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Asymptotic expansions of generalized Nevanlinna functions and their spectral properties

Vladimir Derkach, Seppo Hassi and Henk de Snoo

Abstract. Asymptotic expansions of generalized Nevanlinna functions Q are investigated by means of a factorization model involving a part of the generalized zeros and poles of nonpositive type of the function Q. The main results in this paper arise from the explicit construction of maximal Jordan chains in the root subspace $\mathsf{R}_\infty(S_F)$ of the so-called generalized Friedrichs extension. A classification of maximal Jordan chains is introduced and studied in analytical terms by establishing the connections to the appropriate asymptotic expansions. This approach results in various analytic characterizations of the spectral properties of selfadjoint relations in a Pontryagin space and, conversely, translates spectral theoretical properties into analytic properties of the associated Weyl functions.

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1. Introduction

Let $\mathbf{N}_\kappa$ be the class of generalized Nevanlinna functions, i.e. meromorphic functions on $\mathbb{C} \setminus \mathbb{R}$ with $Q(\bar{z}) = \overline{Q(z)}$ and such that the kernel

$$\mathsf{N}_Q(z,\lambda) = \frac{Q(z) - \overline{Q(\lambda)}}{z - \bar{\lambda}}, \quad z, \lambda \in \rho(Q), \quad z \neq \bar{\lambda},$$

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has κ negative squares on the domain of holomorphy $\rho(Q)$ of Q, see [20]. If the function $Q \in \mathbf{N}_\kappa$ belongs to the subclass $\mathbf{N}_{\kappa,-2n}$, $n \in \mathbb{N}$, (see [6]) then it admits the following asymptotic expansion

$$Q(z) = \gamma - \sum_{j=1}^{2n+1} \frac{s_{j-1}}{z^j} + o\left(\frac{1}{z^{2n+1}}\right), \quad z\widehat{\rightarrow}\infty, \tag{1.1}$$

where $\gamma, s_j \in \mathbb{R}$ and $z\widehat{\rightarrow}\infty$ means that z tends to ∞ nontangentially ($0 < \varepsilon < \arg z < \pi - \varepsilon < 0$). Asymptotic expansions for $Q \in \mathbf{N}_\kappa$ of the form (1.1) (with $\gamma = 0$) were introduced in [21]. They naturally appear, for instance, in the indefinite moment problem considered in [22]. The expansion (1.1) is equivalent to the following operator representation of the function $Q \in \mathbf{N}_{\kappa,-2n}$:

$$Q(z) = \gamma + [(A - z)^{-1}\omega, \omega], \tag{1.2}$$

where $\omega \in \operatorname{dom} A^n$ and A is a selfadjoint operator in a Pontryagin space $\mathfrak{H}$; see [21, Satz 1.10] and Corollary 3.4 below. The representation 1.2 can be taken to be minimal in the sense that ω is a cyclic vector for A, i.e.,

$$\mathfrak{H} = \overline{\operatorname{span}}\, \{\, (A - z)^{-1}\omega : \, z \in \rho(A)\, \},$$

in which case the negative index $\operatorname{sq}_-(\mathfrak{H})$ of $\mathfrak{H}$ is equal to κ. The representation (1.2) shows that ∞ is a generalized zero of the function $Q(z) - \gamma$, or equivalently, that ∞ is a generalized pole of the function $Q_\infty(z) = -1/(Q(z) - \gamma)$. This means that the underlying symmetric operator S is nondensely defined in $\mathfrak{H}$ with

$$\operatorname{dom} S = \{\, f \in \operatorname{dom} A : \, [f, \omega] = 0\, \} \tag{1.3}$$

and that

$$S_F = S \,\widehat{+}\, (\{0\} \times \operatorname{span}\{\omega\}) \tag{1.4}$$

is a selfadjoint extensions of S in $\mathfrak{H}$ with $\infty \in \sigma_p(S_F)$. Here $\widehat{+}$ stands for the componentwise sum in the Cartesian product $\mathfrak{H} \times \mathfrak{H}$. In other words, the extension S_F is multivalued and, in fact, can be interpreted as the generalized Friedrichs extension of S, see [5] and the references therein. It follows from (1.1) and (1.2) that

$$s_0 = [\omega, \omega] \in \mathbb{R}.$$

If $\kappa > 0$ then it is possible that $s_0 \leq 0$, in which case ∞ is a generalized pole of nonpositive type (GPNT) of the function Q_∞, cf. [23]. More precisely, if ∞ is a GPNT of Q_∞ with multiplicity $\kappa_\infty := \kappa_\infty(Q_\infty)$ (see (2.2) below for the definition), then in (1.1) one automatically has

$$s_0 = \cdots = s_j = 0, \quad \text{for every } j < 2\kappa_\infty - 2.$$

Furthermore, if m is the first nonnegative index in (1.1) such that $s_m \neq 0$ (if exists), then, equivalently, the function Q_∞ admits an asymptotic expansion of the form

$$Q_\infty(z) = p_{m+1}z^{m+1} + \cdots + p_{2\ell+1}z^{2\ell+1} + o\left(z^{2\ell+1}\right), \quad z\widehat{\rightarrow}\infty, \tag{1.5}$$

where $p_{m+1} = 1/s_m$, $p_i \in \mathbb{R}$, $i = m+1, \dots, 2\ell+1$, and the integers m, n, and ℓ are connected by $\ell = m - n$ with $m \geq 2\ell$; see Theorem 5.4 below for further details. It turns out that (1.5) holds for some $\ell \leq 0$ if and only if ∞ is a regular critical point of S_F, or equivalently, if and only if the corresponding root subspace

$$\mathsf{R}_\infty(S_F) = \{\, h \in \mathfrak{H} : \{0, h\} \in S_F^k \text{ for some } k \in \mathbb{N}\,\}$$

of the generalized Friedrichs extensions S_F in (1.4) is nondegenerate. In this case the GPNT ∞ of Q_∞ as well as the corresponding root subspace $\mathsf{R}_\infty(S_F)$ are shortly called *regular*. On the other hand, if ∞ is a singular critical point of S_F, then in (1.5) $\ell > 0$ and, moreover, the minimal integer ℓ such that the expansion (1.5) exists coincides with the dimension κ_∞^0 of the *isotropic subspace* of the root subspace $\mathsf{R}_\infty(S_F)$, see Theorem 5.6. In this case the GPNT ∞ of Q_∞ and the corresponding root subspace $\mathsf{R}_\infty(S_F)$ are shortly called *singular* with *the index of singularity* κ_∞^0.

The above mentioned results reflect the close connections between the asymptotic expansions (1.1), (1.5), and the root subspace $\mathsf{R}_\infty(S_F)$ of S_F. The given assertions are examples of the results in the present paper which have been derived by means of the factorization model of the function Q_∞ recently constructed by the authors in [9]. This model is based on the following "proper" factorization of the function $Q_\infty \in \mathbf{N}_\kappa$:

$$Q_\infty(z) = q(z) q^\sharp(z) Q_0(z), \tag{1.6}$$

where q is a (monic) polynomial, $q^\sharp(z) = \overline{q(\bar{z})}$, and $Q_0 \in \mathbf{N}_{\kappa'}$ such that

$$\kappa_\infty(Q_0) = 0 \quad \text{and} \quad \kappa' = \kappa - \deg q,$$

see Lemma 4.3 below. Such a factorization for Q_∞ is in general not unique, but the factorization model based on such a factorization carries the complete information about the root subspace $\mathsf{R}_\infty(S_F)$ of S_F.

