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Generalized Gaussian bridges

Tommi Sottinen*, Adil Yazigi

Department of Mathematics and Statistics, University of Vaasa, P.O. Box 700, FIN-65101 Vaasa, Finland

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Abstract

A generalized bridge is a stochastic process that is conditioned on N linear functionals of its path. We consider two types of representations: orthogonal and canonical. The orthogonal representation is constructed from the entire path of the process. Thus, the future knowledge of the path is needed. In the canonical representation the filtrations of the bridge and the underlying process coincide. The canonical representation is provided for prediction-invertible Gaussian processes. All martingales are trivially prediction-invertible. A typical non-semimartingale example of a prediction-invertible Gaussian process is the fractional Brownian motion. We apply the canonical bridges to insider trading.

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1. Introduction

Let $X = (X_t)_{t\in[0,T]}$ be a continuous Gaussian process with positive definite covariance function R, mean function m of bounded variation, and $X_0 = m(0)$. We consider the conditioning, or bridging, of X on N linear functionals $\mathbf{G}_T = [G_T^i]_{i=1}^N$ of its paths:

$$\mathbf{G}_T(X) = \int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \left[\int_0^T g_i(t)\,\mathrm{d}X_t\right]_{i=1}^N. \tag{1.1}$$

* Corresponding author. Tel.: +358 294498317.
E-mail addresses: tommi.sottinen@uva.fi, tommi.sottinen@uwasa.fi (T. Sottinen), adil.yazigi@uwasa.fi (A. Yazigi).

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We assume, without any loss of generality, that the functions g_i are linearly independent. Indeed, if this is not the case then the linearly dependent, or redundant, components of $\mathbf{g}$ can simply be removed from the conditioning (1.2) without changing it.

The integrals in the conditioning (1.1) are the so-called abstract Wiener integrals (see Definition 2.5 later). The abstract Wiener integral $\int_0^T g(t)\,\mathrm{d}X_t$ will be well-defined for functions or generalized functions g that can be approximated by step functions in the inner product $\langle\!\langle\!\langle \cdot, \cdot \rangle\!\rangle\!\rangle$ defined by the covariance R of X by bilinearly extending the relation $\langle\!\langle\!\langle 1_{[0,t)}, 1_{[0,s)} \rangle\!\rangle\!\rangle = R(t,s)$. This means that the integrands g are equivalence classes of Cauchy sequences of step functions in the norm $|\!|\!| \cdot |\!|\!|$ induced by the inner product $\langle\!\langle\!\langle \cdot, \cdot \rangle\!\rangle\!\rangle$. Recall that for the case of Brownian motion we have $R(t,s) = t \wedge s$. Therefore, for the Brownian motion, the equivalence classes of step functions are simply the space $L^2([0,T])$.

Informally, the generalized Gaussian bridge $X^{\mathbf{g};\mathbf{y}}$ is (the law of) the Gaussian process X conditioned on the set

$$\left\{ \int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \mathbf{y} \right\} = \bigcap_{i=1}^N \left\{ \int_0^T g_i(t)\,\mathrm{d}X_t = y_i \right\}. \tag{1.2}$$

The rigorous definition is given in Definition 1.3 later.

For the sake of convenience, we will work on the canonical filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\Omega = C([0,T])$, $\mathscr{F}$ is the Borel σ-algebra on $C([0,T])$ with respect to the supremum norm, and $\mathbb{P}$ is the Gaussian measure corresponding to the Gaussian coordinate process $X_t(\omega) = \omega(t)$: $\mathbb{P} = \mathbb{P}[X \in \cdot\,]$. The filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$ is the intrinsic filtration of the coordinate process X that is augmented with the null-sets and made right-continuous.

Definition 1.3. The *generalized bridge measure* $\mathbb{P}^{\mathbf{g};\mathbf{y}}$ is the regular conditional law

$$\mathbb{P}^{\mathbf{g};\mathbf{y}} = \mathbb{P}^{\mathbf{g};\mathbf{y}}[X \in \cdot\,] = \mathbb{P}\left[X \in \cdot \;\middle|\; \int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \mathbf{y} \right].$$

A *representation of the generalized Gaussian bridge* is any process $X^{\mathbf{g};\mathbf{y}}$ satisfying

$$\mathbb{P}\left[X^{\mathbf{g};\mathbf{y}} \in \cdot\, \right] = \mathbb{P}^{\mathbf{g};\mathbf{y}}[X \in \cdot\,] = \mathbb{P}\left[X \in \cdot \;\middle|\; \int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \mathbf{y} \right].$$

Note that the conditioning on the $\mathbb{P}$-null-set (1.2) in Definition 1.3 is not a problem, since the canonical space of continuous processes is a Polish space and all Polish spaces are Borel spaces and thus admit regular conditional laws, cf. [20, Theorems A1.2 and 6.3]. Also, note that as a *measure* $\mathbb{P}^{\mathbf{g};\mathbf{y}}$ the generalized Gaussian bridge is unique, but it has several different *representations* $X^{\mathbf{g};\mathbf{y}}$. Indeed, for any representation of the bridge one can combine it with any $\mathbb{P}$-measure-preserving transformation to get a new representation.

In this paper we provide two different representations for $X^{\mathbf{g};\mathbf{y}}$. The first representation, given by Theorem 3.1, is called the *orthogonal representation*. This representation is a simple consequence of orthogonal decompositions of Hilbert spaces associated with Gaussian processes and it can be constructed for any continuous Gaussian process for any conditioning functionals. The second representation, given by Theorem 4.25, is called the *canonical representation*. This representation is more interesting but also requires more assumptions. The canonical representation is dynamically invertible in the sense that the linear spaces $\mathscr{L}_t(X)$ and $\mathscr{L}_t(X^{\mathbf{g};\mathbf{y}})$ (see Definition 2.1 later) generated by the process X and its bridge representation $X^{\mathbf{g};\mathbf{y}}$ coincide for all times $t \in [0,T)$. This means that at every time point $t \in [0,T)$ the bridge and the underlying process can be constructed from each others without knowing the future-time development of the underlying process or the bridge. A typical example of a non-semimartingale Gaussian process for which we can provide the canonically represented generalized bridge is the fractional Brownian motion.

The canonically represented bridge $X^{\mathbf{g};\mathbf{y}}$ can be interpreted as the original process X with an added "information drift" that bridges the process at the final time T. This dynamic drift interpretation should turn out to be useful in applications. We give one such application in connection to insider trading in Section 5. This application is, we must admit, a bit classical.

On earlier work related to bridges, we would like to mention first Alili [1], Baudoin [5], Baudoin and Coutin [6] and Gasbarra et al. [13]. In [1] generalized Brownian bridges were considered. It is our opinion that our article extends [1] considerably, although we do not consider the "non-canonical representations" of [1]. Indeed, Alili [1] only considered Brownian motion. Our investigation extends to a large class of non-semimartingale Gaussian processes. Also, Alili [1] did not give the canonical representation for bridges, i.e. the solution to Eq. (4.9) was not given. We solve Eq. (4.9) in (4.14). The article [5] is, in a sense, more general than this article, since we condition on fixed values $\mathbf{y}$, but in [5] the conditioning is on a probability law. However, in [5] only the Brownian bridge was considered. In that sense our approach is more general. In [6, 13] (simple) bridges were studied in a similar Gaussian setting as in this article. In this article we generalize the results of [6] and [13] to generalized bridges. Second, we would like to mention the articles [9,11,14,17] that deal with Markovian and Lévy bridges and [12] that studies generalized Gaussian bridges in the semimartingale context and their functional quantization.

This paper is organized as follows. In Section 2 we recall some Hilbert spaces related to Gaussian processes. In Section 3 we give the orthogonal representation for the generalized bridge in the general Gaussian setting. Section 4 deals with the canonical bridge representation. First we give the representation for Gaussian martingales. Then we introduce the so-called prediction-invertible processes and develop the canonical bridge representation for them. Then we consider invertible Gaussian Volterra processes, such as the fractional Brownian motion, as examples of prediction-invertible processes. Finally, in Section 5 we apply the bridges to insider trading. Indeed, the bridge process can be understood from the initial enlargement of filtration point of view. For more information on the enlargement of filtrations we refer to [10,19].

2. Abstract Wiener integrals and related Hilbert spaces

In this section $X = (X_t)_{t\in[0,T]}$ is a continuous (and hence separable) Gaussian process with positive definite covariance R, mean zero and $X_0 = 0$.

Definitions 2.1 and 2.2 give us two central separable Hilbert spaces connected to separable Gaussian processes.

Definition 2.1. Let $t \in [0, T]$. The *linear space* $\mathscr{L}_t(X)$ is the Gaussian closed linear subspace of $L^2(\Omega, \mathscr{F}, \mathbb{P})$ generated by the random variables $X_s,\ s \le t$, i.e. $\mathscr{L}_t(X) = \overline{\mathrm{span}}\{X_s; s \le t\}$, where the closure is taken in $L^2(\Omega, \mathscr{F}, \mathbb{P})$.

The linear space is a Gaussian Hilbert space with the inner product $\mathbb{C}\mathrm{ov}[\cdot,\cdot]$. Note that since X is continuous, R is also continuous, and hence $\mathscr{L}_t(X)$ is separable, and any orthogonal basis $(\xi_n)_{n=1}^{\infty}$ of $\mathscr{L}_t(X)$ is a collection of independent standard normal random variables. (Of course, since we chose to work on the canonical space, $L^2(\Omega, \mathscr{F}, \mathbb{P})$ is itself a separable Hilbert space.)

Definition 2.2. Let $t \in [0, T]$. The *abstract Wiener integrand space* $\Lambda_t(X)$ is the completion of the linear span of the indicator functions $1_s := 1_{[0,s)},\ s \le t$, under the inner product $\langle\!\langle\!\langle \cdot,\cdot \rangle\!\rangle\!\rangle$

extended bilinearly from the relation

$$\langle\!\langle\!\langle 1_s, 1_u \rangle\!\rangle\!\rangle = R(s,u).$$

The elements of the abstract Wiener integrand space are equivalence classes of Cauchy sequences $(f_n)_{n=1}^\infty$ of piecewise constant functions. The equivalence of $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ means that

$$|\!|\!|\!| f_n - g_n |\!|\!|\!| \to 0, \quad \text{as } n \to \infty,$$

where $|\!|\!|\!| \cdot |\!|\!|\!| = \sqrt{\langle\!\langle\!\langle \cdot, \cdot \rangle\!\rangle\!\rangle}$.

