Factorized sectorial relations, their maximal-sectorial extensions, and form sums

FACTORIZED SECTORIAL RELATIONS, THEIR MAXIMAL SECTORIAL EXTENSIONS, AND FORM SUMS

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Dedicated to the memory of R.G. Douglas
with admiration for his contributions to mathematics

Abstract. In this paper sectorial operators, or more generally, sectorial relations and their maximal sectorial extensions in a Hilbert space $\mathfrak{H}$ are considered. The particular interest is in sectorial relations S, which can be expressed in the factorized form

$$S = T^*(I + iB)T \quad \text{or} \quad S = T(I + iB)T^*,$$

where B is a bounded selfadjoint operator in a Hilbert space $\mathfrak{K}$ and $T : \mathfrak{H} \to \mathfrak{K}$ or $T : \mathfrak{K} \to \mathfrak{H}$, respectively, is a linear operator or a linear relation which is not assumed to be closed. Using the specific factorized form of S, a description of all the maximal sectorial extensions of S is given with a straightforward construction of the extreme extensions S_F, the Friedrichs extension, and S_K, the Kreĭn extension of S, which uses the above factorized form of S. As an application of this construction the form sum of maximal sectorial extensions of two sectorial relations is treated.

1. Introduction

Factorizations and decompositions of operators play a fundamental role in functional analysis and operator theory. A well-known example is the "Douglas lemma" formulated in [8, Theorem 1] which makes a connection between range inclusion, factorization, and ordering of operators. The importance of this connection is reflected by the remarkable number of applications as well as its usage in the literature where this result plays a central role. The present paper is not aimed to study factorizations on such a general level; it is limited to unbounded nonnegative and sectorial operators, or more generally to sectorial relations S, which admit a factorization of the form $S = T^*(I + iB)T$ or $S = T(I + iB)T^*$, where T is a linear relation and $B \in \mathbf{B}(\mathfrak{H})$ is selfadjoint. The main interest here is in the case where the (linear) relation T is not closed and, therefore, S need not be a maximal sectorial object. This leads to the extension problem for S. Namely $H = T^*(I + iB)T^{**}$ or $H = T^{**}(I + iB)T^*$, respectively, is a maximal sectorial extension of S and it is natural to ask whether this H is the only maximal sectorial extension of S. However, since T is not closed and no further conditions are required on T, the relation S and its closure can have positive defect. This yields immediately the problem "what are the Friedrichs and the Kreĭn (maximal sectorial) extensions of S?" In

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order to answer these questions some background definitions and facts on general sectorial operators and relations are first recalled.

A (linear) relation S in a Hilbert space $\mathfrak{H}$ is said to be *sectorial* with vertex at the origin and semi-angle α, $\alpha \in [0, \pi/2)$, if

$$|\mathrm{Im}\,(h', h)| \leq (\tan\alpha)\mathrm{Re}\,(h', h), \quad \{h, h'\} \in H.$$

Clearly, the closure of a sectorial relation is also sectorial. A sectorial relation S in a Hilbert space $\mathfrak{H}$ is said to be *maximal sectorial* if the existence of a sectorial relation $\widetilde{S}$ in $\mathfrak{H}$ with $S \subset \widetilde{S}$ implies $\widetilde{S} = S$. A maximal sectorial relation is automatically closed.

A sectorial relation S generates a sectorial form, which in general is nondensely defined but closable as stated in the next lemma; for a proof see [18, Theorem VI.1.27], [15, Lemma 7.1].

Lemma 1.1. *Let S be a sectorial relation in a Hilbert space $\mathfrak{H}$. Then the form $\mathfrak{t}_S$ given by*

$$\mathfrak{t}_S[\varphi, \psi] = (\varphi', \psi), \quad \{\varphi, \varphi'\}, \{\psi, \psi'\} \in S,$$

with $\mathrm{dom}\,\mathfrak{t}_S = \mathrm{dom}\,S$ *is well-defined, sectorial, and closable.*

According to the first representation theorem the closure of the form $\mathfrak{t}_S$ determines a unique maximal sectorial relation, which is the *Friedrichs extension* S_F of S; for the densely defined case see [18, VI, Theorem 2.1] for the nondensely defined case see [21], and for the linear relation case see [2, 3]; a recent treatment in the general case can be found in [15, Section 7]. The closure of the form $\mathfrak{t}_S$ is denoted by $\mathfrak{t}_{S_F}$. According to the first representation theorem the domain of S is a core for the closed form $\mathfrak{t}_{S_F}$. It is a consequence of the first representation theorem that there is a one-to-one correspondence between all maximal sectorial relations H in $\mathfrak{H}$ and all closed sectorial forms $\mathfrak{t}$ (not necessarily densely defined) in $\mathfrak{H}$; cf. [18, VI, Theorem 2.7], [15, Theorem 4.3]. This correspondence is denoted by $\mathfrak{t} \to H =: H_\mathfrak{t}$; cf. Lemma 1.1 when $S = H$ is maximal sectorial and $\mathfrak{t}_H$ stands for the closure of $\mathfrak{t}_S$.

All maximal sectorial relations H admit a factorization which uses the real part $(\mathfrak{t}_H)_r$ of the associated closed form $\mathfrak{t}_H$. The real part is a closed nonnegative form and by the first representation theorem there is a unique nonnegative selfadjoint relation H_r corresponding to the closed nonnegative form $(\mathfrak{t}_H)_r$. The present formulation for the induced factorization for H is taken from [15, Theorem 6.2], for the densely defined case; see [18, VI, Theorem 3.2].

Lemma 1.2. *Let H be a maximal sectorial relation and let the closed sectorial form $\mathfrak{t}_H$ correspond to H. Let $(\mathfrak{t}_H)_r$ be the corresponding closed nonnegative form and let H_r be the corresponding nonnegative selfadjoint relation. Then there exists a unique selfadjoint operator $B \in \mathbf{B}(\mathfrak{H})$, which is zero on*

$$\mathfrak{H} \ominus \overline{\mathrm{ran}}\,(H_r)_s^{\frac{1}{2}} = \ker\, H_r \oplus \mathrm{mul}\, H_r, \tag{1.1}$$

with $\|B\| = \tan\alpha$*, such that the form $\mathfrak{t}_H$ is given by*

$$\mathfrak{t}_H[h, k] = ((I + iB)(H_r)_s^{\frac{1}{2}} h, (H_r)_s^{\frac{1}{2}} k), \quad h, k \in \mathrm{dom}\,\mathfrak{t}_H = \mathrm{dom}\, H_r^{\frac{1}{2}}.$$

The maximal sectorial relation H corresponding to $\mathfrak{t}_H$ is given by

$$H = (H_r)^{\frac{1}{2}}(I + iB)(H_r)^{\frac{1}{2}}. \tag{1.2}$$

The orthogonal operator part of H is given by

$$H_{\mathrm{s}} = (H_r)_{\mathrm{s}}^{\frac{1}{2}}(I+iB)(H_r)_{\mathrm{s}}^{\frac{1}{2}}, \tag{1.3}$$

where $(H_r)_{\mathrm{s}} = P_{\overline{\mathrm{dom}}\,H} H_r$ is the operator part of (H_r).

It is the purpose of this paper to study properties of relations S of the form $T^*(I+iB)T$ or $T(I+iB)T^*$ when T is not assumed to be closed and to apply these properties in the study of form sums and sums of sectorial relations. In this case S is sectorial, but typically it is not maximal sectorial. By Lemma 1.1 it induces, in general, a nondensely defined sectorial form, which admits a closure that is again a sectorial form. By the first representation theorem (see [15], [18]) this closed sectorial form corresponds to a maximal sectorial relation which, in addition, extends S. This extension determines (the sectorial version of) the Friedrichs extension S_F of S, analogous to the case where S is nonnegative. Since with S also S^{-1} is sectorial (the sectorial version of) the Kreĭn extension of S can be introduced as $((S^{-1})_F)^{-1}$. The Friedrichs extension and the Kreĭn extension are maximal sectorial extensions of S, which are in addition extremal. In the nonnegative case all nonnegative selfadjoint extensions of S are between S_F and S_K. In the sectorial case there is a version of this property for their real parts (obtained via the real part of the corresponding forms); see [15, Theorem 7.6] and [3, Theorem 3] for a related result.

In Section 2 some basic properties of sectorial relations of the form

$$S = T^*(I+iB)T \quad \text{and} \quad S' = T(I+iB)T^*$$

are studied. In particular, it is shown when the maximal sectorial extension

$$H = T^*(I+iB)T^{**}$$

coincides with the Friedrichs extension S_F of S (Theorem 2.4) and when

$$H' = T^{**}(I+iB)T^*$$

coincides with the Kreĭn extension $(S')_K$ of S' (Theorem 2.6). To give a complete picture of the situation the case $S = T^*(I+iB)T$ is investigated in detail in Section 2.2 by giving a general procedure that leads to the description of the Friedrichs extension S_F and the Kreĭn extension S_K of S and, in fact, all the extremal extensions of S combined with their associated closed sectorial forms; see Theorem 2.9 and Proposition 2.8.

In Section 3 a particular case of a sectorial relation with the factorization $S' = T(I+iB)T^*$ is investigated. The choice for S' treated here occurs when studying the form sums $\mathfrak{t}_1 + \mathfrak{t}_2$ of two closed sectorial (in particular nonnegative) forms in a Hilbert space $\mathfrak{H}$. To explain this let H_1 and H_2 be the maximal sectorial relations in $\mathfrak{H}$ associated with $\mathfrak{t}_1$ and $\mathfrak{t}_2$, respectively. Since the sum $\mathfrak{t}_1 + \mathfrak{t}_2$ is a closed form in $\mathfrak{H}$, there is again an associated maximal sectorial relation $\widehat{H}$ that corresponds to $\mathfrak{t}_1 + \mathfrak{t}_2$; cf. [18, Chapter VI]. In a natural way $\widehat{H}$ can be seen as a maximal sectorial extension of the operator-like sum $H_1 + H_2$ of the maximal sectorial relations H_1 and H_2; for this reason $\widehat{H}$ is called the *form sum extension* of $H_1 + H_2$. To investigate the form sum extension $\widehat{H}$ of $H_1 + H_2$ the Friedrichs and the Kreĭn extension of the sum $H_1 + H_2$ will be constructed; see Theorems 3.2 and 3.3. This leads to a description of all maximal sectorial extensions that are extremal in Proposition 3.4. It turns out that the form sum extension $\widehat{H}$ of $H_1 + H_2$ need not be extremal; a characterization for this is given in Theorem 3.5.

For the treatment in Section 3 the factorized form of H_1+H_2 is again playing a key role. Indeed, according to Lemma 1.2 H_1 and H_2 as maximal sectorial relations admit the factorizations

$$H_j = A_j^{\frac{1}{2}}(I+iB_j)A_j^{\frac{1}{2}},$$

where A_j (the real part of H_j), $j=1,2$, are nonnegative selfadjoint relations in $\mathfrak{H}$ and B_j, $j=1,2$, are bounded selfadjoint operators in $\mathfrak{H}$. This yields the following factorization of H_1+H_2:

$$H_1+H_2 = A_1^{\frac{1}{2}}(I+iB_1)A_1^{\frac{1}{2}} + A_2^{\frac{1}{2}}(I+iB_1)A_2^{\frac{1}{2}} = \Phi\left(I_{\mathfrak{H}^2}+i(B_1\oplus B_2)\right)\Phi^*,$$

where Φ stands for the row operator (or relation) from $\mathfrak{H}\times\mathfrak{H}$ to $\mathfrak{H}$ formally defined by

$$\Phi = \begin{pmatrix} A_1^{\frac{1}{2}} & A_2^{\frac{1}{2}} \end{pmatrix}$$

and whose adjoint Φ^* is the column operator (or relation) formally given by

$$\Phi^* = \begin{pmatrix} A_1^{\frac{1}{2}} \\ A_2^{\frac{1}{2}} \end{pmatrix} : \mathfrak{H}\to\mathfrak{H}\times\mathfrak{H}.$$

Hence H_1+H_2 is a sectorial relation which admits a factorization of the form $S=T(I+iB)T^*$ with $T=\Phi$ and $B=B_1\oplus B_2$. Even in the case that H_1 and H_2 are densely defined operators, the operator T is typically neither closed nor closable; it can even be singular (cf. [16]) if for instance $\operatorname{dom} A_1^{\frac{1}{2}}\cap\operatorname{dom} A_2^{\frac{1}{2}}=\{0\}$.

