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SQUARE-INTEGRABLE SOLUTIONS AND WEYL FUNCTIONS FOR SINGULAR CANONICAL SYSTEMS

JUSSI BEHRNDT, SEPPO HASSI, HENK DE SNOO, AND RUDI WIETSMA

Abstract. Boundary value problems for a singular canonical system of differential equations $Jf'(t) - H(t)f(t) = \lambda\Delta(t)f(t)$, $t \in \imath$ and $\lambda \in \mathbb{C}$, are studied in the associated Hilbert space $L^2_\Delta(\imath)$. With the help of a general monotonicity principle for nondecreasing matrix functions the square-integrable solutions are specified. This yields a direct treatment of defect numbers of the minimal relation and simultaneously makes it possible to assign certain boundary values to the elements of the maximal relation induced by the system of differential equations in $L^2_\Delta(\imath)$. The investigation of boundary value problems for these systems and their spectral theory can be carried out by means of abstract boundary triplet techniques. The paper makes explicit the construction and properties of boundary triplets and Weyl functions for singular canonical system. Furthermore, the Weyl functions are shown to have a property similar to that of the classical Titchmarsh-Weyl coefficients for singular Sturm-Liouville operators: they single out the square-integrable solutions of the corresponding homogeneous system of canonical differential equations.

1. Introduction

One of the central objects in the theory of singular Sturm-Liouville differential expressions is the Titchmarsh-Weyl function m introduced in the classical works E.C. Titchmarsh [61, 62] and H. Weyl [63]. If $\varphi(\cdot,\lambda)$ and $\psi(\cdot,\lambda)$, $\lambda \in \mathbb{C}$, form a fundamental system of solutions of the differential equation

$$-(pu')' + qu = \lambda ru, \qquad 1/p, q, r \in L^1_{\text{loc}}(0,\infty) \text{ real}, \quad r \geq 0, \tag{1.1}$$

and the differential expression is regular at the left endpoint 0 and in the limit-point case at the singular endpoint $+\infty$, then the Titchmarsh-Weyl function $m : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ has the property that

$$\varphi(\cdot,\lambda) + m(\lambda)\psi(\cdot,\lambda) \in L^2_r(0,\infty) \tag{1.2}$$

for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Here $L^2_r(0,\infty)$ denotes the weighted L^2-space consisting of (equivalence classes of) complex valued measurable functions f on $(0,\infty)$ such that $|f|^2 r \in L^1(0,\infty)$. Roughly speaking (1.2) states that the function m singles out the square-integrable solutions of (1.1). This fact has direct consequences for the differential operators associated with the differential expression (1.1) in $L^2_r(0,\infty)$: the minimal operator has deficiency indices $(1,1)$ and the defect elements are given by (1.2). There are many other connections between the Titchmarsh-Weyl function m and the corresponding Sturm-Liouville differential operators. Probably the most important fact is that the spectral properties of all selfadjoint realizations are completely encoded in m and its behaviour close to the singularities on the real line.

The present paper is devoted to the study of more general systems of ordinary differential equations the so-called *canonical systems* of differential equations. These systems are of the form

$$Jf'(t) - H(t)f(t) = \lambda\Delta(t)f(t), \qquad \lambda \in \mathbb{C}, \tag{1.3}$$

where J is a skewadjoint and unitary $n \times n$ matrix, and H and Δ are locally integrable $n \times n$ matrix functions defined on an open interval $\imath = (a,b)$ such that $H(t)$ is selfadjoint and $\Delta(t) \geq 0$. The fundamental matrix $Y(\cdot,\lambda)$ of the canonical system (1.3) consists of n linearly independent solutions which are locally absolutely continuous $n \times 1$ vector functions on $\imath$. For each $\lambda \in \mathbb{C}^+$ or $\lambda \in \mathbb{C}^-$ the $n \times n$ matrix function

$$D(\cdot, \lambda) = Y(\cdot, \lambda)^*(-iJ)Y(t, \lambda) \tag{1.4}$$

is monotonically nondecreasing or nonincreasing, respectively, on $\imath$. According to a general monotonicity principle the limits $D(a, \lambda)$ and $D(b, \lambda)$ when t tends to a and b exist as selfadjoint relations (multivalued operators) in $\mathbb{C}^n$; cf. [3, 4]. The spectra of these selfadjoint relations consist of n eigenvalues on the extended real line. One of the main ingredients for the theory developed in the present paper is the fact that the eigenspaces of $D(a, \lambda)$ and $D(b, \lambda)$ are intimately connected with the square-integrable solutions of (1.3). Here square-integrability of a vector function f means that $\int_\imath f(s)^*\Delta(s)f(s)\, ds$ is finite, that is, f belongs to the Hilbert space $L^2_\Delta(\imath)$. If the Sturm-Liouville problem (1.1) is rewritten as a canonical system, then the function (1.4) is a 2×2 matrix function and Weyl's limit-point and limit-circle classification of a singular endpoint b reduces to the question whether the limit $D(b, \lambda)$ is a selfadjoint relation with one-dimensional multivalued part or whether it is an ordinary 2×2 matrix, respectively; cf. Examples 2.12 and 4.22.

Similarly as in Sturm-Liouville theory one associates minimal and maximal operators or, more precisely, minimal and maximal relations to the canonical system in the Hilbert space $L^2_\Delta(\imath)$. The maximal relation $T_{\max}$ is the adjoint of the closed symmetric minimal relation $T_{\min}$. The minimal relation is not necessarily densely defined; both $T_{\min}$ and $T_{\max}$ are in general multivalued. The number of square-integrable solutions in the upper- and lower-halfplane coincide with the defect numbers of the minimal relation. In this sense the extension theory of symmetric relations is the natural framework for boundary value problems involving canonical systems of differential equations. For this purpose the abstract concept of boundary triplets and their Weyl functions from [15, 16] is used. With the help of a boundary triplet all selfadjoint extensions of the underlying symmetric operator or relation can be parameterized efficiently and their spectral properties can be described with the help of the associated Weyl function.

The main aim of the paper is to study the square-integrable solutions of canonical systems and to define a matrix valued analog M of the Titchmarsh-Weyl coefficient from singular Sturm-Liouville theory. It will be shown that this function singles out the square-integrable fundamental solutions in the sense that in analogy to (1.2) formulas of the type

$$\gamma(\lambda)\eta = Y(\cdot, \lambda) \begin{pmatrix} \eta \\ M(\lambda)\eta \end{pmatrix}, \qquad \eta \in \mathbb{C}^m, \;\; \lambda \in \mathbb{C} \setminus \mathbb{R},$$

hold, where $\gamma(\lambda)$ is a map from $\mathbb{C}^m$ into the defect subspace $\ker\,(T_{\max} - \lambda)$. By a decomposition of elements of the maximal relation which separates the behaviour at one endpoint of $\imath$ from the behaviour at the other endpoint, boundary values will be assigned to elements in the maximal relation. These boundary values will be used to obtain boundary triplets for the maximal relation. It will be shown that the Weyl function corresponding to such a boundary triplet singles out precisely the square-integrable solutions of the canonical system; cf. Section 5 and 6.

The study of square-integrable solutions of canonical systems of differential equations or of related (systems of) differential equations has a long history. In general two points of view have been developed: the function-theoretic point of view and the functional-analytic point of view. The functional-analytic approach was for a long time restricted to Hilbert space operators which are densely defined; the introduction of linear relations (multivalued operators) meant that this restriction need no longer be imposed. The approach to general canonical systems via the extension theory of linear relations goes back to B.C. Orcutt [49] and I.S. Kac [33, 34]; it was rediscovered in [42]; see also [10, 17, 18, 24, 41]. The treatment of the square-integrable solutions via the general monotonicity principle in [3, 4] was inspired by the work of F.V. Atkinson [2] and of H.-D. Niessen and A. Schneider [46, 56]. Incidentally, the general monotonicity principle itself depends very much on the framework of linear relations. The application of the general monotonicity principle makes it possible to obtain easily some results going back to S.A. Orlov [50]. The connection between the Titchmarsh-Weyl coefficient and the square-integrable solutions was investigated by D.B. Hinton and A. Schneider [27, 28] in a special case. In the present paper it is shown that the theory of boundary triplets, including its recent extension to the case of not necessarily equal defect numbers, provides the functional-analytic framework to connect square-integrability with Weyl functions (or Titchmarsh-Weyl coefficients).

The class of canonical systems of differential equations contains large classes of linear ordinary differential equations studied in the literature. There has been an extension of canonical systems to so-called S-hermitian systems, but H. Langer and R. Mennicken [40] have shown how S-hermitian systems can be reduced to canonical systems. The class of S-hermitian systems was studied extensively by A. Schneider [56, 57, 58, 59, 60], and by H.-D. Niessen [46, 47, 48]; see also [53, 54, 55]. A function-theoretic approach to canonical systems can be found in the works of D.B. Hinton and J.K. Shaw [29, 30, 31], V.I. Kogan and F.S. Rofe-Beketov [38], A.M. Krall [39], H. Langer and R. Mennicken [40], and S.A. Orlov [50]. Schneider [57] has shown how large classes of differential expressions can be written in terms of canonical and S-hermitian systems (see also [49]); this includes ordinary differential operators [8, 9, 35, 37] and pairs of ordinary differential operators [5, 12, 13, 51].

The contents of the paper are now outlined. In Section 2 a number of elementary results concerning canonical systems is reviewed. Proofs are included for completeness. The square-integrable solutions of the canonical system are considered in Section 3. The main ideas here are a general monotonicity principle (cf. [3, 4]) and a construction of square-integrable solutions of the corresponding inhomogeneous canonical system (cf. [46]). In Section 4 the maximal and minimal relations associated to the canonical system are constructed in the sense of Orcutt and a decomposition of the maximal relation is proved in terms of solutions which are square-integrable near the endpoints (cf. [27]). Furthermore, special forms of the minimal and maximal relation are obtained in the case that the endpoints of the interval are quasiregular or in the limit-point case. Boundary triplets and Weyl functions in the general case of equal defect numbers are considered in Section 5; special attention is paid to the limit-point and quasiregular case. Section 6 contains the treatment of boundary triplets and Weyl functions for the case of unequal defect numbers. Finally, the appendix contains a very brief introduction to linear relations in Hilbert spaces making the paper self-contained.

2. Preliminaries concerning canonical systems

This section provides a short introduction into the theory of canonical systems of differential equations. Besides some elementary statements on the properties of solutions also the notions of a singular, a quasiregular and a regular endpoint are explained, the concept of definiteness of canonical systems is briefly reviewed and a cut-off technique for solutions is provided. For a more detailed treatment of canonical systems the reader is referred to, e.g., the monograph [2].

2.1. **Notations.** Let $\imath = (a, b) \subset \mathbb{R}$ be an open interval and let $n, m \in \mathbb{N}$. The linear space $\mathcal{L}^1_{\rm loc}(\imath)$ of locally integrable $n \times m$ matrix functions on $\imath$ consists of all measurable $n \times m$ matrix functions F defined almost everywhere on $\imath$ such that for each compact subinterval $I \subset \imath$

$$\int_I |F(s)|\, ds < \infty.$$

Here $|F(s)|$ denotes the norm of $F(s)$ in $\mathbb{C}^{n\times m}$. A function $F \in \mathcal{L}^1_{\rm loc}(\imath)$ is said to be *integrable at the left endpoint* a or *integrable at the right endpoint* b if for some $c \in \imath$

$$\int_a^c |F(s)|\, ds \ < \infty \quad \text{or} \quad \int_c^b |F(s)|\, ds \ < \infty,$$

respectively. In the notation of the function spaces the sizes n and m are suppressed; for instance, the space of locally integrable functions on $\imath$ with values in $\mathbb{C}^n$ will be denoted by $\mathcal{L}^1_{\rm loc}(\imath)$. The space of locally absolutely continuous functions on $\imath$ with values in $\mathbb{C}^n$ is denoted by $AC_{\rm loc}(\imath)$. It is well known (see, e.g., [26]) that a vector function f belongs to $AC_{\rm loc}(\imath)$ if and only if there exists a vector function $h \in \mathcal{L}^1_{\rm loc}(\imath)$ such that for some $c \in \imath$

$$f(t) = \int_c^t h(s)\, ds, \quad t \in \imath.$$

The derivative $h \in \mathcal{L}^1_{\rm loc}(\imath)$ of $f \in AC_{\rm loc}(\imath)$ will be denoted by f'.

Let $\Delta \in \mathcal{L}^1_{\rm loc}(\imath)$ be an $n \times n$ matrix function such that $\Delta(t) \geq 0$ for almost every $t \in \imath$ and let $\mathcal{L}^2_\Delta(\imath)$ denote the linear space of all measurable functions f with values in $\mathbb{C}^n$ which are *square-integrable (with respect to* Δ*)*, that is, $\int_\imath f(s)^*\Delta(s)f(s)\, ds < \infty$. Here and in the following $\psi^*\phi$ denotes the inner product of $\phi, \psi \in \mathbb{C}^n$. Note that

$$(f, g)_\Delta = \int_\imath g(s)^*\Delta(s)f(s)\, ds, \quad f, g \in \mathcal{L}^2_\Delta(\imath),$$

defines a semidefinite inner product on $\mathcal{L}^2_\Delta(\imath)$. The corresponding seminorm will be denoted by $\|\cdot\|_\Delta$. Observe that the identity $\int_\imath f(s)^*\Delta(s)f(s)\, ds = 0$ is equivalent to $\Delta(t)f(t) = 0$ for almost every $t \in \imath$.

The space $\mathcal{L}^2_{\Delta,\rm loc}(\imath)$ consists of all functions that are square-integrable (with respect to Δ) for each compact subinterval $I \subset \imath$, i.e., $\int_I f(s)^*\Delta(s)f(s)\, ds < \infty$. Note that if $f \in \mathcal{L}^2_\Delta(\imath)$, then $\Delta f \in \mathcal{L}^1_{\rm loc}(\imath)$ as follows from the Cauchy-Schwarz inequality and $\Delta \in \mathcal{L}^1_{\rm loc}(\imath)$. A function $f \in \mathcal{L}^2_{\Delta,\rm loc}(\imath)$ is said to be *square-integrable (with respect to* Δ*) at the left endpoint* a or *square-integrable (with respect to* Δ*) at the right endpoint* b if for some $c \in \imath$

$$\int_a^c f(s)^*\Delta(s)f(s)\, ds \ < \infty \quad \text{or} \quad \int_c^b f(s)^*\Delta(s)f(s)\, ds \ < \infty,$$

respectively. A function $f \in \mathcal{L}^2_{\Delta,\rm loc}(\imath)$ belongs to $\mathcal{L}^2_\Delta(\imath)$ if and only if f is square-integrable (with respect to Δ) at both endpoints of $\imath$.

The space $\mathcal{L}^2_\Delta(\imath)$ has the following *approximation property*: each element of the seminormed space $\mathcal{L}^2_\Delta(\imath)$ can be approximated by square-integrable functions with compact support. To see this, let I_m, $m \in \mathbb{N}$, be a sequence of monotonously increasing compact intervals such that $\imath = \cup_{m=1}^\infty I_m$. For $f \in \mathcal{L}^2_\Delta(\imath)$ put $f_m(t) = f(t)$ for $t \in I_m$ and $f(t) = 0$ elsewhere. Then $f_m \in \mathcal{L}^2_\Delta(\imath)$, f_m has support in I_m, and

$$\|f - f_m\|^2_\Delta = \int_\imath (f(s) - f_m(s))^*\Delta(s)(f(s) - f_m(s))\, ds \to 0, \quad m \to \infty, \tag{2.1}$$

as follows from the monotone convergence theorem.

2.2. **Canonical systems of differential equations.** Let $\imath = (a, b) \subset \mathbb{R}$ be an open, not necessarily bounded, interval and let $n \in \mathbb{N}$. Let H and Δ be $n \times n$ matrix functions defined almost everywhere on $\imath$ such that

$$H, \Delta \in \mathcal{L}^1_{\rm loc}(\imath), \quad H(t) = H(t)^*, \quad \text{and} \quad \Delta(t) \geq 0, \tag{2.2}$$

for almost every $t \in \imath$. Furthermore, let J be an $n \times n$ matrix which satisfies

$$J^* = J^{-1} = -J. \tag{2.3}$$

Since the $n \times n$ matrix J is skewadjoint and unitary, the $n \times n$ matrix $-iJ$ is selfadjoint and unitary, and hence 1 and -1 are the only possible eigenvalues of $-iJ$. In the following the multiplicity of the eigenvalues 1 and -1 of $-iJ$ will be denoted by i^+ and i^-, respectively, so that $n = \mathsf{i}^+ + \mathsf{i}^-$.

An (inhomogeneous) *canonical system* of order n is a system of (inhomogeneous) differential equations of the form

$$Jf'(t) - H(t)f(t) = \lambda\Delta(t)f(t) + \Delta(t)g(t), \quad t \in \imath, \quad \lambda \in \mathbb{C}, \tag{2.4}$$

where g is a locally square-integrable function with values in $\mathbb{C}^n$. A function f with values in $\mathbb{C}^n$ is said to be a *solution* of (the inhomogeneous canonical system) (2.4) if f belongs to $AC_{\rm loc}(\imath)$ and the equation (2.4) holds for almost every $t \in \imath$. Observe that f is a solution of

(2.4), then f is also a solution of (2.4) where $g \in \mathcal{L}^2_{\Delta,\mathrm{loc}}(\imath)$ is replaced by $\widetilde{g} \in \mathcal{L}^2_{\Delta,\mathrm{loc}}(\imath)$ with $\Delta(g - \widetilde{g}) = 0$.

Lemma 2.1. *Assume that* $\lambda, \mu \in \mathbb{C}$ *and that* $g, k \in \mathcal{L}^2_{\Delta,\mathrm{loc}}(\imath)$. *Let* $f, h \in AC_{\mathrm{loc}}(\imath)$ *be solutions of the inhomogeneous equations*

$$Jf'(t) - H(t)f(t) = \lambda\Delta(t)f(t) + \Delta(t)g(t)$$

and

$$Jh'(t) - H(t)h(t) = \mu\Delta(t)h(t) + \Delta(t)k(t),$$

respectively. Then for every compact interval $[\alpha, \beta] \subset \imath$*:*

$$\begin{aligned} h(\beta)^*Jf(\beta) - h(\alpha)^*Jf(\alpha) - \int_\alpha^\beta \big(h(s)^*\Delta(s)g(s) - k(s)^*\Delta(s)f(s)\big)\, ds \\ = (\lambda - \bar{\mu}) \int_\alpha^\beta h(s)^*\Delta(s)f(s)ds. \end{aligned}$$

Proof. The assumptions that J is skewadjoint and that $H(t)$ and $\Delta(t)$ are selfadjoint almost everywhere on $\imath$ lead to the identity

$$\begin{aligned} (h^*Jf)' &= h^*(Jf') - (Jh')^*f \\ &= h^*(\lambda\Delta f + \Delta g + Hf) - (\mu\Delta h + \Delta k + Hh)^*f \\ &= h^*\Delta g - k^*\Delta f + (\lambda - \bar{\mu})h^*\Delta f, \end{aligned}$$

which is valid almost everywhere on $\imath$. Integration over the interval $[\alpha, \beta]$ completes the argument. $\square$

For $\lambda = \bar{\mu}$ the formula in Lemma 2.1 reduces to Lagrange's (or Green's) formula:

$$h(\beta)^*Jf(\beta) - h(\alpha)^*Jf(\alpha) = \int_\alpha^\beta \big(h(s)^*\Delta(s)g(s) - k(s)^*\Delta(s)f(s)\big)\, ds.$$

The *homogeneous canonical system* of order n

$$Jf'(t) - H(t)f(t) = \lambda\Delta(t)f(t), \quad t \in \imath, \quad \lambda \in \mathbb{C}, \tag{2.5}$$

has n linearly independent solutions $f \in AC_{\mathrm{loc}}(\imath)$ for every fixed $\lambda \in \mathbb{C}$. A *fundamental matrix* of the canonical system (2.4) is an $n \times n$ matrix function $Y(\cdot, \lambda)$ whose columns are formed by the linearly independent solutions of the homogeneous equation (2.5) and which is fixed by the initial condition

$$Y(c_0, \lambda) = I_n \tag{2.6}$$

for some $c_0 \in \imath$. If for each $\lambda \in \mathbb{C}$ the same initial point $c_0 \in \imath$ is used in (2.6), then the function $\lambda \mapsto Y(t, \lambda)$ is entire for each $t \in \imath$. The following result is a homogeneous version of Lemma 2.1.

Corollary 2.2. *Let* $Y(\cdot, \lambda)$ *be a fundamental matrix of the canonical system* (2.4)*. Then for every compact interval* $[\alpha, \beta] \subset \imath$ *and all* $\lambda, \mu \in \mathbb{C}$*:*

$$Y(\beta, \mu)^*JY(\beta, \lambda) - Y(\alpha, \mu)^*JY(\alpha, \lambda) = (\lambda - \bar{\mu}) \int_\alpha^\beta Y(s, \mu)^*\Delta(s)Y(s, \lambda)ds.$$

Consequently, any fundamental matrix $Y(\cdot, \lambda)$ satisfies

$$Y(t, \bar{\lambda})^*JY(t, \lambda) = J = Y(t, \lambda)JY(t, \bar{\lambda})^*, \quad t \in \imath, \tag{2.7}$$

so that $Y(t, \lambda)$ is invertible for all $t \in \imath$ and

$$Y(t, \lambda)^{-1} = -JY(t, \bar{\lambda})^*J, \quad Y(t, \bar{\lambda})^{-*} = -JY(t, \lambda)J, \quad t \in \imath. \tag{2.8}$$

Remark 2.3. Observe that the canonical system (2.4) depends on the choice of basis for $\mathbb{C}^n$. If U is a unitary $n \times n$ matrix, then the matrix functions H_0 and Δ_0 defined by

$$H_0(t) = UH(t)U^*, \quad \Delta_0(t) = U\Delta(t)U^*, \quad t \in \imath,$$

satisfy the conditions (2.2) and J_0 defined by

$$J_0 = UJU^*,$$

satisfies the conditions (2.3). For $g \in \mathcal{L}^2_{\Delta,\mathrm{loc}}(\imath)$ and a solution f of (2.4), define the functions $f_0(t) = Uf(t)$ and $g_0(t) = Ug(t)$. Then $g_0 \in \mathcal{L}^2_{\Delta_0,\mathrm{loc}}(\imath)$ and f_0 is a solution of the inhomogeneous equation

$$J_0 f'(t) - H_0(t)f(t) = \lambda\Delta_0(t)f(t) + \Delta_0(t)g(t), \quad t \in \imath.$$

The preceding remark shows that one can transform the canonical system (2.4) into an equivalent canonical system (2.4) by transforming, for instance, J into a specific form. Hence the following well known fact is useful.

Lemma 2.4. *Let X be a selfadjoint $2m \times 2m$ matrix which has m positive and m negative eigenvalues (counted with multiplicities). Then there exists a (nonunique) invertible $2m \times 2m$ matrix V such that*

$$X = V^* \begin{pmatrix} 0 & -iI_m \\ iI_m & 0 \end{pmatrix} V.$$

If, in addition, the matrix X is unitary, then the matrix V is unitary.

In particular, if one has a canonical system (2.4) with $n = 2m$ and $\mathrm{i}^+ = \mathrm{i}^- = m$, then Lemma 2.4 (applied to iJ) implies the existence of a unitary $n \times n$ matrix U such that

$$J = U^* \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} U \quad \text{and} \quad UJU^* = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}. \tag{2.9}$$

Hence, in these cases the canonical system is equivalent to a so-called *Hamiltonian system*, see, e.g., [29].

2.3. **Regular and singular endpoints of canonical systems.** The following definition gives a classification for the endpoints of the canonical system (2.4).

Definition 2.5. An endpoint of the interval $\imath$ is said to be a *quasiregular* endpoint of the canonical system (2.4) if the locally integrable functions H and Δ in (2.2) are integrable up to that endpoint. A finite quasiregular endpoint is called *regular*. An endpoint is said to be *singular* when it is not regular. The canonical system (2.4) is called *regular* if both endpoints are regular; otherwise it is called *singular*.

It will turn out that for a regular system all solutions of the homogeneous equation (2.5) are square-integrable, whereas for a singular system not all such solutions are necessarily square-integrable. The following result implies that if the inhomogeneous term $g \in \mathcal{L}^2_{\Delta,\mathrm{loc}}(\imath)$ is square-integrable at a quasiregular endpoint, then every solution of the inhomogeneous equation has a continuous extension to that endpoint, so that it is square-integrable there.

Proposition 2.6. *Assume that the endpoint a or b of the canonical system* (2.4) *is quasiregular and that $g \in \mathcal{L}^2_{\Delta,\mathrm{loc}}(\imath)$ is square-integrable (with respect to Δ) at a or b, respectively. Then each solution f of* (2.4) *is square-integrable (with respect to Δ) at a or at b, and the limits*

$$f(a) := \lim_{t \downarrow a} f(t) \quad \text{or} \quad f(b) := \lim_{t \uparrow b} f(t), \tag{2.10}$$

exist, respectively.

Moreover, for each $\gamma \in \mathbb{C}^n$ there exists a unique solution f of (2.4) *such that $f(a) = \gamma$ or $f(b) = \gamma$, respectively.*

Proof. It suffices to consider the case of the endpoint b. With $c \in (a,b)$ fixed, any solution f of (2.4) satisfies

$$f(t) = f(c) + \int_c^t J^{-1}(\lambda\Delta(s) + H(s)) f(s)\, ds + \int_c^t J^{-1}\Delta(s) g(s)\, ds. \tag{2.11}$$

Note that both integrals on the righthand side exist since $(\lambda\Delta + H)f \in \mathcal{L}^1_{\mathrm{loc}}(\imath)$ for any $f \in AC_{\mathrm{loc}}(\imath)$ and $\Delta g \in \mathcal{L}^1_{\mathrm{loc}}(\imath)$ for $g \in \mathcal{L}^2_{\Delta,\mathrm{loc}}(\imath)$.

Hence, for $t \geq c$, it follows that

$$|f(t)| \leq \left(|f(c)| + \int_c^t |\Delta(s)g(s)|\ ds\right) + \int_c^t |\lambda\Delta(s) + H(s)|\, |f(s)|\, ds.$$

Since the first term on the righthand side is nondecreasing it follows from Gronwall's inequality (cf. [11, Chapter 1, Problem 1]) that

$$|f(t)| \leq \left(|f(c)| + \int_c^t |\Delta(s)g(s)|\ ds\right) e^{\int_c^t |\lambda\Delta(s)+H(s)|\, ds}.$$

Furthermore, as g is square integrable (with respect to Δ) at b it follows that Δg is integrable on (c,b). Since b is a quasiregular endpoint also $\lambda\Delta + H$ is integrable on (c,b) and hence the solution f is bounded on (c,b). Then it is clear from (2.11) that the limit $f(b) := \lim_{t\uparrow b} f(t)$ exists. Moreover, the local boundedness of the solutions shows that

$$\int_c^b f(s)^*\Delta(s)f(s)\, ds \leq M^2 \int_c^b |\Delta(s)|\, ds < \infty$$

and hence f is square-integrable with respect to Δ at b. As a consequence of the existence of the limit at the endpoint b observe that

$$f(t) = f(b) - \int_t^b J^{-1}(\lambda\Delta(s) + H(s)) f(s)\, ds - \int_t^b J^{-1}\Delta(s)g(s)\, ds,$$

and thus

$$|f(t)| \leq \left(|f(b)| + \int_t^b |\Delta(s)g(s)|\ ds\right) e^{\int_t^b |\lambda\Delta(s)+H(s)|\, ds}.$$

In particular, for solutions f of the corresponding homogeneous equation (2.5) it follows that the mapping $f \mapsto f(b)$ is injective, and hence surjective. Therefore, for each $\gamma \in \mathbb{C}^n$ there exists a unique solution f of (2.4) such that $f(b) = \gamma$. ☐

Note that the condition that $g \in \mathcal{L}^2_{\Delta,\mathrm{loc}}(\imath)$ is square-integrable at some endpoint is only used to obtain that $\Delta g \in \mathcal{L}^1_{\mathrm{loc}}(\imath)$ is integrable at that endpoint.

Corollary 2.7. *Assume that the endpoints a and b of the canonical system* (2.4) *are quasiregular and that $g \in \mathcal{L}^2_\Delta(\imath)$. Then each solution f of* (2.4) *belongs to $\mathcal{L}^2_\Delta(\imath)$ and both limits in* (2.10) *exist.*

The next statement is a direct consequence of Proposition 2.6 and identity (2.7).

Corollary 2.8. *Assume that the endpoint a or b of the canonical system* (2.4) *is quasiregular and let $Y(\cdot,\lambda)$ be a fundamental matrix of the canonical system* (2.4)*. Then $Y(\cdot,\lambda)\phi$ is square integrable (with respect to Δ) at a or b for every $\phi \in \mathbb{C}^n$ and $Y(\cdot,\lambda)$ admits a unique continuous extension to a or b such that $Y(a,\lambda)$ or $Y(b,\lambda)$ is invertible, respectively. In particular, the point c_0 in* (2.6) *can be chosen to be a or b, respectively.*

2.4. **Definiteness of the canonical system.** Let $\jmath \subset \imath$ be a nonempty interval. The canonical system (2.4) is said to be *definite on* $\jmath$ if for each $\lambda \in \mathbb{C}$ and for each nontrivial solution f of the corresponding homogeneous equation (2.5) on $\imath$ the condition

$$0 < \int_\jmath f(s)^* \Delta(s) f(s)\, ds \leq \infty$$

holds. If H and Δ are integrable on $\imath$ (in particular, if the canonical system is regular on $\imath$), then the above integral is necessarily finite; see Corollary 2.8.

Lemma 2.9. *If the canonical system* (2.4) *is definite on* $\jmath$*, then it is also definite on every interval* $\widetilde{\jmath}$ *with the property that* $\jmath \subset \widetilde{\jmath} \subset \imath$.