Remark 2.3. (i) The elements of $\Lambda_t(X)$ cannot in general be identified with functions as pointed out e.g. by Pipiras and Taqqu [22] for the case of fractional Brownian motion with Hurst index $H > 1/2$. However, if R is of bounded variation one can identity the function space $|\Lambda_t|(X) \subset \Lambda_t(X)$:

$$|\Lambda_t|(X) = \left\{ f \in \mathbb{R}^{[0,t]};\ \int_0^t \int_0^t |f(s)f(u)|\,|R|(\mathrm{d}s, \mathrm{d}u) < \infty \right\}.$$

(ii) While one may want to interpret that $\Lambda_s(X) \subset \Lambda_t(X)$ for $s \le t$ it may happen that $f \in \Lambda_t(X)$, but $f 1_s \notin \Lambda_s(X)$. Indeed, it may be that $|\!|\!|\!| f 1_s |\!|\!|\!| > |\!|\!|\!| f |\!|\!|\!|$. See Bender and Elliott [7] for an example in the case of fractional Brownian motion.

The space $\Lambda_t(X)$ is isometric to $\mathscr{L}_t(X)$. Indeed, the relation

$$\mathscr{I}_t^X[1_s] := X_s, \quad s \le t, \tag{2.4}$$

can be extended linearly into an isometry from $\Lambda_t(X)$ onto $\mathscr{L}_t(X)$.

Definition 2.5. The isometry $\mathscr{I}_t^X : \Lambda_t(X) \to \mathscr{L}_t(X)$ extended from the relation (2.4) is the *abstract Wiener integral*. We denote

$$\int_0^t f(s)\,\mathrm{d}X_s := \mathscr{I}_t^X[f].$$

Let us end this section by noting that the abstract Wiener integral and the linear spaces are now connected as

$$\mathscr{L}_t(X) = \{\mathscr{I}_t[f];\ f \in \Lambda_t(X)\}.$$

In the special case of the Brownian motion this relation reduces to the well-known Itô isometry with

$$\mathscr{L}_t(W) = \left\{ \int_0^t f(s)\,\mathrm{d}W_s;\ f \in L^2([0,t]) \right\}.$$

3. Orthogonal generalized bridge representation

Denote by $\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle$ the matrix

$$\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle_{ij} := \langle\!\langle\!\langle g_i, g_j \rangle\!\rangle\!\rangle := \mathbb{C}\mathrm{ov}\left[\int_0^T g_i(t)\,\mathrm{d}X_t, \int_0^T g_j(t)\,\mathrm{d}X_t \right].$$

Note that $\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle$ does not depend on the mean of X nor on the conditioned values $\mathbf{y}$: $\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle$ depends only on the conditioning functions $\mathbf{g} = [g_i]_{i=1}^N$ and the covariance R. Also, since $g_1, \ldots, g_N$ are linearly independent and R is positive definite, the matrix $\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle$ is invertible.

Theorem 3.1. *The generalized Gaussian bridge $X^{\mathbf{g};\mathbf{y}}$ can be represented as*

$$X_t^{\mathbf{g};\mathbf{y}} = X_t - \langle\!\langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle\!\rangle^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1} \left(\int_0^T \mathbf{g}(u)\, \mathrm{d}X_u - \mathbf{y} \right). \tag{3.2}$$

Moreover, $X^{\mathbf{g};\mathbf{y}}$ is a Gaussian process with

$$\mathbb{E}\left[X_t^{\mathbf{g};\mathbf{y}}\right] = m(t) - \langle\!\langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle\!\rangle^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1} \left(\int_0^T \mathbf{g}(u)\, \mathrm{d}m(u) - \mathbf{y} \right),$$
$$\mathbb{C}\mathrm{ov}\left[X_t^{\mathbf{g};\mathbf{y}}, X_s^{\mathbf{g};\mathbf{y}}\right] = \langle\!\langle\!\langle 1_t, 1_s \rangle\!\rangle\!\rangle - \langle\!\langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle\!\rangle^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1} \langle\!\langle\!\langle 1_s, \mathbf{g} \rangle\!\rangle\!\rangle.$$

Proof. It is well-known (see, e.g., [24, p. 304]) from the theory of multivariate Gaussian distributions that conditional distributions are Gaussian with

$$\mathbb{E}\left[X_t \,\middle|\, \int_0^T \mathbf{g}(u)\mathrm{d}X_u = \mathbf{y}\right] = m(t) + \langle\!\langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle\!\rangle^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1} \left(\mathbf{y} - \int_0^T \mathbf{g}(u)\, \mathrm{d}m(u) \right),$$
$$\mathbb{C}\mathrm{ov}\left[X_t, X_s \,\middle|\, \int_0^T \mathbf{g}(u)\, \mathrm{d}X_u = \mathbf{y}\right] = \langle\!\langle\!\langle 1_t, 1_s \rangle\!\rangle\!\rangle - \langle\!\langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle\!\rangle^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1} \langle\!\langle\!\langle 1_s, \mathbf{g} \rangle\!\rangle\!\rangle.$$

The claim follows from this. □

Corollary 3.3. *Let X be a centered Gaussian process with $X_0 = 0$ and let m be a function of bounded variation. Denote $X^{\mathbf{g}} := X^{\mathbf{g};\mathbf{0}}$, i.e., $X^{\mathbf{g}}$ is conditioned on $\{\int_0^T \mathbf{g}(t)\mathrm{d}X_t = \mathbf{0}\}$. Then*

$$(X+m)_t^{\mathbf{g};\mathbf{y}} = X_t^{\mathbf{g}} + \left(m(t) - \langle\!\langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle\!\rangle^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1} \int_0^T \mathbf{g}(u)\, \mathrm{d}m(u) \right) + \langle\!\langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle\!\rangle^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1} \mathbf{y}.$$

Remark 3.4. Corollary 3.3 tells us how to construct, by adding a deterministic drift, a general bridge from a bridge that is constructed from a centered process with conditioning $\mathbf{y} = \mathbf{0}$. So, in what follows, we shall almost always assume that the process X is centered, i.e. $m(t) = 0$, and all conditionings are with $\mathbf{y} = \mathbf{0}$.

Example 3.5. Let X be a zero mean Gaussian process with covariance function R. Consider the conditioning on the final value and the average value:

$$X_T = 0,$$
$$\frac{1}{T}\int_0^T X_t\, \mathrm{d}t = 0.$$

This is a generalized Gaussian bridge. Indeed,

$$X_T = \int_0^T 1\, \mathrm{d}X_t =: \int_0^T g_1(t)\, \mathrm{d}X_t,$$
$$\frac{1}{T}\int_0^T X_t\, \mathrm{d}t = \int_0^T \frac{T-t}{T}\, \mathrm{d}X_t =: \int_0^T g_2(t)\, \mathrm{d}X_t.$$

Now,

$$\begin{aligned}
⟪1_t, g_1⟫ &= \mathbb{E}[X_t X_T] = R(t,T), \\
⟪1_t, g_2⟫ &= \mathbb{E}\left[X_t \frac{1}{T}\int_0^T X_s \,\mathrm{d}s\right] = \frac{1}{T}\int_0^T R(t,s)\,\mathrm{d}s, \\
⟪g_1, g_1⟫ &= \mathbb{E}[X_T X_T] = R(T,T), \\
⟪g_1, g_2⟫ &= \mathbb{E}\left[X_T \frac{1}{T}\int_0^T X_s \,\mathrm{d}s\right] = \frac{1}{T}\int_0^T R(T,s)\,\mathrm{d}s, \\
⟪g_2, g_2⟫ &= \mathbb{E}\left[\frac{1}{T}\int_0^T X_s \,\mathrm{d}s \,\frac{1}{T}\int_0^T X_u \,\mathrm{d}u\right] = \frac{1}{T^2}\int_0^T\int_0^T R(s,u)\,\mathrm{d}u\mathrm{d}s, \\
|⟪\mathbf{g}⟫| &= \frac{1}{T^2}\int_0^T\int_0^T R(T,T)R(s,u) - R(T,s)R(T,u)\,\mathrm{d}u\,\mathrm{d}s
\end{aligned}$$

and

$$⟪\mathbf{g}⟫^{-1} = \frac{1}{|⟪\mathbf{g}⟫|}\begin{bmatrix} ⟪g_2, g_2⟫ & -⟪g_1, g_2⟫ \\ -⟪g_1, g_2⟫ & ⟪g_1, g_1⟫ \end{bmatrix}.$$

Thus, by Theorem 3.1,

$$\begin{aligned}
X_t^{\mathbf{g}} &= X_t - \frac{⟪1_t, g_1⟫\, ⟪g_2, g_2⟫ - ⟪1_t, g_2⟫\, ⟪g_1, g_2⟫}{|⟪\mathbf{g}⟫|}\int_0^T g_1(t)\,\mathrm{d}X_t \\
&\quad - \frac{⟪1_t, g_2⟫\, ⟪g_1, g_1⟫ - ⟪1_t, g_1⟫\, ⟪g_1, g_2⟫}{|⟪\mathbf{g}⟫|}\int_0^T g_2(t)\,\mathrm{d}X_t \\
&= X_t - \frac{\int_0^T\int_0^T R(t,T)R(s,u) - R(t,s)R(T,s)\mathrm{d}s\,\mathrm{d}u}{\int_0^T\int_0^T R(T,T)R(s,u) - R(T,s)R(T,u)\,\mathrm{d}s\,\mathrm{d}u} X_T \\
&\quad - \frac{T\int_0^T R(T,T)R(t,s) - R(t,T)R(T,s)\mathrm{d}s}{\int_0^T\int_0^T R(T,T)R(s,u) - R(T,s)R(T,u)\,\mathrm{d}s\,\mathrm{d}u}\int_0^T \frac{T-t}{T}\,\mathrm{d}X_t.
\end{aligned}$$

Remark 3.6. (i) Since Gaussian conditionings are projections in Hilbert space to a subspace, it is well-known that they can be done iteratively. Indeed, let $X^n := X^{g_1,\ldots,g_n;y_1,\ldots,y_n}$ and let $X^0 := X$ be the original process. Then the orthogonal generalized bridge representation X^N can be constructed from the rule

$$X_t^n = X_t^{n-1} - \frac{⟪1_t, g_n⟫_{n-1}}{⟪g_n, g_n⟫_{n-1}}\left[\int_0^T g_n(u)\,\mathrm{d}X_u^{n-1} - y_n\right],$$

where $⟪\cdot,\cdot⟫_{n-1}$ is the inner product in $\mathscr{L}_T(X^{n-1})$.

