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Cospectral Trees Indistinguishable by Scattering

Mats-Erik Pistol[1] · Vyacheslav Pivovarchik[2,3]

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Abstract
Let v_1 and v_2 be two distinct vertices of an equilateral tree T_0. Let $\phi_N^{(i)}$ $(i = 1, 2)$ be the characteristic functions of the Sturm-Liouville problem on T_0 rooted at v_i with Neumann conditions at the root and let $\phi_D^{(i)}$ $(i = 1, 2)$ be the characteristic functions of the Sturm-Liouville problem on T_0 with Dirichlet conditions at the root. We prove that if attaching any tree to T_0 at the vertices v_1 and v_2 leads to cospectral trees and $d(v_1) = d(v_2)$ then $\phi_N(\lambda)^{(1)} \equiv \phi_N(\lambda)^{(2)}$ and $\phi_D(\lambda)^{(1)} \equiv \phi_D(\lambda)^{(1)}$ (which means that the scattering is the same at v_1 and v_2).

Keywords Sturm-Liouville equation · Eigenvalue · spectrum · Equilateral tree · Dirichlet boundary condition · Neumann boundary condition · Root

1 Introduction

In quantum graph theory, i.e. in the theory of quantum mechanical equations considered on metric graph domains, the problem of recovering the shape of a graph was considered in [1], and [4]. It was shown in [4] that if the lengths of the edges are rationally independent then the spectrum of the spectral Sturm-Liouville problem on a graph with standard boundary conditions (Neuman-Kirchhoff at interior vertices and Neumann at the pendant vertices) uniquely determines the shape of this graph.

In [1], it was shown that if one considers graphs having edges whose quotient is a rational number, then there exist co-spectral quantum graphs.

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V. Pivovarchik.: These authors contributed equally to this work.

✉ Vyacheslav Pivovarchik
vpivovarchik@gmail.com

Mats-Erik Pistol
mats-erik.pistol@ftf.lth.se

[1] Department of Solid State Physics, Lund University, Street, Lund S-221 00, Sweden

[2] Department of Higher Mathematics and Statistics, South Ukrainian National Pedagogical University, Staroportofrankovskaya str. 26, Odesa 65020, Ukraine

[3] Department of Mathematics and Statistics, University of Vaasa, PO Box 700, Vaasa 65101, Finland

In [6] it was shown that if we have the Neumann problem with zero potential then the finite interval does not have any cospectral partners. In [2] it was shown that if the graph is a simple connected equilateral graph where the number of vertices less or equal 5 and the potentials on the edges are real L_2 functions then the spectrum of the Sturm-Liouville problem with standard conditions at the vertices uniquely determines the shape of the graph. For trees the minimal number of vertices in a cospectral pair is 9 (see [12] and [11]). If the number of vertices does not exceed 8 then the characteristic polynomial determines the graph [2]. It should be mentioned that there exist cospectral graphs in the case of the Dirichlet conditions at the pendant vertices also (see, e.g. [3, 12]).

If one spectrum does not determine uniquely the shape of a graph then we put the question whether two spectra do it, where the two spectra are obtained under different boundary conditions.

In [10] it was shown how to find the shape of a tree using two spectra: the spectrum of the Neumann problem and the spectrum of the Dirichlet problem, i.e. the problem in which the Dirichlet condition is imposed at a particular vertex called the root. This method works for an arbitrarily large number of vertices. If the solution is not unique, using this method one can find all the solutions.

In [9] it was shown how to find the shape of a graph using the S-function and the eigenvalues of the scattering problem on a graph which consists of an equilateral compact subgraph with a lead attached to it at a vertex which we call the root. The potential on the lead was assumed to be identically zero and therefore the Jost function can be expressed via the characteristic functions of the Dirichlet and Neumann problems. Thus this scattering inverse problem and the spectral inverse problem by two spectra have the same given data. However, it was found in [12] that there are pairs of non-isomorphic graphs that are cospectral also under very general boundary conditions.

