit expressions for the Ando-Radon-Nikodym derivative.

Corollary 4.2. *Let $A, B \in \mathbf{B}(\mathfrak{E})$ be a pair of nonnegative operators with representing maps Φ, Ψ be given by* (2.1) *and* (2.10) *and let the pair of contractions $C_\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{H})$ and $C_\Psi \in \mathbf{B}(\mathfrak{E},\mathfrak{K})$ be defined by* (4.3) – (4.3)*. Then the Ando-Radon-Nikodym derivative of B with respect to A in* (4.5) *in* (4.5) *is given by*

$$\mathcal{R}(\Phi,\Psi) = \mathcal{Q}(C_\Phi^{-1})^*C_\Psi^*C_\Psi C_\Phi^{-1}\mathcal{Q},$$

while the minimal Ando-Radon-Nikodym derivative has the form

$$\mathcal{R}_{\rm op}(\Phi,\Psi) = \mathcal{Q}(C_\Phi^*)^{-1}C_\Psi^*C_\Psi C_\Phi^{-1} = ((C_\Phi^*)^{-1})_{\rm reg}C_\Psi^*C_\Psi C_\Phi^{-1},$$

being considered as a nonnegative operator in $\overline{\operatorname{ran}}\,\Phi$.

The Ando-Radon-Nikodym derivative has been defined by the nonnegative selfadjoint relation $\mathcal{R}(\Phi,\Psi) = \mathcal{Q}R(\Phi,\Psi)^*R(\Phi,\Psi)\mathcal{Q} \in \mathbf{L}(\mathfrak{H})$. Clearly this derivative depends on the chosen representing maps Φ and Ψ. However, with $\Phi = UA^{\frac{1}{2}}$ and $\Psi = VB^{\frac{1}{2}}$ and denoting the Radon-Nikodym derivative of $B^{\frac{1}{2}}$ with respect to $A^{\frac{1}{2}}$ by $R(A^{\frac{1}{2}},B^{\frac{1}{2}})$, one obtains that

$$\mathcal{R}(\Phi,\Psi) = (U^{-1})^*\mathcal{R}(A^{\frac{1}{2}},B^{\frac{1}{2}})U^{-1},$$

see Lemma 7.1. Therefore, one sees the transformation formula

$$\Phi^*\mathcal{R}(\Phi,\Psi)\Phi = A^{\frac{1}{2}}\mathcal{R}(A^{\frac{1}{2}},B^{\frac{1}{2}})A^{\frac{1}{2}}.$$

Likewise, one sees the following formulas

$$\mathcal{R}(\Phi,\Psi)^{\frac{1}{2}}\Phi = (U^{-1})^*\mathcal{R}(A^{\frac{1}{2}},B^{\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}, \quad \Phi^*\mathcal{R}(\Phi,\Psi)^{\frac{1}{2}} = A^{\frac{1}{2}}\mathcal{R}(A^{\frac{1}{2}},B^{\frac{1}{2}})^{\frac{1}{2}}U^{-1},$$

which (again using Lemma 7.1) results in the transformation formula

$$\operatorname{clos}(\Phi^*\mathcal{R}(\Phi,\Psi)^{\frac{1}{2}})\,\mathcal{R}(\Phi,\Psi)^{\frac{1}{2}}\Phi = \operatorname{clos}(A^{\frac{1}{2}}\mathcal{R}(A^{\frac{1}{2}},B^{\frac{1}{2}})^{\frac{1}{2}})\,\mathcal{R}(A^{\frac{1}{2}},B^{\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

5. Ando-Lebesgue decompositions

Let $\mathfrak{E}$ be a Hilbert space and let $A, B \in \mathbf{B}(\mathfrak{E})$ be nonnegative operators. Moreover, let the representing maps $\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{H})$ and $\Psi \in \mathbf{B}(\mathfrak{E},\mathfrak{K})$ be given by (2.1) and (2.10). In this section a special sum decomposition of the operator B will be constructed whose summands are Ando-regular and Ando-singular with respect to A by choosing a special nonnegative contraction in Proposition 2.1. The main result of T. Ando [2, Theorem 2], see also [15, 25], can be reformulated as follows.

Theorem 5.1. *Let $A, B \in \mathbf{B}(\mathfrak{E})$ be a pair of nonnegative operators and assume that the representing maps Φ, Ψ are given by* (2.1) *and* (2.10)*, respectively. Let P be the orthogonal projection onto $\mathfrak{D}(\Phi,\Psi)^\perp$. Then*

$$B = B_{\mathrm{Reg}} + B_{\mathrm{Sing}}, \quad B_{\mathrm{Reg}} = \Psi^*(I-P)\Psi, \quad B_{\mathrm{Sing}} = \Psi^* P \Psi, \tag{5.1}$$

is a sum decomposition of B where $B_{\mathrm{Reg}}, B_{\mathrm{Sing}} \in \mathbf{B}(\mathfrak{E})$ are nonnegative. Furthermore, B_{Reg} is Ando-regular with respect to A and B_{Sing} is Ando-singular with respect to A.