(ii) If $g_j = 1_{t_j},\ j = 1,\ldots,N$, then the corresponding generalized bridge is a *multibridge*. That is, it is pinned down to values y_j at points t_j. For the multibridge $X^N = X^{1_{t_1},\ldots,1_{t_N};y_1,\ldots,y_N}$ the orthogonal bridge decomposition can be constructed from the iteration

$$\begin{aligned}
X_t^0 &= X_t, \\
X_t^n &= X_t^{n-1} - \frac{R_{n-1}(t,t_n)}{R_{n-1}(t_n,t_n)}\left[X_{t_n}^{n-1} - y_n\right],
\end{aligned}$$

where

$$R_0(t,s) = R(t,s),$$
$$R_n(t,s) = R_{n-1}(t,s) - \frac{R_{n-1}(t,t_n)R_{n-1}(t_n,s)}{R_{n-1}(t_n,t_n)}.$$

4. Canonical generalized Bridge representation

The problem with the orthogonal bridge representation (3.2) of $X^{\mathbf{g};\mathbf{y}}$ is that in order to construct it at any point $t \in [0,T)$ one needs the whole path of the underlying process X up to time T. In this section we construct a bridge representation that is canonical in the following sense:

Definition 4.1. The bridge $X^{\mathbf{g};\mathbf{y}}$ is of *canonical representation* if, for all $t \in [0,T)$, $X_t^{\mathbf{g};\mathbf{y}} \in \mathscr{L}_t(X)$ and $X_t \in \mathscr{L}_t(X^{\mathbf{g};\mathbf{y}})$.

Example 4.2. Consider the classical Brownian bridge. That is, condition the Brownian motion W with $\mathbf{g} = g = 1$. Now, the orthogonal representation is

$$W_t^1 = W_t - \frac{t}{T} W_T.$$

This is not a canonical representation, since the future knowledge W_T is needed to construct W_t^1 for any $t \in (0,T)$. A canonical representation for the Brownian bridge is, by calculating the $\ell_{\mathbf{g}}^*$ in Theorem 4.12,

$$W_t^1 = W_t - \int_0^t \int_0^s \frac{1}{T-u}\,\mathrm{d}W_u\,\mathrm{d}s$$
$$= (T-t)\int_0^t \frac{1}{T-s}\,\mathrm{d}W_s.$$

Remark 4.3. Since the conditional laws of Gaussian processes are Gaussian and Gaussian spaces are linear, the assumptions $X_t^{\mathbf{g};\mathbf{y}} \in \mathscr{L}_t(X)$ and $X_t \in \mathscr{L}_t(X^{\mathbf{g};\mathbf{y}})$ of Definition 4.1 are the same as assuming that $X_t^{\mathbf{g};\mathbf{y}}$ is $\mathscr{F}_t^X$-measurable and X_t is $\mathscr{F}_t^{X^{\mathbf{g};\mathbf{y}}}$-measurable (and, consequently, $\mathscr{F}_t^X = \mathscr{F}_t^{X^{\mathbf{g};\mathbf{y}}}$). This fact is very special to Gaussian processes. Indeed, in general conditioned processes such as generalized bridges are not linear transformations of the underlying process.

We shall require that the restricted measures $\mathbb{P}_t^{\mathbf{g};\mathbf{y}} := \mathbb{P}^{\mathbf{g};\mathbf{y}}|\mathscr{F}_t$ and $\mathbb{P}_t := \mathbb{P}|\mathscr{F}_t$ are equivalent for all $t < T$ (they are obviously singular for $t = T$). To this end we assume that the matrix

$$\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle_{ij}(t) := \mathbb{E}\left[\left(G_T^i(X) - G_t^i(X)\right)\left(G_T^j(X) - G_t^j(X)\right)\right]$$
$$= \mathbb{E}\left[\int_t^T g_i(s)\,\mathrm{d}X_s \int_t^T g_j(s)\,\mathrm{d}X_s\right] \tag{4.4}$$

is invertible for all $t < T$.

Remark 4.5. On notation: in the previous section we considered the matrix $\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle$, but from now on we consider the function $\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle(\cdot)$. Their connection is of course $\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle = \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle(0)$. We hope that this overloading of notation does not cause confusion to the reader.

Gaussian martingales

We first construct the canonical representation when the underlying process is a continuous Gaussian martingale M with strictly increasing bracket $\langle M\rangle$ and $M_0 = 0$. Note that the bracket is strictly increasing if and only if the covariance R is positive definite. Indeed, for Gaussian martingales we have $R(t,s) = \mathbb{V}\mathrm{ar}(M_{t\wedge s}) = \langle M\rangle_{t\wedge s}$.

Define a Volterra kernel

$$\ell_{\mathbf{g}}(t,s) := -\mathbf{g}^\top(t)\,\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(t)\,\mathbf{g}(s). \tag{4.6}$$

Note that the kernel $\ell_{\mathbf{g}}$ depends on the process M through its covariance $\langle\!\langle\!\langle \cdot,\cdot\rangle\!\rangle\!\rangle$, and in the Gaussian martingale case we have

$$\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle_{ij}(t) = \int_t^T g_i(s)g_j(s)\,\mathrm{d}\langle M\rangle_s.$$

Lemma 4.7 is the key observation in finding the canonical generalized bridge representation. Actually, it is a multivariate version of Proposition 6 of [13].

Lemma 4.7. *Let $\ell_{\mathbf{g}}$ be given by (4.6) and let M be a continuous Gaussian martingale with strictly increasing bracket $\langle M\rangle$ and $M_0 = 0$. Then the Radon–Nikodym derivative $\mathrm{d}\mathbb{P}^{\mathbf{g}}_t/\mathrm{d}\mathbb{P}_t$ can be expressed in the form*

$$\frac{\mathrm{d}\mathbb{P}^{\mathbf{g}}_t}{\mathrm{d}\mathbb{P}_t} = \exp\left\{\int_0^t\int_0^s \ell_{\mathbf{g}}(s,u)\,\mathrm{d}M_u\mathrm{d}M_s \;-\frac{1}{2}\int_0^t\left(\int_0^s \ell_{\mathbf{g}}(s,u)\,\mathrm{d}M_u\right)^2\mathrm{d}\langle M\rangle_s\right\}$$

for all $t \in [0,T)$.

Proof. Let

$$p(\mathbf{y};\boldsymbol{\mu},\boldsymbol{\Sigma}) := \frac{1}{(2\pi)^{N/2}|\boldsymbol{\Sigma}|^{1/2}}\exp\left\{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})\right\}$$

be the Gaussian density on $\mathbb{R}^N$ and let

$$\alpha^{\mathbf{g}}_t(\mathrm{d}\mathbf{y}) := \mathbb{P}\left[\mathbf{G}_T(M)\in\mathrm{d}\mathbf{y}\,\middle|\,\mathscr{F}^M_t\right]$$

be the conditional law of the conditioning functionals $\mathbf{G}_T(M) = \int_0^T \mathbf{g}(s)\,\mathrm{d}M_s$ given the information $\mathscr{F}^M_t$.

First, by Bayes' formula, we have

$$\frac{\mathrm{d}\mathbb{P}^{\mathbf{g}}_t}{\mathrm{d}\mathbb{P}_t} = \frac{\mathrm{d}\alpha^{\mathbf{g}}_t}{\mathrm{d}\alpha^{\mathbf{g}}_0}(\mathbf{0}).$$

Second, by the martingale property, we have

$$\frac{\mathrm{d}\alpha^{\mathbf{g}}_t}{\mathrm{d}\alpha^{\mathbf{g}}_0}(\mathbf{0}) = \frac{p\Big(\mathbf{0};\mathbf{G}_t(M),\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle(t)\Big)}{p\Big(\mathbf{0};\mathbf{G}_0(M),\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle(0)\Big)},$$

where we have denoted $\mathbf{G}_t(M) = \int_0^t \mathbf{g}(s)\,\mathrm{d}M_s$.

Third, denote

$$\frac{p\Big(\mathbf{0};\mathbf{G}_t(M),\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle(t)\Big)}{p\Big(\mathbf{0};\mathbf{G}_0(M),\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle(0)\Big)} =: \left(\frac{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(0)}{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(t)}\right)^{\frac{1}{2}} \exp\left\{F(t,\mathbf{G}_t(M)) - F(0,\mathbf{G}_0(M))\right\},$$

with

$$F(t,\mathbf{G}_t(M)) = -\frac{1}{2}\left(\int_0^t \mathbf{g}(s)\,\mathrm{d}M_s\right)^{\top} \langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(0)\left(\int_0^t \mathbf{g}(s)\,\mathrm{d}M_s\right).$$

Then, straightforward differentiation yields

$$\int_0^t \frac{\partial F}{\partial s}(s,\mathbf{G}_s(M))\,\mathrm{d}s = -\frac{1}{2}\int_0^t\left(\int_0^s \ell_{\mathbf{g}}(s,u)\,\mathrm{d}M_u\right)^2 \mathrm{d}\langle M\rangle_s,$$

$$\int_0^t \frac{\partial F}{\partial x}(s,\mathbf{G}_s(M))\,\mathrm{d}M_s = \int_0^t\int_0^s \ell_{\mathbf{g}}(s,u)\,\mathrm{d}M_u\,\mathrm{d}M_s,$$

$$-\frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s,\mathbf{G}_s(M))\,\mathrm{d}\langle M\rangle_s = \log\left(\frac{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(t)}{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(0)}\right)^{\frac{1}{2}}$$

and the form of the Radon–Nikodym derivative follows by applying the Itô formula. □

Corollary 4.8. *The canonical bridge representation $M^{\mathbf{g}}$ satisfies the stochastic differential equation*

$$\mathrm{d}M_t = \mathrm{d}M_t^{\mathbf{g}} - \int_0^t \ell_{\mathbf{g}}(t,s)\,\mathrm{d}M_s^{\mathbf{g}}\,\mathrm{d}\langle M\rangle_t, \tag{4.9}$$

where $\ell_{\mathbf{g}}$ is given by (4.6). Moreover $\langle M\rangle = \langle M^{\mathbf{g}}\rangle$.