Proof. Let the assumptions of the statement hold, then any nontrivial solution f of (2.5) on $\imath$ satisfies

$$0 < \int_\jmath f(s)^* \Delta(s) f(s)\, ds \leq \int_{\widetilde{\jmath}} f(s)^* \Delta(s) f(s)\, ds.$$

Hence, the canonical system is definite on $\jmath$. □

An equivalent statement for definiteness on $\jmath$ is that for each $\lambda \in \mathbb{C}$ and each solution f of (2.5)

$$\int_\jmath f(s)^* \Delta(s) f(s)\, ds = 0 \qquad \text{implies} \qquad f(t) = 0, \quad t \in \jmath.$$

According to the existence and uniqueness theorem for linear systems of differential equations the conclusion $f(t) = 0$, $t \in \jmath$, implies that $f(t) = 0$, $t \in \imath$. The next lemma shows that it suffices to check the definiteness condition for only one $\lambda \in \mathbb{C}$.

Lemma 2.10. *The canonical system* (2.4) *is definite on the interval* $\jmath \subset \imath$ *if and only if for some* $\lambda_0 \in \mathbb{C}$ *and for each solution* f *of* $Jf' - Hf = \lambda_0 \Delta f$ *the condition*

$$\int_\jmath f(s)^* \Delta(s) f(s)\, ds = 0$$

implies $f(t) = 0$ *for* $t \in \jmath$*, and thus* $f(t) = 0$ *for* $t \in \imath$.

Proof. ($\Rightarrow$) This implication is clear.

($\Leftarrow$) Choose any $\lambda \in \mathbb{C}$ and let f be a solution of $Jf' - Hf = \lambda \Delta f$ with $\int_\jmath f^*(s)\Delta(s) f(s)\, ds = 0$ or, equivalently, $\Delta(t) f(t) = 0$ for almost all $t \in \jmath$. Thus f is also a solution of $Jf' - Hf = \lambda_0 \Delta f$ with $\int_\jmath f^*(s)\Delta(s) f(s)\, ds = 0$. By assumption this implies that $f(t) = 0$ for $t \in \jmath$, and hence for $t \in \imath$. Therefore the system is definite. □

It follows from Lemma 2.10 that the canonical system (2.4) is definite on the interval $\jmath \subset \imath$ if and only if for each solution f of $Jf' - Hf = 0$ the condition $\Delta f = 0$ on $\jmath$ implies that $f(t) = 0$ for $t \in \jmath$, and thus $f(t) = 0$ for $t \in \imath$. In particular, if there exists a nonempty interval $\jmath \subset \imath$ such that $\Delta(t)$ has full rank n for almost all $t \in \jmath$, then the canonical system (2.4) is definite on the interval $\jmath \subset \imath$.

The following result will be used frequently in the rest of this paper; a proof is provided for completeness.

Proposition 2.11. *The canonical system* (2.4) *is definite on* $\imath$ *if and only if there exists a compact interval* $I \subset \imath$ *such that the canonical system* (2.4) *is definite on the interval* I.

Proof. Necessity follows from Lemma 2.9. Hence assume that the canonical system (2.4) is definite on $\imath$. Fix some $\lambda_0 \in \mathbb{C}$ and introduce for each compact subinterval $\jmath$ of $\imath$ the subset $d(\jmath)$ of $\mathbb{C}^n$ by

$$d(\jmath) = \left\{ \phi \in \mathbb{C}^n : |\phi| = 1, \int_\jmath \phi^* Y(s,\lambda_0)^* \Delta(s) Y(s,\lambda_0) \phi\, ds = 0 \right\}.$$

Clearly, $d(\jmath)$ is compact and $\jmath \subset \widetilde{\jmath}$ implies $d(\widetilde{\jmath}) \subset d(\jmath)$. Now choose an increasing sequence of compact intervals $\jmath_m \subset \imath$, $m \in \mathbb{N}$, such that their union equals the interval $\imath$. Then

$$\bigcap_{m\in\mathbb{N}} d(\jmath_m) = \emptyset. \tag{2.12}$$

To see this, assume that there exists an element $\phi \in \mathbb{C}^n$ with $|\phi| = 1$, such that

$$\int_{\jmath_m} \phi^* Y(s,\lambda_0)^* \Delta(s) Y(s,\lambda_0)\phi \, ds = 0$$

for every m. Then by monotone convergence $\int_\imath \phi^* Y(s,\lambda_0)^* \Delta(s) Y(s,\lambda_0)\phi \, ds = 0$. Since the canonical system (2.4) is definite, this implies that $Y(\cdot,\lambda)\phi = 0$, which leads to $\phi = 0$, a contradiction. Therefore, the identity (2.12) is valid. Since each of the sets $d(\jmath_m)$ in (2.12) is compact it follows that there exists a compact interval $\jmath_k$ such that $d(\jmath_k) = \emptyset$. Hence $I = \jmath_k$ satisfies the requirements. □

The notion of definiteness can be found in [21, p. 249, p. 300] and [49]. Proposition 2.11 can be found in [48, Hilfsatz (3.1)] and [38]; for a more abstract treatment see [4].

Example 2.12 (Weighted Sturm-Liouville equations)**.** Let $\imath \subset \mathbb{R}$ be an open interval. Let $1/p, q, r \in \mathcal{L}^1_{\rm loc}(\imath)$ be real-valued functions, assume $r(t) \geq 0$ for almost all $t \in \imath$, and define the 2×2 matrix J and the 2×2 matrix functions H and Δ by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad H(t) = \begin{pmatrix} -q(t) & 0 \\ 0 & 1/p(t) \end{pmatrix}, \quad \Delta(t) = \begin{pmatrix} r(t) & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.13}$$

Let f be a solution of $Jf' - Hf = 0$ which satisfies $\Delta f = 0$, so that in components

$$-f_2' + qf_1 = 0, \quad f_1' - (1/p)f_2 = 0, \quad rf_1 = 0.$$

Assume that there exists a nonempty interval $\jmath \subset \imath$ such that $r(t) > 0$, $t \in \jmath$. Then $f_1(t) = 0$ and, hence, also $f_2(t) = 0$, when $t \in \jmath$. Therefore the corresponding system is definite on $\jmath$ and, thus, on $\imath$.

Remark 2.13. A stronger form of definiteness is obtained when for all compact intervals $I \subset \imath$ the inequality

$$0 < \int_I f(s)^* \Delta(s) f(s)\, ds \tag{2.14}$$

is satisfied for any nontrivial solution f of (2.5); see [2, 29, 30, 31, 52, 56]. To see that this kind of definiteness is stronger than the present notion of definiteness consider the following example. Define the nonnegative locally integrable matrix function Δ such that $\Delta(t)$ is invertible for t on a compact interval $[\alpha, \beta] \subset \imath$ and such that $\Delta(t) = 0$ on the complement. The canonical system (2.4) is clearly definite on $\imath$ whereas (2.14) is not satisfied for any interval contained in the complement of $[\alpha, \beta]$.

2.5. **Localization of solutions.** If the canonical system (2.4) is definite, then a solution of the inhomogeneous canonical system can be localized at one endpoint, in the sense that it can be made trivial at the other endpoint. First some preliminary results of general nature will be stated.

Lemma 2.14. *Let the canonical system* (2.4) *be definite and assume that its endpoints a and b are quasiregular. Then for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$ the $2n \times 2n$-matrix*

$$\begin{pmatrix} Y(a,\lambda) & Y(a,\bar\lambda) \\ Y(b,\lambda) & Y(b,\bar\lambda) \end{pmatrix} \tag{2.15}$$

is invertible.

Proof. It follows from Corollaries 2.2 and 2.8 that

$$\begin{pmatrix} Y(a,\lambda)^* & Y(b,\lambda)^* \\ Y(a,\bar\lambda)^* & Y(b,\bar\lambda)^* \end{pmatrix} \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} Y(a,\lambda) & Y(a,\bar\lambda) \\ Y(b,\lambda) & Y(b,\bar\lambda) \end{pmatrix}$$
$$= (\lambda - \bar\lambda) \int_a^b \begin{pmatrix} Y(s,\lambda)^*\Delta(s)Y(s,\lambda) & 0 \\ 0 & -Y(s,\bar\lambda)^*\Delta(s)Y(s,\bar\lambda) \end{pmatrix} ds.$$

By definiteness (see Lemma 2.10) the matrix on the righthand side is invertible, which implies the invertibility of the matrix in (2.15) for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. □

In particular, the assumptions in Lemma 2.14 imply that for $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$\frac{Y(b,\lambda)^* J Y(b,\lambda) - Y(a,\lambda)^* J Y(a,\lambda)}{\lambda - \bar\lambda}$$

is positive definite. The following two results are also immediate consequences of Lemma 2.14.

Corollary 2.15. *Let the canonical system* (2.4) *be regular and definite. Then for all $\gamma_a, \gamma_b \in \mathbb{C}^n$ and every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exist solutions $f_\lambda \in \mathcal{L}^2_\Delta(\imath)$ and $f_{\bar\lambda} \in \mathcal{L}^2_\Delta(\imath)$ of the homogeneous equation* (2.5) *for λ and $\bar\lambda$, respectively, such that*

$$f_\lambda(a) + f_{\bar\lambda}(a) = \gamma_a, \quad f_\lambda(b) + f_{\bar\lambda}(b) = \gamma_b.$$

Observe that the function $f = f_\lambda + f_{\bar\lambda}$ with $f_\lambda, f_{\bar\lambda} \in \mathcal{L}^2_\Delta(\imath)$ as in Corollary 2.15 is a solution of the equation

$$Jf' - Hf = \lambda\Delta f + \Delta g, \quad \text{where} \quad g = \bar\lambda f_{\bar\lambda} - \lambda f_{\bar\lambda}.$$

This implies the following statement; cf. [49].

Corollary 2.16. *Let the canonical system* (2.4) *be regular and definite. Then for all $\gamma_a, \gamma_b \in \mathbb{C}^n$ there exist an element $g \in \mathcal{L}^2_\Delta(\imath)$ and a solution $f \in \mathcal{L}^2_\Delta(\imath)$ of* (2.4) *which satisfies the boundary conditions*

$$f(a) = \gamma_a, \quad f(b) = \gamma_b.$$

Note that the conclusions in Corollaries 2.15 and 2.16 remain valid under the more general conditions that H and Δ are integrable on $\imath$. In this case $f(a)$ and $f(b)$ denote the limits in (2.10).

Proposition 2.17. *Let the canonical system* (2.4) *be definite, let $g \in \mathcal{L}^2_{\Delta,\mathrm{loc}}(\imath)$ and let $f \in AC_{\mathrm{loc}}(\imath)$ be a solution of the inhomogeneous equation* (2.4)*. Then there exists a compact interval $[\alpha,\beta] \subseteq \imath$, $f_a \in AC_{\mathrm{loc}}(\imath)$ and $g_a \in \mathcal{L}^2_{\Delta,\mathrm{loc}}(\imath)$ satisfying*

$$Jf_a'(t) - H(t)f_a(t) = \lambda\Delta(t)f_a(t) + \Delta(t)g_a(t)$$

such that

$$f_a(t) = \begin{cases} f(t), & t \in (a,\alpha], \\ 0, & t \in [\beta,b), \end{cases} \qquad g_a(t) = \begin{cases} g(t), & t \in (a,\alpha], \\ 0, & t \in [\beta,b). \end{cases}$$

Similarly, there exists a compact interval $[\alpha,\beta] \subseteq \imath$, $f_b \in AC_{\mathrm{loc}}(\imath)$ and $g_b \in \mathcal{L}^2_{\Delta,\mathrm{loc}}(\imath)$ satisfying

$$Jf_b'(t) - H(t)f_b(t) = \lambda\Delta(t)f_b(t) + \Delta(t)g_b(t)$$

such that

$$f_b(t) = \begin{cases} 0, & t \in (a,\alpha], \\ f(t), & t \in [\beta,b), \end{cases} \qquad g_b(t) = \begin{cases} 0, & t \in (a,\alpha], \\ g(t), & t \in [\beta,b). \end{cases}$$

Proof. According to Proposition 2.11 there exists a compact interval $[\alpha,\beta] \subset \imath$ such that the canonical system (2.4) is definite on $[\alpha,\beta]$. In particular, the points α and β are regular endpoints for the canonical system (2.4) restricted to (α,β). Hence Corollary 2.16 implies that for $f(\alpha) \in \mathbb{C}^n$ there exists a function $k \in \mathcal{L}^2_\Delta(\alpha,\beta)$ and an $h \in AC_{\mathrm{loc}}(\alpha,\beta)$ satisfying

$$Jh'(t) - H(t)h(t) = \lambda\Delta(t)h(t) + \Delta(t)k(t), \quad h(\alpha) = f(\alpha), \quad h(\beta) = 0$$

on (α,β). Hence the functions f_a and g_a defined by

$$f_a(t)=\begin{cases} f(t), & t\in(a,\alpha],\\ h(t), & t\in(\alpha,\beta),\\ 0, & t\in[\beta,b),\end{cases} \quad g_a(t)=\begin{cases} g(t), & t\in(a,\alpha],\\ k(t), & t\in(\alpha,\beta),\\ 0, & t\in[\beta,b).\end{cases}$$

satisfy the asserted properties. A similar argument shows the existence of the functions f_b and g_b with the asserted properties. □

In particular, when f is a solution of the homogeneous system (2.5), then f can be localized as indicated above. The following restatement of this fact in terms of matrix functions (groupings of column vector functions) is useful.

Corollary 2.18. *Let the canonical system* (2.4) *be definite and let* $Y(\cdot,\lambda)$ *be a corresponding fundamental matrix. Then there exits a compact interval* $[\alpha,\beta]\subseteq\imath$*, a* $n\times n$ *matrix function* $Y_a(\cdot,\lambda)\in AC_{\rm loc}(\imath)$ *and a* $n\times n$ *matrix functions* $Z_a(\cdot,\lambda)$ *whose columns belong to* $\mathcal{L}^2_\Delta(\imath)$*, satisfying*

$$JY_a'(t,\lambda)\phi-H(t)Y_a(t,\lambda)\phi=\lambda\Delta(t)Y_a(t,\lambda)\phi+\Delta(t)Z_a(t,\lambda)\phi,\quad \phi\in\mathbb{C}^n,$$

such that

$$Y_a(t,\lambda)=\begin{cases} Y(t,\lambda), & t\in(a,\alpha],\\ 0, & t\in[\beta,b),\end{cases} \quad Z_a(t,\lambda)=\begin{cases} 0, & t\in(a,\alpha],\\ 0, & t\in[\beta,b).\end{cases}$$

Similarly, there exists a compact interval $[\alpha,\beta]\subseteq\imath$*, a* $n\times n$ *matrix function* $Y_b(\cdot,\lambda)\in AC_{\rm loc}(\imath)$ *and a* $n\times n$ *matrix functions* $Z_b(\cdot,\lambda)$ *whose columns belong to* $\mathcal{L}^2_\Delta(\imath)$*, satisfying*

$$JY_b'(t,\lambda)\phi-H(t)Y_b(t,\lambda)\phi=\lambda\Delta(t)Y_b(t,\lambda)\phi+\Delta(t)Z_b(t,\lambda)\phi,\quad \phi\in\mathbb{C}^n,$$

such that

$$Y_b(t,\lambda)=\begin{cases} 0, & t\in(a,\alpha],\\ Y(t,\lambda), & t\in[\beta,b),\end{cases} \quad Z_b(t,\lambda)=\begin{cases} 0, & t\in(a,\alpha],\\ 0, & t\in[\beta,b).\end{cases}$$

With $\phi\in\mathbb{C}^n$, observe that the function $Y_a(\cdot,\lambda)\phi$ belongs to $\mathcal{L}^2_\Delta(\imath)$ if and only if $Y(\cdot,\lambda)\phi$ is square-integrable at a, and, likewise, that the function $Y_b(\cdot,\lambda)\phi$ belongs to $\mathcal{L}^2_\Delta(\imath)$ if and only if $Y(\cdot,\lambda)\phi$ is square-integrable at b.

3. Square-integrable solutions of singular canonical systems

This section is concerned with the square-integrability of the solutions of the homogeneous canonical system (2.5). These solutions are studied in terms of a monotone matrix function on $\imath$ which by a general monotonicity principle from [3] admits limits at the endpoints of $\imath$ in the sense of linear relations (multivalued operators). The number of square-integrable solutions at the endpoints coincides with the multiplicity of the finite eigenvalues of the limits. One of the advantages of this abstract geometric approach and point of view is that it provides a very simple interpretation of the constructions from [2, 46, 56].

3.1. Monotonicity properties.

For a fundamental matrix $Y(\cdot,\lambda)$ of the canonical system (2.4) introduce the $n\times n$ matrix function $D(\cdot,\lambda)$ on $\imath$ by

$$D(t,\lambda)=Y(t,\lambda)^*(-iJ)Y(t,\lambda),\quad t\in\imath,\quad \lambda\in\mathbb{C}. \tag{3.1}$$

Observe that the function $t\mapsto D(t,\lambda)$, $t\in\imath$, is locally absolutely continuous for every $\lambda\in\mathbb{C}$. Moreover, for all $t\in\imath$ and $\lambda\in\mathbb{C}$ the matrix $D(t,\lambda)$ is selfadjoint and invertible, and the identities (2.8) imply

$$D(t,\lambda)^{-1}=JD(t,\bar\lambda)J^*,\quad t\in\imath,\quad \lambda\in\mathbb{C}. \tag{3.2}$$

Furthermore, it follows from Corollary 2.2 that

$$D(\beta,\lambda)-D(\alpha,\lambda)=2\,\mathrm{Im}\,\lambda\int_\alpha^\beta Y(s,\lambda)^*\Delta(s)Y(s,\lambda)\,ds,\quad \lambda\in\mathbb{C}, \tag{3.3}$$

holds for any compact interval $[\alpha,\beta] \subset \imath$. Hence the matrix function $D(\cdot,\lambda)$ is constant for $\lambda \in \mathbb{R}$, and only the case $\lambda \in \mathbb{C}\setminus\mathbb{R}$ will be of interest in the following. The statements in the next proposition are a direct consequence of (3.1), (3.3) and the fact that $Y(t,\lambda)$ is invertible for all $t \in \imath$.

Proposition 3.1. *For $\lambda \in \mathbb{C}_+$ or $\lambda \in \mathbb{C}_-$ the $n \times n$ matrix function $D(\cdot,\lambda)$ is nondecreasing or nonincreasing on $\imath$, respectively, and the numbers of positive and negative eigenvalues of $D(t,\lambda)$, $t \in \imath$, coincide with the multiplicities i^+ and i^- of the eigenvalues 1 and -1 of $-iJ$, respectively.*

The monotonicity of the functions $D(\cdot,\lambda)$ means that for each $\phi \in \mathbb{C}^n$ the limit as $t \to a$ or $t \to b$ of $\phi^* D(t,\lambda)\phi$ exists as a real number or as $\pm\infty$. Therefore, it is natural to define domains associated with the endpoint a by

$$\begin{aligned} \mathfrak{D}(a,\lambda) &= \big\{\phi \in \mathbb{C}^n : \lim_{t\downarrow a} \phi^* D(t,\lambda)\phi > -\infty\big\}, \quad \lambda \in \mathbb{C}_+ \\ \mathfrak{D}(a,\lambda) &= \big\{\phi \in \mathbb{C}^n : \lim_{t\uparrow a} \phi^* D(t,\lambda)\phi < \infty\big\}, \quad \lambda \in \mathbb{C}_-, \end{aligned} \tag{3.4}$$

and with the endpoint b by

$$\begin{aligned} \mathfrak{D}(b,\lambda) &= \big\{\phi \in \mathbb{C}^n : \lim_{t\uparrow b} \phi^* D(t,\lambda)\phi < \infty\big\}, \quad \lambda \in \mathbb{C}_+, \\ \mathfrak{D}(b,\lambda) &= \big\{\phi \in \mathbb{C}^n : \lim_{t\uparrow b} \phi^* D(t,\lambda)\phi > -\infty\big\}, \quad \lambda \in \mathbb{C}_-. \end{aligned} \tag{3.5}$$

The following theorem, which is an immediate consequence of [3, Theorem 3.1, Corollary 3.6], explains the limits of the function $D(\cdot,\lambda)$ in terms of linear relations (in the sense of multivalued operators) which are selfadjoint; see Section 7 for a short introduction.

Theorem 3.2. *For every $\lambda \in \mathbb{C}\setminus\mathbb{R}$ there exist selfadjoint relations $D(a,\lambda)$ and $D(b,\lambda)$ which are the limits of $D(\cdot,\lambda)$ in the resolvent sense, i.e.,*

$$(D(a,\lambda)-\mu)^{-1} = \lim_{t\downarrow a}(D(t,\lambda)-\mu)^{-1}, \quad (D(b,\lambda)-\mu)^{-1} = \lim_{t\uparrow b}(D(t,\lambda)-\mu)^{-1},$$

for every $\mu \in \mathbb{C}\setminus\mathbb{R}$. In terms of these limits the space $\mathbb{C}^n$ allows the orthogonal decompositions:

$$\mathbb{C}^n = \begin{cases} \operatorname{dom} D(a,\lambda) \oplus \operatorname{mul} D(a,\lambda) = \mathfrak{D}(a,\lambda) \oplus \operatorname{mul} D(a,\lambda); \\ \operatorname{dom} D(b,\lambda) \oplus \operatorname{mul} D(a,\lambda) = \mathfrak{D}(b,\lambda) \oplus \operatorname{mul} D(b,\lambda). \end{cases}$$

The graphs of the selfadjoint limit relations $D(a,\lambda)$ and $D(b,\lambda)$ decompose accordingly:

$$D(a,\lambda) = D(a,\lambda)_s \mathbin{\widehat{\oplus}} \big(\{0\} \times \operatorname{mul} D(a,\lambda)\big),$$

$$D(b,\lambda) = D(b,\lambda)_s \mathbin{\widehat{\oplus}} \big(\{0\} \times \operatorname{mul} D(b,\lambda)\big),$$

where $D(a,\lambda)_s$ and $D(b,\lambda)_s$ are (the graphs of) selfadjoint operators in $\mathfrak{D}(a,\lambda)$ and $\mathfrak{D}(b,\lambda)$, respectively, and $\widehat{\oplus}$ denotes the orthogonal sum of subspaces in $\mathbb{C}^n \times \mathbb{C}^n$. Moreover,

$$\begin{aligned} D(a,\lambda)_s\phi &= \lim_{t\downarrow a} D(t,\lambda)\phi, \quad \phi \in \mathfrak{D}(a,\lambda), \\ D(b,\lambda)_s\phi &= \lim_{t\uparrow b} D(t,\lambda)\phi, \quad \phi \in \mathfrak{D}(b,\lambda). \end{aligned} \tag{3.6}$$

The monotonicity of the $n \times n$ matrix function $D(\cdot,\lambda)$ implies that the limit relation $D(a,\lambda)$ and $D(b,\lambda)$ from Theorem 3.2 satisfy the inequalities

$$\begin{aligned} (\psi,\phi) &\le (D(t,\lambda)\phi,\phi) \quad \text{for all} \quad \{\phi,\psi\} \in D(a,\lambda), \quad \lambda \in \mathbb{C}_+, \\ (D(t,\lambda)\phi,\phi) &\le (\psi,\phi) \quad \text{for all} \quad \{\phi,\psi\} \in D(a,\lambda), \quad \lambda \in \mathbb{C}_-, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} (D(t,\lambda)\phi,\phi) &\le (\psi,\phi) \quad \text{for all} \quad \{\phi,\psi\} \in D(b,\lambda), \quad \lambda \in \mathbb{C}_+, \\ (\psi,\phi) &\le (D(t,\lambda)\phi,\phi) \quad \text{for all} \quad \{\phi,\psi\} \in D(b,\lambda), \quad \lambda \in \mathbb{C}_-, \end{aligned} \tag{3.8}$$

hold for $t \in \imath$. For $\phi \in \operatorname{dom} D(a,\lambda) = \mathfrak{D}(a,\lambda)$ the inequalities (3.7) reduce to

$$(3.9) \qquad \begin{aligned} (D(a,\lambda)_s\phi,\phi) &\le (D(t,\lambda)\phi,\phi), \qquad \lambda \in \mathbb{C}_+,\\ (D(t,\lambda)\phi,\phi) &\le (D(a,\lambda)_s\phi,\phi), \qquad \lambda \in \mathbb{C}_-, \end{aligned}$$

and, analogously, for $\phi \in \operatorname{dom} D(b,\lambda) = \mathfrak{D}(b,\lambda)$ the inequalities (3.8) reduce to

$$(3.10) \qquad \begin{aligned} (D(t,\lambda)\phi,\phi) &\le (D(b,\lambda)_s\phi,\phi), \qquad \lambda \in \mathbb{C}_+,\\ (D(b,\lambda)_s\phi,\phi) &\le (D(t,\lambda)\phi,\phi), \qquad \lambda \in \mathbb{C}_-. \end{aligned}$$

In particular, if $\operatorname{mul} D(a,\lambda) = \operatorname{mul} D(b,\lambda) = \{0\}$, then the inequalities

$$(3.11) \qquad \begin{aligned} D(a,\lambda) \le D(t,\lambda) \le D(b,\lambda), \qquad \lambda \in \mathbb{C}_+,\\ D(a,\lambda) \ge D(t,\lambda) \ge D(b,\lambda), \qquad \lambda \in \mathbb{C}_-, \end{aligned}$$

are satisfied for $t \in \imath$.

Using the limit relations from Theorem 3.2, the identity (3.2) can be extended to the endpoints of the interval $\imath$.

Corollary 3.3. *The limit relations $D(a,\lambda)$ and $D(b,\lambda)$ satisfy*

$$D(a,\lambda)^{-1} = JD(a,\bar\lambda)J^*, \quad D(b,\lambda)^{-1} = JD(b,\bar\lambda)J^*.$$

Proof. It suffices to show that the limit values of $D(t,\lambda)^{-1}$ coincide with the selfadjoint relations $D(a,\lambda)^{-1}$ and $D(b,\lambda)^{-1}$, respectively. Let A be the resolvent limit of $D(t,\lambda)^{-1}$ as t tends to a. Then by (A.1):

$$\begin{aligned} (A-\zeta)^{-1} &= \lim_{t\downarrow a} (D(t,\lambda)^{-1}-\zeta)^{-1}\\ &= \lim_{t\downarrow a}\left(-\frac{1}{\zeta^2}\left(D(t,\lambda)-\frac{1}{\zeta}\right)^{-1}-\frac{1}{\zeta}\right)\\ &= -\frac{1}{\zeta^2}\left(D(a,\lambda)-\frac{1}{\zeta}\right)^{-1}-\frac{1}{\zeta}, \end{aligned}$$

for $\zeta \in \mathbb{C}\setminus\mathbb{R}$. Hence using (A.1) once more, the above identity shows that the limit A satisfies $A = D(a,\lambda)^{-1}$. For the endpoint b a similar argument can be used. ☐

Remark 3.4. Note that any two fundamental matrices $Y_1(\cdot,\lambda)$ and $Y_2(\cdot,\lambda)$ of the canonical system (2.4) are related via

$$Y_1(\cdot,\lambda) = Y_2(\cdot,\lambda)X(\lambda), \qquad \text{where} \quad X(\lambda) = Y_2(c,\lambda)^{-1}Y_1(c,\lambda)$$

and c is an arbitrary fixed point in $\imath$. This implies that the associated matrix functions $D_1(\cdot,\lambda)$ and $D_2(\cdot,\lambda)$ in (3.1) are connected via $D_1(\cdot,\lambda) = X^*(\lambda)D_2(\cdot,\lambda)X(\lambda)$, where $X(\lambda)$ invertible. This identity is preserved in the limits $t \to a$ and $t \to b$. Therefore, the dimensions of the eigenspaces corresponding to the positive, negative, zero and infinite eigenvalues of the selfadjoint relations $D(a,\lambda)$ and $D(b,\lambda)$ do not depend on the chosen fundamental matrix $Y(\cdot,\lambda)$.

3.2. **Decompositions in terms of the eigenspaces of the limit relations.** Denote the eigenspaces of the selfadjoint relation $D(a,\lambda)$ corresponding to the positive, negative, zero, and infinite eigenvalues by

$$\mathcal{A}^+(\lambda), \quad \mathcal{A}^-(\lambda), \quad \mathcal{A}^0(\lambda), \quad \mathcal{A}^\infty(\lambda),$$

and denote the corresponding dimensions by

$$\mathsf{a}^+(\lambda), \quad \mathsf{a}^-(\lambda), \quad \mathsf{a}^0(\lambda), \quad \mathsf{a}^\infty(\lambda).$$

Likewise, denote the eigenspaces of the selfadjoint relation $D(b,\lambda)$ corresponding to the positive, negative, zero, and infinite eigenvalues by

$$\mathcal{B}^+(\lambda), \quad \mathcal{B}^-(\lambda), \quad \mathcal{B}^0(\lambda), \quad \mathcal{B}^\infty(\lambda),$$

and denote the corresponding dimensions by

$$\mathsf{b}^+(\lambda), \quad \mathsf{b}^-(\lambda), \quad \mathsf{b}^0(\lambda), \quad \mathsf{b}^\infty(\lambda).$$

Then the spaces $\mathfrak{D}(a,\lambda)$ and $\mathfrak{D}(b,\lambda)$ allow the decompositions:

$$\begin{aligned}\mathfrak{D}(a,\lambda) &= \mathcal{A}^+(\lambda) \oplus \mathcal{A}^-(\lambda) \oplus \mathcal{A}^0(\lambda),\\ \mathfrak{D}(b,\lambda) &= \mathcal{B}^+(\lambda) \oplus \mathcal{B}^-(\lambda) \oplus \mathcal{B}^0(\lambda),\end{aligned} \tag{3.12}$$

and, moreover,

$$\mathfrak{D}(a,\lambda) \cap \mathfrak{D}(b,\lambda) = \mathcal{A}^\infty(\lambda)^\perp \cap \mathcal{B}^\infty(\lambda)^\perp = \big(\mathcal{A}^\infty(\lambda) + \mathcal{B}^\infty(\lambda)\big)^\perp. \tag{3.13}$$

Furthermore, the identities

$$\begin{aligned}\mathcal{A}^+(\lambda) &= J\mathcal{A}^+(\bar\lambda), \quad \mathcal{A}^-(\lambda) = J\mathcal{A}^-(\bar\lambda), \quad \mathcal{A}^\infty(\lambda) = J\mathcal{A}^0(\bar\lambda),\\ \mathcal{B}^+(\lambda) &= J\mathcal{B}^+(\bar\lambda), \quad \mathcal{B}^-(\lambda) = J\mathcal{B}^-(\bar\lambda), \quad \mathcal{B}^\infty(\lambda) = J\mathcal{B}^0(\bar\lambda),\end{aligned} \tag{3.14}$$

follow from Corollary 3.3.