For some general developments on the notions of Friedrichs and Kreĭn extensions the reader is referred to see [1, 6, 7, 10, 11, 18, 19] in the case of nonnegative operators and relations and [2, 15, 18, 21] in the case of sectorial relations. Treatments of extremal extensions can be found in [3, 5, 13], while construction of factorizations for these extensions have been treated in [5, 13, 20, 22, 23, 24] and the notion of form sums appears in [9, 12, 14, 24]. Throughout this paper [15] will be used as a standard reference for various concepts and results on sectorial relations and their extensions; therein one can also find a more detailed description on the literature and developments in this area. As another general overview on sectorial relations we would like to mention the survey paper of Yu.M. Arlinskiĭ [4].

Finally it should be pointed out that the results in Section 2 apply in particular to the factorized nonnegative relations of the form

$$S=T^*T \quad\text{or}\quad S=TT^*,$$

where T is a linear relation or operator which is not assumed to be closed. The special case where $S=T^*T$ is a densely defined nonnegative operator and the densely defined operator T is not closed has been recently investigated in [24]. Similarly, the results in Section 3 extend the earlier results concerning the sum of nonnegative relations obtained in [12] and [14].

2. Some characteristic properties of $T^*(I+iB)T$ and $T(I+iB)T^*$

In this section the class of linear relations S in a Hilbert space $\mathfrak{H}$ which admit a factorization of the form

$$S=T^*(I+iB)T \quad\text{or}\quad S=T(I+iB)T^* \tag{2.1}$$

will be studied; here B is a bounded operator in a Hilbert space $\mathfrak{K}$ and T is a linear operator or a linear relation (not necessarily closed) from $\mathfrak{H}$ to $\mathfrak{K}$ or from $\mathfrak{K}$ to $\mathfrak{H}$, respectively. This class contains all densely defined, not necessarily closed, sectorial relations, but also a wide class of multivalued sectorial relations; for instance Lemma 1.2 shows that all maximal sectorial relations admit a factorization of the form (2.1) with T a closed operator or a closed relation; see (1.2), (1.3). Conversely, if T is closed then the relation S in (2.1) is maximal sectorial. In the case that T is not closed the relation S need not be maximal sectorial, but it has maximal sectorial extensions.

2.1. **Some basic properties of T^*CT.** To study operators and relations S determined by the factorization (2.1), the following observations concerning products of the form T^*CT are helpful.

Lemma 2.1. *Let T be a relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$, let $C \in \mathbf{B}(\mathfrak{K})$ and let the linear relation W in $\mathfrak{H}$ be defined as the product*

$$W = T^*CT.$$

Then the following statements hold:

(i) *If C has the property*

$$(Cf, f) = 0 \quad \Rightarrow \quad f = 0 \tag{2.2}$$

then for each $\varphi' \in \operatorname{ran} W$ there is precisely one $\alpha \in \mathfrak{K}$ such that for any $\varphi \in \mathfrak{H}$ with $\{\varphi, \varphi'\} \in W$ one has

$$\{\varphi, \alpha\} \in T \quad \text{and} \quad \{C\alpha, \varphi'\} \in T^*, \tag{2.3}$$

in which case

$$(\varphi', \varphi) = (C\alpha, \alpha). \tag{2.4}$$

Moreover, for every $\{\varphi, \varphi'\} \in W$ the element $\varphi \in \mathfrak{H}$ is uniquely determined modulo $\ker T$. In particular, W satisfies the following identities

$$\operatorname{mul} W = \operatorname{mul} T^* \quad \text{and} \quad \ker W = \ker T. \tag{2.5}$$

(ii) *If for any sequence (f_n) the operator C satisfies the property*

$$\lim_{n\to\infty} (Cf_n, f_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} f_n = 0, \tag{2.6}$$

then the following implication is also true

$$T \quad \text{is closed} \quad \Rightarrow \quad W \quad \text{is closed}.$$

In particular, the closure of W satisfies $W^{**} \subset T^*CT^{**}$ and

$$\operatorname{mul} W^{**} = \operatorname{mul} W = \operatorname{mul} T^*, \quad \ker T \subset \ker W^{**} \subset \ker T^{**}.$$

Proof. (i) Let $\varphi' \in \operatorname{ran} W$. Then for any $\varphi \in \mathfrak{H}$ such that $\{\varphi, \varphi'\} \in W$ there exists $\alpha \in \mathfrak{K}$ such that (2.3) holds and consequently (2.4) is satisfied, too. To see the uniqueness properties of α and φ assume that also $\{\varphi_0, \varphi'\} \in W$ with $\varphi_0 \in \mathfrak{H}$. Then analogously there exists an element $\alpha_0 \in \mathfrak{K}$ such that

$$\{\varphi_0, \alpha_0\} \in T, \quad \{C\alpha_0, \varphi'\} \in T^*,$$

which via (2.3) leads to

$$\{\varphi - \varphi_0, \alpha - \alpha_0\} \in T, \quad \{C(\alpha - \alpha_0), 0\} \in T^*.$$

Hence $(C(\alpha - \alpha_0), \alpha - \alpha_0) = 0$ and now the assumption in (i) implies that $\alpha = \alpha_0$, i.e., α is unique. Moreover, one concludes that $\{\varphi - \varphi_0, 0\} \in T$, which proves the claimed uniqueness of φ and the equality $\ker W = \ker T$.

To see that $\operatorname{mul} W = \operatorname{mul} T^*$, assume that $\{0, \varphi'\} \in W$. Then it follows from (2.3) and (2.4) that $\alpha = 0$, which implies that $\operatorname{mul} W \subset \operatorname{mul} T^*$. The reverse inclusion is trivial and hence (2.5) is shown.

(ii) Assume that T is closed. To see that W is closed, let $\{\varphi_n, \varphi'_n\} \in W$ converge to $\{\varphi, \varphi'\} \in \mathfrak{H}$. Then there exists a sequence of vectors $\alpha_n \in \mathfrak{K}$ such that

$$\{\varphi_n, \alpha_n\} \in T \quad \text{and} \quad \{C\alpha_n, \varphi'_n\} \in T^*,$$

and it follows that

$$(C\alpha_n, \alpha_n) = (\varphi'_n, \varphi_n) \to (\varphi', \varphi).$$

Consequently,

$$(C(\alpha_n - \alpha_m, \alpha_n - \alpha_m) \to 0, \quad n, m \to \infty,$$

and now the assumption in (ii) shows that (α_n) is a Cauchy sequence in $\mathfrak{K}$. Hence, α_n converges to some α in $\mathfrak{K}$ and one concludes that $\{\varphi, \alpha\} \in T$ and $\{C\alpha, \varphi'\} \in T^*$. Thus $\{\varphi, \varphi'\} \in W$ and W is closed.

Finally, the inclusion $W \subset T^*CT^{**}$ is clearly true and since T^{**} is closed, also T^*CT^{**} is closed by the property (2.6). Therefore,

$$W \subset W^{**} \subset T^*CT^{**}.$$

By the statement (i) this leads to $\ker T \subset \ker W^{**} \subset \ker T^*CT^{**} = \ker T^{**}$ and

$$\operatorname{mul} T^* = \operatorname{mul} W \subset \operatorname{mul} W^{**} \subset \operatorname{mul} T^*CT^{**} = \operatorname{mul} T^*,$$

so that $\operatorname{mul} W = \operatorname{mul} W^{**} = \operatorname{mul} T^*$. This completes the proof. □

By changing the roles of T and T^* in Lemma 2.1 leads to the following result.

Corollary 2.2. *Let T be a relation from a Hilbert space $\mathfrak{K}$ to a Hilbert space $\mathfrak{H}$, let $C \in \mathbf{B}(\mathfrak{K})$ and let the linear relation W in $\mathfrak{H}$ be defined as the product*

$$W = TCT^*.$$

Then:

(i) *the assumption* (2.2) *implies that W satisfies the properties in part (i) in Lemma 2.1 with the roles of T and T^* interchanged.*
(ii) *If* (2.6) *holds, then $W^{**} \subset T^{**}CT^*$ and if T is closed then also W is closed. Moreover,*

$$\ker W^{**} = \ker W = \ker T^*, \quad \operatorname{mul} T \subset \operatorname{mul} W^{**} \subset \operatorname{mul} T^{**}.$$

Proof. (i) This assertion is proved by interchanging the roles of T and T^* in the proof of Lemma 2.1.

(ii) The statement with T closed is obtained by applying part (ii) of Lemma 2.1 to T^* instead of T. As to the remaining assertions observe that $W \subset W^{**} \subset T^{**}CT^*$ and hence $\operatorname{mul} T = \operatorname{mul} W \subset \operatorname{mul} W^{**} \subset \operatorname{mul} T^{**}CT^* = \operatorname{mul} T^{**}$. Moreover,

$$\ker T^* = \ker W \subset \ker W^{**} \subset \ker T^*CT^{**} = \ker T^*,$$

and thus $\ker W = \ker K^{**} = \ker T^*$. □

In particular, all (positively or negatively) definite operators C satisfy the assumption (i) in Lemma 2.1 and all uniformly definite operators C satisfy the assumption (ii) in Lemma 2.1. Of course there are many other operators where assumption (i) or (ii) in Lemma 2.1 is satisfied. Notice that if C satisfies the assumption (i) or (ii) in Lemma 2.1, then the same is true also for the following operators

$$C^*; \quad \eta C \ (0 \neq \eta \in \mathbb{C}); \quad X^*CX,$$

where X is a bounded operator with bounded inverse. In the present paper Lemma 2.1 is applied to a special class of sectorial relations.

Proposition 2.3. *Let T be a linear relation and let $C = I+iB$ for some selfadjoint operator $B \in \mathbf{B}(\mathfrak{K})$. Then*

$$S = T^*(I+iB)T \quad \textit{and} \quad S' = T(I+iB)T^*,$$

with T from $\mathfrak{H}$ to $\mathfrak{K}$ or from $\mathfrak{K}$ to $\mathfrak{H}$, respectively, are sectorial relations in $\mathfrak{H}$ with vertex at the origin and semi-angle at most $\arctan \|B\|$, and S admits the properties (i) and (ii) in Lemma 2.1 while S' admits the properties in Corollary 2.2.

If, in addition, the relation T is closed, i.e. $T = T^{**}$, then S and S' as well as their adjoints are maximal sectorial with

$$S^* = T^*(I-iB)T, \quad (S')^* = T(I-iB)T^*.$$

Proof. Since B is selfadjoint one concludes that for all $\{\varphi, \varphi'\} \in S$:

$$|\mathrm{Im}\,(\varphi', \varphi)| = |(B\alpha, \alpha)| \leq \|B\|(\alpha, \alpha) = \|B\|\mathrm{Re}\,(\varphi', \varphi);$$

cf. the beginning of the proof of Lemma 2.1. Hence S is sectorial with vertex at the origin and semi-angle at most $\arctan \|B\|$. The argument concerning S' remains the same.

The properties for S in Lemma 2.1 and for S' in Corollary 2.2 follow from that fact that the real part of $C = I+iB$ as the identity operator is boundedly invertible.

Finally, if T is closed then also $S = T^*(I+iB)T$ and $S' = T(I+iB)T^*$ are closed by Lemma 2.1. The fact that S, S' are maximal sectorial can be found in [17]. Then also their adjoints are maximal sectorial and since $S^* = (T^*(I+iB)T)^* \supset T^*(I-iB)T$, where $T^*(I-iB)T$ is maximal sectorial (again see [17]), equality $S^* = T^*(I-iB)T$ prevails. The equality $(S')^* = T(I-iB)T^*$ is now obtained by changing the roles of T and T^*. □

It is a consequence of Lemma 1.2 that a set $\mathcal{D}$ is a core for the form $\mathfrak{t}_H$ precisely when $\mathcal{D}$ is a core for its real part $(\mathfrak{t}_H)_r$. This observation combined with Lemmas 1.1, 1.2, and 2.1 leads to a characterization concerning the factorization (2.1) of S and its Friedrichs extension S_F.