Attaching a lead at different interior vertices may lead to different scattering data (S-functions and eigenvalues). However, an example of a tree with the same M-function for two different vertices was found in [12] (which means that the S-function at these vertices and the eigenvalues are the same if we attach any compact graph at these vertices).

In the present paper we show that if we have a tree such that attaching an interval at two different vertices of the same degree give cospectral graphs then the M-function at these vertices must be the same. It has previously been shown that attaching any compact graph to a graph having two vertices with the same M-function results in cospectral graphs.

2 Statement of the Problem and Auxiliary Results

Let T be a metric equilateral tree with p vertices and $g = p - 1$ edges each of length l.

We choose the vertex v_0 as the root and direct all the edges away from the root.

Let us describe the *Neumann* spectral problem on this tree. We consider the Sturm-Liouville equations on the edges

$$-y_j'' + q_j(x)y_j = \lambda y_j, \quad j = 1, 2, ..., g \tag{1}$$

where $q_j \in L_2(0, l)$ are real.

If an edge e_j is incident with a pendant vertex which is not the root then we impose the Dirichlet boundary conditions at the pendant vertex

$$y_j(l) = 0. \tag{2}$$

At each interior vertex, v_i, we impose the continuity conditions

$$y_j(l) = y_k(0) \tag{3}$$

for the incoming edge e_j and for all e_k outgoing from v_i and Kirchhoff's conditions

$$y_j'(l) = \sum_k y_k'(0) \tag{4}$$

where the sum is taken over all edges e_k outgoing from v_i. At the root we have the continuity conditions

$$y_j(0) = y_k(0) \tag{5}$$

for the edges e_j and e_k outgoing from v_0 and Kirchhoff's condition

$$\sum_k y_k'(0) = 0 \tag{6}$$

where the sum is taken over all edges e_k outgoing from v_0

We will call problem (1)–(6) the Neumann problem. We will call standard (or generalized Neumann) conditions (3), (4) and (4), (5). Standard conditions are Neumann at pendant vertices.

Let us describe the corresponding Dirichlet problem. It consists of equations (1)–(4) and of equation

$$y_j(0) = 0 \tag{7}$$

for all edges e_j incident with the root. We illustrate the situations in Fig. 1.

Denote by $s_j(\lambda, x)$ the solution of (1) which satisfies the conditions $s_j(\lambda, 0) = s_j'(\lambda, 0) - 1 = 0$ and by $c_j(\lambda, x)$ the solution which stisfies $c_j(\lambda, 0) - 1 = c_j'(\lambda, 0) = 0$. Substituting

$$y_j(\lambda, x) = A_j s_j(\lambda, x) + B_j c_j(\lambda, x) \tag{8}$$

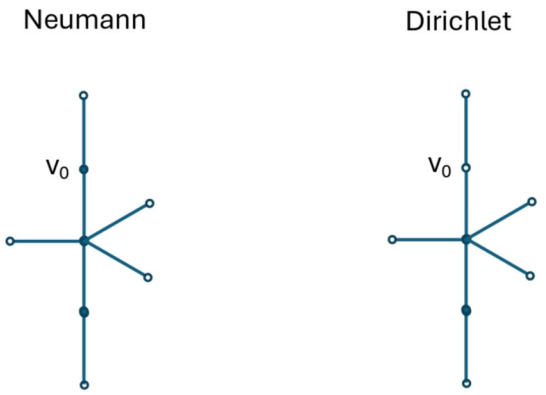

Fig. 1 Hollow circles mean Dirichlet boundary conditions. Solid circles mean generalized Neumann boundary conditions.

into equations (2)–(6) we obtain a system of $2g$ homogeneous linear algebraic equations with unknowns A_j and B_j. We call characteristic function of the Neumann problem the determinant $\psi_N(\lambda)$ of the coefficient matrix of this system. The zeros of $\psi_N(\lambda)$ are the eigenvalues of problem (1)–(6). Substituting (8) into (2)–(4), (7) we again obtain a system of $2g$ homogeneous linear algebraic equations. We call characteristic function the determinant $\psi_D(\lambda)$ of coefficient matrix of this system.