The next lemma shows how the dimensions of the eigenspaces of $D(a,\lambda)$ and $D(b,\lambda)$ are related to the numbers i^+ and i^- of positive and negative eigenvalues of the matrix $D(t,\lambda)$, $t \in \imath$. The results in the following lemma can be derived from the continuous dependence of the eigenvalues of $D(t,\lambda)$ on t; cf. [2, 48, 56] and [3, 4] for a general approach. If, e.g., $\lambda \in \mathbb{C}_+$ and t tends to b, then roughly speaking some of the positive eigenvalues of $D(t,\lambda)$ can move to $+\infty$ and some of the negative eigenvalues can move to 0. If t tends ot a or $\lambda \in \mathbb{C}_-$ similar phenomena appear.

Lemma 3.5. *The identities*

$$\begin{aligned}\mathsf{a}^+(\lambda) + \mathsf{a}^0(\lambda) = \mathsf{i}^+ &= \mathsf{b}^+(\lambda) + \mathsf{b}^\infty(\lambda),\\ \mathsf{a}^-(\lambda) + \mathsf{a}^\infty(\lambda) = \mathsf{i}^- &= \mathsf{b}^-(\lambda) + \mathsf{b}^0(\lambda),\end{aligned} \qquad \lambda \in \mathbb{C}_+,$$

and

$$\begin{aligned}\mathsf{a}^+(\lambda) + \mathsf{a}^\infty(\lambda) = \mathsf{i}^+ &= \mathsf{b}^+(\lambda) + \mathsf{b}^0(\lambda),\\ \mathsf{a}^-(\lambda) + \mathsf{a}^0(\lambda) = \mathsf{i}^- &= \mathsf{b}^-(\lambda) + \mathsf{b}^\infty(\lambda),\end{aligned} \qquad \lambda \in \mathbb{C}_-,$$

hold. In particular,

$$\mathsf{a}^+(\lambda),\, \mathsf{b}^+(\lambda) \le \mathsf{i}^+, \quad \mathsf{a}^-(\lambda),\, \mathsf{b}^-(\lambda) \le \mathsf{i}^-, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{3.15}$$

Remark 3.6. Equality may happen in the inequalities (3.15). If the endpoint a is quasiregular, see Definition 2.5, then it follows from the definition in (3.1) and Corollary 2.8 that $\mathsf{a}^0(\lambda) = \mathsf{a}^\infty(\lambda) = 0$ and hence $\mathsf{a}^+(\lambda) = \mathsf{i}^+$, $\mathsf{a}^-(\lambda) = \mathsf{i}^-$. Likewise, if the endpoint b is quasiregular, then $\mathsf{b}^0(\lambda) = \mathsf{b}^\infty(\lambda) = 0$ and $\mathsf{b}^+(\lambda) = \mathsf{i}^+$, $\mathsf{b}^-(\lambda) = \mathsf{i}^-$.

Note that Lemma 3.5 provides lower bounds for the dimensions of the spaces $\mathfrak{D}(a,\lambda)$ and $\mathfrak{D}(b,\lambda)$, respectively,

$$\dim \mathfrak{D}(a,\lambda) = \begin{cases} \mathsf{i}^+ + \mathsf{a}^-(\lambda) \ge \mathsf{i}^+, & \lambda \in \mathbb{C}_+,\\ \mathsf{i}^- + \mathsf{a}^+(\lambda) \ge \mathsf{i}^-, & \lambda \in \mathbb{C}_-,\end{cases} \tag{3.16}$$

and that

$$\dim \mathfrak{D}(b,\lambda) = \begin{cases} \mathsf{i}^- + \mathsf{b}^+(\lambda) \ge \mathsf{i}^-, & \lambda \in \mathbb{C}_+,\\ \mathsf{i}^+ + \mathsf{b}^-(\lambda) \ge \mathsf{i}^+, & \lambda \in \mathbb{C}_-.\end{cases} \tag{3.17}$$

Under an additional condition Lemma 3.5 leads to a direct sum decomposition of $\mathbb{C}^n$ in terms of the eigenspaces of $D(a,\lambda)$ and $D(b,\lambda)$.

Proposition 3.7. *For $\lambda \in \mathbb{C}_+$ equivalent are*

(i) $\mathcal{A}^0(\lambda) \cap \mathcal{B}^0(\lambda) = \{0\}$;
(ii) $\mathbb{C}^n = (\mathcal{A}^+(\lambda) \oplus \mathcal{A}^0(\lambda)) + (\mathcal{B}^-(\lambda) \oplus \mathcal{B}^0(\lambda))$, *direct sums;*
(iii) $\mathbb{C}^n = (\mathcal{A}^-(\lambda) \oplus \mathcal{A}^\infty(\lambda)) + (\mathcal{B}^+(\lambda) \oplus \mathcal{B}^\infty(\lambda))$, *direct sums.*

For $\lambda \in \mathbb{C}_-$ equivalent are

(i)′ $\mathcal{A}^0(\lambda) \cap \mathcal{B}^0(\lambda) = \{0\}$;

(ii)′ $\mathbb{C}^n = (\mathcal{A}^-(\lambda) \oplus \mathcal{A}^0(\lambda)) + (\mathcal{B}^+(\lambda) \oplus \mathcal{B}^0(\lambda))$, *direct sums;*
(iii)′ $\mathbb{C}^n = (\mathcal{A}^+(\lambda) \oplus \mathcal{A}^\infty(\lambda)) + (\mathcal{B}^-(\lambda) \oplus \mathcal{B}^\infty(\lambda))$, *direct sums.*

Proof. Only the statements for $\lambda \in \mathbb{C}_+$ will be shown. A similar reasoning applies for $\lambda \in \mathbb{C}_-$.

(i) ⇒ (iii) Assume that $\phi \in \mathbb{C}^n$ is orthogonal to the set on the righthand side of (iii), that is,

$$\phi \in \big(\mathcal{A}^-(\lambda) \oplus \mathcal{A}^\infty(\lambda)\big)^\perp \cap \big(\mathcal{B}^+(\lambda) \oplus \mathcal{B}^\infty(\lambda)\big)^\perp = \big(\mathcal{A}^+(\lambda) \oplus \mathcal{A}^0(\lambda)\big) \cap \big(\mathcal{B}^-(\lambda) \oplus \mathcal{B}^0(\lambda)\big)$$

and hence $(D(b,\lambda)_s\phi,\phi) \le 0 \le (D(a,\lambda)_s\phi,\phi)$. On the other hand, for $\lambda \in \mathbb{C}_+$ the function $D(\cdot,\lambda)$ is monotonically increasing,

$$(D(a,\lambda)_s\phi,\phi) \le (D(t,\lambda)\phi,\phi) \le (D(b,\lambda)_s\phi,\phi), \qquad t \in \imath;$$

cf. (3.9) and (3.10). Hence $(D(a,\lambda)_s\phi,\phi) = 0 = (D(b,\lambda)_s\phi,\phi)$, so that $\phi \in \mathcal{A}^0(\lambda) \cap \mathcal{B}^0(\lambda)$ and assumption (i) implies $\phi = 0$. This shows that $\mathbb{C}^n$ can be written as in (iii). The fact that the sum is direct follows from a dimension argument, see Lemma 3.5.

(iii) ⇒ (ii) It follows from (iii) that $(\mathcal{A}^+(\lambda) \oplus \mathcal{A}^0(\lambda)) \cap (\mathcal{B}^-(\lambda) \oplus \mathcal{B}^0(\lambda))$ is trivial, hence the sum in (ii) is direct. Lemma 3.5 and a dimension argument imply (ii).

(ii) ⇒ (i) If (i) would not be true, then the sum in (ii) would not be direct. □

Assume now that for some $\lambda \in \mathbb{C}_+$ the condition

$$\mathcal{A}^0(\lambda) \cap \mathcal{B}^0(\lambda) = \{0\} = \mathcal{A}^0(\bar\lambda) \cap \mathcal{B}^0(\bar\lambda) \tag{3.18}$$

holds, or, equivalently, that

$$\mathcal{A}^\infty(\lambda) \cap \mathcal{B}^\infty(\lambda) = \{0\} = \mathcal{A}^\infty(\bar\lambda) \cap \mathcal{B}^\infty(\bar\lambda) \tag{3.19}$$

holds, see (3.14). Then by Proposition 3.7 there exist skewprojections $P_a(\lambda)$, $P_b(\lambda)$, $P_a(\bar\lambda)$, and $P_b(\bar\lambda)$ with

$$P_a(\lambda) + P_b(\lambda) = I = P_a(\bar\lambda) + P_b(\bar\lambda), \tag{3.20}$$

such that for $\lambda \in \mathbb{C}_+$

$$\begin{aligned} \operatorname{ran} P_a(\lambda) &= \ker\, P_b(\lambda) = \mathcal{A}^+(\lambda) \oplus \mathcal{A}^0(\lambda),\\ \ker\, P_a(\lambda) &= \operatorname{ran} P_b(\lambda) = \mathcal{B}^-(\lambda) \oplus \mathcal{B}^0(\lambda), \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} \operatorname{ran} P_a(\bar\lambda) &= \ker\, P_b(\bar\lambda) = \mathcal{A}^-(\bar\lambda) \oplus \mathcal{A}^0(\bar\lambda),\\ \ker\, P_a(\bar\lambda) &= \operatorname{ran} P_b(\bar\lambda) = \mathcal{B}^+(\bar\lambda) \oplus \mathcal{B}^0(\bar\lambda), \end{aligned} \tag{3.22}$$

hold.

Lemma 3.8. *Assume that the condition* (3.18) *holds for some* $\mu \in \mathbb{C} \setminus \mathbb{R}$ *and let* $P_a(\lambda)$ *and* $P_b(\lambda)$ *be the skewprojections in* $\mathbb{C}^n$ *defined in* (3.21) *and* (3.22) *for* $\lambda \in \{\mu,\bar\mu\}$. *Then for* $\lambda \in \{\mu,\bar\mu\}$ *the following hold:*

(i) $P_a(\lambda)^*D(t,\lambda)P_a(\lambda) \ge 0$ *and* $P_b(\lambda)^*D(t,\lambda)P_b(\lambda) \le 0$ *for all* $t \in \imath$;
(ii) $P_a(\bar\lambda)^*JP_a(\lambda) = 0$ *and* $P_b(\bar\lambda)^*JP_b(\lambda) = 0$;
(iii) $JP_a(\lambda) = P_b(\bar\lambda)^*J$ *and* $JP_b(\lambda) = P_a(\bar\lambda)^*J$;
(iv) $P_b(\lambda)JP_a(\bar\lambda)^* + P_a(\lambda)JP_b(\bar\lambda)^* = J$;
(v) $P_a(\bar\lambda)^*D(t,\bar\lambda)P_a(\bar\lambda) \le 0$ *and* $P_b(\bar\lambda)^*D(t,\bar\lambda)P_b(\bar\lambda) \ge 0$ *for all* $t \in \imath$.

Proof. (i), (ii) For $\lambda \in \mathbb{C}_+$ the inequality (3.11) yields

$$\big(D(t,\lambda)P_a(\lambda)\phi, P_a(\lambda)\phi\big) \ \ge\ \big(D(a,\lambda)_s P_a(\lambda)\phi, P_a(\lambda)\phi\big), \quad t \in \imath, \quad \phi \in \mathbb{C}^n,$$

and since $P_a(\lambda)\phi \in \mathcal{A}^+(\lambda) \oplus \mathcal{A}^0(\lambda)$, it follows that the term on the righthand side is nonnegative. A similar argument applies for $\lambda \in \mathbb{C}_-$ and the endpoint is b.

(iii) It suffices to show the first identity, which follows from (ii):

$$\begin{aligned} JP_a(\lambda) &= \big(P_a(\bar\lambda)^* + P_b(\bar\lambda)^*\big)\, JP_a(\lambda) = P_b(\bar\lambda)^*JP_a(\lambda)\\ &= P_b(\bar\lambda)^*J(P_a(\lambda) + P_b(\lambda)) = P_b(\bar\lambda)^*J. \end{aligned}$$

(iv) This follows from (iii)

$$\begin{aligned}\left(P_b(\lambda)JP_a(\bar\lambda)^* + P_a(\lambda)JP_b(\bar\lambda)^*\right) J &= P_b(\lambda)JJP_b(\lambda) + P_a(\lambda)JJP_a(\lambda)\\ &= -\left(P_b(\lambda) + P_a(\lambda)\right) = -I.\end{aligned}$$

(v) For $\phi, \psi \in \mathbb{C}^n$ the identity $(P_a(\bar\lambda)^* JP_a(\lambda)\phi, \psi) = (JP_a(\lambda)\phi, P_a(\bar\lambda)\psi)$ together with (3.14) and the definition of $P_a(\lambda)$ and $P_a(\bar\lambda)$ implies $P_a(\bar\lambda)^* JP_a(\lambda) = 0$. Similar considerations apply for the point b. □

3.3. **Square-integrable solutions of the homogeneous and inhomogeneous equation.** The square-integrability of the solutions of the canonical system (2.4) is intimately related to the limit relations $D(a,\lambda)$ and $D(b,\lambda)$ and their domains $\mathfrak{D}(a,\lambda)$ and $\mathfrak{D}(b,\lambda)$. In fact, it follows from (3.3), (3.4) and (3.5) that

$$\begin{aligned}\mathfrak{D}(a,\lambda) &= \left\{ \phi \in \mathbb{C}^n : \int_a^c \phi^* Y(s,\lambda)^* \Delta(s) Y(s,\lambda)\phi \, ds < \infty \right\},\\ \mathfrak{D}(b,\lambda) &= \left\{ \phi \in \mathbb{C}^n : \int_c^b \phi^* Y(s,\lambda)^* \Delta(s) Y(s,\lambda)\phi \, ds < \infty \right\},\end{aligned} \tag{3.23}$$

and these equalities do not depend on the choice of $c \in \imath$. Hence, $\phi \in \mathfrak{D}(a,\lambda)$ or $\phi \in \mathfrak{D}(b,\lambda)$ if and only if $Y(\cdot,\lambda)\phi$ is a solution of (2.5) which is square-integrable at a or b, respectively. Therefore, the number of solutions which are square-integrable at a or b coincides with the dimension of $\mathfrak{D}(a,\lambda)$ or $\mathfrak{D}(b,\lambda)$, respectively. In particular, the number of solutions of (2.5) which are square-integrable on $\imath$ coincides with the dimension of $\mathfrak{D}(a,\lambda) \cap \mathfrak{D}(b,\lambda)$. Under the assumption (3.18) this dimension will be specified in Theorem 3.10 below. Incidentally, the usual condition of definiteness of the canonical system implies condition (3.18).

Lemma 3.9. *Assume that the canonical system* (2.4) *is definite on* $\imath$. *Then the condition* (3.18) *is satisfied for all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and let $\phi \in \mathcal{A}^0(\lambda) \cap \mathcal{B}^0(\lambda)$. Then

$$(D(a,\lambda)_s\phi, \phi) = 0 = (D(b,\lambda)_s\phi, \phi)$$

and the monotonicity of $D(\cdot,\lambda)$ implies $(D(t,\lambda)\phi,\phi) = 0$ for $t \in \imath$. Therefore, (3.3) and (3.23) yield

$$\int_\imath \phi^* Y(s,\lambda)^* \Delta(s) Y(s,\lambda)\phi \, ds = 0.$$

Since the canonical system is assumed to be definite this implies $\phi = 0$. □

Theorem 3.10. *Assume that the condition* (3.18) *holds for some* $\mu \in \mathbb{C} \setminus \mathbb{R}$. *Then the numbers of linearly independent solutions of* (2.5) *which are square-integrable (with respect to* Δ*) at both endpoints* a *and* b *are for* $\lambda \in \{\mu, \bar\mu\}$ *given by*

$$\dim\left(\mathfrak{D}(a,\lambda) \cap \mathfrak{D}(b,\lambda)\right) = \begin{cases} \mathsf{a}^-(\lambda) + \mathsf{b}^+(\lambda), & \lambda \in \mathbb{C}_+,\\ \mathsf{a}^+(\lambda) + \mathsf{b}^-(\lambda), & \lambda \in \mathbb{C}_-.\end{cases}$$

In particular, if the canonical system (2.4) *is definite on* $\imath$, *then the preceding equality holds for all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. Recall that

$$\mathfrak{D}(a,\lambda) \cap \mathfrak{D}(b,\lambda) = \left(\mathcal{A}^\infty(\lambda) + \mathcal{B}^\infty(\lambda)\right)^\perp; \tag{3.24}$$

cf. (3.13). Moreover, $\dim\left(\mathcal{A}^\infty(\lambda) + \mathcal{B}^\infty(\lambda)\right) = \mathsf{a}^\infty(\lambda) + \mathsf{b}^\infty(\lambda)$ for $\lambda \in \{\mu, \bar\mu\}$, see (3.18) and (3.19). Hence (3.24) implies that for $\lambda \in \{\mu, \bar\mu\}$

$$\dim\left(\mathfrak{D}(a,\lambda) \cap \mathfrak{D}(b,\lambda)\right) = n - \mathsf{a}^\infty(\lambda) - \mathsf{b}^\infty(\lambda) = \mathsf{i}^+ + \mathsf{i}^- - \mathsf{a}^\infty(\lambda) - \mathsf{b}^\infty(\lambda)$$

and the statement follows from Lemma 3.5. □

Assume that the condition (3.18) holds for some $\mu \in \mathbb{C} \setminus \mathbb{R}$, or, more specifically, that the canonical system (2.4) is definite on $\imath$. Then Lemma 3.9 ensures that the skewprojections $P_a(\lambda)$, $P_b(\lambda)$ in (3.20)-(3.22) exist for $\lambda \in \{\mu, \bar\mu\}$. These projections lead to solutions of (2.5) which are square-integrable near the endpoints: for each $c \in \imath$ one has $P_a(\lambda)\phi \in \mathfrak{D}(a, \lambda)$, $P_b(\lambda)\phi \in \mathfrak{D}(b, \lambda)$, and

$$Y(\cdot, \lambda)P_a(\lambda)\phi \in \mathcal{L}^2_\Delta(a, c), \quad Y(\cdot, \lambda)P_b(\lambda)\phi \in \mathcal{L}^2_\Delta(c, b), \quad \phi \in \mathbb{C}^n. \tag{3.25}$$

These functions provide i^+ or i^- square-integrable solutions at a and i^- or i^+ square-integrable solutions at b if $\lambda \in \mathbb{C}_+$ or $\lambda \in \mathbb{C}_-$, respectively; see Lemma 3.5, (3.21) and (3.22).

For a function $g \in \mathcal{L}^2_\Delta(\imath)$ define the functions $\mathcal{G}(\lambda)g$, $\lambda \in \{\mu, \bar\mu\}$, by

$$\begin{aligned}(\mathcal{G}(\lambda)g)(t) &= Y(t, \lambda)P_a(\lambda)J \int_t^b P_b(\bar\lambda)^* Y(s, \bar\lambda)^*\Delta(s)g(s)\, ds \\ &\quad - Y(t, \lambda)P_b(\lambda)J \int_a^t P_a(\bar\lambda)^* Y(s, \bar\lambda)^*\Delta(s)g(s)\, ds.\end{aligned} \tag{3.26}$$

It follows from (3.25) that the integrals, and hence the function $\mathcal{G}(\lambda)g$ is well defined for $\lambda \in \{\mu, \bar\mu\}$. In the next proposition it is shown that the constructions in [46, 56] given for a definite canonical system remain valid under the weaker geometric condition (3.18). Since it is fundamental for the rest of the paper a full proof is included for the convenience of the reader.

Proposition 3.11. *Assume that the condition* (3.18) *holds for some* $\mu \in \mathbb{C} \setminus \mathbb{R}$ *and let* $g, k \in \mathcal{L}^2_\Delta(\imath)$. *Then for* $\lambda \in \{\mu, \bar\mu\}$

(i) $\mathcal{G}(\lambda)g \in AC_{\mathrm{loc}}(\imath)$ *is a solutions of* (2.4)*;*
(ii) $\mathcal{G}(\lambda)g \in \mathcal{L}^2_\Delta(\imath)$ *and* $\|\mathcal{G}(\lambda)g\|_\Delta \leq (1/\mathrm{Im}\,\lambda)\|g\|_\Delta$*;*
(iii) $(\mathcal{G}(\lambda)g, k)_\Delta = (g, \mathcal{G}(\bar\lambda)k)_\Delta$.

In particular, if the canonical system (2.4) *is definite on* $\imath$, *then the preceding statement holds for all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. The following notation will be useful in this proof. For a compact interval $I \subset \imath$ let

$$(f, g)_{\Delta, I} = \int_I g(s)^*\Delta(s)f(s)\, ds, \quad f, g \in \mathcal{L}^2_{\Delta,\mathrm{loc}}(\imath),$$

and denote the corresponding seminorm by $\|\cdot\|_{\Delta, I}$.

Step 1. For any $g \in \mathcal{L}^2_\Delta(\imath)$ the integrands in the definition of $\mathcal{G}(\lambda)g$ are square-integrable near the respective endpoints, so that the function $\mathcal{G}(\lambda)g$ belongs to $AC_{\mathrm{loc}}(\imath)$. The function $\mathcal{G}(\lambda)g$ can be written as $(\mathcal{G}(\lambda)g)(t) = Y(t, \lambda)(\mathcal{F}(\lambda)g)(t)$, where the function $\mathcal{F}(\lambda)g$ is defined by

$$\begin{aligned}(\mathcal{F}(\lambda)g)(t) &= P_a(\lambda)J \int_t^b P_b(\bar\lambda)^* Y(s, \bar\lambda)^*\Delta(s)g(s)\, ds \\ &\quad - P_b(\lambda)J \int_a^t P_a(\bar\lambda)^* Y(s, \bar\lambda)^*\Delta(s)g(s)\, ds.\end{aligned} \tag{3.27}$$

Therefore, it is clear that

$$(\mathcal{G}(\lambda)g)'(t) = Y'(t, \lambda)(\mathcal{F}(\lambda)g)(t) + Y(t, \lambda)(\mathcal{F}(\lambda)g)'(t). \tag{3.28}$$

Observe that with (3.27), Lemma 3.8 (iv), and the identity (2.7)

$$\begin{aligned}Y(t, \lambda)(\mathcal{F}(\lambda)g)'(t) &= -Y(t, \lambda)\left[P_a(\lambda)JP_b(\bar\lambda)^* + P_b(\lambda)JP_a(\bar\lambda)^*\right] Y(t, \bar\lambda)^*\Delta(t)g(t) \\ &= -Y(t, \lambda)JY(t, \bar\lambda)^*\Delta(t)g(t) = -J\Delta(t)g(t).\end{aligned} \tag{3.29}$$

Hence, due to (3.28) and (3.29), and the definition of $Y(\cdot, \lambda)$, it follows that

$$\begin{aligned}J(\mathcal{G}(\lambda)g)' - H(\mathcal{G}(\lambda)g) &= \left[JY'(\cdot, \lambda) - HY(\cdot, \lambda)\right](\mathcal{F}(\lambda)g) + \Delta g \\ &= \lambda\Delta Y(\cdot, \lambda)\mathcal{F}(\lambda)g + \Delta g = \lambda\Delta(\mathcal{G}(\lambda)g) + \Delta g,\end{aligned}$$

which completes the proof of (i).

Step 2. Assume that $g \in \mathcal{L}^2_\Delta(\imath)$ has compact support and let $I = [\alpha, \beta] \subset \imath$ be any compact interval containing the support of g. By Step 1 the function $\mathcal{G}(\lambda)g$ is a solution of (2.4). Hence, it follows from Lemma 2.1 (with $\mu = \lambda$) that

$$\begin{aligned}(\lambda - \bar\lambda)\|\mathcal{G}(\lambda)g\|^2_{\Delta,I} &= (\mathcal{G}(\lambda)g, g)_{\Delta,I} - (g, \mathcal{G}(\lambda)g)_{\Delta,I} \\ &\quad + (\mathcal{G}(\lambda)g)(\beta)^* J(\mathcal{G}(\lambda)g)(\beta) - (\mathcal{G}(\lambda)g)(\alpha)^* J(\mathcal{G}(\lambda)g)(\alpha).\end{aligned} \tag{3.30}$$

From the definition of $\mathcal{G}(\lambda)g$ in (3.26) one obtains that

$$(\mathcal{G}(\lambda)g)(\alpha) = Y(\alpha, \lambda)P_a(\lambda)\gamma_\alpha, \quad (\mathcal{G}(\lambda)g)(\beta) = Y(\beta, \lambda)P_b(\lambda)\gamma_\beta,$$

for some $\gamma_\alpha, \gamma_\beta \in \mathbb{C}^n$. Therefore, it follows from Lemma 3.8 (i) that

$$\frac{(\mathcal{G}(\lambda)g)(\alpha)^* J(\mathcal{G}(\lambda)g)(\alpha)}{\lambda - \bar\lambda} = \gamma_\alpha^* P_a(\lambda)^* \frac{D(\alpha, \lambda)}{2\mathrm{Im}\,\lambda} P_a(\lambda)\gamma_\alpha \geq 0$$

and

$$\frac{(\mathcal{G}(\lambda)g)(\beta)^* J(\mathcal{G}(\lambda)g)(\beta)}{\lambda - \bar\lambda} = \gamma_\beta^* P_b(\lambda)^* \frac{D(\beta, \lambda)}{2\mathrm{Im}\,\lambda} P_b(\lambda)\gamma_\beta \leq 0$$

hold. It follows from these inequalities and (3.30) that

$$\|\mathcal{G}(\lambda)g\|^2_{\Delta,I} \leq \frac{(\mathcal{G}(\lambda)g, g)_{\Delta,I} - (g, \mathcal{G}(\lambda)g)_{\Delta,I}}{\lambda - \bar\lambda} \leq \frac{1}{\mathrm{Im}\,\lambda}\|\mathcal{G}(\lambda)g\|_{\Delta,I}\|g\|_{\Delta,I},$$

which leads to

$$\|\mathcal{G}(\lambda)g\|_{\Delta,I} \leq \frac{1}{\mathrm{Im}\,\lambda}\|g\|_{\Delta,I}.$$

Observe that $\|g\|_{\Delta,I} = \|g\|_\Delta$ since g has support in I. Hence

$$\|\mathcal{G}(\lambda)g\|_{\Delta,I} \leq \frac{1}{\mathrm{Im}\,\lambda}\|g\|_\Delta$$

holds for any compact interval I containing the support of g. Let I_m be a monotonically increasing sequence of compact intervals such that their union equals $\imath$. Then the monotone convergence theorem implies

$$\|\mathcal{G}(\lambda)g\|_\Delta \leq \frac{1}{\mathrm{Im}\,\lambda}\|g\|_\Delta \tag{3.31}$$

for all $g \in \mathcal{L}^2_\Delta(\imath)$ with compact support. In particular, $\mathcal{G}(\lambda)g \in \mathcal{L}^2_\Delta(\imath)$.

Step 3. Let $g \in \mathcal{L}^2_\Delta(\imath)$ and let I_m be a monotonically increasing sequence of compact intervals such that their union equals $\imath$. Denote by $g_m \in \mathcal{L}^2_\Delta(\imath)$ the function that equals g on I_m and is 0 outside I_m. It follows from the Cauchy-Schwarz inequality and (2.1) that for each fixed $t \in \imath$

$$\int_t^b P_b(\bar\lambda)^* Y(s, \bar\lambda)^* \Delta(s) g_m(s)\, ds \;\to \int_t^b P_b(\bar\lambda)^* Y(s, \bar\lambda)^* \Delta(s) g(s)\, ds$$

and

$$\int_a^t P_a(\bar\lambda)^* Y(s, \bar\lambda)^* \Delta(s) g_m(s)\, ds \;\to \int_a^t P_a(\bar\lambda)^* Y(s, \bar\lambda)^* \Delta(s) g(s)\, ds$$

as $m \to \infty$. Therefore $(\mathcal{G}(\lambda)g_m)(t)$ tends to $(\mathcal{G}(\lambda)g)(t)$ for each fixed $t \in \imath$. Hence for almost every $t \in \imath$

$$(\mathcal{G}(\lambda)g_m)(t)^* \Delta(t)(\mathcal{G}(\lambda)g_m)(t) \to (\mathcal{G}(\lambda)g)(t)^* \Delta(t)(\mathcal{G}(\lambda)g)(t). \tag{3.32}$$

It follows from (3.31) in Step 2 that

$$\begin{aligned}\int_\imath (\mathcal{G}(\lambda)g_m)(s)^* \Delta(s)(\mathcal{G}(\lambda)g_m)(s)\, ds &\leq \frac{1}{\mathrm{Im}\,\lambda}\int_\imath g_m(s)^* \Delta(s) g_m(s)\, ds \\ &\leq \frac{1}{\mathrm{Im}\,\lambda}\int_\imath g(s)^* \Delta(s) g(s)\, ds < \infty\end{aligned} \tag{3.33}$$

for all $m \in \mathbb{N}$. Since the functions $(\mathcal{G}(\lambda)g_m)^* \Delta(\mathcal{G}(\lambda)g_m)$ are nonnegative, it follows from (3.32) and (3.33) in connection with Fatou's lemma (cf. [26]) that

$$\int_\imath (\mathcal{G}(\lambda)g)(s)^* \Delta(s)(\mathcal{G}(\lambda)g)(s)\, ds \leq \frac{1}{\mathrm{Im}\,\lambda}\int_\imath g(s)^* \Delta(s) g(s)\, ds\, (< \infty).$$

Hence it has been shown that for every $g \in \mathcal{L}^2_\Delta(\imath)$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the function $\mathcal{G}(\lambda)g$ belongs to $\mathcal{L}^2_\Delta(\imath)$ and that

$$\|\mathcal{G}(\lambda)g\|_\Delta \leq \frac{1}{\operatorname{Im}\lambda}\|g\|_\Delta. \tag{3.34}$$

This completes the proof of (ii).