Proof. The short proof here uses the corresponding decomposition of the linear relation $L(\Phi,\Psi)$ in (3.1) generated by the representing maps Φ and Ψ; cf. [8, Theorem 4.1], [10, Theorem 8.2]. Let P be the orthogonal projection onto $\mathfrak{D}(\Phi,\Psi)^\perp = \operatorname{mul} L(\Phi,\Psi)^{**}$. Then the Lebesgue decomposition of $L(\Phi,\Psi)$, see (7.2) in Subsection 7.1, reads as

$$\begin{aligned} L(\Phi,\Psi) &= (I-P)L(\Phi,\Psi) + PL(\Phi,\Psi) \\ &= L(\Phi,(I-P)\Psi) + L(\Phi,P\Psi) = L(\Phi,\Psi_{\mathrm{reg}}) + L(\Phi,\Psi_{\mathrm{sing}}), \end{aligned}$$

where one has defined

$$\Psi_{\mathrm{reg}} = (I-P)\Psi, \quad \Psi_{\mathrm{sing}} = P\Psi.$$

The Lebesgue decomposition of Ψ with respect to Φ is now given by

$$\Psi = \Psi_{\mathrm{reg}} + \Psi_{\mathrm{sing}},$$

see Section 7.2 and [10, 12]. From this decomposition one obtains a decomposition of B:

$$B = \Psi^*\Psi = \Psi^*\Psi_{\mathrm{reg}} + \Psi^*\Psi_{\mathrm{sing}} = \Psi^*(I-P)\Psi + \Psi^* P\Psi$$

into nonnegative components. Therefore, by Lemma 3.3, B_{Reg} is Ando-regular with respect to A and B_{Sing} is Ando-singular with respect to A. □

It is clear that B_{Reg} and B_{Sing} are mutually singular; see Corollary 2.3. Note that B is Ando-regular with respect to A if and only if $B = B_{\mathrm{Reg}}$ and B is Ando-singular with respect to A if and only if $B = B_{\mathrm{Sing}}$.

Definition 5.2. Let $A, B \in \mathbf{B}(\mathfrak{E})$ be a pair of nonnegative operators and assume that the representing maps Φ, Ψ are given by (2.1) and (2.10). Then the decomposition (5.1) is called the *Ando-Lebesgue decomposition* of B with respect to A.

The regular part B_{Reg} of B in Theorem 5.1 has the following characterization in terms of a maximality property.

Proposition 5.3. *Let $A, B \in \mathbf{B}(\mathfrak{E})$ be nonnegative and let $C \in \mathbf{B}(\mathfrak{E})$ be a nonnegative operator, which is Ando-regular with respect to A. If $C \le B$, then $C \le B_{\mathrm{Reg}}$.*

Proof. Write $C = \Xi^*\Xi$ for some $\Xi \in \mathbf{B}(\mathfrak{E},\mathfrak{L})$ where $\mathfrak{L}$ is a Hilbert space. Then $C \le B$ implies $\|\Xi\eta\| \le \|\Psi\eta\|$, $\eta \in \mathfrak{E}$. Therefore it follows from [12, Corollary 6.10] that $\|\Xi\eta\| \le \|\Psi_{\mathrm{reg}}\eta\|$, $\eta \in \mathfrak{E}$. This implies that $C \le B_{\mathrm{Reg}}$. □

The regular and singular parts B_{Reg} and B_{Sing} in (5.1) are given in terms of the representing maps Φ and Ψ in (2.1) and (2.10). From Proposition 5.3 it is clear that B_{Reg} and B_{Sing} do not depend on the choice of representing maps Φ and Ψ. Consequently, one may also use the pair $A^{\frac12}, B^{\frac12}$ as representing maps. This leads to the following observation:

$$B_{\mathrm{Reg}} = B^{\frac12}(I-P')B^{\frac12}, \quad B_{\mathrm{Sing}} = B^{\frac12}P'B^{\frac12}. \tag{5.2}$$

Here P' is the orthogonal projection from $\mathfrak{E}$ onto $\mathfrak{D}(A^{\frac{1}{2}}, B^{\frac{1}{2}})^{\perp}$ and

$$P' = V^* P V, \tag{5.3}$$

see also (2.6) and Subsection 7.1. However, in the last context, with the pair $A^{\frac{1}{2}}, B^{\frac{1}{2}}$ as representing maps, there is another way to consider the regular part $B_{\rm Reg}$ of B with respect to A, namely via the notion of parallel difference and the ensuing limit procedure below; see Subsection 7.4.

Proposition 5.4. *Let $A, B \in \mathbf{B}(\mathfrak{H})$ be a pair of nonnegative operators. Then*

$$B_{\rm Reg} = \lim_{n\to\infty} nA : B, \quad \textit{strongly.} \tag{5.4}$$

Proof. It suffices to express $B_{\rm Reg}$ as a parallel difference involving A and B:

$$B_{\rm Reg} = (A : B) \div B, \tag{5.5}$$

as this implies (5.4). For this purpose, apply Lemma 7.4, with $T = L(A^{\frac{1}{2}}, B^{\frac{1}{2}})$ and P' the orthogonal projection onto $\operatorname{mul} L(A^{\frac{1}{2}}, B^{\frac{1}{2}})^{**}$, to an element $\{f, f'\}$ of the form

$$\{f, f'\} = \{A^{\frac{1}{2}}\eta, B^{\frac{1}{2}}\eta\}, \quad \eta \in \mathfrak{E}.$$

In the right-hand side of (7.20) parametrize the running elements $\{g, g'\} \in T$ and $h \in \operatorname{dom} T$ as follows

$$\{g, g'\} = \{A^{\frac{1}{2}}\zeta, B^{\frac{1}{2}}\zeta\}, \quad \zeta \in \mathfrak{E}, \quad \text{and} \quad h = A^{\frac{1}{2}}\omega, \quad \omega \in \mathfrak{E}.$$

