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Inverse Sturm-Liouville problem of recovering coefficients in boundary conditions

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Abstract. We consider the spectral Sturm-Liouville problem on an equilateral star graph of three edges with the standard conditions at the interior vertex and the Robin conditions at the pendant vertices. It is shown that when the potentials are zeros on the edges, the asymptotics of the eigenvalues determine the shape of the graph and the constants in the Roben conditions. This method can be applied for the case of more complicated equilateral simple connected graphs.

Анотація. Розглянута спектральна задача Штурма-Ліувілля на рівнобічному зірковому графі з трьома ребрами з стандартними умовами у центральній вершині та умовами Робена у висячих вершинах. Показано, що, у випадку нульових потенціалів на ребрах асимптотика власних значень визначає форму графа та сталі в умовах Робена. Цей метод може бути застосований у випадку більш складних зв'язаних рівнобічних графів.

1. Introduction

There are three types of inverse problems in the theory of quantum graphs.

The first problem, where the shape of a graph and conditions at the vertices are known. We need to find potentials of the Sturm-Liouville equations on the edges. For the investigations dedicated to this problem see, e.g. [2,9].

In the second problem, conditions at the vertices and spectrum are known. We need to find the shape of the graph. For example, works [5,7,8,

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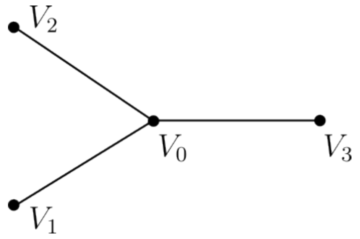
10,11] are concerned with this question. The works in this direction started in the famous article by M. Kac «Can one hear the shape of a drum?» [6].

In the third problem, the shape of the graph and the spectrum are known, and we want to find the coefficients in the conditions at the vertices, see e.g. [1]. Our paper is about the third type of inverse problems.

Let us consider boundary value problem on a star equilateral metric graph, which has three edges of length one each:



$$-y_j'' = \lambda^2\, y_j, \quad x \in [0;1], \quad j = 1,2,3, \tag{1.1}$$

$$y_1(1) = y_2(1) = y_3(1), \tag{1.2}$$

$$y_1'(1) + y_2'(1) + y_3'(1) = 0, \tag{1.3}$$

$$y_1'(0) - b_1 y_1(0) = 0, \tag{1.4}$$

$$y_2'(0) - b_2 y_2(0) = 0, \tag{1.5}$$

$$y_3'(0) - b_3 y_3(0) = 0. \tag{1.6}$$

The boundary condition $y'(0) + by(0) = 0$, where $b \in \mathbb{R}$, is called the *Robin condition.*

In classical mechanics, this condition has the following physical meaning. Let us imagine that the end of a string has a weightless ring which can move without friction in the direction orthogonal to the equilibrium position of the string. The string is stretched and can move in the transverse direction.

Vibrations of such a string can be described by equation

$$\frac{\partial^2 U}{\partial x^2} + \rho(x)\frac{\partial^2 U}{\partial t^2} = 0, \quad x \in [0;l], \tag{1.7}$$

where $\rho(x)$ is the linear density of the string, x is the spatial coordinate, t is time and $U(x,t)$ is the transverse displacement of the point on the string with the coordinate x at the time t. If we look for a solution of (1.7) of the form

$$U(s,t) = v(s) \cdot e^{i\lambda t}, \quad s \in [0;l],$$

then we obtain

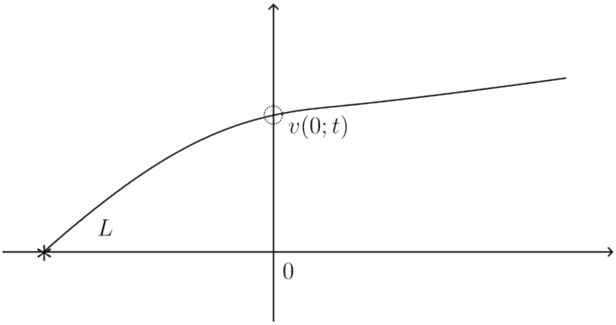
$$\frac{\partial^2 v}{\partial s^2} + \lambda^2 \rho(s) v = 0. \tag{1.8}$$