A major part of the results presented in this paper is associated with the structure of the root subspace $\mathsf{R}_\infty(S_F)$ of S_F in a model space and the various connections to the asymptotic expansions (1.1) and (1.5). By using the factorization model based on a proper factorization (1.6) of Q_∞ maximal Jordan chains in $\mathsf{R}_\infty(S_F)$ are constructed in explicit terms. Their construction leads to *three different types of maximal Jordan chains* in $\mathsf{R}_\infty(S_F)$. Each of these three types of maximal Jordan chains admits its own characteristic features, reflecting various properties of the root subspace $\mathsf{R}_\infty(S_F)$. The construction shows explicitly, for instance, when the root subspace $\mathsf{R}_\infty(S_F)$ is regular and when it is singular. The length of the maximal Jordan chain as well as the *signature of the root subspace* $\mathsf{R}_\infty(S_F)$ can be easily read off from their construction. In the case that the root subspace $\mathsf{R}_\infty(S_F)$ is regular, the three types of maximal Jordan chain can be characterized by their length. The first type of maximal Jordan chain is of length $2k+1$, where $k = \deg q = \kappa_\infty(Q_\infty)$, and the second and third type of maximal Jordan chains are of length $2k$ and $2k-1$, respectively. The classification of these maximal Jordan chains remains the same in the case when the root subspace $\mathsf{R}_\infty(S_F)$ is singular. In that case the index of singularity κ_∞^0 as introduced above enters to the formulas, while the difference $\kappa_-(\mathsf{R}_\infty(S_F)) - \kappa_+(\mathsf{R}_\infty(S_F))$ of the negative and the positive index of $\mathsf{R}_\infty(S_F)$ remains unaltered, see Theorem 4.12. All of these facts can be translated into the analytical properties of the functions Q_∞ and $Q = \gamma - 1/Q_\infty$ via the asymptotic expansions (1.1) and (1.5), and conversely.

The classification of maximal Jordan chains in $\mathsf{R}_\infty(S_F)$ motivates an analogous *classification of generalized zeros and poles of nonpositive type* of the function $Q \in \mathbf{N}_\kappa$, which turns out to be connected with the characterization of the multiplicities of GZNT and GPNT of the function Q due to H. Langer in [24]; see Subsection 3.2 for the definitions of generalized zeros and poles of types (T1)–(T3). This induces a classification for the asymptotic expansions for the functions Q and Q_∞; see Theorems 5.3 and 5.4. Some further characterizations of the three different types of generalized zeros and poles are obtained by means of the factorized integral representations of the functions Q and Q_∞, which are based of their canonical factorizations, see [11]; for definitions, see Subsection 2.1, cf. also [5]. In particular, Theorem 6.1 and Theorem 6.3 extend some earlier results by the authors in [6] (where $\kappa = 1$) and in [8], from the regular case to the singular case in an explicit manner involving the index of singularity κ_∞^0, which is characterized in Theorem 5.6 below.

The construction of the maximal Jordan chains in $\mathsf{R}_\infty(S_F)$ using the factorization model for Q_∞ in (1.6) is carried out in Section 4. The most careful treatment of the model is required in the construction of maximal Jordan chains which are of the third type (T3). The reason is that the factorization of Q_∞ does not produce a minimal model for the function Q_∞ directly. In the minimal factorization model the maximal Jordan chains of type (T3) are roughly speaking the shortest ones, cf. (4.21), (4.24), (4.28); see also Theorem 5.3 and Theorem 6.3. The results in Lemma 4.11 and part (iii) of Theorem 4.12 characterize maximal Jordan chains of type (T3). In this case the underlying symmetric relation $S(Q)$ is multivalued (before the auxiliary part of the space is factored out). This statement is true more generally: for an arbitrary $\mathbf{N}_\kappa$-function Q the occurrence of generalized zeros and poles of type (T3) in $\mathbb{R} \cup \{\infty\}$ is an indication that point spectrum $\sigma_p(S(Q))$ of $S(Q)$ is nonempty, see Lemma 6.4 below, which by part (i) of Theorem 4.6 is equivalent to $S(Q)$ being not simple. In fact, the existence of maximal Jordan chains of type (T3) or, equivalently, the existence of GZNT and GPNT of type (T3) can be used to give criteria for minimality of various factorization models for $\mathbf{N}_\kappa$-functions, see Propositions 6.6 and 6.7 below.

The topics considered in this paper have connections to some other recent studies involving asymptotic expansions of $\mathbf{N}_\kappa$-functions, see in particular [6], [8], [9], [12], [13], [15], and their canonical factorization, see e.g. [3], [5], [7], [11], [14]. For instance, in [13] the authors investigate the subclass of $\mathbf{N}_\kappa$-functions with $\kappa = \kappa_\infty(Q)$ and extend some results e.g. from [6], [8]. General operator models based on the canonical factorization of $\mathbf{N}_\kappa$-functions have been introduced in [3]; for another model not using the canonical factorization of Q, see [18]. The construction of a minimal canonical factorization model by using reproducing kernel Pontryagin space methods has been recently worked out in [12], cf. also [3, Theorem 4.1].

Some of the results in the present paper can be naturally augmented by the results which can be found from [15], where characteristic properties of the generalized zeros and poles of $\mathbf{N}_\kappa$-functions have been studied with the aid of their operator representations.

The present paper forms a continuation of the paper [9], where the details concerning the construction of the announced factorization model can be found. Some basic definitions and concepts which will be used throughout the paper are given in Section 2. In Section 3 some additions concerning the subclasses $\mathbf{N}_{\kappa,-\ell}$ as introduced in [6] are given, including a proof for [6, Proposition 6.2] as announced in that paper, cf. Theorem 3.3 below; see also Theorem 5.4 for an extension of these results. Asymptotic expansions are introduced in Section 3 and a classification of generalized zeros and poles is given. In Section 4 the main ingredients concerning the factorization model are given and the construction of maximal Jordan chains in $\mathsf{R}_\infty(S_F)$ is carried out. The connection between the properties of the root subspace $\mathsf{R}_\infty(S_F)$ and the asymptotic expansions of the form (1.1) and (1.5) is investigated in Section 5. Finally, in Section 6 the classification of GZNT and GPNT is connected with factorized integral representations of the functions Q and $Q_\infty(z)$. In this section also the generalized zeros and poles of nonpositive type of $\mathbf{N}_\kappa$-functions which belong to $\mathbb{R}$ are briefly treated and some consequences as announced above are established.

2. Preliminaries

2.1. Canonical factorization of $Q \in \mathbf{N}_\kappa$

The notions of generalized poles and generalized zeros of nonpositive type were introduced in [23]. The following definitions are based on [24]. A point $\alpha \in \mathbb{R}$ is called a *generalized pole* of nonpositive type (GPNT) of the function $Q \in \mathbf{N}_\kappa$ with multiplicity $\kappa_\alpha(Q)$ if

$$-\infty < \lim_{z \widehat{\to} \alpha} (z-\alpha)^{2\kappa_\alpha+1} Q(z) \le 0, \quad 0 < \lim_{z \widehat{\to} \alpha} (z-\alpha)^{2\kappa_\alpha-1} Q(z) \le \infty. \tag{2.1}$$

Similarly, the point ∞ is called a generalized pole of nonpositive type (GPNT) of Q with multiplicity $\kappa_\infty(Q)$ if

$$0 \le \lim_{z \widehat{\to} \infty} \frac{Q(z)}{z^{2\kappa_\infty+1}} < \infty, \quad -\infty \le \lim_{z \widehat{\to} \infty} \frac{Q(z)}{z^{2\kappa_\infty-1}} < 0. \tag{2.2}$$

A point $\beta \in \mathbb{R}$ is called a *generalized zero* of nonpositive type (GZNT) of the function $Q \in \mathbf{N}_\kappa$ if β is a generalized pole of nonpositive type of the function $-1/Q$. The multiplicity $\pi_\beta(Q)$ of the GZNT β of Q can be characterized by the inequalities:

$$0 < \lim_{z \widehat{\to} \beta} \frac{Q(z)}{(z-\beta)^{2\pi_\beta+1}} \le \infty, \quad -\infty < \lim_{z \widehat{\to} \beta} \frac{Q(z)}{(z-\beta)^{2\pi_\beta-1}} \le 0. \tag{2.3}$$

Similarly, the point ∞ is called a generalized zero of nonpositive type (GZNT) of Q with multiplicity $\pi_\infty(Q)$ if

$$-\infty \le \lim_{z \widehat{\to} \infty} z^{2\pi_\infty+1}Q(z) < 0, \quad 0 \le \lim_{z \widehat{\to} \infty} z^{2\pi_\infty-1}Q(z) < \infty. \tag{2.4}$$