Then the identity (7.20) reads as

$$\begin{aligned}
\|(I-P')B^{1/2}\eta\|^2 &= \sup_{\omega\in\mathfrak{E}} \left\{ \inf_{\zeta\in\mathfrak{E}} \left\{ \|B^{\frac{1}{2}}\zeta\|^2 + \|A^{\frac{1}{2}}(\omega-\zeta)\|^2 \right\} - \|A^{\frac{1}{2}}(\eta-\omega)\|^2 \right\} \\
&= \sup_{\omega\in\mathfrak{E}} \left\{ \inf_{\zeta\in\mathfrak{E}} \{ (B\zeta,\zeta) + (A(\omega-\zeta), \omega-\zeta) \} - (A(\eta-\omega), \eta-\omega) \right\} \\
&= \sup_{\omega\in\mathfrak{E}} \big\{ ((A:B)\omega,\omega) - (A(\eta-\omega),\eta-\omega) \big\}.
\end{aligned}$$

The left-hand side equals $(B_{\rm Reg}\eta, \eta)$ (see (5.2)), and therefore it follows from (7.19) that (5.5) holds. $\square$

The background of the above argument is that the regular part $B_{\rm Reg}$ is a shorted operator in the following sense. If P' is the orthogonal projection from $\mathfrak{E}$ onto the closed linear subspace $\operatorname{mul} L(A^{\frac{1}{2}}, B^{\frac{1}{2}})^{**} = \mathfrak{D}(A^{\frac{1}{2}}, B^{\frac{1}{2}})^{\perp}$, then the orthogonal projection $I - P'$ maps onto the closure of the pre-image $B^{-\frac{1}{2}} \operatorname{ran} A^{\frac{1}{2}}$; cf. (1.1). In this sense $B^{\frac{1}{2}}(I-P')B^{\frac{1}{2}}$ is a shortening of B to the operator range $\operatorname{ran} A^{\frac{1}{2}}$; see [21, 22]. The shorted operator is the maximum of all operators dominated by B, which are almost dominated by A. Note that the sequence of operators $nA : B \in \mathbf{B}(\mathfrak{E})$ in Proposition 5.4 consists of nonnegative operators, which satisfy

(a) $nA : B \leq A$;
(b) $nA : B \leq (n+1)A : B$;
(c) $nA : B \to B_{\rm Reg}$ strongly.

6. Ando-Lebesgue-type decompositions

Let $A, B \in \mathbf{B}(\mathfrak{E})$ be a pair of nonnegative operators. In Definition 5.2 the special Ando-Lebesgue decomposition of B with respect to A was introduced. It is now quite natural to define the more general notion of Ando-Lebesgue-type decomposition.

Definition 6.1. Let $A, B \in \mathbf{B}(\mathfrak{E})$ be a pair of nonnegative operators. Then B is said to have an *Ando-Lebesgue-type decomposition with respect to* A if

$$B = B_1 + B_2,$$

where $B_1, B_2 \in \mathbf{B}(\mathfrak{E})$ are nonnegative operators, such that B_1 is Ando-regular with respect to A, while B_2 is Ando-singular with respect to A.

In this section all Ando-Lebesgue-type decompositions of B with respect to A will be determined by specifying the nonnegative contractions in Proposition 2.1. The following explicit description of the Ando-Lebesgue-type decompositions of B with respect to A seems to be new.

Theorem 6.2. *Let $A, B \in \mathbf{B}(\mathfrak{E})$ be a pair of nonnegative operators and assume that the representing maps Φ, Ψ are given by* (2.1) *and* (2.10)*. Assume that B has a sum decomposition*

$$B = B_1 + B_2,$$

where the summands B_1 and B_2 are given by

$$B_1 = \Psi^*(I-K)\Psi, \quad B_2 = \Psi^* K \Psi, \tag{6.1}$$

with a nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$. Then B_1 is Ando-regular with respect to A if and only if

$$\mathfrak{D}(\Phi, (I-K)^{\frac{1}{2}}\Psi) \text{ is dense in } \mathfrak{K}. \tag{6.2}$$

Likewise, B_2 is Ando-singular with respect to Φ if and only if

$$\operatorname{ran} \Phi^* \cap \operatorname{ran} (K^{\frac{1}{2}}\Psi)^* = \{0\}. \tag{6.3}$$

Moreover, B_1 and B_2 are mutually singular if and only if the nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$ is an orthogonal projection.

Proof. By the sum decomposition in Proposition 2.1 it suffices to determine the nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$ such that $\Psi^*(I-K)\Psi$ is Ando-regular with respect to A and $\Psi^* K \Psi$ is Ando-singular with respect to A. Now recall from Lemma 3.3 that $\Psi^*(I-K)\Psi$ is Ando-regular with respect to A if and only if $(I-K)^{\frac{1}{2}}\Psi$ is regular with respect to Φ. This is the case if and only if the relation

$$L(\Phi, (I-K)^{\frac{1}{2}}\Psi) \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$$

is regular or, equivalently, (6.2) holds. Likewise, by Lemma 3.3, $\Psi^* K \Psi$ is Ando-singular with respect to A if and only if $K^{\frac{1}{2}}\Psi$ is regular with respect to Φ. This is the case if and only if the relation

$$L(\Phi, K^{\frac{1}{2}}\Psi) \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$$

is singular or, equivalently, (6.3) holds. Finally, for the characterization of mutual singular summands, apply Corollary 2.3. $\square$

The main results for the Lebesgue-type decompositions of a nonnegative operator $B \in \mathbf{B}(\mathfrak{E})$ in terms of a nonnegative operator $A \in \mathbf{B}(\mathfrak{E})$ are Theorem 5.1 and Theorem 6.2, stated in terms of their representing maps Φ and Ψ in (2.1) and (2.10). It may be helpful to compare these decompositions of the nonnegative form $(B\cdot,\cdot)$ with respect to the nonnegative form $(A\cdot,\cdot)$ with the analogous decompositions for the corresponding representing maps Φ and Ψ. With any nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$ one can decompose Ψ and $\Psi^*\Psi$ as follows

$$\Psi = (I-K)\Psi + K\Psi, \quad \Psi^*\Psi = \Psi^*(I-K)\Psi + \Psi^*K\Psi. \tag{6.4}$$