If the left end of the string is free to move in the direction orthogonal to the equilibrium position of the string, then we face the Neumann condition

$$\frac{\partial v}{\partial s}|_{s=0} = 0.$$

But let us assume that this ring is connected to a certain fixed point $x = -L$ by an elastic weightless thread as is shown on the figure below. Then,

$$\frac{v(0;t)}{L} = \frac{\partial v}{\partial s}|_{s=0}. \tag{1.9}$$



Using the Liouville transformation [4]

$$x = \int_0^s \rho^{\frac{1}{2}}(r)dr,$$
$$y(\lambda,\ x) = \rho^{\frac{1}{4}}(s(x))v(\lambda,\ s(x))$$

and substituting $v(\lambda,\ s(x)) = \rho^{-\frac{1}{4}}(s(x))y(\lambda,\ x)$ in equation (1.8) and (1.9), we arrive at:

$$\begin{gathered} -y'' + q(x)y = \lambda^2 y, \\ y'(0) + by(0) = 0 \end{gathered} \tag{1.10}$$

where

$$b = -\frac{1}{4}\rho^{-\frac{3}{4}}(0) - \frac{\rho(0)^{\frac{1}{4}}}{L}.$$

In quantum mechanics, Robin‘s condition (1.10) describes the partial passage of a particle through the point $x = 0$.

2. Eigenvalue asymptotics

Let us return to problem (1.1)-(1.6). We express the general solution of equation (1.1) in the following form:

$$y_j = A_j \frac{\sin \lambda x}{\lambda} + B_j \cos \lambda x, \tag{2.1}$$

where A_j, B_j – real numbers. Substituting (2.1) into (1.4)-(1.6) we arrive at $A_j = b_j B_j$. Thus,

$$y_j = b_j B_j \frac{\sin \lambda x}{\lambda} + B_j \cos \lambda x. \tag{2.2}$$

Substituting (2.2) into (1.2)-(1.3), we obtain the following system of equations for B_1, B_2, B_3:

$$\begin{cases} B_1(b_1 \frac{\sin\lambda}{\lambda} + \cos\lambda) = B_2(b_2 \frac{\sin\lambda}{\lambda} + \cos\lambda), \\ B_2(b_2 \frac{\sin\lambda}{\lambda} + \cos\lambda) = B_3(b_3 \frac{\sin\lambda}{\lambda} + \cos\lambda), \\ B_1(b_1 \cos\lambda - \lambda\sin\lambda) + B_2(b_2\cos\lambda - \lambda\sin\lambda) \\ \qquad\qquad\qquad\qquad + B_3(b_3\cos\lambda - \lambda\sin\lambda) = 0. \end{cases}$$

If the determinant of this system is not equal to zero, then the solution is unique, namely $B_1 = B_2 = B_3 = 0$. In order to find nontrivial solutions we look for λ such that the determinant equals zero:

$$\det \begin{pmatrix} b_1 \frac{\sin\lambda}{\lambda} + \cos\lambda & -(b_2 \frac{\sin\lambda}{\lambda} + \cos\lambda) & 0 \\ 0 & b_2 \frac{\sin\lambda}{\lambda} + \cos\lambda & -(b_3 \frac{\sin\lambda}{\lambda} + \cos\lambda) \\ b_1\cos\lambda - \lambda\sin\lambda & b_2\cos\lambda - \lambda\sin\lambda & b_3\cos\lambda - \lambda\sin\lambda \end{pmatrix} = 0. \tag{2.3}$$

We call the determinant in (2.3) the *characteristic function.* This way we have the following equation:

$$\begin{aligned} &-3\lambda \sin\lambda \cos^2\lambda + (b_1+b_2+b_3)(\cos^3\lambda - 2\cos\lambda\sin^2\lambda) + \\ &+ 3(b_1b_2 + b_2b_3 + b_1b_3)\frac{(\cos^2\lambda\sin\lambda - \sin^3\lambda)}{\lambda} + \\ &+ 3b_1b_2b_3 \frac{\sin^2\lambda\cos\lambda}{\lambda^2} = 0. \end{aligned} \tag{2.4}$$

The leading term of this equation, as $\lambda \to +\infty$, is $-3\lambda\sin\lambda\cos^2\lambda$. The zeros of this term are:

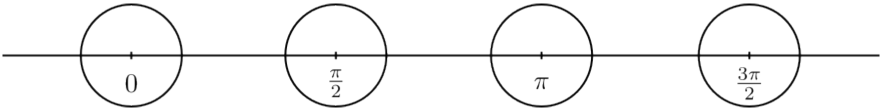
$$\widetilde{\lambda_k} = \frac{\pi k}{2}, \ k \in \mathbb{Z}.$$

The zeros $\widetilde{\lambda}_{2k}$ with $k \in \mathbb{Z}\backslash\{0\}$ are simple, while the zeros $\widetilde{\lambda}_{2k-1}$ with $k \in \mathbb{Z}$ and λ_0 are double.

