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PASSIVE SYSTEMS WITH A NORMAL MAIN OPERATOR AND QUASI-SELFADJOINT SYSTEMS

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Abstract. Passive systems $\tau = \{T, \mathfrak{M}, \mathfrak{N}, \mathfrak{H}\}$ with $\mathfrak{M}$ and $\mathfrak{N}$ as an input and output space and $\mathfrak{H}$ as a state space are considered in the case that the main operator on the state space is normal. Basic properties are given and a general unitary similarity result involving some spectral theoretic conditions on the main operator is established. A passive system τ with $\mathfrak{M} = \mathfrak{N}$ is said to be quasi-selfadjoint if $\operatorname{ran}(T - T^*) \subset \mathfrak{N}$. The subclass $\mathbf{S}^{qs}(\mathfrak{N})$ of the Schur class $\mathbf{S}(\mathfrak{N})$ is the class formed by all transfer functions of quasi-selfadjoint passive systems. The subclass $\mathbf{S}^{qs}(\mathfrak{N})$ is characterized and minimal passive quasi-selfadjoint realizations are studied. The connection between the transfer function belonging to the subclass $\mathbf{S}^{qs}(\mathfrak{N})$ and the Q-function of T is given.

1. Introduction

Let $\mathfrak{M}, \mathfrak{N}$, and $\mathfrak{H}$ be separable Hilbert spaces and let

$$T = \begin{pmatrix} D & C \\ B & A \end{pmatrix} : \begin{pmatrix} \mathfrak{M} \\ \mathfrak{H} \end{pmatrix} \to \begin{pmatrix} \mathfrak{N} \\ \mathfrak{H} \end{pmatrix} \tag{1.1}$$

be a bounded linear operator. Here and in the following, it will be tacitly assumed that the spaces in the righthand side are orthogonal sums: $\mathfrak{M} \oplus \mathfrak{H}$ and $\mathfrak{N} \oplus \mathfrak{H}$. The system of equations

$$\begin{cases} h_{k+1} = Ah_k + B\xi_k, \\ \sigma_k = Ch_k + D\xi_k, \end{cases} \qquad k \geq 0, \tag{1.2}$$

describes the evolution of a *linear discrete time-invariant system* $\tau = \{T, \mathfrak{M}, \mathfrak{N}, \mathfrak{H}\}$. The Hilbert spaces $\mathfrak{M}$ and $\mathfrak{N}$ are called the input and the output spaces, respectively, and the Hilbert space $\mathfrak{H}$ is called the state space. The operators A, B, C, and D are called the *main operator*, the *control operator*, the *observation operator*, and the *feedthrough operator* of τ, respectively. The subspaces

$$\mathfrak{H}^c = \overline{\operatorname{span}}\, \{ A^n B\mathfrak{M} : \, n \in \mathbb{N}_0\} \text{ and } \mathfrak{H}^o = \overline{\operatorname{span}}\, \{ A^{*n} C^* \mathfrak{N} : \, n \in \mathbb{N}_0\} \tag{1.3}$$

are called the *controllable* and *observable* subspaces of $\tau = \{T, \mathfrak{M}, \mathfrak{N}, \mathfrak{H}\}$, respectively. If $\mathfrak{H}^c = \mathfrak{H}$ ($\mathfrak{H}^o = \mathfrak{H}$) then the system τ is said to be *controllable* (*observable*), and *minimal* if τ is both controllable and observable. If $\mathfrak{H} = \operatorname{clos}\{\mathfrak{H}^c + \mathfrak{H}^o\}$ then the system τ is said to be a *simple*. Two discrete-time systems $\tau_1 = \{T_1, \mathfrak{M}, \mathfrak{N}, \mathfrak{H}_1\}$ and

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$\tau_2 = \{T_2, \mathfrak{M}, \mathfrak{N}, \mathfrak{H}_2\}$ are *unitarily similar* if there exists a unitary operator U from $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ such that

$$A_2 = UA_1U^*, \quad B_2 = UB_1, \quad C_2 = C_1U^*, \text{ and } D_2 = D_1. \tag{1.4}$$

If the linear operator T is contractive (isometric, co-isometric, unitary), then the corresponding discrete-time system is said to be *passive* (*isometric, co-isometric, conservative*). The *transfer function*

$$\Theta(\lambda) := D + \lambda C(I - \lambda A)^{-1}B, \quad \lambda \in \mathbb{D}, \tag{1.5}$$

of the passive system τ in (1.2) belongs to the *Schur class* $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$, i.e., $\Theta(\lambda)$ is holomorphic in the unit disk $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and its values are contractive linear operators from $\mathfrak{M}$ into $\mathfrak{N}$. Every operator-valued function $\Theta(\lambda)$ from the Schur class $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$ can be realized as the transfer function of a passive system, which can be chosen as observable co-isometric (controllable isometric, simple conservative, passive minimal). Moreover two isometric and controllable (co-isometric and observable, simple conservative) systems having the same transfer function are unitarily similar. D.Z. Arov [10] has shown that two minimal passive systems τ_1 and τ_2 with the same transfer function $\Theta(\lambda)$ are only *weakly similar*, i.e., there is a closed densely defined operator $Z : \mathfrak{H}_1 \to \mathfrak{H}_2$ such that Z is invertible, Z^{-1} is densely defined, and

$$ZA_1f = A_2Zf, \quad C_1f = C_2Zf, \quad f \in \operatorname{dom} Z, \quad \text{and} \quad ZB_1 = B_2. \tag{1.6}$$

Weak similarity preserves neither the dynamical properties of the system nor the spectral properties of its main operator A. In [14], [15] necessary and sufficient conditions have been established for minimal passive systems with the same transfer function to be (unitarily) similar. In [5] a parametrization of the contractive block-operator matrices in (1.1) was used to establish some new aspects and some explicit formulas for the interplay between the system τ, its transfer function $\Theta(\lambda)$, and the Sz.-Nagy–Foiaş characteristic function of the contraction A.

In this paper the same approach is applied to study passive systems with a normal main operator, including the class of *passive quasi-selfadjoint* systems (*pqs*-systems for short), as defined in the paper. Furthermore, using the famous Mergelyan's theorem from complex analysis a general unitary similarity result is proved for such systems.

The passive system $\tau = \{T, \mathfrak{N}, \mathfrak{N}, \mathfrak{H}\}$ is called a *pqs*-system if the operator

$$T = \begin{pmatrix} D & C \\ B & A \end{pmatrix} : \begin{pmatrix} \mathfrak{N} \\ \mathfrak{H} \end{pmatrix} \to \begin{pmatrix} \mathfrak{N} \\ \mathfrak{H} \end{pmatrix} \tag{1.7}$$

is a *quasi-selfadjoint contraction* (*qsc*-operator for short), i.e., T is a contraction and $\operatorname{ran}(T - T^*) \subset \mathfrak{N}$, cf. [4]. This last condition is equivalent to $A = A^*$ and $C = B^*$. If τ is a *pqs*-system, then the transfer function (1.5) of τ takes the form

$$\Theta(\lambda) = W(\lambda) + D,$$

where the function $W(\lambda)$ is a Herglotz-Nevanlinna function defined on $\operatorname{Ext}\{(-\infty, -1] \cup [1, \infty)\}$. The subclass $\mathbf{S}^{qs}(\mathfrak{N})$ of the Schur class $\mathbf{S}(\mathfrak{N})$ of $\mathbf{L}(\mathfrak{N})$-valued functions is the class of all transfer functions of *pqs*-systems $\tau = \{T; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}\}$. A necessary and sufficient condition for the function $\Theta(\lambda)$ to be in the class $\mathbf{S}^{qs}$ is given, the minimal *pqs*-systems with the given operator-valued function $\Theta(\lambda)$ from the class $\mathbf{S}^{qs}$ is constructed using operator representations of Herglotz-Nevanlinna functions. Moreover, a necessary and sufficient condition for the function $\Theta(\lambda) \in \mathbf{S}^{qs}$ to be inner (co-inner) is proved and connections with *pqs*-system and other minimal systems with the same transfer function are established. Also it is shown that if, for instance, $\Theta(\lambda) \in \mathbf{S}^{qs}(\mathfrak{N})$ and $\varphi_\Theta(\lambda) = 0$ ($\psi_\Theta(\lambda) = 0$) then $\Theta(\lambda)$ is inner (co-inner). A matrix form of the inner function from the class $\mathbf{S}^{qs}(\mathfrak{N})$ when $\dim \mathfrak{N} < \infty$ is also given, and in the case of scalar functions from the class $\mathbf{S}^{qs}(\mathfrak{N})$ a minimal representation is obtained by means of Jacobi matrices.

2. Preliminaries

Let $\mathfrak{M}$ and $\mathfrak{N}$ be Hilbert spaces and let $\Theta(\lambda)$ belong to the Schur class $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$. The notation $\Theta(\xi)$, $\xi \in \mathbb{T}$, stands for the non-tangential strong limit value of $\Theta(\lambda)$ which exist almost everywhere on $\mathbb{T}$, cf. [27]. A function $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ is said to be *inner* if $\Theta^*(\xi)\Theta(\xi) = I_\mathfrak{M}$ for almost all $\xi \in \mathbb{T}$, and it is said to be *co-inner* if $\Theta(\xi)\Theta^*(\xi) = I_\mathfrak{N}$ for almost all $\xi \in \mathbb{T}$. A function $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ is said to be *bi-inner* if it is both inner and co-inner.

2.1. **Contractions and their defect operators.** Let $A \in \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ be a contraction, in other words, let $\|Af\| \le \|f\|$ for all $f \in \mathfrak{H}$, or equivalently $I - A^*A \ge 0$. The selfadjoint operator $D_A = (I - A^*A)^{1/2}$ is said to be the *defect operator* of A. Observe that $\ker D_A = \ker D_A^2 = \ker (I - A^*A)$, and that

$$\ker (I - A^*A) = \{ f \in \mathfrak{H} : \|Af\| = \|f\| \}. \tag{2.1}$$

Clearly, any contraction A satisfies

$$I \ge A^*A \ge \cdots \ge A^{*n}A^n \ge 0, \quad n \in \mathbb{N},$$

in other words the sequence $\|A^n f\|$ with $f \in \mathfrak{H}$ is monotonically nonincreasing. In particular, the strong limit

$$S_A = s - \lim A^{*n}A^n, \tag{2.2}$$

exists as an operator in $\mathbf{L}(\mathfrak{H}_1)$, cf. [25, p. 261]. The defect operators D_A and D_{A^*} satisfy the following commutation relation:

$$AD_A = D_{A^*}A, \quad D_A A^* = A^* D_{A^*}. \tag{2.3}$$

Let $\mathfrak{D}_A$ stand for the closure of the range $\operatorname{ran} D_A$. Then

$$\begin{pmatrix} -A & D_{A^*} \\ D_A & A^* \end{pmatrix} : \begin{pmatrix} \mathfrak{D}_A \\ \mathfrak{H}_2 \end{pmatrix} \to \begin{pmatrix} \mathfrak{D}_{A^*} \\ \mathfrak{H}_1 \end{pmatrix} \tag{2.4}$$

is unitary. Define the subspaces $\mathfrak{H}_{A,0}$ and $\mathfrak{H}_{A,1}$ by

$$\mathfrak{H}_{A,1} = \{ f \in \mathfrak{H} : \|f\| = \|A^n f\| = \|A^{*n} f\|, \, n \in \mathbb{N} \}, \quad \mathfrak{H}_{A,0} = \mathfrak{H} \ominus \mathfrak{H}_{A,1}.$$

Then $\mathfrak{H} = \mathfrak{H}_{A,0} \oplus \mathfrak{H}_{A,1}$ is a canonical orthogonal decomposition of $\mathfrak{H}$ such that

$$A = A_0 \oplus A_1, \quad A_j = A{\upharpoonright}\mathfrak{H}_{A,j}, \quad j = 0, 1, \tag{2.5}$$

where $\mathfrak{H}_{A,0}$ and $\mathfrak{H}_{A,1}$ reduce A, A_0 is a completely non-unitary contraction, and A_1 is a unitary operator. The function

$$\Phi_A(\lambda) := \big(-A + \lambda D_{A^*}(I_\mathfrak{H} - \lambda A^*)^{-1} D_A\big){\upharpoonright}\mathfrak{D}_A, \quad \lambda \in \mathbb{D}, \tag{2.6}$$

is the Sz.-Nagy–Foiaş characteristic function of the contraction A. It belongs to the Schur class $\mathbf{S}(\mathfrak{D}_A, \mathfrak{D}_{A^*})$; cf. [27]. In fact, a straightforward calculation using the identities $(I - \lambda A)^{-1} = I + \lambda(I - \lambda A)^{-1}A$ and (2.3) yields

$$D^2_{\Phi_A(\lambda)} = (1 - \lambda\bar{\lambda})D_A(I - \bar{\lambda}A)^{-1}(I - \lambda A^*)^{-1}D_A{\upharpoonright}\mathfrak{D}_A, \tag{2.7}$$

which shows that $\Phi_A(\lambda)$ is contractive for $\lambda \in \mathbb{D}$. Note also that $\Phi_{A^*}(\lambda)$ is the transfer function of the conservative system

$$\Sigma = \{T, \mathfrak{D}_{A^*}, \mathfrak{D}_A, \mathfrak{H}\}, \tag{2.8}$$

where

$$T = \begin{pmatrix} -A^* & D_A \\ D_{A^*} & A \end{pmatrix} : \begin{pmatrix} \mathfrak{D}_{A^*} \\ \mathfrak{H} \end{pmatrix} \to \begin{pmatrix} \mathfrak{D}_A \\ \mathfrak{H} \end{pmatrix}. \tag{2.9}$$

Let $A \in \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ be a contraction and let Σ be the corresponding conservative system in (2.8), (2.9). Then the controllable and observable subspaces, as defined in (1.3), are given by

$$\mathfrak{H}^c_\Sigma = \overline{\operatorname{span}}\,\{A^n D_{A^*}\mathfrak{D}_{A^*} : n \in \mathbb{N}_0\}, \quad \mathfrak{H}^o_\Sigma = \overline{\operatorname{span}}\,\{A^{*n} D_A\mathfrak{D}_A : n \in \mathbb{N}_0\}. \tag{2.10}$$

Observe that A is completely nonunitary if and only if Σ is minimal. Since clearly $\Phi_A(\lambda)^* = \Phi_{A^*}(\bar{\lambda})$, one has also

$$D^2_{\Phi_A(\lambda)^*} = (1 - \bar{\lambda}\lambda)D_{A^*}(I - \lambda A^*)^{-1}(I - \bar{\lambda}A)^{-1}D_{A^*}{\upharpoonright}\mathfrak{D}_{A^*}. \tag{2.11}$$

Observe that if $\xi \in \mathbb{T} := \{\xi \in \mathbb{C} : |\xi| = 1\}$ belongs to the resolvent set of A, then (2.7) and (2.11) show that $\Phi_A(\xi)$ is a unitary operator; cf. [27, p. 239].