Step 4. Finally, let the functions $g, k \in \mathcal{L}^2_\Delta(\imath)$ have compact support in $I = [\alpha, \beta] \subset \imath$. Since the functions $\mathcal{G}(\lambda)g$ and $\mathcal{G}(\bar\lambda)k$ are solutions of the inhomogeneous canonical system (2.4) with λ and $\bar\lambda$, and with g replaced by g and k, respectively, it follows from Lemma 2.1 (with $\mu = \bar\lambda$) that

$$\begin{aligned}(\mathcal{G}(\lambda)g, k)_{\Delta,I} &- (g, \mathcal{G}(\bar\lambda)k)_{\Delta,I} \\ &= (\mathcal{G}(\bar\lambda)k)(\alpha)^* J\mathcal{G}(\lambda)g(\alpha) - (\mathcal{G}(\bar\lambda)k)(\beta)^* J\mathcal{G}(\lambda)g(\beta).\end{aligned} \tag{3.35}$$

From the definition of $\mathcal{G}(\lambda)g$ in (3.26) one obtains that

$$(\mathcal{G}(\lambda)g)(\alpha) = Y(\alpha,\lambda)P_a(\lambda)\gamma_{g,\alpha}, \quad (\mathcal{G}(\lambda)g)(\beta) = Y(\beta,\lambda)P_b(\lambda)\gamma_{g,\beta},$$
$$(\mathcal{G}(\bar\lambda)k)(\alpha) = Y(\alpha,\bar\lambda)P_a(\bar\lambda)\gamma_{k,\alpha}, \quad (\mathcal{G}(\bar\lambda)k)(\beta) = Y(\beta,\bar\lambda)P_b(\bar\lambda)\gamma_{k,\beta},$$

where $\gamma_{g,\alpha}$, $\gamma_{g,\beta}$, $\gamma_{k,\alpha}$, $\gamma_{k,\beta} \in \mathbb{C}^n$. Therefore (2.7) and Lemma 3.8 (v) imply that

$$(\mathcal{G}(\bar\lambda)k)(\alpha)^* J\mathcal{G}(\lambda)g(\alpha) = 0, \quad (\mathcal{G}(\bar\lambda)k)(\beta)^* J\mathcal{G}(\lambda)g(\beta) = 0.$$

It follows from these identities and (3.35) that

$$(\mathcal{G}(\lambda)g, k)_{\Delta,I} = (g, \mathcal{G}(\bar\lambda)k)_{\Delta,I}$$

for all functions $g, k \in \mathcal{L}^2_\Delta(\imath)$ with compact support on $I = [\alpha, \beta] \subset \imath$. Therefore

$$(\mathcal{G}(\lambda)g, k)_\Delta = (g, \mathcal{G}(\bar\lambda)k)_\Delta \tag{3.36}$$

for all functions $g, k \in \mathcal{L}^2_\Delta(\imath)$ with compact support. Now let g, k be any functions in $\mathcal{L}^2_\Delta(\imath)$ and approximate them by square-integrable functions with compact support. Then it follows from the approximation property (2.1), (3.34), and (3.36) that $(\mathcal{G}(\lambda)g, k)_\Delta = (g, \mathcal{G}(\bar\lambda)k)_\Delta$. This completes the proof of (iii). □

4. Maximal and minimal relations for singular canonical system

In this section the maximal and minimal relation associated with the definite canonical system (2.4) in the Hilbert space $L^2_\Delta(\imath)$ are investigated. The approach to canonical systems via linear relations goes back to [49], see also [33, 34] and [24, 42]. The minimal relation is closed and symmetric, and that its adjoint is the maximal relation. Hence the defect numbers of the minimal relation are constant in the upper halfplane and in the lower halfplane, which is equivalent to the number of square-integrable solutions of (2.5) being constant in each halfplane. Furthermore, the technique from Section 2.5 is applied to obtain a decomposition of the maximal relation in terms of cut-off solutions of the homogeneous equation (2.5) which is inspired by the treatment in [27]. If, in addition, the endpoints of $\imath$ are quasiregular or in the limit-point case (see Definition 4.18) this yields special forms of the maximal and minimal relations, and their defect spaces. It is stressed that from now on the canonical system is assumed to be definite.

4.1. **Maximal and minimal relations associated to singular canonical systems.** The semidefinite space $\mathcal{L}^2_\Delta(\imath)$ as considered in the previous sections gives rise to the Hilbert space $L^2_\Delta(\imath)$ which consists of the equivalence classes of elements from $\mathcal{L}^2_\Delta(\imath)$ with respect to the seminorm. The scalar product in $L^2_\Delta(\imath)$ is denoted by $(\cdot,\cdot)_\Delta$. For more information concerning these spaces, see [32] and the expositions in [1, Sections 1.4 and 8.6] and [19, p.1350].

In the Hilbert space $L^2_\Delta(\imath)$ the canonical system (2.4) induces the *maximal relation* $T_{\max}$, defined by

$$T_{\max} = \{ \{f, g\} \in L^2_\Delta(\imath) \times L^2_\Delta(\imath) : Jf' - Hf = \Delta g \}.$$

The corresponding *minimal relation* $T_{\min}$ is defined in terms of $T_{\max}$ by

$$T_{\min} = T_{\max} \cap T^*_{\max}. \tag{4.1}$$

The definition of $T_{\max}$ needs to be explained: an element $\{f,g\} \in L^2_\Delta(\imath) \times L^2_\Delta(\imath)$ belongs to $T_{\max}$ if and only if the equivalence class f contains a locally absolutely continuous representative $\widetilde{f}$ such that the inhomogeneous equation $J\widetilde{f}'(t) - H(t)\widetilde{f}(t) = \Delta(t)\widetilde{g}(t)$ is satisfied for almost every $t \in \imath$. Here $\widetilde{g}$ is any representative of $g \in L^2_\Delta(\imath)$ (observe that $\Delta\widetilde{g}$ is independent of the representative of g).

Due to the standing assumption that the canonical system (2.4) is definite, the following useful property holds. A proof is included for completeness; cf. [49]

Lemma 4.1. *If $\{f,g\} \in T_{\max}$, then the equivalence class f has a unique locally absolutely continuous representative.*

Proof. Let $\{f,g\} \in T_{\max}$ and let $\tilde{f}_1$ and $\tilde{f}_2$ be locally absolutely continuous representatives of f. Then $J(\tilde{f}_1 - \tilde{f}_2)' - H(\tilde{f}_1 - \tilde{f}_2) = 0$ holds and

$$\int_\imath (\tilde{f}_1 - \tilde{f}_2)(s)^* \Delta(s)(\tilde{f}_1 - \tilde{f}_2)(s)\, ds = 0.$$

Therefore, by Lemma 2.10 it follows that $\tilde{f}_1(t) = \tilde{f}_2(t)$ for all $t \in \imath$. □

The eigenspace of $T_{\max}$ at $\lambda \in \mathbb{C}$ is defined by $\mathfrak{N}_\lambda(T_{\max}) = \ker(T_{\max} - \lambda)$. With $\mathfrak{N}_\lambda(T_{\max})$ one associates the subspace

$$\widehat{\mathfrak{N}}_\lambda(T_{\max}) = \big\{\{f_\lambda, \lambda f_\lambda\} : f_\lambda \in \mathfrak{N}_\lambda(T_{\max})\big\}, \quad \lambda \in \mathbb{C}.$$

If $\{f_\lambda, \lambda f_\lambda\} \in \widehat{\mathfrak{N}}_\lambda(T_{\max})$, then by definition there exists a unique representative $\widetilde{f}_\lambda \in AC_{\mathrm{loc}}(\imath)$ of f_λ such that $J\widetilde{f}_\lambda' - H\widetilde{f}_\lambda = \lambda\Delta\widetilde{f}_\lambda$. In other words, $\widetilde{f}_\lambda$ is a square-integrable solution of the homogeneous equation (2.5). Conversely, every square-integrable solution of the homogeneous equation (2.5) is the unique representative in $AC_{\mathrm{loc}}(\imath)$ of its equivalence class. Therefore, the eigenspace $\mathfrak{N}_\lambda(T_{\max})$ of $T_{\max}$ is made up of the (equivalence classes of) square-integrable solutions of the homogeneous equation (2.5):

$$\mathfrak{N}_\lambda(T_{\max}) = \big\{\, Y(\cdot,\lambda)\phi : \phi \in \mathfrak{D}(a,\lambda) \cap \mathfrak{D}(b,\lambda) \,\big\}; \tag{4.2}$$

cf. (3.23) and Theorem 3.10. Clearly, the preceding identity shows that

$$\dim \mathfrak{N}_\lambda(T_{\max}) \leq n. \tag{4.3}$$

In particular, the eigenspace $\mathfrak{N}_\lambda(T_{\max})$ and, hence, also the space $\widehat{\mathfrak{N}}_\lambda(T_{\max})$ is closed for every $\lambda \in \mathbb{C}$.

In order to show the connection between the minimal and maximal relation, the operator $\mathcal{G}(\lambda)$ defined in (3.26) on the seminormed space $\mathcal{L}^2_\Delta(\imath)$ needs to be lifted to an operator on the Hilbert space $L^2_\Delta(\imath)$. Therefore let $g \in L^2_\Delta(\imath)$ and let $\widetilde{g} \in \mathcal{L}^2_\Delta(\imath)$ be an element in the equivalence class g. Since the canonical system (2.4) is definite on $\imath$, Proposition 3.11 implies that $\mathcal{G}(\lambda)\widetilde{g}$ belongs to $AC_{\mathrm{loc}}(\imath) \cap \mathcal{L}^2_\Delta(\imath)$ and satisfies

$$J(\mathcal{G}(\lambda)\widetilde{g})' - H(\mathcal{G}(\lambda)\widetilde{g}) = \lambda\Delta(\mathcal{G}(\lambda)\widetilde{g}) + \Delta\widetilde{g} \tag{4.4}$$

for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The definition of the operator $\mathcal{G}(\lambda)$ in (3.26) implies that $\mathcal{G}(\lambda)\widetilde{g}$ remains the same when $\widetilde{g} \in \mathcal{L}^2_\Delta(\imath)$ is replaced by $\widetilde{h} \in \mathcal{L}^2_\Delta(\imath)$ which is in the same equivalence class; since then $\Delta(\widetilde{g} - \widetilde{h}) = 0$. Denote by f the equivalence class in $L^2_\Delta(\imath)$ to which $\mathcal{G}(\lambda)\widetilde{g} \in \mathcal{L}^2_\Delta(\imath)$ belongs and set

$$G(\lambda)g := f. \tag{4.5}$$

Clearly, this procedure defines an operator $G(\lambda)$ in $L^2_\Delta(\imath)$. Moreover, by (4.4) and Lemma 4.1 $\mathcal{G}(\lambda)\widetilde{g}$ is the unique representative of $G(\lambda)g$ that belongs to $AC_{\mathrm{loc}}(\imath)$. Hence the following result is obtained by reformulation Proposition 3.11 into the context of the Hilbert space $L^2_\Delta(\imath)$. Observe that the definiteness of the canonical system implies that the statements in Proposition 3.11 hold for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proposition 4.2. *Let $G(\lambda)$ be the linear mapping in $L^2_\Delta(\imath)$ defined in (4.5) for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then $G(\lambda)$ is a bounded everywhere defined operator in $L^2_\Delta(\imath)$, $G(\lambda)^* = G(\bar\lambda)$ and*

$$\{G(\lambda)g, (I + \lambda G(\lambda))g\} \in T_{\max}, \quad g \in L^2_\Delta(\imath).$$

As a consequence of the preceding preparations, the abstract result given in Proposition A.2 shows that the minimal and maximal relations are each others adjoints. This leads to a von Neumann decomposition of the maximal relation in terms of the minimal relation and the defect subspaces of the maximal relation; cf. [46, 49] for the corresponding decomposition of the domains.

Theorem 4.3. *The minimal relation $T_{\min}$ is a closed symmetric relation in $L^2_\Delta(\imath)$ and $T^*_{\min} = T_{\max}$ holds. Moreover, $T_{\max}$ has the following componentwise sum decomposition:*

$$T_{\max} = T_{\min} \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(T_{\max}) \,\widehat{+}\, \widehat{\mathfrak{N}}_{\bar\lambda}(T_{\max}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad \textit{direct sums.}$$

Proof. Since the eigenspace $\mathfrak{N}_\lambda(T_{\max})$ is closed, it follows from Proposition 4.2 that the operator $G(\lambda)$ in (4.5) satisfies the assumptions of Proposition A.2. Hence, the relation $T_{\min} = T_{\max} \cap T^*_{\max}$ in (4.1) is closed and it is the adjoint of $T_{\max}$. The asserted decomposition of $T_{\max}$ is therefore just the von Neumann decomposition for $T^*_{\min} = T_{\max}$; cf. Proposition A.1. □

Example 4.4 (Weighted Sturm-Liouville equations)**.** Let $1/p, q, r \in \mathcal{L}^1_{\mathrm{loc}}(\imath)$ be real-valued functions and assume that there exists an interval $\jmath \subset \imath$ such that $r(t) > 0$ for $t \in \jmath$. Then the associated canonical system with $n = 2$ and with J, H, and Δ defined by (2.13) is definite; cf. Example 2.12. Define the space $\mathcal{L}^2_r(\imath)$ of all measurable functions φ for which

$$\int_\imath \varphi(s)^* r(s) \varphi(s)\, ds < \infty.$$

The corresponding semi-inner product is denoted by $(\cdot,\cdot)_r$ and the corresponding Hilbert space of equivalence classes of elements from $\mathcal{L}^2_r(\imath)$ is denoted by $L^2_r(\imath)$. For $\widetilde{f} \in \mathcal{L}^2_\Delta(\imath)$ write

$$\widetilde{f}(t) = \begin{pmatrix} \widetilde{f}_1(t) \\ \widetilde{f}_2(t) \end{pmatrix};$$

then it is clear that

$$(\widetilde{f}, \widetilde{f})_\Delta = \int_\imath \left(\widetilde{f}_1(s)^* \ \widetilde{f}_2(s)^*\right) \begin{pmatrix} r(s) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{f}_1(s) \\ \widetilde{f}_2(s) \end{pmatrix} ds = (\widetilde{f}_1, \widetilde{f}_1)_r.$$

Hence the mapping R taking $\widetilde{f} \in \mathcal{L}^2_\Delta(\imath)$ to $\widetilde{f}_1 \in \mathcal{L}^2_r(\imath)$ is an isometry in the sense of the semi-inner products. It is clear that this mapping is onto, since each function in $\mathcal{L}^2_r(\imath)$ can be seen as the first component of an element in $\mathcal{L}^2_\Delta(\imath)$ with the understanding that the second component can be any measurable function. Furthermore, it is clear that R induces an isometry, again denoted by R, from $L^2_\Delta(\imath)$ onto $L^2_r(\imath)$.

In the Hilbert space $L^2_r(\imath)$ define the maximal relation $\mathcal{T}_{\max}$ as follows:

$$\mathcal{T}_{\max} = \left\{ \{F, G\} \in L^2_r(\imath) \times L^2_r(\imath) : -(pF')' + qF = rG \right\},$$

in the sense that there exist representatives $\widetilde{F}$ and $\widetilde{G} \in \mathcal{L}^2_r(\imath)$ of F and G, respectively, such that $\widetilde{F} \in AC_{\mathrm{loc}}(\imath)$, $p\widetilde{F}' \in AC_{\mathrm{loc}}(\imath)$, and

$$-(p\widetilde{F}')' + q\widetilde{F} = r\widetilde{G}.$$

It is clear that if $\{f, g\} \in T_{\max}$, then there exist representatives $\widetilde{f}, \widetilde{g} \in \mathcal{L}^2_\Delta(\imath)$ with $\widetilde{f} \in AC_{\mathrm{loc}}(\imath)$ and $\widetilde{g}$, such that

$$J\widetilde{f}' - H\widetilde{f} = \Delta\widetilde{g},$$

which leads to the equations

$$-\widetilde{f}_2' + q\widetilde{f}_1 = r\widetilde{g}_1 \qquad \text{and} \qquad \widetilde{f}_1' - (1/p)\widetilde{f}_2 = 0.$$

Hence, the pair $\{\widetilde{f}_1, \widetilde{g}_1\}$ in $\mathcal{L}^2_r(\imath) \times \mathcal{L}^2_r(\imath)$ generates an element in $\mathfrak{T}_{\max}$ and, moreover, each element in $\mathfrak{T}_{\max}$ is obtained in this way. Hence $\{f, g\} \mapsto \{Rf, Rg\}$ maps $T_{\max}$ bijectively onto $\mathfrak{T}_{\max}$. In particular, R maps ker $(T_{\max} - \lambda)$ one-to-one onto ker $(\mathfrak{T}_{\max} - \lambda)$. Since the functions p, q, and r are real it follows that the defect numbers are equal.

Remark 4.5. In the rest of this paper the distinction between equivalence classes and their representatives will not be made explicit as long as no confusion arises. In particular, to all elements $\{f, g\} \in T_{\max}$ one can associate unique boundary values, in the extended complex plane, by means of the limits to the boundary points of the unique locally absolutely continuous representative of f, see Lemma 4.1.

4.2. **Defect numbers of the minimal relation.** Since $T_{\min}$ is symmetric, it follows from the general theory of linear relations that the defect numbers of $T_{\min}$ are constant in the upper halfplane and in the lower halfplane; see Section 7. Hence

$$\begin{aligned} n_+(T_{\min}) &= \dim \mathfrak{N}_\lambda(T_{\max}), \quad \lambda \in \mathbb{C}_-, \\ n_-(T_{\min}) &= \dim \mathfrak{N}_\lambda(T_{\max}), \quad \lambda \in \mathbb{C}_+. \end{aligned}$$

On the other hand, it follows from (4.2) and Theorem 3.10 that

$$\dim \mathfrak{N}_\lambda(T_{\max}) = \begin{cases} \mathsf{a}^-(\lambda) + \mathsf{b}^+(\lambda), & \lambda \in \mathbb{C}_+, \\ \mathsf{a}^+(\lambda) + \mathsf{b}^-(\lambda), & \lambda \in \mathbb{C}_-, \end{cases} \tag{4.6}$$

where $\mathsf{a}^\pm(\lambda)$ and $\mathsf{b}^\pm(\lambda)$ are the dimensions of the eigenspaces of the limit relations $D(a, \lambda)$ and $D(b, \lambda)$ corresponding to the positive and negative eigenvalues; cf. Section 3.2. The preceding observations lead to the following proposition.

Proposition 4.6. *The following statements hold:*

(i) $\mathsf{a}^-(\lambda) + \mathsf{b}^+(\lambda)$ *is constant for* $\lambda \in \mathbb{C}_+$*;*
(ii) $\mathsf{a}^+(\lambda) + \mathsf{b}^-(\lambda)$ *is constant for* $\lambda \in \mathbb{C}_-$*.*

The above proposition is based on the connection of the numbers $\mathsf{a}^+(\lambda)$, $\mathsf{a}^-(\lambda)$, $\mathsf{b}^+(\lambda)$, $\mathsf{b}^-(\lambda)$ (which have been defined strictly in terms of the canonical system) to the defect numbers of a symmetric relation in a Hilbert space; a different proof of Proposition 4.6 can be found in [38]. In addition, the following proposition gives similar results concerning the dimensions of the individual eigenspaces of the limit relations $D(a, \lambda)$ and $D(b, \lambda)$. These results can be seen as consequences of Proposition 4.6 and hence are based on general principles, see [3, 4] or [50, 56] for a different point of view. Statement (ii) of Proposition 4.7 is known as Weyl's first theorem; cf. [25].

Proposition 4.7. *The following statements hold:*

(i) $\mathsf{a}^+(\lambda)$*,* $\mathsf{a}^-(\lambda)$*,* $\mathsf{b}^+(\lambda)$*, and* $\mathsf{b}^-(\lambda)$ *are constant for* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*;*
(ii) $\mathsf{a}^0(\lambda)$*,* $\mathsf{a}^\infty(\lambda)$*,* $\mathsf{b}^0(\lambda)$*, and* $\mathsf{b}^\infty(\lambda)$ *are constant for* $\lambda \in \mathbb{C}_+$ *and* $\lambda \in \mathbb{C}_-$*;*
(iii) $\mathsf{a}^0(\lambda) = \mathsf{a}^\infty(\bar{\lambda})$ *and* $\mathsf{b}^0(\lambda) = \mathsf{b}^\infty(\bar{\lambda})$ *for* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*.*

Proof. Since the canonical system (2.4) is assumed to be definite on $\imath$, it follows from Proposition 2.11 that there exists a compact interval $[c, d] \subset \imath$ such that the canonical system is definite on the interval $[c, d]$. Hence the canonical system is also definite on the interval $(a, d]$ and on the interval $[c, b)$.

(i) As the canonical system is definite on $(a, d]$, Proposition 4.6 may be applied when the underlying interval is (a, d). This leads to

$$\begin{aligned} &\mathsf{a}^-(\lambda) + \mathsf{d}^+(\lambda) \quad \text{constant for} \quad \lambda \in \mathbb{C}_+, \\ &\mathsf{a}^+(\lambda) + \mathsf{d}^-(\lambda) \quad \text{constant for} \quad \lambda \in \mathbb{C}_-, \end{aligned}$$

with an obvious interpretation of the quantities $\mathsf{d}^+(\lambda)$ and $\mathsf{d}^-(\lambda)$. Since d is a regular endpoint for the interval (a, d), one has $\mathsf{d}^+(\lambda) = \mathsf{i}^+$ and $\mathsf{d}^-(\lambda) = \mathsf{i}^-$; see Remark 3.6. Hence $\mathsf{a}^-(\lambda)$ is constant on $\mathbb{C}_+$ and $\mathsf{a}^+(\lambda)$ is constant on $\mathbb{C}_-$. Consequently, (3.14) implies that $\mathsf{a}^-(\lambda)$ and

$\mathsf{a}^+(\lambda)$ are constant on $\mathbb{C}\setminus\mathbb{R}$. Similar arguments show that $\mathsf{b}^+(\lambda)$ and $\mathsf{b}^-(\lambda)$ are also constant on $\mathbb{C}\setminus\mathbb{R}$.

(ii) & (iii) These statements follow from (i) and Lemma 3.5. □

Proposition 4.7 leads to the following definition.

Definition 4.8. The quantities $\mathsf{a}^+(\lambda)$, $\mathsf{a}^-(\lambda)$, $\mathsf{b}^+(\lambda)$, and $\mathsf{b}^-(\lambda)$ (being independent of $\lambda\in\mathbb{C}\setminus\mathbb{R}$) will be written as

$$\mathsf{a}^+,\quad \mathsf{a}^-,\quad \mathsf{b}^+,\quad \text{and}\quad \mathsf{b}^-,$$

respectively, in the rest of the paper.

Consequently, the defect numbers of $T_{\min}$, see (4.6), can be written as

$$n_+(T_{\min}) = \mathsf{a}^+ + \mathsf{b}^-,\quad n_-(T_{\min}) = \mathsf{a}^- + \mathsf{b}^+, \tag{4.7}$$

so that, in particular, by the von Neumann decomposition in Theorem 4.3

$$\dim\big(T_{\max}/T_{\min}\big) = n_+(T_{\min}) + n_-(T_{\min}) = \mathsf{a}^+ + \mathsf{a}^- + \mathsf{b}^+ + \mathsf{b}^-. \tag{4.8}$$

4.3. **The Lagrange identity and decompositions via localized solutions.** In the following it is convenient to make use of the notation

$$\langle\{f,g\},\{h,k\}\rangle_\Delta := (g,h)_\Delta - (f,k)_\Delta,\qquad \{f,g\},\{h,k\}\in T_{\max}.$$

With this notation an element $\{f_0,g_0\}$ belongs to $T_{\min}$ if and only if

$$\langle\{f,g\},\{f_0,g_0\}\rangle_\Delta = 0$$

for all $\{f,g\}\in T_{\max}$; cf. (A.2).

Lemma 4.9. *For $\{f,g\},\{h,k\}\in T_{\max}$ the limits*

$$[f,h](a) := \lim_{t\downarrow a} h(t)^*Jf(t),\quad [f,h](b) := \lim_{x\uparrow b} h(t)^*Jf(t) \tag{4.9}$$

exist and the Lagrange identity

$$\langle\{f,g\},\{h,k\}\rangle_\Delta = [f,h](b) - [f,h](a) \tag{4.10}$$

holds.

Proof. Let $I=[\alpha,\beta]\subset\imath$ be any compact interval. Then for $\{f,g\},\{h,k\}\in T_{\max}$ one has by Lemma 2.1

$$\int_\alpha^\beta h(s)^*\Delta(s)g(s)\,ds - \int_\alpha^\beta k(s)^*\Delta(s)f(s)\,ds = h(\beta)^*Jf(\beta) - h(\alpha)^*Jf(\alpha).$$

Since $f,g,h,k\in L^2_\Delta(\imath)$ the limits as $\alpha\to a$ and $\beta\to b$ in (4.9) exist and the identity (4.10) follows. □

The next proposition provides a characterization of the minimal relation.

Proposition 4.10. *The minimal relation $T_{\min}$ admits the representation*

$$T_{\min} = \big\{\,\{f,g\}\in T_{\max} : [f,h](a) = 0 = [f,h](b)\ \text{ for all } h\in\operatorname{dom}T_{\max}\,\big\}.$$

Proof. Note first that $T_{\min}\subset T_{\min}^* = T_{\max}$ implies $\{f,g\}\in T_{\min}$ if and only if $\{f,g\}\in T_{\max}$ and $(g,h)_\Delta = (f,k)_\Delta$ for all $\{h,k\}\in T_{\max}$. Hence Lemma 4.9 implies

$$T_{\min} = \big\{\,\{f,g\}\in T_{\max} : [f,h](a) = [f,h](b)\ \text{ for all } h\in\operatorname{dom}T_{\max}\,\big\}. \tag{4.11}$$

It remains to show that an element $\{f,g\}$ from the righthand side of (4.11) satisfies

$$[f,h](a)=0\quad\text{and}\quad [f,h](b)=0\quad\text{for all}\quad h\in\operatorname{dom}T_{\max}.$$

To see this, let $\{h,k\}\in T_{\max}$ be arbitrary, then by Proposition 2.17 there exists an element $\{h_a,k_a\}\in T_{\max}$ such that h_a coincides with h in a neighborhood of a and h_a is zero in a neighborhood of b. Consequently, by (4.11),

$$[f,h](a) = [f,h_a](a) = [f,h_a](b) = 0.$$

A similar argument shows $[f,h](b)=0$ for all $\{h,k\}\in T_{\max}$. □

The von Neumann decomposition of $T_{\max}$ in Theorem 4.3 will be supplemented by a decomposition of $T_{\max}$ in terms of localized versions of the fundamental solutions, see Definition 4.11 below. Therefore denote by $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ the eigenspaces of the nonzero finite eigenvalues of the selfadjoint limit relations $D(a,\lambda)$ and $D(b,\lambda)$, respectively, i.e.

$$\mathcal{A}(\lambda) = \mathcal{A}^+(\lambda) \oplus \mathcal{A}^-(\lambda), \quad \mathcal{B}(\lambda) = \mathcal{B}^+(\lambda) \oplus \mathcal{B}^-(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.12}$$

Recall that the dimensions of $\mathcal{A}^\pm(\lambda)$ and $\mathcal{B}^\pm(\lambda)$ do not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and that they are denoted by $\mathsf{a}^\pm$ and $\mathsf{b}^\pm$; cf. Definition 4.8. This implies

$$\dim \mathcal{A}(\lambda) = \mathsf{a}^+ + \mathsf{a}^-, \quad \dim \mathcal{B}(\lambda) = \mathsf{b}^+ + \mathsf{b}^-, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.13}$$

The cut-off functions $Y_a(\cdot,\lambda)$ and $Y_b(\cdot,\lambda)$ from Corollary 2.18 lead to the following definition.

Definition 4.11. Let $Y_a(\cdot,\lambda)$, $Y_b(\cdot,\lambda)$, $Z_a(\cdot,\lambda)$, and $Z_b(\cdot,\lambda)$ be the $n\times n$ matrix functions from Corollary 2.18 and let $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ be as in (4.12). For $\phi_a \in \mathcal{A}(\lambda)$ and $\phi_b \in \mathcal{B}(\lambda)$ define

$$\mathcal{Y}_a(\cdot,\lambda)\phi_a := \{Y_a(\cdot,\lambda)\phi_a, \lambda Y_a(\cdot,\lambda)\phi_a + Z_a(\cdot,\lambda)\phi_a\},$$
$$\mathcal{Y}_b(\cdot,\lambda)\phi_b := \{Y_b(\cdot,\lambda)\phi_b, \lambda Y_b(\cdot,\lambda)\phi_b + Z_b(\cdot,\lambda)\phi_b\}.$$

Note, that if $[\alpha,\beta]$ is a compact subinterval as in Corollary 2.18, then

$$\mathcal{Y}_a(t,\lambda)\phi_a = \begin{cases} \{Y(t,\lambda)\phi_a, \lambda Y(t,\lambda)\phi_a\}, & a < t \le \alpha, \\ \{0,0\}, & \beta \le t < b, \end{cases} \tag{4.14}$$

$$\mathcal{Y}_b(t,\lambda)\phi_b = \begin{cases} \{0,0\}, & a < t \le \alpha, \\ \{Y(t,\lambda)\phi_b, \lambda Y(t,\lambda)\phi_b\}, & \beta \le t < b. \end{cases} \tag{4.15}$$

Theorem 4.12. *For $\lambda \in \mathbb{C}\setminus\mathbb{R}$ the maximal relation $T_{\max}$ has the following componentwise sum decomposition:*

$$T_{\max} = T_{\min} \,\widehat{+}\, \{\, \mathcal{Y}_a(\cdot,\lambda)\phi_a : \phi_a \in \mathcal{A}(\lambda)\,\} \,\widehat{+}\, \{\, \mathcal{Y}_b(\cdot,\lambda)\phi_b : \phi_b \in \mathcal{B}(\lambda)\,\}, \tag{4.16}$$

where the sums are direct and $\mathcal{A}(\lambda)$, $\mathcal{B}(\lambda)$ are as in (4.12).