Theorem 2.4. *Let S be a not necessarily closed sectorial relation in the Hilbert space $\mathfrak{H}$. Then the following assertions are equivalent:*

(i) $\mathrm{mul}\, S = \mathrm{mul}\, S^*$;
(ii) *there exists a Hilbert space $\mathfrak{K}$, a linear relation $T : \mathfrak{H} \to \mathfrak{K}$ with $\mathrm{dom}\, T = \mathrm{dom}\, S$ and a selfadjoint operator $B \in \mathbf{B}(\mathfrak{K})$, such that*

$$S = T^*(I+iB)T \quad \textit{and} \quad S_F = T^*(I+iB)T^{**}. \tag{2.7}$$

Moreover, in (ii) $T : \mathfrak{H} \to \mathfrak{K}$ can be assumed to be a closable operator.

Proof. (i) ⇒ (ii) Assume that S is a sectorial relation such that $\mathrm{mul}\, S = \mathrm{mul}\, S^*$. Let S_F be the Friedrichs extension of S associated with the closure of the form $\mathfrak{t}_S$ defined in Lemma 1.1. By Lemma 1.2 S_F admits the factorization (1.2) with $(S_F)_r^{\frac{1}{2}}$

and $B \in \mathbf{B}(\mathfrak{H})$), while its operator part is factorized as in (1.3) using the operator part of $(S_F)_r^{\frac{1}{2}}$. Now introduce the operator T as the following restriction:

$$T := ((S_F)_r^{\frac{1}{2}})_{\rm s} \upharpoonright \operatorname{dom} S. \tag{2.8}$$

Recall that $\operatorname{dom} S$ is a core for the forms $\mathfrak{t}_{S_F}$ and $(\mathfrak{t}_{S_F})_r$. Consequently, $\operatorname{dom} S$ is also a core for the operator part, i.e., $\operatorname{clos} T = ((S_F)_r^{\frac{1}{2}})_{\rm s}$. In particular, T is closable. Moreover,

$$T^* = (((S_F)_r^{\frac{1}{2}})_{\rm s})^* = (S_F)_r^{\frac{1}{2}}, \tag{2.9}$$

where the adjoint is taken in $\mathfrak{H}$; notice that $(\operatorname{dom} T)^\perp = \operatorname{mul} S_F = \operatorname{mul} (S_F)_r = \operatorname{mul} (S_F)_r^{\frac{1}{2}}$.

We claim that $S = T^*(I + iB)T$. In fact, by the definition of T one has $(\operatorname{dom} S)^\perp = (\operatorname{dom} T)^\perp = \operatorname{mul} T^*$ and hence the assumption $\operatorname{mul} S = \operatorname{mul} S^*$ yields

$$\operatorname{mul} S = \operatorname{mul} T^* = \operatorname{mul} S_F.$$

This identity combined with the inclusion $S \subset S_F$ and the identities (2.8) and (2.9) shows that

$$S = \{\{f, f'\} \in S_F : f \in \operatorname{dom} S\} = T^*(I + iB)T.$$

(ii) $\Rightarrow$ (i) By Proposition 2.3 every relation S of the form (2.7) is sectorial. Clearly,

$$S \subset S^{**} \subset T^*(I + iB)T^{**},$$

and by the assumption $S_F = T^*(I+iB)T^{**}$. Since the domain of S is a core for the closed form $\mathfrak{t}_{S_F}$, one has $\operatorname{mul} S_F = \operatorname{mul} S^*$. On the other hand, by Lemma 2.1 (i) (cf. Proposition 2.3) S and S_F in (2.7) satisfy $\operatorname{mul} S = \operatorname{mul} T^*$ and $\operatorname{mul} S_F = \operatorname{mul} T^*$. Therefore, $\operatorname{mul} S = \operatorname{mul} S^*$ holds.

The last assertion is clear from the proof (i) $\Rightarrow$ (ii). □

In the case that S is densely defined Theorem 2.4 gives the following result.

Corollary 2.5. *Let S be a densely defined sectorial operator in the Hilbert space $\mathfrak{H}$. Then there exists a Hilbert space $\mathfrak{K}$, a closable operator $T : \mathfrak{H} \to \mathfrak{K}$ with* $\operatorname{dom} T = \operatorname{dom} S$ *and a selfadjoint operator $B \in \mathbf{B}(\mathfrak{K})$, such that*

$$S = T^*(I + iB)T \quad \textit{and} \quad S_F = T^*(I + iB)T^{**}.$$

Proof. If S is densely defined, then $\operatorname{mul} S \subset \operatorname{mul} S^* = (\operatorname{dom} S)^\perp = \{0\}$ and now the statement follows from Theorem 2.4. □

Corollary 2.5 extends [24, Theorem 5.3]: if $S \geq 0$ is a densely defined operator then there is a closable operator T in $\mathfrak{H}$ such that

$$S = T^*T \quad \text{and} \quad S_F = T^*T^{**};$$

in [24] these factorizations for $S \geq 0$ were constructed in another way.

Theorem 2.4 involves the Friedrichs extension S_F of S. There is a similar result for the Kreĭn extension S_K of S. The Kreĭn extension in the nonnegative case was introduced and studied in [19]. Following the approach used in the nonnegative case in [1, 7] this extension is defined for a sectorial relation S using the inverse S^{-1} by the formula

$$S_K = ((S^{-1})_F)^{-1};$$

cf. [3, Definition 2], [15, Definition 7.4]. This leads to the following analog of Theorem 2.4.

Theorem 2.6. *Let S be a not necessarily closed sectorial relation in the Hilbert space $\mathfrak{H}$. Then the following assertions are equivalent:*

(i) $\ker S = \ker S^*$;

(ii) *there exists a Hilbert space $\mathfrak{K}$, a linear relation $T : \mathfrak{K} \to \mathfrak{H}$ with* $\operatorname{ran} T = \operatorname{ran} S$ *and a selfadjoint operator $B \in \mathbf{B}(\mathfrak{K})$, such that*

$$S = T(I + iB)T^* \quad \text{and} \quad S_K = T^{**}(I + iB)T^*. \tag{2.10}$$

Moreover, in (ii) the inverse $T^{-1} : \mathfrak{H} \to \mathfrak{K}$ can be assumed to be a closable operator.

Proof. (i) $\Rightarrow$ (ii) Assume that S is a sectorial relation such that $\ker S = \ker S^*$ and consider its inverse S^{-1}. By the assumption one has $\operatorname{mul} S^{-1} = \operatorname{mul}(S^{-1})^*$ and hence by Theorem 2.4 there exist a linear relation $\widetilde{T} : \mathfrak{H} \to \mathfrak{K}$, which can be assume to be closable, and a selfadjoint operator $\widetilde{B} \in \mathbf{B}(\mathfrak{K})$ such that

$$S^{-1} = \widetilde{T}^*(I + i\widetilde{B})\widetilde{T}, \quad (S^{-1})_F = \widetilde{T}^*(I + i\widetilde{B})\widetilde{T}^{**}.$$

Passing to the inverses one obtains

$$S = \widetilde{T}^{-1}(I + i\widetilde{B})^{-1}(\widetilde{T}^*)^{-1}, \quad S_K = (\widetilde{T}^{**})^{-1}(I + i\widetilde{B})^{-1}(\widetilde{T}^*)^{-1}.$$

Since $(I + i\widetilde{B})^{-1} = (I + \widetilde{B}^2)^{-\frac{1}{2}}(I - i\widetilde{B})(I + \widetilde{B}^2)^{-\frac{1}{2}}$, this yields

$$S = T(I - i\widetilde{B})T^*, \quad S_K = T^{**}(I - i\widetilde{B})T^*,$$

where $T = \widetilde{T}^{-1}(I+\widetilde{B}^2)^{-\frac{1}{2}}$ and $T^* = (I+\widetilde{B}^2)^{-\frac{1}{2}}(\widetilde{T}^{-1})^*$; note that $(\widetilde{T}^{-1})^* = (\widetilde{T}^*)^{-1}$. By construction $\operatorname{ran} T = \operatorname{dom} \widetilde{T} = \operatorname{dom} S^{-1} = \operatorname{ran} S$. Since $(I + \widetilde{B}^2)^{\frac{1}{2}}$ is bounded with bounded inverse one has $\operatorname{clos} T^{-1} = (I+\widetilde{B}^2)^{\frac{1}{2}}(\operatorname{clos} \widetilde{T})$ and thus T^{-1} is closable precisely when $\widetilde{T}$ is closable. Therefore the assertions in (ii) hold and one has the factorizations (2.10) with $T = \widetilde{T}^{-1}(I + \widetilde{B}^2)^{-\frac{1}{2}}$ and $B = -\widetilde{B}$.

(ii) $\Rightarrow$ (i) By Proposition 2.3 every relation S of the form (2.10) is sectorial. Clearly,

$$S \subset S^{**} \subset T^{**}(I + iB)T^*,$$

and by the assumption $S_K = T^{**}(I + iB)T^*$. Since the range of S is a core for the closed form $\mathfrak{t}_{(S^{-1})_F}$, one has $\ker S_K = \ker S^*$. On the other hand, by Proposition 2.3 (or Corollary 2.2) S and S_K in (2.10) satisfy $\ker S = \ker T^*$ and $\ker S_K = \ker T^*$. Therefore, $\ker S = \ker S^*$ holds. □

It is clear that there is an analog of Corollary 2.5 concerning the factorization $T(I + iB)T^*$ whose formulation is left to the reader. In what follows the purpose is to offer a construction for maximal sectorial extensions, in particular, for the Friedrichs extension and the Kreĭn extension, for sectorial relations S and S' which admit a factorization as in Proposition 2.3 without any additional conditions as in Theorems 2.4 and 2.6. In the next section attention is limited to the case $S = T^*(I + iB)T$. On the other hand, in Section 3 a special case where S admits a factorization $S = T(I + iB)T^*$ is treated by investigating the form sum of two maximal sectorial relations.

2.2. **Maximal sectorial extensions of $T^*(I + iB)T$ with nonclosed T.** In Lemma 2.1 it has been shown that the relation $T^*(I+iB)T$, when T is not necessarily closed, is still sectorial. The purpose in this section is to show that $T^*(I+iB)T$ has maximal sectorial extensions and, in particular, to describe all of them. It is clear that every maximal sectorial extension of $T^*(I+iB)T$ is also an extension of the closure $\operatorname{clos}(T^*(I+iB)T)$. On the other hand,

$$\operatorname{clos}(T^*(I+iB)T) \subset T^*(I+iB)T^{**}, \tag{2.11}$$

since by Proposition 2.3 the relation $T^*(I+iB)T^{**}$ is closed and, in fact, a maximal sectorial relation in $\mathfrak{H}$. Hence, it is clear that without any additional assumptions on $T^*(I+iB)T$ the relation on the right-hand side of (2.11) is one of the maximal sectorial extensions of $S := T^*(I+iB)T$. Under the additional condition $\operatorname{mul} S = \operatorname{mul} S^*$ one has $S_F = T^*(I+iB)T^{**}$; see Theorem 2.4. In what follows this additional condition will not be assumed.

The aim now is to describe all extremal maximal sectorial extensions of $S = T^*(I+iB)T$, including the Friedrichs extension S_F, using the given factorized form of S. The purpose is to incorporate explicitly the prescribed structure of $S = T^*(I+iB)T$ in the construction of maximal sectorial extensions of $T^*(I+iB)T$. The approach presented here has the advantage that it prevents the construction of an auxiliary Hilbert space when compared with the procedure appearing in [15] for a sectorial relations S without additional information on its structure.

Recall from Lemma 2.1 that for each $\varphi', \psi' \in \operatorname{ran} S$ there exist unique elements $\alpha, \beta \in \mathfrak{K}$ with

$$\{\varphi,\alpha\} \in T, \quad \{(I+iB)\alpha, \varphi'\} \in T^*, \quad \{\psi,\beta\} \in T, \quad \{(I+iB)\beta, \psi'\} \in T^*. \tag{2.12}$$

Next introduce the linear subspace $\mathfrak{M}_0$ of the Hilbert space $\mathfrak{K}$ via

$$\mathfrak{M}_0 = \{\alpha \in \mathfrak{K} :\ \alpha \in \operatorname{ran} T, (I+iB)\alpha \in \operatorname{dom} T^*\}, \tag{2.13}$$

and let $\mathfrak{M}$ be the closure of $\mathfrak{M}_0$ in $\mathfrak{K}$. Moreover, let B_m be the compression of B to $\mathfrak{M}$:

$$B_m := P_{\mathfrak{M}} B \restriction \mathfrak{M} \in \mathbf{B}(\mathfrak{M}). \tag{2.14}$$

Then B_m is a selfadjoint operator in $\mathfrak{M}$. Next we construct a pair of relations $Q \subset T$ and $J \subset Q^*$, which will be used to describe the minimal and maximal and, in fact, all extremal maximal sectorial extensions of $T^*(I+iB)T$.