Let us assume that the root v_0 is an interior vertex. We divide our tree T into two subtrees T_1 and T_2 having $\mathbf{v}$ as the only common vertex. (We say that T_1 and T_2 are *complementary subtrees* of T. Denote by $\psi_N^{(i)}(\lambda)$ the characteristic function of the Neumann problem on T_i and by $\psi_D^{(i)}(\lambda)$ the characteristic function of the Dirichlet problem on T_i.

The following theorem was proved in [7].

Theorem 2.1 *Let the root* $\mathbf{v}$ *of a tree* T *be an interior vertex. Let* T_1 *and* T_2 *be complementary subtrees of* T. *Then with the same orientation of the graph and the subgraphs edges described above,*

$$\psi_N(z) = \psi_N^{(1)}(z)\psi_D^{(2)}(z) + \psi_D^{(1)}(z)\psi_N^{(2)}(z), \tag{9}$$

$$\psi_D(z) = \psi_D^{(1)}(z)\psi_D^{(2)}(z).$$

3 Main Results

Let T_0 and $\tilde{T}$ be equilateral trees with the lengths of the edges equal l, the same real potential on the edges $q(x) \in L_2(0, l)$ symmetric with respect to the midpoint of an edge ($q(l-x) \overset{a.e.}{=} q(x)$). Let v_1 and v_2 be distinct interior vertices of T_0 of degrees $d(v_1)$ and $d(v_2)$, respectively. Let v_0 be a vertex of $\tilde{T}$ (the root of $\tilde{T}$). We attach $\tilde{T}$ to the tree T_0 in such a way that v_0 coinciding with v_i ($i = 1, 2$) is the only common vertex of $\tilde{T}$ and T_0. Thus we obtain two trees T_1 and T_2.

Then by Theorem 2.1

$$\phi_i(\lambda) = \tilde{\phi}_N(\lambda)\phi_D^{(i)}(\lambda) + \tilde{\phi_D}(\lambda)\phi_N^{(i)}(\lambda) \tag{10}$$

where $\phi_i(\lambda)$ is the characteristic function of problem (1)-(6) on T_i $(i = 1, 2)$, $\phi_N^{(1)}(\lambda) \equiv \phi_N^{(2)}(\lambda)$ is the characteristic function of (1)–(6) on T_0 (the functions $\phi_N^{(1)}(\lambda)$ and $\phi_N^{(2)}(\lambda)$ are the same because they correspond to the same (standard) conditions at v_1 and v_2), $\phi_D^{(i)}(\lambda)$ $(i = 1, 2)$ is the Dirichlet characteristic function (characteristic function of problem (1)–(4), (7) on T_0 with the Dirichlet condition at the root v_i, $\tilde{\phi}_N(\lambda)$ is the Neumann characteristic functions on $\tilde{T}$ and $\tilde{\phi}_D(\lambda)$ is the Dirichlet characteristic function with the Dirichlet condition at v_0.

The following result which follows from (10) is known even for more general case (see [5]).

Theorem 3.1 *Suppose* $\phi_N^{(1)}(\lambda) \equiv \phi_N^{(2)}(\lambda)$ *and* $\phi_D^{(1)}(\lambda) \equiv \phi_D^{(2)}(\lambda)$. *Then attaching a tree* $\tilde{T}$ *by its root to the vertices* v_1 *and* v_2 *and imposing the same conditions at all the other vertices of* $\tilde{T}$ *we obtain co-spectral trees.*

Theorem 3.2 *Let the equilateral trees* T_1 *and* T_2 *(constructed by attaching a tree* $\tilde{T}$ *to interior vertices* v_1 *and* v_2 *of a tree* T_0. *Let the problems (1)–(4), (7) with the same real symmetric* $L_2(0, l)$*-potential on the edges be co-spectral. Then* $\phi_1(\lambda) \equiv \phi_2(\lambda)$ *if and only if* $d(v_1) = d(v_2)$.