Proof. The equations (4.14) and (4.15) show that $\mathcal{Y}_a(t,\lambda)\phi_a, \mathcal{Y}_b(t,\lambda)\phi_b \in L^2_\Delta(\imath) \times L^2_\Delta(\imath)$ for all $\phi_a \in \mathcal{A}(\lambda)$ and $\phi_b \in \mathcal{B}(\lambda)$, see (3.23) and (3.12). Consequently, $\mathcal{Y}_a(t,\lambda)\phi_a, \mathcal{Y}_b(t,\lambda)\phi_b \in T_{\max}$ for all $\phi_a \in \mathcal{A}(\lambda)$ and $\phi_b \in \mathcal{B}(\lambda)$, see Definition 4.11. I.e. by the preceding arguments the righthand side in (4.16) is contained in $T_{\max}$.

As to the reverse inclusion define $f(\lambda)$ and $h(\lambda)$ by

$$f(\lambda) = \mathcal{Y}_a(\cdot,\lambda)\phi_a + \mathcal{Y}_b(\cdot,\lambda)\phi_b, \quad h(\lambda) = \mathcal{Y}_a(\cdot,\lambda)\psi_a + \mathcal{Y}_b(\cdot,\lambda)\psi_b$$

for $\phi_a, \psi_a \in \mathcal{A}(\lambda)$ and $\phi_b, \psi_b \in \mathcal{B}(\lambda)$. Then, see (4.9),

$$[f(\lambda), h(\lambda)](b) = [\mathcal{Y}_b(\cdot,\lambda)\phi_b, \mathcal{Y}_b(\cdot,\lambda)\psi_b](b) = i \lim_{t\uparrow b} \psi_b^* Y(t,\lambda)^*(-iJ)Y(t,\lambda)\phi_b$$

and a similar result holds at the endpoint a. Hence by Theorem 3.2

$$[f(\lambda), h(\lambda)](b) = i\psi_b^* D(b,\lambda)_s \phi_b, \quad [f(\lambda), h(\lambda)](a) = i\psi_a^* D(a,\lambda)_s \phi_a.$$

Consequently, $[f(\lambda), h(\lambda)](b) = 0$ for arbitrary $h(\lambda)$ if and only if $\phi_b \in \mathcal{A}^0(\lambda)$, i.e. $\phi_b = 0$. A similar statement holds for the limit at a. Therefore $f(\lambda) \in T_{\min}$ if and only if $\phi_a = 0 = \phi_b$, see Proposition 4.10. These arguments show that the righthand side of (4.16) is an extensions of $T_{\min}$ of dimension $\dim \mathcal{A}(\lambda) + \dim \mathcal{B}(\lambda) = \mathsf{a}^+ + \mathsf{a}^- + \mathsf{b}^+ + \mathsf{b}^-$. Hence the statement follows from (4.8). ☐

The next statement can be obtained with the same arguments as in the proof of Theorem 4.12 by computing $\langle \{f,g\}, \mathcal{Y}_a(\cdot,\lambda)\chi_a\rangle_\Delta$ and $\langle \{f,g\}, \mathcal{Y}_b(\cdot,\lambda)\chi_b\rangle_\Delta$, respectively, for $\{f,g\} \in T_{\max}$, $\chi_a \in \mathcal{A}(\lambda)$, $\chi_b \in \mathcal{B}(\lambda)$. It shows, in particular, how ϕ_a and ϕ_b in (4.16) can be obtained in terms of the elements in $T_{\max}$.

Corollary 4.13. *Let $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and decompose $\{f,g\}$ according to Theorem 4.12 in the form*

$$\{f,g\} = \{f_0,g_0\} + \mathcal{Y}_a(\cdot,\lambda)\phi_a + \mathcal{Y}_b(\cdot,\lambda)\phi_b,$$

with $\{f_0,g_0\}\in T_{\min}$, $\phi_a \in \mathcal{A}(\lambda)$, and $\phi_b \in \mathcal{B}(\lambda)$. Then

$$(D(a,\lambda)_s\phi_a,\chi_a) = \big[f, Y(\cdot,\lambda)\chi_a\big](a), \quad (D(b,\lambda)_s\phi_b,\chi_b) = \big[f, Y(\cdot,\lambda)\chi_b\big](b)$$

hold for all $\chi_a \in \mathcal{A}(\lambda)$ and $\chi_b \in \mathcal{B}(\lambda)$.

4.4. **Quasiregular endpoints and singular endpoints in the limit-point case.** The maximal and minimal relations $T_{\max}$ and $T_{\min}$ have special properties when one or both of the endpoints of the interval $\imath$ on which the canonical system (2.4) is considered are quasiregular or in the limit-point case; cf. Definitions 2.5 and 4.18.

Recall from Remark 3.6 that if the endpoint a is quasiregular, then $\mathsf{a}^+ = \mathsf{i}^+$ and $\mathsf{a}^- = \mathsf{i}^-$, and

$$\mathcal{A}^0(\lambda) = \mathcal{A}^\infty(\lambda) = \{0\}, \quad \mathcal{A}^+(\lambda)\oplus\mathcal{A}^-(\lambda) = \mathbb{C}^n, \quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{4.17}$$

Similarly, if the endpoint b is quasiregular, then $\mathsf{b}^+ = \mathsf{i}^+$ and $\mathsf{b}^- = \mathsf{i}^-$, and

$$\mathcal{B}^0(\lambda) = \mathcal{B}^\infty(\lambda) = \{0\}, \quad \mathcal{B}^+(\lambda)\oplus\mathcal{B}^-(\lambda) = \mathbb{C}^n, \quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{4.18}$$

In the case of a quasiregular endpoint $T_{\max}$ and $T_{\min}$ take a special form. The following proposition shows these forms in the case that the endpoint a is quasiregular, if the endpoint b is quasiregular similar results hold.

Proposition 4.14. *Assume that the endpoint a is quasiregular. Then for $\lambda\in\mathbb{C}\setminus\mathbb{R}$ the maximal relation $T_{\max}$ has the componentwise sum decomposition*

$$T_{\max} = T_{\min} \,\widehat{+}\, \big\{\, \mathcal{Y}_a(\cdot,\lambda)\phi_a : \phi_a\in\mathbb{C}^n \,\big\} \,\widehat{+}\, \big\{\, \mathcal{Y}_b(\cdot,\lambda)\phi_b : \phi_b\in\mathcal{B}(\lambda)\,\big\},$$

where the sums are direct. Moreover, the minimal relation admits the representation

$$T_{\min} = \big\{\, \{f,g\}\in T_{\max} : f(a)=0,\ [f,h](b)=0 \text{ for all } h\in\operatorname{dom}T_{\max}\,\big\}.$$

In particular, the mapping $\Gamma : T_{\max}\to\mathbb{C}^n$, $\{f,g\}\mapsto f(a)$ is well defined and onto.

Proof. The form of $T_{\max}$ is a consequence of Theorem 4.12 and (4.17). Since a is quasiregular, it follows that $f(a)$ exists for every $\{f,g\}\in T_{\max}$ by Proposition 2.6, see Remark 4.5. Hence $\Gamma\mathcal{Y}_a(\cdot,\lambda)\phi_a = Y(a,\lambda)\phi_a$, $\phi_a\in\mathbb{C}^n$, which shows the surjectivity of Γ, because $Y(a,\lambda)$ is invertible. Finally, for $f\in\operatorname{dom}T_{\min}$ and $\mathcal{Y}_a(\cdot,\lambda)\phi_a\in T_{\max}$ it follows from Definition 4.11 and Proposition 4.10 that

$$0 = [f, Y_a(\cdot,\lambda)\phi_a](a) = \phi_a^* Y(a,\lambda)^* J f(a), \qquad \phi_a\in\mathbb{C}^n.$$

Since $Y(a,\lambda)$ is invertible one concludes $f(a)=0$ and hence $T_{\min}$ has the indicated form. □

Observe, that if in Proposition 4.14 $\{f,g\}\in T_{\max}$ is decomposed as

$$\{f,g\} = \{f_0,g_0\} + \mathcal{Y}_a(\cdot,\lambda)\phi_a + \mathcal{Y}_b(\cdot,\lambda)\phi_b \tag{4.19}$$

with $\{f_0,g_0\}\in T_{\min}$, $\phi_a\in\mathbb{C}^n$, and $\phi_b\in\mathcal{B}(\lambda)$, then $\phi_a = Y(a,\lambda)^{-1}f(a)$.

The following simple lemma is inspired by [27, Section 4].

Lemma 4.15. *Let the endpoint a be quasiregular. Then the defect numbers are given by*

$$n_+(T_{\min}) = \mathsf{i}^+ + \mathsf{b}^- \qquad \text{and} \qquad n_-(T_{\min}) = \mathsf{i}^- + \mathsf{b}^+.$$

In particular, if the defect numbers coincide, then $\mathsf{b}^+=\mathsf{b}^-$ if and only if $\mathsf{i}^+=\mathsf{i}^-$, in which case $n = 2\mathsf{i}^+ = 2\mathsf{i}^-$.

Proof. The quasiregularity of a yields $\mathsf{a}^+ = \mathsf{i}^+$ and $\mathsf{a}^- = \mathsf{i}^-$, see Remark 3.6. Hence the first statement follows directly from (4.7). The other statements are clear. □

The preceding result shows that $\mathsf{i}^+ \leq n_+(T_{\min}) \leq n$ and $\mathsf{i}^- \leq n_-(T_{\min}) \leq n$, which implies that

$$n \leq n_+(T_{\min}) + n_-(T_{\min}) \leq 2n.$$

The above simple inequality which actually holds if either of the two endpoints is quasiregular, goes back to Atkinson; cf. [2, Theorem 9.11.1] and also [45].

Proposition 4.16. *Let the endpoints a and b be quasiregular. Then the defect numbers are equal and $n_+(T_{\max}) = n_-(T_{\max}) = n = \mathsf{i}^+ + \mathsf{i}^-$ holds. Then for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the maximal relation $T_{\max}$ has the componentwise sum decomposition*

$$T_{\max} = T_{\min} \mathbin{\widehat{+}} \{\, \mathcal{Y}_a(\cdot,\lambda)\phi_a : \phi_a \in \mathbb{C}^n \,\} \mathbin{\widehat{+}} \{\, \mathcal{Y}_b(\cdot,\lambda)\phi_b : \phi_b \in \mathbb{C}^n \,\},$$

where the sums are direct. Moreover, the minimal relation $T_{\min}$ is given by

$$T_{\min} = \{\, \{f,g\} \in T_{\max} : f(a) = f(b) = 0 \,\}$$

and the space $\mathfrak{N}_\lambda(T_{\max})$, characterized in (4.2), has the form

$$\mathfrak{N}_\lambda(T_{\max}) = \{\, Y(\cdot,\lambda)\phi : \phi \in \mathbb{C}^n \,\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

In particular, the mapping $\Gamma : T_{\max} \to \mathbb{C}^{2n}$, $\{f,g\} \mapsto \{f(a), f(b)\}$ is well defined and onto.

Proof. The statements concerning the defect numbers are direct consequences of Lemma 4.15 and Remark 3.6. The characterization of $T_{\max}$ $T_{\min}$ is obtained from Proposition 4.14 (applied to a and b). Since a and b are quasiregular, it follows from (4.17) and (4.18) that $\mathfrak{D}(a,\lambda) = \mathbb{C}^n = \mathfrak{D}(b,\lambda)$, see (3.12). Hence $\mathfrak{D}(a,\lambda) \cap \mathfrak{D}(b,\lambda) = \mathbb{C}^n$, which together with (4.2) leads to the given form of $\mathfrak{N}_\lambda(T_{\max})$. The statement concerning Γ follows from similar arguments as in Proposition 4.14. □

Remark 4.17. Note that canonical systems (2.4) having maximal defect numbers (n,n) were called quasiregular canonical systems in [42]. In their paper these systems are characterized by means of trace condition, see [42, Theorem 5.16].

As the complete opposite of a quasiregular endpoint the concept of an endpoint in the limit-point case is introduced in the next definition. In Example 4.22 below the connection to Weyl's limit-circle and limit-point classification for the special case of Sturm-Liouville differential expression is explained.

Definition 4.18. The endpoint a or b of the interval $\imath$ is said to be in the *limit-point case* if

$$\mathsf{a}^+ = \mathsf{a}^- = 0 \qquad \text{or} \qquad \mathsf{b}^+ = \mathsf{b}^- = 0,$$

respectively.

Observe that a is in the limit-point case if and only if

$$\mathcal{A}^+(\lambda) = \mathcal{A}^-(\lambda) = \{0\}, \quad \mathcal{A}^0(\lambda) \oplus \mathcal{A}^\infty(\lambda) = \mathbb{C}^n, \quad \lambda \in \mathbb{C}_\pm. \tag{4.20}$$

Likewise, b is in the limit-point case if and only if

$$\mathcal{B}^+(\lambda) = \mathcal{B}^-(\lambda) = \{0\}, \quad \mathcal{B}^0(\lambda) \oplus \mathcal{B}^\infty(\lambda) = \mathbb{C}^n, \quad \lambda \in \mathbb{C}_\pm. \tag{4.21}$$

If an endpoint is in the limit-point case, $T_{\max}$ and $T_{\min}$ take a special form. The following proposition shows these forms in the case that the endpoint b is in the limit-point case, if a is in the limit-point case a similar result holds.

Proposition 4.19. *Assume that the endpoint b is in the limit-point case. Then for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the maximal relation $T_{\max}$ has the componentwise sum decomposition*

$$T_{\max} = T_{\min} \mathbin{\widehat{+}} \{\, \mathcal{Y}_a(\cdot,\lambda)\phi_a : \phi_a \in \mathcal{A}(\lambda) \,\},$$

where the sums are direct. Moreover, the minimal relation admits the representation

$$T_{\min} = \{\, \{f,g\} \in T_{\max} : [f,h](a) = 0 \ \text{for all} \ h \in \operatorname{dom} T_{\max} \,\}.$$

Proof. If b is in the limit-point case, then $\mathcal{B}(\lambda) = \{0\}$, see (4.12) and (4.21). Hence the representation of $T_{\max}$ follows from Theorem 4.12. Now $\{f,g\} \in T_{\max}$ belongs to $T_{\min}$ if and only if for all $\{h_0,k_0\} \in T_{\min}$ and $\phi_a \in \mathcal{A}(\lambda)$

$$0 = \langle \{f,g\}, \{h_0,k_0\} + \mathcal{Y}_a(\cdot,\lambda)\phi_a \rangle_\Delta = \langle \{f,g\}, \mathcal{Y}_a(\cdot,\lambda)\phi_a \rangle_\Delta = -[f, Y(\cdot,\lambda)\phi_a](a),$$

which implies the representation for $T_{\min}$. □

Since $T_{\max} = T_{\min}^*$, see Theorem 4.3, the above statement has the following consequence.

Corollary 4.20. *If both endpoints a and b are in the limit-point case, then $T_{\min} = T_{\max}$ is selfadjoint.*

Finally, consider the case that one endpoint is quasiregular and one endpoint is in the limit-point case; cf. Proposition 4.14 and 4.19.

Proposition 4.21. *Let the endpoint a be quasiregular and let the endpoint b be in the limit-point case. Assume that the defect numbers are equal or, equivalently, that $\mathsf{i}^+ = \mathsf{i}^-$, in which case $n = 2\mathsf{i}^+ = 2\mathsf{i}^-$. Then for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the maximal relation $T_{\max}$ has the componentwise sum decomposition*

$$T_{\max} = T_{\min} \,\widehat{+}\, \{\, \mathcal{Y}_a(\cdot,\lambda)\phi_a : \phi_a \in \mathbb{C}^n \,\},$$

where the sums are direct. Moreover, the minimal relation admits the representation

$$T_{\min} = \big\{ \{f,g\} \in T_{\max} : f(a) = 0 \big\},$$

and the space $\mathfrak{N}_\lambda(T_{\max})$, characterized in (4.2)*, is given by*

$$\mathfrak{N}_\lambda(T_{\max}) = \{\, Y(\cdot,\lambda)\phi : \phi \in \mathcal{B}^0(\lambda) \,\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where $\dim \mathcal{B}^0(\lambda) = \mathsf{i}^+ = \mathsf{i}^-$.

Proof. The statement on the defect numbers follows directly from Lemma 4.15 and Definition 4.18. The expression for $T_{\min}$ follows from the formulas for $T_{\min}$ in Propositions 4.14 and 4.19. Furthermore, as a is quasiregular and b is in the limit-point case $\mathfrak{D}(a,\lambda) = \mathbb{C}^n$ and $\mathfrak{D}(b,\lambda) = \mathcal{B}^0(\lambda)$, see (3.12), (4.17) and (4.21). Hence

$$\mathfrak{D}(a,\lambda) \cap \mathfrak{D}(b,\lambda) = \mathbb{C}^n \cap \mathcal{B}^0(\lambda) = \mathcal{B}^0(\lambda),$$

which together with (4.2) the stated expression for $\mathfrak{N}_\lambda(T_{\max})$. For $\dim \mathcal{B}^0(\lambda)$, see Lemma 3.5. □

Example 4.22 (Weighted Sturm-Liouville equations)**.** Assume that the endpoint a for the weighted Sturm-Liouville equation in Examples 2.12 and 4.4 is regular. Since the corresponding matrix J has the form

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

it is clear that $\mathsf{i}^+ = \mathsf{i}^- = 1$, so that $\mathsf{a}^+ = \mathsf{a}^- = 1$; cf. Remark 3.6. Since the defect numbers are equal (see Example 4.4) it follows from Lemma 4.15 that $\mathsf{b}^+ = \mathsf{b}^-$. Since $\mathsf{b}^+ + \mathsf{b}^- \leq 2$ there are two cases:

(1) $\mathsf{b}^+ = \mathsf{b}^- = 0$;
(2) $\mathsf{b}^+ = \mathsf{b}^- = 1$;

In particular, the defect numbers are either 1 or 2, see Lemma 4.15. The first case corresponds to the usual limit-point case since the defect numbers are $(1,1)$, i.e., for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists (up to scalar multiples) one solution of the homogeneous equation which is square-integrable at the singular endpoint b, see [61, 62, 63] and e.g., [11, 25, 45]. The second case is the limit-circle case since the defect numbers are $(2,2)$; it corresponds to a 2×2 canonical system whose H and Δ are integrable on $\imath$; cf. [20].

4.5. **An alternative characterization of the minimal relation.** Recall that by Proposition 4.10 $T_{\min}$ consists, roughly speaking, of all elements $\{f,g\} \in T_{\max}$ of which the first component vanishes at the endpoints of $\imath$. Let T_0 be the restriction of the maximal relation $T_{\max}$ to the elements where there first component has compact support in $\imath$,

$$T_0 := \big\{ \{f,g\} \in T_{\max} : f \text{ has compact support} \big\}.$$

More precisely, an element $\{f,g\} \in L^2_\Delta(\imath) \times L^2_\Delta(\imath)$ belongs to T_0 if and only if the equivalence class f contains a locally absolutely continuous representative $\tilde{f}$ with compact support such that the inhomogeneous equation $J\widetilde{f}'(t) - H(t)\widetilde{f}(t) = \Delta(t)\widetilde{g}(t)$ is satisfied for almost every $t \in \imath$. Here $\widetilde{g}$ is any representative of $g \in L^2_\Delta(\imath)$.

The following proposition offers a different characterization of the minimal relation $T_{\min}$ which is of independent interest; cf. [49].

Proposition 4.23. *The minimal relation $T_{\min}$ is the closure of T_0 in $L^2_\Delta(\imath)$.*

Proof. Observe first that the inclusion $T_0 \subset T_{\min}$ follows immediately from Proposition 4.10. Therefore Theorem 4.3 implies that $T_0 \subset T_{\min} = T^*_{\max}$, which leads to

$$T_{\max} = T^*_{\min} \subset T^*_0.$$

Hence to prove the statement in the proposition it suffices to show that $T^*_0 \subset T_{\max}$.

For this, let $\{f,g\} \in T^*_0$ so that $f,g \in L^2_\Delta(\imath)$. From the theory of differential equations it follows that there exists a locally absolutely continuous function $\varphi \in AC_{\rm loc}(\imath)$ which is a solution of

$$J\varphi'(t) - H(t)\varphi(t) = \Delta(t)g(t). \tag{4.22}$$

Now let $[\alpha,\beta] \subset \imath$ be an arbitrary compact interval which contains the compact subinterval I on which the canonical system is definite; cf. Proposition 2.11. Since the system is also definite on $\jmath := (\alpha,\beta)$ the maximal and minimal relation $T_{\max}(\jmath)$ and $T_{\min}(\jmath)$ associated to the restricted system are well defined and have the properties shown in the previous subsections. Then it is clear that (the restriction of) $\{\varphi,g\}$ belongs to $T_{\max}(\jmath)$ as $\varphi,g \in L^2_\Delta(\jmath)$. Now let $\{h,k\} \in T_0$ and assume that the support of h is contained in $\jmath$. Note that, in particular, it follows that $\Delta k = 0$ outside the compact interval $[\alpha,\beta]$. Therefore, as $\{f,g\} \in T^*_0$ it follows

$$\int_\alpha^\beta h(s)^*\Delta(s)g(s)\,ds = \int_\alpha^\beta k(s)^*\Delta(s)f(s)\,ds.$$

However, $\{\varphi,g\} \in T_{\max}(\jmath)$ also implies that

$$\int_\alpha^\beta h(s)^*\Delta(s)g(s)\,ds = \int_\alpha^\beta k(s)^*\Delta(s)\varphi(s)\,ds,$$

since (the restriction of) $\{h,k\}$ is an element in $T_{\min}(\jmath)$; cf. Proposition 4.10. Combining these identities shows that

$$\int_\alpha^\beta k(s)^*\Delta(s)(f(s)-\varphi(s))\,ds = 0. \tag{4.23}$$

Note that each element in $T_{\min}(\jmath)$ can be seen as a restriction of an element in T_0 whose first component has support in $\jmath$. Therefore it follows that (4.23) holds for all $k \in \operatorname{ran} T_{\min}(\jmath)$, so that by Theorem 4.3 (applied to the interval $\jmath$), $f-\varphi \in (\operatorname{ran} T_{\min}(\jmath))^\perp = \ker T_{\max}(\jmath)$. Hence, there exists a constant $c_\jmath$ and a measurable function $\omega_\jmath$ on $\jmath$ for which

$$f(t)-\varphi(t) = Y(t,0)c_\jmath + \omega_\jmath(t) \quad \text{and} \quad \Delta(t)\omega_\jmath(t) = 0 \tag{4.24}$$

for almost all $t \in \jmath$. Since the canonical system is definite on every interval $\jmath$ which contains I, see Proposition 2.11, it follows that the constant $c_\jmath$ in (4.24) does not depend on the choice of the interval $\jmath$, i.e., $c_\jmath = c$. To see this, let $\widetilde{\jmath} \subset \imath$ be an interval that contains $\jmath$ and let $c_{\widetilde{\jmath}}$ and $\omega_{\widetilde{\jmath}}$ be such that

$$f(t)-\varphi(t) = Y(t,0)c_{\widetilde{\jmath}} + \omega_{\widetilde{\jmath}}(t) \quad \text{and} \quad \Delta(t)\omega_{\widetilde{\jmath}}(t) = 0$$

for almost all $t \in \tilde{\jmath}$. Hence $Y(\cdot,0)c_\jmath - Y(\cdot,0)c_{\tilde{\jmath}} = \omega_{\tilde{\jmath}} - \omega_\jmath$ is a solution of the homogeneous equation on $\jmath$ such that $\Delta(t)(\omega_{\tilde{\jmath}} - \omega_\jmath)(t) = 0$ for almost all $t \in \jmath$. By definiteness $\omega_{\tilde{\jmath}}(t) = \omega_\jmath(t)$ for all $t \in \jmath$ and hence $c_\jmath = c_{\tilde{\jmath}}$.

Therefore, for any interval $\jmath \subset \imath$ which contains the compact interval I in Proposition 2.11, it follows that the function

$$f - \varphi - Y(\cdot,0)c \tag{4.25}$$

is a null-function with respect to Δ on the interval $\jmath$. Therefore the function in (4.25) is a null-function with respect to Δ on the interval $\imath$. Now the function $\varphi + Y(\cdot,0)c$ solves the equation (4.22) and it belongs to the same equivalence class as f. Since by assumption $f \in L^2_\Delta(\imath)$ it follows that $\{f,g\} \in T_{\max}$. Hence $T_0^* \subset T_{\max}$. □

5. Boundary triplets and Weyl functions for singular canonical systems with equal defect numbers

Boundary triplets and associated Weyl functions provide an efficient abstract tool for the description of the spectral properties of the closed extensions of a symmetric operator or relation with equal defect numbers, see, e.g., [6, 7, 15, 16, 22, 36] and Section 5.1 below for a brief summary. The aim of this section is to show how boundary triplets for singular canonical system with equal defect numbers can be chosen and to interpret the corresponding Weyl function as an analytic object that specifies the square-integrable solutions of the underlying homogeneous canonical differential equation. Besides the general singular case also the quasiregular and limit-point case is discussed in detail. As in Section 4 the canonical system is assumed to be definite in the following.

5.1. **Boundary triplets in case of equal defect numbers.** In this subsection S stands for a closed symmetric relation with equal, not necessarily finite, defect numbers $n_\pm(S) = \dim\ker\,(S^* \pm i)$ in a Hilbert space $(\mathfrak{H}, (\cdot,\cdot)_\mathfrak{H})$. The following definitions and basic facts are taken from [15, 16, 22].

Definition 5.1. A *boundary triplet* $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for the adjoint relation S^* consists of an auxiliary Hilbert space $(\mathcal{H}, (\cdot,\cdot)_\mathcal{H})$ and two mappings $\Gamma_0, \Gamma_1 : S^* \to \mathcal{H}$ such that the abstract Lagrange or Green's identity

$$(f',g)_\mathfrak{H} - (f,g')_\mathfrak{H} = (\Gamma_1\widehat{f}, \Gamma_0\widehat{g})_\mathcal{H} - (\Gamma_0\widehat{f}, \Gamma_1\widehat{g})_\mathcal{H} \tag{5.1}$$

holds for all $\widehat{f} = \{f,f'\}$, $\widehat{g} = \{g,g'\} \in S^*$ and such that the mapping $\Gamma : \widehat{f} \to \{\Gamma_0\widehat{f}, \Gamma_1\widehat{f}\}$ from S^* to $\mathcal{H} \times \mathcal{H}$ is surjective.

For a vector $\phi \in \mathcal{H} \times \mathcal{H}$ the first component in $\mathcal{H} \times \{0\}$ and second component in $\{0\} \times \mathcal{H}$ is denoted by ϕ_0 and ϕ_1, respectively, sometimes also by $[\phi]_0$ and $[\phi]_1$, respectively. In particular, the following notation will be used:

$$\phi = \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} [\phi]_0 \\ [\phi]_1 \end{pmatrix} \quad \text{and} \quad \phi = \{\phi_0, \phi_1\} = \{[\phi]_0, [\phi]_1\}. \tag{5.2}$$

If $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for S^*, then $\dim \mathcal{H} = n_\pm(S)$ and $S = \ker\Gamma$. Moreover, the relations A_0 and A_1 defined by

$$A_0 = \ker\Gamma_0, \quad A_1 = \ker\Gamma_1, \tag{5.3}$$

are selfadjoint extensions of S such that

$$A_0 \cap A_1 = S, \quad A_0 \,\widehat{+}\, A_1 = S^*, \tag{5.4}$$

where the last sum is componentwise. Conversely, for any two selfadjoint extensions A_0 and A_1 of S with the properties (5.4), there exists a boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for S^* such that (5.3) holds. In particular, a boundary triplet is not unique if the defect numbers $n_\pm(S)$ of S are not equal to zero.

Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*, then S^* has with respect to the selfadjoint extension $A_0 = \ker \Gamma$ the direct sum decomposition

$$S^* = A_0 \widehat{+} \widehat{\mathfrak{N}}_\lambda(S^*), \quad \lambda \in \rho(A_0), \quad \text{direct sum}, \tag{5.5}$$

where the eigenspace $\widehat{\mathfrak{N}}_\lambda(S^*)$ is defined by

$$\widehat{\mathfrak{N}}_\lambda(S^*) = \big\{ \{f_\lambda, \lambda f_\lambda\} : f_\lambda \in \mathfrak{N}_\lambda(S^*) \big\}, \quad \mathfrak{N}_\lambda(S^*) = \ker (S^* - \lambda). \tag{5.6}$$

Definition 5.2. Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* with $A_0 = \ker \Gamma_0$. The associated *γ-field* is defined by

$$\gamma(\lambda) = \{ \{\Gamma_0 \widehat{f}_\lambda, f_\lambda\} : \widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(S^*)\}, \quad \lambda \in \rho(A_0),$$

and the associated *Weyl function* is defined by

$$M(\lambda) = \{ \{\Gamma_0 \widehat{f}_\lambda, \Gamma_1 \widehat{f}_\lambda\} : \widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(S^*)\}, \quad \lambda \in \rho(A_0).$$

Denote by π_1 the orthogonal projection in $\mathfrak{H} \oplus \mathfrak{H}$ onto the first component. The following result follows from the decomposition (5.5) and the properties of the boundary mappings; it will be used frequently in this section.