Lemma 2.7. *Associate with $T^*(I+iB)T$ the subspace $\mathfrak{M}_0$ of $\mathfrak{K}$ in* (2.13) *and the compression B_m in* (2.14) *and define the linear relation Q from $\mathfrak{H}$ to $\mathfrak{M}$ and the linear relation J from $\mathfrak{M}$ to $\mathfrak{H}$ via*

$$\begin{aligned} Q &= \{\{\varphi,\alpha\} \in T :\ \alpha \in \mathfrak{M}_0\}, \\ J &= \{\{(I+iB_m)\alpha, \varphi'\} :\ \alpha \in \mathfrak{M}_0,\ \{(I+iB)\alpha, \varphi'\} \in T^*\}. \end{aligned}$$

Then $Q \subset J^*$, or equivalently, $J \subset Q^*$, and Q is a closable operator with dense range in $\mathfrak{M}$, while J is densely defined and satisfies $\operatorname{mul} J = \operatorname{mul} J^{**} = \operatorname{mul} T^*$. *Moreover, one has the equality*

$$T^*(I+iB)T = J(I+iB_m)Q.$$

Proof. It is first shown that $Q \subset J^*$. For this let $\{\varphi,\alpha\} \in Q$ and $\{(I+iB_m)\beta, \psi'\} \in J$. Then $\alpha, \beta \in \mathfrak{M}_0$ and they correspond to some $\{\varphi,\varphi'\}, \{\psi,\psi'\} \in T^*(I+iB)T$ via (2.12). In particular, $\{\varphi,\alpha\} \in T$ and hence

$$(\psi', \varphi) - ((I+iB_m)\beta, \alpha) = (\psi', \varphi) - ((I+iB)\beta, \alpha) = 0,$$

where the last equality follows from (2.12). Hence $Q \subset J^*$ and, equivalently, $J \subset Q^*$.

Next it is shown that the set $(I+iB_m)(\mathfrak{M}_0)$ is dense in $\mathfrak{M}$. Assume conversely that there exists $\beta \in \mathfrak{M}$ such that $((I+iB_m)\alpha, \beta) = 0$ for all $\alpha \in \mathfrak{M}_0$. Let $\alpha_n \in \mathfrak{M}_0$ be a sequence such that $\alpha_n \to \beta$ (in $\mathfrak{K}$). Then

$$0 = ((I+iB_m)\alpha_n, \beta) = ((I+iB)\alpha_n, \beta)$$

and by taking limit this leads to

$$0 = \lim_{n\to\infty} ((I+iB)\alpha_n, \beta) = ((I+iB)\beta, \beta),$$

which implies that $\beta = 0$. Consequently, J is densely defined in $\mathfrak{M}$ and hence its adjoint J^* is an operator. Since $Q \subset J^*$, the relation Q is a closable operator. Furthermore, by definition, $\operatorname{ran} Q$ is dense in $\mathfrak{M}$.

Now consider the multivalued parts of J and its closure J^{**}. The inclusion $\operatorname{mul} T^* \subset \operatorname{mul} J$ follows from the definition of J and clearly $\operatorname{mul} J \subset \operatorname{mul} J^{**}$. On the other hand, if $\psi' \in \operatorname{mul} J^{**}$ then there are sequences $\{\psi_n, \beta_n\} \in T$ and $\{(I+iB)\beta_n, \psi_n'\} \in T^*$ such that $(I+iB_m)\beta_n \to 0$ and $\psi_n' \to \psi'$. Then necessarily $\beta_n \to 0$ in $\mathfrak{M}$ since B_m is selfadjoint and hence $(I+iB_m)$ is boundedly invertible in $\mathfrak{M}$. Then $(I+iB)\beta_n \to 0$ in $\mathfrak{K}$ and consequently $\{0, \psi'\} \in T^*$, i.e. $\psi' \in \operatorname{mul} T^*$. Hence, $\operatorname{mul} J^{**} \subset \operatorname{mul} T^*$ and the equalities $\operatorname{mul} J = \operatorname{mul} J^{**} = \operatorname{mul} T^*$ follow.

Finally, the last identity is shown. The inclusion $T^*(I+iB)T \subset J(I+iB_m)Q$ follows directly from (2.12) and the definitions of Q and J. The reverse inclusion $J(I+iB_m)Q \subset T^*(I+iB)T$ is clear from the definitions of Q and J. □

It follows from Lemma 2.7 that J^* is a closed operator from $\mathfrak{H}$ into $\mathfrak{M}$ and its domain is dense in $(\operatorname{mul} J^{**})^\perp = \overline{\operatorname{dom}}\, T$. Moreover, by definition the domain of the restriction $Q \subset J^*$ is given by $\operatorname{dom} Q = \operatorname{dom}(T^*(I+iB)T)$; cf. (2.13). The next result characterizes a class of closed sectorial forms generated by linear operators K lying between these two operators.

Proposition 2.8. *Let the notation be as in Lemma 2.7 and let K be a linear operator satisfying*

$$Q \subset K \subset J^*.$$

Then the form induced by K:

$$\mathfrak{t}_K[h,k] = \langle (I+iB_m)Kh, Kk\rangle, \quad h,k \in \operatorname{dom} K,$$

is closable. The closure of the form $\mathfrak{t}$ is given by

$$\mathfrak{t}_{K^{**}}[h,k] = \langle (I+iB_m)K^{**}h, K^{**}k\rangle, \quad h,\, k \in \operatorname{dom} K^{**}, \tag{2.15}$$

and the corresponding maximal sectorial relation $K^*(I+iB_m)K^{**}$ is an extension of the sectorial relation $T^*(I+iB)T$.

Proof. Clearly K is closable and its closure K^{**} satisfies

$$Q \subset K \subset K^{**} \subset J^*, \quad J \subset J^{**} \subset K^*.$$

Hence, the form $\mathfrak{t}_K$ is also closable and its closure is determined by K^{**} as in (2.15). By Proposition 2.3 $K^*(I+iB_m)K^{**}$ is maximal sectorial and it clearly corresponds to the closed form $\mathfrak{t}_{K^{**}}$ in (2.15); cf. Lemma 1.2. Furthermore, since $J \subset K^*$ and $Q \subset K^{**}$ it follows from Lemma 2.7 that

$$T^*(I+iB)T = J(I+iB_m)Q \subset K^*(I+iB_m)K^{**},$$

which proves the last statement. □

It is clear from Proposition 2.8 that

$$K_1 \subset K_2 \quad \Longleftrightarrow \quad \mathfrak{t}_{K_1} \subset \mathfrak{t}_{K_2}$$

and that these forms are closed precisely when the operators K_1 and K_2 are closed. The next result shows that the minimal choice $K_1 = Q^{**}$ in fact corresponds to the Friedrichs extension and the maximal choice $K_2 = J^*$ corresponds to the Kreĭn extension of $T^*(I+iB)T$. Therefore the above procedure in this sense covers the extreme maximal sectorial extensions of $T^*(I+iB)T$.

Theorem 2.9. *Let $S = T^*(I+iB)T$, B_m, Q, and J be as in Lemma 2.7. Then the following statements hold.*

(i) *The Friedrichs extension S_F of S is given by*

$$S_F = Q^*(I+iB_m)Q^{**}$$

and the corresponding closed form $\mathfrak{t}_F$ is given by

$$\mathfrak{t}_{S_F}[h,k] = ((I+iB_m)Q^{**}h, Q^{**}k)\,, \quad h,\, k \in \operatorname{dom} Q^{**}.$$

(ii) *The Kreĭn extension S_K of S is given by*

$$S_K = J^{**}(I+iB_m)J^*$$

and the corresponding closed form $\mathfrak{t}_{S_K}$ is given by

$$\mathfrak{t}_{S_K}[h,k] = ((I+iB_m)J^*h, J^*k)\,, \quad h,\, k \in \operatorname{dom} J^*.$$

In particular, S_K is an operator if and only if T is densely defined. Therefore, $S = T^*(I+iB)T$ admits a maximal sectorial operator extension, precisely when T is densely defined; here T need not be a closable operator, and it can even be multivalued.

Proof. (i) According to Proposition 2.8 $H = Q^*(I+iB_m)Q^{**}$ is a maximal sectorial extension of S. In order to show that it coincides with S_F it suffices to prove that $\operatorname{dom} H \subset \operatorname{dom} \mathfrak{t}_{S_F}$; see e.g. [15, Theorem 7.3]. Let $h \in \operatorname{dom} H$. Then $\{h, h'\} \in Q^*(I+iB_m)Q^{**}$ for some $h' \in \mathfrak{H}$. In particular, $h \in \operatorname{dom} Q^{**}$ and $\{h, Q^{**}h\}$ can be approximated by a sequence of elements

$$\{\varphi_n, \alpha_n\} \in Q,$$

where $\alpha_n \in \mathfrak{M}_0$ and $\{(I+iB_m)\alpha_n, \varphi_n'\} \in J \subset Q^*$ such that

$$\varphi_n \to h \text{ in } \mathfrak{H}, \quad \alpha_n \to Q^{**}h \text{ in } \mathfrak{M}; \tag{2.16}$$

see Lemma 2.7 and (2.13). Hence (α_n) is a Cauchy sequence in $\mathfrak{M}$ and this yields

$$((\varphi_n' - \varphi_m', \varphi_n - \varphi_m) = ((I+iB_m)(\alpha_n - \alpha_m), \alpha_n - \alpha_m) \to 0, \quad n, m \to \infty. \tag{2.17}$$

Since $\{\varphi_n, \varphi_n'\} \in J(I+iB_m)Q = S$ by Lemma 2.7, it follows from (2.16) and (2.17) by the definition of the form $\mathfrak{t}_{S_F}$ that $h \in \operatorname{dom} \mathfrak{t}_{S_F}$; cf. e.g. [15, Eq. (7.2)]. Hence $\operatorname{dom} H \subset \operatorname{dom} \mathfrak{t}_{S_F}$ and the claim $H = S_F$ is proved.

(ii) Likewise $H = J^{**}(I+iB_m)J^*$ is a maximal sectorial extension of S by Proposition 2.8. To show that $H = S_K$, it suffices to prove that $\operatorname{ran} H \subset \operatorname{dom} \mathfrak{t}_{(S^{-1})_F}$; see [15, Theorem 7.5]. Let $h' \in \operatorname{ran} H$. Then $\{h, h'\} \in J^{**}(I+iB_m)J^*$ for some $h \in \mathfrak{H}$, and

$$\{(I+iB_m)J^*h, h'\} \in J^{**}.$$

This element can be approximated by a sequence of elements

$$\{(I+iB_m)\alpha_n, \varphi_n'\} \in J,$$

where $\alpha_n \in \mathfrak{M}_0$ and $\{\varphi_n, \alpha_n\} \in Q \subset J^*$ for some $\varphi_n \in \operatorname{dom} T$, such that

$$\varphi_n' \to h' \text{ in } \mathfrak{H}, \quad (I + iB_m)\alpha_n \to (I + iB_m)J^*h \text{ in } \mathfrak{M}; \tag{2.18}$$

see (2.13) and Lemma 2.7. Since B_m is bounded and selfadjoint in $\mathfrak{M}$, the operator $I + iB_m$ is bounded with bounded inverse and, therefore, (2.18) is equivalent to

$$\varphi_n' \to h' \text{ in } \mathfrak{H}, \quad \alpha_n = J^*\varphi_n \to J^*h \text{ in } \mathfrak{M}. \tag{2.19}$$

In particular, (α_n) is a Cauchy sequence in $\mathfrak{M}$ and again (2.17) follows. Since $\{\varphi_n, \varphi_n'\} \in J(I + iB_m)Q = S$ (see Lemma 2.7), it follows from (2.17) and (2.19) that $h' \in \operatorname{dom} \mathfrak{t}_{(S^{-1})_F}$. Therefore, $\operatorname{ran} H \subset \operatorname{dom} \mathfrak{t}_{(S^{-1})_F}$ and $H = S_K$ is proved.