The first identity in (6.4) is a Lebesgue-type decomposition of Ψ with respect to Φ precisely when

$$(I-K)\Psi \text{ is regular and } K\Psi \text{ is singular,}$$

with respect to Φ, while for the Lebesgue-type decompositions of the form $\Psi^*\Psi$ with respect to $\Phi^*\Phi$ one needs

$$(I-K)^{\frac{1}{2}}\Psi \text{ is regular and } K^{\frac{1}{2}}\Psi \text{ is singular,}$$

with respect to Φ. Clearly, these descriptions are the same when $K \in \mathbf{B}(\mathfrak{K})$ is an orthogonal projection as, for instance, in Theorem 5.1. In the general case of Theorem 6.2 there is no equivalence: one can only say that $(I-K)\Psi$ is regular with respect to Φ implies that $(I-K)^{\frac{1}{2}}\Psi$ is regular with respect to Φ, and, likewise, that $K^{\frac{1}{2}}\Psi$ is singular with respect to Φ implies that $K\Psi$ is singular with respect to Φ.

The following uniqueness result goes back to Ando [2]; similar results in a more general setting can be found in [7, 8, 12].

Theorem 6.3. *Let $A, B \in \mathbf{B}(\mathfrak{E})$ be a pair of nonnegative operators and assume that the representing maps Φ, Ψ are given by* (2.1) *and* (2.10)*. Then the following statements are equivalent:*

(i) *B admits a unique Ando-Lebesgue-type decomposition with respect to A given by* (5.1)*;*
(ii) *$\mathfrak{D}(\Phi, \Psi)$ is closed.*

Proof. The operator B has a unique Ando-Lebesgue-type decomposition with respect to A if and only if the operator range relation $L(\Phi, \Psi)$ has a unique Lebesgue-type decomposition. This last assertion is the case if and only if $\mathfrak{D}(\Phi, \Psi)$ is closed; cf. [10, Corollary 8.6] □

It should be noted that when the conditions (i) and (ii) do not hold, then there exist infinitely many Ando-Lebesgue-type decompositions of B with respect to A, where the summands B_1 and B_2 in Theorem 6.2 are mutually singular as well infinitely many such decompositions, where B_1 and B_2 are not mutually singular. This follows from [12, Theorems 5.7, 6.9] by means of constructions given in [12, Lemma 5.5, Corollary 5.8].

The results in this section for the pair $A, B \in \mathbf{B}(\mathfrak{E})$ of nonnegative operators are formulated in terms of their representing maps Φ, Ψ given by (2.1) and (2.10). Recall that with $\Phi = UA^{\frac{1}{2}}$ and $\Psi = VB^{\frac{1}{2}}$ one has the decomposition (2.6) instead of (6.1), where the nonnegative contraction K and K' are connected by

$$K' = V^*KV \quad \text{and} \quad K = VK'V^*,$$

see also (5.3). As noted before, the nonnegative contractions K and K' are not unique, but their compressions to $\overline{\operatorname{ran}}\,\Psi$ and $\overline{\operatorname{ran}}\,B^{\frac{1}{2}}$, respectively, are uniquely determined. So without loss of generality one could require that $\operatorname{ran} K \subset \overline{\operatorname{ran}}\,\Psi$ and $\operatorname{ran} K' \subset \overline{\operatorname{ran}}\,B^{\frac{1}{2}}$. In this situation one observes

$$\Phi = UA^{\frac{1}{2}}, \quad (I-K)^{\frac{1}{2}}\Psi = V(I-K')^{\frac{1}{2}}B^{\frac{1}{2}}, \quad K^{\frac{1}{2}}\Psi = VK'^{\frac{1}{2}}B^{\frac{1}{2}}.$$

In fact, by Lemma 7.1 and Corollary 7.2 $L(\Phi,(I-K)^{\frac{1}{2}}\Psi)$ is regular if and only if $L(A^{\frac{1}{2}},(I-K')^{\frac{1}{2}}B^{\frac{1}{2}})$ is regular. Likewise, one sees that $L(\Phi,K^{\frac{1}{2}}\Psi)$ is singular if and only if $L(A^{\frac{1}{2}},K'^{\frac{1}{2}}B^{\frac{1}{2}})$ is singular. Consequently, the Ando-regularity of the corresponding terms in (2.6) and (6.1) happen simultaneously and, likewise, this is the situation for the Ando-singularity of the corresponding terms in (2.6) and (6.1). For completeness it is remarked that the condition $\mathfrak{D}(\Phi,\Psi)$ is closed in Theorem 6.3 is equivalent to the condition $\mathfrak{D}(A^{\frac{1}{2}},B^{\frac{1}{2}})$ is closed; cf. Corollary 7.2.

7. Appendix

This appendix contains some useful facts concerning linear relations and other notions which are in the background of this paper.