Proof. The claim follows by using Girsanov's theorem. □

Remark 4.10. (i) Note that for all $\varepsilon > 0$,

$$\int_0^{T-\varepsilon}\int_0^t \ell_{\mathbf{g}}(t,s)^2\,\mathrm{d}\langle M\rangle_s\,\mathrm{d}\langle M\rangle_t < \infty.$$

In view of (4.9) this means that the processes M and $M^{\mathbf{g}}$ are equivalent in law on $[0, T-\varepsilon]$ for all $\varepsilon > 0$. Indeed, Eq. (4.9) can be viewed as the *Hitsuda representation* between two equivalent Gaussian processes, cf. Hida and Hitsuda [16]. Also note that

$$\int_0^{T}\int_0^t \ell_{\mathbf{g}}(t,s)^2\,\mathrm{d}\langle M\rangle_s\,\mathrm{d}\langle M\rangle_t = \infty$$

meaning that the measures $\mathbb{P}$ and $\mathbb{P}^{\mathbf{g}}$ are singular on $[0, T]$.

(ii) In the case of the Brownian bridge, cf. Example 4.2, the item (i) above can be clearly seen. Indeed,

$$\ell_{\mathbf{g}}(t,s) = \frac{1}{T-t}$$

and $\mathrm{d}\langle W\rangle_s = \mathrm{d}s$.

(iii) In the case of $\mathbf{y} \neq \mathbf{0}$, the formula (4.9) takes the form

$$\mathrm{d}M_t = \mathrm{d}M_t^{\mathbf{g};\mathbf{y}} + \left(\mathbf{g}^{\top}(t)\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(t)\mathbf{y} - \int_0^t \ell_{\mathbf{g}}(t,s)\,\mathrm{d}M_s^{\mathbf{g};\mathbf{y}}\right)\mathrm{d}\langle M\rangle_t. \tag{4.11}$$

Next we solve the stochastic differential equation (4.9) of Corollary 4.8. In general, solving a Volterra–Stieltjes equation like (4.9) in a closed form is difficult. Of course, the general theory of Volterra equations suggests that the solution will be of the form (4.14) of Theorem 4.12, where $\ell^*_{\mathbf{g}}$ is the resolvent kernel of $\ell_{\mathbf{g}}$ determined by the resolvent equation (4.15). Also, the general theory suggests that the resolvent kernel can be calculated implicitly by using the Neumann series. In our case the kernel $\ell_{\mathbf{g}}$ factorizes in its argument. This allows us to calculate the resolvent $\ell^*_{\mathbf{g}}$ explicitly as (4.13). (We would like to point out that a similar SDE was treated in [2,15].)

Theorem 4.12. *Let $s \le t \in [0, T]$. Define the Volterra kernel*

$$\begin{aligned} \ell^*_{\mathbf{g}}(t,s) &:= -\ell_{\mathbf{g}}(t,s)\frac{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(t)}{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(s)} \\ &= |\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(t)\mathbf{g}^\top(t)\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(t)\,\frac{\mathbf{g}(s)}{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(s)}. \end{aligned} \tag{4.13}$$

Then the bridge $M^{\mathbf{g}}$ has the canonical representation

$$\mathrm{d}M^{\mathbf{g}}_t = \mathrm{d}M_t - \int_0^t \ell^*_{\mathbf{g}}(t,s)\,\mathrm{d}M_s\,\mathrm{d}\langle M\rangle_t, \tag{4.14}$$

i.e., (4.14) is the solution to (4.9).

Proof. Eq. (4.14) is the solution to (4.9) if the kernel $\ell^*_{\mathbf{g}}$ satisfies the *resolvent equation*

$$\ell_{\mathbf{g}}(t,s) + \ell^*_{\mathbf{g}}(t,s) = \int_s^t \ell_{\mathbf{g}}(t,u)\ell^*_{\mathbf{g}}(u,s)\,\mathrm{d}\langle M\rangle_u. \tag{4.15}$$

This is well-known if $\mathrm{d}\langle M\rangle_u = \mathrm{d}u$, cf. e.g. Riesz and Sz.-Nagy [23]. In the $\mathrm{d}\langle M\rangle$ case the resolvent equation can be derived as in the classical $\mathrm{d}u$ case. We show the derivation here, for the convenience of the reader:

Suppose (4.14) is the solution to (4.9). This means that

$$\begin{aligned} \mathrm{d}M_t &= \left(\mathrm{d}M_t - \int_0^t \ell^*_{\mathbf{g}}(t,s)\,\mathrm{d}M_s\,\mathrm{d}\langle M\rangle_t\right) \\ &\quad - \int_0^t \ell_{\mathbf{g}}(t,s)\left(\mathrm{d}M_s - \int_0^s \ell^*_{\mathbf{g}}(s,u)\,\mathrm{d}M_u\,\mathrm{d}\langle M\rangle_s\right)\mathrm{d}\langle M\rangle_t, \end{aligned}$$

or, in the integral form, by using Fubini's theorem,

$$\begin{aligned} M_t &= M_t - \int_0^t\int_s^t \ell^*_{\mathbf{g}}(u,s)\ \mathrm{d}\langle M\rangle_u \mathrm{d}M_s - \int_0^t\int_s^t \ell_{\mathbf{g}}(u,s)\ \mathrm{d}\langle M\rangle_u \mathrm{d}M_s \\ &\quad + \int_0^t\int_s^t\int_u^s \ell_{\mathbf{g}}(s,v)\ell^*_{\mathbf{g}}(v,u)\mathrm{d}\langle M\rangle_v\,\mathrm{d}\langle M\rangle_u \mathrm{d}M_s. \end{aligned}$$

The resolvent criterion (4.15) follows by identifying the integrands in the $\mathrm{d}\langle M\rangle_u\mathrm{d}M_s$-integrals above.

Finally, let us check that the resolvent equation (4.15) is satisfied with $\ell_{\mathbf{g}}$ and $\ell^*_{\mathbf{g}}$ defined by (4.6) and (4.13), respectively:

$$\begin{aligned} &\int_s^t \ell_{\mathbf{g}}(t,u)\ell^*_{\mathbf{g}}(u,s)\,\mathrm{d}\langle M\rangle_u \\ &\quad = -\int_s^t \mathbf{g}^\top(t)\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(t)\mathbf{g}(u)\,|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(u)\mathbf{g}^\top(u)\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(u)\frac{\mathbf{g}(s)}{|\langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle|(s)}\,\mathrm{d}\langle M\rangle_u \end{aligned}$$

$$
\begin{aligned}
&= -\mathbf{g}^\top(t)\langle\langle\langle\mathbf{g}\rangle\rangle\rangle^{-1}(t)\frac{\mathbf{g}(s)}{|\langle\langle\langle\mathbf{g}\rangle\rangle\rangle|(s)}\int_s^t \mathbf{g}(u)|\langle\langle\langle\mathbf{g}\rangle\rangle\rangle|(u)\mathbf{g}^\top(u)\langle\langle\langle\mathbf{g}\rangle\rangle\rangle^{-1}(u)\,\mathrm{d}\langle M\rangle_u \\
&= \mathbf{g}^\top(t)\langle\langle\langle\mathbf{g}\rangle\rangle\rangle^{-1}(t)\frac{\mathbf{g}(s)}{|\langle\langle\langle\mathbf{g}\rangle\rangle\rangle|(s)}\int_s^t \langle\langle\langle\mathbf{g}\rangle\rangle\rangle^{-1}(u)|\langle\langle\langle\mathbf{g}\rangle\rangle\rangle|(u)\mathrm{d}\langle\langle\langle\mathbf{g}\rangle\rangle\rangle(u) \\
&= \mathbf{g}^\top(t)\langle\langle\langle\mathbf{g}\rangle\rangle\rangle^{-1}(t)\frac{\mathbf{g}(s)}{|\langle\langle\langle\mathbf{g}\rangle\rangle\rangle|(s)}\Big(|\langle\langle\langle\mathbf{g}\rangle\rangle\rangle|(t) - |\langle\langle\langle\mathbf{g}\rangle\rangle\rangle|(s)\Big) \\
&= \mathbf{g}^\top(t)\langle\langle\langle\mathbf{g}\rangle\rangle\rangle^{-1}(t)\mathbf{g}(s)\frac{|\langle\langle\langle\mathbf{g}\rangle\rangle\rangle|(t)}{|\langle\langle\langle\mathbf{g}\rangle\rangle\rangle|(s)} - \mathbf{g}^\top(t)\langle\langle\langle\mathbf{g}\rangle\rangle\rangle^{-1}(t)\mathbf{g}(s) \\
&= \ell^*_{\mathbf{g}}(t,s) + \ell_{\mathbf{g}}(t,s),
\end{aligned}
$$

since

$$\mathrm{d}\langle\langle\langle\mathbf{g}\rangle\rangle\rangle(t) = -\mathbf{g}^\top(t)\mathbf{g}(t)\mathrm{d}\langle M\rangle_t.$$

So, the resolvent equation (4.15) holds. □

Gaussian prediction-invertible processes

To construct a canonical representation for bridges of Gaussian non-semimartingales is problematic, since we cannot apply stochastic calculus to non-semimartingales. In order to invoke the stochastic calculus we need to associate the Gaussian non-semimartingale with some martingale. A natural martingale associated with a stochastic process is its prediction martingale:

For a (Gaussian) process X its *prediction martingale* is the process $\hat{X}$ defined as

$$\hat{X}_t = \mathbb{E}\left[X_T|\mathscr{F}_t^X\right].$$

Since for Gaussian processes $\hat{X}_t \in \mathscr{L}_t(X)$, we may write, at least informally, that

$$\hat{X}_t = \int_0^t p(t,s)\,\mathrm{d}X_s,$$

where the abstract kernel p depends also on T (since $\hat{X}$ depends on T). In Definition 4.16 we assume that the kernel p exists as a real, and not only formal, function. We also assume that the kernel p is invertible.