The last statement follows from the minimality of S_K, which implies in particular that $\operatorname{dom} \mathfrak{t}_H \subset \operatorname{dom} \mathfrak{t}_{S_K}$: if H is any maximal sectorial operator extension of S, then H and, therefore, also S_K is densely defined; notice that $\operatorname{mul} S_K = \operatorname{mul} J^{**} = \operatorname{mul} T^*$. □

The maximal sectorial extensions $K^*(I + iB_m)K^{**}$ of the sectorial relation S as described in Proposition 2.8 with B_m as in (2.21) and $Q \subset K \subset J^*$ can be characterized among all maximal sectorial extensions of S. The main ingredient in Proposition 2.8 is that the maximal sectorial extensions of S of the form $T^*(I + iB_m)T$ with B_m as in (2.21) and T an arbitrary closed linear operator satisfying $Q \subset T \subset J^*$ can be identified as the class of all *extremal sectorial extensions of* S; for details see [15, Theorems 8.4, 8.5].

This subsection is finished with an example illustrating some special choices for T with descriptions of the mappings Q and J appearing in the description of the maximal sectorial extensions S_F and S_K of the sectorial relation $S = T^*(I + iB)T$.

Example 2.10. (a) Let T be an operator and consider the form

$$\mathfrak{t}[h,k] = ((I + iB)Th, Tk), \quad h, k \in \operatorname{dom} T.$$

Then this form is T is closable (closed) if and only if T is closable (closed, respectively), in which case the closures are related by

$$\widetilde{\mathfrak{t}}[h,k] = ((I + iB)T^{**}h, T^{**}k), \quad h, k \in \operatorname{dom} T^{**},$$

and one has the equalities $Q^{**} = J^* = T^{**}$ and, consequently,

$$S_F = S_K = T^*(I + iB)T^{**},$$

which is an operator if and only if T is densely defined.

(b) Let T be a singular operator (or singular relation); for definitions see e.g. [16]. Then $\overline{\operatorname{ran}}\, T = \operatorname{mul} T^{**}$ and $\overline{\operatorname{dom}}\, T = \ker T^{**}$. In this case $\mathfrak{M} = \{0\}$ and hence,

$$Q = 0 \restriction \operatorname{dom}(T^*(I + iB)T) = 0 \restriction \ker T, \qquad \operatorname{dom} J = \{0\}, \quad \operatorname{mul} J = \operatorname{mul} T^*,$$

so $\operatorname{dom} Q = \ker Q$ while J is a pure relation. Consequently,

$$S_F = Q^*Q^{**}, \quad S_K = J^{**}J^*$$

are nonnegative selfadjoint relations with $\operatorname{dom} S_F = \ker S_F = \overline{\ker T}$, $\operatorname{dom} S_K = \ker S_K = \overline{\operatorname{dom}}\, T$. If, in addition, T is densely defined, then $S_K = 0$ is a selfadjoint operator, while S_F is an operator if and only if $\ker T$ is dense in $\mathfrak{H}$.

(c) Let T be a densely defined (not necessarily closable) operator or relation. Then $J : \mathfrak{M} \to \mathfrak{H}$ is densely defined and since $\operatorname{mul} J = \operatorname{mul} J^{**} = \operatorname{mul} T^* = \{0\}$, the Krein extension S_K is a densely defined maximal sectorial operator:

$$S_K = J^{**}(I + iB_m)J^*;$$

cf. Theorem 2.9.

2.3. **Connection to the abstract construction.** In this section the explicit construction of maximal sectorial extensions for $S = T^*(I + iB)T$ that was using the factorized form of S is connected with the construction appearing in the abstract setting where the specific form of S is taken into account.

The starting point here follows the construction presented in [15]. With any sectorial relation S in $\mathfrak{H}$ introduce the range space $\operatorname{ran} S$ in $\mathfrak{K}$ and provide it with a new inner product. Let $\{\varphi, \varphi'\}, \{\psi, \psi'\} \in S$ and define

$$\langle \varphi', \psi' \rangle_S = \frac{1}{2} \left((\varphi', \psi) + (\varphi, \psi') \right). \tag{2.20}$$

Note that if $\{\varphi_0, \varphi'\}, \{\psi_0, \psi'\} \in S$ the inner product remains the same. Due to the definition of $\{\varphi, \varphi'\}, \{\psi, \psi'\} \in S$ one sees that

$$\langle \varphi', \varphi' \rangle_S = \operatorname{Re} (\varphi', \varphi).$$

Now sectoriality of S combined with an application of the Cauchy-Schwarz inequality (see [15] for details) shows that the isotropic part of $\operatorname{ran} S$ with respect to the inner product $\langle \cdot, \cdot \rangle_S$ is given by

$$\mathfrak{R}_0 = \{ \varphi' \in \operatorname{ran} S : (\varphi', \varphi) = 0 \text{ for some } \varphi \text{ with } \{\varphi, \varphi'\} \in S \},$$

in particular, $\mathfrak{R}_0 = \operatorname{ran} S \cap \operatorname{mul} S^*$. Let $(\mathfrak{H}_S, \langle \cdot, \cdot \rangle_S)$ be the Hilbert space completion of $\operatorname{ran} S / \mathfrak{R}_0$ with respect to the inner product generated on the factor space by (2.20). Define the symmetric form $\mathfrak{b}$ on $\operatorname{dom} \mathfrak{b} = \operatorname{ran} S / \mathfrak{R}_0$ by

$$\mathfrak{b}[[\varphi'], [\psi']] = \frac{i}{2} \left((\varphi, \psi') - (\varphi', \psi) \right), \quad \{\varphi, \varphi'\}, \{\psi, \psi'\} \in S.$$

Note that this definition is correct as seen by checking it for $\{\varphi_0, \varphi'\}, \{\psi_0, \psi'\} \in S$. It follows from [15] that $\mathfrak{b}$ is a bounded everywhere defined symmetric form on $\operatorname{ran} S / \mathfrak{R}_0$. Therefore its closure, also denoted by $\mathfrak{b}$, is an everywhere defined bounded symmetric form on $\mathfrak{H}_S$. Hence there exists a bounded selfadjoint operator $B_S \in \mathbf{B}(\mathfrak{H}_S)$ such that

$$\mathfrak{b}[[\varphi'], [\psi']] = \langle B_S [\varphi'], [\psi'] \rangle_S, \quad \{\varphi, \varphi'\}, \{\psi, \psi'\} \in S. \tag{2.21}$$

Now the prescribed form $T^*(I + iB)T$ of S will be incorporated in the above abstract construction. For this purpose recall that for each $\varphi', \psi' \in \operatorname{ran} S$ there exists unique elements $\alpha, \beta \in \mathfrak{K}$ with

$$\{\varphi, \alpha\} \in T, \quad \{(I + iB)\alpha, \varphi'\} \in T^*, \quad \{\psi, \beta\} \in T, \quad \{(I + iB)\beta, \psi'\} \in T^*. \tag{2.22}$$

see (2.12). This leads to

$$\langle \varphi', \psi' \rangle_S = (\alpha, \beta),$$

showing again that the definition is independent of the particular first entries in $\{\varphi, \varphi'\}, \{\psi, \psi'\} \in S$. Furthermore, (2.22) implies that

$$(\varphi', \varphi) = 0 \quad \Leftrightarrow \quad (\alpha, \alpha) + i(B\alpha, \alpha) = 0 \quad \Leftrightarrow \quad \alpha = 0.$$

Thus $\mathfrak{R}_0 = \operatorname{mul} T^* = \operatorname{mul} S$ and on $\operatorname{ran} S / \mathfrak{R}_0$ one has

$$\langle [\varphi'], [\psi'] \rangle_S = (\alpha, \beta), \quad \langle [\varphi'], [\varphi'] \rangle_S = (\alpha, \alpha). \tag{2.23}$$

Furthermore, it follows from (2.21) that the bounded symmetric form $\mathfrak{b}$ defined on $\operatorname{dom} \mathfrak{b} = \operatorname{ran} S / \mathfrak{R}_0$ satisfies

$$\mathfrak{b}[[\varphi'], [\psi']] = (B\alpha, \beta), \quad \{\varphi, \varphi'\}, \{\psi, \psi'\} \in S.$$

In other words,

$$\langle B_S[\varphi'], [\psi']\rangle_S = (B\alpha, \beta), \quad \{\varphi, \varphi'\}, \{\psi, \psi'\} \in S. \tag{2.24}$$

Now consider the linear space $\mathfrak{M}_0 \subset \mathfrak{K}$ defined in (2.13),

$$\mathfrak{M}_0 = \{\alpha \in \mathfrak{K} : \alpha \in \operatorname{ran} T, (I + iB)\alpha \in \operatorname{dom} T^*\},$$

equipped with the original topology of $\mathfrak{K}$. Moreover, define the mapping $\imath_0$ from $\mathfrak{M}_0$ onto $\operatorname{ran} S/\mathfrak{R}_0$ by

$$\imath_0\alpha = [\varphi'].$$

It follows from (2.23) that $\imath_0$ is an isometry. Hence the closure $\imath$ is a closed isometric operator from the Hilbert space $\mathfrak{M}$, the closure of $\mathfrak{M}_0$, onto the Hilbert space $\mathfrak{H}_S$. Moreover, (2.24) shows that

$$B_m := P_{\mathfrak{M}} B \upharpoonright \mathfrak{M} = \imath^* B_S\, \imath \in \mathbf{B}(\mathfrak{M}).$$

This gives the connection between the space $\mathfrak{H}_S$ and the operator B_S appearing in the abstract construction in [15] and the compression B_m of the prescribed operator B to the subspace $\mathfrak{M}$.

Remark 2.11. The relations $\widetilde{Q} = \imath Q$ from $\mathfrak{H}$ to $\mathfrak{H}_S$ and $\widetilde{J} = J\imath^*$ from $\mathfrak{H}_S$ to $\mathfrak{H}$ are the abstract counterparts of Q and J occurring in [15] when constructing maximal sectorial extensions for a sectorial relation S.

3. **Form sums of maximal sectorial relations**

As indicated in Section 1 the treatment of the sum of two closed sectorial forms gives rise to the notion of form sum extension of the sum of the representing maximal sectorial relations H_1 and H_2. In order to the study the form sum extension more closely one needs to study the class of all maximal sectorial extensions of the sum $H_1 + H_2$.

Let H_1 and H_2 be maximal sectorial relations in a Hilbert space $\mathfrak{H}$. Then the sum $H_1 + H_2$ is a sectorial relation in $\mathfrak{H}$ with

$$\operatorname{dom}(H_1 + H_2) = \operatorname{dom} H_1 \cap \operatorname{dom} H_2,$$

so that the sum is not necessarily densely defined. In particular, $H_1 + H_2$ and its closure need not be operators. In fact, one sees that

$$\operatorname{mul}(H_1 + H_2) = \operatorname{mul} H_1 + \operatorname{mul} H_2. \tag{3.1}$$

To describe the class of maximal sectorial extension of H_1+H_2 some basic notations are fixed in Section 3.1. The Friedrichs extension and Kreĭn extension of $H_1 + H_2$ and, more generally, all extremal maximal sectorial extensions of $H_1 + H_2$ and their factorizations are then described in Section 3.2 and finally in Section 3.3 the form sum extension of $H_1 + H_2$ and its relation to the extremal maximal sectorial extensions of $H_1 + H_2$ will be investigated.