7.1. **Linear relations.** The linear relations from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ are denoted by $\mathbf{L}(\mathfrak{H},\mathfrak{K})$. A linear relation $\mathbf{L}(\mathfrak{H},\mathfrak{K})$ is closed if its graph in $\mathfrak{H}\times\mathfrak{K}$ is closed. If $H\in\mathbf{L}(\mathfrak{H},\mathfrak{K})$, then $\operatorname{dom} H$, $\operatorname{ran} H$, $\ker H$, and $\operatorname{mul} H$ stand for the domain, range, kernel, and multivalued part of H. For $H,K\in\mathbf{L}(\mathfrak{H},\mathfrak{K})$ the componentwise sum (i.e. the linear span of the graphs of H and K) is denoted by $H \,\widehat{+}\, K$, while the usual sum is denoted by $H+K$:

$$H+K=\bigl\{\{\eta,\eta'+\eta''\}:\,\{\eta,\eta'\}\in H,\,\{\eta,\eta''\}\in K\bigr\}.$$

The sum $H+K$ is called strict if in the above sum the element $\eta'+\eta''$ uniquely determines η' and η''. In other words, the sum $H+K$ is strict if and only if the decomposition $\operatorname{mul}(H+K)=\operatorname{mul} H+\operatorname{mul} K$ is direct. The product of $H\in\mathbf{L}(\mathfrak{H},\mathfrak{K})$ and $K\in\mathbf{B}(\mathfrak{K},\mathfrak{L})$ is denoted by $KH\in\mathbf{L}(\mathfrak{H},\mathfrak{L})$. The adjoint of $H\in\mathbf{L}(\mathfrak{H},\mathfrak{K})$ is a closed linear relation in $\mathbf{L}(\mathfrak{K},\mathfrak{H})$ defined by

$$H^*=\bigl\{\{\varphi,\psi\}\in\mathfrak{K}\times\mathfrak{H}:\,(\psi,\eta)=(\varphi,\zeta)\text{ for all }\{\eta,\zeta\}\in H\bigr\},$$

so that

$$(\operatorname{dom} H)^\perp=\operatorname{mul} H^* \quad\text{and}\quad (\operatorname{ran} H)^\perp=\ker H^*.$$

It is not difficult to see that H^{**} is the closure of H in $\mathfrak{H}\times\mathfrak{K}$. The adjoint of the sum of relations behaves as usual: $H^*+K^*\subset(H+K)^*$ with equality if for instance H or K belongs to $\mathbf{B}(\mathfrak{H},\mathfrak{K})$. For the product it is useful to recall that

$$K^*H^*\subset(HK)^*,$$

with equality if $H\in\mathbf{B}(\mathfrak{H},\mathfrak{K})$.

Let $H\in\mathbf{L}(\mathfrak{H},\mathfrak{K})$ be a linear relation and define its *regular (closable) part* $H_{\rm reg}$ and *singular part* $H_{\rm sing}$ by

$$H_{\rm reg}=QH,\quad H_{\rm sing}=(I-Q)H, \tag{7.1}$$

where Q stands for the orthogonal projection onto $\overline{\operatorname{dom}}\,H^*$. Then H is regular if and only if $H=H_{\rm reg}$ or, equivalently, if $\operatorname{mul} H^{**}=\{0\}$ and H is singular if and only if $H = H_{\mathrm{sing}}$; see Section 3. In particular, this leads to the following sum decomposition

$$H = H_{\mathrm{reg}} + H_{\mathrm{sing}}, \tag{7.2}$$

which is called the *Lebesgue decomposition of the linear relation* H; see [7, 9, 12]. In the special case when H is closed, the regular part is usually called the *orthogonal operator part* of H and then H decomposes as the orthogonal sum (in the graph sense) as follows

$$H = H_{\mathrm{op}} \mathbin{\widehat{\oplus}} (\{0\} \times \operatorname{mul} H). \tag{7.3}$$

Further information about linear relations can be found in [4].

The following observations deal with a special class of transformations of a linear relation $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ by means of bounded linear operators. Let $L \in \mathbf{B}(\mathfrak{H}, \mathfrak{L})$, and $R \in \mathbf{B}(\mathfrak{K}, \mathfrak{R})$, then it follows from the definition of multiplication that one can write $RT \in \mathbf{L}(\mathfrak{H}, \mathfrak{R})$ and $TL^{-1} \in \mathbf{L}(\mathfrak{L}, \mathfrak{K})$ as

$$RT = \big\{ \{\eta, R\eta'\} : \{\eta, \eta'\} \in T \big\} \quad \text{and} \quad TL^{-1} = \big\{ \{L\eta, \eta'\} : \{\eta, \eta'\} \in T \big\}. \tag{7.4}$$

Consequently, the linear relation RTL^{-1} belongs to $\mathbf{L}(\mathfrak{L}, \mathfrak{R})$ and can be written as

$$RTL^{-1} = \big\{ \{L\eta, R\eta'\} : \{\eta, \eta'\} \in T \big\}.$$

From this representation one sees immediately that

$$\operatorname{dom} RTL^{-1} = L \operatorname{dom} T \quad \text{and} \quad \operatorname{ran} RTL^{-1} = R \operatorname{ran} T. \tag{7.5}$$

In the special case that L is the orthogonal projection onto $\overline{\operatorname{dom}}\, T$ and R is the orthogonal projection onto $\overline{\operatorname{ran}}\, T$, one sees that $RTL^{-1} = T$. The following results for the adjoints are now easily obtained; cf. [4, Proposition 1.3.9].