Definition 4.16. A Gaussian process X is *prediction-invertible* if there exists a kernel p such that its prediction martingale $\hat{X}$ is continuous, can be represented as

$$\hat{X}_t = \int_0^t p(t,s)\,\mathrm{d}X_s,$$

and there exists an inverse kernel p^{-1} such that, for all $t \in [0,T]$, $p^{-1}(t,\cdot) \in L^2([0,T],\mathrm{d}\langle\hat{X}\rangle)$ and X can be recovered from $\hat{X}$ by

$$X_t = \int_0^t p^{-1}(t,s)\,\mathrm{d}\hat{X}_s.$$

Remark 4.17. In general it seems to be a difficult problem to determine whether a Gaussian process is prediction-invertible or not. In the discrete time non-degenerate case all Gaussian processes are prediction-invertible. In continuous time the situation is more difficult, as Example 4.18 illustrates. Nevertheless, we can immediately see that if the centered Gaussian process X with covariance R is prediction-invertible, then the covariance must satisfy the relation

$$R(t,s) = \int_0^{t\wedge s} p^{-1}(t,u)\, p^{-1}(s,u)\, \mathrm{d}\langle \hat{X}\rangle_u,$$

where the bracket $\langle \hat{X}\rangle$ can be calculated as the variance of the conditional expectation:

$$\langle \hat{X}\rangle_u = \mathbb{V}\mathrm{ar}\left(\mathbb{E}\left[X_T|\mathscr{F}_u\right]\right).$$

However, this criterion does not seem to be very helpful in practice.

Example 4.18. Consider the Gaussian slope $X_t = t\xi,\ t \in [0,T]$, where ξ is a standard normal random variable. Now, if we consider the "raw filtration" $\mathscr{G}_t^X = \sigma(X_s; s \le t)$, then X is not prediction invertible. Indeed, then $\hat{X}_0 = 0$ but $\hat{X}_t = X_T$, if $t \in (0,T]$. So, $\hat{X}$ is not continuous. On the other hand, the augmented filtration is simply $\mathscr{F}_t^X = \sigma(\xi)$ for all $t \in [0,T]$. So, $\hat{X} = X_T$. Note, however, that in both cases the slope X can be recovered from the prediction martingale: $X_t = \frac{t}{T}\hat{X}_t$.

In order to represent abstract Wiener integrals of X in terms of Wiener–Itô integrals of $\hat{X}$ we need to extend the kernels p and p^{-1} to linear operators:

Definition 4.19. Let X be prediction-invertible. Define operators P and P^{-1} by extending linearly the relations

$$\mathrm{P}[1_t] = p(t,\cdot),$$
$$\mathrm{P}^{-1}[1_t] = p^{-1}(t,\cdot).$$

Now the following lemma is obvious.

Lemma 4.20. *Let f be such a function that $\mathrm{P}^{-1}[f] \in L^2([0,T],\mathrm{d}\langle\hat{X}\rangle)$ and let $\hat{g} \in L^2([0,T],\mathrm{d}\langle\hat{X}\rangle)$. Then*

$$\int_0^T f(t)\,\mathrm{d}X_t = \int_0^T \mathrm{P}^{-1}[f](t)\,\mathrm{d}\hat{X}_t, \tag{4.21}$$

$$\int_0^T \hat{g}(t)\,\mathrm{d}\hat{X}_t = \int_0^T \mathrm{P}[\hat{g}](t)\,\mathrm{d}X_t. \tag{4.22}$$

Remark 4.23. (i) Eqs. (4.21) or (4.22) can actually be taken as the definition of the Wiener integral with respect to X.

(ii) The operators P and P^{-1} depend on T.

(iii) If $p^{-1}(\cdot,s)$ has bounded variation, we can represent P^{-1} as

$$\mathrm{P}^{-1}[f](s) = f(s)p^{-1}(T,s) + \int_s^T \left(f(t) - f(s)\right)\, p^{-1}(\mathrm{d}t,s).$$

A similar formula holds for P also, if $p(\cdot,s)$ has bounded variation.

(iv) Let $⟪\mathbf{g}⟫^X(t)$ denote the remaining covariance matrix with respect to X, i.e.,

$$⟪\mathbf{g}⟫^X_{ij}(t) = \mathbb{E}\left[\int_t^T g_i(s)\,\mathrm{d}X_s \int_t^T g_j(s)\,\mathrm{d}X_s\right].$$

Let $⟪\hat{\mathbf{g}}⟫^{\hat{X}}(t)$ denote the remaining covariance matrix with respect to $\hat{X}$, i.e.,

$$⟪\hat{\mathbf{g}}⟫^{\hat{X}}_{ij}(t) = \int_t^T \hat{g}_i(s)\hat{g}_j(s)\,\mathrm{d}\langle\hat{X}\rangle_s.$$

Then

$$\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle_{ij}^{X}(t) = \langle\!\langle\!\langle \mathrm{P}^{-1}[\mathbf{g}] \rangle\!\rangle\!\rangle_{ij}^{\hat{X}}(t) = \int_t^T \mathrm{P}^{-1}[g_i](s)\mathrm{P}^{-1}[g_j](s)\,\mathrm{d}\langle \hat{X} \rangle_s.$$

Now, let $X^{\mathbf{g}}$ be the bridge conditioned on $\int_0^T \mathbf{g}(s)\,\mathrm{d}X_s = \mathbf{0}$. By Lemma 4.20 we can rewrite the conditioning as

$$\int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \int_0^T \mathrm{P}^{-1}[\mathbf{g}](t)\,\mathrm{d}\hat{X}(t) = \mathbf{0}. \tag{4.24}$$

With this observation the following theorem, that is the main result of this article, follows.

Theorem 4.25. *Let X be prediction-invertible Gaussian process. Assume that, for all $t \in [0, T]$ and $i = 1, \ldots, N$, $g_i 1_t \in \Lambda_t(X)$. Then the generalized bridge $X^{\mathbf{g}}$ admits the canonical representation*

$$X_t^{\mathbf{g}} = X_t - \int_0^t \int_s^t p^{-1}(t,u)\mathrm{P}\left[\hat{\ell}_{\hat{\mathbf{g}}}^{*}(u,\cdot)\right](s)\,\mathrm{d}\langle \hat{X} \rangle_u\,\mathrm{d}X_s, \tag{4.26}$$

where

$$\hat{g}_i = \mathrm{P}^{-1}[g_i],$$
$$\hat{\ell}_{\hat{\mathbf{g}}}^{*}(u,v) = |\langle\!\langle\!\langle \hat{\mathbf{g}} \rangle\!\rangle\!\rangle^{\hat{X}}|(u)\hat{\mathbf{g}}^{\top}(u)(\langle\!\langle\!\langle \hat{\mathbf{g}} \rangle\!\rangle\!\rangle^{\hat{X}})^{-1}(u)\,\frac{\hat{\mathbf{g}}(v)}{|\langle\!\langle\!\langle \hat{\mathbf{g}} \rangle\!\rangle\!\rangle^{\hat{X}}|(v)},$$
$$\langle\!\langle\!\langle \hat{\mathbf{g}} \rangle\!\rangle\!\rangle_{ij}^{\hat{X}}(t) = \int_t^T \hat{g}_i(s)\hat{g}_j(s)\,\mathrm{d}\langle \hat{X} \rangle_s = \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle_{ij}^{X}(t).$$

Proof. Since $\hat{X}$ is a Gaussian martingale and because of the equality (4.24) we can use Theorem 4.12. We obtain

$$\mathrm{d}\hat{X}_s^{\hat{\mathbf{g}}} = \mathrm{d}\hat{X}_s - \int_0^s \hat{\ell}_{\hat{\mathbf{g}}}^{*}(s,u)\,\mathrm{d}\hat{X}_u\,\mathrm{d}\langle \hat{X} \rangle_s.$$

Now, by using the fact that X is prediction invertible, we can recover X from $\hat{X}$, and consequently also $X^{\mathbf{g}}$ from $\hat{X}^{\hat{\mathbf{g}}}$ by operating with the kernel p^{-1} in the following way:

$$\begin{aligned} X_t^{\mathbf{g}} &= \int_0^t p^{-1}(t,s)\,\mathrm{d}\hat{X}_s^{\hat{\mathbf{g}}} \\ &= X_t - \int_0^t p^{-1}(t,s)\left(\int_0^s \hat{\ell}_{\hat{\mathbf{g}}}^{*}(s,u)\,\mathrm{d}\hat{X}_u\right)\mathrm{d}\langle \hat{X} \rangle_s. \end{aligned} \tag{4.27}$$

The representation (4.27) is a canonical representation already but it is written in terms of the prediction martingale $\hat{X}$ of X. In order to represent (4.27) in terms of X we change the Wiener integral in (4.27) by using Fubini's theorem and the operator P:

$$\begin{aligned} X_t^{\mathbf{g}} &= X_t - \int_0^t p^{-1}(t,s)\int_0^s \mathrm{P}\left[\hat{\ell}_{\hat{\mathbf{g}}}^{*}(s,\cdot)\right](u)\,\mathrm{d}X_u\,\mathrm{d}\langle \hat{X} \rangle_s \\ &= X_t - \int_0^t \int_s^t p^{-1}(t,u)\mathrm{P}\left[\hat{\ell}_{\hat{\mathbf{g}}}^{*}(u,\cdot)\right](s)\,\mathrm{d}\langle \hat{X} \rangle_u\,\mathrm{d}X_s. \quad \square \end{aligned}$$

Remark 4.28. Recall that, by assumption, the processes X^{g} and X are equivalent on $\mathscr{F}_t$, $t < T$. So, the representation (4.26) is an analogue of the Hitsuda representation for prediction-invertible processes. Indeed, one can show, just like in [25,26], that a zero mean Gaussian process $\tilde{X}$ is equivalent in law to the zero mean prediction-invertible Gaussian process X if it admits the representation

$$\tilde{X}_t = X_t - \int_0^t f(t,s)\,\mathrm{d}X_s$$

where

$$f(t,s) = \int_s^t p^{-1}(t,u)\mathrm{P}\left[v(u,\cdot)\right](s)\,\mathrm{d}\langle\hat{X}\rangle_u$$

for some Volterra kernel $v \in L^2([0,T]^2, \mathrm{d}\langle\hat{X}\rangle \otimes \mathrm{d}\langle\hat{X}\rangle)$.