It was shown in [23] that for $Q \in \mathbf{N}_\kappa$ the total number (counting multiplicities) of poles (zeros) in $\mathbb{C}_+$ and generalized poles (zeros) of nonpositive type in $\mathbb{R} \cup \{\infty\}$ is equal to κ. Let $\alpha_1, \dots, \alpha_l$ ($\beta_1, \dots, \beta_m$) be all the generalized poles (zeros) of nonpositive type in $\mathbb{R}$ and the poles (zeros) in $\mathbb{C}_+$ with multiplicities $\kappa_1, \dots, \kappa_l$ ($\pi_1, \dots, \pi_m$). Then the function Q admits a canonical factorization of the form

$$Q(z) = r(z)r^\sharp(z)Q_{00}(z), \quad Q_{00} \in \mathbf{N}_0, \quad r = \frac{\widetilde{p}}{\widetilde{q}}, \tag{2.5}$$

where $\widetilde{p}(z) = \prod_{j=1}^m (z-\beta_j)^{\pi_j}$ and $\widetilde{q}(z) = \prod_{j=1}^l (z-\alpha_j)^{\kappa_j}$ are relatively prime polynomials of degree $\kappa - \pi_\infty(Q)$ and $\kappa - \kappa_\infty(Q)$, respectively; see [11], [5]. It follows from (2.5) that the function Q admits the (factorized) integral representation

$$Q(z) = r(z)r^\sharp(z)\left(a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\rho(t)\right), \quad r = \frac{\widetilde{p}}{\widetilde{q}}, \tag{2.6}$$

where $a \in \mathbb{R}$, $b \ge 0$, and $\rho(t)$ is a nondecreasing function satisfying the integrability condition

$$\int_{\mathbb{R}} \frac{d\rho(t)}{t^2+1} < \infty. \tag{2.7}$$

2.2. The subclasses $\mathbf{N}_{\kappa,1}$ and $\mathbf{N}_{\kappa,0}$

A function $Q \in \mathbf{N}_\kappa$ is said to belong to the subclass $\mathbf{N}_{\kappa,1}$, if

$$\lim_{z \widehat{\to} \infty} \frac{Q(z)}{z} = 0 \text{ and } \int_\eta^\infty \frac{|\operatorname{Im} Q(iy)|}{y}\, dy < \infty,$$

with $\eta > 0$ large enough. Similarly $Q \in \mathbf{N}_\kappa$ is said to belong to the subclass $\mathbf{N}_{\kappa,0}$, if

$$\lim_{z \widehat{\to} \infty} \frac{Q(z)}{z} = 0 \text{ and } \limsup_{z \widehat{\to} \infty} |z \operatorname{Im} Q(z)| < \infty,$$

see [5]. In the following theorems the subclasses $\mathbf{N}_{\kappa,1}$ and $\mathbf{N}_{\kappa,0}$ are characterized both in terms of the integral representation (2.6) and in terms of operator representations of the form (1.2). Let E_t be a spectral function of a selfadjoint operator A in a Pontryagin space $\mathfrak{H}$, see [1]. Denote by $\mathfrak{H}_\ell := \mathfrak{H}_\ell(A)$, $\ell \in \mathbb{N}$, the set of all elements $h \in \mathfrak{H}$ such that $\int_\Delta |t|^\ell d[E_t h, h] < \infty$ for some neighborhood Δ of $\pm\infty$. Moreover, let $\mathfrak{H}_{-\ell}(A)$, $\ell \in \mathbb{N}$, be the corresponding dual spaces. Here, for instance, $\mathfrak{H}_{-1}(A)$ can be identified as the set of all generalized elements obtained by completing $\mathfrak{H}$ with respect to the inner product $\int_\Delta (1+|t|)^{-1} d[E_t h, h] < \infty$ with some neighbourhood Δ of $\pm\infty$. The operator A admits a natural continuation $\widetilde{A}$ from $\mathfrak{H}$ into $\mathfrak{H}_{-1}$, see [5] for further details. The classes $\mathbf{N}_{\kappa,1}$ and $\mathbf{N}_{\kappa,0}$ are characterized in the following two theorems, see [5].

Theorem 2.1. ([5]) *For $Q \in \mathbf{N}_\kappa$ the following statements are equivalent:*

(i) *Q belongs to $\mathbf{N}_{\kappa,1}$;*
(ii) *$Q(z) = \gamma + [(\widetilde{A} - z)^{-1}\omega, \omega]$, $z \in \rho(A)$, for some selfadjoint operator A in a Pontryagin space $\mathfrak{H}$, a cyclic vector $\omega \in \mathfrak{H}_{-1}$, and $\gamma \in \mathbb{R}$;*
(iii) *Q has the integral representation (2.6) with $\deg \widetilde{q} - \deg \widetilde{p} = \pi_\infty(Q) > 0$, or with $\deg \widetilde{p} = \deg \widetilde{q}$ ($\pi_\infty(Q) = 0$), $b = 0$, and*

$$\int_{\mathbb{R}} (1 + |t|)^{-1} d\rho(t) < \infty. \tag{2.8}$$

Theorem 2.2. ([5]) *For $Q \in \mathbf{N}_\kappa$ the following statements are equivalent:*
(i) *Q belongs to $\mathbf{N}_{\kappa,0}$;*
(ii) *$Q(z) = \gamma + O(1/z)$, $z \widehat{\rightarrow} \infty$;*
(iii) *$Q(z) = \gamma + [(A - z)^{-1}\omega, \omega]$, $z \in \rho(A)$, for some selfadjoint operator A in a Pontryagin space $\mathfrak{H}$, a cyclic vector $\omega \in \mathfrak{H}$, and $\gamma \in \mathbb{R}$;*
(iv) *Q has the integral representation (2.6) with $\deg \widetilde{q} - \deg \widetilde{p} = \pi_\infty(Q) > 0$, or with $\deg \widetilde{p} = \deg \widetilde{q}$ ($\pi_\infty(Q) = 0$), $b = 0$, and*

$$\int_{\mathbb{R}} d\rho(t) < \infty. \tag{2.9}$$

Remark 2.3. If $Q \in \mathbf{N}_{\kappa,0}$ then the operator representation of Q in part (iii) of Theorem 2.2 implies that

$$\lim_{z \widehat{\rightarrow} \infty} -z(Q(z) - \gamma) = [\omega, \omega].$$

Hence, the statement (ii) in Theorem 2.2 can be strengthened in the sense that for every function $Q \in \mathbf{N}_{\kappa,0}$ there are real numbers γ and s_0, such that

$$Q(z) = \gamma - \frac{s_0}{z} + o\left(\frac{1}{z}\right), \quad z \widehat{\rightarrow} \infty. \tag{2.10}$$

3. Asymptotic expansions of generalized Nevanlinna functions

Asymptotic expansions of generalized Nevanlinna functions (as in (2.10)) can be used for studying operator and spectral theoretical properties of selfadjoint extensions of symmetric operators in Pontryagin and Hilbert spaces, see [20], [16]. In this section a subdivision of the class $\mathbf{N}_\kappa$ of generalized Nevanlinna functions is given along the lines of [16], [6]. Moreover, a classification for generalized zeros of nonpositive type is introduced and interpreted via asymptotic expansions.

3.1. The subclasses $\mathbf{N}_{\kappa,-\ell}$ of generalized Nevanlinna functions

Definition 3.1. A function $Q \in \mathbf{N}_\kappa$ is said to belong to the subclass $\mathbf{N}_{\kappa,-2n}$, $n \in \mathbb{N}$, if there are real numbers γ and $s_0, \dots, s_{2n-1}$ such that the function

$$\widetilde{Q}(z) = z^{2n}\left(Q(z) - \gamma + \sum_{j=1}^{2n} \frac{s_{j-1}}{z^j}\right) \tag{3.1}$$

is $O(1/z)$ as $z\widehat{\to}\infty$. Moreover, $Q \in \mathbf{N}_\kappa$ is said to belong to the subclass $\mathbf{N}_{\kappa,-2n+1}$ if the function $\widetilde{Q}$ in (3.1) belongs to $\mathbf{N}_{\kappa',1}$ for some $\kappa' \in \mathbb{N}$.