Proof We use representations [8], Theorem 6.4.2:

$$\phi_i = s^{p-g+r}(\lambda, l)\, P_i(c(\lambda, l)) \quad (i = 1, 2)$$

where p is the number of vertices, g is the number of edges, r is the number of pendant vertices with Dirichlet conditions, $P_i(z) = det(-\lambda\hat{D}_i + \hat{A}_i)$, $D_i = diag\{d(v_1), d(v_2), ..., d(v_p)\}$ are the diagonal degree matrices, A_i are the adjacency matrices, $-\lambda\hat{D}_i + \hat{A}_i$ are the prime submatrices of $-\lambda D_i + A_i$, obtained by deleting the columns and rows corresponding to the pendant vertices.

The trees T_1 and T_2 are co-spectral what means that the sets of zeros of $P_1(z)$ and $P_2(z)$ coincide. The leading term of the polynomial P_1 is

$$(d(v_1) + d(v_0))d(v_2)\prod_{i=3}^{p} d(v_i)z^p$$

while the leading term of the polynomial P_2 is

$$d(v_1)(d(v_2) + d(v_0))\prod_{i=3}^{p} d(v_i)z^p.$$

Thus, $P_1(z) = C\,P_2(z)$ where $C = \frac{d(v_2)(d(v_0)+d(v_1))}{d(v_1)(d(v_0)+d(v_2))}$ and, therefore, $\phi_1(z) = C\phi_2(z)$. We conclude that $\phi_1(\lambda) \equiv \phi_2(\lambda)$ if and only if $d(v_1) = d(v_2)$. □

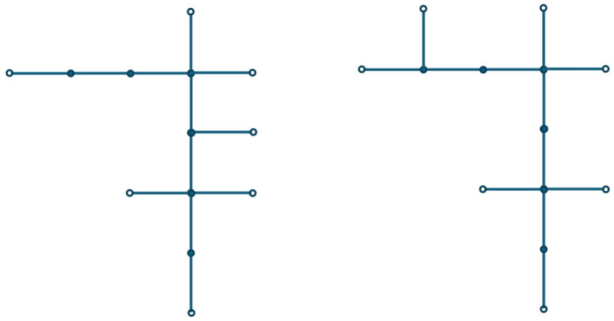

Fig. 2 Trees obtained by attaching P_2 to the tree T_0 of Fig. 3 at the vertices v_1 and v_2, respectively.

Theorem 3.3 *Let the trees T_1 and T_2 be co-spectral, i.e. the sets of zeros of $\phi_1(z)$ and $\phi_2(z)$ coincide and let $d(v_1) = d(v_2)$. Then $\phi_N^{(1)}(\lambda) \equiv \phi_N^{(2)}(\lambda)$ and $\phi_D^{(1)}(\lambda) \equiv \phi_D^{(2)}(\lambda)$.*

Proof First, since conditions are standard at both vertices v_1 and v_2, $\phi_N^{(1)}(\lambda)$ and $\phi_N^{(2)}(\lambda)$ are equal to the determinant of coefficient matrix of one system of equations. Thus, we conclude that $\phi_N^{(1)}(\lambda) \equiv \phi_N^{(2)}(\lambda)$.

By Theorem 3.2 we have $\phi_1(\lambda) \equiv \phi_2(\lambda)$. Due to (10) we obtain

$$\phi_1(\lambda) \equiv \phi_2(\lambda) = \tilde{\phi}_N(\lambda)\phi_D^{(1)}(\lambda) + \tilde{\phi_D}(\lambda)\phi_N^{(1)}(\lambda) = \tilde{\phi}_N(\lambda)\phi_D^{(2)}(\lambda) + \tilde{\phi_D}(\lambda)\phi_N^{(2)}(\lambda) \tag{11}$$

and, consequently,

$$\phi_D^{(1)}(\lambda) = \frac{\phi_1(\lambda) - \tilde{\phi}_D(\lambda)\phi_N^{(1)}(\lambda)}{\tilde{\phi}_N(\lambda)} = \frac{\phi_2(\lambda) - \tilde{\phi}_D(\lambda)\phi_N^{(2)}(\lambda)}{\tilde{\phi}_N(\lambda)} \equiv \phi_D^{(2)}(\lambda).$$

□

Proposition 3.4 *The M-functions of the problems on T_0 with the roots at v_1 and v_2 are the same. This allows computer search for vertices on trees with equal M-functions to be quite fast. Previously it was necessary to attach intervals of arbitrary length to confirm that two vertices have the same M-function [12] but we find here that for trees it is enough to attach intervals of length one.*

Remark 3.5 If $d(v_1) \neq d(v_2)$ then co-spectrality of T_1 and T_2 does not imply $\phi_1(\lambda) \equiv \phi_2(\lambda)$.