We will use Rouche‘s theorem for the following functions:

$$\begin{aligned} F(\lambda) &= 3\lambda \sin\lambda \cos^2\lambda, \\ f(\lambda) &= (b_1+b_2+b_3)(\cos^3\lambda - 2\cos\lambda\sin^2\lambda) + \\ &\quad + 3(b_1b_2+b_2b_3+b_1b_3)\frac{(\cos^2\lambda\sin\lambda - \sin^3\lambda)}{\lambda} + \\ &\quad + 3b_1b_2b_3\frac{\sin^2\lambda\cos\lambda}{\lambda^2}. \end{aligned}$$

We consider circles with radii $0 < r < \frac{\pi}{4}$ centred at points $\frac{\pi k}{2}$ (see figure below). There is one simple zero $\widetilde{\lambda}_{2k}$ inside each circle C_{2k}, $k = 1, 2, \ldots$ and are double zeros $\widetilde{\lambda}^{(i)}_{2k-1}$, $i = 1, 2$ inside each circle C_{2k-1} and C_0.



We will show that for large k the inequality

$$\begin{aligned} |3\lambda\sin\lambda\cos^2\lambda| > \Big|&(b_1+b_2+b_3)(\cos^3\lambda - 2\cos\lambda\sin^2\lambda) + \\ &+ 3(b_1b_2+b_2b_3+b_1b_3)\frac{(\cos^2\lambda\sin\lambda - \sin^3\lambda)}{\lambda} + \\ &+ 3b_1b_2b_3\frac{\sin^2\lambda\cos\lambda}{\lambda^2}\Big| \end{aligned}$$

holds on C_k. Since $\sin\lambda$ and $\cos\lambda$ are periodic functions,

$$c_1 < |\sin\lambda| < d_1, \qquad c_2 < |\cos\lambda| < d_2,$$

where c_1, d_1, c_2, d_2 are positive constants independent of k.

On our circles we have

$$\frac{\pi k}{2} - r < |\lambda| < \frac{\pi k}{2} + r$$

for $k > \frac{2r}{\pi}$.

We estimate $|F(\lambda)|$ from below and $|f(\lambda)|$ from above. Note that

$$|F(\lambda)| = |3\lambda\sin\lambda\cos^2\lambda| > 3\left(\frac{\pi k}{2} - r\right)c_1c_2^2, \tag{2.5}$$

$$\begin{aligned} |f(\lambda)| &< |b_1+b_2+b_3|(d_2^3 + 2d_1^2d_2) + \\ &\quad + 3|b_1b_2+b_2b_3+b_1b_3|\left(\frac{d_1d_2^2}{\frac{\pi k}{2} - r} + \frac{d_1^3d_2}{(\frac{\pi k}{2} - r)c_2}\right) + \end{aligned} \tag{2.6}$$

$$+3|b_1b_2b_3|\frac{d_1^2d_2}{(\frac{\pi k}{2}-r)^2}.$$

Clearly, there exists k_0 such that

$$3\left(\frac{\pi k}{2}-r\right)c_1c_2^2 > |b_1+b_2+b_3|(d_2^3+2d_1^2d_2)+$$
$$+3|b_1b_2+b_2b_3+b_1b_3|\left(\frac{d_1d_2^2}{\frac{\pi k}{2}-r}+\frac{d_1^3d_2}{(\frac{\pi k}{2}-r)c_2}\right)+$$
$$+3|b_1b_2b_3|\frac{d_1^2d_2}{(\frac{\pi k}{2}-r)^2}$$

for all $\lambda \in C_k$ and all $k > k_0$. Then from (2.5) and (2.6) we derive

$$|F(\lambda)| > |f(\lambda)|.$$

By Rouche‘s theorem we conclude that there is exactly one zero of the function $F(\lambda)+f(\lambda)$ inside each circle C_{2k}, $k > k_0$ and exactly two (or one double) zeros inside each circles C_{2k-1} for $k > k_0$.