The next lemma clarifies the above definition of the subclasses $\mathbf{N}_{\kappa,-\ell}$, $\ell \in \mathbb{N}$.

Lemma 3.2. *If the function Q belongs to the subclass $\mathbf{N}_{\kappa,-2n}$ ($\mathbf{N}_{\kappa,-2n+1}$) for some $n \in \mathbb{N}$, then the function $\widetilde{Q}$ in* (3.1) *belongs to the subclass $\mathbf{N}_{\kappa',0}$ (resp. $\mathbf{N}_{\kappa',1}$) with $\kappa' \le \kappa$. Moreover, the following inclusions are satisfied*

$$\cdots \subset \mathbf{N}_{\kappa,-2n-1} \subset \mathbf{N}_{\kappa,-2n} \subset \mathbf{N}_{\kappa,-2n+1} \subset \cdots \subset \mathbf{N}_{\kappa,0} \subset \mathbf{N}_{\kappa,1}. \tag{3.2}$$

Proof. Rewrite the expression for the function $\widetilde{Q}$ in (3.1) in the form $\widetilde{Q}(z) = z^{2n}\widehat{Q}(z)$. Then $\widehat{Q}(z)$ as a sum of two generalized Nevanlinna functions is also a generalized Nevanlinna function, and therefore, in view of (2.5), $\widetilde{Q}$ is a generalized Nevanlinna function, too. Next it is shown that the inequality $\kappa(\widetilde{Q}) \le \kappa(Q)$ is satisfied. First observe that the condition $\widetilde{Q}(z) = O(1)$ and hence, in particular, the condition $\widetilde{Q}(z) = O(1/z)$ as $z\widehat{\to}\infty$ implies that

$$\kappa_\infty(\widetilde{Q}) = 0, \tag{3.3}$$

cf. (2.2), (2.4). Clearly, $\kappa_\alpha(\widetilde{Q}) = \kappa_\alpha(Q)$ for every $\alpha \ne 0, \infty$, while for $\alpha = 0$ one derives from (2.1) the estimate

$$\kappa_0(\widetilde{Q}) \le \kappa_0(Q). \tag{3.4}$$

Therefore, one can conclude from (3.3) and (3.4) that $\kappa(\widetilde{Q}) \le \kappa(Q)$. Now by Theorem 2.2 the condition $\widetilde{Q}(z) = O(1/z)$, $z\widehat{\to}\infty$, is equivalent to $\widetilde{Q} \in \mathbf{N}_{\kappa',0}$ with $\kappa' \le \kappa$, which proves the first statement for the subclasses $\mathbf{N}_{\kappa,-2n}$. If $Q \in \mathbf{N}_{\kappa,-2n+1}$, then $\widetilde{Q}(z) = O(1)$ and since $\kappa(\widetilde{Q}) \le \kappa(Q)$, one actually has $\widetilde{Q} \in \mathbf{N}_{\kappa',1}$ for $\kappa' \le \kappa$.

Since $\mathbf{N}_{\kappa,0} \subset \mathbf{N}_{\kappa,1}$ the inclusions $\mathbf{N}_{\kappa,-2n} \subset \mathbf{N}_{\kappa,-2n+1}$, $n \in \mathbb{N}$, follow from the first part of the lemma. Now let $Q \in \mathbf{N}_{\kappa,-2n-1}$. Then by definition

$$z^2\widetilde{Q}(z) + zs_{2n} + s_{2n+1} \in \mathbf{N}_{\kappa',1}, \tag{3.5}$$

where $\widetilde{Q}$ is as in (3.1) and $\kappa' \le \kappa$. It is clear from (3.5) (see Theorem 2.1) that $\widetilde{Q}(z) = O(1/z)$ as $z\widehat{\to}\infty$. Hence, $Q \in \mathbf{N}_{\kappa,-2n}$ and this proves the remaining inclusions in (3.2). □

The subclasses $\mathbf{N}_{\kappa,-\ell}$, $\ell \in \mathbb{N}$, are now characterized by means of the operator and the integral representation of Q in (1.2) and (2.6), respectively.

Theorem 3.3. *For $Q \in \mathbf{N}_\kappa$ the following statements are equivalent:*

(i) $Q \in \mathbf{N}_{\kappa,-\ell}$, $\ell \in \mathbb{N}$;
(ii) $Q(z) = \gamma + [(A-z)^{-1}\omega, \omega]$, $z \in \rho(A)$, *for some selfadjoint operator A in a Pontryagin space $\mathfrak{H}$, a cyclic vector $\omega \in \mathfrak{H}_\ell$, and $\gamma \in \mathbb{R}$;*

(iii) *Q has an integral representation* (2.6) *with* $\pi_\infty(Q) = \deg \widetilde{q} - \deg \widetilde{p} \geq 0$ *(and* $b = 0$ *if* $\pi_\infty(Q) = 0$*), such that*

$$\int_{\mathbb{R}} (1 + |t|)^{\ell - 2\pi_\infty} d\rho(t) < \infty. \tag{3.6}$$

Proof. (i) ⇒ (iii) Let $Q \in \mathbf{N}_{\kappa,-\ell}$, where ℓ is either $2n$ or $2n-1$, $n \in \mathbb{N}$. In view of (3.2) and Theorems 2.1, 2.2 one has $\pi_\infty(Q) = \deg \widetilde{q} - \deg \widetilde{p} \geq 0$, and if $\pi_\infty(Q) = 0$ then $b = 0$ and (2.8) or (2.9) is satisfied. By Lemma 3.2 the function $\widetilde{Q}$ in (3.1) belongs to $\mathbf{N}_{\kappa',2n-\ell}$ with $\kappa' \leq \kappa$. Hence, $\widetilde{Q}$ admits the factorization

$$\widetilde{Q} = \widetilde{r}(z)\widetilde{r}^\sharp(z)\left(\widetilde{a} + \widetilde{b}z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\right) d\widetilde{\rho}(t), \quad \widetilde{r} = \frac{\widetilde{p}_2}{\widetilde{q}_2}, \tag{3.7}$$

where $\widetilde{p}_2$ and $\widetilde{q}_2$ are the polynomials associated to $\widetilde{Q}$, cf. (2.5). Moreover, the inequality $\pi_\infty(\widetilde{Q}) = \deg \widetilde{q}_2 - \deg \widetilde{p}_2 \geq 0$ holds by Theorems 2.1 and 2.2. On the other hand, it follows from (2.6) and (3.1) that $\widetilde{Q}$ admits also the representation

$$\widetilde{Q}(z) = z^{2n} r(z) r^\sharp(z)\left(a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\right) d\rho(t) + p_1(z), \tag{3.8}$$

where p_1 is a polynomial with $\deg p_1 \leq 2n$. An application of the generalized Stieltjes inversion formula (see [19]) shows that the measures $d\widetilde{\rho}(t)$ in (3.7) and $d\rho(t)$ in (3.8) are connected by

$$|\widetilde{r}(t)|^2 d\widetilde{\rho}(t) = t^{2n} |r(t)|^2 \, d\rho(t). \tag{3.9}$$

Therefore, if $\widetilde{Q} \in \mathbf{N}_{\kappa',1} \setminus \mathbf{N}_{\kappa',0}$ so that $\ell = 2n-1$, then $\deg \widetilde{p}_2 = \deg \widetilde{q}_2$ and $d\widetilde{\rho}(t)$ satisfies the condition (2.8) in Theorem 2.1. The condition (3.6) follows now from (3.9). If $\widetilde{Q} \in \mathbf{N}_{\kappa',0}$ so that $\ell = 2n$, then either $\deg \widetilde{p}_2 = \deg \widetilde{q}_2$ in which case $d\widetilde{\rho}(t)$ satisfies the condition (2.9) in Theorem 2.2, or $\pi_\infty(\widetilde{Q}) = \deg \widetilde{q}_2 - \deg \widetilde{p}_2 > 0$ in which case $d\widetilde{\rho}(t)$ satisfies the condition (2.7). In both cases

$$\int_{|t|>M} |\widetilde{r}(t)|^2 d\widetilde{\rho}(t) < \infty, \qquad \text{for } M > 0 \text{ large enough.}$$

Hence, again the condition (3.6) follows from (3.9).