4 Examples

1. Consider the graphs of Fig. 2.

They are obtained by attaching one edge to the tree of Fig. 3 at the vertices v_1 and v_2.

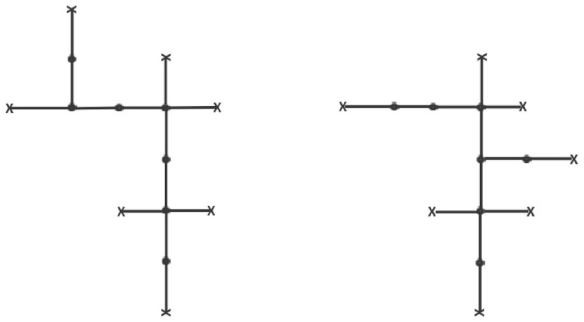

Fig. 3 Tree T_0 corresponding to Examples 1 and 2.

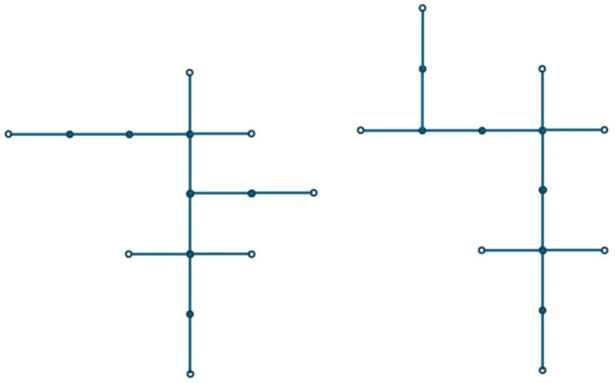

Fig. 4 The tree T_0 of Fig. 3 with P_3 attached to the vertices v_1 and v_2, respectively.

Thus, in our terms $\tilde{T}$ is P_2 and T_0 is the tree of Fig. 3 and therefore

$$\phi_N^{(1)}(\lambda) = \phi_N^{(2)}(\lambda) = s^5(\lambda, l)(256c^6(\lambda, l) - 192c^4(\lambda, l) + 36c^2(\lambda, l) - 1.$$

The characteristic function of the Sturm-Liouville problem with the Dirichlet boundary conditions at the pendant vertices on both graphs T_1 and T_2 is the same:

$$\phi_1(\lambda) = \phi_2(\lambda) = s^6(\lambda, l)(384c^6(\lambda, l) - 256c^4(\lambda, l) + 42c^2(\lambda, l) - 1).$$

2. For the trees of Fig. 4 we have $\tilde{T} = P_3$ and T_0 is the tree of Fig. 3. Calculations show that

$$\phi_1(\lambda) \equiv \phi_2(\lambda) = -s^6(\lambda, l)(-768c^7(\lambda, l) + 640c^5(\lambda, l) - 148c^3(\lambda, l) + 8c(\lambda, l)),$$

the same for both trees.

3. Consider the trees presented in Fig. 34 b) of [12], see Fig. 5. For these graphs $\phi_1(z) = (-24c^3(\lambda, l) + 4c(\lambda, l))s^5(\lambda, l)$, $\phi_2(z) = (-30c^3(\lambda, l) + 5c(\lambda, l))s^5(\lambda, l)$. These trees can be obtained by attaching an edge (graph P_2) to the tree of Fig. 6 at v_1 and v_2, respectively. For both these trees $\phi_N^{(1)}(\lambda) \equiv \phi_N^{(2)} = (-20c^3(\lambda, l) + 4c(\lambda, l)s^4(\lambda, l)$.