It seems that, except in [13], the prediction-invertible Gaussian processes have not been studied at all. Therefore, we give a class of prediction-invertible processes that is related to a class that has been studied in the literature: the Gaussian Volterra processes. See, e.g., Alòs et al. [3], for a study of stochastic calculus with respect to Gaussian Volterra processes.

Definition 4.29. V is an *invertible Gaussian Volterra process* if it is continuous and there exist Volterra kernels k and k^{-1} such that

$$V_t = \int_0^t k(t,s)\,\mathrm{d}W_s, \tag{4.30}$$

$$W_t = \int_0^t k^{-1}(t,s)\,\mathrm{d}V_s. \tag{4.31}$$

Here W is the standard Brownian motion, $k(t,\cdot) \in L^2([0,t]) = \Lambda_t(W)$ and $k^{-1}(t,\cdot) \in \Lambda_t(V)$ for all $t \in [0,T]$.

Remark 4.32. (i) The representation (4.30), defining a Gaussian Volterra process, states that the covariance R of V can be written as

$$R(t,s) = \int_0^{t\wedge s} k(t,u)k(s,u)\,\mathrm{d}u.$$

So, in some sense, the kernel k is the square root, or the Cholesky decomposition, of the covariance R.

(ii) The inverse relation (4.31) means that the indicators 1_t, $t \in [0,T]$, can be approximated in $L^2([0,t])$ with linear combinations of the functions $k(t_j,\cdot)$, $t_j \in [0,t]$. I.e., the indicators 1_t belong to the image of the operator K extending the kernel k linearly as discussed below.

Precisely as with the kernels p and p^{-1}, we can define the operators K and K^{-1} by linearly extending the relations

$$\mathrm{K}[1_t] := k(t,\cdot) \quad \text{and} \quad \mathrm{K}^{-1}[1_t] := k^{-1}(t,\cdot).$$

Then, just like with the operators P and P^{-1}, we have

$$\int_0^T f(t)\,\mathrm{d}V_t = \int_0^T \mathrm{K}[f](t)\,\mathrm{d}W_t,$$
$$\int_0^T g(t)\,\mathrm{d}W_t = \int_0^T \mathrm{K}^{-1}[g](t)\,\mathrm{d}V_t.$$

The connection between the operators K and K^{-1} and the operators P and P^{-1} are

$$\mathrm{K}[g] = k(T,\cdot)\mathrm{P}^{-1}[g],$$
$$\mathrm{K}^{-1}[g] = k^{-1}(T,\cdot)\mathrm{P}[g].$$

So, invertible Gaussian Volterra processes are prediction-invertible and the following corollary to Theorem 4.25 is obvious:

Corollary 4.33. *Let V be an invertible Gaussian Volterra process and let $\mathrm{K}[g_i] \in L^2([0,T])$ for all $i = 1,\dots,N$. Denote*

$$\tilde{\mathbf{g}}(t) := \mathrm{K}[\mathbf{g}](t).$$

Then the bridge $V^{\mathbf{g}}$ admits the canonical representation

$$V_t^{\mathbf{g}} = V_t - \int_0^t \int_s^t k(t,u)\mathrm{K}^{-1}\left[\tilde{\ell}^*_{\tilde{\mathbf{g}}}(u,\cdot)\right](s)\,\mathrm{d}u\,\mathrm{d}V_s, \tag{4.34}$$

where

$$\tilde{\ell}_{\tilde{\mathbf{g}}}(u,v) = |\langle\!\langle\!\langle\tilde{\mathbf{g}}\rangle\!\rangle\!\rangle^W|(u)\tilde{\mathbf{g}}^\top(u)(\langle\!\langle\!\langle\tilde{\mathbf{g}}\rangle\!\rangle\!\rangle^W)^{-1}(u)\,\frac{\tilde{\mathbf{g}}(v)}{|\langle\!\langle\!\langle\tilde{\mathbf{g}}\rangle\!\rangle\!\rangle^W|(v)},$$
$$\langle\!\langle\!\langle\tilde{\mathbf{g}}\rangle\!\rangle\!\rangle^W_{ij}(t) = \int_t^T \tilde{g}_i(s)\tilde{g}_j(s)\,\mathrm{d}s = \langle\!\langle\!\langle\mathbf{g}\rangle\!\rangle\!\rangle^V_{ij}(t).$$

Example 4.35. The *fractional Brownian motion $B = (B_t)_{t\in[0,T]}$* with Hurst index $H \in (0,1)$ is a centered Gaussian process with $B_0 = 0$ and covariance function

$$R(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Another way of defining the fractional Brownian motion is that it is the unique centered Gaussian H-self-similar process with stationary increments normalized so that $\mathbb{E}[B_1^2] = 1$.

It is well-known that the fractional Brownian motion is an invertible Gaussian Volterra process with

$$\mathrm{K}[f](s) = c_H s^{\frac{1}{2}-H}\mathrm{I}^{H-\frac{1}{2}}_{T-}\left[(\,\cdot\,)^{H-\frac{1}{2}} f\right](s), \tag{4.36}$$

$$\mathrm{K}^{-1}[f](s) = \frac{1}{c_H} s^{\frac{1}{2}-H}\mathrm{I}^{\frac{1}{2}-H}_{T-}\left[(\,\cdot\,)^{H-\frac{1}{2}} f\right](s). \tag{4.37}$$

Here $\mathrm{I}_{T-}^{H-\frac{1}{2}}$ and $\mathrm{I}_{T-}^{\frac{1}{2}-H}$ are the *Riemann–Liouville fractional integrals* over $[0, T]$ of order $H-\frac{1}{2}$ and $\frac{1}{2}-H$, respectively:

$$\mathrm{I}_{T-}^{H-\frac{1}{2}}[f](t) = \begin{cases} \dfrac{1}{\Gamma\left(H-\frac{1}{2}\right)} \displaystyle\int_t^T \dfrac{f(s)}{(s-t)^{\frac{3}{2}-H}}\,\mathrm{d}s, & \text{for } H > \dfrac{1}{2}, \\ \dfrac{-1}{\Gamma\left(\frac{3}{2}-H\right)} \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_t^T \dfrac{f(s)}{(s-t)^{H-\frac{1}{2}}}\,\mathrm{d}s, & \text{for } H < \dfrac{1}{2}, \end{cases}$$

and c_H is the normalizing constant

$$c_H = \left(\frac{2H\Gamma\left(H+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-H\right)}{\Gamma(2-2H)} \right)^{\frac{1}{2}}.$$

Here

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,\mathrm{d}t$$

is the Gamma function. For the proofs of these facts and for more information on the fractional Brownian motion we refer to the monographs by Biagini et al. [8] and Mishura [21].

One can calculate the canonical representation for generalized fractional Brownian bridges by using the representation (4.34) by plugging in the operators K and K^{-1} defined by (4.36) and (4.37), respectively. Unfortunately, even for a simple bridge the formula becomes very complicated. Indeed, consider the standard fractional Brownian bridge B^1, i.e., the conditioning is $g(t) = 1_T(t)$. Then

$$\tilde{g}(t) = \mathrm{K}[1_T](t) = k(T, t)$$

is given by (4.36). Consequently,

$$\langle\!\langle\!\langle \tilde{g} \rangle\!\rangle\!\rangle^W(t) = \int_t^T k(T, s)^2\,\mathrm{d}s,$$

$$\tilde{\ell}_{\tilde{g}}^*(u, v) = k(T, u) \frac{k(T, v)}{\int_v^T k(T, w)^2\,\mathrm{d}w}.$$

We obtain the canonical representation for the fractional Brownian bridge:

$$B_t^1 = B_t - \int_0^t \int_s^t k(t, u) k(T, u) \mathrm{K}^{-1} \left[\frac{k(T, \cdot)}{\int_\cdot^T k(T, w)^2\,\mathrm{d}w} \right](s)\,\mathrm{d}u\,\mathrm{d}B_s.$$

This representation can be made "explicit" by plugging in the definitions (4.36) and (4.37). It seems, however, very difficult to simplify the resulting formula.

5. Application to insider trading

We consider insider trading in the context of initial enlargement of filtrations. Our approach here is motivated by Amendiger [4] and Imkeller [18], where only one condition was used. We extend that investigation to multiple conditions although otherwise our investigation is less general than in [4].

Consider an insider who has at time $t = 0$ some insider information of the evolution of the price process of a financial asset S over a period $[0, T]$. We want to calculate the additional expected utility for the insider trader. To make the maximization of the utility of terminal wealth reasonable we have to assume that our model is arbitrage-free. In our Gaussian realm this boils down to assuming that the (discounted) asset prices are governed by the equation

$$\frac{\mathrm{d}S_t}{S_t} = a_t \, \mathrm{d}\langle M\rangle_t + \mathrm{d}M_t, \tag{5.1}$$

where $S_0 = 1$, M is a continuous Gaussian martingale with strictly increasing $\langle M\rangle$ with $M_0 = 0$, and the process a is $\mathbb{F}$-adapted satisfying $\int_0^T a_t^2 \, \mathrm{d}\langle M\rangle_t < \infty$ $\mathbb{P}$-a.s.