A contraction A in a Hilbert space $\mathfrak{H}$ is said to belong to the *classes* $C_{0\cdot}$ or $C_{\cdot 0}$ if

$$s - \lim_{n\to\infty} A^n = 0 \quad \text{or} \quad s - \lim_{n\to\infty} A^{*n} = 0,$$

respectively. By definition, $C_{00} := C_{0\cdot} \cap C_{\cdot 0}$. Hence $A \in C_{00}$ precisely when

$$s - \lim_{n\to\infty} A^n = s - \lim_{n\to\infty} A^{*n} = 0. \tag{2.12}$$

Observe that $A \in C_{00}$ implies that A is completely nonunitary, cf. (2.5). The completely non-unitary part of a contraction A belongs to the class $C_{\cdot 0}$, $C_{0\cdot}$, or C_{00} if and only if its characteristic function $\Phi_A(\lambda)$ in (2.6) is inner, co-inner, or bi-inner, respectively; cf. [27, Theorem VI.2.3]. It follows from (2.2) that $S_A = 0$ implies $A \in C_{0\cdot}$ and that $S_{A^*} = 0$ implies $A \in C_{\cdot 0}$.

A contraction A is said to be *strict* if $\|Af\| < \|f\|$ for all nontrivial $f \in \mathfrak{H}_1$. Note that in view of (2.1) a contraction A is strict if and only if $\ker D_A = \ker D^2_A = \ker (I - A^*A) = \{0\}$. Finally, a passive system $\tau = \{T; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}\}$ is said to be *strongly stable* or *strongly co-stable* if the main operator A belongs to the class $C_{0\cdot}$ or $C_{\cdot 0}$, respectively; see [11], [16].

2.2. **Some properties of normal contractions.** An operator $A \in \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ is said to be *normal* if $A^*A = AA^*$, or equivalently, if $\|Af\| = \|A^*f\|$ for all $f \in \mathfrak{H}$, cf. [25, p. 281]. It is clear from $\mathfrak{H} = \mathfrak{H}_{A,0} \oplus \mathfrak{H}_{A,1}$ and the orthogonal decomposition in (2.5) that a contraction A is normal if and only if its completely nonunitary part A_0 is normal in $\mathfrak{H}_{A,0}$. If A is a normal contraction then, parallel to (2.1), one has

$$\begin{aligned}&\ker\,(I-(A^*A)^n)=\ker\,(I-A^{*n}A^n)=\{\,f\in\mathfrak{H}:\,\|A^nf\|=\|f\|\,\}\\&=\ker\,(I-(AA^*)^n)=\ker\,(I-A^nA^{*n})=\{\,f\in\mathfrak{H}:\,\|A^{*n}f\|=\|f\|\,\}.\end{aligned} \tag{2.13}$$

Moreover, if A is a normal contraction, then the defect operators D_A and D_{A^*} satisfy $D_A=D_{A^*}$ and $\mathfrak{D}_A=\mathfrak{D}_{A^*}$; in addition, (2.3) reads as

$$AD_A=D_AA,\quad A^*D_A=D_AA^*. \tag{2.14}$$

Lemma 2.1. *Let $A\in\mathbf{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ be a normal contraction. Then the strong limit S_A satisfies $S_A=S_{A^*}$ and*

$$S_A(I-A^*A)=0. \tag{2.15}$$

If, in addition, A is strict, then $S_A=0$.

Proof. If A is normal, then (2.2) implies that $S_A=S_{A^*}$ and

$$S_AA^*A=(s-\lim A^{*n}A^n)A^*A=s-\lim A^{*(n+1)}A^{n+1}=S_A,$$

which leads to (2.15). □

Proposition 2.2. *Let $A\in\mathbf{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ be a normal contraction. Then the following statements are equivalent:*

(i) *$A\in C_{00}$;*
(ii) *A is completely non-unitary;*
(iii) *A is strict.*

Moreover, the characteristic function $\Phi_A(\lambda)$ of A in (2.6) is bi-inner.

Proof. (i) ⇒ (ii) This implication is a general fact for not necessarily normal contractions.

(ii) ⇒ (iii) Let A be completely non-unitary. Assume that A is not strict. Then there exists an element $0\neq f_0\in\mathfrak{H}_1$ such that $\|Af_0\|=\|f_0\|$. Since $\ker\,(I-A^*A)\subset\ker\,(I-(A^*A)^n)$, $n\in\mathbb{N}$, it follows from (2.1) and (2.13) that $\|f_0\|=\|A^nf_0\|=\|A^{*n}f_0\|>0$. This contradicts the fact that A is completely nonunitary.

(iii) ⇒ (i) Let A be strict, so that $\ker\,(I-A^*A)=\{0\}$. Then Lemma 2.1 implies that $S_A=0$, which leads to (2.12), so that $A\in C_{00}$.

Observe that if A is normal then $\mathfrak{H}_{A,1}=\ker D_A$ as was just shown above. The completely non-unitary part A_0 of A is normal and satisfies $\ker D_{A_0}=\{0\}$. Thus $A_0\in C_{00}$, i.e., $\Phi_A(\lambda)$ is bi-inner; cf. [27, Theorem VI.2.3]. □

If a contraction A is normal, then its controllable and observable subspaces coincide, which leads to the following observation.

Proposition 2.3. *Let $A\in\mathbf{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ be a normal contraction, and let Σ be the corresponding conservative system in (2.8), (2.9). Then $\mathfrak{H}^c_\Sigma=\mathfrak{H}^o_\Sigma$ and the following statements are equivalent:*

(i) *Σ is simple;*
(ii) *Σ is controllable;*
(iii) *Σ is observable;*
(iv) *Σ is minimal.*

Proof. Since A is normal, it follows that $D_{A^{*n}} = D_{A^n}$ for all $n \in \mathbb{N}_0$. Hence the identities

$$(\mathfrak{H}_\Sigma^c)^\perp = \bigcap_{n=0}^{\infty} \ker\, (D_{A^*}A^{*n}) = \bigcap_{n=1}^{\infty} \ker\, D_{A^{*n}}, \tag{2.16}$$

$$(\mathfrak{H}_\Sigma^o)^\perp = \bigcap_{n=0}^{\infty} \ker\, (D_{A}A^{n}) = \bigcap_{n=1}^{\infty} \ker\, D_{A^{n}} \tag{2.17}$$

imply that $(\mathfrak{H}_\Sigma^c)^\perp = (\mathfrak{H}_\Sigma^o)^\perp$, or equivalently, $\mathfrak{H}_\Sigma^c = \mathfrak{H}_\Sigma^o$. This identity implies the equivalence of (i), (ii), (iii), and (iv). □

The following corollary is based on the fact that a contraction A is completely nonunitary if and only if the corresponding system Σ in (2.8), (2.9) is minimal.

Corollary 2.4. *Let A be a normal contraction. Then the statements (i)–(iii) in Proposition 2.2 and the statements (i)–(iv) in Proposition 2.3 are all equivalent.*

2.3. **Parametrization of block operators.** For a proof and some history of the following theorem, see [5].

Theorem 2.5. *Let $\mathfrak{M}, \mathfrak{N}$, $\mathfrak{H}$, and $\mathfrak{K}$ be Hilbert spaces. The operator matrix T in* (1.1) *is a contraction if and only if T is of the form*

$$T = \begin{pmatrix} -KA^*M + D_{K^*}XD_M & KD_A \\ D_{A^*}M & A \end{pmatrix}, \tag{2.18}$$

where $A \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, $M \in \mathbf{L}(\mathfrak{M}, \mathfrak{D}_{A^*})$, $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$, and $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$ are contractions, all uniquely determined by T. Furthermore, the following equality holds for all $h \in \mathfrak{M}$,$f \in \mathfrak{H}$:

$$\begin{aligned} &\left\| \begin{pmatrix} h \\ f \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} -KA^*M + D_{K^*}XD_M & KD_A \\ D_{A^*}M & A \end{pmatrix} \begin{pmatrix} h \\ f \end{pmatrix} \right\|^2 \\ &\qquad = \|D_K(D_Af - A^*Mh) - K^*XD_Mh\|^2 + \|D_XD_Mh\|^2. \end{aligned} \tag{2.19}$$

Corollary 2.6. *Let $A \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be a contraction. Assume that $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$, $M \in \mathbf{L}(\mathfrak{M}, \mathfrak{D}_{A^*})$, and $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$ are contractions. Then the operator T in* (2.18) *is:*

(i) *isometric if and only if $D_XD_M = 0$ and $D_KD_A = 0$;*
(ii) *co-isometric if and only if $D_{X^*}D_{K^*} = 0$ and $D_{M^*}D_{A^*} = 0$.*

Let $\tau = \{T; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}\}$ be a passive system and let (2.18) be the representation of the block operator T in (1.1). Define for $\lambda \in \mathbb{D}$ the following operator-valued holomorphic functions

$$\varphi(\lambda) := \begin{pmatrix} -D_XD_M \\ D_K\Phi_{A^*}(\lambda)M - K^*XD_M \end{pmatrix} : \mathfrak{M} \to \begin{pmatrix} \mathfrak{D}_M \\ \mathfrak{D}_K \end{pmatrix}, \tag{2.20}$$

and

$$\psi(\lambda) := \begin{pmatrix} D_{K^*}D_{X^*} & K\Phi_{A^*}(\lambda)D_{M^*} - D_{K^*}XM^* \end{pmatrix} : \begin{pmatrix} \mathfrak{D}_{K^*} \\ \mathfrak{D}_{M^*} \end{pmatrix} \to \mathfrak{N}. \tag{2.21}$$

Theorem 2.7 ([5]). *Let $\tau = \{T; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}\}$ be a passive system and let* (2.18) *be the representation of the block operator T in* (1.1). *Then the transfer function $\Theta(\lambda)$ of τ and the characteristic function $\Phi_{A^*}(\lambda)$ of A^* (see* (2.6)*) are connected via*

$$\Theta(\lambda) = K\Phi_{A^*}(\lambda)M + D_{K^*}XD_M, \quad \lambda \in \mathbb{D}; \tag{2.22}$$

in particular, $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$. In addition, the identities

$$\left\|D_{\Theta(\lambda)}h\right\|^2 = \left\|D_{\Phi_{A^*(\lambda)}}Mh\right\|^2 + \|\varphi(\lambda)h\|^2, \quad h \in \mathfrak{M}, \tag{2.23}$$

$$\left\|D_{\Theta^*(\lambda)}g\right\|^2 = \left\|D_{\Phi_{A}(\bar{\lambda})}K^*g\right\|^2 + \|\psi^*(\lambda)g\|^2, \quad g \in \mathfrak{N}, \tag{2.24}$$

hold and the functions $\varphi(\lambda)$ and $\psi(\lambda)$ in (2.20) *and* (2.21) *are Schur functions.*

3. Passive systems with a normal main operator

Let τ be a passive system of the form (1.1). If its main operator A is normal, then many properties of τ and its transfer function simplify.

3.1. **Basic properties.** The controllable and observable subspaces of the passive system in (1.1) are defined in (1.3). Let the block matrix T have the parametrization (2.18), so that $A^nB = A^nD_{A^*}M$ and $A^{*n}C^* = A^{*n}D_AK^*$. If, in addition, A is normal it follows that $D_{A^*} = D_A$ and then (2.14) implies

$$A^nB = D_AA^nM, \quad A^{*n}C^* = A^{*n}D_AK^*.$$

Hence, if A is normal, then $\mathfrak{H}^c$ and $\mathfrak{H}^o$ have the form:

$$\mathfrak{H}^c = \overline{\operatorname{span}}\{\, D_AA^nM\mathfrak{M} :\, n \in \mathbb{N}_0\}, \quad \mathfrak{H}^o = \overline{\operatorname{span}}\{\, D_AA^{*n}K^*\mathfrak{N} :\, n \in \mathbb{N}_0\}, \tag{3.1}$$

or, equivalently,

$$(\mathfrak{H}^c)^\perp = \bigcap_{n=0}^{\infty} \ker\,(M^*A^{*n}D_A), \qquad (\mathfrak{H}^o)^\perp = \bigcap_{n=0}^{\infty} \ker\,(KA^nD_A), \tag{3.2}$$

Let the subspaces $\mathfrak{H}^c_N$ and $\mathfrak{H}^o_N$ be defined by

$$\mathfrak{H}^c_N = \overline{\operatorname{span}}\,\{\, A^nM\mathfrak{M} :\, n \in \mathbb{N}_0\,\}, \quad \mathfrak{H}^o_N = \overline{\operatorname{span}}\,\{\, A^{*n}K^*\mathfrak{N} :\, n \in \mathbb{N}_0\,\}, \tag{3.3}$$

or, equivalently, by

$$(\mathfrak{H}^c_N)^\perp = \bigcap_{n=0}^{\infty} \ker\,(M^*A^{*n}), \qquad (\mathfrak{H}^o_N)^\perp = \bigcap_{n=0}^{\infty} \ker\,(KA^n). \tag{3.4}$$

Lemma 3.1. *Let $\tau = \{T; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}\}$ be a passive system with T of the form* (2.18) *with some contractions $A \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, $M \in \mathbf{L}(\mathfrak{M}, \mathfrak{D}_{A^*})$, $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$, and $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$. Assume that A is normal.*

(i) *If $\mathfrak{H}^c_N$ is invariant under A^*, then $\mathfrak{H} \ominus \mathfrak{H}^c_N \subset \mathfrak{H} \ominus \mathfrak{H}^c$.*
(ii) *If $\mathfrak{H}^o_N$ be invariant under A, then $\mathfrak{H} \ominus \mathfrak{H}^o_N \subset \mathfrak{H} \ominus \mathfrak{H}^o$.*

Proof. (i) Assume that $\mathfrak{H}_N^c$ is invariant under A^* or, equivalently, that $\mathfrak{H} \ominus \mathfrak{H}_N^c$ is invariant under A. Hence, if $f \in \mathfrak{H} \ominus \mathfrak{H}_N^c$ then f and Af both belong to $\ker (M^*A^{*n})$ for all $n \in \mathbb{N}_0$. Thus, in particular, $D_A^2 f = (I - A^*A)f \in \ker M^*$. Moreover, if $p(t)$ is a polynomial then $p(D_A^2)f \in \ker M^*$. Since there exists a sequence of polynomials $\{p_m(t)\}_{m=1}^\infty$ such that the sequence $\{p_m(D_A^2)\}$ converges uniformly to D_A, it follows that $D_A f \in \ker M^*$. Furthermore, the sequence $\{p_m(D_A^2)A^{*n}\}$ converges uniformly to $D_A A^{*n}$ for all $n \in \mathbb{N}$. Since $p_m(D_A^2)A^{*n} f \in \ker M^*$ for all $n \in \mathbb{N}_0$, one concludes that $D_A A^{*n} f \in \ker M^*$ for all $n \in \mathbb{N}_0$. It follows that

$$\mathfrak{H} \ominus \mathfrak{H}_N^c \subset \bigcap_{n=0}^\infty \ker (M^* D_A A^{*n}) = \mathfrak{H} \ominus \overline{\operatorname{span}} \{ D_A A^n M \mathfrak{M} : n \in \mathbb{N}_0 \} = \mathfrak{H} \ominus \mathfrak{H}^c.$$

(ii) The proof of (ii) is similar to the proof of (i). □

Proposition 3.2. *Let $\tau = \{T; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}\}$ be a passive system where T is of the form* (2.18) *with contractions $A \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, $M \in \mathbf{L}(\mathfrak{M}, \mathfrak{D}_{A^*})$, $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$, and $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$. Assume that A is normal.*

(i) *τ is controllable if and only if*

$$\ker D_A = \{0\} \quad \textit{and} \quad \operatorname{ran} D_A \cap (\mathfrak{H} \ominus \mathfrak{H}_N^c) = \{0\}. \tag{3.5}$$

In particular, if $\ker D_A = \{0\}$ *and $\mathfrak{H}_N^c = \mathfrak{H}$, then τ is controllable; if τ is controllable and $\mathfrak{H}_N^c$ is invariant under A^*, then* $\ker D_A = \{0\}$ *and $\mathfrak{H}_N^c = \mathfrak{H}$.*

(ii) *τ is observable if and only if*

$$\ker D_A = \{0\} \quad \textit{and} \quad \operatorname{ran} D_A \cap (\mathfrak{H} \ominus \mathfrak{H}_N^o) = \{0\}; \tag{3.6}$$

In particular, if $\ker D_A = \{0\}$ *and $\mathfrak{H}_N^o = \mathfrak{H}$, then τ is observable; if τ is observable and $\mathfrak{H}_N^o$ is invariant under A, then* $\ker D_A = \{0\}$ *and $\mathfrak{H}_N^o = \mathfrak{H}$.*

(iii) *τ is simple if and only if*

$$\ker D_A = \{0\} \quad \textit{and} \quad \operatorname{ran} D_A \cap \mathfrak{H} \ominus (\mathfrak{H}_N^c + \mathfrak{H}_N^o) = \{0\}. \tag{3.7}$$

In particular, if $\ker D_A = \{0\}$ *and* $\mathfrak{H} = \operatorname{clos} \{\mathfrak{H}_N^c + \mathfrak{H}_N^o\}$, *then τ is simple; if τ is simple, $\mathfrak{H}_N^c$ is invariant under A^*, and $\mathfrak{H}_N^o$ is invariant under A, then* $\ker D_A = \{0\}$ *and* $\mathfrak{H} = \operatorname{clos} \{\mathfrak{H}_N^c + \mathfrak{H}_N^o\}$.