Lemma 7.1. *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, $L \in \mathbf{B}(\mathfrak{H}, \mathfrak{L})$, and $R \in \mathbf{B}(\mathfrak{K}, \mathfrak{R})$. Then the relation $S = RTL^{-1} \in \mathbf{L}(\mathfrak{L}, \mathfrak{R})$ satisfies*

$$S^* = (L^{-1})^* T^* R^* \quad \text{and} \quad S^{**} = RT^{**}L^{-1}. \tag{7.6}$$

Moreover, if L^*L is the orthogonal projection onto $\overline{\operatorname{dom}}\, T$ and R^*R is the orthogonal projection onto $\overline{\operatorname{ran}}\, T$, then $T = R^* S (L^{-1})^*$ and

$$T^* = L^{-1} S^* R \quad \text{and} \quad T^{**} = R^* S^{**} (L^{-1})^*. \tag{7.7}$$

In particular, when these conditions are satisfied, the nonnegative selfadjoint relations $S^*S^{**} \in \mathbf{L}(\mathfrak{L})$ and $S^{**}S^* \in \mathbf{L}(\mathfrak{R})$ are given by

$$S^*S^{**} = (L^{-1})^* T^* T^{**} L^{-1} \quad \text{and} \quad S^{**}S^* = RT^{**}T^*R^*. \tag{7.8}$$

Proof. Taking adjoints in the products RT and TL^{-1} leads to $T^*R^* \subset (RT)^*$ and $(L^{-1})^*T^* \subset (TL^{-1})^*$. By means of (7.4) it is not difficult to see that these inclusions are actually identities:

$$T^*R^* = (RT)^* \quad \text{and} \quad (L^{-1})^*T^* = (TL^{-1})^*. \tag{7.9}$$

Likewise, taking adjoints in the products T^*R^* and $(L^{-1})^*T^*$ which occur in (7.9) gives $RT^{**} \subset (RT)^{**}$ and $T^{**}L^{-1} \subset (TL^{-1})^{**}$. Again, by means of (7.4) it is straightforward to see that

$$RT^{**} = (RT)^{**} \quad \text{and} \quad T^{**}L^{-1} = (TL^{-1})^{**}. \tag{7.10}$$

By successively applying (7.9) and (7.10) one obtains the required identities in (7.6), respectively.

From $S = RTL^{-1}$ it is clear that $R^*S(L^{-1})^* = R^*RT(L^*L)^{-1}$ in the sense of linear relations. If L^*L and R^*R are orthogonal projections onto $\overline{\operatorname{dom}}\, T$ and $\overline{\operatorname{ran}}\, T$, then $R^*S(L^{-1})^* = T$. The identities in (7.8) follow in a similar way from (7.9). □

The following consequences of Lemma 7.1 are quite useful to observe.

Corollary 7.2. *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, $L \in \mathbf{B}(\mathfrak{H}, \mathfrak{L})$, and $R \in \mathbf{B}(\mathfrak{K}, \mathfrak{R})$, and define $S = RTL^{-1}$. Assume that L^*L is the orthogonal projection onto $\overline{\operatorname{dom}}\, T$ and R^*R is the orthogonal projection onto $\overline{\operatorname{ran}}\, T$. Then*

(a) *S is regular if and only if T is regular;*
(b) *S is singular if and only if T is singular;*
(c) *$\operatorname{dom} S^*$, $\operatorname{dom} S^{**}$, $\operatorname{dom} T^*$, and $\operatorname{dom} T^{**}$ are simultaneously closed subspaces.*

Proof. (a) Assume that T is regular and let $\{0, \chi\} \in S^{**}$. Then there exists a sequence $\{\eta_n, \eta_n'\} \in T$ such that $\{L\eta_n, R\eta_n'\} \to \{0, \chi\}$. Then clearly $\chi \in \overline{\operatorname{ran}}\, R$. Moreover $\eta_n = L^*L\eta_n \to 0$ and, likewise, $\eta_n' = R^*R\eta_n' \to R^*\chi$. Since T is regular, it follows that $\chi \in \ker R^*$, and therefore $\chi = 0$. Thus S is regular. The reverse statement follows by symmetry.

(b) Assume that T is singular, so that $T^{**} = \mathfrak{X} \times \mathfrak{Y}$. Therefore, $S^{**} = L\mathfrak{X} \times R\mathfrak{Y}$; here $L\mathfrak{X}$ and $R\mathfrak{Y}$ are closed linear subspaces of $\mathfrak{H}$ and $\mathfrak{K}$, respectively. Again the reverse statement holds by symmetry.

(c) Recall that $\operatorname{dom} H$ of a closed linear relation H is closed if and only if $\operatorname{dom} H^*$ is closed, see e.g. [4, Theorem 1.3.5]. On the other hand, using (7.5) one concludes from (7.6) and (7.7) that $\operatorname{dom} S^{**}$ and $\operatorname{dom} T^{**}$ are connected by

$$\operatorname{dom} S^{**} = L \operatorname{dom} T^{**} \quad \text{and} \quad \operatorname{dom} T^{**} = L^* \operatorname{dom} S^{**}.$$

This leads to (c). □

Corollary (7.2) shows that the choice of the (minimal or non-minimal) representing maps Ψ and Φ for the forms (2.1) and (2.11) do not affect the regularity or singularity of the linear relation $L(\Phi, \Psi)$ in (3.1); see Section 3 and Subsection 7.2 below. This implies also that the Lebesgue-type decompositions of the linear relation $L(\Phi, \Psi)$ obey the same transforms determined by a pair of partial isometries which connect two different representing maps for the forms (2.1) and (2.11); see the discussion after Theorem 6.3.

Notice also that the statement (c) in Corollary (7.2) concerns the uniqueness of Lebesgue-type decompositions by the characterization in Theorem 6.3.