Assuming that the trading ends at time $T - \varepsilon$, the insider knows some functionals of the return over the interval $[0, T]$. If $\varepsilon = 0$ there is obviously arbitrage for the insider. The insider information will define a collection of functionals $G_T^i(M) = \int_0^T g_i(t) \, \mathrm{d}M_t$, where $g_i \in L^2([0, T], \mathrm{d}\langle M\rangle)$, $i = 1, \ldots, N$, such that

$$\int_0^T \mathbf{g}(t) \frac{\mathrm{d}S_t}{S_t} = \mathbf{y} = [y_i]_{i=1}^N, \tag{5.2}$$

for some $\mathbf{y} \in \mathbb{R}^N$. This is equivalent to the conditioning of the Gaussian martingale M on the linear functionals $\mathbf{G}_T = [G_T^i]_{i=1}^N$ of the log-returns:

$$\mathbf{G}_T(M) = \int_0^T \mathbf{g}(t) \, \mathrm{d}M_t = \left[\int_0^T g_i(t) \, \mathrm{d}M_t\right]_{i=1}^N .$$

Indeed, the connection is

$$\int_0^T \mathbf{g}(t) \, \mathrm{d}M_t = \mathbf{y} - \langle\!\langle\!\langle a, \mathbf{g}\rangle\!\rangle\!\rangle =: \mathbf{y}',$$

where

$$\langle\!\langle\!\langle a, \mathbf{g}\rangle\!\rangle\!\rangle = [\langle\!\langle\!\langle a, g_i\rangle\!\rangle\!\rangle]_{i=1}^N = \left[\int_0^T a_t g_i(t) \, \mathrm{d}\langle M\rangle_t\right]_{i=1}^N .$$

As the natural filtration $\mathbb{F}$ represents the information available to the ordinary trader, the insider trader's information flow is described by a larger filtration $\mathbb{G} = (\mathscr{G}_t)_{t\in[0,T]}$ given by

$$\mathscr{G}_t = \mathscr{F}_t \vee \sigma(G_T^1, \ldots, G_T^N).$$

Under the augmented filtration $\mathbb{G}$, M is no longer a martingale. It is a Gaussian semimartingale with the *semimartingale decomposition*

$$\mathrm{d}M_t = \mathrm{d}\tilde{M}_t + \left(\int_0^t \ell_{\mathbf{g}}(t, s) \, \mathrm{d}M_s - \mathbf{g}^\top(t) \langle\!\langle\!\langle \mathbf{g}\rangle\!\rangle\!\rangle^{-1}(t) \mathbf{y}'\right) \mathrm{d}\langle M\rangle_t, \tag{5.3}$$

where $\tilde{M}$ is a continuous $\mathbb{G}$-martingale with bracket $\langle M\rangle$, and which can be constructed through the formula (4.11).

In this market, we consider the portfolio process π defined on $[0, T - \varepsilon] \times \Omega$ as the fraction of the total wealth invested in the asset S. So the dynamics of the discounted value process associated to a self-financing strategy π is defined by $V_0 = v_0$ and

$$\frac{\mathrm{d}V_t}{V_t} = \pi_t \frac{\mathrm{d}S_t}{S_t}, \quad \text{for } t \in [0, T-\varepsilon],$$

or equivalently by

$$V_t = v_0 \exp\left(\int_0^t \pi_s \,\mathrm{d}M_s + \int_0^t \left(\pi_s a_s - \frac{1}{2}\pi_s^2\right) \mathrm{d}\langle M\rangle_s\right). \tag{5.4}$$

Let us denote by $\langle\!\langle\!\langle \cdot, \cdot \rangle\!\rangle\!\rangle_\varepsilon$ and $|\!|\!|\!| \cdot |\!|\!|\!|_\varepsilon$ the inner product and the norm on $L^2([0, T-\varepsilon], \mathrm{d}\langle M\rangle)$.

For the ordinary trader, the process π is assumed to be a non-negative $\mathbb{F}$-progressively measurable process such that

(i) $\mathbb{P}[|\!|\!|\!|\pi|\!|\!|\!|_\varepsilon^2 < \infty] = 1$.
(ii) $\mathbb{P}[\langle\!\langle\!\langle \pi, f \rangle\!\rangle\!\rangle_\varepsilon < \infty] = 1$, for all $f \in L^2([0, T-\varepsilon], \mathrm{d}\langle M\rangle)$.

We denote this class of portfolios by $\Pi(\mathbb{F})$. By analogy, the class of the portfolios disposable to the insider trader shall be denoted by $\Pi(\mathbb{G})$, the class of non-negative $\mathbb{G}$-progressively measurable processes that satisfy the conditions (i) and (ii) above.

The aim of both investors is to maximize the expected utility of the terminal wealth $V_{T-\varepsilon}$, by finding an optimal portfolio π on $[0, T-\varepsilon]$ that solves the optimization problem

$$\max_\pi \mathbb{E}\left[U(V_{T-\varepsilon})\right].$$

Here, the utility function U will be the logarithmic utility function, and the utility of the process (5.4) valued at time $T-\varepsilon$ is

$$\begin{aligned} \log V_{T-\varepsilon} &= \log v_0 + \int_0^{T-\varepsilon} \pi_s \,\mathrm{d}M_s + \int_0^{T-\varepsilon} \left(\pi_s a_s - \frac{1}{2}\pi_s^2\right) \mathrm{d}\langle M\rangle_s \\ &= \log v_0 + \int_0^{T-\varepsilon} \pi_s \,\mathrm{d}M_s + \frac{1}{2}\int_0^{T-\varepsilon} \pi_s \left(2a_s - \pi_s\right) \mathrm{d}\langle M\rangle_s \\ &= \log v_0 + \int_0^{T-\varepsilon} \pi_s \,\mathrm{d}M_s + \frac{1}{2}\langle\!\langle\!\langle \pi, 2a - \pi \rangle\!\rangle\!\rangle_\varepsilon. \end{aligned} \tag{5.5}$$

From the ordinary trader's point of view M is a martingale. So, $\mathbb{E}\left(\int_0^{T-\varepsilon} \pi_s \,\mathrm{d}M_s\right) = 0$ for every $\pi \in \Pi(\mathbb{F})$ and, consequently,

$$\mathbb{E}\left[U(V_{T-\varepsilon})\right] = \log v_0 + \frac{1}{2}\mathbb{E}\left[\langle\!\langle\!\langle \pi, 2a - \pi \rangle\!\rangle\!\rangle_\varepsilon\right].$$

Therefore, the ordinary trader, given $\Pi(\mathbb{F})$, will solve the optimization problem

$$\max_{\pi \in \Pi(\mathbb{F})} \mathbb{E}\left[U(V_{T-\varepsilon})\right] = \log v_0 + \frac{1}{2} \max_{\pi \in \Pi(\mathscr{F})} \mathbb{E}\left[\langle\!\langle\!\langle \pi, 2a - \pi \rangle\!\rangle\!\rangle_\varepsilon\right]$$

over the term $\langle\!\langle\!\langle \pi, 2a - \pi \rangle\!\rangle\!\rangle_\varepsilon = 2\langle\!\langle\!\langle \pi, a \rangle\!\rangle\!\rangle_\varepsilon - |\!|\!|\!|\pi|\!|\!|\!|_\varepsilon^2$. By using the polarization identity we obtain

$$\langle\!\langle\!\langle \pi, 2a - \pi \rangle\!\rangle\!\rangle_\varepsilon = |\!|\!|\!|a|\!|\!|\!|_\varepsilon^2 - |\!|\!|\!|\pi - a|\!|\!|\!|_\varepsilon^2 \le |\!|\!|\!|a|\!|\!|\!|_\varepsilon^2.$$

Thus, the maximum is obtained with the choice $\pi_t = a_t$ for $t \in [0, T-\varepsilon]$, and maximal expected utility value is

$$\max_{\pi \in \Pi(\mathbb{F})} \mathbb{E}\left[U(V_{T-\varepsilon})\right] = \log v_0 + \frac{1}{2}\mathbb{E}\left[|||a|||_{\varepsilon}^2\right].$$

From the insider trader's point of view the process M is not a martingale under his information flow $\mathbb{G}$. The insider can update his utility of terminal wealth (5.5) by considering (5.3), where $\tilde{M}$ is a continuous $\mathbb{G}$-martingale. This gives

$$\begin{aligned}\log V_{T-\varepsilon} &= \log v_0 + \int_0^{T-\varepsilon} \pi_s \,\mathrm{d}\tilde{M}_s + \frac{1}{2}\langle\langle\langle \pi, 2a - \pi\rangle\rangle\rangle_{\varepsilon} \\ &\quad + \left\langle\!\!\left\langle\!\!\left\langle \pi, \int_0^{\cdot} \ell_{\mathbf{g}}(\cdot, t)\,\mathrm{d}M_t - \mathbf{g}^{\top}\langle\langle\langle \mathbf{g}\rangle\rangle\rangle^{-1}\mathbf{y}' \right\rangle\!\!\right\rangle\!\!\right\rangle_{\varepsilon}.\end{aligned}$$

Now, the insider maximizes the expected utility over all $\pi \in \Pi(\mathbb{G})$:

$$\begin{aligned}\max_{\pi \in \Pi(\mathbb{G})} \mathbb{E}\left[\log V_{T-\varepsilon}\right] &= \log v_0 + \frac{1}{2} \max_{\pi \in \Pi(\mathbb{G})} \mathbb{E} \\ &\quad \times \left[\left\langle\!\!\left\langle\!\!\left\langle \pi, 2\left(a + \int_0^{\cdot} \ell_{\mathbf{g}}(\cdot, t)\,\mathrm{d}M_t - \mathbf{g}^{\top}\langle\langle\langle \mathbf{g}\rangle\rangle\rangle^{-1}\mathbf{y}'\right) - \pi \right\rangle\!\!\right\rangle\!\!\right\rangle_{\varepsilon}\right].\end{aligned}$$