Proposition 5.3. *The restriction $\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^*)$, $\lambda \in \rho(A_0)$, of the mapping Γ_0 to $\widehat{\mathfrak{N}}_\lambda(S^*)$ is a bijective mapping onto $\mathcal{H}$. In particular, $\gamma(\lambda) \in \mathbf{B}(\mathcal{H}, \mathfrak{H})$ is a bounded linear operator from $\mathcal{H}$ to $\mathfrak{H}$, given by*

$$\gamma(\lambda) = \pi_1 (\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^*))^{-1}, \quad \lambda \in \rho(A_0).$$

The values $M(\lambda)$ of the Weyl function M are in $\mathbf{B}(\mathcal{H})$ and are given by

$$M(\lambda) = \Gamma_1 (\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^*))^{-1}, \qquad \lambda \in \rho(A_0).$$

Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* with associated γ-field γ and Weyl function M. Then the γ-field satisfies the identity

$$\gamma(\lambda) = (I + (\lambda - \mu)(A_0 - \lambda)^{-1})\gamma(\mu), \quad \lambda, \mu \in \rho(A_0), \tag{5.7}$$

which, in particular, shows that γ is a holomorphic on $\rho(A_0)$. The Weyl function and the γ-field are related via the identity

$$\frac{M(\lambda) - M(\mu)^*}{\lambda - \bar{\mu}} = \gamma(\mu)^* \gamma(\lambda), \qquad \lambda, \mu \in \rho(A_0). \tag{5.8}$$

In particular, since $\gamma(\lambda)$ is injective and maps onto $\mathfrak{N}_\lambda(S^*)$, (5.8) shows that M is a Nevanlinna function with the additional property $0 \in \rho(\operatorname{Im} M(\lambda))$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Remark 5.4. The γ-field and Weyl function are defined on the set $\rho(A_0)$ which contains $\mathbb{C} \setminus \mathbb{R}$. However, due to the holomorphy of the functions γ and M it is sufficient (and in the case of canonical systems in the present paper more convenient) to consider only the values $\gamma(\lambda)$ and $M(\lambda)$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Boundary triplets are particularly convenient for the parametrization and description of the extensions H of S which satisfy $S \subset H \subset S^*$. More precisely, the mapping

$$\Theta \mapsto A_\Theta := \big\{ \widehat{f} \in S^* : \{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\} \in \Theta \big\} = \ker (\Gamma_1 - \Theta \Gamma_0) \tag{5.9}$$

establishes a bijective correspondence between the closed linear relations Θ in $\mathcal{H}$ and the closed extensions $A_\Theta \subset S^*$ of S. Furthermore, $A_{\Theta^*} = (A_\Theta)^*$ holds and, in particular, the closed extension A_Θ of S in (5.9) is symmetric or selfadjoint if and only if the relation Θ is symmetric or selfadjoint, respectively.

Let Θ be a closed relation in $\mathcal{H}$ and let A_Θ be the corresponding extension of S in (5.9). With the help of the Weyl function the spectral properties of A_Θ can be described. For instance, a point $\lambda \in \rho(A_0)$ belongs to $\rho(A_\Theta)$ if and only if $0 \in \rho(\Theta - M(\lambda))$, and similar correspondences hold for the spectral subsets of A_Θ; see [16, Proposition 1.6]. Furthermore, for all $\lambda \in \rho(A_0) \cap \rho(A_\Theta)$ Kreĭn's formula for the resolvents for the canonical extensions of S holds,

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(M(\lambda) - \Theta\big)^{-1} \gamma(\bar{\lambda})^*.$$

Recall that a relation Θ in $\mathcal{H}$ is selfadjoint if and only if there exists a *Nevanlinna pair* $\{\Phi,\Psi\}$, i.e.,

$$\Phi,\Psi\in\mathbf{B}(\mathcal{H}),\quad \Phi\Psi^*=\Psi\Phi^*\quad\text{and}\quad 0\in\rho(\Psi\pm i\Phi), \tag{5.10}$$

such that Θ can be written in the form

$$\Theta=\bigl\{\{h,h'\}\in\mathcal{H}\times\mathcal{H}:\,\Phi h+\Psi h'=0\bigr\}=\bigl\{\{\Psi^*k,-\Phi^*k\}:k\in\mathcal{H}\bigr\}. \tag{5.11}$$

In the case $n=\dim\mathcal{H}<\infty$ the condition $0\in\rho(\Psi\pm i\Phi)$ in (5.10) can be replaced by the equivalent condition that the rank of the $n\times 2n$ matrix $[\Phi\,;\Psi]$ is maximal. In terms of this parametrization one has

$$A_\Theta=\bigl\{\,\widehat{f}\in S^*:\Phi\Gamma_0\widehat{f}+\Psi\Gamma_1\widehat{f}=0\,\bigr\}=\ker\,(\Phi\Gamma_0+\Psi\Gamma_1), \tag{5.12}$$

and Kreĭn's formula reads as

$$(A_\Theta-\lambda)^{-1}=(A_0-\lambda)^{-1}-\gamma(\lambda)\Psi^*\bigl(M(\lambda)\Psi^*+\Phi^*\bigr)^{-1}\gamma(\bar\lambda)^*$$

for all $\lambda\in\rho(A_\Theta)\cap\rho(A_0)$.

All possible boundary triplets associated to S^* can be described as follows; cf. [16, Proposition 1.7]. For a given boundary triplet $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for S^*, a Hilbert space $\mathcal{H}'$, and a block operator matrix $W=(W_{ij})_{i,j=0}^1\in\mathbf{B}(\mathcal{H}\times\mathcal{H},\mathcal{H}'\times\mathcal{H}')$, with the properties

$$W\begin{pmatrix}0 & -iI_{\mathcal{H}'}\\ iI_{\mathcal{H}'} & 0\end{pmatrix}W^*=\begin{pmatrix}0 & -iI_{\mathcal{H}}\\ iI_{\mathcal{H}} & 0\end{pmatrix} \tag{5.13}$$

and

$$W^*\begin{pmatrix}0 & -iI_{\mathcal{H}}\\ iI_{\mathcal{H}} & 0\end{pmatrix}W=\begin{pmatrix}0 & -iI_{\mathcal{H}'}\\ iI_{\mathcal{H}'} & 0\end{pmatrix}, \tag{5.14}$$

the triplet $\{\mathcal{H}',\Gamma_0^W,\Gamma_1^W\}$ defined by

$$\begin{pmatrix}\Gamma_0^W\{f,g\}\\ \Gamma_1^W\{f,g\}\end{pmatrix}=\begin{pmatrix}W_{00} & W_{01}\\ W_{10} & W_{11}\end{pmatrix}\begin{pmatrix}\Gamma_0\{f,g\}\\ \Gamma_1\{f,g\}\end{pmatrix},\quad \{f,g\}\in S^*, \tag{5.15}$$

is also a boundary triplet for S^*. Conversely, for each pair of boundary triplets $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ and $\{\mathcal{H}',\Gamma_0',\Gamma_1'\}$ for S^* there exists an operator W with the above mentioned properties such that $\Gamma_0'=\Gamma_0^W$ and $\Gamma_1'=\Gamma_1^W$ hold.

If $\{\mathcal{H}^W,\Gamma_0^W,\Gamma_1^W\}$ is a boundary triplet for S^* which is connected with the boundary triplet $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ via (5.15), then the corresponding γ-field γ_W and Weyl function M_W satisfy the identities

$$\gamma_W(\lambda)=\gamma(\lambda)\bigl(W_{00}+W_{01}M(\lambda)\bigr)^{-1}, \tag{5.16}$$

and

$$M_W(\lambda)=\bigl(W_{10}+W_{11}M(\lambda)\bigr)\bigl(W_{00}+W_{01}M(\lambda)\bigr)^{-1}, \tag{5.17}$$

for all $\lambda\in\rho(A_0)\cap\rho(A_0^W)$, where $A_0^W=\ker\Gamma_0^W$. In particular,

$$W=\begin{pmatrix}0 & I_{\mathcal{H}}\\ -I_{\mathcal{H}} & 0\end{pmatrix}$$

satisfies (5.13) and (5.14), the corresponding boundary triplet via (5.15) is given by

$$\Gamma_0^W\{f,g\}=\Gamma_1\{f,g\},\quad \Gamma_1^W\{f,g\}=-\Gamma_0\{f,g\},$$

and the associated γ-field and Weyl function are given by

$$\gamma_W(\lambda)=\gamma(\lambda)M(\lambda)^{-1},\quad M_W(\lambda)=-M(\lambda)^{-1},\quad \lambda\in\rho(A_0)\cap\rho(A_1).$$

5.2. **Canonical systems with quasiregular endpoints.** In this subsection the abstract concepts of boundary triplets and Weyl functions is illustrated for the canonical system (2.4) in the case that both its endpoints are quasiregular. Then the defect numbers of the associated minimal relation $T_{\min}$ from Section 4.1 are equal to n and that each element $\{f,g\} \in T_{\max}$ admits boundary values $f(a), f(b) \in \mathbb{C}^n$ at the endpoints of the interval $\imath$; cf. Propositions 4.16.

In the next theorem a boundary triplet for $T_{\max}$ is given and its corresponding γ-field and Weyl function are obtained.

Theorem 5.5. *Assume that both endpoints a and b are quasiregular endpoints for the canonical system* (2.4)*. Then $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ with*

$$\Gamma_0\{f,g\} := \frac{1}{\sqrt{2}}(f(a)+f(b)), \quad \Gamma_1\{f,g\} := -\frac{J}{\sqrt{2}}(f(a)-f(b)),$$

is a boundary triplet for $T_{\max}$. Moreover, the γ-field γ and the Weyl function M associated to $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ have the form

$$\gamma(\lambda) = \sqrt{2}Y(\cdot,\lambda)\big(Y(a,\lambda)+Y(b,\lambda)\big)^{-1}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

and

$$M(\lambda) = -J\big(Y(a,\lambda)-Y(b,\lambda)\big)\big(Y(a,\lambda)+Y(b,\lambda)\big)^{-1}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

Proof. Since the endpoints a and b are quasiregular, the Lagrange identity (4.10) reduces to

$$\big\langle \{f,g\},\{h,k\}\big\rangle_\Delta = h(b)^*Jf(b) - h(a)^*Jf(a), \quad \{f,g\},\{h,k\} \in T_{\max}.$$

Now a straight-forward calculation shows that the boundary mappings Γ_0 and Γ_1 satisfy the abstract Lagrange identity (5.1). The surjectivity of the mapping $\Gamma = (\Gamma_0,\Gamma_1)^\top : T_{\max} \to \mathbb{C}^n \times \mathbb{C}^n$ follows from Proposition 4.16. Hence $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $T_{\max}$.

To obtain the expressions for the associated γ-field and Weyl function recall that

$$\mathfrak{N}_\lambda(T_{\max}) = \{\, Y(\cdot,\lambda)\phi : \phi \in \mathbb{C}^n \,\}, \qquad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

see Proposition 4.16. Hence for $\widehat{f}_\lambda = \{Y(\cdot,\lambda)\phi, \lambda Y(\cdot,\lambda)\phi\}$, $\phi \in \mathbb{C}^n$, one has

$$\Gamma_0\widehat{f}_\lambda = \frac{1}{\sqrt{2}}(Y(a,\lambda)+Y(b,\lambda))\phi, \quad \Gamma_1\widehat{f}_\lambda = -\frac{J}{\sqrt{2}}(Y(a,\lambda)-Y(b,\lambda))\phi,$$

which leads to

$$\gamma(\lambda) = \left\{ \left\{ \frac{1}{\sqrt{2}}(Y(a,\lambda)+Y(b,\lambda))\phi, Y(\cdot,\lambda)\phi \right\} : \phi \in \mathbb{C}^n \right\}$$

and

$$M(\lambda) = \left\{ \left\{ \frac{1}{\sqrt{2}}(Y(a,\lambda)+Y(b,\lambda))\phi, -\frac{J}{\sqrt{2}}(Y(a,\lambda)-Y(b,\lambda))\phi \right\} : \phi \in \mathbb{C}^n \right\},$$

see Definition 5.2. These identities together with Proposition 5.3 yield the formulas for the γ-field and the Weyl function. □

Let $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be the boundary triplet for $T_{\max}$ from Theorem 5.5. Then the selfadjoint relations $A_0 = \ker \Gamma_0$ and $A_1 = \ker \Gamma_1$ are given by

$$A_i = \ker \Gamma_i = \big\{\{f,g\} \in T_{\max} : f(a) = (-1)^{i+1}f(b)\big\}, \qquad i = 0,1.$$

All other selfadjoint extensions of $T_{\min}$ in $L^2_\Delta(\imath)$ can be described via (5.9) or (5.12) with the help of selfadjoint relations Θ in $\mathbb{C}^n$ or Nevanlinna pairs $\{\Phi,\Psi\}$. The next corollary is a direct consequence of Theorem 5.5 and (5.12).

Corollary 5.6. *Assume that a and b are quasiregular endpoints for the canonical system* (2.4) *and let Θ be a selfadjoint relation in $\mathbb{C}^n$ represented by a Nevanlinna pair $\{\Phi,\Psi\}$ of $n\times n$ matrices in the form* (5.11)*. Then*

$$A_\Theta = \big\{\{f,g\} \in T_{\max} : \Phi(f(a)+f(b)) = \Psi J(f(a)-f(b))\big\} \tag{5.18}$$

is a selfadjoint realization of the canonical system (2.4) *in $L^2_\Delta(\imath)$, and conversely, each selfadjoint realization of the canonical system can be written in the form* (5.18)*.*

The selfadjoint relation A_Θ in (5.18) can also be written as

$$A_\Theta = \{ \{f,g\} \in T_{\max} : Uf(a) + Vf(b) = 0 \},$$

where $U = \Phi - \Psi J$ and $V = \Phi + \Psi J$ are $n \times n$ matrices satisfying

$$UJU^* = VJV^*, \quad \operatorname{rank}[U;V] = n,$$

see [21, p. 250], [49, Theorem 2.9]. Note that the γ-field and Weyl function in Theorem 5.5 are connected by

$$\gamma(\lambda) = \sqrt{2}Y(\cdot,\lambda)\big(Y(a,\lambda) - Y(b,\lambda)\big)^{-1} JM(\lambda), \quad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

and that the invertibility of the matrices $Y(a,\lambda) \pm Y(b,\lambda)$ follows directly from Lemma 2.14. Formulas for the Weyl function M as in Theorem 5.5 can be found in the literature; cf. [41] where the notion of Q-function is used. However, other forms may occur due to a different choice of the boundary triplet. One special case of interest may be mentioned in particular, namely when $n = 2m$ and J is of the form

$$J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}. \tag{5.19}$$

Decompose the vectors $\phi \in \mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^m$ into two components $[\phi]_0, [\phi]_1 \in \mathbb{C}^m$ as in (5.2) and let the fundamental matrix be decomposed accordingly into $m \times m$ block form:

$$Y(\cdot,\lambda) = \begin{pmatrix} Y_{00}(\cdot,\lambda) & Y_{01}(\cdot,\lambda) \\ Y_{10}(\cdot,\lambda) & Y_{11}(\cdot,\lambda) \end{pmatrix}.$$

In order to apply the abstract transformation results from Section 5.1, define the $4m \times 4m$ matrix W by

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} I_m & 0 & 0 & -I_m \\ I_m & 0 & 0 & I_m \\ 0 & I_m & I_m & 0 \\ 0 & -I_m & I_m & 0 \end{pmatrix}, \tag{5.20}$$

so that W satisfies (5.13) and (5.14). Let $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be the boundary triplet in Theorem 5.5. If this boundary triplet is transformed by (5.15), where W is as in (5.20), then the following result is directly obtained.

Corollary 5.7. *Assume that a and b are quasiregular endpoints for the canonical system* (2.4) *and that J is of the form* (5.19). *Then $\{\mathbb{C}^{2m}, \Gamma_0, \Gamma_1\}$ with*

$$\Gamma_0\{f,g\} := \begin{pmatrix} [f(a)]_0 \\ [f(b)]_0 \end{pmatrix}, \quad \Gamma_1\{f,g\} := \begin{pmatrix} [f(a)]_1 \\ -[f(b)]_1 \end{pmatrix},$$

is a boundary triplet for $T_{\max}$. Moreover, the γ-field γ and the Weyl function M associated to $\{\mathbb{C}^{2m}, \Gamma_0, \Gamma_1\}$ have the form

$$\gamma(\lambda) = Y(\cdot,\lambda) \begin{pmatrix} Y_{00}(a,\lambda) & Y_{01}(a,\lambda) \\ Y_{00}(b,\lambda) & Y_{01}(b,\lambda) \end{pmatrix}^{-1}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

and

$$M(\lambda) = \begin{pmatrix} Y_{10}(a,\lambda) & Y_{11}(a,\lambda) \\ -Y_{10}(b,\lambda) & -Y_{11}(b,\lambda) \end{pmatrix} \begin{pmatrix} Y_{00}(a,\lambda) & Y_{01}(a,\lambda) \\ Y_{00}(b,\lambda) & Y_{01}(b,\lambda) \end{pmatrix}^{-1}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

For a Nevanlinna pair $\{\Phi, \Psi\}$ of $m \times m$ matrices define the relation $T'_{\max}$ by

$$T'_{\max} = \{ \{f,g\} \in T_{\max} : \Phi[f(b)]_0 + \Psi[f(b)]_1 = 0 \},$$

and the linear relation $T'_{\min}$ by

$$T'_{\min} = \{ \{f,g\} \in T_{\max} : f(a) = 0,\, \Phi[f(b)]_0 + \Psi[f(b)]_1 = 0 \}.$$

Then $T'_{\min}$ is closed and symmetric with defect numbers (m,m) and its adjoint is given by $T'_{\max}$, see [14]. Here $T'_{\max}$ can be interpreted as a restriction of $T_{\max}$ by means of a selfadjoint boundary condition at b.

The defect subspaces of $T'_{\max}$ have the form

$$\mathfrak{N}_\lambda(T'_{\max}) = \big\{\, Y(\cdot,\lambda)\phi : \Phi[Y(b,\lambda)\phi]_0 + \Psi[Y(b,\lambda)\phi]_1 = 0,\ \phi \in \mathbb{C}^{2m} \,\big\}$$

for $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Note that the condition $\Phi[Y(b,\lambda)\phi]_0 + \Psi[Y(b,\lambda)\phi]_1 = 0$ is equivalent to

$$\big(\Phi Y_{01}(b,\lambda) + \Psi Y_{11}(b,\lambda)\big)\phi_1 = -\big(\Phi Y_{00}(b,\lambda) + \Psi Y_{10}(b,\lambda)\big)\phi_0. \tag{5.21}$$

It is not difficult to verify that $\{\mathbb{C}^m, \Gamma'_0, \Gamma'_1\}$ with

$$\Gamma'_0\{f,g\} := [f(a)]_0, \quad \Gamma'_1\{f,g\} := [f(a)]_1, \quad \{f,g\} \in T'_{\max},$$

is a boundary triplet for $T'_{\max}$. Under the assumption $Y(a,\lambda) = I$ it follows from Definition 5.2 that for $\lambda \in \mathbb{C}\setminus\mathbb{R}$ the corresponding γ-field and Weyl function are given by

$$\begin{aligned} \gamma'(\lambda) &= \big\{\{\phi_0, Y(\cdot,\lambda)\phi\} : \Phi[Y(b,\lambda)\phi]_0 + \Psi[Y(b,\lambda)\phi]_1 = 0,\ \phi \in \mathbb{C}^{2m} \,\big\}, \\ M'(\lambda) &= \big\{\{\phi_0, \phi_1\} : \Phi[Y(b,\lambda)\phi]_0 + \Psi[Y(b,\lambda)\phi]_1 = 0,\ \phi \in \mathbb{C}^{2m} \,\big\}, \end{aligned}$$

and they are connected via

$$\gamma'(\lambda) = Y(\cdot,\lambda)\begin{pmatrix} I \\ M'(\lambda)\end{pmatrix}, \qquad \lambda \in \mathbb{C}\setminus\mathbb{R};$$

cf. Proposition 5.3. With the help of (5.21) one also obtains

$$M'(\lambda) = -\big(\Phi Y_{01}(b,\lambda) + \Psi Y_{11}(b,\lambda)\big)^{-1}\big(\Phi Y_{00}(b,\lambda) + \Psi Y_{10}(b,\lambda)\big), \quad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

5.3. **Canonical systems in the limit-point case.** One of the main motivations for the introduction of abstract γ-fields and Weyl functions has been the Titchmarsh-Weyl theory for Sturm-Liouville equations in the limit-point case. In this subsection the corresponding limit-point case for canonical systems is treated. This treatment is of independent interest, but also serves as an introduction to the case of general singular canonical systems. Let $T_{\max}$ and $T_{\min}$ be the maximal and minimal relation associated to the canonical system (2.4) on $\imath$ and assume that the endpoint a is quasiregular and that the endpoint b is in the limit-point case. Furthermore, suppose that the defect numbers of $T_{\min}$ are equal, so that $\mathsf{i}^+ = \mathsf{i}^-$ and $n = 2m$, where $m := \mathsf{i}^+$; cf. Proposition 4.21. In particular, there exists a $2m \times 2m$ unitary matrix U, which satisfies (2.9):

$$UJU^* = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}, \tag{5.22}$$

see Lemma 2.4. Recall that $[\phi]_0$, $[\phi]_1$ denote the first and second component of $\phi \in \mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^m$.

Theorem 5.8. *Assume that a is a quasiregular endpoint, that b is a singular endpoint which is in the limit-point case, and that the defect numbers of $T_{\min}$ are equal. Let U be a unitary $2m \times 2m$ matrix such that (5.22) holds. Then $\{\mathbb{C}^m, \Gamma_0, \Gamma_1\}$ with*

$$\Gamma_0\{f,g\} := [Uf(a)]_0, \quad \Gamma_1\{f,g\} := [Uf(a)]_1,$$

is a boundary triplet for $T_{\max}$. Moreover, the γ-field γ and the Weyl function M associated to $\{\mathbb{C}^m, \Gamma_0, \Gamma_1\}$ have the form

$$\gamma(\lambda) = \big\{\, \{[UY(a,\lambda)\phi]_0, Y(\cdot,\lambda)\phi\} : \phi \in \mathcal{B}^0(\lambda) \,\big\}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

and

$$M(\lambda) = \big\{\, \{[UY(a,\lambda)\phi]_0, [UY(a,\lambda)\phi]_1\} : \phi \in \mathcal{B}^0(\lambda) \,\big\}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

Proof. Since the endpoint a is quasiregular the elements $\{f,g\}, \{h,k\} \in T_{\max}$ have boundary values $f(a), h(a) \in \mathbb{C}^n$ which are of the form $f(a) = Y(a,\lambda)\phi_a$ and $h(a) = Y(a,\lambda)\psi_a$, where $\phi_a, \psi_a \in \mathbb{C}^n$, respectively, see Proposition 4.14 and the observations following it; cf. (4.19). Moreover, according to Proposition 4.19 $\{f,g\}, \{h,k\} \in T_{\max}$ admit the decompositions

$$\{f,g\} = \{f_0, g_0\} + \mathcal{Y}_a(\cdot,\lambda)\phi_a, \quad \{h,k\} = \{h_0, k_0\} + \mathcal{Y}_a(\cdot,\lambda)\psi_a,$$

where $\{f_0, g_0\}, \{h_0, k_0\} \in T_{\min}$. Therefore the Lagrange identity has the form

$$\begin{aligned}\big\langle \{f,g\},\{h,k\}\big\rangle_\Delta &= \big\langle \{f_0,g_0\} + \mathcal{Y}_a(\cdot,\lambda)\phi_a, \{h_0,k_0\} + \mathcal{Y}_a(\cdot,\lambda)\psi_a\big\rangle_\Delta \\ &= \big\langle \mathcal{Y}_a(\cdot,\lambda)\phi_a, \mathcal{Y}_a(\cdot,\lambda)\psi_a\big\rangle_\Delta \\ &= -\big[Y(\cdot,\lambda)\phi_a, Y(\cdot,\lambda)\psi_a\big](a) \\ &= -h(a)^* J f(a)\end{aligned}$$

and from (5.22) one obtains

$$\begin{aligned}-h(a)^* J f(a) &= -(Uh(a))^* \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} Uf(a) \\ &= [Uh(a)]_0^* [Uf(a)]_1 - [Uh(a)]_1^* [Uf(a)]_0.\end{aligned}$$

Hence the abstract Lagrange identity (5.1) holds. The surjectivity of the mapping $\Gamma = (\Gamma_0, \Gamma_1)^\top : T_{\max} \to \mathbb{C}^m \times \mathbb{C}^m$ is a consequence of Proposition 4.14. Thus $\{\mathbb{C}^m, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $T_{\max}$.

To obtain expressions for the associated γ-field and Weyl function recall that

$$\mathfrak{N}_\lambda(T_{\max}) = \{\, Y(\cdot,\lambda)\phi : \phi \in \mathcal{B}^0(\lambda)\,\},$$

where $\dim \mathcal{B}^0(\lambda) = m$; see Proposition 4.21. Hence for $\widehat{f}_\lambda = \{Y(\cdot,\lambda)\phi, \lambda Y(\cdot,\lambda)\phi\}$, $\phi \in \mathcal{B}^0(\lambda)$, one has

$$\Gamma_0 \widehat{f}_\lambda = [UY(a,\lambda)\phi]_0, \quad \Gamma_1 \widehat{f}_\lambda = [UY(a,\lambda)\phi]_1, \quad \phi \in \mathcal{B}^0(\lambda).$$

Hence the statements on the γ-field and the Weyl function follows directly from Definition 5.2. □

Remark 5.9. Observe the analogy between the boundary triplet and the formulas for the γ-field and the Weyl function in Theorem 5.8 (with $U = I_n$) and the boundary triplet $\{\mathbb{C}^m, \Gamma_0', \Gamma_1'\}$, γ-field γ', and Weyl function M' below Corollary 5.7.

Let $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be the boundary triplet for $T_{\max}$ from Theorem 5.8. Then the selfadjoint relations $A_0 = \ker \Gamma_0$ and $A_1 = \ker \Gamma_1$ are given by

$$A_i = \ker \Gamma_i = \big\{\{f,g\} \in T_{\max} : [Uf(a)]_i = 0\big\}, \qquad i = 0, 1.$$

In the next corollary the selfadjoint realizations of the canonical system in the limit-point case are described with the help of Nevanlinna pairs $\{\Phi, \Psi\}$; cf. (5.11) and (5.12).

Corollary 5.10. *Assume that a is a quasiregular endpoint, that b is a singular endpoint which is in the limit-point case, and that the defect numbers of $T_{\min}$ are equal. Moreover, let U be a unitary $2m \times 2m$ matrix such that* (5.22) *holds and let Θ be a selfadjoint relation in $\mathbb{C}^m$ represented by a Nevanlinna pair of $m \times m$ matrices $\{\Phi, \Psi\}$ as in* (5.11)*. Then*

$$A_\Theta = \big\{\{f,g\} \in T_{\max} : \Phi[Uf(a)]_0 + \Psi[Uf(a)]_1 = 0\big\} \tag{5.23}$$

is a selfadjoint realization of the canonical system (2.4) *in $L^2_\Delta(\imath)$, and conversely, each selfadjoint realization of the canonical system can be written in the form* (5.23)*.*

The next theorem, which is a simple consequence of the previous theorem and Proposition 5.3, shows that the Weyl function M singles out the square-integrable solutions of the homogeneous canonical differential equation (2.5).

Theorem 5.11. *Let $\{\mathbb{C}^m, \Gamma_0, \Gamma_1\}$ be the boundary triplet for $T_{\max}$ from Theorem 5.8 and let γ and M be the associated γ-field and the Weyl function. Then*

$$\gamma(\lambda)\eta = Y(\cdot,\lambda) Y(a,\lambda)^{-1} U^{-1} \begin{pmatrix} \eta \\ M(\lambda)\eta \end{pmatrix}$$

holds for all $\eta \in \mathbb{C}^m$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. Since the γ-field is defined everywhere on $\mathbb{C}^m$ the mapping $\phi \mapsto [UY(a,\lambda)\phi]_0$ is an isomorphism from $\mathcal{B}^0(\lambda)$ onto $\mathbb{C}^m$; cf. Proposition 5.3. Hence for every $\eta \in \mathbb{C}^m$ there exists a unique $\phi \in \mathcal{B}^0(\lambda)$ such that $\eta = [UY(a,\lambda)\phi]_0$. Making use of the form of the Weyl function M from Theorem 5.8 and Proposition 5.3 one concludes

$$\begin{aligned}\gamma(\lambda)[UY(a,\lambda)\phi]_0 &= Y(\cdot,\lambda)\phi = Y(\cdot,\lambda)Y(a,\lambda)^{-1}U^{-1}\begin{pmatrix}[UY(a,\lambda)\phi]_0\\ [UY(a,\lambda)\phi]_1\end{pmatrix}\\ &= Y(\cdot,\lambda)Y(a,\lambda)^{-1}U^{-1}\begin{pmatrix}[UY(a,\lambda)\phi]_0\\ M(\lambda)\,[UY(a,\lambda)\phi]_0\end{pmatrix},\end{aligned}$$

which completes the proof. □

Example 5.12 (Weighted Sturm-Liouville equations)**.** Consider the Sturm-Liouville equation from Examples 2.12, 4.4, and 4.22 on the interval $\imath = (0,\infty)$ and assume $r(t) > 0$ for $t \in \imath$. Then the corresponding canonical system is definite and $\mathcal{T}_{\max}$ is (the graph of) an operator. Let the Sturm-Liouville expression

$$\ell = \frac{1}{r}\left(-\frac{d}{dt}\,p\,\frac{d}{dt} + q\right)$$

be regular at 0 and in the limit-point case at ∞. Then $\mathsf{a}^+ = \mathsf{a}^- = 1$ and $\mathsf{b}^+ = \mathsf{b}^- = 0$, and the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ in Theorem 5.8 (here $U = I_2$) is given by

$$\Gamma_0\{f_1, \mathcal{T}_{\max} f_1\} = f_1(0), \quad \Gamma_1\{f_1, \mathcal{T}_{\max} f_1\} = (pf_1')(0), \quad f_1 \in \operatorname{dom}\mathcal{T}_{\max}.$$

The selfadjoint realizations A_0 and A_1 coincide with the Sturm-Liouville operators corresponding to Dirichlet and Neumann boundary conditions at 0, respectively. Let

$$Y(t,\lambda) = \begin{pmatrix}u_1(t,\lambda) & v_1(t,\lambda)\\ u_2(t,\lambda) & v_2(t,\lambda)\end{pmatrix}, \qquad \lambda \in \mathbb{C}\setminus\mathbb{R},\ t \in (0,\infty),$$

be a fundamental matrix of the corresponding canonical system with $Y(0,\lambda) = I_2$. Then $u_1(\cdot,\lambda)$ and $v_1(\cdot,\lambda)$ are solutions of the differential equation $\ell f = \lambda f$ which satisfy the boundary conditions $u_1(0,\lambda) = (pv_1)'(0,\lambda) = 1$ and $(pu_1)'(0,\lambda) = v_1(0,\lambda) = 0$. In this situation Theorem 5.11 implies

$$u_1(\cdot,\lambda) + M(\lambda)v_1(\cdot,\lambda) \in L^2(0,\infty), \qquad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

i.e., the Weyl function M coincides with the classical Titchmarsh-Weyl coefficient associated to the singular Sturm-Liouville expression which combines the solutions $u_1(\cdot,\lambda)$ and $v_1(\cdot,\lambda)$ to a square-integrable solution; cf. [61, 62, 63] and [11, 25, 45].