3.1. **Pairs of maximal sectorial relations.**

According to (1.2) the maximal sectorial relations H_1 and H_2 are decomposed as follows

$$H_j = A_j^{\frac{1}{2}}(I + iB_j)A_j^{\frac{1}{2}}, \quad 1 \leq j \leq 2, \tag{3.2}$$

where A_j (the real part of H_j), $1 \leq j \leq 2$, are nonnegative selfadjoint relations in $\mathfrak{H}$ and B_j, $1 \leq j \leq 2$, are (unique) bounded selfadjoint operators in $\mathfrak{H}$; see (1.1) in Lemma 1.2. Furthermore, if A_1 and A_2 are decomposed as

$$A_j = A_{js} \oplus A_{j\infty}, \quad 1 \leq j \leq 2,$$

where $A_{j\infty} = \{0\} \times \operatorname{mul} A_j$, $1 \leq j \leq 2$, A_{js}, $1 \leq j \leq 2$ are densely defined nonnegative selfadjoint operators (defined as orthogonal complements in the graph sense), then the uniquely determined square roots of A_j, $1 \leq j \leq 2$ are given by

$$A_j^{\frac{1}{2}} = A_{js}^{\frac{1}{2}} \oplus A_{j\infty}, \quad 1 \leq j \leq 2.$$

Associated with H_1 and H_2 is the relation Φ from $\mathfrak{H} \times \mathfrak{H}$ to $\mathfrak{H}$, defined by

$$\Phi = \left\{ \{\{f_1, f_2\}, f_1' + f_2'\} : \{f_j, f_j'\} \in A_j^{\frac{1}{2}}, 1 \leq j \leq 2 \right\}. \tag{3.3}$$

Clearly, Φ is a relation whose domain and multivalued part are given by

$$\operatorname{dom} \Phi = \operatorname{dom} A_1^{\frac{1}{2}} \times \operatorname{dom} A_2^{\frac{1}{2}}, \quad \operatorname{mul} \Phi = \operatorname{mul} H_1 + \operatorname{mul} H_2.$$

The relation Φ is not necessarily densely defined in $\mathfrak{H} \times \mathfrak{H}$, so that in general Φ^* is a relation as $\operatorname{mul} \Phi^* = (\operatorname{dom} \Phi)^{\perp}$. Furthermore, the adjoint Φ^* of Φ is the relation from $\mathfrak{H}$ to $\mathfrak{H} \times \mathfrak{H}$, given by

$$\Phi^* = \left\{ \{h, \{h_1', h_2'\}\} : \{h, h_j'\} \in A_j^{\frac{1}{2}}, 1 \leq j \leq 2 \right\}. \tag{3.4}$$

The identity (3.4) shows that the (orthogonal) operator part $(\Phi^*)_s$ of Φ^* is given by:

$$\begin{aligned} (\Phi^*)_s &= \left\{ \{h, \{h_1', h_2'\}\} : \{h, h_j'\} \in A_{js}^{\frac{1}{2}}, 1 \leq j \leq 2 \right\} \\ &= \left\{ \left\{ h, \{A_{1s}^{\frac{1}{2}} h, A_{2s}^{\frac{1}{2}} h\} \right\} : h \in \operatorname{dom} A_1^{\frac{1}{2}} \cap \operatorname{dom} A_2^{\frac{1}{2}} \right\}. \end{aligned} \tag{3.5}$$

The identities (3.4) and (3.5) show that

$$\operatorname{dom} \Phi^* = \operatorname{dom} A_1^{\frac{1}{2}} \cap \operatorname{dom} A_2^{\frac{1}{2}}, \ \operatorname{ran} (\Phi^*)_s = \mathfrak{F}_0, \ \operatorname{mul} \Phi^* = \operatorname{mul} H_1 \times \operatorname{mul} H_2,$$

where the subspace $\mathfrak{F}_0 \subset \mathfrak{H} \times \mathfrak{H}$ is defined by

$$\mathfrak{F}_0 = \left\{ \left\{ A_{1s}^{\frac{1}{2}} h, A_{2s}^{\frac{1}{2}} h \right\} : h \in \operatorname{dom} A_1^{\frac{1}{2}} \cap \operatorname{dom} A_2^{\frac{1}{2}} \right\}. \tag{3.6}$$

The closure of $\mathfrak{F}_0$ in $\mathfrak{H} \times \mathfrak{H}$ will be denoted by $\mathfrak{F}$. Define the relation Ψ from $\mathfrak{H}$ to $\mathfrak{H} \times \mathfrak{H}$ by

$$\Psi = \left\{ \left\{ h, \left\{ A_{1s}^{\frac{1}{2}} h, A_{2s}^{\frac{1}{2}} h \right\} \right\} : h \in \operatorname{dom} H_1 \cap \operatorname{dom} H_2 \right\} \subset \mathfrak{H} \times (\mathfrak{H} \times \mathfrak{H}). \tag{3.7}$$

It follows from this definition that

$$\operatorname{dom} \Psi = \operatorname{dom} H_1 \cap \operatorname{dom} H_2, \quad \operatorname{ran} \Psi = \mathfrak{E}_0, \quad \operatorname{mul} \Psi = \{0\},$$

where the space $\mathfrak{E}_0 \subset \mathfrak{H} \times \mathfrak{H}$ is defined by

$$\mathfrak{E}_0 = \left\{ \left\{ A_{1s}^{\frac{1}{2}} f, A_{2s}^{\frac{1}{2}} f \right\} : f \in \operatorname{dom} H_1 \cap \operatorname{dom} H_2 \right\}. \tag{3.8}$$

Observe that $\mathfrak{E}_0 \subset \mathfrak{F}_0$. The closure of $\mathfrak{E}_0$ in $\mathfrak{H} \times \mathfrak{H}$ will be denoted by $\mathfrak{E}$. Hence,

$$\mathfrak{E} \subset \mathfrak{F}. \tag{3.9}$$

Comparison of (3.5) and (3.7) shows

$$\Psi \subset (\Phi^*)_s,$$

and thus the operator Ψ is closable. It follows from $\overline{\mathrm{dom}}\,\Psi^* = (\mathrm{mul}\,\Psi^{**})^\perp$ and $\mathrm{mul}\,\Psi^* = (\mathrm{dom}\,\Psi)^\perp$ that

$$\overline{\mathrm{dom}}\,\Psi^* = \mathfrak{H}, \quad \mathrm{mul}\,\Psi^* = (\mathrm{dom}\,H_1 \cap \mathrm{dom}\,H_2)^\perp.$$

Next define the relation K from $\mathfrak{H} \times \mathfrak{H}$ to $\mathfrak{H}$ by

$$\begin{aligned} K \;=\; & \{\{\{(I+iB_1)A_{1s}^{\frac{1}{2}}f, (I+iB_2)A_{2s}^{\frac{1}{2}}f\}, f_1'+f_2'\} : \\ & \{(I+iB_1)A_{1s}^{\frac{1}{2}}f, f_1'\} \in A_1^{\frac{1}{2}}, \{(I+iB_2)A_{2s}^{\frac{1}{2}}f, f_2'\} \in A_2^{\frac{1}{2}}\} \\ & \subset (\mathfrak{H} \times \mathfrak{H}) \times \mathfrak{H}. \end{aligned} \tag{3.10}$$

Clearly, the domain and multivalued part of K are given by

$$\mathrm{dom}\,K = \mathfrak{D}_0, \quad \mathrm{mul}\,K = \mathrm{mul}\,(H_1 + H_2),$$

where

$$\mathfrak{D}_0 = \left\{ \{(I+iB_1)A_{1s}^{1/2}f, (I+iB_2)A_{2s}^{1/2}f\} : \, f \in \mathrm{dom}\,H_1 \cap \mathrm{dom}\,H_2 \right\}. \tag{3.11}$$

The closure of $\mathfrak{D}_0$ in $\mathfrak{H} \times \mathfrak{H}$ will be denoted by $\mathfrak{D}$.

Lemma 3.1. *The relations K, Φ, and Ψ satisfy the following inclusions:*

$$K \subset \Phi \subset \Psi^*, \quad \Psi \subset \Phi^* \subset K^*. \tag{3.12}$$

Proof. To see this note that $K \subset \Phi$ follows from (3.3) and (3.10), and that $\Psi \subset \Phi^*$ follows from (3.4) and (3.7). Therefore, also $\Phi^* \subset K^*$ and $\Phi \subset \Phi^{**} \subset \Psi^*$. □

3.2. **The Friedrichs and the Kreĭn extensions of $H_1 + H_2$.** Let H_1 and H_2 be maximal sectorial relations in a Hilbert space $\mathfrak{H}$. Since (the closure of) the sectorial sum $H_1 + H_2$ has equal defect numbers, (the closure of) the sum $H_1 + H_2$ has maximal sectorial extensions in $\mathfrak{H}$. Two of them, the Friedrichs extension and the Kreĭn extension $(H_1 + H_2)_F$ and $(H_1 + H_2)_K$ as maximal sectorial relations have factorizations as H_1 and H_2 in (3.2). A natural problem is to express such factorizations in terms of the initial relations H_1 and H_2.

Introduce the orthogonal sum of the operators B_1 and B_2 in $\mathfrak{H} \times \mathfrak{H}$ by

$$B_\oplus := B_1 \oplus B_2 = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}.$$

This shorthand notation is used to shorten some of the forthcoming formulas. The aim in the description of $(H_1 + H_2)_F$ and $(H_1 + H_2)_K$ is to keep the presentation as explicit as possible by incorporating the initial data on the factorizations (3.2) of H_1 and H_2 directly via the mappings Φ, Ψ, and K in Subsection 3.1.

Now proceed to the construction of the Friedrichs extension for the sum $H_1 + H_2$.

Theorem 3.2. *Let H_1 and H_2 be maximal sectorial and let Ψ be defined by* (3.7). *The Friedrichs extension of $H_1 + H_2$ is given by*

$$(H_1 + H_2)_F = \Psi^*(I + iB_\oplus)\Psi^{**}$$

and the corresponding form is given by

$$\mathfrak{t}_F[f,g] = ((I + iB_\oplus)\Psi^{**}f, \Psi^{**}g), \quad f, \, g \in \mathrm{dom}\,\mathfrak{t}_F = \mathrm{dom}\,\Psi^{**}.$$

Proof. First it is shown that the relation $\Psi^*(I+iB_\oplus)\Psi^{**}$ extends the relation $S:=H_1+H_2$. Let $\{h,h_1'+h_2'\}\in H_1+H_2$ for some $\{h,h_1'\}\in H_1$ and $\{h,h_2'\}\in H_2$. Thus,

$$\{h,\{A_{1s}^{1/2}h,A_{2s}^{1/2}h\}\in\Psi\subset\Psi^{**},$$

and also

$$\{\{(I+iB_1)A_1^{1/2}h,(I+iB_2)A_2^{1/2}h\},h_1'+h_2'\}\in K\subset\Psi^*,$$

as can be verified directly

$$\langle\{\{(I+iB_1)A_{1s}^{1/2}h,(I+iB_2)A_2^{1/2}h\},h_1'+h_2'\},\{\varphi,\{A_{1s}^{1/2}\varphi,A_{2s}^{1/2}\varphi\}\}\rangle=0,$$

for all $\varphi\in\operatorname{dom}S=\operatorname{dom}H_1\cap\operatorname{dom}H_2$. Therefore $S\subset\Psi^*(I+iB_\oplus)\Psi^{**}$.

Now let $\{f,g\}\in\Psi^*(I+iB_\oplus)\Psi^{**}$, so that $\{f,h\}\in\Psi^{**}$ and $\{(I+iB_\oplus)h,g\}\in\Psi^*$ for some $h\in\mathfrak{H}\times\mathfrak{H}$. Since Ψ^{**} is the closure of Ψ there exists a sequence of elements $f_n\in\operatorname{dom}\Psi=\operatorname{dom}S$ such that

$$f_n\to f,\quad \Psi f_n\to h,\quad \text{as } n\to\infty. \tag{3.13}$$

It follows from $\{f,h\}\in\Psi^{**}$ and $\{(I+iB_\oplus)h,g\}\in\Psi^*$ that

$$(g,f)=(h,h)+i(B_\oplus h,h),$$

which implies that

$$\operatorname{Re}(g,f)=(h,h). \tag{3.14}$$

Similarly it follows from $\{f,h\}\in\Psi^{**}$ and

$$\{\{(I+iB_1)A_{1s}^{1/2}f_n,(I+iB_2)A_{2s}^{1/2}f_n\},(H_{1s}+H_{2s})f_n\}\in\Psi^*$$

that

$$((H_{1s}+H_{2s})f_n,f)=(\{A_{1s}^{1/2}f_n,A_{2s}^{1/2}f_n\},h)+i(\{B_1A_{1s}^{1/2}f_n,B_2A_{2s}^{1/2}f_n\},h). \tag{3.15}$$

Likewise, it follows from $\{f_n,\{A_{1s}^{1/2}f_n,A_{2s}^{1/2}f_n\}\}\in\Psi\subset\Psi^{**}$ and $\{(I+iB_\oplus)h,g\}\in\Psi^*$ that

$$(g,f_n)=(h,\{A_{1s}^{1/2}f_n,A_{2s}^{1/2}f_n\})+i(B_\oplus h,\{A_{1s}^{1/2}f_n,A_{2s}^{1/2}f_n\}). \tag{3.16}$$