(iv) *τ is minimal if and only if*

$$\begin{aligned} \ker D_A = \{0\}, \quad \operatorname{ran} D_A \cap (\mathfrak{H} \ominus \mathfrak{H}_N^c) = \{0\}, \\ \textit{and} \quad \operatorname{ran} D_A \cap (\mathfrak{H} \ominus \mathfrak{H}_N^o) = \{0\}. \end{aligned} \tag{3.8}$$

In particular, if $\ker D_A = \{0\}$ *and $\mathfrak{H} = \mathfrak{H}_N^c = \mathfrak{H}_N^o$, then τ is minimal; if τ is minimal, $\mathfrak{H}_N^c$ is invariant under A^*, and $\mathfrak{H}_N^o$ is invariant under A, then* $\ker D_A = \{0\}$ *and $\mathfrak{H} = \mathfrak{H}_N^c = \mathfrak{H}_N^o$.*

Proof. (i) Assume that (3.5) holds. Let $f \in (\mathfrak{H}^c)^\perp$. It follows from (3.2) and (3.4) that $D_A f \in \bigcap_{n=0}^\infty \ker (M^*A^{*n}) = (\mathfrak{H}_N^c)^\perp$. The second condition in (3.5) shows that $D_A f = 0$ and the first condition in (3.5) yields $f = 0$. Therefore, $\mathfrak{H}^c = \mathfrak{H}$ and τ is controllable.

Now assume that τ is controllable, i.e. $\mathfrak{H}^c = \mathfrak{H}$. Then (3.2) implies that $\ker D_A = \{0\}$. Furthermore, if $D_A f \in (\mathfrak{H}_N^c)^\perp$, then (3.4) implies that $D_A f \in \bigcap_{n=0}^\infty \ker (M^*A^{*n})$.

By (3.2) this leads to $f \in (\mathfrak{H}^c)^\perp$ and hence $f = 0$ by controllability of τ. This shows that (3.5) is satisfied.

If ker $D_A = \{0\}$ and $\mathfrak{H}_N^c = \mathfrak{H}$, then (3.5) is satisfied. It follows that τ is controllable.

If $\mathfrak{H}_N^c$ is invariant under A^*, then $\mathfrak{H} \ominus \mathfrak{H}_N^c \subset \mathfrak{H} \ominus \mathfrak{H}^c$ by Lemma 3.1. Hence, if in addition, τ is controllable, it follows that $\mathfrak{H}_N^c = \mathfrak{H}$; moreover, it follows that ker $D_A = \{0\}$.

(ii) The proof is completely analogous to the proof for part (i).

(iii) If τ is simple then it immediately follows from (3.1) that ker $D_A = \{0\}$. Moreover, it is clear from (3.2) and (3.4) that $D_A f \in \mathfrak{H} \ominus (\mathfrak{H}_N^c + \mathfrak{H}_N^o)$ if and only if $f \in \mathfrak{H} \ominus (\mathfrak{H}^c + \mathfrak{H}^o)$. Now the statement is obtained as in part (i).

(iv) This is obvious from the definition of minimality. □

Corollary 3.3. *Let the main operator A of the passive system $\tau = \{T; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}\}$ be normal and let the system be simple. Then the system τ is strongly stable and strongly co-stable.*

Proof. Since τ is simple and A is normal, Proposition 3.2 shows that ker $D_A = \{0\}$ or, equivalently, that the contraction A is strict. Hence Lemma 2.2 implies that $A \in C_{00}$. Therefore τ is strongly stable and strongly co-stable. □

3.2. **Defect functions.** Associated with $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ are the right and left *defect functions* (or *spectral factors*) $\varphi_\Theta(\lambda)$ and $\psi_\Theta(\lambda)$, which satisfy

$$\varphi_\Theta^*(\xi)\varphi_\Theta(\xi) \le I_{\mathfrak{M}} - \Theta^*(\xi)\Theta(\xi), \quad \psi_\Theta(\xi)\psi_\Theta^*(\xi) \le I_{\mathfrak{N}} - \Theta(\xi)\Theta^*(\xi), \tag{3.9}$$

almost everywhere on $\mathbb{T}$. These operator-valued Schur functions are (up to a constant unitary factor) uniquely determined by the following maximality property: if $\widetilde{\varphi}(\lambda)$ and $\widetilde{\psi}(\lambda)$ are operator-valued Schur functions for which

$$\widetilde{\varphi}^*(\xi)\widetilde{\varphi}(\xi) \le I_{\mathfrak{M}} - \Theta^*(\xi)\Theta(\xi), \quad \widetilde{\psi}(\xi)\widetilde{\psi}^*(\xi) \le I_{\mathfrak{M}} - \Theta(\xi)\Theta^*(\xi), \tag{3.10}$$

then they are dominated by $\varphi_\Theta(\lambda)$ and $\psi_\Theta(\lambda)$ in the following sense:

$$\widetilde{\varphi}^*(\xi)\widetilde{\varphi}(\xi) \le \varphi_\Theta^*(\xi)\varphi_\Theta(\xi), \quad \widetilde{\psi}(\xi)\widetilde{\psi}^*(\xi) \le \psi_\Theta(\xi)\psi_\Theta^*(\xi), \tag{3.11}$$

almost everywhere on the unit circle $\mathbb{T}$; cf. [17], [19], [20], [21], [22].

Note that it follows from Theorem 2.7 that the functions $\varphi(\lambda)$ and $\psi(\lambda)$ satisfy the inequalities

$$\varphi(\xi)^*\varphi(\xi) \le \varphi_\Theta^*(\xi)\varphi_\Theta(\xi), \quad \psi(\xi)\psi^*(\xi) \le \psi_\Theta(\xi)\psi_\Theta^*(\xi), \tag{3.12}$$

for almost all $\xi \in \mathbb{T}$.

Proposition 3.4. *Let $\tau = \{T; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}\}$ be a passive system with a normal main operator A and let $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ be its transfer function. If τ is simple and $\varphi_\Theta(\lambda) = 0$ ($\psi_\Theta(\lambda) = 0$), then $\Theta(\lambda)$ is inner (co-inner, respectively).*

Proof. By Corollary 3.3 one has $A \in C_{00}$ and, in particular, A is completely non-unitary. Therefore, $\Phi_A(\lambda)$ and $\Phi_{A^*}(\lambda)$ are bi-inner. On the other hand, if $\varphi_\Theta(\lambda) = 0$ ($\psi_\Theta(\lambda) = 0$), then (3.12) shows that $\varphi(\xi) = 0$ ($\psi(\xi) = 0$) for almost all $\xi \in \mathbb{T}$. Now (2.23) ((2.24), respectively) yields that $D_{\Theta(\xi)} = 0$ ($D_{\Theta^*(\xi)} = 0$) almost everywhere on $\mathbb{T}$, i.e. $\Theta(\lambda)$ is inner (co-inner). □

3.3. **Unitary similarity.** Recall that two passive systems $\tau_j = \{T_j; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}_j\}$, $j = 1,2$, are said to be unitarily similar if there is a unitary operator $U : \mathfrak{H}_1 \to \mathfrak{H}_2$, such that (1.4) holds. In particular, in this case the spectra of the corresponding main operators A_1 and A_2 coincide. It is clear that if the systems τ_1 and τ_2 are unitarily similar then they have the same transfer function. However, two minimal passive systems τ_1 and τ_2 with the same transfer function $\Theta(\lambda)$ are in general not unitarily similar; such systems are only weakly similar as shown in D.Z. Arov [10], see (1.6). In the case of passive systems with normal main operators the following sufficient spectral-theoretic condition can be established.

Theorem 3.5. *Let $\tau_1 = \{T_1; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}_1\}$ and $\tau_2 = \{T_2; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}_2\}$ be two minimal passive systems whose transfer functions coincide in some neighborhood of zero. Let the main operator A_k be normal and let $C_k = SB_k^*$, $k = 1,2$, with S bounded and injective. Then, if the spectrum $\sigma(A_k)$ of A_k, $k = 1,2$, does not contain interior points and $\rho(A_1) \cap \rho(A_2)$ is a connected set in $\mathbb{C}$, the systems τ_1 and τ_2 are unitarily similar.*

Proof. Assume that the transfer functions $\Theta_1(\lambda)$ and $\Theta_2(\lambda)$ of τ_1 and τ_2 coincide in some neighborhood of zero. Since $\Theta_1(\lambda)$ and $\Theta_2(\lambda)$ are holomorphic on $\mathbb{D}$ it follows that $\Theta_1(\lambda) = \Theta_2(\lambda)$ for all $\lambda \in \mathbb{D}$. The definition (1.5) implies that $D_1 = \Theta_1(0) = \Theta_2(0) = D_2$ and that

$$\sum_{m=0}^{\infty} \lambda^m C_1 A_1^m B_1 = \sum_{m=0}^{\infty} \lambda^m C_2 A_2^m B_2, \quad \lambda \in \mathbb{D}.$$

Since $C_k = SB_k^*$, $k = 1,2$, where S is bounded and injective, the previous equality yields

$$B_1^* A_1^m B_1 = B_2^* A_2^m B_2, \quad m \in \mathbb{N}_0. \tag{3.13}$$

Now define the relation Z_0 by

$$Z_0 = \left\{ \left\{ \sum_{j=0}^{m} A_1^j B_1 u_j, \sum_{j=0}^{m} A_2^j B_2 u_j \right\} : \, u_0, u_1, \ldots, u_m \in \mathfrak{M}, \; m \in \mathbb{N}_0 \right\}. \tag{3.14}$$

Clearly Z_0 is linear and

$$\operatorname{dom} Z_0 = \operatorname{span}\{ A_1^n B_1 \mathfrak{M} : \, n \in \mathbb{N}_0\}, \quad \operatorname{ran} Z_0 = \operatorname{span}\{ A_2^n B_2 \mathfrak{M} : \, n \in \mathbb{N}_0\}.$$

Furthermore, it follows from (3.13) that

$$\left\{ \sum_{j=0}^{m} A_2^{*j} B_2 v_j, \sum_{j=0}^{m} A_1^{*j} B_1 v_j \right\} \in Z_0^*, \quad v_1, \ldots, v_m \in \mathfrak{M}, \quad m \in \mathbb{N}_0, \tag{3.15}$$

so that

$$\operatorname{span}\{ A_2^{*n} B_2 \mathfrak{M} : \, n \in \mathbb{N}_0\} \subset \operatorname{dom} Z_0^*, \quad \operatorname{span}\{ A_1^{*n} B_1 \mathfrak{M} : \, n \in \mathbb{N}_0\} \subset \operatorname{ran} Z_0^*.$$

Due to the controllability and observability conditions (note that $C_k^* = B_k S^*$), it follows from (3.14) and (3.15) that both Z_0 and Z_0^* have dense domains and dense ranges. In particular, Z_0 and Z_0^* are (graphs of) operators, and, in fact, $\operatorname{mul} Z_0^{**} = (\operatorname{dom} Z_0^*)^\perp$ implies that Z_0 is a closable operator, i.e., its closure Z_0^{**} is (the graph of) an operator; cf. [11].