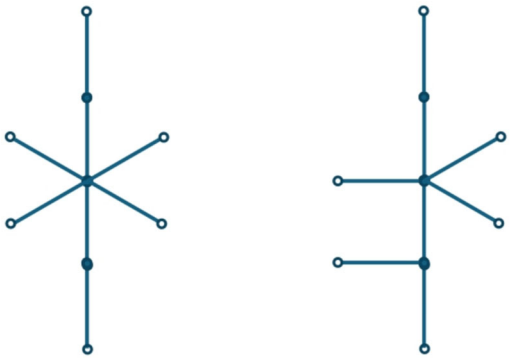

Fig. 5 The trees obtained by attaching P_2 to the vertices v_1 and v_2 to the tree of Fig. 6.

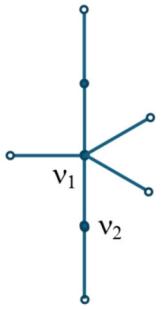

Fig. 6 The tree T_0 of Example 3.

For the first tree $\phi_D^{(1)} = 4c^2(\lambda, l)s^5(\lambda, l)$ and and for the second tree $\phi_D^{(2)}(\lambda) = (10c^2(\lambda, l) - 1)s^5(\lambda, l)$. The reason of this difference is caused by the difference in degrees $d(v_1) = 5 \neq 2 = d(v_2)$.

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References

1. von Below, J.: Can one hear the shape of a network? In F. Ali Mehmeti, J. von Below, and S. Nicaise, editors, *Partial Differential Equations on Multistructures (Proc. Luminy 1999)*, volume 219 of *Lect. Notes Pure Appl. Math.*, pages 19–36, New York, Marcel Dekker (2001).
2. Chernyshenko, A., Pivovarchik, V.: Recovering the shape of a quantum graph. Integr. Equ. Oper. Theory **92**, 23 (2020)
3. Dudko, A., Lesechko, O., Pivovarchik, V.: Recovering the shape of an equilateral quantum tree with the Dirichlet conditions at the pendant vertices. Opuscula Math. **44**(5), 689–705 (2024). https://doi.org/10.7494/OpMath.2024.44.5.689
4. Gutkin, B., Smilansky, U.: Can one hear the shape of a graph? J. Phys. A Math. Gen. **34**, 6061–6068 (2001)
5. Kurasov, P., Muller, J.: Isospectral graphs via inner symmetries. St. Petersburg Mathematical Journal (2024), 287-309
6. Kurasov, P., Naboko, S.: Rayleigh estimates for differential operators on graphs. J. Spectr. Theory **4**(2), 211–219 (2014). https://doi.org/10.4171/JST
7. Law, C.-K., Pivovarchik, V.: *Characteristic functions of quantum graphs*. J. Phys A: Math. Theor., **42** (2009) 035302, 11pp. https://doi.org/10.1088/1751-8113/42/3/035302.
8. Möller, M., Pivovarchik, V.: Direct and inverse finite-dimensional spectral problems on graphs Operator Theory: Advances and Applications, 283. Birkhäuser/Springer, 2020. 359 pp. ISBN: 978-3-030-60483-7; 978-3-030-60484-4 https://www.springer.com/gp/book/9783030604837
9. Mugnolo, D., Pivovarchik, V.: Distinguishing co-spectral quantum graphs by scattering, J. Phys. A: Math. Theor., Vol. 56, issue 9, (2023) https://doi.org/10.1088/1751-8121/acbb44, arXiv: 2211.05465
10. Pivovarchik, V.: Recovering the shape of a quantum tree by two spectra. Integr. Equ. Oper. Theory (2024) 96:11 https://doi.org/10.1007]. arXiv:2301.05939
11. Pivovarchik, V., Chernyshenko, A.: Cospectral Quantum Graphs with Dirichlet Conditions at Pendant Vertices, Ukrainian Math. J. vol 75 (2023), 439 - 455. Cospectral quantum graphs. 2022 arXiv:2112.14235
12. Pistol, M.-E.: Generating isospectral but not isomorphic quantum graphs. arXiv: 2104.12885

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