(ii) ⇔ (iii) Let E_t be the spectral function of a selfadjoint operator A in the minimal representation (1.2) of Q. It follows from (1.2), (2.6), and the generalized Stieltjes inversion formula that

$$d[E_t\omega, \omega] = |r(t)|^2 \, d\rho(t), \quad t \in \Delta,$$

in some neighbourhood Δ of $\pm\infty$. This implies that

$$\int_{\mathbb{R}} (1 + |t|)^{\ell - 2\pi_\infty} \, d\rho(t) < \infty \;\text{ if and only if }\; \int_{\Delta} (1 + |t|)^{\ell} \, d[E_t\omega, \omega] < \infty,$$

i.e., $\omega \in \mathfrak{H}_\ell$, which proves the equivalence of (ii) and (iii).

(ii) $\Rightarrow$ (i) First consider the case $\ell = 2n$. Then $\omega \in \mathfrak{H}_\ell$ means that $\omega \in \operatorname{dom} A^n$. Define the function $\widetilde{Q}$ in (3.1) by setting

$$s_j = [A^j\omega, \omega], \quad s_{n+j} = [A^j\omega, A^n\omega], \quad j = 0, \dots, n. \tag{3.10}$$

Then a straightforward calculation shows that $\widetilde{Q}$ admits the operator representation

$$\widetilde{Q}(z) = [(A - z)^{-1}\omega', \omega'], \quad \omega' = A^n\omega \in \mathfrak{H}. \tag{3.11}$$

Therefore, $\widetilde{Q}(z) = O(1/z)$ and $Q \in \mathbf{N}_{\kappa,-\ell}$.

Now let $\ell = 2n - 1$. Then $\omega \in \mathfrak{H}_\ell$ means that $\omega' := A^n\omega \in \mathfrak{H}_{-1}$. Hence $s_{2n-1} := [\widetilde{A}\omega', \omega']$ is well defined. Moreover, by defining $s_0, \dots, s_{2n-2}$ as in (3.10) it follows that the function $\widetilde{Q}$ in (3.1) admits the operator representation

$$\widetilde{Q}(z) = [(\widetilde{A} - z)^{-1}\omega', \omega'], \quad \omega' = A^n\omega \in \mathfrak{H}_{-1}. \tag{3.12}$$

Hence, by Theorem 2.1 $\widetilde{Q} \in \mathbf{N}_{\kappa',1}$ for some $\kappa' \in \mathbb{N}$ and thus $Q \in \mathbf{N}_{\kappa,-\ell}$. This completes the proof. □

From Theorem 3.3 one obtains the following result of M.G. Kreĭn and H. Langer, see [21, Satz 1.10]

Corollary 3.4. ([21]) *The function $Q \in \mathbf{N}_\kappa$ admits an operator representation $Q(z) = \gamma + [(A - z)^{-1}\omega, \omega]$ with $\gamma \in \mathbb{R}$ and $\omega \in \mathfrak{H}_{2n}$ $(= \operatorname{dom} A^n)$ if and only if there are real numbers γ and $s_0, \dots, s_{2n}$, such that*

$$Q(z) = \gamma - \sum_{j=1}^{2n+1} \frac{s_{j-1}}{z^j} + o\left(\frac{1}{z^{2n+1}}\right), \quad z \widehat{\rightarrow} \infty. \tag{3.13}$$

In this case the numbers $s_0, \dots, s_{2n}$ are given by (3.10).

Proof. The proof of Theorem 3.3 shows that the condition $\omega \in \operatorname{dom} A^n$ is equivalent to the operator representation (3.11) of the function $\widetilde{Q}(z)$ in (3.1). Now by applying (2.10) in Remark 2.3 to the function $\widehat{Q}(z)$ in (3.11) and taking into account (3.1) the equivalence to the expansion (3.13) follows. □

The criterion of M.G. Kreĭn and H. Langer in Corollary 3.4 does not hold in the case of an odd number $\ell = 2n - 1$. However, it is clear that if $\omega \in \mathfrak{H}_{2n-1}$ then the analog of the expansion (3.13) exists.

Corollary 3.5. *If the function $Q \in \mathbf{N}_\kappa$ admits an operator representation $Q(z) = \gamma + [(A - z)^{-1}\omega, \omega]$ with $\gamma \in \mathbb{R}$ and $\omega \in \mathfrak{H}_{2n-1}$ $(= \operatorname{dom} |A|^{n-1/2})$ then there are real numbers γ and $s_0, \dots, s_{2n}$, such that*

$$Q(z) = \gamma - \sum_{j=1}^{2n} \frac{s_{j-1}}{z^j} + o\left(\frac{1}{z^{2n}}\right), \quad z \widehat{\rightarrow} \infty. \tag{3.14}$$

Proof. Since $\omega \in \mathfrak{H}_{2n-1}$ the operator representation (3.12) in the proof of Theorem 3.3 shows that $\widetilde{Q}(z) = o(1)$. The expansion (3.14) for the function Q follows now from the definition of $\widetilde{Q}$ in (3.1). □

It is emphasized that the existence of the expansion (3.14) does not imply that $\omega \in \mathfrak{H}_{2n-1}$ or, equivalently, that $\omega' := A^n\omega$ belongs to $\mathfrak{H}_{-1}$. In this case $[\widetilde{A}\omega', \omega']$ need not be defined and hence it cannot coincide with the coefficient s_{2n-1} in (3.14).

3.2. A classification of generalized zeros of nonpositive type

For what follows it will be useful to give a classification for generalized zeros and poles of nonpositive type of a function $Q \in \mathbf{N}_\kappa$.

Let ∞ be a GZNT of Q with multiplicity $\pi_\infty > 0$. It follows from (2.4) that precisely one of the following three cases can occur:

(T1) $-s_{2\pi_\infty} := \lim_{z \widehat{\to} \infty} z^{2\pi_\infty+1}Q(z) < 0, \quad \lim_{z \widehat{\to} \infty} z^{2\pi_\infty-1}Q(z) = 0;$
(T2) $\lim_{z \widehat{\to} \infty} z^{2\pi_\infty+1}Q(z) = \infty, \quad \lim_{z \widehat{\to} \infty} z^{2\pi_\infty-1}Q(z) = 0;$
(T3) $\lim_{z \widehat{\to} \infty} z^{2\pi_\infty+1}Q(z) = \infty, \quad -s_{2\pi_\infty-2} := \lim_{z \widehat{\to} \infty} z^{2\pi_\infty-1}Q(z) > 0.$

In these cases ∞ is said to be a generalized zero of type (T1), (T2), or (T3), respectively; the shorter notations GZNT1, GZNT2, and GZNT3 are used accordingly. The corresponding classification for a finite generalized zero $\beta \in \mathbb{R}$ of Q is defined analogously:

(T1) $\lim_{z \widehat{\to} \beta} \dfrac{Q(z)}{(z-\beta)^{2\pi_\beta+1}} > 0, \quad \lim_{z \widehat{\to} \beta} \dfrac{Q(z)}{(z-\beta)^{2\pi_\beta-1}} = 0;$
(T2) $\lim_{z \widehat{\to} \beta} \dfrac{Q(z)}{(z-\beta)^{2\pi_\beta+1}} = \infty, \quad \lim_{z \widehat{\to} \beta} \dfrac{Q(z)}{(z-\beta)^{2\pi_\beta-1}} = 0;$
(T3) $\lim_{z \widehat{\to} \beta} \dfrac{Q(z)}{(z-\beta)^{2\pi_\beta+1}} = \infty, \quad \lim_{z \widehat{\to} \beta} \dfrac{Q(z)}{(z-\beta)^{2\pi_\beta-1}} < 0.$

A generalized pole of nonpositive type $\beta \in \mathbb{R} \cup \{\infty\}$ of Q is said to be of type (T1), (T2) or (T3), if β is a generalized zero of nonpositive type of the function $-1/Q$ which is of type (T1), (T2) or (T3), respectively.