5.4. **General canonical systems with equal defect numbers.** In this subsection boundary mappings for the maximal relation $T_{\max}$ associated to the canonical system (2.4), see Section 4.1, are given under the assumption that the defect numbers of the minimal relation $T_{\min}$ are equal, that is,

$$m := \mathsf{a}^- + \mathsf{b}^+ = \mathsf{a}^+ + \mathsf{b}^- \tag{5.24}$$

holds; cf. (4.7).

Fix some $\lambda_0 \in \mathbb{C}\setminus\mathbb{R}$ and consider the matrices $D(a,\lambda_0)_s$ and $D(b,\lambda_0)_s$ from Theorem 3.2 which have a^+ positive and a^- negative eigenvalues, and b^+ positive and b^- negative eigenvalues, respectively. Their restrictions to the corresponding positive eigenspaces $\mathcal{A}^+(\lambda_0)$, $\mathcal{B}^+(\lambda_0)$ and negative eigenspaces $\mathcal{A}^-(\lambda_0)$, $\mathcal{B}^-(\lambda_0)$ will be denoted by $D(a,\lambda_0)^+$, $D(b,\lambda_0)^+$, $D(a,\lambda_0)^-$, and $D(b,\lambda_0)^-$, respectively. Recall that $\mathcal{A}(\lambda_0) = \mathcal{A}^+(\lambda_0) \oplus \mathcal{A}^-(\lambda_0)$ and $\mathcal{B}(\lambda_0) = \mathcal{B}^+(\lambda_0) \oplus \mathcal{B}^-(\lambda_0)$; cf. (4.12). As a consequence of the assumption (5.24), Lemma 2.4 implies that there exists a (nonunique) invertible $2m \times 2m$ matrix V in $\mathcal{A}(\lambda_0) \times \mathcal{B}(\lambda_0)$ such that

$$V^*\begin{pmatrix}0 & -iI_m\\ iI_m & 0\end{pmatrix}V = \begin{pmatrix}-D(a,\lambda_0)^+ & 0 & 0 & 0\\ 0 & -D(a,\lambda_0)^- & 0 & 0\\ 0 & 0 & D(b,\lambda_0)^+ & 0\\ 0 & 0 & 0 & D(b,\lambda_0)^-\end{pmatrix}. \tag{5.25}$$

The next theorem gives a description of the boundary triplets for general singular canonical systems with equal defect numbers. Roughly speaking the Lagrange identity (4.10) will be rewritten with the help of the decomposition in Theorem 4.12, the matrices $D(a,\lambda_0)^\pm$ and $D(b,\lambda_0)^\pm$ and the identity (5.25). The formulas for the boundary mappings in Theorem 5.13 below can be written in a more explicit form by constructing V and applying Corollary 4.13, see also Section 5.5. As in (5.2) the components of a vector $\psi \in \mathbb{C}^{2m}$ with respect to the decomposition $\mathbb{C}^{2m} = \mathbb{C}^m \times \mathbb{C}^m$ will written as $[\psi]_0$ and $[\psi]_1$.

Theorem 5.13. *Assume that the defect numbers of $T_{\min}$ are equal. Fix $\lambda_0 \in \mathbb{C}\setminus\mathbb{R}$ and decompose $\{f,g\} \in T_{\max}$ according to Theorem 4.12 in the form*

$$\{f,g\} = \{f_0,g_0\} + \mathcal{Y}_a(\cdot,\lambda_0)\phi_a + \mathcal{Y}_b(\cdot,\lambda_0)\phi_b,$$

with $\{f_0,g_0\} \in T_{\min}$, $\phi_a \in \mathcal{A}(\lambda_0)$, $\phi_b \in \mathcal{B}(\lambda_0)$. Then the following statements hold:

(i) *if V is a matrix which satisfies* (5.25)*, then $\{\mathbb{C}^m,\Gamma_0,\Gamma_1\}$, with*

$$\Gamma_0\{f,g\} = \left[V\begin{pmatrix}\phi_a\\ \phi_b\end{pmatrix}\right]_0 \quad \text{and} \quad \Gamma_1\{f,g\} = \left[V\begin{pmatrix}\phi_a\\ \phi_b\end{pmatrix}\right]_1,$$

is a boundary triplet for $T_{\max}$.

(ii) *if $\{\mathbb{C}^m,\Gamma_0,\Gamma_1\}$ is a boundary triplet for $T_{\max}$, then there exists a (nonunique) matrix V which satisfies* (5.25) *such that Γ_0 and Γ_1 have the form in (i).*

Proof. (i) Decompose $\{f,g\},\{h,k\} \in T_{\max}$ in the form

$$\begin{aligned}\{f,g\} &= \{f_0,g_0\} + \mathcal{Y}_a(\cdot,\lambda_0)\phi_a + \mathcal{Y}_b(\cdot,\lambda_0)\phi_b,\\ \{h,k\} &= \{h_0,k_0\} + \mathcal{Y}_a(\cdot,\lambda_0)\psi_a + \mathcal{Y}_b(\cdot,\lambda_0)\psi_b,\end{aligned} \tag{5.26}$$

with $\{f_0,g_0\},\{h_0,k_0\} \in T_{\min}$, $\phi_a,\psi_a \in \mathcal{A}(\lambda_0)$ and $\phi_b,\psi_b \in \mathcal{B}(\lambda_0)$. Then the Lagrange identity (4.10) becomes

$$\begin{aligned}\langle\{f,g\},\{h,k\}\rangle_\Delta &= \langle \mathcal{Y}_a(\cdot,\lambda_0)\phi_a + \mathcal{Y}_b(\cdot,\lambda_0)\phi_b, \mathcal{Y}_a(\cdot,\lambda_0)\psi_a + \mathcal{Y}_b(\cdot,\lambda_0)\psi_b\rangle_\Delta\\ &= \big[Y(\cdot,\lambda_0)\phi_b, Y(\cdot,\lambda_0)\psi_b\big](b) - \big[Y(\cdot,\lambda_0)\phi_a, Y(\cdot,\lambda_0)\psi_a\big](a).\end{aligned}$$

In a similar way as in the proof of Theorem 4.12 one concludes from (4.9), (3.1), and (3.6) that

$$\begin{aligned}&\lim_{t\uparrow b}\psi_b^* Y(t,\lambda_0)^* J Y(t,\lambda_0)\phi_b - \lim_{t\downarrow a}\psi_a^* Y(t,\lambda_0)^* J Y(t,\lambda_0)\phi_a\\ &= i\lim_{t\uparrow b}\psi_b^* D(t,\lambda_0)\phi_b - i\lim_{t\downarrow a}\psi_a^* D(t,\lambda_0)\phi_a = i\psi_b^* D(b,\lambda_0)_s\phi_b - i\psi_a^* D(a,\lambda_0)_s\phi_a\\ &= i\psi_b^*\begin{pmatrix}D(b,\lambda_0)^+ & \\ 0 & D(b,\lambda_0)^-\end{pmatrix}\phi_b - i\psi_a^*\begin{pmatrix}D(a,\lambda_0)^+ & 0\\ 0 & D(a,\lambda_0)^-\end{pmatrix}\phi_a\end{aligned}$$

Combing the previous two identities with the identity (5.25) and the definition of Γ_0 and Γ_1 ones gets

$$\begin{aligned}&\langle\{f,g\},\{h,k\}\rangle_\Delta\\ &= i\begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix}^*\begin{pmatrix}-D(a,\lambda_0)^+ & 0 & 0 & 0\\ 0 & -D(a,\lambda_0)^- & 0 & 0\\ 0 & 0 & D(b,\lambda_0)^+ & 0\\ 0 & 0 & 0 & D(b,\lambda_0)^-\end{pmatrix}\begin{pmatrix}\phi_a\\ \phi_b\end{pmatrix}\\ &= \begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix}^* V^*\begin{pmatrix}0 & I_m\\ -I_m & 0\end{pmatrix}V\begin{pmatrix}\phi_a\\ \phi_b\end{pmatrix}\\ &= \left(\begin{pmatrix}0 & I_m\\ -I_m & 0\end{pmatrix}\begin{pmatrix}\Gamma_0\{f,g\}\\ \Gamma_1\{f,g\}\end{pmatrix},\begin{pmatrix}\Gamma_0\{h,k\}\\ \Gamma_1\{h,k\}\end{pmatrix}\right)\\ &= \big(\Gamma_1\{f,g\},\Gamma_0\{h,k\}\big) - \big(\Gamma_0\{f,g\},\Gamma_1\{h,k\}\big).\end{aligned}$$

Since $\dim\mathcal{A}(\lambda_0)\times\mathcal{B}(\lambda_0) = 2m$ and V is invertible, the mapping $\Gamma = (\Gamma_0,\Gamma_1)^\top : T_{\max} \to \mathbb{C}^m\times\mathbb{C}^m$ is onto. Hence $\{\mathbb{C}^m,\Gamma_0,\Gamma_1\}$ is a boundary triplet for $T_{\max}$.

(ii) Suppose that $\{\mathbb{C}^m, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $T_{\max}$ and let V be a fixed matrix that satisfies (5.25). Then there exists a unique $2m \times 2m$ matrix W such that (5.13) and (5.14) hold with $I_m = I_{\mathcal{H}} = I_{\mathcal{H}'}$ and

$$\begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} \{f, g\} = WV \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix}, \qquad \{f, g\} \in T_{\max}.$$

It is not difficult to check that the matrix $\widetilde{V} := WV$ also satisfies (5.25) which implies (ii). □

For completeness the analogue of Corollary 5.6 and 5.10 is stated in the general case.

Corollary 5.14. *Let $\{\mathbb{C}^m, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $T_{\max}$ from Theorem 5.13 and let Θ be a selfadjoint relation in $\mathbb{C}^m$ represented by a Nevanlinna pair of $n \times n$ matrices $\{\Phi, \Psi\}$ as in* (5.11)*. Then*

$$A_\Theta = \left\{ \{f, g\} \in T_{\max} : \Phi \left[V \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix} \right]_0 + \Psi \left[V \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix} \right]_1 = 0 \right\} \tag{5.27}$$

is a selfadjoint realization of the canonical system in $L^2_\Delta(\imath)$, and conversely, each selfadjoint realization of the canonical system can be written in the form (5.27)*.*

To derive the formulas for the corresponding γ-field and the Weyl function, the m-dimensional space $\mathfrak{D}(a, \lambda) \cap \mathfrak{D}(b, \lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, will be identified with a subspace of the $2m$-dimensional space $\mathcal{A}(\lambda_0) \times \mathcal{B}(\lambda_0)$ with $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ fixed. Recall that

$$\widehat{\mathfrak{N}}_\lambda(T_{\max}) = \big\{ \{Y(\cdot, \lambda)\phi, \lambda Y(\cdot, \lambda)\phi\} : \phi \in \mathfrak{D}(a, \lambda) \cap \mathfrak{D}(b, \lambda) \big\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

see (4.2). It follows from Theorem 4.12 that for $\phi \in \mathfrak{D}(a, \lambda) \cap \mathfrak{D}(b, \lambda)$ there exist unique $\{f_0(\lambda), g_0(\lambda)\} \in T_{\min}$, $\phi_a(\lambda) \in \mathcal{A}(\lambda_0)$ and $\phi_b(\lambda) \in \mathcal{B}(\lambda_0)$ such that

$$\widehat{f}_\lambda = \{Y(\cdot, \lambda)\phi, \lambda Y(\cdot, \lambda)\phi\} = \{f_0(\lambda), g_0(\lambda)\} + \mathcal{Y}_a(\cdot, \lambda_0)\phi_a(\lambda) + \mathcal{Y}_b(\cdot, \lambda_0)\phi_b(\lambda) \tag{5.28}$$

holds. Hence the mapping

$$Z(\lambda) : \mathfrak{D}(a, \lambda) \cap \mathfrak{D}(b, \lambda) \to \mathcal{A}(\lambda_0) \times \mathcal{B}(\lambda_0), \qquad \phi \mapsto \begin{pmatrix} \phi_a(\lambda) \\ \phi_b(\lambda) \end{pmatrix},$$

is injective and $\operatorname{ran} Z(\lambda)$ is an m-dimensional subspace of the $2m$-dimensional space $\mathcal{A}(\lambda_0) \times \mathcal{B}(\lambda_0)$.

Proposition 5.15. *Assume that the defect numbers of $T_{\min}$ are equal, let V be a matrix which satisfies* (5.25)*, and let $\{\mathbb{C}^m, \Gamma_0, \Gamma_1\}$ be the corresponding boundary triplet for $T_{\max}$ from Theorem 5.13. Then the associated γ-field γ and the Weyl function M have the form*

$$\gamma(\lambda) = \big\{ \{[VZ(\lambda)\phi]_0, Y(\cdot, \lambda)\phi\} : \phi \in \mathfrak{D}(a, \lambda) \cap \mathfrak{D}(b, \lambda) \big\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

and

$$M(\lambda) = \big\{ \{[VZ(\lambda)\phi]_0, [VZ(\lambda)\phi]_1\} : \phi \in \mathfrak{D}(a, \lambda) \cap \mathfrak{D}(b, \lambda) \big\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Proof. Decompose $\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(T_{\max})$ in the form (5.28) with $\phi \in \mathfrak{D}(a, \lambda) \cap \mathfrak{D}(b, \lambda)$ and $\phi_a(\lambda) \in \mathcal{A}(\lambda_0)$, $\phi_b(\lambda) \in \mathcal{B}(\lambda_0)$. Then the definition of the mappings Γ_i in Theorem 5.13 shows that

$$\Gamma_i \widehat{f}_\lambda = \left[V \begin{pmatrix} \phi_a(\lambda) \\ \phi_b(\lambda) \end{pmatrix} \right]_i = [VZ(\lambda)\phi]_i, \quad i = 0, 1.$$

Now the expressions for the γ-field and Weyl function follow from Definition 5.2. □

The following statement shows that also in the general singular case with equal defect numbers the Weyl function associated to a boundary triplet singles out the square-integrable solutions of the homogeneous canonical differential equation. Here the inverse mapping $Z(\lambda)^{-1} : \operatorname{ran} Z(\lambda) \to \mathfrak{D}(a, \lambda) \cap \mathfrak{D}(b, \lambda)$ will be used.

Theorem 5.16. *Let $\{\mathbb{C}^m, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $T_{\max}$ from Theorem 5.13 and let γ and M be the associated γ-field and Weyl function from Proposition 5.15. Then*

$$\gamma(\lambda)\eta = Y(\cdot,\lambda)Z(\lambda)^{-1}V^{-1}\begin{pmatrix}\eta\\ M(\lambda)\eta\end{pmatrix}$$

holds for all $\eta \in \mathbb{C}^m$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Proof. Let $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Since the γ-field is defined everywhere on $\mathbb{C}^m$ the mapping $\phi \mapsto [VZ(\lambda)\phi]_0$ is an isomorphism from $\mathfrak{D}(a,\lambda)\cap\mathfrak{D}(b,\lambda)$ onto $\mathbb{C}^m$; cf. Proposition 5.3. Hence for every $\eta \in \mathbb{C}^m$ there exists a unique $\phi \in \mathfrak{D}(a,\lambda)\cap\mathfrak{D}(b,\lambda)$ such that $\eta = [VZ(\lambda)\phi]_0$. Now Proposition 5.15 implies

$$\gamma(\lambda)\,[VZ(\lambda)\phi]_0 = Y(\cdot,\lambda)\phi = Y(\cdot,\lambda)Z(\lambda)^{-1}V^{-1}VZ(\lambda)\phi,$$

where $Z(\lambda)\phi \in \mathcal{A}(\lambda_0)\times\mathcal{B}(\lambda_0)$. Making use of the Weyl function M in Proposition 5.15 and Proposition 5.3 one obtains

$$\begin{aligned}\gamma(\lambda)\,[VZ(\lambda)\phi]_0 &= Y(\cdot,\lambda)Z(\lambda)^{-1}V^{-1}\begin{pmatrix}[VZ(\lambda)\phi]_0\\ [VZ(\lambda)\phi]_1\end{pmatrix}\\ &= Y(\cdot,\lambda)Z(\lambda)^{-1}V^{-1}\begin{pmatrix}[VZ(\lambda)\phi]_0\\ M(\lambda)[VZ(\lambda)\phi]_0\end{pmatrix},\end{aligned}$$

which completes the proof. □

The result in Theorem 5.16 holds for all boundary triplets for $T_{\max}$: if W is a matrix which satisfies (5.13) and (5.14), then WV satisfies (5.25) and hence the γ-field γ_W and the Weyl function M_W associate to WV via the boundary triplet in Theorem 5.13 satisfy by Theorem 5.16

$$\gamma_W(\lambda)\eta = Y(\cdot,\lambda)Z(\lambda)^{-1}(WV)^{-1}\begin{pmatrix}\eta\\ M_W(\lambda)\eta\end{pmatrix}$$

for all $\eta \in \mathbb{C}^m$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

5.5. **Boundary triplets in terms of the limit relations.** In this subsection boundary triplets for singular canonical systems with equal defect numbers (m,m) in two special cases are expressed in terms of the limit relations $D(a,\lambda_0)$ and $D(b,\lambda_0)$, $\lambda_0 \in \mathbb{C}\setminus\mathbb{R}$. More specifically, these boundary triplets are obtained by constructing a $2m\times 2m$ matrix V satisfying (5.25), see Theorem 5.13, in terms of the restrictions $D(a,\lambda_0)^+$, $D(a,\lambda_0)^-$, $D(b,\lambda_0)^+$, and $D(b,\lambda_0)^-$ of $D(a,\lambda_0)_s$ and $D(b,\lambda_0)_s$, respectively, cf. Section 5.4.

Define the $2m\times 2m$ matrix C by

$$C = \begin{pmatrix}(D(a,\lambda_0)^+)^{\frac12} & 0 & 0 & 0\\ 0 & (-D(a,\lambda_0)^-)^{\frac12} & 0 & 0\\ 0 & 0 & (D(b,\lambda_0)^+)^{\frac12} & 0\\ 0 & 0 & 0 & (-D(b,\lambda_0)^-)^{\frac12}\end{pmatrix}.$$

Furthermore, define the $2m\times 2m$ matrix S and the unitary $2m\times 2m$ matrix U by

$$S = \begin{pmatrix}0 & I_{\mathsf{a}^-} & 0 & 0\\ 0 & 0 & I_{\mathsf{b}^+} & 0\\ I_{\mathsf{a}^+} & 0 & 0 & 0\\ 0 & 0 & 0 & I_{\mathsf{b}^-}\end{pmatrix}, \quad U = \frac{1}{\sqrt2}\begin{pmatrix}I_{\mathsf{a}^-} & 0 & I_{\mathsf{a}^+} & 0\\ 0 & I_{\mathsf{b}^+} & 0 & I_{\mathsf{b}^-}\\ iI_{\mathsf{a}^-} & 0 & -iI_{\mathsf{a}^+} & 0\\ 0 & iI_{\mathsf{b}^+} & 0 & -iI_{\mathsf{b}^-}\end{pmatrix}.$$

Then it is not difficult to check that the matrix $V := USC$ satisfies (5.25). This matrix can be computed for the general case of equal defect numbers of $T_{\min}$. In the next corollary the special case $\mathsf{a}^+ = \mathsf{a}^-$, or equivalently $\mathsf{b}^+ = \mathsf{b}^-$, is considered. In this situation one has

$$V = \frac{1}{\sqrt2}\begin{pmatrix}(D(a,\lambda_0)^+)^{\frac12} & (-D(a,\lambda_0)^-)^{\frac12} & 0 & 0\\ 0 & 0 & (D(b,\lambda_0)^+)^{\frac12} & (D(b,\lambda_0)^-)^{\frac12}\\ -i(D(a,\lambda_0)^+)^{\frac12} & i(-D(a,\lambda_0)^-)^{\frac12} & 0 & 0\\ 0 & 0 & i(D(b,\lambda_0)^+)^{\frac12} & -i(-D(b,\lambda_0)^-)^{\frac12}\end{pmatrix}.$$

In the following the elements $\phi_a \in \mathcal{A}(\lambda_0)$ and $\phi_b \in \mathcal{B}(\lambda_0)$ are decomposed in $\phi_a^\pm \in \mathcal{A}^\pm(\lambda_0)$ and $\phi_b^\pm \in \mathcal{B}^\pm(\lambda_0)$, respectively.

Corollary 5.17. *Suppose, in addition to* (5.24)*, that* $\mathsf{a}^+ = \mathsf{a}^-$ *or, equivalently,* $\mathsf{b}^+ = \mathsf{b}^-$ *holds, and decompose* $\{f,g\} \in T_{\max}$ *according to Theorem 4.12 in the form*

$$\{f,g\} = \{f_0, g_0\} + \mathcal{Y}_a(\cdot,\lambda_0)\phi_a + \mathcal{Y}_b(\cdot,\lambda_0)\phi_b,$$

with $\{f_0,g_0\} \in T_{\min}$*,* $\phi_a \in \mathcal{A}(\lambda_0)$*, and* $\phi_b \in \mathcal{B}(\lambda_0)$*. Then* $\{\mathbb{C}^m, \Gamma_0, \Gamma_1\}$*, with*

$$\Gamma_0\{f,g\} = \frac{1}{\sqrt{2}} \begin{pmatrix} (D(a,\lambda_0)^+)^{\frac12}\phi_a^+ + (-D(a,\lambda_0)^-)^{\frac12}\phi_a^- \\ (D(b,\lambda_0)^+)^{\frac12}\phi_b^+ + (D(b,\lambda_0)^-)^{\frac12}\phi_b^- \end{pmatrix}$$

and

$$\Gamma_1\{f,g\} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i(D(a,\lambda_0)^+)^{\frac12}\phi_a^+ + i(-D(a,\lambda_0)^-)^{\frac12}\phi_a^- \\ i(D(b,\lambda_0)^+)^{\frac12}\phi_b^+ - i(-D(b,\lambda_0)^-)^{\frac12}\phi_b^- \end{pmatrix},$$

is a boundary triplet for $T_{\max}$*.*

If the endpoint a is quasiregular and b is in the limit-point case, then the boundary triplet in Corollary 5.17 can be transformed into the one in Theorem 5.8.

Similar considerations as above show that in the special case $\mathsf{a}^+ = \mathsf{b}^+$, or equivalently $\mathsf{a}^- = \mathsf{b}^-$, the matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} (D(a,\lambda_0)^+)^{\frac12} & 0 & (D(b,\lambda_0)^+)^{\frac12} & 0 \\ 0 & (-D(a,\lambda_0)^-)^{\frac12} & 0 & (-D(b,\lambda_0)^-)^{\frac12} \\ -i(D(a,\lambda_0)^+)^{\frac12} & 0 & i(D(b,\lambda_0)^+)^{\frac12} & 0 \\ 0 & i(-D(a,\lambda_0)^-)^{\frac12} & 0 & -i(-D(b,\lambda_0)^-)^{\frac12} \end{pmatrix}$$

satisfies (5.25). This leads to the following corollary, which can be regarded as a generalization of the quasiregular case from Section 5.2.

Corollary 5.18. *Suppose, in addition to* (5.24)*, that* $\mathsf{a}^+ = \mathsf{b}^+$ *or, equivalently,* $\mathsf{a}^- = \mathsf{b}^-$ *holds, and decompose* $\{f,g\} \in T_{\max}$ *according to Theorem 4.12 in the form*

$$\{f,g\} = \{f_0, g_0\} + \mathcal{Y}_a(\cdot,\lambda_0)\phi_a + \mathcal{Y}_b(\cdot,\lambda_0)\phi_b,$$

with $\{f_0,g_0\} \in T_{\min}$*,* $\phi_a \in \mathcal{A}(\lambda_0)$*, and* $\phi_b \in \mathcal{B}(\lambda_0)$*. Then* $\{\mathbb{C}^m, \Gamma_0, \Gamma_1\}$*, with*

$$\Gamma_0\{f,g\} = \frac{1}{\sqrt{2}} \begin{pmatrix} (D(a,\lambda_0)^+)^{\frac12}\phi_a^+ + (D(b,\lambda_0)^+)^{\frac12}\phi_b^+ \\ (-D(a,\lambda_0)^-)^{\frac12}\phi_a^- + (-D(b,\lambda_0)^-)^{\frac12}\phi_b^- \end{pmatrix}$$

and

$$\Gamma_1\{f,g\} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i(D(a,\lambda_0)^+)^{\frac12}\phi_a^+ + i(D(b,\lambda_0)^+)^{\frac12}\phi_b^+ \\ i(-D(a,\lambda_0)^-)^{\frac12}\phi_a^- - i(-D(b,\lambda_0)^-)^{\frac12}\phi_b^- \end{pmatrix},$$

is a boundary triplet for $T_{\max}$*.*

Remark 5.19. The boundary triplets in Corollaries 5.17 and 5.18 may be written in a more explicit form by expressing $\phi_a^\pm$ and $\phi_b^\pm$ in terms of $\{f,g\} \in T_{\max}$. More precisely, if $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ is fixed and $\{f,g\} \in T_{\max}$ is decomposed in the form

$$\{f,g\} = \{f_0,g_0\} + \mathcal{Y}_a(\cdot,\lambda_0)\begin{pmatrix}\phi_a^+ \\ \phi_a^-\end{pmatrix} + \mathcal{Y}_b(\cdot,\lambda_0)\begin{pmatrix}\phi_b^+ \\ \phi_b^-\end{pmatrix},$$

where $\{f_0,g_0\} \in T_{\min}$, then it follows that

$$(D(a,\lambda_0)_s^\pm \phi_a^\pm, \chi_a^\pm) = \big[f, Y(\cdot,\lambda_0)\chi_a^\pm\big](a),$$
$$(D(b,\lambda_0)_s^\pm \phi_b^\pm, \chi_b^\pm) = \big[f, Y(\cdot,\lambda_0)\chi_b^\pm\big](b),$$

hold for all $\chi_a^\pm \in \mathcal{A}(\lambda_0)^\pm$ and $\chi_b^\pm \in \mathcal{B}(\lambda_0)^\pm$, respectively; cf. Corollary 4.13. Therefore, by introducing bases in $\mathcal{A}(\lambda_0)^\pm$ and $\mathcal{B}(\lambda_0)^\pm$ the elements $\phi_a^\pm$ and $\phi_b^\pm$ can be computed in terms of $\{f,g\}$.

6. Boundary triplets and Weyl functions for singular canonical systems with unequal defect numbers.

The notion of boundary triplets can be extended to symmetric operators and relations with unequal defect numbers; cf. [43] and [44]. In this section the definition and some properties of such boundary triplets and the associated γ-fields and Weyl functions are briefly recalled and the class of boundary triplets for singular canonical systems with unequal defect numbers is characterized. Furthermore it is shown that also in the general singular case with unequal defect numbers the Weyl function singles out the square-integrable solutions of the homogeneous canonical differential equation.

6.1. **Boundary triplets in the case of unequal defect numbers.** Let S be a closed symmetric relation with unequal defect numbers $n_+(S) < n_-(S)$ in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot)_{\mathfrak{H}})$. The following definition of a boundary triplet for this case is taken from [43]. The range $\mathcal{H}_0$ of the first boundary mapping will be decomposed in subspaces $\mathcal{H}_1 \oplus \mathcal{H}_2$ and the orthogonal projections in $\mathcal{H}_0$ onto $\mathcal{H}_1$ and $\mathcal{H}_2$ will be denoted by P_1 and P_2, respectively.

Definition 6.1. A *boundary triplet* $\{\mathcal{H}_0 \times \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for the adjoint relation S^* consists of an auxiliary Hilbert space $(\mathcal{H}_0, (\cdot,\cdot)_{\mathcal{H}_0})$ which decomposes into the orthogonal sum $\mathcal{H}_0 = \mathcal{H}_1 \oplus \mathcal{H}_2$, and two mappings $\Gamma_j : S^* \to \mathcal{H}_j$, $j = 0, 1$, such that

$$(f', g)_{\mathfrak{H}} - (f, g')_{\mathfrak{H}} = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g})_{\mathcal{H}_0} - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g})_{\mathcal{H}_0} + i(P_2\Gamma_0 \widehat{f}, P_2\Gamma_0 \widehat{g})_{\mathcal{H}_2}$$

holds for all $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in S^*$ and the mapping $\Gamma : \widehat{f} \to \{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\}$ from S^* to $\mathcal{H}_0 \times \mathcal{H}_1$ is surjective.

Boundary triplets in the case of unequal defect numbers have similar properties as boundary triplets for symmetric relations with equal defect numbers in Section 5.1. In the following some basic facts from [43] are recalled for the convenience of the reader. If $\{\mathcal{H}_0 \times \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a boundary triplet for S^*, then

$$\dim \mathcal{H}_0 = n_-(S) \qquad \text{and} \qquad \dim \mathcal{H}_1 = n_+(S).^1$$

Furthermore, the closed extension $A_0 = \ker \Gamma_0$ is maximal symmetric and the same holds for A_1^*, where $A_1 = \ker \Gamma_1$. The mapping $\Theta \mapsto A_\Theta$ in (5.9) establishes a bijective correspondence between the closed linear relations in $\mathcal{H}_0 \times \mathcal{H}_1$ and the closed extensions $A_\Theta \subset S^*$ of S. In particular, the maximal symmetric, maximal dissipative or maximal accumulative extensions A_Θ can be described with the help of similar properties of the relation $\Theta \subset \mathcal{H}_0 \times \mathcal{H}_1$; cf. [43, Proposition 3.9]. Moreover, if $\{\mathcal{H}_0' \times \mathcal{H}_1', \Gamma_0', \Gamma_1'\}$ is a second boundary triplet for S^* then there exists a block operator matrix W with similar properties as (5.13) and (5.14) such that $(\Gamma_0', \Gamma_1')^\top = W(\Gamma_0, \Gamma_1)^\top$ holds, see [43, Proposition 3.12] for details.