Next it is shown that under the assumptions on the main operators A_k, $k = 1,2$, the mapping Z_0 becomes isometric. Since A is contractive the spectrum $\sigma(A_k)$ is a compact subset of the closed unit disk. The union $\sigma(A_1)\cup\sigma(A_2)$ is also compact and, in addition, does not have interior points. Indeed, this follows immediately from the fact that the sets $\sigma(A_1)$ and $\sigma(A_2)$ are closed and do not have interior points. Furthermore, by assumption $\mathbb{C}\setminus(\sigma(A_1)\cup\sigma(A_2)) = \rho(A_1)\cap\rho(A_2)$ is connected. Therefore, according to Mergelyan's theorem (see e.g. [26, Theorem 20.5]) every continuous complex-valued function on $\sigma(A_1)\cup\sigma(A_2)$ can be uniformly approximated on $\sigma(A_1)\cup\sigma(A_2)$ by complex polynomials. Since for every $n,m \in \mathbb{N}_0$ the function $f_{n,m}(z) = \bar{z}^n z^m$ is continuous on $\mathbb{C}$, there exists a sequence $\{P_j^{n,m}(z) : j \in \mathbb{N}_0\}$ of polynomials converging uniformly on $\sigma(A_1)\cup\sigma(A_2)$ to $f_{n,m}(z)$. It follows from (3.13) that for every $n,k,j \in \mathbb{N}_0$ one has

$$B_1^* P_j^{n,m}(A_1) B_1 = B_2^* P_j^{n,m}(A_2) B_2. \tag{3.16}$$

The functional calculus for normal operators shows that $f_{n,m}(A_k) = A_k^{*n} A_k^m$, $k = 1,2$, and therefore taking strong limits in (3.16) yields

$$B_1^* A_1^{*n} A_1^m B_1 = B_2^* A_2^{*n} A_2^m B_2, \quad m,n \in \mathbb{N}_0. \tag{3.17}$$

These identities imply that

$$\left\| \sum_{j=0}^m A_1^j B_1 u_j \right\|^2 = \left\| \sum_{j=0}^m A_2^j B_2 u_j \right\|^2, \quad u_0, u_1, \ldots, u_m \in \mathfrak{M}, \quad m \in \mathbb{N}_0,$$

and, therefore, the operator Z_0 in (3.14) is isometric. Since Z_0 is densely defined with dense range, its closure Z is unitary. The identities $ZA_1 = A_2 Z$ and $ZB_1 = B_2$ are immediate from (3.14), while (3.15) shows that $Z_0^* B_2 = B_1$ which gives the identity $C_2 Z = C_1$. Therefore, the systems τ_1 and τ_2 are unitarily similar; cf. (1.4). □

Corollary 3.6. *Let $\tau_1 = \{T_1; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}_1\}$ and $\tau_2 = \{T_2; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}_2\}$ be two minimal passive systems such that A_k is selfadjoint ($A_k = A_k^*$) or skew-symmetric ($A_k = -A_k^*$) and $C_k = SB_k^*$, $k = 1,2$, with S bounded and injective. Then τ_1 and τ_2 are unitarily similar if and only if their transfer functions coincide in some neighborhood of zero.*

Corollary 3.7. *Let $\tau_1 = \{T_1; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}_1\}$ and $\tau_2 = \{T_2; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}_2\}$ be two minimal passive systems such that A_k is normal and has a discrete spectrum, and $C_k = SB_k^*$, $k = 1,2$, with S bounded and injective. Then τ_1 and τ_2 are unitarily similar if and only if their transfer functions coincide in some neighborhood of zero.*

Corollary 3.8. *Let $\tau_1 = \{T_1; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}_1\}$ and $\tau_2 = \{T_2; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}_2\}$ be two minimal passive systems with a finite-dimensional state space $\mathfrak{H}_k$ such that A_k is normal and $C_k = SB_k^*$, $k = 1,2$, with S bounded and injective. Then τ_1 and τ_2 are unitarily similar if and only if their transfer functions coincide in some neighborhood of zero.*

Remark 3.9. (i) The proof of Theorem 3.5 uses that fact that the operators $f_{n,m}(A) = A^{*n} A^m$, $m,n \in \mathbb{N}_0$, can be approximated by a sequence of polynomials in A. If, in particular, the adjoint A^* of a bounded operator A can be approximated by a sequence $P_n(A)$, $n \in \mathbb{N}_0$, of polynomials in A, i.e.,

$$A^* = s - \lim_{n\to\infty} P_n(A), \tag{3.18}$$

then the same is true for all of the operators $f_{n,m}(A) = A^{*n}A^m$, $m, n \in \mathbb{N}_0$. By taking strong limits in $AP_n(A) = P_n(A)A$ one obtains from (3.18) the identity $AA^* = A^*A$. Therefore, the condition (3.18) implies that A is a normal operator.

(ii) If A is a normal operator, then $A^* = f(A)$ with $f(z) = \bar{z}$ by the functional calculus for normal operators. The function $f(z)$ does not satisfy the Cauchy-Riemann equations, so it is nowhere holomorphic. Consequently, if $\sigma(A)$ has interior points, the adjoint A^* cannot satisfy the condition (3.18), as one would get a uniform approximation for $f(z)$ on $\sigma(A)$ via polynomials $P_n(z)$.

(iii) If A is a normal operator on a finite-dimensional space, then it has $n = \dim \mathfrak{H}$ eigenvalues and it is unitarily similar to a diagonal matrix. Therefore, if A has d nonreal eigenvalues then by standard interpolation one finds a polynomial Q (say, of degree at most $d-1$ when using only the nonreal spectral points) such that $A^* = Q(A)$. If there are two normal operators A_1 and A_2 on $\mathfrak{H}_k$, $n_k = \dim \mathfrak{H}_k < \infty$, $k = 1, 2$, then together they have at most $n_1 + n_2$ different nonreal eigenvalues and one can find a polynomial P (of degree at most $n_1 + n_2 - 1$) such that $A_1^* = P(A_1)$ and $A_2^* = P(A_2)$. Then $f_{n,m}(A_k) = A_k^{*n} A_k^m = P(A_k)^n A_k^m$, $n, m \in \mathbb{N}_0$, is also a polynomial in A_k, $k = 1, 2$. So, in the proof of Theorem 3.5 no limit procedure is needed in the case of finite-dimensional state spaces.

(iv) Finally, note that the criterion for unitary similarity of minimal passive systems with the same transfer function which has been established in [14] is essentially of different nature than the above spectral theoretical sufficient condition in Theorem 3.5.

4. Passive quasi-selfadjoint systems

4.1. **Quasi-selfadjoint contractions and associated passive systems.** Let $\mathcal{H}$ be a Hilbert space. A linear operator $T \in \mathbf{L}(\mathcal{H})$ is said to be a *quasi-selfadjoint contraction* (*qsc*-operator for short) if

$$\operatorname{dom} T = \mathcal{H}, \ \|T\| \leq 1, \ \text{and } \ker\,(T - T^*) \neq \{0\}.$$

The next theorem is a consequence of Theorem 2.5; see [4].

Theorem 4.1. *Let T be a qsc-operator in the Hilbert space $\mathcal{H}$ and let $\mathfrak{N}$ be a subspace in $\mathcal{H}$ such that* $\operatorname{ran}(T - T^*) \subset \mathfrak{N}$. *Then with respect to the decomposition $\mathcal{H} = \mathfrak{N} \oplus \mathfrak{H}$, where $\mathfrak{H} = \mathcal{H} \ominus \mathfrak{N}$, the operator T has the following block form*

$$T = \begin{pmatrix} -KAK^* + D_{K^*}XD_{K^*} & KD_A \\ D_AK^* & A \end{pmatrix} : \begin{pmatrix} \mathfrak{N} \\ \mathfrak{H} \end{pmatrix} \to \begin{pmatrix} \mathfrak{N} \\ \mathfrak{H} \end{pmatrix}, \tag{4.1}$$

where $A = P_{\mathfrak{H}} T \upharpoonright \mathfrak{H}$ is a selfadjoint contraction and $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$, $X \in \mathbf{L}(\mathfrak{D}_{K^*})$ are contractions.

The system $\tau = \{T; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}\}$ is said to be *passive quasi-selfadjoint* (τ is a *pqs*-system for short) if T in (1.7) is a contraction and if $\operatorname{ran}(T - T^*) \subset \mathfrak{N}$. It follows that T is a *qsc*-operator in $\mathfrak{N} \oplus \mathfrak{H}$ and that $A = A^*$ and $C = B^*$. Moreover, according to Theorem 4.1, B, C, and D have the form

$$B = D_AK^*, \quad C = KD_A, \quad D = -KAK^* + D_{K^*}XD_{K^*}, \tag{4.2}$$

where $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$ and $X \in \mathbf{L}(\mathfrak{D}_K^*)$ are contractions. For a *pqs*-system the controllable and observable subspaces coincide, see (3.1):

$$\mathfrak{H}^c = \mathfrak{H}^o = \overline{\operatorname{span}}\, \{\, D_A A^n K^* \mathfrak{N} : \, n \in \mathbb{N}_0 \,\} \subset \mathfrak{D}_A. \tag{4.3}$$

4.2. **Minimal representations of *pqs*-systems and unitary similarity.** A *pqs*-system can always be reduced to a minimal *pqs*-system.

Proposition 4.2. *Let $\tau = \{T; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}\}$ be a pqs-system of the form* (1.7) *and let B, C, and D be given by* (4.2) *with some contractions K and X. Define the system*

$$\tau_s = \{T_s; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}^s\}, \tag{4.4}$$

where the subspace $\mathfrak{H}^s$ is given by

$$\mathfrak{H}^s = \overline{\operatorname{span}}\, \{\, A^n K^* \mathfrak{N} : \, n \in \mathbb{N}_0 \,\}, \tag{4.5}$$

and where the operator T_s is given by

$$T_s = \begin{pmatrix} D & C{\upharpoonright}\,\mathfrak{H}^s \\ B & A{\upharpoonright}\,\mathfrak{H}^s \end{pmatrix} : \begin{pmatrix} \mathfrak{N} \\ \mathfrak{H}^s \end{pmatrix} \to \begin{pmatrix} \mathfrak{N} \\ \mathfrak{H}^s \end{pmatrix}. \tag{4.6}$$

Then τ_s is a minimal pqs-system and the transfer functions of the systems τ and τ^s coincide. Moreover, the system τ is minimal if and only if

(i) *$\|Af\| < \|f\|$ for all $f \in \mathfrak{H} \backslash \{0\}$,*
(ii) *$\mathfrak{H}^s = \mathfrak{H}$.*

In this case the system τ is strongly stable and strongly co-stable.

Proof. The subspace $\mathfrak{H}^s$ in (4.5) reduces A and therefore it also reduces $D_A = (I_{\mathfrak{H}} - A^2)^{1/2}$. Furthermore, $\overline{\operatorname{ran}}\, K^* \subset \mathfrak{H}^s$. Let $A_s = A{\upharpoonright}\,\mathfrak{H}^s$, then $D_{A_s} = D_A{\upharpoonright}\,\mathfrak{H}^s$ and, hence, $D_A K^* = D_{A_s} K^*$. Define the operator C_s by

$$C_s = C{\upharpoonright}\,\mathfrak{H}^s = K D_{A_s}.$$

Then T_s in (4.6) is a *qsc*-operator in $\mathfrak{N} \oplus \mathfrak{H}^s$. Since $\operatorname{ran} K^* \subset \mathfrak{D}_A \cap \mathfrak{H}^s$, one has $\mathfrak{D}_{A_s} = \mathfrak{H}^s$. Now the construction shows that the system τ^s in (4.4) is minimal. Clearly, the transfer functions of τ and τ^s coincide.

As to the minimality of τ observe that $\mathfrak{H}^s = \mathfrak{H}_N^c = \mathfrak{H}_N^o$, since $A = A^*$; see (3.3). Hence, the characteristic properties (i) and (ii) for minimality of a *pqs*-system τ are obtained from Proposition 3.2.

The last statement holds by Corollary 3.3. □

It is a consequence of Theorem 3.5 that within the class of *pqs*-systems the following unitary similarity criterion holds; see Corollary 3.6.

Proposition 4.3. *Let $\tau_1 = \{T_1; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}_1\}$ and $\tau_2 = \{T_2; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}_2\}$ be two minimal pqs-systems. Then τ_1 and τ_2 are unitarily similar if and only if their transfer functions coincide in some neighborhood of zero.*

4.3. **Transfer functions of** *pqs***-systems.** Let $\tau = \{T; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}\}$ be a *pqs*-system of the form (1.7) and assume that T is represented in the form (4.1). Then the transfer function $\Theta(\lambda)$ of τ has the form

$$\Theta(\lambda) = K\Phi_A(\lambda)K^* + D_{K^*}XD_{K^*}, \quad \lambda \in \mathbb{D}, \tag{4.7}$$

where $\Phi_A(\lambda)$ is the characteristic function of the selfadjoint contraction A; see (2.6). The function $\Phi_A(\lambda)$ is holomorphic on $\mathbb{T} \setminus \{-1, 1\}$ and, in fact, it belongs to Herglotz-Nevanlinna class on Ext $\{(-\infty, -1] \cup [1, \infty)\}$. Furthermore, $\Phi_A(\lambda)$ has nontangential strong limit values $\Phi_A(\pm 1) = \pm I_{\mathfrak{D}_A}$; see e.g. [4, Theorem 2.3]. Consequently, the limit value $\Phi_A(\xi)$ is unitary for every $\xi \in \mathbb{T}$ (see (2.7), (2.11)), in particular, $\Phi_A(\lambda)$ is bi-inner. It follows from (4.7) that $\Theta(\lambda)$, initially defined on $\mathbb{D}$, admits a holomorphic continuation onto Ext $\{(-\infty, -1] \cup [1, \infty)\}$. Furthermore, $\Theta(\lambda)$ has nontangential strong limit values $\Theta(\pm 1)$ at ± 1 which are given by

$$\Theta(1) = KK^* + D_{K^*}XD_{K^*}, \quad \Theta(-1) = -KK^* + D_{K^*}XD_{K^*}. \tag{4.8}$$

Define the function $W(\lambda)$ by

$$W(\lambda) = \Theta(\lambda) - \Theta(0), \quad \lambda \in \text{Ext}\,\{(-\infty, -1] \cup [1, \infty)\}. \tag{4.9}$$

Since

$$\Theta(0) = -KAK^* + D_{K^*}XD_{K^*}, \tag{4.10}$$

it follows that

$$W(\lambda) = \lambda K\left(I - \lambda A\right)^{-1} D_A^2 K^*, \quad \lambda \in \text{Ext}\,\{(-\infty, -1] \cup [1, \infty)\}. \tag{4.11}$$

Hence $W^*(\bar{\lambda}) = W(\lambda)$ and

$$\frac{W(\lambda) - W^*(\xi)}{\lambda - \bar{\xi}} = \begin{cases} KD_A(I - \lambda A)^{-1}(I - \bar{\xi}A)^{-1}D_AK^*, \ \bar{\xi} \neq \lambda, \\ KD_A(I - \lambda A)^{-2}D_AK^*, \ \bar{\xi} = \lambda. \end{cases}$$

Therefore $W(\lambda)$ is an operator-valued Herglotz-Nevanlinna function with a holomorphic continuation onto Ext $\{(-\infty, -1] \cup [1, \infty)\}$. From (4.8) one sees that the strong limit values $W(\pm 1)$ exist and that they are given by

$$W(1) = K(I + A)K^*, \quad W(-1) = -K(I - A)K^*.$$

Hence,

$$\frac{W(1) + W(-1)}{2} = KAK^*, \quad I - \frac{W(1) - W(-1)}{2} = I - KK^* = D_{K^*}^2 \geq 0.$$

Since X in (4.10) is a contraction in $\mathfrak{D}_{K^*}$, these identities show that

$$\Theta(0) \in \mathbf{B}\left(-\frac{W(1) + W(-1)}{2},\ I - \frac{W(1) - W(-1)}{2}\right). \tag{4.12}$$

Here $\mathbf{B}(S, R) = \{\, S + R^{1/2}XR^{1/2} \in \mathbf{L}(\mathfrak{N}) : X \text{ a contraction in } \mathbf{L}(\overline{\operatorname{ran}}\, R)\,\}$ stands for the *operator ball* with center $S \in \mathbf{L}(\mathfrak{N})$ and left and right radii $R \geq 0$.

Definition 4.4. Let $\mathfrak{N}$ be a Hilbert space. The *class* $\mathbf{S}^{qs}(\mathfrak{N})$ consists of all $\mathbf{L}(\mathfrak{N})$-valued functions $\Theta(\lambda)$, defined on $\mathbb{D}$, such that

(S1) $W(\lambda) = \Theta(\lambda) - \Theta(0)$ is a Herglotz-Nevanlinna function with a holomorphic continuation onto the domain Ext $\{(-\infty, -1] \cup [1, \infty)\}$;

(S2) the strong limit values $W(\pm 1)$ exist and $W(1) - W(-1) \leq 2I$;
(S3) $\Theta(0)$ belongs to the operator ball in (4.12).

The following proposition is now clear.

Proposition 4.5. *Let $\tau = \{T; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}\}$ be a pqs-system. Then its transfer function $\Theta(\lambda)$ belongs to $\mathbf{S}^{qs}(\mathfrak{N})$.*

5. The class $\mathbf{S}^{qs}$ and its realization via passive systems

5.1. **The realization of the class $\mathbf{S}^{qs}$.** The next theorem is a converse to Proposition 4.5. In its proof a minimal *pqs*-system is constructed explicitly via an operator representation of the Herglotz-Nevanlinna function $W(\lambda) = \Theta(\lambda) - \Theta(0)$.