To give some immediate implications of the above classification consider the generalized zero ∞ of Q. If it is of the first type, then it follows from (T1) that $Q \in \mathbf{N}_{\kappa,-2\pi_\infty}$. Moreover, Q has the following asymptotic expansion:

$$Q(z) = -\frac{s_{2\pi_\infty}}{z^{2\pi_\infty+1}} + o\left(\frac{1}{z^{2\pi_\infty+1}}\right), \quad z \widehat{\to} \infty, \quad s_{2\pi_\infty} > 0. \tag{3.15}$$

If the generalized zero ∞ of Q is of type (T3), then $Q \in \mathbf{N}_{\kappa,-2\pi_\infty-2}$ and Q has the following asymptotic expansion

$$Q(z) = -\frac{s_{2\pi_\infty-2}}{z^{2\pi_\infty-1}} + o\left(\frac{1}{z^{2\pi_\infty-1}}\right), \quad z \widehat{\to} \infty, \quad s_{2\pi_\infty-2} < 0. \tag{3.16}$$

In the case that the generalized zero ∞ is of type (T2) there are two possibilities: either Q belongs to $\mathbf{N}_{\kappa,-2\pi_\infty}$, in which case both of the moments $s_{2\pi_\infty-1}$ and $s_{2\pi_\infty}$ are finite and Q has the asymptotic expansion

$$Q(z) = -\frac{s_{2\pi_\infty-1}}{z^{2\pi_\infty}} - \frac{s_{2\pi_\infty}}{z^{2\pi_\infty+1}} + o\left(\frac{1}{z^{2\pi_\infty+1}}\right), \quad z \widehat{\to} \infty, \quad s_{2\pi_\infty-1} \neq 0, \tag{3.17}$$

or Q belongs to $\mathbf{N}_{\kappa,-2(\pi_\infty-1)} \setminus \mathbf{N}_{\kappa,-2\pi_\infty}$ and it has the asymptotic expansion

$$Q(z) = -\frac{s_{2\pi_\infty-1}}{z^{2\pi_\infty}} + o\left(\frac{1}{z^{2\pi_\infty}}\right), \quad z\widehat{\rightarrow}\infty, \tag{3.18}$$

or

$$Q(z) = o\left(\frac{1}{z^{2\pi_\infty-1}}\right), \quad z\widehat{\rightarrow}\infty. \tag{3.19}$$

Observe, that the expansions (3.17) and (3.18) are also special cases of the expansion (3.19). Hence, if ∞ is a generalized zero of type (T2), then Q has an expansion of the form (3.16), but now with $s_{2\pi_\infty-2} = 0$; however, Q does not have an expansion of the form (3.15). Similar observations remain true for generalized zeros $\beta \in \mathbb{R}$ and poles $\alpha \in \mathbb{R} \cup \{\infty\}$. For instance, to get the analogous expansions for a generalized zero $\beta \in \mathbb{R}$ apply the transform $-Q(1/(z-\beta))$ to the expansions in (3.15)–(3.19); cf. also [15]. The role of the above classification for generalized zeros and poles of nonpositive type will be described in detail in Sections 4–6.

4. An operator model for the generalized Friedrichs extension

4.1. Boundary triplets and Weyl functions

The construction of the model uses the notion of a boundary triplet in a Pontryagin space setting. Let $\mathfrak{H}$ be a Pontryagin space with negative index κ, let S be a closed symmetric relation in $\mathfrak{H}$ with defect numbers (n, n), and let S^* be the adjoint of S. A triplet $\Pi = \{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ is said to be a *boundary triplet* for S^*, if the following two conditions are satisfied:

(i) the mapping $\Gamma : \widehat{f} \to \{\Gamma_0\widehat{f}, \Gamma_1\widehat{f}\}$ from S^* to $\mathbb{C}^n \oplus \mathbb{C}^n$ is surjective;

(ii) the abstract Green's identity

$$[f', g] - [f, g'] = (\Gamma_1\widehat{f}, \Gamma_0\widehat{g}) - (\Gamma_0\widehat{f}, \Gamma_1\widehat{g}) \tag{4.1}$$

holds for all $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in S^*$,

see e.g. [2], [10]. It is easily seen that $A_0 = \ker \Gamma_0$ and $A_1 = \ker \Gamma_1$ are selfadjoint extensions of S. Associated to every boundary triplet there is the Weyl function Q defined by

$$Q(z)\Gamma_0\widehat{f}_z = \Gamma_1\widehat{f}_z, \quad z \in \rho(A_0),$$

where $\widehat{f}_z := \{f_z, zf_z\} \in \widehat{\mathfrak{N}}_z$, and $\mathfrak{N}_z = \ker(S^* - z)$ denotes the defect subspace of S at $z \in \mathbb{C}$. It follows from (4.1) that the Weyl function Q is also a Q-function of the pair $\{S, A_0\}$ in the sense of Kreĭn and Langer, see [20]. If S is *simple*, so that

$$\mathfrak{H} = \overline{\operatorname{span}}\,\{\mathfrak{N}_z : z \in \rho(A_0)\},$$

then the Weyl function Q belongs to the class $\mathbf{N}_\kappa$, otherwise $Q \in \mathbf{N}_{\kappa'}$ with $\kappa' \leq \kappa$. Moreover, if S is simple and H is a selfadjoint extension of S in $\mathfrak{H}$, then the point spectrum of H is also simple, that is, every eigenspace of H is one-dimensional, and if $\alpha \in \mathbb{R} \cup \{\infty\}$, then the root subspace at α is at most $2\kappa + 1$-dimensional.

In the case where S is given by (1.3) one can define a boundary triplet for S^* as follows.

Proposition 4.1. (cf. [5]) *Let A be a selfadjoint operator in a Pontryagin space $\mathfrak{H}$ and let the restriction S of A be defined by* (1.3) *with $\omega \in \mathfrak{H}$. Then the adjoint S^* of S in $\mathfrak{H}$ is of the form*

$$S^* = \{\{f, Af + c\omega\} : f \in \operatorname{dom} A, c \in \mathbb{C}\}$$

and a boundary triplet $\Pi^\infty = \{\mathbb{C}, \Gamma_0^\infty, \Gamma_1^\infty\}$ for S^* is determined by

$$\Gamma_0^\infty \widehat{f} = [f, \omega], \quad \Gamma_1^\infty \widehat{f} = c, \quad \widehat{f} = \{f, Af + c\omega\} \in S^*.$$

The corresponding Weyl function Q_∞ is given by

$$Q_\infty(z) = -\frac{1}{[(A-z)^{-1}\omega, \omega]}, \quad z \in \rho(A). \tag{4.2}$$

4.2. The model operator $S(Q_\infty)$ corresponding to a proper factorization

Operator models for generalized Nevanlinna functions whose only generalized pole of nonpositive type is at ∞ have been constructed in [6] and [14]. Such functions admit a canonical factorization of the form

$$Q_\infty(z) = q(z)q^\sharp(z)Q_0(z), \tag{4.3}$$

where $Q_0 \in \mathbf{N}_0$, $q(z) = z^k + q_{k-1}z^{k-1} + \cdots + q_0$ is a polynomial, and $q^\sharp(z) = \overline{q(\bar{z})}$. In general, models which are based on the canonical factorization of $Q \in \mathbf{N}_\kappa$ are not necessarily minimal, i.e., the underlying model operator $S(Q_\infty)$ need not be simple and it can even be a symmetric relation (multivalued operator). However, with the canonical factorization the nonsimple part of $S(Q_\infty)$ can be easily identified and factored out to produce a simple symmetric operator from $S(Q_\infty)$, cf. [3]. The model constructed for $S(Q_\infty)$ in [6] uses an orthogonal coupling of two symmetric operators. In [9] this model was adapted to the case where the function Q_0 is allowed to be a generalized Nevanlinna function, too. In this case the situation becomes more involved and, in general, one cannot represent $S(Q_\infty)$ as an orthogonal sum of a simple symmetric operator and a selfadjoint relation. However, such a simple orthogonal decomposition for $S(Q_\infty)$ can still be obtained if the factorization (4.3) of Q_∞ is proper. This concept is defined as follows.