The optimal portfolio π for the insider trader can be computed in the same way as for the ordinary trader. We obtain the optimal portfolio

$$\pi_t = a_t + \int_0^t \ell_{\mathbf{g}}(t, s)\,\mathrm{d}M_s - \mathbf{g}^{\top}(t)\langle\langle\langle \mathbf{g}\rangle\rangle\rangle^{-1}(t)\mathbf{y}', \quad t \in [0, T-\varepsilon].$$

Let us then calculate the additional expected logarithmic utility for the insider trader. Since

$$\mathbb{E}\left[\left\langle\!\!\left\langle\!\!\left\langle a, \int_0^{\cdot} \ell_{\mathbf{g}}(\cdot, s)\,\mathrm{d}M_s - \mathbf{g}^{\top}\langle\langle\langle \mathbf{g}\rangle\rangle\rangle^{-1}\mathbf{y}' \right\rangle\!\!\right\rangle\!\!\right\rangle_{\varepsilon}\right] = 0,$$

we obtain that

$$\begin{aligned}\Delta_{T-\varepsilon} &= \max_{\pi \in \Pi(\mathbb{G})} \mathbb{E}\left[U(V_{T-\varepsilon})\right] - \max_{\pi \in \Pi(\mathbb{F})} \mathbb{E}\left[U(V_{T-\varepsilon})\right] \\ &= \frac{1}{2}\mathbb{E}\left[\left|\left|\left| \int_0^{\cdot} \ell_{\mathbf{g}}(\cdot, s)\,\mathrm{d}M_s - \mathbf{g}^{\top}\langle\langle\langle \mathbf{g}\rangle\rangle\rangle^{-1}\mathbf{y}' \right|\right|\right|_{\varepsilon}^2\right].\end{aligned}$$

Now, let us use the short-hand notation

$$\begin{aligned}&\mathbf{G}_t := \int_0^t \mathbf{g}(s)\,\mathrm{d}M_s, \\ &\langle\langle\langle \mathbf{g}\rangle\rangle\rangle(t, s) := \langle\langle\langle \mathbf{g}\rangle\rangle\rangle(t) - \langle\langle\langle \mathbf{g}\rangle\rangle\rangle(s), \\ &\langle\langle\langle \mathbf{g}\rangle\rangle\rangle^{-1}(t, s) := \langle\langle\langle \mathbf{g}\rangle\rangle\rangle^{-1}(t) - \langle\langle\langle \mathbf{g}\rangle\rangle\rangle^{-1}(s).\end{aligned}$$

Then, by expanding the square $|||\cdot|||_{\varepsilon}^2$, we obtain

$$\begin{aligned}2\Delta_{T-\varepsilon} &= \mathbb{E}\left[\left|\left|\left| \int_0^{\cdot} \ell_{\mathbf{g}}(\cdot, s)\,\mathrm{d}M_s - \mathbf{g}^{\top}\langle\langle\langle \mathbf{g}\rangle\rangle\rangle^{-1}\mathbf{y}' \right|\right|\right|_{\varepsilon}^2\right] \\ &= \mathbb{E}\left[|||\mathbf{g}^{\top}\langle\langle\langle \mathbf{g}\rangle\rangle\rangle^{-1}\left(\mathbf{y}' + \mathbf{G}\right)|||_{\varepsilon}^2\right]\end{aligned}$$

$$= \mathbb{E}\left[\int_0^{T-\varepsilon} \mathbf{y}'^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{g}(t)\mathbf{g}^\top(t)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{y}' \,\mathrm{d}\langle M\rangle_t\right]$$
$$+\mathbb{E}\left[\int_0^{T-\varepsilon} \mathbf{G}_t^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{g}(t)\mathbf{g}^\top(t)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{G}_t \,\mathrm{d}\langle M\rangle_t\right].$$

Now the formula $\mathbb{E}[\mathbf{x}^\top \mathbf{A}\mathbf{x}] = \mathrm{Tr}[\mathbf{A}\mathbb{C}\mathrm{ov}\mathbf{x}] + \mathbb{E}[\mathbf{x}]^\top \mathbf{A}\mathbb{E}[\mathbf{x}]$ yields

$$\begin{aligned}
2\Delta_{T-\varepsilon} &= \int_0^{T-\varepsilon} \mathbf{y}'^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{g}(t)\mathbf{g}^\top(t)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{y}' \,\mathrm{d}\langle M\rangle_t \\
&\quad + \int_0^{T-\varepsilon} \mathrm{Tr}\left[\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{g}(t)\mathbf{g}^\top(t)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle(0,t)\right] \mathrm{d}\langle M\rangle_t \\
&= \mathbf{y}'^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(T-\varepsilon,0)\mathbf{y}' \\
&\quad + \int_0^{T-\varepsilon} \mathrm{Tr}\left[\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{g}(t)\mathbf{g}^\top(t)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle(0)\right] \mathrm{d}\langle M\rangle_t \\
&\quad - \int_0^{T-\varepsilon} \mathrm{Tr}\left[\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t)\mathbf{g}(t)\mathbf{g}^\top(t)\right] \mathrm{d}\langle M\rangle_t \\
&= (\mathbf{y} - \langle\!\langle\!\langle \mathbf{g}, a \rangle\!\rangle\!\rangle)^\top \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(T-\varepsilon,0)\,(\mathbf{y} - \langle\!\langle\!\langle \mathbf{g}, a \rangle\!\rangle\!\rangle) \\
&\quad + \mathrm{Tr}\left[\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(T-\varepsilon,0)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle(0)\right] + \log\frac{|\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle|(T-\varepsilon)}{|\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle|(0)}.
\end{aligned}$$

We have proved the following proposition:

Proposition 5.6. *The additional logarithmic utility in the model (5.1) for the insider with information (5.2) is*

$$\begin{aligned}
\Delta_{T-\varepsilon} &= \max_{\pi\in\Pi(\mathbb{G})} \mathbb{E}\left[U(V_{T-\varepsilon})\right] - \max_{\pi\in\Pi(\mathbb{F})} \mathbb{E}\left[U(V_{T-\varepsilon})\right] \\
&= \frac{1}{2}(\mathbf{y} - \langle\!\langle\!\langle \mathbf{g}, a \rangle\!\rangle\!\rangle)^\top \left(\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(T-\varepsilon) - \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(0)\right)(\mathbf{y} - \langle\!\langle\!\langle \mathbf{g}, a \rangle\!\rangle\!\rangle) \\
&\quad + \frac{1}{2}\mathrm{Tr}\left[\left(\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(T-\varepsilon) - \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(0)\right)\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle(0)\right] + \frac{1}{2}\log\frac{|\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle|(T-\varepsilon)}{|\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle|(0)}.
\end{aligned}$$

Example 5.7. Consider the classical Black and Scholes pricing model:

$$\frac{\mathrm{d}S_t}{S_t} = \mu\mathrm{d}t + \sigma\mathrm{d}W_t, \qquad S_0 = 1,$$

where $W = (W_t)_{t\in[0,T]}$ is the standard Brownian motion. Assume that the insider trader knows at time $t = 0$ that the total and the average return of the stock price over the period $[0, T]$ are both zeros and that the trading ends at time $T - \varepsilon$. So, the insider knows that

$$G_T^1 = \int_0^T g_1(t)\,\mathrm{d}W_t = \frac{y_1}{\sigma} - \frac{\mu}{\sigma}\langle\!\langle\!\langle g_1, 1_T \rangle\!\rangle\!\rangle = -\frac{\mu}{\sigma}\langle\!\langle\!\langle g_1, 1_T \rangle\!\rangle\!\rangle$$
$$G_T^2 = \int_0^T g_2(t)\,\mathrm{d}W_t = \frac{y_2}{\sigma} - \frac{\mu}{\sigma}\langle\!\langle\!\langle g_2, 1_T \rangle\!\rangle\!\rangle = -\frac{\mu}{\sigma}\langle\!\langle\!\langle g_2, 1_T \rangle\!\rangle\!\rangle,$$

where

$$g_1(t) = 1_T(t),$$
$$g_2(t) = \frac{T-t}{T}.$$

Then, by Proposition 5.6,

$$\Delta_{T-\varepsilon} = \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 \langle\!\langle\!\langle \mathbf{g}, 1_T \rangle\!\rangle\!\rangle^\top \left(\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(T-\varepsilon) - \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(0)\right) \langle\!\langle\!\langle \mathbf{g}, 1_T \rangle\!\rangle\!\rangle$$
$$+ \frac{1}{2}\mathrm{Tr}\left[\left(\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(T-\varepsilon) - \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(0)\right) \langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle(0)\right] + \frac{1}{2}\log \frac{|\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle|(T-\varepsilon)}{|\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle|(0)},$$

with

$$\langle\!\langle\!\langle \mathbf{g} \rangle\!\rangle\!\rangle^{-1}(t) = \begin{bmatrix} \frac{4}{T}\left(\frac{T}{T-t}\right) & -\frac{6}{T}\left(\frac{T}{T-t}\right)^2 \\ -\frac{6}{T}\left(\frac{T}{T-t}\right)^2 & \frac{12}{T}\left(\frac{T}{T-t}\right)^3 \end{bmatrix}$$

for all $t \in [0, T-\varepsilon]$. We obtain

$$\Delta_{T-\varepsilon} = \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 \left\{3T\left(\frac{T}{\varepsilon}\right)^3 - 6T\left(\frac{T}{\varepsilon}\right)^2 + 4T\left(\frac{T}{\varepsilon}\right) - T\right\}$$
$$+2\left(\frac{T}{\varepsilon}\right)^3 - 3\left(\frac{T}{\varepsilon}\right)^2 + 2\left(\frac{T}{\varepsilon}\right) - 2\log\left(\frac{T}{\varepsilon}\right) - 1.$$

Here it can be nicely seen that $\Delta_0 = 0$ (no trading at all) and $\Delta_T = \infty$ (the knowledge of the final values implies arbitrage).

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