Theorem 5.1. *Let $\mathfrak{N}$ be a Hilbert space and let $\Theta(\lambda) \in \mathbf{S}^{qs}(\mathfrak{N})$. Then $\Theta(\lambda)$ is the transfer function of a minimal pqs-system $\tau = \{T; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}\}$.*

Proof. Assume that $\Theta(\lambda) \in \mathbf{S}^{qs}(\mathfrak{N})$. By the condition (S1) the function

$$\widetilde{W}(z) := -W(1/z), \quad z \in \operatorname{Ext}[-1, 1],$$

is a Herglotz-Nevanlinna function of the class $\mathbf{N}_{\mathfrak{N}}[-1, 1]$ with $\widetilde{W}(\infty) = 0$, see [4]. It follows from the condition (S2) that the strong limit values $\widetilde{W}(\pm 1)$ exist. Then according to [4, Theorem 2.3] there exist a Hilbert space $\widetilde{\mathfrak{H}}$, a selfadjoint contraction $\widetilde{A}$ in $\widetilde{\mathfrak{H}}$, and an operator $\widetilde{G} \in \mathbf{L}(\mathfrak{N}, \mathfrak{D}_{\widetilde{A}})$, such that

$$\widetilde{W}(z) = \widetilde{G}^*(\widetilde{A} - zI)^{-1}(I - \widetilde{A}^2)\widetilde{G},$$

see [4]. It follows that

$$W(-1) = -\widetilde{W}(-1) = -\widetilde{G}^*(I - \widetilde{A})\widetilde{G}, \quad W(1) = -\widetilde{W}(1) = \widetilde{G}^*(I + \widetilde{A})\widetilde{G}.$$

Consequently,

$$\frac{W(1) + W(-1)}{2} = \widetilde{G}^*\widetilde{A}\widetilde{G}, \quad I - \frac{W(1) - W(-1)}{2} = I - \widetilde{G}^*\widetilde{G}.$$

The condition $W(1) - W(-1) \leq 2I$ implies that $\widetilde{G}$ is contractive. The condition (S3) means that $\Theta(0) = -\widetilde{G}^*\widetilde{A}\widetilde{G} + D_{\widetilde{G}^*}\widetilde{X}D_{\widetilde{G}}$ for some contraction $\widetilde{X}$ in the Hilbert space $\mathfrak{D}_{\widetilde{G}}$. Define in the Hilbert space $\widetilde{\mathcal{H}} = \mathfrak{N} \oplus \widetilde{\mathfrak{H}}$ the operator $\widetilde{T}$ by

$$\widetilde{T} = \begin{pmatrix} -\widetilde{G}^*\widetilde{A}\widetilde{G} + D_{\widetilde{G}^*}\widetilde{X}D_{\widetilde{G}} & \widetilde{G}^*D_{\widetilde{A}} \\ D_{\widetilde{A}}\widetilde{G} & \widetilde{A} \end{pmatrix}.$$

Then $\widetilde{T}$ is a *qsc*-operator, $\operatorname{ran}(\widetilde{T} - \widetilde{T}^*) \subset \mathfrak{N}$, and the operator $\widetilde{T}$ defines a *pqs*-system $\widetilde{\tau} = \{\widetilde{T}; \mathfrak{N}, \mathfrak{N}, \widetilde{\mathfrak{H}}\}$; cf. Theorem 4.1. The corresponding transfer function is given by

$$\Theta_{\widetilde{\tau}}(\lambda) = \widetilde{G}^*\left(-\widetilde{A} + \lambda\left(I - \lambda\widetilde{A}\right)^{-1} D^2_{\widetilde{A}}\right)\widetilde{G} + D_{\widetilde{G}^*}\widetilde{X}D_{\widetilde{G}}, \quad \lambda \in \mathbb{D}.$$

Therefore, $\Theta_{\widetilde{\tau}}(\lambda) = \Theta(0) + W(\lambda) = \Theta(\lambda)$, $\lambda \in \mathbb{D}$. This means that the function $\Theta(\lambda)$ can be realized as the transfer function of the *pqs*-system $\widetilde{\tau}$. Finally, replacing $\widetilde{\tau}$ by the system $\widetilde{\tau}^s$, cf. Proposition 4.2, one obtains a minimal *pqs*-system. The corresponding transfer function still coincides with the function $\Theta(\lambda)$. □

Observe that Theorem 5.1 implies that the class $\mathbf{S}^{qs}(\mathfrak{N})$ is a subclass of the Schur class $\mathbf{S}(\mathfrak{N})$. Furthermore, the proof shows that a function $\Theta(\lambda)$ from the class $\mathbf{S}^{qs}(\mathfrak{N})$ admits the integral representation

$$\Theta(\lambda) = \Theta(0) + \lambda \int_{-1}^{1} \frac{1-t^2}{1-t\lambda}\, d\Sigma(t),$$

where $\Sigma(t)$ is a non-decreasing $\mathbf{L}(\mathfrak{N})$-valued function with bounded variation, $\Sigma(-1) = 0$, $\Sigma(1) \leq I_{\mathfrak{N}}$, and

$$\left|\left(\left(\Theta(0) + \int_{-1}^{1} t\, d\Sigma(t)\right) f, g\right)\right|^2 \leq \left(\left(I - \Sigma(1)\right) f, f\right) \left(\left(I - \Sigma(1)\right) g, g\right), \quad f, g \in \mathfrak{N}.$$

Corollary 5.2. *Let $\mathfrak{N}$ be a Hilbert space and let $\Theta(\lambda) \in \mathbf{S}^{qs}(\mathfrak{N})$. If $\varphi_\Theta(\lambda) = 0$ ($\psi_\Theta(\lambda) = 0$) then $\Theta(\lambda)$ is inner (co-inner).*

Proof. By Theorem 5.1 there exists a minimal *pqs*-system $\tau = \{T, \mathfrak{N}, \mathfrak{N}, \mathfrak{H}\}$ with transfer function $\Theta(\lambda)$. Now the statement follows from Proposition 3.4. □

Theorem 5.3. *Let $\mathfrak{N}$ be a Hilbert space and let $\Theta(\lambda) \in \mathbf{S}^{qs}(\mathfrak{N})$. Then:*

(i) *if $\Theta(\lambda)$ is inner then*

$$(5.1)\qquad \begin{aligned} &\left(\frac{\Theta(1) - \Theta(-1)}{2}\right)^2 = \frac{\Theta(1) - \Theta(-1)}{2}, \\ &(\Theta(1) + \Theta(-1))^*(\Theta(1) + \Theta(-1)) = 4I_{\mathfrak{N}} - 2\left(\Theta(1) - \Theta(-1)\right); \end{aligned}$$

(ii) *if $\Theta(\lambda)$ is co-inner then*

$$(5.2)\qquad \begin{aligned} &\left(\frac{\Theta(1) - \Theta(-1)}{2}\right)^2 = \frac{\Theta(1) - \Theta(-1)}{2}, \\ &(\Theta(1) + \Theta(-1))(\Theta(1) + \Theta(-1))^* = 4I_{\mathfrak{N}} - 2\left(\Theta(1) - \Theta(-1)\right); \end{aligned}$$

(iii) *if (5.1) ((5.2)) holds and $\Theta(\xi)$ is isometric (co-isometric) for some $\xi \in \mathbb{T}$, $\xi \neq \pm 1$, then $\Theta(\lambda)$ is inner (co-inner).*

Proof. Since $\Theta(\lambda) \in \mathbf{S}^{qs}(\mathfrak{N})$, it is the transfer function of a minimal *pqs*-system $\tau = \{T, \mathfrak{N}, \mathfrak{N}, \mathfrak{H}\}$. The operator T, being quasi-selfadjoint, has the form (4.1) and $\Theta(\lambda)$ is given by (4.7) with a holomorphic continuation into the domain $\operatorname{Ext}\{(-\infty, -1] \cup [1, \infty)\}$. Since $\Phi_A(\xi)$ is unitary for every $\xi \in \mathbb{T}$, it follows from (2.23) and (2.24) in Theorem 2.7 and the definitions (2.20) and (2.21) that for all $h \in \mathfrak{N}$ and $\xi \in \mathbb{T}$

$$(5.3)\qquad \begin{aligned} &\left\|D_{\Theta(\xi)} h\right\|^2 = ||D_X D_{K^*} h||^2 + \left\|\left(D_K \Phi_A(\xi) K^* - K^* X D_{K^*}\right) h\right\|^2, \\ &\|D_{\Theta^*}(\xi) h\|^2 = ||D_{X^*} D_{K^*} h||^2 + \left\|\left(D_K \Phi_A(\bar{\xi}) K^* - K^* X^* D_{K^*}\right) h\right\|^2. \end{aligned}$$

(i) Suppose that $\Theta(\lambda)$ is inner. Then (5.3) shows that

$$\begin{cases} D_X D_{K^*} = 0, \\ D_K \Phi_A(\xi) K^* = K^* X D_{K^*}, \quad \xi \in \mathbb{T}. \end{cases}$$

The last equality yields that $D_K \Phi_A(\lambda) K^* = K^* X D_{K^*}$ for all $\lambda \in \mathbb{D}$. Since

$$\Phi_A(\lambda) = -A + \sum_{n=0}^{\infty} \lambda^{n+1} A^n D_A^2,$$

it follows that $D_K D_A A^n D_A K^* = 0$, $n \in \mathbb{N}_0$. The minimality of τ implies that $D_K f = 0$ for all $f \in \mathfrak{D}_A$; see (4.3). Hence, K is isometric and D_{K^*} is an orthogonal projector in the subspace $\mathfrak{N}$. Due to the identity $D_X D_{K^*} = 0$, $X \in \mathbf{L}(\mathfrak{D}_{K^*})$ is isometric (here possibly $\mathfrak{D}_{K^*} = \{0\}$). Now from the equalities in (4.8) one obtains

$$KK^* = \frac{\Theta(1) - \Theta(-1)}{2} \tag{5.4}$$

and

$$XD_{K^*} = D_{K^*} X D_{K^*} = \frac{\Theta(1) + \Theta(-1)}{2}. \tag{5.5}$$

These identities together with the equality $X^*X = I_{\mathfrak{D}_{K^*}}$ lead to (5.1).

(ii) The proof is similar to that of (i).

(iii) Assume that (5.1) holds and that $\Theta(\xi)$ is isometric for some $\xi \in \mathbb{T}$, $\xi \neq \pm 1$. Due to (5.1) D_{K^*} is an orthogonal projector in $\mathfrak{N}$ and X is isometric in $\mathfrak{D}_{K^*}$. Hence, $K^* \in \mathbf{L}(\mathfrak{N}, \mathfrak{H})$ is a partial isometry and, moreover, $D_K K^* = K^* D_{K^*} = 0$ and $K^* X D_{K^*} = 0$. Since $\Theta(\xi)$ is isometric, (5.3) gives $D_K \Phi_A(\xi) K^* = 0$. Furthermore, since

$$-A + \xi(I_{\mathfrak{H}} - A^2)(I_{\mathfrak{H}} - \xi A)^{-1} = \bar{\xi} + (\xi - \bar{\xi})(I_{\mathfrak{H}} - \xi A)^{-1},$$

the equality $D_K \Phi_A(\xi) K^* = 0$ with $\xi \neq \bar{\xi}$ implies that $D_K (I_{\mathfrak{H}} - \xi A)^{-1} K^* = 0$, i.e.

$$(I_{\mathfrak{H}} - \xi A)^{-1} \operatorname{ran} K^* \subset \operatorname{ran} K^*.$$

Thus $A(\operatorname{ran} K^*) \subset \operatorname{ran} K^*$ (since $A = A^*$) and hence $D_A(\operatorname{ran} K^*) \subset \operatorname{ran} K^*$, so that

$$\overline{\operatorname{span}} \{A^n D_A K^* \mathfrak{N} :\, n \in \mathbb{N}_0\} \subset \operatorname{ran} K^*. \tag{5.6}$$

Consequently $\operatorname{ran} K^* = \mathfrak{H}$, i.e., $K \in \mathbf{L}(\mathfrak{H}, \mathfrak{N})$ is isometric. Therefore, $D_K \Phi_A(\zeta) K^* = 0$ for all $\zeta \in \mathbb{T}$ and $\Theta(\lambda)$ is inner in view of (5.3). Similarly, if (5.2) holds and $\Theta(\xi)$ is co-isometric for some $\xi \in \mathbb{T}$, $\xi \neq \pm 1$, then $\Theta(\lambda)$ is co-inner. □

Theorem 5.4. *Let the Hilbert space $\mathfrak{N}$ be finite-dimensional and let $\Theta(\lambda)$ be a non-constant inner function from $\mathbf{S}^{qs}(\mathfrak{N})$. Then $\Theta(\lambda)$ is rational and*

$$\Theta(\lambda) = \operatorname{diag}\left(\frac{\lambda - a_1}{1 - \lambda\, a_1}, \frac{\lambda - a_2}{1 - \lambda\, a_2}, \ldots, \frac{\lambda - a_m}{1 - \lambda\, a_m}, X\right)$$

relative to some orthonormal basis in $\mathfrak{N}$. Here the not necessarily distinct numbers $a_1, a_2, \ldots, a_m$ belong to $(-1, 1)$, and X is a constant unitary matrix.