Definition 4.2. ([9]) The factorization $Q_\infty(z) = q(z)q^\sharp(z)Q_0(z)$ is said to be *proper* if q is a divisor of degree $\kappa_\infty(Q_\infty) > 0$ of the polynomial $\widetilde{q}$ in the canonical factorization of the function Q_∞, cf. (2.5) .

Clearly, proper factorizations of $Q_\infty \in \mathbf{N}_\kappa$ always exist, but they are not unique if $\widetilde{q}$ has more than one zero and $\kappa_\infty(Q_\infty) < \kappa$. Proper factorizations $Q_\infty = qq^\sharp Q_0$ can be characterized also without using the canonical factorization of Q_∞.

Lemma 4.3. *Let $Q_\infty \in \mathbf{N}_\kappa$ have a factorization of the form*

$$Q_\infty(z) = q(z)q^\sharp(z)Q_0(z), \quad \deg q = k \geq 1, \tag{4.4}$$

where $q(z)$ is a monic polynomial, and let $\alpha \in \sigma(q)$ be a zero of q with multiplicity k_α. Then the following statements are equivalent:

(i) *the factorization (4.4) of Q_∞ is proper;*

(ii) *the multiplicities $\kappa_\infty(Q_\infty)$ and $\pi_\alpha(Q_\infty)$ satisfy the following relations:*

$$\kappa_\infty(Q_\infty) = \deg q \text{ and } \pi_\alpha(Q_\infty) \geq k_\alpha \text{ for all } \alpha \in \sigma(q); \tag{4.5}$$

(iii) *$\kappa_\infty(Q_0)$ and $\kappa(Q_\infty) = \kappa$ satisfy the following identities*

$$\kappa_\infty(Q_0) = 0 \text{ and } \kappa(Q_\infty) = \deg q + \kappa(Q_0). \tag{4.6}$$

Proof. (i) $\Leftrightarrow$ (ii) In a proper factorization (4.4) $\kappa_\infty(Q_\infty) = \deg q$ and clearly the inequalities in (4.5) just mean that q divides the polynomial $\widetilde{q}$ in the canonical factorization of Q_∞.

(i) $\Rightarrow$ (iii) If the factorization (4.4) is proper, then in the canonical factorization of the function Q_0 the numerator $\widetilde{q}_0$ and denominator $\widetilde{p}_0\,(=\widetilde{p})$ of the corresponding rational factor r_0 are of the same degree $\kappa(Q_0)$, and this implies (4.6).

(iii) $\Rightarrow$ (i) It follows from the second equality in (4.6) that q and the polynomial $\widetilde{p}_0$ in the canonical factorization of the function Q_0 are relatively prime and, therefore, q is a factor of the polynomial $\widetilde{q}$ in the canonical factorization of Q_∞. Moreover, $\pi_\infty(Q_0) = 0$. Now the assumption $\kappa_\infty(Q_0) = 0$ implies that $\kappa_\infty(Q_\infty) = \deg q$. □

The construction of factorization models is now briefly described. Let q be a polynomial as in (4.4) of degree $k = \deg q$. Define the $k \times k$ matrices $\mathcal{B}_q$ and $\mathcal{C}_q$ by

$$\mathcal{B}_q = \begin{pmatrix} q_1 & \dots & q_{k-1} & 1 \\ \vdots & \iddots & 1 & 0 \\ q_{k-1} & \iddots & \iddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}, \quad \mathcal{C}_q = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 1 \\ -q_0 & -q_1 & \dots & -q_{k-1} \end{pmatrix},$$

so that $\sigma(\mathcal{C}_q) = \sigma(q)$. Moreover, let $\mathfrak{H}_q$ be a $2k$-dimensional Pontryagin space defined by

$$(\mathbb{C}^k \oplus \mathbb{C}^k, \langle \mathcal{B}\cdot, \cdot \rangle), \quad \mathcal{B} = \begin{pmatrix} 0 & B_q \\ B_{q^\sharp} & 0 \end{pmatrix}.$$

A general factorization model for functions Q_∞ of the form (4.4) was constructed in [9] and can be applied, in particular, for proper factorizations of Q_∞.

Theorem 4.4. (cf. [9]) *Let $Q_\infty \in \mathbf{N}_\kappa$ be a generalized Nevanlinna function and let*

$$Q_\infty(\lambda) = q(\lambda) q^\sharp(\lambda) Q_0(\lambda), \tag{4.7}$$

be a proper factorization Q_∞, where q is a monic polynomial of degree $k = \deg q \geq 1$. Let S_0 be a closed symmetric relation in a Pontryagin space $\mathfrak{H}_0$ with the boundary triplet $\Pi^0 = \{\mathcal{H}, \Gamma_0^0, \Gamma_1^0\}$ whose Weyl function is Q_0. Then:

(i) *the function Q_0 in (4.7) belongs to the class $\mathbf{N}_{\kappa-k}$;*

(ii) *the linear relation*

$$S(Q_\infty)=\left\{\left\{\begin{pmatrix} f_0\\ f\\ \widetilde{f}\end{pmatrix},\begin{pmatrix} f_0'\\ C_{q^\sharp}f\\ C_q\widetilde{f}+\Gamma_0^0\widehat{f}_0e_k\end{pmatrix}\right\}:\ \begin{matrix}\widehat{f}_0=\{f_0,f_0'\}\in S_0^*,\\ f_1=\Gamma_1^0\widehat{f}_0,\\ \widetilde{f}_1=0\end{matrix}\right\} \tag{4.8}$$

is closed and symmetric in $\mathfrak{H}:=\mathfrak{H}_0\oplus\mathfrak{H}_q$ *and has defect numbers* $(1,1)$*;*

(iii) *the adjoint* $S(Q_\infty)^*$ *of* $S(Q_\infty)$ *is given by*

$$S(Q_\infty)^*=\left\{\left\{\begin{pmatrix} f_0\\ f\\ \widetilde{f}\end{pmatrix},\begin{pmatrix} f_0'\\ C_{q^\sharp}f+\widetilde{\varphi}e_k\\ C_q\widetilde{f}+\Gamma_0^0\widehat{f}_0e_k\end{pmatrix}\right\}:\ \begin{matrix}\widehat{f}_0=\{f_0,f_0'\}\in S_0^*,\\ f_1=\Gamma_1^0\widehat{f}_0,\\ \widetilde{\varphi}\in\mathbb{C}\end{matrix}\right\};$$

(iv) *a boundary triplet* $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ *for* $S(Q_\infty)^*$ *is determined by*

$$\Gamma_0(\widehat{f}_0\oplus\widehat{F})=\widetilde{f}_1,\quad \Gamma_1(\widehat{f}_0\oplus\widehat{F})=\widetilde{\varphi},\quad \widehat{f}_0\oplus\widehat{F}\in S(Q_\infty)^*; \tag{4.9}$$

(v) *the corresponding Weyl function coincides with* Q_∞.