The following definition is a generalization of Definition 5.2. Note that the dimension of the eigenspace $\widehat{\mathfrak{N}}_\lambda(S^*)$ from (5.6) is given by

$$\dim \widehat{\mathfrak{N}}_\lambda(S^*) = \begin{cases} \dim \mathcal{H}_0, & \lambda \in \mathbb{C}_+, \\ \dim \mathcal{H}_1, & \lambda \in \mathbb{C}_-. \end{cases}$$

Definition 6.2. Let $\{\mathcal{H}_0 \times \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*. The associated *γ-field* is defined by

$$\gamma(\lambda) = \begin{cases} \{\, \{\Gamma_0 \widehat{f}_\lambda, f_\lambda\} : \widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(S^*) \,\}, & \lambda \in \mathbb{C}_+, \\ \{\, \{P_1\Gamma_0 \widehat{f}_\lambda, f_\lambda\} : \widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(S^*) \,\}, & \lambda \in \mathbb{C}_-, \end{cases}$$

and the associated *Weyl function* is defined by

$$M(\lambda) = \begin{cases} \{\, \{\Gamma_0 \widehat{f}_\lambda, \Gamma_1 \widehat{f}_\lambda\} : \widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(S^*) \,\}, & \lambda \in \mathbb{C}_+, \\ \left\{ \left\{ P_1\Gamma_0 \widehat{f}_\lambda, \begin{pmatrix} \Gamma_1 \widehat{f}_\lambda \\ iP_2\Gamma_0 \widehat{f}_\lambda \end{pmatrix} \right\} : \widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(S^*) \right\}, & \lambda \in \mathbb{C}_-. \end{cases}$$

[1] Note that in [43] the defect numbers of a closed symmetric relation T are defined as $\widetilde{n}_\pm(T) := \dim \ker (T^* - \lambda)$, $\lambda \in \mathbb{C}_\pm$, whereas in this paper the usual definition $n_\pm(T) = \dim \ker (T^* - \lambda)$, $\lambda \in \mathbb{C}_\mp$, is used; cf. (A.3).

The above definition parallels Definition 5.2 and differs only for $\lambda \in \mathbb{C}_-$ from this definition. Note that for $\lambda \in \mathbb{C}_-$ the element in ran $M(\lambda)$ is decomposed with respect to the decomposition $\mathcal{H}_0 = \mathcal{H}_1 \oplus \mathcal{H}_2$. In [43] the γ-field and Weyl function are formally defined in a slightly different way. The next proposition is the analogue of Proposition 5.3. The orthogonal projection in $\mathfrak{H} \oplus \mathfrak{H}$ onto the first component is denoted by π_1.

Proposition 6.3. *The restriction $\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^*)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, of the mapping Γ_0 to $\widehat{\mathfrak{N}}_\lambda(S^*)$ is a bijective mapping onto $\mathcal{H}_0$ or $\mathcal{H}_1$ if $\lambda \in \mathbb{C}_+$ or $\lambda \in \mathbb{C}_-$, respectively. In particular, $\gamma(\lambda)$ is the graph of a bounded linear operator from $\mathcal{H}_0$ or $\mathcal{H}_1$ to $\mathfrak{N}_\lambda(S^*)$ if $\lambda \in \mathbb{C}_+$ or $\lambda \in \mathbb{C}_-$, respectively, given by*

$$\gamma(\lambda) = \begin{cases} \pi_1(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^*))^{-1}, & \lambda \in \mathbb{C}_+, \\ \pi_1(P_1\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^*))^{-1}, & \lambda \in \mathbb{C}_-. \end{cases}$$

The values $M(\lambda)$ of the Weyl function M are in $\mathbf{B}(\mathcal{H}_0, \mathcal{H}_1)$ or $\mathbf{B}(\mathcal{H}_1, \mathcal{H}_0)$ if $\lambda \in \mathbb{C}_+$ or $\lambda \in \mathbb{C}_-$, respectively, and are given by

$$M(\lambda) = \begin{cases} \Gamma_1(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^*))^{-1}, & \lambda \in \mathbb{C}_+, \\ \begin{pmatrix} \Gamma_1(P_1\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^*))^{-1} \\ iP_2\Gamma_0(P_1\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^*))^{-1} \end{pmatrix}, & \lambda \in \mathbb{C}_-. \end{cases}$$

The analogues of the formulas (5.7), (5.8) and more details on the properties of γ and M can be found in [43].

6.2. **General singular canonical systems with unequal defect numbers.** In this section boundary triplets for singular canonical systems which do not satisfy the assumption $\mathsf{a}^+ + \mathsf{b}^- = \mathsf{a}^- + \mathsf{b}^+$ from Section 5 will be characterized. For brevity only the case $n_+(T_{\min}) < n_-(T_{\min})$, i.e.,

$$m := \mathsf{a}^+ + \mathsf{b}^- < \mathsf{a}^- + \mathsf{b}^+, \tag{6.1}$$

see (4.7), will be discussed. Similar results can be established for $n_+(T_{\min}) > n_-(T_{\min})$. Let r be a positive integer such that

$$m + r = \mathsf{a}^- + \mathsf{b}^+.$$

Before stating an analogue of Theorem 5.13 in the case (6.1) of unequal defect numbers a suitable generalization of the identity (5.25) will be provided. For this fix $\lambda_0 \in \mathbb{C}^+$ and denote by $D(a,\lambda_0)^+$, $D(a,\lambda_0)^-$, $D(b,\lambda_0)^+$ and $D(b,\lambda_0)^-$ the restrictions of $D(a,\lambda_0)_s$ and $D(b,\lambda_0)_s$ onto the subspaces $\mathcal{A}^+(\lambda_0)$, $\mathcal{A}^-(\lambda_0)$, $\mathcal{B}^+(\lambda_0)$ and $\mathcal{B}^-(\lambda_0)$ corresponding to positive and negative eigenvalues, respectively. A variant of Lemma 2.4 shows that there exists an invertible $(2m + r) \times (2m + r)$ matrix V such that

$$V^* \begin{pmatrix} 0 & 0 & -iI_m \\ 0 & I_r & 0 \\ iI_m & 0 & 0 \end{pmatrix} V = \begin{pmatrix} -D(a,\lambda_0)^+ & 0 & 0 & 0 \\ 0 & -D(a,\lambda_0)^- & 0 & 0 \\ 0 & 0 & D(b,\lambda_0)^+ & 0 \\ 0 & 0 & 0 & D(b,\lambda_0)^- \end{pmatrix} \tag{6.2}$$

since the $(2m + r) \times (2m + r)$ matrix on the righthand side has $m + r$ positive and m negative eigenvalues. The vectors $\phi \in \mathbb{C}^{m+r+m} = \mathbb{C}^{m+r} \times \mathbb{C}^m$ will be decomposed into vectors $[\phi]_0 \in \mathbb{C}^{m+r}$ and $[\phi_1] \in \mathbb{C}^m$; cf. (5.2). Furthermore, $P_m[\phi]_0$ and $P_r[\phi]_0$ denote the orthogonal projections of $[\phi]_0$ onto $\mathbb{C}^m \times \{0\}$ and $\{0\} \times \mathbb{C}^r$, respectively. For $\phi, \psi \in \mathbb{C}^{m+r+m}$ one gets the identity

$$\psi^* \begin{pmatrix} 0 & 0 & -iI_m \\ 0 & I_r & 0 \\ iI_m & 0 & 0 \end{pmatrix} \phi = -i\left((P_m[\psi]_0)^*[\phi]_1 - [\psi]_1^*(P_m[\phi]_0)\right) + (P_r[\psi]_0)^* P_r[\phi]_0. \tag{6.3}$$

The next theorem is the analogue of Theorem 5.13 for the case (6.1).

Theorem 6.4. *Let $\lambda_0 \in \mathbb{C}\setminus\mathbb{R}$ be fixed, assume that the defect numbers of $T_{\min}$ satisfy* (6.1)*, and decompose $\{f,g\} \in T_{\max}$ according to Theorem 4.12 in the form*

$$\{f,g\} = \{f_0, g_0\} + \mathcal{Y}_a(\cdot,\lambda_0)\phi_a + \mathcal{Y}_b(\cdot,\lambda_0)\phi_b,$$

with $\{f_0,g_0\} \in T_{\min}$, $\phi_a \in \mathcal{A}(\lambda_0)$, $\phi_b \in \mathcal{B}(\lambda_0)$. Then the following statements hold:

(i) *if V is a matrix which satisfies* (6.2)*, then $\{\mathbb{C}^{m+r}\times\mathbb{C}^m, \Gamma_0, \Gamma_1\}$, with*

$$\Gamma_0\{f,g\} = \left[V\begin{pmatrix}\phi_a\\ \phi_b\end{pmatrix}\right]_0 \quad \text{and} \quad \Gamma_1\{f,g\} = \left[V\begin{pmatrix}\phi_a\\ \phi_b\end{pmatrix}\right]_1,$$

is a boundary triplet for $T_{\max}$.

(ii) *if $\{\mathbb{C}^{m+r}\times\mathbb{C}^m, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $T_{\max}$, then there exists a (nonunique) matrix V which satisfies* (6.2) *such that Γ_0 and Γ_1 have the form in (i).*

Proof. (i) Decompose $\{f,g\},\{h,k\} \in T_{\max}$ in the form (5.26) with $\{f_0,g_0\},\{h_0,k_0\} \in T_{\min}$, $\phi_a,\psi_a \in \mathcal{A}(\lambda_0)$, $\phi_b,\psi_b \in \mathcal{B}(\lambda_0)$. As in the proof of Theorem 5.13 with (5.25) replaced by (6.2) if follows that

$$\begin{aligned}
\langle \{f,g\},\{h,k\}\rangle_\Delta &= \begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix}^* V^* \begin{pmatrix}0 & 0 & I_m\\ 0 & iI_r & 0\\ -I_m & 0 & 0\end{pmatrix} V \begin{pmatrix}\phi_a\\ \phi_b\end{pmatrix}\\
&= \left(\begin{pmatrix}0 & 0 & I_m\\ 0 & iI_r & 0\\ -I_m & 0 & 0\end{pmatrix}\begin{pmatrix}\Gamma_0\{f,g\}\\ \Gamma_1\{f,g\}\end{pmatrix}, \begin{pmatrix}\Gamma_0\{h,k\}\\ \Gamma_1\{h,k\}\end{pmatrix}\right)\\
&= (\Gamma_1\{f,g\},\Gamma_0\{h,k\}) - (\Gamma_0\{f,g\},\Gamma_1\{h,k\}) + i(P_r\Gamma_0\{f,g\},P_r\Gamma_0\{h,k\}),
\end{aligned}$$

where in the first two inner products in $\mathbb{C}^m$ only the first m entries of $\Gamma_0\{h,k\} \in \mathbb{C}^{m+r}$ and $\Gamma_0\{f,g\} \in \mathbb{C}^{m+r}$ appear (see (6.3)). Since V is invertible, the map $\Gamma = (\Gamma_0,\Gamma_1)^\top : T_{\max} \to \mathbb{C}^{m+r}\times\mathbb{C}^m$ is onto.

(ii) This statement can be proved in the same way as Theorem 5.13 (ii). □

Next the γ-field and Weyl function corresponding to the boundary triplet in Theorem 6.4 will be specified and related to the square-integrable solutions of the canonical system. Recall that the dimension of the space $\mathfrak{D}(a,\lambda)\cap\mathfrak{D}(b,\lambda)$ coincides with the defect numbers of $T_{\min}$,

$$\dim\big(\mathfrak{D}(a,\lambda)\cap\mathfrak{D}(b,\lambda)\big) = \begin{cases} m+r, & \lambda\in\mathbb{C}_+,\\ m, & \lambda\in\mathbb{C}_-.\end{cases}$$

As in Section 5.4 the space $\mathfrak{D}(a,\lambda)\cap\mathfrak{D}(b,\lambda)$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, can be identified with subspaces of the $2m+r$-dimensional $\mathcal{A}(\lambda_0)\times\mathcal{B}(\lambda_0)$, where $\lambda_0\in\mathbb{C}\setminus\mathbb{R}$ fixed. Since

$$\widehat{\mathfrak{N}}_\lambda(T_{\max}) = \big\{\,\{Y(\cdot,\lambda)\phi, \lambda Y(\cdot,\lambda)\phi\} : \phi\in\mathfrak{D}(a,\lambda)\cap\mathfrak{D}(b,\lambda)\,\big\}, \quad \lambda\in\mathbb{C}\setminus\mathbb{R},$$

if follows from Theorem 4.12 that for $\phi\in\mathfrak{D}(a,\lambda)\cap\mathfrak{D}(b,\lambda)$ there exist unique $\{f_0(\lambda),g_0(\lambda)\}\in T_{\min}$, $\phi_a=\phi_a(\lambda)\in\mathcal{A}(\lambda_0)$ and $\phi_b=\phi_b(\lambda)\in\mathcal{B}(\lambda_0)$, such that

$$\widehat{f}_\lambda = \{Y(\cdot,\lambda)\phi, \lambda Y(\cdot,\lambda)\phi\} = \{f_0(\lambda),g_0(\lambda)\} + \mathcal{Y}_a(\cdot,\lambda_0)\phi_a(\lambda) + \mathcal{Y}_b(\cdot,\lambda_0)\phi_b(\lambda)$$

holds. Hence the mapping

$$Z(\lambda) : \mathfrak{D}(a,\lambda)\cap\mathfrak{D}(b,\lambda) \to \mathcal{A}(\lambda_0)\times\mathcal{B}(\lambda_0), \qquad \phi\mapsto\begin{pmatrix}\phi_a(\lambda)\\ \phi_b(\lambda)\end{pmatrix},$$

is injective and $\operatorname{ran} Z(\lambda)$ is an $(m+r)$-dimensional subspace of $\mathcal{A}(\lambda_0)\times\mathcal{B}(\lambda_0)$ if $\lambda\in\mathbb{C}_+$ and an m-dimensional subspace of $\mathcal{A}(\lambda_0)\times\mathcal{B}(\lambda_0)$ if $\lambda\in\mathbb{C}_-$. The next proposition is the analogue of Proposition 5.15 for the case of unequal defect numbers. The proof remains the same, except that the definition of the γ-field and Weyl function from Section 6.1 has to be used.

Proposition 6.5. *Assume that the defect numbers of $T_{\min}$ are $n_+(T_{\min}) = m$ and $n_-(T_{\min}) = m + r$, let V be a matrix with which satisfies (6.2), and let $\{\mathbb{C}^{m+r} \times \mathbb{C}^m, \Gamma_0, \Gamma_1\}$ be the corresponding boundary triplet for $T_{\max}$ from Theorem 6.4. Then the associated γ-field γ and Weyl function M have the form*

$$\gamma(\lambda) = \begin{cases} \big\{\{[VZ(\lambda)\phi]_0, Y(\cdot,\lambda)\phi\} : \phi \in \mathfrak{D}(a,\lambda) \cap \mathfrak{D}(b,\lambda)\big\}, & \lambda \in \mathbb{C}_+, \\ \big\{\{P_m[VZ(\lambda)\phi]_0, Y(\cdot,\lambda)\phi\} : \phi \in \mathfrak{D}(a,\lambda) \cap \mathfrak{D}(b,\lambda)\big\}, & \lambda \in \mathbb{C}_-, \end{cases}$$

and

$$M(\lambda) = \begin{cases} \big\{\{[VZ(\lambda)\phi]_0, [VZ(\lambda)\phi]_1\} : \phi \in \mathfrak{D}(a,\lambda) \cap \mathfrak{D}(b,\lambda)\big\}, & \lambda \in \mathbb{C}_+, \\ \left\{\left\{P_m[VZ(\lambda)\phi]_0, \begin{pmatrix} [VZ(\lambda)\phi]_1 \\ iP_r\,[VZ(\lambda)\phi]_0 \end{pmatrix}\right\} : \phi \in \mathfrak{D}(a,\lambda) \cap \mathfrak{D}(b,\lambda)\right\}, & \lambda \in \mathbb{C}_-. \end{cases}$$

The following statement shows that also in the general singular case with unequal defect numbers the Weyl function associated to a boundary triplet singles out the square-integrable solutions of the homogeneous canonical differential equation; cf. Theorem 5.16. As a consequence of the definition of the Weyl function the following matrix $\mathfrak{J}$ appears when $\lambda \in \mathbb{C}_-$:

$$\mathfrak{J} := \begin{pmatrix} I_m & 0 & 0 \\ 0 & 0 & I_m \\ 0 & iI_r & 0 \end{pmatrix}.$$

Theorem 6.6. *Let $\{\mathbb{C}^{m+r} \times \mathbb{C}^m, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $T_{\max}$ from Theorem 6.4 and let γ and M be the associated γ-field and Weyl function from Proposition 6.5. Then*

$$\gamma(\lambda)\eta = Y(\cdot,\lambda)Z(\lambda)^{-1}V^{-1}\begin{pmatrix} \eta \\ M(\lambda)\eta \end{pmatrix}$$

holds for all $\eta \in \mathbb{C}^{m+r}$ and $\lambda \in \mathbb{C}_+$, and

$$\gamma(\lambda)\eta = Y(\cdot,\lambda)Z(\lambda)^{-1}V^{-1}\mathfrak{J}^{-1}\begin{pmatrix} \eta \\ M(\lambda)\eta \end{pmatrix}$$

holds for all $\eta \in \mathbb{C}^m$ and $\lambda \in \mathbb{C}_-$, respectively.

Proof. For $\lambda \in \mathbb{C}_+$ the statement coincides with the one in Theorem 5.16. Hence only the case $\lambda \in \mathbb{C}_-$ will be shown. The same reasoning as in the proof of Theorem 5.16 shows that the mapping $\phi \mapsto P_m[VZ(\lambda)\phi]_0$ is an isomorphism from $\mathfrak{D}(a,\lambda) \cap \mathfrak{D}(b,\lambda)$ onto $\mathbb{C}^m$ and hence for every $\eta \in \mathbb{C}^m$ there exists a unique $\phi \in \mathfrak{D}(a,\lambda) \cap \mathfrak{D}(b,\lambda)$ such that $\eta = P_m[VZ(\lambda)\phi]_0$; cf. Proposition 6.3. Now Proposition 6.5 implies

$$\gamma(\lambda)P_m\,[VZ(\lambda)\phi]_0 = Y(\cdot,\lambda)\phi = Y(\cdot,\lambda)Z(\lambda)^{-1}V^{-1}\mathfrak{J}^{-1}\mathfrak{J}VZ(\lambda)\phi,$$

where $Z(\lambda)\phi \in \mathcal{A}(\lambda_0) \times \mathcal{B}(\lambda_0)$. With the help of Proposition 6.3 and the particular form of the Weyl function from Proposition 6.5 one concludes

$$\begin{aligned} \gamma(\lambda)P_m\,[VZ(\lambda)\phi]_0 &= Y(\cdot,\lambda)Z(\lambda)^{-1}V^{-1}\mathfrak{J}^{-1}\mathfrak{J}\begin{pmatrix} P_m[VZ(\lambda)\phi]_0 \\ P_r[VZ(\lambda)\phi]_0 \\ [VZ(\lambda)\phi]_1 \end{pmatrix} \\ &= Y(\cdot,\lambda)Z(\lambda)^{-1}V^{-1}\mathfrak{J}^{-1}\begin{pmatrix} P_m[VZ(\lambda)\phi]_0 \\ [VZ(\lambda)\phi]_1 \\ iP_r[VZ(\lambda)\phi]_0 \end{pmatrix} \\ &= Y(\cdot,\lambda)Z(\lambda)^{-1}V^{-1}\mathfrak{J}^{-1}\begin{pmatrix} P_m[VZ(\lambda)\phi]_0 \\ M(\lambda)P_m[VZ(\lambda)\phi]_0 \end{pmatrix}, \end{aligned}$$

which completes the proof. □

Appendix A. Some general facts concerning linear relations

This appendix contains a brief outline of linear relations in Hilbert spaces; for more information, see for instance [10, 23]. A (*closed*) *linear relation* A in a Hilbert space $\mathfrak{H}$ is a (closed) linear subspace of the product space $\mathfrak{H}\times\mathfrak{H}$. The elements in a linear relation are usually denoted in the form $\{f,g\}$. The *domain*, *range*, *kernel*, and *multivalued part* of a linear relation A in $\mathfrak{H}$ are defined by

$$\begin{aligned}\operatorname{dom} A &= \big\{f\in\mathfrak{H} :\ \{f,g\}\in A \text{ for some } g\in\mathfrak{H}\big\},\\ \operatorname{ran} A &= \big\{g\in\mathfrak{H} :\ \{f,g\}\in A \text{ for some } f\in\mathfrak{H}\big\},\\ \ker A &= \big\{f\in\mathfrak{H} :\ \{f,0\}\in A\big\},\\ \operatorname{mul} A &= \big\{g\in\mathfrak{H} :\ \{0,g\}\in A\big\},\end{aligned}$$

respectively. A linear relation A is (the graph of) a linear operator if and only if $\operatorname{mul} A$ is trivial. The *inverse* A^{-1} of a linear relation A is defined as $A^{-1}=\{\{k,h\} : \{h,k\}\in A\}$, so that $\operatorname{dom} A^{-1}=\operatorname{ran} A$, $\operatorname{ran} A^{-1}=\operatorname{dom} A$, $\ker A^{-1}=\operatorname{mul} A$, and $\operatorname{mul} A^{-1}=\ker A$. It is not difficult to check that with the above notions the following identity holds:

$$(A^{-1}-\lambda)^{-1} = -\frac{1}{\lambda}-\frac{1}{\lambda^2}\left(A-\frac{1}{\lambda}\right)^{-1}, \quad \lambda\in\mathbb{C},\quad \lambda\neq 0. \tag{A.1}$$

The *resolvent set* $\rho(A)$ of a closed linear relation A is the set of all $\lambda\in\mathbb{C}$ such that $(A-\lambda)^{-1}\in\mathbf{B}(\mathfrak{H})$. Here $\mathbf{B}(\mathfrak{H})=\mathbf{B}(\mathfrak{H},\mathfrak{H})$, where $\mathbf{B}(\mathfrak{H},\mathfrak{K})$ stands for the linear space of bounded everywhere defined operators from the Hilbert space $\mathfrak{H}$ to the Hilbert space $\mathfrak{K}$. The complement of $\rho(A)$ in $\mathbb{C}$ is the *spectrum* $\sigma(A)$ of A. A point $\lambda\in\mathbb{C}$ is said to be an *eigenvalue* of a linear relation A if $\mathfrak{N}_\lambda(A):=\ker(A-\lambda)$ is nontrivial; i.e., $\{f_\lambda,\lambda f_\lambda\}\in A$ for some $f_\lambda\neq 0$. The following notation will be used

$$\widehat{\mathfrak{N}}_\lambda(A)=\{\,\widehat{f}_\lambda=\{f_\lambda,\lambda f_\lambda\} : f_\lambda\in\mathfrak{N}_\lambda(A)\}.$$

The *adjoint* A^* of a linear relation A is defined by

$$A^* := \big\{\{h,k\} : (g,h)=(f,k) \text{ for all } \{f,g\}\in A\big\}. \tag{A.2}$$

If A is a densely defined operator this definition reduces to the usual definition of the adjoint operator. It follows immediately from the definition that A^* is closed and that the identities $(\operatorname{dom} A)^\perp=\operatorname{mul} A^*$ and $(\operatorname{ran} A)^\perp=\ker A^*$ hold. A relation S is said to be *symmetric* if $S\subset S^*$. The *defect subspace* of S is defined by $\mathfrak{N}_\lambda(S^*)=\ker(S^*-\lambda)$. The *defect numbers* of S are defined by

$$\begin{aligned} n_+(S) &= \dim\ker(S^*-\lambda), \quad \lambda\in\mathbb{C}_-,\\ n_-(S) &= \dim\ker(S^*-\lambda), \quad \lambda\in\mathbb{C}_+.\end{aligned} \tag{A.3}$$

They are well defined since the dimension of $\ker(S^*-\lambda)$ is constant for $\lambda\in\mathbb{C}_+$ and for $\lambda\in\mathbb{C}_-$, respectively.

A relation H is said to be *selfadjoint* if $H=H^*$. Each selfadjoint relation H induces an orthogonal decomposition $\mathfrak{H}=\overline{\operatorname{dom}}\,H\oplus\operatorname{mul} H$, where $\overline{\operatorname{dom}}\,H$ stands for the closure of the domain of H in $\mathfrak{H}$. The selfadjoint relation H itself decomposes accordingly

$$H = H_{\rm s}\,\widehat{\oplus}\,H_{\rm mul}$$

where $H_{\rm s}$ and $H_{\rm mul}$ are given by

$$H_{\rm s}=\{\,\{f,g\}\in H : g\in\mathfrak{H}\ominus\operatorname{mul} H\,\},\quad H_{\rm mul}=\{0\}\times\operatorname{mul} H.$$

The above sum is a componentwise sum which is orthogonal, so that $H_{\rm s}$ is a selfadjoint operator in $\overline{\operatorname{dom}}\,H$ and $H_{\rm mul}$ is a purely multivalued selfadjoint relation in $\operatorname{mul} H$.

The symmetric relation S has selfadjoint extensions in $\mathfrak{H}$ if and only if the defect numbers of S are equal. Since $\mathfrak{H}=\operatorname{ran}(S-\lambda)\oplus\ker(S^*-\bar\lambda)$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, the adjoint S^* of S can be decomposed via von Neumann's decomposition.

Proposition A.1. *Let S be a closed symmetric linear relation in a Hilbert space $\mathfrak{H}$ and let $\mu \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$S^* = S \,\widehat{+}\, \widehat{\mathfrak{N}}_\mu(S^*) \,\widehat{+}\, \widehat{\mathfrak{N}}_{\bar\mu}(S^*), \quad \textit{direct sums},$$

where $\widehat{+}$ stands for the componentwise sum in $\mathfrak{H} \times \mathfrak{H}$. The sums are orthogonal when $\mu = \pm i$.

For each symmetric relation one can construct a so-called symmetric bounded right inverse, for instance by means of the above von Neumann decomposition. Conversely, each symmetric bounded right inverse gives rise to a symmetric relation.

Proposition A.2. *Let T be a linear relation in a Hilbert space $\mathfrak{H}$. Let $\mu \in \mathbb{C}_+$ and assume that for $\lambda \in \{\mu, \bar\mu\}$ the eigenspace $\mathfrak{N}_\lambda(T)$ is closed and that there exists a bounded everywhere defined linear operator $G(\lambda)$ such that $G(\lambda)^* = G(\bar\lambda)$ and*

$$\{G(\lambda)g, (I + \lambda G(\lambda))g\} \in T, \quad g \in \mathfrak{H}. \tag{A.4}$$

Then $T_0 := T \cap T^*$ is a closed symmetric relation and $T = T_0^*$.

Proof. Define $H(\lambda)$, $\lambda \in \{\mu, \bar\mu\}$, by

$$H(\lambda) = \big\{\{G(\lambda)g, (I + \lambda G(\lambda))g\} : g \in \mathfrak{H}\big\}, \tag{A.5}$$

so that

$$(H(\lambda) - \lambda)^{-1} = G(\lambda). \tag{A.6}$$

Since $G(\mu)^* = G(\bar\mu)$, the preceding equality shows that $H(\mu)^* = H(\bar\mu)$. By the assumption (A.4) $H(\mu), H(\bar\mu) \subseteq T$, hence

$$T^* \subset H(\mu)^* = H(\bar\mu) \subset T.$$

Therefore, the relation $T_0 = T^* \cap T = T^*$ is closed and symmetric.

Moreover, since $G(\mu)$ is bounded and everywhere defined, (A.6) implies that $\mathrm{ran}\,(H(\mu) - \mu) = \mathfrak{H}$ and hence a direct, algebraic, argument shows that

$$T = H(\mu) \,\widehat{+}\, \widehat{\mathfrak{N}}_\mu(T), \qquad \text{direct sum}. \tag{A.7}$$

It remains to show that T is closed. To see this, assume there is a sequence $\{h_n, k_n\} \in T$ converging to $\{h, k\} \in \mathfrak{H} \times \mathfrak{H}$. By (A.7) there exist $\chi_n \in \mathfrak{H}$ and $\varphi_n \in \mathfrak{N}_\mu(T)$ such that

$$\{h_n, k_n\} = \{G(\mu)\chi_n, (I + \mu G(\mu))\chi_n\} + \{\varphi_n, \mu\varphi_n\}.$$

Hence it follows that $\chi_n = k_n - \mu h_n$ converges to $k - \mu h$ and, therefore, φ_n converges to $h - G(\mu)(k - \mu h)$. Decompose the element $\{h, k\}$ as follows

$$\begin{aligned}\{h, k\} = \{G(\mu)(k - \mu h), &(I + \mu G(\mu))(k - \mu h)\} \\ &+ \{h - G(\mu)(k - \mu h), \mu(h - G(\mu)(k - \mu h))\}.\end{aligned} \tag{A.8}$$

The first element in the righthand side of (A.8) belongs to $H(\mu)$ by (A.5). Since it is assumed that $\mathfrak{N}_\mu(T)$ is closed, the last element in the righthand side of (A.8), being the limit of $\{\varphi_n, \mu\varphi_n\} \in \widehat{\mathfrak{N}}_\mu(T)$, belongs to $\widehat{\mathfrak{N}}_\mu(T)$. Therefore, it follows from (A.7) that $\{h, k\} \in T$. □

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