Proof. Since $\Theta(\lambda) \in \mathbf{S}^{qs}(\mathfrak{N})$, it is the transfer function of a minimal *pqs*-system $\tau = \{T, \mathfrak{N}, \mathfrak{N}, \mathfrak{H}\}$. As T is quasi-selfadjoint, it has the form (4.1), and $\Theta(\lambda)$ is given by (4.7). Since $\mathfrak{N}$ is finite-dimensional and $\Theta(\lambda)$ is inner, $\Theta(\lambda)$ is automatically bi-inner. Then in (3.9) one has $\varphi_\Theta(\lambda) = 0$ and $\psi_\Theta(\lambda) = 0$. Thus by [5, Theorem 1.1] τ is conservative (in fact, this conclusion can be derived also from the proof of Theorem 5.3 above by applying Corollary 2.6). As $\Theta(\lambda)$ is nonconstant, A is non-isometric, K is isometric, and X appearing in (4.1) is unitary in $\mathfrak{D}_{K^*}$. Since τ is minimal, Proposition 4.2 shows that $\mathfrak{D}_A = \mathfrak{H} = \mathfrak{H}^s$. Hence $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$ isometric implies that $\dim \mathfrak{H} \leq \dim \mathfrak{N} < \infty$, so that $\Theta(\lambda)$ is rational. Let $\mathfrak{N}_0 = \operatorname{ran} K$, $\mathfrak{N}_1 = \mathfrak{N} \ominus \mathfrak{N}_0 = \ker K^*$. Then $K^* \mathfrak{N}_0 = \mathfrak{H}$ and

$$\Theta(\lambda)\!\upharpoonright\!\mathfrak{N}_1 = X : \mathfrak{N}_1 \to \mathfrak{N}_1,\ \Theta(\lambda)\!\upharpoonright\!\mathfrak{N}_0 = K\Phi_A(\lambda)K^*\!\upharpoonright\!\mathfrak{N}_0 : \mathfrak{N}_0 \to \mathfrak{N}_0.$$

Suppose that $\dim \mathfrak{H} = m$ and that $a_1, \dots, a_m$ are the eigenvalues of A. Choose an orthonormal basis in $\mathfrak{H}$ consisting of eigenvectors of A. Then K maps this basis onto some orthonormal basis in $\mathfrak{N}_0$. With respect to this basis the matrix $\Theta(\lambda)\upharpoonright\mathfrak{N}_0$ is diagonal with entries $\theta_{kk}(\lambda)$, $k = 1, \dots, m$, and since $\Phi_A(\lambda)$ has the form (2.6), it follows that

$$\Theta_{kk}(\lambda) = -a_k + \frac{(1-a_k^2)\lambda}{1-\lambda\, a_k} = \frac{\lambda - a_k}{1-\lambda\, a_k}.$$

This completes the proof. □

5.2. **Bi-inner dilations of functions from the class $\mathbf{S}^{qs}$.** The function $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ is said to have an *inner dilation* if there exists a function $\Theta_r(\lambda)$ such that

$$\boldsymbol{\Theta}(\lambda) = \begin{pmatrix} \Theta(\lambda) \\ \Theta_r(\lambda) \end{pmatrix} \in \mathbf{S}(\mathfrak{M}, \mathfrak{N} \oplus \mathfrak{L})$$

is inner. The function $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ is said to have a *co-inner dilation* if there exists a function $\Theta_l(\lambda)$ such that

$$\boldsymbol{\Theta}(\lambda) = \begin{pmatrix} \Theta(\lambda) & \Theta_l(\lambda) \end{pmatrix} \in \mathbf{S}(\mathfrak{M} \oplus \mathfrak{K}, \mathfrak{N})$$

is co-inner. The function $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ is said to have a *bi-inner dilation* if there exist functions $\Theta_{11}(\lambda)$, $\Theta_{22}(\lambda)$, and $\Theta_{21}(\lambda)$ such that

$$\boldsymbol{\Theta}(\lambda) = \begin{pmatrix} \Theta(\lambda) & \Theta_{12}(\lambda) \\ \Theta_{21}(\lambda) & \Theta_{22}(\lambda) \end{pmatrix} \in \mathbf{S}(\mathfrak{M} \oplus \mathfrak{K}, \mathfrak{N} \oplus \mathfrak{L}) \tag{5.7}$$

is bi-inner. Recall the following result due to Arov [11]; cf. [16].

Proposition 5.5. *Let $\tau = \{T; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}\}$ be a passive system with transfer function $\Theta(\lambda)$. Then:*

(i) *if τ is strongly stable, then $\Theta(\lambda)$ has an inner dilation;*
(ii) *if τ is strongly co-stable, then $\Theta(\lambda)$ has a co-inner dilation;*
(iii) *if τ is strongly stable and strongly co-stable, then $\Theta(\lambda)$ has a bi-inner dilation.*

In [5] this result was proved using the parametrization in Theorem 2.5 and (2.22). Since a function $\Theta(\lambda) \in \mathbf{S}^{qs}(\mathfrak{N})$ can be realized as the transfer function of a minimal *pqs*-system, it is strongly stable and strongly co-stable by Corollary 3.3. Hence it admits a bi-inner dilation by Proposition 5.5.

Proposition 5.6. *Among the bi-inner dilations of $\Theta(\lambda) \in \mathbf{S}^{qs}(\mathfrak{N})$ there exists a bi-inner dilation from the class $\mathbf{S}^{qs}(\mathfrak{N})$.*

Proof. Since $\Theta(\lambda) \in \mathbf{S}^{qs}(\mathfrak{N})$, it is the transfer function of a minimal *pqs*-system $\tau = \{T, \mathfrak{N}, \mathfrak{N}, \mathfrak{H}\}$ by Theorem 5.1. The operator T, being quasi-selfadjoint, has the form (4.1) and therefore $\Theta(\lambda)$ is given by (4.7). Define the following functions

$$\Theta_{21}(\lambda) := \begin{pmatrix} D_K \Phi_A(\lambda) K^* - K^* X D_{K^*} \\ -D_X D_{K^*} \end{pmatrix} : \mathfrak{N} \to \begin{pmatrix} \mathfrak{D}_K \\ \mathfrak{D}_{K^*} \end{pmatrix}, \quad \lambda \in \mathbb{D},$$

$$\Theta_{12}(\lambda) := \begin{pmatrix} K \Phi_A(\lambda) D_K - D_{K^*} X K & D_{K^*} D_{X^*} \end{pmatrix} : \begin{pmatrix} \mathfrak{D}_K \\ \mathfrak{D}_{K^*} \end{pmatrix} \to \mathfrak{N}, \quad \lambda \in \mathbb{D},$$

$$\Theta_{22}(\lambda) = \begin{pmatrix} K^* X K + D_K \Phi_A(\lambda) D_K & -K^* D_{X^*} \\ D_X K & X^* \end{pmatrix} : \begin{pmatrix} \mathfrak{D}_K \\ \mathfrak{D}_{K^*} \end{pmatrix} \to \begin{pmatrix} \mathfrak{D}_K \\ \mathfrak{D}_{K^*} \end{pmatrix}, \quad \lambda \in \mathbb{D}.$$

Let $\mathfrak{K} = \mathfrak{L} = \mathfrak{D}_K \oplus \mathfrak{D}_{K^*}$ and $\mathfrak{V} = \mathfrak{N} \oplus \mathfrak{L}$, and let $\mathbf{\Theta}(\lambda)$ be defined by (5.7). Furthermore, define the operator $\mathbf{T}$ by

$$\mathbf{T} = \begin{pmatrix} \mathbf{D} & \mathbf{C} \\ \mathbf{B} & \mathbf{A} \end{pmatrix} : \begin{pmatrix} \mathfrak{V} \\ \mathfrak{H} \end{pmatrix} \to \begin{pmatrix} \mathfrak{V} \\ \mathfrak{H} \end{pmatrix},$$

where $\mathbf{A} = A$, $\mathbf{D} = \mathbf{\Theta}(0)$, and

$$\mathbf{B} = D_A \begin{pmatrix} K^* & D_K & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{N} \\ \mathfrak{D}_K \\ \mathfrak{D}_{K^*} \end{pmatrix} \to \mathfrak{H}, \ \mathbf{C} = \mathbf{B}^* = \begin{pmatrix} K \\ D_K \\ 0 \end{pmatrix} D_A : \mathfrak{H} \to \begin{pmatrix} \mathfrak{N} \\ \mathfrak{D}_K \\ \mathfrak{D}_{K^*} \end{pmatrix}.$$

Simple calculations show that the operator $\mathbf{T}$ is unitary and quasi-selfadjoint. Hence, the system $\eta = \{\mathbf{T}; \mathfrak{V}, \mathfrak{V}, \mathfrak{H}\}$ is conservative. Since $A \in C_{00}$, the system η is minimal; see Corollary 2.4 and e.g. [5, Proposition 5.2]. In addition it is easy to see that the transfer function of η coincides with $\mathbf{\Theta}(\lambda)$ and therefore $\mathbf{\Theta}(\lambda) \in \mathbf{S}^{qs}(\mathfrak{V})$ by Proposition 4.5. Since η is minimal and conservative, $\mathbf{\Theta}(\lambda) \in \mathbf{S}(\mathfrak{V})$ is a bi-inner dilation of $\Theta(\lambda) \in \mathbf{S}(\mathfrak{N})$ in view of [5, Corollary 5.3]. □

Remark 5.7. With straightforward calculations it is easy to see directly that $\mathbf{\Theta}(\lambda) \in \mathbf{S}(\mathfrak{V})$ as defined explicitly in Proposition 5.6 is bi-inner; cf. [5, Proposition 7.1]. Furthermore, using the explicit formula of $\mathbf{\Theta}(\lambda)$ one can also calculate the function $\mathbf{W}(\lambda) := \mathbf{\Theta}(\lambda) - \mathbf{\Theta}(0)$ as introduced in Definition 4.4. Observe, that

$$\mathbf{W}(\lambda) : \begin{pmatrix} \mathfrak{N} \\ \mathfrak{D}_K \\ \mathfrak{D}_{K^*} \end{pmatrix} \to \begin{pmatrix} \mathfrak{N} \\ \mathfrak{D}_K \\ \mathfrak{D}_{K^*} \end{pmatrix}$$

and that

$$\mathbf{\Theta}(0) = \begin{pmatrix} -KAK^* + D_{K^*} X D_{K^*} & -KAD_K - D_{K^*} XK & D_{K^*} D_{X^*} \\ -D_K AK^* - K^* X D_{K^*} & K^* XK - D_K AD_K & -K^* D_{X^*} \\ -D_X D_{K^*} & D_X K & X^*. \end{pmatrix}.$$

Thus,

$$\mathbf{W}(\lambda) = \begin{pmatrix} \lambda K D_A^2 (I_{\mathfrak{H}} - \lambda A)^{-1} K^* & \lambda K D_A^2 (I_{\mathfrak{H}} - \lambda A)^{-1} D_K & 0 \\ \lambda D_K D_A^2 (I_{\mathfrak{H}} - \lambda A)^{-1} K^* & \lambda D_K D_A^2 (I_{\mathfrak{H}} - \lambda A)^{-1} D_K & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which implies that the Nevanlinna kernel

$$\frac{\mathbf{W}(\lambda) - \mathbf{W}^*(\lambda)}{\lambda - \bar{\lambda}}$$

is given by

$$\begin{pmatrix} KD_A^2 (I_{\mathfrak{H}} - \bar{\lambda} A)^{-1} (I_{\mathfrak{H}} - \lambda A)^{-1} K^* & KD_A^2 (I_{\mathfrak{H}} - \bar{\lambda} A)^{-1} (I_{\mathfrak{H}} - \lambda A)^{-1} D_K & 0 \\ D_K D_A^2 (I_{\mathfrak{H}} - \bar{\lambda} A)^{-1} (I_{\mathfrak{H}} - \lambda A)^{-1} K^* & D_K D_A^2 (I_{\mathfrak{H}} - \bar{\lambda} A)^{-1} (I_{\mathfrak{H}} - \lambda A)^{-1} D_K & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $\mathbf{W}(\lambda)$ is a Herglotz-Nevanlinna function defined on $\operatorname{Ext}\{(-\infty, -1] \cup [1, \infty)\}$, which reflects the defining properties of the class $\mathbf{S}^{qs}(\mathfrak{N})$ in Definition 4.4.

6. Minimal systems with transfer functions of the class $\mathbf{S}^{qs}$

In this section the class of minimal systems $\dot{\tau} = \{T, \mathfrak{N}, \mathfrak{N}, \mathfrak{H}\}$ of the form (1.2) are considered. The system $\dot{\tau}$ is not assumed to be passive, so that the bounded operator T in (1.1) need not be contractive. The next result can be seen as an extension of the unitary similarity result for *pqs*-systems in Proposition 4.3.

Theorem 6.1. *Let* $\Theta(\lambda) \in \mathbf{S}^{qs}(\mathfrak{N})$ *and let*

$$\dot{\tau} = \left\{ \begin{pmatrix} \dot{D} & \dot{C} \\ \dot{B} & \dot{A} \end{pmatrix}; \mathfrak{N}, \mathfrak{N}, \mathfrak{H} \right\}. \tag{6.1}$$

be a minimal, not necessarily passive, system whose transfer function coincides with $\Theta(\lambda)$ *in some neighborhood of zero. Then there exists a positive selfadjoint operator* S *in* $\mathfrak{H}$ *such that*

$$\dot{A}^* S = S\dot{A}, \quad \dot{C}^* = S\dot{B},$$

and such that

$$\widehat{\tau}_0 = \left\{ \begin{pmatrix} D & \dot{C}S^{-1/2} \\ S^{1/2}\dot{B} & S^{1/2}\dot{A}S^{-1/2} \end{pmatrix}; \mathfrak{N}, \mathfrak{N}, \mathfrak{H} \right\}$$

is a minimal pqs-system with transfer function $\Theta(\lambda)$. *Moreover,* $\dot{\tau}$ *becomes a pqs-system with respect to the inner product in* $\operatorname{dom} S^{1/2} \subset \mathfrak{H}$ *given by*

$$(\varphi, \psi)_{S^{1/2}} = (S^{1/2}\varphi, S^{1/2}\psi)_{\mathfrak{H}}, \quad \varphi, \psi \in \operatorname{dom} S^{1/2}.$$

Furthermore, if $\Theta(\lambda)$ *is inner (co-inner), then all minimal passive realizations of* $\Theta(\lambda)$ *are unitarily similar isometric (co-isometric) pqs-systems.*

Proof. Since $\Theta(\lambda) \in \mathbf{S}^{qs}(\mathfrak{N})$, it is the transfer function of a minimal *pqs*-system τ_0 of the form

$$\tau_0 = \left\{ \begin{pmatrix} D & C \\ B & A \end{pmatrix}; \mathfrak{N}, \mathfrak{N}, \overset{\circ}{\mathfrak{H}} \right\}. \tag{6.2}$$

Here A is a selfadjoint contraction in the Hilbert space $\overset{\circ}{\mathfrak{H}}$ and $C = B^*$. Furthermore, $\Theta(\lambda)$ has the form

$$\Theta(\lambda) = W(\lambda) + D,$$

where $W(\lambda) = \lambda B^*(I - \lambda A)^{-1}B$, $\lambda \in \mathbb{D}$, is a Herglotz-Nevanlinna function on the domain $\operatorname{Ext}\{(-\infty, -1] \cup [1, \infty)\}$ and $W(0) = 0$. Since the transfer function of $\dot{\tau}$ in (6.1) is of the form

$$\Theta(\lambda) = D + \lambda\dot{C}(I - \lambda\dot{A})^{-1}\dot{B},$$

one concludes that $\dot{D} = D$ and in some neighborhood of zero

$$\dot{C}(I - \lambda\dot{A})^{-1}\dot{B} = B^*(I - \lambda A)^{-1}B. \tag{6.3}$$

Consequently,

$$\dot{C}\dot{A}^k\dot{B} = B^*A^kB, \quad k \in \mathbb{N}_0. \tag{6.4}$$

Define the linear operator Y by

$$Y = \left\{ \left\{ \sum_{k=0}^{n} \dot{A}^k\dot{B}u_k, \sum_{k=0}^{n} A^kBu_k \right\} : u_0, u_1, ..., u_n \subset \mathfrak{N}, \quad n \in \mathbb{N}_0 \right\}. \tag{6.5}$$

It follows from (3.13) that

$$\left\{\sum_{k=0}^{m} A^k B v_k, \sum_{k=0}^{m} \dot{A}^{*k}\dot{C}^* v_k\right\} \in Y^*, \quad v_1,\dots,v_m \in \mathfrak{N}, \quad m \in \mathbb{N}_0. \tag{6.6}$$