A combination of (3.15) and (3.16) leads to

$$\begin{aligned}\operatorname{Re}(((H_{1s}+H_{2s})f_n,f)+(g,f_n)) &= \operatorname{Re}((h,\{A_{1s}^{1/2}f_n,A_{2s}^{1/2}f_n\})\\ &\quad+(\{A_{1s}^{1/2}f_n,A_{2s}^{1/2}f_n\},h)).\end{aligned} \tag{3.17}$$

This leads to the following identity

$$\begin{aligned}\|h-\Psi f_n\|^2 &= \|h\|^2-(h,\{A_{1s}^{1/2}f_n,A_{2s}^{1/2}f_n\})\\ &\quad-(\{A_{1s}^{1/2}f_n,A_{2s}^{1/2}f_n\},h)+\|\{A_{1s}^{1/2}f_n,A_{2s}^{1/2}f_n\}\|^2\\ &= \|h\|^2-\operatorname{Re}(h,\{A_{1s}^{1/2}f_n,A_{2s}^{1/2}f_n\})\\ &\quad-\operatorname{Re}(\{A_{1s}^{1/2}f_n,A_{2s}^{1/2}f_n\},h)+\|\{A_{1s}^{1/2}f_n,A_{2s}^{1/2}f_n\}\|^2\\ &= \operatorname{Re}(g-(H_{1s}+H_{2s})f_n,f-f_n),\end{aligned}$$

where (3.14), and (3.17) have been used, respectively. Therefore (3.13) implies that

$$f_n\to f,\quad \operatorname{Re}(g-(H_{1s}+H_{2s})f_n,f-f_n)\to 0. \tag{3.18}$$

Since $f_n\in\operatorname{dom}S$, it follows from (3.18) and the definition of S_F that $\{f,g\}\in S_F$. Hence, $\Psi^*(I+iB)\Psi^{**}\subset S_F$, and since $\Psi^*(I+iB)\Psi^{**}$ and S_F are both maximal sectorial, the identity $\Psi^*(I+iB)\Psi^{**}=S_F$ follows. The statement concerning the associated closed form $\mathfrak{t}_F$ follows from the first representation theorem and the definition of S_F; cf. [15, Theorem 5.1]. □

Next the construction of the Kreĭn extension for the sum $H_1 + H_2$ is given.

Theorem 3.3. *Let H_1 and H_2 be maximal sectorial and let K be defined by* (3.10). *The Kreĭn extension of $H_1 + H_2$ is given by*

$$(H_1 + H_2)_K = K^{**}(I + iB_\oplus)K^*.$$

If, in addition, $\mathfrak{E} = \operatorname{clos}\mathfrak{E}_0$ and $\mathfrak{D} = \operatorname{clos}\mathfrak{D}_0$ (see (3.8), (3.11)*) satisfy the equality $\mathfrak{E} = \mathfrak{D}$ then the corresponding closed sectorial form is given by*

$$\mathfrak{t}_K[f, g] = ((I + iB_\oplus)(K^*)_s f, (K^*)_s g), \quad f, g \in \operatorname{dom}\mathfrak{t}_K = \operatorname{dom} K^*.$$

Proof. Assume that $\{f, f_1'+f_2'\} \in S = H_1+H_2$, with $\{f, f_1'\} \in H_1$ and $\{f, f_2'\} \in H_2$. This implies that

$$\{\{(I + iB_1)A_{1s}^{1/2} f, (I + iB_2)A_{2s}^{1/2} f\}, f_1' + f_2'\} \in K \subset K^{**}.$$

Moreover,

$$\{f, \{A_{1s}^{1/2} f, A_{2s}^{1/2} f\}\} \in K^*,$$

as can be verified directly

$$\langle\{f, \{A_{1s}^{1/2} f, A_{2s}^{1/2} f\}\}, \{\{(I + iB_1)A_{1s}^{1/2}\varphi, (I + iB_2)A_{2s}^{1/2}\varphi\}, f_1' + f_2'\}\rangle = 0,$$

for all $\varphi \in \operatorname{dom} S = \operatorname{dom} H_1 \cap \operatorname{dom} H_2$. Therefore $S \subset K^{**}(I + iB_\oplus)K^*$.

Now assume that $\{f, g\} \in K^{**}(I + iB_\oplus)K^*$. This means that $\{f, h\} \in K^*$ and $\{(I + iB_\oplus)h, g\} \in K^{**}$ for some $h \in \mathfrak{H} \times \mathfrak{H}$. Since K^{**} is the closure of K there exists a sequence of elements $\{\varphi_n, \varphi_n'\} \in K$ with

$$\{\varphi_n, \varphi_n'\} \to \{(I + iB_\oplus)h, g\} \in K^{**}, \quad \text{as } n \to \infty.$$

Clearly,

$$\varphi_n = \{(I + iB_1)A_{1s}^{1/2} f_n, (I + iB_2)A_{2s}^{1/2} f_n\}, \quad \varphi_n' = f_{n1}' + f_{n2}'$$

for some $\{f_n, f_{n1}'\} \in H_1$ and $\{f_n, f_{n2}'\} \in H_2$. Therefore,

$$\{A_{1s}^{1/2} f_n, A_{2s}^{1/2} f_n\} \to h, \quad f_{n1}' + f_{n2}' \to g, \quad \text{as} \quad n \to \infty. \tag{3.19}$$

It follows from $\{f, h\} \in K^*$ and $\{(I + iB_\oplus)h, g\} \in K^{**}$ that

$$(g, f) = (h, h) + i(B_\oplus h, h),$$

which implies that

$$\operatorname{Re}(g, f) = (h, h). \tag{3.20}$$

On the other hand, $\{f, h\} \in K^*$ and $\{\varphi_n, \varphi_n'\} \in K^{**}$ leads to

$$(f_{n1}' + f_{n2}', f) = (\{A_{1s}^{1/2} f_n, A_{2s}^{1/2} f_n\}, h) + i(B_\oplus\{A_{1s}^{1/2} f_n, A_{2s}^{1/2} f_n\}, h). \tag{3.21}$$

Similarly it follows from $\{f_n, \{A_{1s}^{1/2} f_n, A_{2s}^{1/2} f_n\}\} \in K^*$ and $\{(I + iB_\oplus)h, g\} \in K^{**}$ that

$$(g, f_n) = (h, \{A_{1s}^{1/2} f_n, A_{2s}^{1/2} f_n\}) + i(B_\oplus h, \{A_{1s}^{1/2} f_n, A_{2s}^{1/2} f_n\}). \tag{3.22}$$

Now a combination of (3.21) and (3.22) shows that

$$\begin{aligned} \operatorname{Re}((f_{n1}' + f_{n2}', f) + (g, f_n)) \;=\;& \operatorname{Re}((h, \{A_{1s}^{1/2} f_n, A_{2s}^{1/2} f_n\}) \\ & +(\{A_{1s}^{1/2} f_n, A_{2s}^{1/2} f_n\}, h)). \end{aligned} \tag{3.23}$$

This leads to the following identity

$$\begin{aligned}\|h-\{A_{1s}^{1/2}f_n, A_{2s}^{1/2}f_n\}\|^2 &= \|h\|^2-(h,\{A_1^{1/2}f_n, A_2^{1/2}f_n\})\\ &\quad -(\{A_1^{1/2}f_n, A_2^{1/2}f_n\},h)+\|\{A_1^{1/2}f_n, A_2^{1/2}f_n\}\|^2\\ &= \|h\|^2-\mathrm{Re}\,(h,\{A_1^{1/2}f_n, A_2^{1/2}f_n\})\\ &\quad -\mathrm{Re}\,(\{A_{1s}^{1/2}f_n, A_{2s}^{1/2}f_n\},h)+\|\{A_{1s}^{1/2}f_n, A_{2s}^{1/2}f_n\}\|^2\\ &= \mathrm{Re}\,(g-(f'_{n1}+f'_{n2}), f-f_n),\end{aligned}$$

where (3.20), and (3.23) have been used, respectively. Therefore (3.19) implies that

$$f'_{n1}+f'_{n2}\to g, \quad \mathrm{Re}\,(g-(f'_{n1}+f'_{n2}), f-f_n)\to 0. \tag{3.24}$$

Since $\{f_n, f'_{n1}+f'_{n2}\}\in S$, the relation (3.24) implies that $\{f,g\}\in S_K$. Hence, $K^{**}(I+iB_\oplus)K^*\subset S_K$, and since $K^{**}(I+iB_\oplus)K^*$ and S_K are both maximal sectorial (see Proposition 2.3), the identity $K^{**}(I+iB)K^*=S_K$ follows.

As to the statement concerning the form $\mathfrak{t}_K$ observe that

$$\mathfrak{D}_0=\mathrm{dom}\,K=(I+iB_\oplus)\mathfrak{E}_0; \tag{3.25}$$

see (3.8), (3.11). Therefore, the assumption $\mathfrak{E}=\mathfrak{D}$ implies that $\mathfrak{D}=\overline{\mathrm{dom}}\,K$ is invariant under the selfadjoint operator $B_\oplus$. Then also $\mathrm{mul}\,K^*=\mathfrak{D}^\perp$ is invariant under $B_\oplus$ and hence it follows from [15, Theorem 5.1] that $K^{**}(I+iB_\oplus)K^*=((K^*)_s)^*(I+iB_\oplus)(K^*)_s$ and that the corresponding closed form $\mathfrak{t}_K$ is determined by the operator part $(K^*)_s$ of K^*. □

The product $K^{**}(I+iB_\oplus)K^*$ is a maximal sectorial relation whose multivalued part is given by $\mathrm{mul}\,K^{**}$. Therefore, it follows from Theorem 3.3 that

$$\mathrm{mul}\,(H_1+H_2)_K=\mathrm{mul}\,K^{**}.$$

Recall from [3, Theorem 1] (cf. [15, Theorem 7.6]) that the Kreĭn extension S_K has the largest form domain among all maximal sectorial extensions of a sectorial relation S. In particular, this implies that the relation S is "*sectorially closable*", i.e., S has a maximal sectorial operator extension if and only if the Kreĭn extension S_K is an operator, which in the present case holds for $S=H_1+H_2$ if and only if the relation K is a closable operator or, equivalently, K^* is densely defined.

Likewise, the product $\Psi^*(I+iB_\oplus)\Psi^{**}$ is a maximal sectorial relation whose multivalued part is given by $\mathrm{mul}\,\Psi^*=(\mathrm{dom}\,\Psi)^\perp$, so that it follows from Theorem 3.2 that

$$\mathrm{mul}\,(H_1+H_2)_F=(\mathrm{dom}\,H_1\cap\mathrm{dom}\,H_2)^\perp.$$

Hence, when H_1+H_2 is densely defined, then H_1+H_2 is automatically an operator and all maximal sectorial extensions are operators. The orthogonal operator part of $\Psi^*(I+iB_\oplus)\Psi^{**}$ is the maximal sectorial operator corresponding to the closed form

$$((I+iB_\oplus)\Psi^{**}h,\Psi^{**}k), \quad h,k\in\mathrm{dom}\,\Psi^{**}.$$

The description of the closed sectorial form $\mathfrak{t}_K$ associated with the Kreĭn extension $(H_1+H_2)_K$ in Theorem 3.3 is stated under the additional condition $\mathfrak{E}=\mathfrak{D}$. When this condition fails to hold the description of the form $\mathfrak{t}_K$ becomes more involved and will be treated elsewhere; see [17]. The form $\mathfrak{t}_K$ can be used to give a complete description of all *extremal maximal sectorial extensions* of the sum H_1+H_2. Namely, a maximal sectorial extension H of a sectorial relation S is extremal precisely when the corresponding closed sectorial form $\mathfrak{t}_H$ is a restriction of the closed sectorial form $\mathfrak{t}_K$ generated by the Kreĭn extension S_K of S; see e.g. [15, Definition 7.7, Theorems 8.4, 8.5]. Therefore, Theorem 3.3 implies the following description of all extremal maximal sectorial extensions of $H_1 + H_2$.