Since the radius r can be chosen arbitrary small we conclude that

$$\lambda_{2k} = \pi k + o(1),\ k \to +\infty. \tag{2.7}$$

Let us express λ_{2k} in the following form

$$\lambda_{2k} = \pi k + \frac{x_1}{k} + \frac{x_2}{k^3} + \frac{x_3}{k^5} + o\left(\frac{1}{k^5}\right). \tag{2.8}$$

In order to simplify calculations we rewrite (2.4) as follows

$$-3\lambda\sin\lambda\cos\lambda + (b_1+b_2+b_3)(\cos^2\lambda - 2\sin^2\lambda)+$$
$$+3(b_1b_2+b_2b_3+b_1b_3)\left(\frac{\cos\lambda\sin\lambda}{\lambda}-\frac{\sin^3\lambda}{\lambda\cos\lambda}\right)+$$
$$+3b_1b_2b_3\frac{\sin^2\lambda}{\lambda^2}=0. \tag{2.9}$$

This equation is equivalent to (2.4) if $\lambda \neq \pi(k-\frac{1}{2})$.

Substituting (2.8) into (2.9), and using expansions $\cos x \approx 1-\frac{x^2}{2}+\frac{x^4}{24}$, $\sin x \approx x - \frac{x^3}{6}+\frac{x^5}{120}$ for $x \to 0$, we obtain

$$3x_1\pi - (b_1+b_2+b_3)+$$
$$+\frac{1}{k^2}\Big[3x_2\pi - 2\pi {x_1}^3+$$
$$+3{x_1}^2+3{x_1}^2(b_1+b_2+b_3)-3\frac{x_1}{\pi}(b_1b_2+b_2b_3+b_1b_3)\Big]+$$

$$
\begin{aligned}
&+\frac{1}{k^4}\Big[3\pi x_3-6\pi {x_1}^2x_2+\frac{2}{5}\pi {x_1}^5+6x_1x_2-2{x_1}^4-\\
&-(b_1+b_2+b_3)\left(-6x_1x_2+\frac{2}{3}x_1^4+\frac{{x_1}^3}{3}\right)-\\
&-3(b_1b_2+b_2b_3+b_1b_3)\left(\frac{x_2}{\pi}-\frac{5{x_1}^3}{3\pi}-\frac{{x_1}^2}{\pi^2}\right)-3b_1b_2b_3\frac{{x_1}^2}{\pi^2}\Big]\\
=\;&o\left(\frac{1}{k^4}\right). \qquad (2.10)
\end{aligned}
$$

This means that

$$b_1+b_2+b_3=3x_1\pi. \qquad (2.11)$$

Let us first consider the case when $x_1\neq 0$. Substituting (2.11) into (2.10), we obtain

$$b_1b_2+b_2b_3+b_1b_3=\frac{x_2}{x_1}\pi^2+\pi x_1+\frac{7}{3}\pi^2x_1^2 \qquad (2.12)$$

and

$$
\begin{aligned}
b_1b_2b_3=\;&\frac{\pi^3x_3}{x_1^2}+\frac{5}{3}\pi^3x_2-\frac{8}{15}\pi^3x_1^3+\frac{2\pi^2x_2}{x_1}+\\
&+\pi^2x_1^2-\frac{\pi^3x_1^2}{3}-\frac{\pi^3x_2^2}{x_1^3}+\frac{5\pi^3x_2}{3}+\pi x_1.
\end{aligned}
\qquad (2.13)
$$

Now we consider separately the case, when $x_1=0$. From (2.11) we obtain that $b_1+b_2+b_3=0$. Therefore, equation (2.4) can be written as follows:

$$\sin\lambda\Big[3\lambda\cos^2\lambda-3(b_1b_2+b_2b_3+b_1b_3)\frac{\cos^2\lambda-\sin^2\lambda}{\lambda}-3b_1b_2b_3\frac{\sin\lambda\cos\lambda}{\lambda^2}\Big]=0.$$

The set of zeros in λ of the left hand-side of this equation consists of two subsequences. The first one is the sequence of zeros of the function $\sin\lambda$. This sequence give us no information about the coefficients b_1, b_2, b_3. Thus, we consider the following equation:

$$
\begin{aligned}
&3\lambda\cos^2\lambda-3(b_1b_2+b_2b_3+b_1b_3)\frac{\cos^2\lambda-\sin^2\lambda}{\lambda}-\\
&\qquad\qquad-3b_1b_2b_3\frac{\sin\lambda\cos\lambda}{\lambda^2}=0.
\end{aligned}
\qquad (2.14)
$$