By minimality of $\dot{\tau}$ and τ_0, Y and its adjoint Y^* have a dense domain and dense range. In particular, the operator Y is closable and the closure $\overline{Y} = Y^{**}$ is a densely defined operator with dense range. Definition 6.5 and the relations (6.4), (6.6) yield the equalities

$$\begin{cases} A\overline{Y}u = \overline{Y}\dot{A}u, \ u \in \operatorname{dom}\overline{Y}, \\ \dot{C}u = B^*\overline{Y}u, \ u \in \operatorname{dom}\overline{Y}, \\ Y\dot{B} = B, \end{cases} \tag{6.7}$$

which means that the systems $\dot{\tau}$ and τ_0 are weakly similar; see (1.6), cf. also [10]. Now define a positive selfadjoint operator S by $S = Y^*\overline{Y}$. Since $Y^*B = \dot{C}^*$ and $Y\dot{B} = B$, one obtains $S\dot{B} = \dot{C}^*$. Represent $\overline{Y}$ in the form $\overline{Y} = US^{1/2}$, where $U : \mathfrak{H} \to \overset{0}{\mathfrak{H}}$ is an isometry and $\operatorname{dom} S^{1/2} = \operatorname{dom}\overline{Y}$. Because $\overline{\operatorname{ran}}\, Y = \overset{0}{\mathfrak{H}}$, the operator U is a unitary from $\mathfrak{H}$ onto $\overset{0}{\mathfrak{H}}$. Define a selfadjoint contraction $\widehat{A}$ in $\mathfrak{H}$ by the equality:

$$\widehat{A} = U^{-1}AU.$$

Then the equality $\overline{Y}\dot{A} = A\overline{Y}$ gives

$$S^{1/2}\dot{A}v = \widehat{A}S^{1/2}v, \ v \in \operatorname{dom} S^{1/2},$$

or, equivalently,

$$\widehat{A}v = S^{1/2}\dot{A}S^{-1/2}v, \ v \in \operatorname{dom} S^{-1/2}.$$

Let $\widehat{B} = U^{-1}B$. From $Y\dot{B} = B$ one obtains $S^{1/2}\dot{B} = \widehat{B}$ and the relation $S\dot{B} = \dot{C}^*$ yields $S^{-1/2}\dot{C}^* = \widehat{B}$. Therefore $\widehat{B}^*v = \dot{C}S^{-1/2}v$, $v \in \operatorname{dom} S^{-1/2}$. The system

$$\widehat{\tau}_0 = \left\{\begin{pmatrix} D & \widehat{B}^* \\ \widehat{B} & \widehat{A}\end{pmatrix}; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}\right\}$$

is unitarily equivalent to τ_0. Hence $\widehat{\tau}_0$ is a minimal pqs-system with the transfer function $\Theta(\lambda)$. Let $v \in \operatorname{dom} S$, $u \in \operatorname{dom} S^{1/2}$. Then

$$\begin{aligned} \left(S^{1/2}\dot{A}v, S^{1/2}u\right) &= \left(\widehat{A}S^{1/2}v, S^{1/2}u\right) = \left(S^{1/2}v, \widehat{A}S^{1/2}u\right) \\ &= \left(S^{1/2}v, S^{1/2}\dot{A}u\right) = \left(\dot{A}^*Sv, u\right). \end{aligned} \tag{6.8}$$

It follows that the vector $S^{1/2}\dot{A}v$ belongs to $\operatorname{dom} S^{1/2}$ and

$$S\dot{A}v = \dot{A}^*Sv, \ v \in \operatorname{dom} S.$$

Consider $\operatorname{dom} S^{1/2}$ as a pre-Hilbert space equipped with the inner product $(f,g)_{S^{1/2}} := (S^{1/2}f, S^{1/2}g)$ and let $\mathfrak{H}_{S^{1/2}}$ be the completion of $\operatorname{dom} S^{1/2}$ with respect to this inner product. Since $S^{1/2}\dot{A}v = \widehat{A}S^{1/2}v$, $v \in \operatorname{dom} S^{1/2}$, one obtains from (6.8)

$$(\dot{A}u, v)_{S^{1/2}} = (S^{1/2}\dot{A}v, S^{1/2}u) = (S^{1/2}v, S^{1/2}\dot{A}u) = (v, \dot{A}u)_{S^{1/2}},$$

for all $v, u \in \operatorname{dom} S^{1/2}$. Thus, the operator $\dot{A}$ is symmetric with respect to the inner product $(\cdot,\cdot)_{S^{1/2}}$. Furthermore, the operator $\dot{T}$ defined via the block formula

$$\dot{T} = \begin{pmatrix} D & \dot{C} \\ \dot{B} & \dot{A} \end{pmatrix} : \begin{pmatrix} \mathfrak{N} \\ \mathfrak{H}_{S^{1/2}} \end{pmatrix} \to \begin{pmatrix} \mathfrak{N} \\ \mathfrak{H}_{S^{1/2}} \end{pmatrix}$$

is a *qsc*-operator. In fact, since $\dot{C}^* = S^{1/2}\widehat{B}$ and $S^{1/2}\dot{B} = \widehat{B}$, the operator $\dot{B} : \mathfrak{N} \to \mathfrak{H}_{S^{1/2}}$ is the adjoint of the operator $\dot{C} : \operatorname{dom} S^{1/2} \subset \mathfrak{H}_{S^{1/2}} \to \mathfrak{N}$. Because the operator matrix

$$\widehat{T} = \begin{pmatrix} D & \widehat{B}^* \\ \widehat{B} & \widehat{A} \end{pmatrix} : \begin{pmatrix} \mathfrak{N} \\ \mathfrak{H} \end{pmatrix} \to \begin{pmatrix} \mathfrak{N} \\ \mathfrak{H} \end{pmatrix}$$

is a contraction, one obtains with $f \in \operatorname{dom} S^{1/2}$ and $u \in \mathfrak{N}$

$$\begin{aligned} &\left\|\dot{A}f + \dot{B}u\right\|^2_{S^{1/2}} + \left\|\dot{C}f + Du\right\|^2 \\ &= \left\|S^{1/2}(\dot{A}f + \dot{B}u)\right\|^2 + \left\|\dot{C}f + Du\right\|^2 \\ &= \left\|\widehat{A}S^{1/2}f + \widehat{B}u\right\|^2 + \left\|\widehat{B}^*S^{1/2}f + Du\right\|^2 \\ &\le ||S^{1/2}f||^2 + ||u||^2 = \|f\|^2_{S^{1/2}} + ||u||^2. \end{aligned}$$

Therefore, after a renormalization of the state space by means of the operator S the system $\dot{\tau}$ becomes a *pqs*-system.

Finally, if the function $\Theta(\lambda) \in \mathbf{S}^{qs}(\mathfrak{N})$ is inner (co-inner) then $\varphi_\Theta(\xi) = 0$ ($\psi_\Theta(\xi) = 0$) for almost all $\xi \in \mathbb{T}$, see (3.9), and now it follows from [5, Theorem 1.1] that every two minimal passive realizations of the inner (co-inner) function $\Theta(\lambda) \in \mathbf{S}^{qs}(\mathfrak{N})$ are unitarily similar isometric (co-isometric) *pqs*-systems. □

7. The class $\mathbf{S}^{qs}$ and Q-functions of quasi-selfadjoint contractions

7.1. **Hermitian contractions.** Let A_0 be a Hermitian contraction in the Hilbert space $\mathcal{H}$ with $\operatorname{dom} A_0 = \mathfrak{H}$ and let $\mathfrak{N} = \mathcal{H} \ominus \mathfrak{H}$. Then $A := P_{\mathfrak{H}} A_0 \restriction \mathfrak{H}$ is selfadjoint in the Hilbert space $\mathfrak{H}$. It follows that there is a contraction $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$ such that

$$A_0 = A + KD_A. \tag{7.1}$$

The operator A_0 is said to be *simple* if there is no nonzero subspace in $\operatorname{dom} A_0$ which is invariant under A_0; cf. [24]. Since A_0 is Hermitian, simplicity of A_0 is equivalent to A_0 being completely nonselfadjoint, i.e., A_0 has no selfadjoint part. Note that A_0 is simple if and only if the subspace $\mathfrak{H}_0^s$ in (4.5) coincides with $\mathfrak{H}$; cf. [4, Lemma 3.2]. For a Hermitian contraction $A_0 = A + KD_A$ the restriction $A_0 \restriction \mathfrak{H}_0^s$ is called the *simple part of* A_0.

A *qsc*-operator T is said to be a *quasi-selfadjoint contractive extension* or *qsc-extension* of a Hermitian contraction A_0 if

$$A_0 \subset T \text{ and } A_0 \subset T^*,$$

or, equivalently, if $\operatorname{dom} A_0 \subset \ker\,(T - T^*)$, cf. [6], [8]. A *qsc*-operator T has always Hermitian restrictions A_0 for which T is a *qsc*-extension. Namely, with a subspace

$\mathfrak{N} \supset \operatorname{ran}(T - T^*)$ define

$$\operatorname{dom} A_0 = \mathfrak{H} := \mathcal{H} \ominus \mathfrak{N}, \quad A_0 = T \upharpoonright \operatorname{dom} A_0.$$

Then $\operatorname{dom} A_0 \subset \ker(T - T^*)$.

With A_0 and $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$ as in (7.1), the formula (4.1) provides a one-to one correspondence between all contractions $X \in \mathbf{L}(D_{K^*})$ and all *qsc*-extensions of A. The operator form of all *qsc*-extensions with their resolvents was obtained in [7], [8]. Clearly, the subspaces $\mathfrak{H}'$ and $\mathfrak{H}'' = \mathfrak{H} \ominus \mathfrak{H}'$, where

$$\mathfrak{H}' := \overline{\operatorname{span}} \left\{ (T - zI)^{-1}\mathfrak{N} : \, |z| > 1 \right\} = \overline{\operatorname{span}} \left\{ T^n \mathfrak{N} : \, n \in \mathbb{N}_0 \right\}, \tag{7.2}$$

are invariant with respect to T and T^*, respectively. The inclusion $\mathfrak{N} \subset \mathfrak{H}'$ implies that $\mathfrak{H}'' \subset \mathfrak{N}^{\perp} = \operatorname{dom} A \subset \ker(T - T^*)$. Therefore the restriction of T^* to $\mathfrak{H}''$ is a selfadjoint operator in $\mathfrak{H}''$. The restriction $T \upharpoonright \mathfrak{H}' \, (= P_{\mathfrak{H}'} T \upharpoonright \mathfrak{H}')$ is called the $\mathfrak{N}$-*minimal part of* T. Moreover, T is said to be $\mathfrak{N}$-*minimal* if the equality $\mathfrak{H} = \mathfrak{H}'$ holds [4]. The subspaces $\mathfrak{H}'$ and $\mathfrak{H}^s$ of $\mathcal{H} = \mathfrak{N} \oplus \mathfrak{H}_0$ defined in (7.2) and (4.5), respectively, are connected by $\mathfrak{H}' = \mathfrak{N} \oplus \mathfrak{H}^s$. Every *qsc*-extension T of the Hermitian contraction A_0 in $\mathcal{H}$ with $\operatorname{dom} A_0 = \mathfrak{H}$ generates a *pqs*-system in the following manner: let (1.7) be the block-operator representation of T, then the system $\tau = \{T; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}\}$ is a *pqs*-system.

Proposition 7.1. *Let A_0 be a Hermitian contraction in $\mathcal{H} = \mathfrak{N} \oplus \mathfrak{H}$ with $\mathfrak{H} = \operatorname{dom} A_0$, let T be a qsc-extension of A in $\mathcal{H}$, and let τ be the pqs-system generated by T with the state space $\mathfrak{H}$ and the input and the output space $\mathfrak{N}$. Then the following statements are equivalent:*

(i) *the Hermitian contraction A_0 is simple;*
(ii) *the qsc-extension T of A_0 is $\mathfrak{N}$-minimal;*
(iii) *the pqs-system τ is minimal.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) was proved in [4]. For the proof of (i) $\Leftrightarrow$ (iii), observe that the simplicity of A_0 is equivalent to $\mathfrak{H}_0^s = \mathfrak{H}$. Since $\operatorname{ran} K^* \subset \mathfrak{D}_{A_0}$ and $\mathfrak{D}_{A_0}$ is invariant under A_0, the inclusion $\mathfrak{H}_0^s \subset \mathfrak{D}_{A_0}$ holds. Hence, $\mathfrak{H}_0^s = \mathfrak{H}$ implies in fact that $\ker D_{A_0} = \{0\}$, i.e., $\|Af\| < \|f\|$ holds for all $f \in \mathfrak{H} \setminus \{0\}$. Now the assertion follows from Proposition 4.2. □

7.2. **Transfer functions and Q-functions.** Let T be a *qsc*-operator in a separable Hilbert space $\mathcal{H}$ and let $\mathfrak{N}$ be a subspace of $\mathfrak{H}$ such that $\mathfrak{N} \supset \operatorname{ran}(T - T^*)$. The operator-valued function

$$Q_T(z) = P_{\mathfrak{N}}(T - zI)^{-1} \upharpoonright \mathfrak{N}, \quad |z| > 1, \tag{7.3}$$

is said to be a *Q-function* of T, cf. [4]. Analytical properties of the Q-function of *qsc*-operators and its applications to the parametrization of the resolvents of all *qsc*-extenions of corresponding Hermitian contraction were established in [4].

Let $\mathfrak{N}$ be a Hilbert space. An operator-valued function $Q(z)$ with values in $\mathbf{L}(\mathfrak{N})$ and holomorphic outside $\mathbb{D}$ is said to belong to the class $\mathbf{Q}(\mathfrak{N})$ (see [4, Section 6]) if

(S1) $Q(z)$ has the asymptotic expansion

$$Q(z) = -\frac{1}{z} I + \frac{1}{z^2} F + o\left(\frac{1}{z^2}\right), \; z \to \infty; \tag{7.4}$$

(S2) the kernel

$$\frac{Q(z) - Q^*(\xi) - Q^*(\xi)(F - F^*)Q(z)}{z - \bar{\xi}}$$

is nonnegative;

(S3) the kernel

$$\frac{(1-z^2)Q(z) - (1-\bar{\xi}^2)Q^*(\xi) - (1-z\bar{\xi})Q^*(\xi)(F-F^*)Q(z) - (z-\bar{\xi})I}{z-\bar{\xi}}$$

is nonnegative;

(S4) there exist a complex number z_0, $|z_0| > 1$, and a vector $f \in \mathfrak{N}$, such that

$$\frac{Q(z_0) - Q^*(z_0) - Q^*(z_0)(F - F^*)Q(z_0)}{z_0 - \bar{z}_0} \neq Q^*(z_0)Q(z_0)f.$$

If T is a *qsc*-operator in the Hilbert space $\mathfrak{H}$, $\mathfrak{N}$ is a subspace of $\mathfrak{H}$ such that $\operatorname{ran}(T - T^*) \subset \mathfrak{N}$, and $Q_T(z)$ is its Q-function defined by (7.3), then the function $Q_T(z)$ belongs to the class $\mathbf{Q}(\mathfrak{N})$, see [4]. The converse statement is also true.

Theorem 7.2 ([4])**.** *Let $Q(z)$ belong to $\mathbf{Q}(\mathfrak{N})$. Then there exist Hilbert spaces $\mathcal{H} \supset \mathfrak{N}$, $\mathfrak{N} \neq \mathcal{H}$, and a $\mathfrak{N}$-minimal qsc-operator T in $\mathfrak{H}$, such that $\mathfrak{N} \supset \ker\,(T - T^*)$, and*

$$Q(z) = P_{\mathfrak{N}}(T - zI)^{-1}{\upharpoonright}\,\mathfrak{N}, \quad |z| > 1.$$

The next proposition gives connections between the transfer function of a *pqs*-system τ and the Q-function of the corresponding *qsc*-operator T.