7.2. **Operator range relations.** Let $\Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $\Psi \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ and associate with these operators the linear relation $L(\Phi, \Psi) \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, given by (3.1). Then $L(\Phi, \Psi)$ is called an *operator range relation* in $\mathbf{L}(\mathfrak{H}, \mathfrak{K})$ and any operator range relation in $\mathbf{L}(\mathfrak{H}, \mathfrak{K})$ is of this form; see [4, Theorem 1.10.1]. Note that its domain and range are given by

$$\operatorname{dom} L(\Phi, \Psi) = \operatorname{ran} \Phi \quad \text{and} \quad \operatorname{ran} L(\Phi, \Psi) = \operatorname{ran} \Psi,$$

while its kernel and multivalued part are given by

$$\ker L(\Phi, \Psi) = \Phi(\ker \Psi) \quad \text{and} \quad \operatorname{mul} L(\Phi, \Psi) = \Psi(\ker \Phi).$$

It is clear from (3.1) that

$$L(\Phi, \Psi)^* = \big\{ \{k, h\} \in \mathfrak{K} \times \mathfrak{H} : \Psi^* k = \Phi^* h \big\}.$$

Hence, with the linear subspaces $\mathfrak{D}(\Phi,\Psi)$ and $\mathfrak{R}(\Phi,\Psi)$ defined by (3.2) it follows that the domain and the range of $L(\Phi,\Psi)^*$ are given by

$$\operatorname{dom} L(\Phi,\Psi)^* = \mathfrak{D}(\Phi,\Psi), \quad \operatorname{ran} L(\Phi,\Psi)^* = \mathfrak{R}(\Phi,\Psi),$$

and, likewise, the kernel and multivalued part of $L(\Phi,\Psi)^*$ are given by

$$\ker\, L(\Phi,\Psi)^* = \ker\, \Psi^*, \quad \operatorname{mul} L(\Phi,\Psi)^* = \ker\, \Phi^*. \tag{7.11}$$

Thus the criteria for $L(\Phi,\Psi)$ to be regular or singular in Definition 3.1 are straightforward.

Lemma 7.3. *Let $\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{H})$, $\Psi_1 \in \mathbf{B}(\mathfrak{E},\mathfrak{K})$, $\Psi_2 \in \mathbf{B}(\mathfrak{E},\mathfrak{K})$, and assume that*

$$\Psi = \Psi_1 + \Psi_2 \quad \textit{and} \quad \Psi(\ker\, \Phi) = \Psi_1(\ker\, \Phi) + \Psi_2(\ker\, \Phi). \tag{7.12}$$

Then there is the identity

$$L(\Phi,\Psi) = L(\Phi,\Psi_1) + L(\Phi,\Psi_2). \tag{7.13}$$

The sum in (7.13) *is strict if and only if*

$$\Psi_1(\ker\, \Phi) \cap \Psi_2(\ker\, \Phi) = \{0\}.$$

In particular, if $\Psi_1(\ker\, \Phi) = \{0\}$ or $\Psi_2(\ker\, \Phi) = \{0\}$, then the second identity in (7.12) *is automatically satisfied.*

Proof. In general, there is the inclusion

$$L(\Phi,\Psi_1+\Psi_2) \subset L(\Phi,\Psi_1) + L(\Phi,\Psi_2).$$

Moreover, there is domain equality

$$\operatorname{dom} L(\Phi,\Psi) = \operatorname{dom} L(\Phi,\Psi_1) = \operatorname{dom} L(\Phi,\Psi_2),$$

and there is also the equality involving multivalued parts

$$\operatorname{mul} L(\Phi,\Psi_1+\Psi_2) = \operatorname{mul} L(\Phi,\Psi_1) + \operatorname{mul} L(\Phi,\Psi_2). \tag{7.14}$$

Hence from the classical Arens result in [3] it follows that (7.13) holds. The sum in (7.13) is strict (i.e., the sum in (7.14) is direct) if and only if

$$\Psi_1(\ker\, \Phi) \cap \Psi_2(\ker\, \Phi) = \{0\}. \qquad \square$$

7.3. **The Radon-Nikodym derivative.** Let $\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{H})$, $\Psi \in \mathbf{B}(\mathfrak{E},\mathfrak{K})$, and let $L(\Phi,\Psi) \in \mathbf{L}(\mathfrak{H},\mathfrak{K})$ be the corresponding operator range relation in (3.1). In this subsection the relevant facts concerning Radon-Nikodym derivatives are recalled from [10]. First observe that by Douglas lemma [5] the inequalities

$$\Phi^*\Phi \le \Phi^*\Phi + \Psi^*\Psi, \quad \Psi^*\Psi \le \Phi^*\Phi + \Psi^*\Psi$$

hold precisely when there exists a pair of contractions $C_\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{H})$ and $C_\Psi \in \mathbf{B}(\mathfrak{E},\mathfrak{K})$, such that

$$\Phi = C_\Phi\,(\Phi^*\Phi + \Psi^*\Psi)^{\frac{1}{2}}, \quad \Psi = C_\Psi\,(\Phi^*\Phi + \Psi^*\Psi)^{\frac{1}{2}}. \tag{7.15}$$

Here the contractions C_A and C_B are uniquely determined by the conditions

$$\ker\, (\Phi^*\Phi + \Psi^*\Psi) \subset \ker\, C_\Phi, \quad \ker\, (\Phi^*\Phi + \Psi^*\Psi) \subset \ker\, C_\Psi, \tag{7.16}$$

respectively; in what follows this will be tactitly assumed. In this case (7.15) implies that $\ker\, C_\Phi^* = \ker\, \Phi^*$ and $\ker\, C_\Psi^* = \ker\, \Psi^*$ and, equivalently,

$$\overline{\operatorname{ran}}\, C_\Phi = \overline{\operatorname{ran}}\, \Phi \quad \text{and} \quad \overline{\operatorname{ran}}\, C_\Psi = \overline{\operatorname{ran}}\, \Psi.$$