Proposition 3.4. *Let H_1 and H_2 be maximal sectorial relations in a Hilbert space $\mathfrak{H}$ and assume that $\mathfrak{E} = \operatorname{clos}\mathfrak{E}_0$ and $\mathfrak{D} = \operatorname{clos}\mathfrak{D}_0$ (see* (3.8), (3.11)*) satisfy the equality $\mathfrak{E} = \mathfrak{D}$. Then the following statements are equivalent:*

(i) *$\widetilde{H}$ is an extremal maximal sectorial extension of $H_1 + H_2$;*

(ii) *$\widetilde{H} = T_{\mathfrak{L}}^*(I + iB_\oplus)T_{\mathfrak{L}}^{**}$, where $T_{\mathfrak{L}}$ is the restriction of the operator part $(K^*)_s$ to a linear subspace $\mathfrak{L}$ satisfying*

$$\operatorname{dom}(H_1 + H_2) \subset \mathfrak{L} \subset \operatorname{dom} K^*.$$

3.3. **The form sum construction.** The maximal sectorial relations H_1 and H_2 generate the following closed sectorial form

$$((I + iB_1)A_{1s}^{\frac{1}{2}}h, A_{1s}^{\frac{1}{2}}k) + ((I + iB_2)A_{2s}^{\frac{1}{2}}h, A_{2s}^{\frac{1}{2}}k), \quad h, k \in \operatorname{dom} A_1^{\frac{1}{2}} \cap \operatorname{dom} A_2^{\frac{1}{2}}. \tag{3.26}$$

Observe that the restriction of this form to $\operatorname{dom}\Psi^{**}$ is equal to

$$(\Psi^{**}h, \Psi^{**}k) = ((I + iB_1)A_{1s}^{\frac{1}{2}}h, A_{1s}^{\frac{1}{2}}k) + ((I + iB_2)A_{2s}^{\frac{1}{2}}h, A_{2s}^{\frac{1}{2}}k), \quad h, k \in \operatorname{dom}\Psi^{**},$$

since $\Psi^{**} \subset (\Phi^*)_s$, cf. (3.5). Thus, the form in (3.26) has a natural domain which is in general larger than $\operatorname{dom}\Psi^{**}$.

Theorem 3.5. *Let H_1 and H_2 be maximal sectorial and let Φ be defined by* (3.3)*. The maximal sectorial relation*

$$\Phi^{**}(I + iB_\oplus)\Phi^*$$

is an extension of the relation $H_1 + H_2$, which corresponds to the closed sectorial form in (3.26)*.*

Assume, in addition, that $\mathfrak{E} = \operatorname{clos}\mathfrak{E}_0$ and $\mathfrak{D} = \operatorname{clos}\mathfrak{D}_0$ (see (3.8), (3.11)*) satisfy the equality $\mathfrak{E} = \mathfrak{D}$ and let $\mathfrak{F} = \operatorname{clos}\mathfrak{F}_0$ be defined by* (3.6)*. Then the following statements are equivalent:*

(i) *$\Phi^{**}(I + iB_\oplus)\Phi^*$ is extremal;*

(ii) *$\mathfrak{E} = \mathfrak{F}$.*

Proof. By [15, Theorem 5.1] the form sum (3.26) can be written as

$$((I + iB_\oplus)(\Phi^*)_s\, h, (\Phi^*)_s\, k), \quad h, k \in \operatorname{dom}(\Phi^*)_s = \operatorname{dom} A_1^{\frac{1}{2}} \cap \operatorname{dom} A_2^{\frac{1}{2}},$$

so that $\Phi^{**}(I + iB_\oplus)\Phi^*$ is the maximal sectorial relation in $\mathfrak{H}$ which corresponds to (3.26) via the first representation theorem, since $\operatorname{mul}\Phi^* = \operatorname{mul} H_1 \times \operatorname{mul} H_2$ is clearly invariant under $B_\oplus$, when B_1 and B_2 are the unique operators as described in Lemma 1.2.

To show that $\Phi^{**}(I + iB_\oplus)\Phi^*$ extends $H_1 + H_2$, let $\{h, h_1' + h_2'\} \in (H_1 + H_2)$ for some $\{h, h_1'\} \in H_1$ and $\{h, h_2'\} \in H_2$, so that $h \in \operatorname{dom} H_1 \cap \operatorname{dom} H_2$. Clearly, $\{h, \{A_{1s}^{\frac{1}{2}}h, A_{2s}^{\frac{1}{2}}h\}\} \in \Phi^*$. Moreover,

$$\{\{(I + iB_1)A_{1s}^{\frac{1}{2}}h, (I + iB_2)A_{2s}^{\frac{1}{2}}h\}, h_1' + h_2'\} \in \Phi^{**},$$

as can be verified directly:

$$(h_1' + h_2', \varphi) - (\{(I + iB_1)A_{1s}^{\frac{1}{2}}h, (I + iB_2)A_{2s}^{\frac{1}{2}}h\}, \{A_{1s}^{\frac{1}{2}}\varphi, A_{2s}^{\frac{1}{2}}\varphi\}) = 0$$

for all $\varphi \in \operatorname{dom} H_1 \cap \operatorname{dom} H_2$. Therefore $\{h, h_1'+h_2'\} \in \Phi^{**}(I+iB_\oplus)\Phi^*$. Hence $H_1+H_2 \subset \Phi^{**}(I+iB_\oplus)\Phi^*$. This proves the first statement.

Now the equivalence in the second statement will be proved.

(i) $\Rightarrow$ (ii) Since $\mathfrak{E} \subset \mathfrak{F}$ by (3.9) it is enough to prove the inclusion $\mathfrak{F} \subset \mathfrak{E}$. Assume that the form sum extension of H_1+H_2 is extremal. Then by Proposition 3.4 there exists a subspace $\mathfrak{L}$ such that

$$((\Phi^*)_s)^*(I+iB_\oplus)(\Phi^*)_s = \Phi^{**}(I+iB_\oplus)\Phi^* = T_{\mathfrak{L}}^*(I+iB_\oplus)T_{\mathfrak{L}}^{**}. \tag{3.27}$$

Let $P_{\mathfrak{E}}$ be the orthogonal projection of $\mathfrak{H}\times\mathfrak{H}$ onto $\mathfrak{E}$. By (3.12) $(\Phi^*)_s \subset \Phi^* \subset K^*$ and therefore $P_{\mathfrak{E}}(\Phi^*)_s \subset P_{\mathfrak{E}}K^* = (K^*)_s$, since by assumption $\mathfrak{E} = \mathfrak{D} = \operatorname{d\overline{om}} K$. Moreover, $\operatorname{dom} P_{\mathfrak{E}}(\Phi^*)_s = \operatorname{dom}(\Phi^*)_s = \operatorname{dom} T_{\mathfrak{L}}^{**}$ and since $P_{\mathfrak{E}}(\Phi^*)_s$ and $T_{\mathfrak{L}}^{**}$ are restrictions of the operator $(K^*)_s$ it follows that

$$P_{\mathfrak{E}}(\Phi^*)_s = T_{\mathfrak{L}}^{**}, \quad ((\Phi^*)_s)^* P_{\mathfrak{E}} = T_{\mathfrak{L}}^*.$$

The assumption $\mathfrak{E} = \mathfrak{D}$ also implies that $\mathfrak{E} = \operatorname{d\overline{om}} K$ is invariant under $B_\oplus$; see (3.25). Now one obtains from (3.27) the equalities

$$\begin{aligned} ((\Phi^*)_s)^*(I+iB_\oplus)(\Phi^*)_s &= T_{\mathfrak{L}}^*(I+iB_\oplus)T_{\mathfrak{L}}^{**} \\ &= ((\Phi^*)_s)^* P_{\mathfrak{E}}(I+iB_\oplus)T_{\mathfrak{L}}^{**} \\ &= ((\Phi^*)_s)^*(I+iB_\oplus)P_{\mathfrak{E}}T_{\mathfrak{L}}^{**} \\ &= ((\Phi^*)_s)^*(I+iB_\oplus)T_{\mathfrak{L}}^{**}. \end{aligned}$$

Hence for every $f \in \operatorname{dom}((\Phi^*)_s)^*(I+iB_\oplus)(\Phi^*)_s$ one has

$$(I+iB_\oplus)((\Phi^*)_s f - T_{\mathfrak{L}}^{**} f) \in \ker\, ((\Phi^*)_s)^*.$$

Since $(\Phi^*)_s f - T_{\mathfrak{L}}^{**} f \in \mathfrak{F} = \overline{\operatorname{ran}}\,(\Phi^*)_s = (\ker\,((\Phi^*)_s)^*)^\perp$, see (3.5), (3.6), this implies that

$$((I+iB_\oplus)((\Phi^*)_s f - T_{\mathfrak{L}}^{**} f), (\Phi^*)_s f - T_{\mathfrak{L}}^{**} f) = 0$$

and thus $(\Phi^*)_s f - T_{\mathfrak{L}}^{**} f = 0$. Therefore $(\Phi^*)_s(\operatorname{dom}((\Phi^*)_s)^*(I+iB_\oplus)(\Phi^*)_s) \subset \operatorname{ran} T_{\mathfrak{L}}^{**} \subset \mathfrak{E}$. Since $\operatorname{dom}((\Phi^*)_s)^*(I+iB_\oplus)(\Phi^*)_s$ is a core for the corresponding closed form, or equivalently, the closure of $(\Phi^*)_s \restriction \operatorname{dom}((\Phi^*)_s)^*(I+iB_\oplus)(\Phi^*)_s$ is equal to $(\Phi^*)_s$, the claim follows: $\mathfrak{F} = \overline{\operatorname{ran}}\,(\Phi^*)_s \subset \mathfrak{E}$.

(ii) $\Rightarrow$ (i) Assume that $\mathfrak{E} = \mathfrak{F}$. Then $\mathfrak{F}_0 = \operatorname{ran}(\Phi^*)_s \subset \mathfrak{E}$ and the equalities $\mathfrak{E} = \mathfrak{D} = \operatorname{d\overline{om}} K$ combined with $(\Phi^*)_s \subset \Phi^* \subset K^*$ imply that $(\Phi^*)_s \subset P_{\mathfrak{E}}K^* = (K^*)_s$. Therefore $(\Phi^*)_s = T_{\mathfrak{L}}$ with the choice

$$\mathfrak{L} = \operatorname{dom}(\Phi^*)_s = \operatorname{dom} A_1^{\frac12} \cap \operatorname{dom} A_2^{\frac12}.$$

Hence,

$$\Phi^{**}(I+iB_\oplus)\Phi^* = ((\Phi^*)_s)^*(I+iB_\oplus)(\Phi^*)_s = T_{\mathfrak{L}}^*(I+iB_\oplus)T_{\mathfrak{L}}^{**},$$

which shows that $\Phi^{**}(I+iB_\oplus)\Phi^*$ is extremal, cf. Proposition 3.4. □

The maximal sectorial relation $\Phi^{**}(I+iB_\oplus)\Phi^*$ naturally extends the factorized sectorial relation $\Phi(I+iB_\oplus)\Phi^* = H_1+H_2$ and, as indicated in Section 1 it is called the *form sum extension of the sectorial relation* H_1+H_2 (induced by the form (3.26)). Its multivalued part is given by $\operatorname{mul}\Phi^{**}(I+iB_\oplus)\Phi^* = \operatorname{mul}\Phi^{**} = (\operatorname{dom}\Phi^*)^\perp$, so that

$$\operatorname{mul}\Phi^{**}(I+iB_\oplus)\Phi^* = (\operatorname{dom} A_1^{\frac12} \cap \operatorname{dom} A_2^{\frac12})^\perp.$$

In particular, the form sum extension of H_1+H_2 or, equivalently, the closure of Φ, is an operator precisely when $\operatorname{dom} A_1^{\frac{1}{2}} \cap \operatorname{dom} A_2^{\frac{1}{2}}$ is dense in $\mathfrak{H}$. The orthogonal operator part of $\Phi^{**}(I+iB_{\oplus})\Phi^*$ is the maximal sectorial operator which corresponds to the form sum (3.26) restricted to the closure of $\operatorname{dom} H_1^{\frac{1}{2}} \cap \operatorname{dom} H_2^{\frac{1}{2}}$. As a comparison with $(H_1+H_2)_K$ recall that $\Phi^* \subset K^*$ by Lemma 3.1 and that H_1+H_2 is "sectorially closable" if and only if $(H_1+H_2)_K$ is an operator, or, equivalently, K^* is densely defined (see Section 3.2). In particular, if the form sum is densely defined then also $(H_1+H_2)_K$ is a densely defined operator.

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