Proposition 7.3. *Let $\tau = \{T; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}\}$ be a pqs-system. Then the transfer function $\Theta(\lambda)$ of τ and the Q-function of the qsc-operator T are connected by the following relations*

$$Q(z) = \left(\Theta\left(\frac{1}{z}\right) - zI_{\mathfrak{N}}\right)^{-1}, \ |z| > 1; \quad \Theta(\lambda) = \frac{1}{\lambda}I + Q^{-1}\left(\frac{1}{\lambda}\right), \ \lambda \in \mathbb{D}. \tag{7.5}$$

Proof. With $W(z) = \Theta(1/z) - zI$, $|z| > 1$, the resolvent $(T - zI)^{-1}$ has the form

$$\begin{pmatrix} W^{-1}(z) & -W^{-1}(z)C(A - zI)^{-1} \\ -(A - zI)^{-1}BW^{-1}(z) & (A - zI)^{-1}\left(I + BW^{-1}(z)C(A - zI)^{-1}\right) \end{pmatrix}.$$

Here the Schur-Frobenius formula has been applied; cf. [4, Section 2.4]. It follows that

$$P_{\mathfrak{N}}(T - zI)^{-1}{\upharpoonright}\,\mathfrak{N} = W^{-1}(z).$$

Thus, the relations (7.5) hold. □

Observe that Theorem 5.1 and Theorem 7.2 imply that $\Theta(\lambda)$ belongs to $\mathbf{S}^{qs}(\mathfrak{N})$ if and only if

$$Q(z) = \left(\Theta\left(\frac{1}{z}\right) - zI_{\mathfrak{N}}\right)^{-1}, \quad |z| > 1,$$

belongs to $\mathbf{Q}(\mathfrak{N})$.

7.3. **Scalar functions of the class $\mathbf{S}^{qs}$ and Jacobi matrices.** Let $l_2(\mathbb{N})$ and $l_2(\mathbb{N}_0)$ be the Hilbert spaces of square summable complex-valued sequences

$$x = \{x_1, x_2, \dots, x_k, \dots\}, \quad x = \{x_0, x_1, \dots, x_k, \dots\},$$

considered as semi-infinite vector-columns, with the inner product given by

$$(x, y) = \sum_{k=1}^{\infty} x_k \bar{y}_k, \quad (x, y) = \sum_{k=0}^{\infty} x_k \bar{y}_k,$$

respectively. Clearly $\mathbb{C} \oplus l_2(\mathbb{N}) = l_2(\mathbb{N}_0)$. Define the vectors $\{\delta_k\}$, $k \in \mathbb{N}_0$, by

$$\delta_0 = (1, 0, 0, \dots)^T, \quad \delta_k = (0, \dots, 0, 1, 0, 0, \dots)^T, \quad k \in \mathbb{N},$$

so that 1 is the $(k+1)$-st entry. Then the vectors $\{\delta_k\}$ form an orthonormal basis in $l_2(\mathbb{N}_0)$.

Theorem 7.4. *Let the scalar function $\Theta(\lambda)$ belong to the class $\mathbf{S}^{qs}$.*

(i) *If $\Theta(\lambda)$ is rational with n poles then any minimal pqs-system $\tau = \{T, \mathbb{C}, \mathbb{C}, \mathfrak{H}\}$ with the transfer function $\theta(\lambda)$ is unitarily equivalent to the pqs-system*

$$\tau_0 = \{T_0, \mathbb{C}, \mathbb{C}, \mathbb{C}^n\},$$

where the operator T_0 in the Hilbert space $\mathbb{C} \oplus \mathbb{C}^n = \mathbb{C}^{n+1}$ with respect to the canonical basis $\{\delta_k\}_{k=0}^n$ is given by the three-diagonal Jacobi matrix

$$T_0 = \begin{pmatrix} \Theta(0) & a_0 & 0 & 0 & \cdot & \cdot & \cdot \\ a_0 & b_1 & a_1 & 0 & \cdot & \cdot & \cdot \\ 0 & a_1 & b_2 & a_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot & 0 & a_{n-1} & b_n \end{pmatrix}. \tag{7.6}$$

(ii) *If $\Theta(\lambda)$ is not rational then any minimal pqs-system $\tau = \{T, \mathbb{C}, \mathbb{C}, \mathfrak{H}\}$ with the transfer function $\Theta(\lambda)$ is unitarily equivalent to the pqs-system*

$$\tau_0 = \{T_0, \mathbb{C}, \mathbb{C}, l_2(\mathbb{N})\},$$

where the operator T_0 in the Hilbert space $\mathbb{C} \oplus l_2(\mathbb{N}) = l_2(\mathbb{N}_0)$ with respect to the canonical basis $\{\delta_k\}_{k=0}^{\infty}$ is given by the semi-infinite three-diagonal Jacobi matrix

$$T_0 = \begin{pmatrix} \Theta(0) & a_0 & 0 & 0 & 0 & \cdot & \cdot \\ a_0 & b_1 & a_1 & 0 & 0 & \cdot & \cdot \\ 0 & a_1 & b_2 & a_2 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \tag{7.7}$$

In both cases $a_k > 0$ and $b_k \in \mathbb{R}$ for all relevant k, and these numbers are uniquely determined by the function $\Theta(\lambda)$.

Proof. Let $\Theta(\lambda) \in \mathbf{S}^{qs}$. By Proposition 7.3 the function $Q(z) = (\Theta(1/z) - z)^{-1}$, $|z| > 1$, belongs to the class $\mathbf{Q}$. Without loss of generality it can be assumed that $\operatorname{Im} \Theta(0) \geq 0$. Since the function $w(\lambda) = \Theta(\lambda) - \Theta(0)$ belongs to the Herglotz-Nevanlinna class, the function $Q(z)$ has a holomorphic continuation to the lower half-plane. If $\Theta(\lambda)$ is rational with n poles $\{\mu_k\}$, then these poles are simple and belong to $(-\infty,-1)\cup(1,\infty)$. It follows that

$$\Theta\left(\frac{1}{z}\right)=a\,\frac{\prod\limits_{k=1}^{n}(z-\lambda_k)}{\prod\limits_{k=1}^{n}(z-\mu_k)},\quad \operatorname{Im} a\geq 0,$$

and $Q(z)$ is rational with poles of total multiplicity $n+1$, in the upper half-plane.

Let $\tau=\{T,\mathbb{C},\mathbb{C},\mathfrak{H}\}$ be a minimal *pqs*-system with the transfer function $\Theta(\lambda)$. Then $Q(z)$ is the Q-function of the corresponding *qsc*-operator T, i.e.,

$$Q(z)=\left((T-zI)^{-1}\bar{1},\bar{1}\right),\quad |z|>1,$$

where

$$\bar{1}=\begin{pmatrix}1\\ \mathbf{0}\end{pmatrix},\quad 1\in\mathbb{C},\quad \mathbf{0}\in\mathfrak{H}\quad\text{(null-vector)}.$$

The operator T in the Hilbert space $\mathcal{H}=\mathbb{C}\oplus\mathfrak{H}$ is dissipative ($\operatorname{Im} T(f,f)\geq 0$ for all $f\in\mathcal{H}$), $\dim\operatorname{ran}(T-T^*)\leq 1$, and by Proposition 7.1 the operator T is $\mathbb{C}$-minimal, i.e., $\overline{\operatorname{span}}\,\{T^n\bar{1}:\,n\in\mathbb{N}_0\}=\mathcal{H}$. If $\operatorname{Im}\Theta(0)=0$, then T is selfadjoint with the cyclic vector $\bar{1}$. If $\operatorname{Im}\Theta(0)\neq 0$, then T is a prime dissipative operator with a rank-one imaginary part and, moreover, $\operatorname{ran}(T-T^*)=\mathbb{C}\oplus\{0\}$.

Note that T is unitarily equivalent to T_0 given by (7.6) (in the case of a rational $Q(z)$) or by (7.7) (in the opposite case) with respect the orthonormal basis $\{\delta_k\}$, i.e., there exists a unitary operator $U\in\mathbf{L}(\mathcal{H},\mathbb{C}^{n+1})$ or $U\in\mathbf{L}(\mathcal{H},l_2(\mathbb{N}_0))$ such that $UT=T_0U$; cf. [9, Theorem 2.10, Theorem 6.1, Remark 6.2]. Moreover,

$$U\bar{1}=\delta_0. \tag{7.8}$$

Thus, for $|z|>1$ it follows that $Q(z)=((T_0-zI)^{-1}\delta_0,\delta_0)$. The entries of the matrix T_0 can be found by the continued-fraction expansion of the function $Q(z)$:

$$Q(z)=\frac{-1}{z-\Theta(0)}\,+\,\frac{-a_0^2}{z-b_1}\,+\,\frac{-a_1^2}{z-b_2}\,+\ldots+\,\frac{-a_{n-1}^2}{z-b_n}\,+\ldots,$$

Note that T_0 is selfadjoint with the cyclic vector δ_0 if $\operatorname{Im}\Theta(0)=0$, and that T_0 is a prime dissipative operator with a rank-one imaginary part and $\operatorname{ran}(T_0-T_0^*)=\operatorname{span}\{\delta_0\}$ if $\operatorname{Im}\Theta(0)\neq 0$. From (7.7) it follows that

$$T_0=\begin{pmatrix}D_0 & B_0^*\\ B_0 & A_0\end{pmatrix}:\begin{pmatrix}\mathbb{C}\\ \mathbb{C}^n\end{pmatrix}\to\begin{pmatrix}\mathbb{C}\\ \mathbb{C}^n\end{pmatrix}\ \text{ or }\ T_0=\begin{pmatrix}D_0 & B_0^*\\ B_0 & A_0\end{pmatrix}:\begin{pmatrix}\mathbb{C}\\ l_2(\mathbb{N})\end{pmatrix}\to\begin{pmatrix}\mathbb{C}\\ l_2(\mathbb{N})\end{pmatrix},$$

respectively, where $D_0=\Theta(0)$,

$$B_0 1=\begin{pmatrix}a_0\\0\\0\\ \vdots\end{pmatrix},\quad B_0^*\begin{pmatrix}x_1\\x_2\\x_3\\ \vdots\end{pmatrix}=a_0x_1\delta_0,$$

and

$$A_0\begin{pmatrix}x_1\\x_2\\ \vdots\end{pmatrix}=\begin{pmatrix}b_1 & a_1 & 0 & 0 & \cdot & \cdot\\ a_1 & b_2 & a_2 & 0 & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\end{pmatrix}\begin{pmatrix}x_1\\x_2\\ \vdots\end{pmatrix}.$$

Decompose T according to $\mathcal{H} = \mathbb{C} \oplus \mathfrak{H}$:

$$T = \begin{pmatrix} D & B^* \\ B & A \end{pmatrix} : \begin{pmatrix} \mathbb{C} \\ \mathfrak{H} \end{pmatrix} \to \begin{pmatrix} \mathbb{C} \\ \mathfrak{H} \end{pmatrix}.$$

Because (7.8) holds, the unitary operator U takes the following block operator matrix form

$$U = \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} \mathbb{C} \\ \mathfrak{H} \end{pmatrix} \to \begin{pmatrix} \mathbb{C} \\ \mathbb{C}^n \end{pmatrix} \quad \text{or} \quad U = \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} \mathbb{C} \\ \mathfrak{H} \end{pmatrix} \to \begin{pmatrix} \mathbb{C} \\ l_2(\mathbb{N}) \end{pmatrix},$$

respectively. Hence, $VA = A_0V$, $VB = B_0$, i.e., the *pqs*-systems τ and τ_0 are unitarily equivalent. □

Here is a simple example to illustrate the situation.

Example 7.5. Consider the scalar-valued function $\Theta(\lambda)$ defined by

$$\Theta(\lambda) = d + \frac{1 - \sqrt{1-\lambda^2}}{2\lambda}, \quad \lambda \in \operatorname{Ext}\{(-\infty, -1] \cup [1, +\infty)\}.$$

Then $\Theta(0) = d$ and $W(\lambda) = \Theta(\lambda) - \Theta(0)$ is given by

$$W(\lambda) = \frac{1 - \sqrt{1-\lambda^2}}{2\lambda}, \quad \lambda \in \operatorname{Ext}\{(-\infty, -1] \cup [1, +\infty)\}.$$

Clearly $W(\lambda)$ is a Herglotz-Nevanlinna function. It follows that

$$-2W\left(\frac{1}{z}\right) = \sqrt{z^2 - 1} - z = \frac{1}{\pi} \int\limits_{-1}^{1} \frac{\sqrt{1-t^2}\, dt}{t - z}, \quad z \in \operatorname{Ext}[-1, 1].$$

Hence $W(1) = -W(-1) = 1/2$. Moreover, $\Theta(\lambda)$ belongs to the class $\mathbf{S}^{qs}$ if and only if $|d| \le 1/2$; see (4.12). Assume that this condition is satisfied. Consider the weighted Hilbert space $L_2([-1, 1], \rho(t))$ with the weight function

$$\rho(t) = \frac{2}{\pi}\sqrt{1-t^2}, \quad t \in [-1, 1].$$

Define the operator A in $L_2([-1, 1], \rho(t))$ by

$$(Af)(t) = tf(t), \quad f(t) \in L_2([-1, 1], \rho(t)).$$

Then A is a selfadjoint contraction. The function $e_0(t) = 1$, $t \in [-1, 1]$ belongs to $L_2([-1, 1], \rho(t))$ and $\|e_0\| = 1$. Define the operator $B : \mathbb{C} \to L_2([-1, 1], \rho(t))$ by

$$Bc = \frac{1}{2}\, c\, e_0(t), \quad c \in \mathbb{C}.$$

Then

$$B^* f(t) = \frac{1}{\pi} \int\limits_{-1}^{1} f(t)\sqrt{1-t^2}\, dt, \quad f(t) \in L_2([-1, 1], \rho(t)).$$

Let D be the multiplication by d in the space $\mathbb{C}$. One can check that

$$T = \begin{pmatrix} D & B^* \\ B & A \end{pmatrix} : \begin{pmatrix} \mathbb{C} \\ L_2([-1, 1], \rho(t)) \end{pmatrix} \to \begin{pmatrix} \mathbb{C} \\ L_2([-1, 1], \rho(t)) \end{pmatrix}$$

is a *qsc*-operator. Moreover, the corresponding *pqs*-system

$$\tau = \{T; \mathbb{C}, \mathbb{C}, L_2\left([-1,1], \rho(t)\right)\}$$

is minimal and has the transfer function $\Theta(\lambda)$. Since the operator A is unitarily equivalent to the Jacobi matrix (see [18])

$$A_0 = \begin{pmatrix} 0 & 1/2 & 0 & 0 & \cdot & \cdot & \cdot \\ 1/2 & 0 & 1/2 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

the unitarily equivalent three-diagonal minimal *pqs*-system is of the form

$$T_0 = \begin{pmatrix} d & 1/2 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1/2 & 0 & 1/2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1/2 & 0 & 1/2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

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