The connection between the linear relations $L(\Phi,\Psi)$ and $L(C_\Phi,C_\Psi)$ in $\mathbf{L}(\mathfrak{H},\mathfrak{K})$ is given by

$$L(\Phi,\Psi)^{**}=L(C_\Phi,C_\Psi)=C_\Psi C_\Phi^{-1} \tag{7.17}$$

so that $L(C_\Phi,C_\Psi)$ is automatically closed; cf. [10, Theorem 4.4]. Due to (7.17) and $\operatorname{mul}(C_\Phi)^{-1}=\ker C_\Phi$, the relation $L(\Phi,\Psi)$ is closable if and only if the product relation $C_\Psi(C_\Phi)^{-1}$ is an operator. In the present case, where Ψ is regular with respect to Φ, the closed linear operator in (7.17) is the *Radon-Nikodym derivative* of Ψ with respect to Φ:

$$R(\Phi,\Psi):=L(C_\Phi,C_\Psi)\in\mathbf{L}(\mathfrak{H},\mathfrak{K})$$

If $\mathcal{P}$ is the orthogonal projection from $\mathfrak{E}$ onto $\ker C_\Phi$, then $R(\Phi,\Psi)$ may be rewritten as

$$R(\Phi,\Psi)=C_\Psi(C_\Phi)^{-1}=C_\Psi\,(I-\mathcal{P})(C_\Phi)^{-1}=C_\Psi\,((C_\Phi)^{-1})_{\rm reg}, \tag{7.18}$$

The Radon-Nikodym $R(\Phi,\Psi)$ is the smallest of all closed linear operators $C\in\mathbf{L}(\mathfrak{H},\mathfrak{K})$ with the property that $\Psi=C\Phi$; see [10, Theorem 7.4]. Observe, that

$$\operatorname{dom}R(\Phi,\Psi)=\operatorname{ran}C_\Phi,\quad \operatorname{ran}R(\Phi,\Psi)=\operatorname{ran}C_\Psi.$$

It is clear that $\Phi\in\mathbf{B}(\mathfrak{E},\mathfrak{H})$ and $\Psi\in\mathbf{B}(\mathfrak{E},\mathfrak{K})$ are (contractively) dominated by $(\Phi^*\Phi+\Psi^*\Psi)^{\frac12}$ and due to (7.15) the operators C_Φ and C_Ψ or, more precisely, their restrictions to $\overline{\operatorname{ran}}\,(\Phi^*\Phi+\Psi^*\Psi)^{\frac12}$, can be interpreted as the Radon-Nikodym derivatives of Φ and Ψ with respect to $(\Phi^*\Phi+\Psi^*\Psi)^{\frac12}$, respectively; cf. [10, Lemma 7.3]. Thus, under the assumption that $L(\Phi,\Psi)$ is closable or, equivalently, that $\ker C_\Phi\subset\ker C_\Psi$, it follows that the expression $C_\Psi(C_\Phi)^{-1}$ is the analog of the usual formula from measure theory; see [10, Theorem 7.7 & (7.5)-(7.6)].

Under the assumption that $\Psi\in\mathbf{B}(\mathfrak{E},\mathfrak{K})$ is regular with respect to $\Phi\in\mathbf{B}(\mathfrak{E},\mathfrak{H})$ it follows that $R(\Phi,\Psi)\in\mathbf{L}(\mathfrak{H},\mathfrak{K})$ is a closed linear operator, which is in general unbounded. The corresponding adjoint relation $R(\Phi,\Psi)^*\in\mathbf{L}(\mathfrak{K},\mathfrak{H})$ is densely defined as follows from

$$(\operatorname{dom}R(\Phi,\Psi)^*)^\perp=\operatorname{mul}R(\Phi,\Psi)^{**}=\{0\},$$

(since Ψ is regular with respect to Φ; cf. (3.2)). Moreover, (7.11) implies that

$$\operatorname{mul}R(\Phi,\Psi)^*=\operatorname{mul}L(\Phi,\Psi)^*=\ker\Phi^*.$$

7.4. **Parallel sums and differences.** Recall that for a pair of nonnegative operators $A,B\in\mathbf{B}(\mathfrak{K})$ the *parallel sum* $A:B\in\mathbf{B}(\mathfrak{K})$ is a nonnegative operator that is defined by

$$((A:B)\eta,\eta)=\inf\{(A(h+\eta),h+\eta)+(Bh,h):\,h\in\mathfrak{K}\},\quad \eta\in\mathfrak{K},$$

(and polarization), cf. [6]. If, for instance, $\operatorname{ran}(A+B)$ is closed, then

$$A:B=A(A+B)^{-1}B.$$

The *parallel difference* $(A:B)\div B\in\mathbf{B}(\mathfrak{K})$ of $A:B$ and A is a nonnegative operator, given by

$$((A:B)\div A)\eta,\eta)=\sup_{\omega\in\mathfrak{H}}\big\{((A:B)\,\omega,\omega)-(A(\eta-\omega),\eta-\omega)\big\},\quad \eta\in\mathfrak{H}. \tag{7.19}$$

It can be characterized as the strong limit of a nondecreasing sequence

$$(A:B)\div A=\lim_{n\to\infty}nA:B,\quad\text{strongly}.$$

The details concerning parallel sums and differences can be found e.g. in [21].

The following result is useful to decide when a relation is regular; cf. [9, Lemma 5.3, Theorem 5.4]. Its formulation is suitable for an application to operator range relations.

Lemma 7.4. *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be a linear relation and let P be the orthogonal projection onto* $\operatorname{mul} T^{**}$. *If $\{f, f'\} \in T$, then there is the identity*

$$\|(I-P)f'\|^2 = \sup_{h \in \operatorname{dom} T} \left\{ \inf_{\{g,g'\} \in T} \left\{ \|g'\|^2 + \|h-g\|^2 \right\} - \|f-h\|^2 \right\}. \tag{7.20}$$

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