The left hand-side of this equation is an analytic function everywhere, except at $\lambda=0$. The set of zeros of this function may be presented as a union of two subsequences. So, similarly to (2.7), using Rouche‘s theorem we conclude that for both subsequences one has

$$\lambda_{2k-1}=\pi(k-\frac{1}{2})+o(1).$$

Let us look for λ_{2k-1} in the following form

$$\lambda_{2k-1}=\pi(k-\frac{1}{2})+\frac{y_1}{\pi(k-\frac{1}{2})}+\frac{y_2}{\pi^3k^3}+o(\frac{1}{k^3}). \tag{2.15}$$

Substituting (2.15) into (2.14) we obtain:

$$\frac{y_1^2}{\pi(k-\frac{1}{2})}+\frac{2y_1y_2-\frac{1}{3}y_1^4+y_1^3}{\pi^3k^3}+$$
$$+(b_1b_2+b_2b_3+b_1b_3)\left(\frac{1}{\pi(k-\frac{1}{2})}-\frac{2y_1^2+y_1}{\pi^3k^3}\right)+$$
$$+b_1b_2b_3\frac{y_1}{\pi^3k^3}=o\left(\frac{1}{k^3}\right). \tag{2.16}$$

Thus,

$$b_1b_2+b_2b_3+b_1b_3=-y_1^2. \tag{2.17}$$

Substituting (2.17) into (2.16) and assuming that $y_1\neq 0$, we arrive at:

$$b_1b_2b_3=-2y_2-\frac{5}{3}y_1^3-2y_1^2.$$

If $y_1=0$, then $b_1+b_2+b_3=0$ and $b_1b_2+b_2b_3+b_1b_3=0$ imply

$$b_1=b_2=b_3=0.$$

3. INVERSE PROBLEMS

First of all let us consider the following inverse problem. The shape of the graph is known. We need to find the constants b_1, b_2 and b_3 using the spectrum of problem (1.1)-(1.6) and, thus, the coefficients x_j in the decomposition (2.8). To do it, in the case of $x_1\neq 0$ we first use equations (2.11)-(2.13) to find $b_1+b_2+b_3$, $b_1b_2+b_2b_3+b_1b_3$ and $b_1b_2b_3$. By Vieta‘s theorem, it remains to solve the cubic equation

$$z^3-3x_1\pi z^2+\left(\frac{x_2}{x_1}\pi^2+\pi x_1+\frac{7}{3}\pi^2x_1^2\right)z-$$
$$-\frac{\pi^3x_3}{x_1^2}-\frac{5}{3}\pi^3x_2+\frac{8}{15}\pi^3x_1^3-\frac{2\pi^2x_2}{x_1}-\pi^2x_1^2+$$
$$+\frac{\pi^3x_1^2}{3}+\frac{\pi^3x_2^2}{x_1^3}-\frac{5\pi^3x_2}{3}-\pi x_1=0.$$

The solutions of this equation are nothing but the values of b_1, b_2 and b_3.

In the case of $x_1=0$ the equation which is to be solved is

$$z^3-y_1^2z+2y_2+\frac{5}{3}y_1^3+2y_1^2=0.$$

To conclude we note that using this method, we can also solve a more complicated inverse problem. Suppose we know the spectrum of problem (1.1)-(1.6). We need to find the shape of the graph and the constants in Robin‘s conditions. Assume that the graph is simple connected and equilateral. The spectrum consists of subsequence $\lambda_k^1 = \pi(k-1)+o(1)$ and two subsequences $\lambda_k^i = \pi(k-\frac{1}{2})+o(1)$ $(i=2,3)$. We conclude that the leading term as $\lambda \to +\infty$ of our characteristic function is $C\lambda\sin\lambda\cos^2\lambda$ where $C \neq 0$ is a constant. Then we evaluate

$$\begin{aligned} C\lambda\sin\lambda\cos^2\lambda &= C\cos^2\lambda(1-\cos^2\lambda)\left(\frac{\sin\lambda}{\lambda}\right)^{-1} \\ &= -\frac{1}{3}C\phi_8(\cos\lambda)\left(\frac{\sin\lambda}{\lambda}\right)^{-1}, \end{aligned}$$

where $\phi_8(z) = 3z^4 - 3z^2$ is the characteristic polynomial of the graph G_8 computed in [3]. It means that $C = 1$ and our graph is nothing but the star graph with three edges. If so, we can find b_1, b_2 and b_3 according to the procedure described above.

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