Proof. Since the factorization (4.7) is proper the statement (i) is immediate from the equality (4.6) in Lemma 4.3. All the other statements are contained in [9]. □

In fact, the statement (iv) in Theorem 4.4 can be obtained directly also from Proposition 4.1, since $S(Q_\infty)$ in (4.8) is a restriction of the selfadjoint relation

$$A(Q_\infty)=\left\{\left\{\begin{pmatrix} f_0\\ f\\ \widetilde{f}\end{pmatrix},\begin{pmatrix} f_0'\\ C_{q^\sharp}f\\ C_q\widetilde{f}+\Gamma_0^0\widehat{f}_0e_k\end{pmatrix}\right\}:\ \begin{matrix}\widehat{f}_0=\{f_0,f_0'\}\in S_0^*,\\ f_1=\Gamma_1^0\widehat{f}_0\end{matrix}\right\} \tag{4.10}$$

to the subspace $\mathfrak{H}\ominus\omega_0$, where $\omega_0=\operatorname{col}(0,e_k,0)$; compare (1.3). The generalized Friedrichs extension of $S(Q_\infty)$ is given by

$$S_F(Q_\infty)=\left\{\left\{\begin{pmatrix} f_0\\ f\\ \widetilde{f}\end{pmatrix},\begin{pmatrix} f_0'\\ C_{q^\sharp}f+\widetilde{\varphi}e_k\\ C_q\widetilde{f}+\Gamma_0^0\widehat{f}_0e_k\end{pmatrix}\right\}:\ \begin{matrix}\widehat{f}_0=\{f_0,f_0'\}\in S_0^*,\\ f_1=\Gamma_1^0\widehat{f}_0,\\ \widetilde{f}_1=0,\ \widetilde{\varphi}\in\mathbb{C}\end{matrix}\right\}. \tag{4.11}$$

According to (4.2) in Proposition 4.1 the Weyl function $Q_\infty(z)$ corresponding to the boundary triplet (4.9) is of the form

$$Q_\infty(z)=-\frac{1}{[(A(Q_\infty)-z)^{-1}\omega_0,\omega_0]}.$$

Thus the function $Q=-1/Q_\infty$ has the representation

$$Q(z)=[(A(Q_\infty)-z)^{-1}\omega,\omega], \tag{4.12}$$

which is, however, not necessarily minimal, since $\operatorname{mul}S$ and $\ker(S-\alpha)$, $\alpha\in\sigma(q)$, can be nontrivial. The following lemma describes these subspaces.

Lemma 4.5. ([9]) *Under the assumptions of Theorem* 4.4 *let* S_0 *be a simple closed symmetric operator in the Pontryagin space* $\mathfrak{H}_0$*, let* $A_i^0=\ker\Gamma_i^0(\supset S_0)$*,* $i=0,1$*, and let* k_α *be the multiplicity of* $\alpha\in\mathbb{C}$ *as a zero of the polynomial* q*. Then:*

(i) $\operatorname{mul} S(Q_\infty)$ *is nontrivial if and only if* $\operatorname{mul} A_1^0$ *is nontrivial and in this case*

$$\operatorname{mul} S(Q_\infty) = \{\, (g, 0, \Gamma_0^0 \widehat{g} e_k)^\top : \widehat{g} = \{0, g\} \in A_1^0 \,\}; \tag{4.13}$$

(ii) *if* $\operatorname{mul} S(Q_\infty)$ *is nontrivial, then it is spanned by a positive vector;*

(iii) *if* $\operatorname{mul} A_0^0$ *is nontrivial, then it is spanned by a positive vector;*

(iv) $\sigma_p(S(Q_\infty)) = \sigma_p(A_0^0) \cap \sigma(q^\sharp)$ *and for* $\alpha \in \sigma_p(A_0^0) \cap \sigma(q^\sharp)$ *one has*

$$\ker(S(Q_\infty) - \alpha) = \{\, (g_0, \Gamma_1^0 \widehat{g}_0 \Lambda|_{\lambda=\alpha}, 0)^\top : g_0 \in \ker(A_0^0 - \alpha) \,\}; \tag{4.14}$$

where $\Lambda = (1, \lambda, \ldots, \lambda^{k-1})$, $\lambda \in \mathbb{C}$;

(v) *if* $\ker(S(Q_\infty) - \alpha)$ *or, equivalently,* $\ker(A_0^0 - \alpha)$ *is nontrivial, then it is spanned by a positive vector.*

It follows from (ii) and (iv) that the linear relation $S(Q_\infty)$ can be decomposed into a direct sum of an operator S' with an empty point spectrum and a selfadjoint part in a Hilbert space which is the sum of $\operatorname{mul} S(Q_\infty)$ and $\ker(S(Q_\infty) - \alpha)$, $\alpha \in \sigma_p(A_0^0) \cap \sigma(q^\sharp)$. The next theorem shows that the reduced operator S' is simple.

Theorem 4.6. *Let the assumptions of Theorem* 4.4 *be satisfied and let* $S(Q_\infty)$, $A(Q_\infty)$, *and* $S_F(Q_\infty)$ *be given by* (4.8), (4.10), *and* (4.11), *respectively. Then:*

(i) $S(Q_\infty)$ *is simple if and only if* $\sigma_p(S(Q_\infty)) = \emptyset$. *In this case the linear relations* $S = S(Q_\infty)$, $A = A(Q_\infty)$, *and* $S_F = S_F(Q_\infty)$ *satisfy the equalities* (1.3) *and* (1.4) *with* $\omega = \omega_0$ *and the operator representation* (4.12) *of* $Q = -1/Q_\infty$ *is minimal.*

(ii) *If* $S(Q_\infty)$ *is not simple, then the subspace*

$$\mathfrak{H}'' = \operatorname{span}\{\, \operatorname{mul} S(Q_\infty), \ker(S(Q_\infty) - \alpha) : \alpha \in \sigma_p(A_0^0) \cap \sigma(q) \,\} \tag{4.15}$$

is positive and reducing for $S(Q_\infty)$. *The simple part of* $S(Q_\infty)$ *coincides with the restriction* S' *of* $S(Q_\infty)$ *to* $\mathfrak{H}' := \mathfrak{H} \ominus \mathfrak{H}''$. *The compressions* S', A', *and* S_F' *of* $S(Q_\infty)$, $A(Q_\infty)$, *and* $S_F(Q_\infty)$ *to the subspace* $\mathfrak{H}'$ *satisfy the equalities* (1.3) *and* (1.4), *with* $\omega \in \mathfrak{H}'$ *given by*

$$\omega = \begin{cases} \omega_0, & \text{if } k > 1, \\ (g, -1/\overline{\Gamma_0 \widehat{g}}, \Gamma_0 \widehat{g})^\top, & \text{if } k = 1, \end{cases}$$

and the function $Q = -1/Q_\infty$ *admits the minimal representation*

$$Q(\lambda) = -1/Q_\infty(\lambda) = [(A' - \lambda)^{-1} \omega, \omega].$$

4.3. The root subspace of the generalized Friedrichs extension at ∞.

Let the assumptions of Theorem 4.4 be satisfied and let $S(Q_\infty)$, $A(Q_\infty)$, $S_F(Q_\infty)$ and S', A', S_F' be the same as in Theorem 4.6. Since S' is a simple symmetric operator in the Pontryagin space $\mathfrak{H}'$ its selfadjoint extension S_F' has a simple point spectrum. In particular, the multivalued part $\operatorname{mul} S_F'$ of S_F' is at most one-dimensional. The corresponding root subspace

$$\mathsf{R}_\infty(S_F') = \operatorname{span}\{\, g \in \mathfrak{H}' : \{0, g\} \in {S_F'}^k, \text{ for some } k \in \mathbb{N} \,\}$$

is at most $2\kappa+1$-dimensional. The root subspace $\mathsf{R}_\infty(S'_F)$ is spanned by the vectors ω_j which form a maximal Jordan chain in $\mathsf{R}_\infty(S'_F)$:

$$\mathsf{R}_\infty(S'_F) = \operatorname{span}\{\,\omega_j \in \mathfrak{H}' : \{\omega_{j-1},\omega_j\} \in S'_F, \quad j = 0,\dots,\nu-1\,\},$$

where $\nu = \dim \mathsf{R}_\infty(S'_F)$ and $\omega_{-1} = 0$.

The main properties of the root subspace $\mathsf{R}_\infty(S'_F)$ of the selfadjoint extension S'_F of S' are given in Lemmas 4.7–4.11 below. The proofs of these lemmas are constructive, since they are based on the factorization model for $S(Q_\infty)$ given in Theorem 4.4, cf. also [9]. As a consequence a detailed description for the structure of the corresponding Jordan chains in the factorization model is obtained. It turns out that three different types of maximal Jordan chains can appear in this model. In each of these cases maximal Jordan chains in $\mathsf{R}_\infty(S_F)$ can be regular or singular and the signature of $\mathsf{R}_\infty(S_F)$ has its own specific nature in each case.

The first lemma concerns the dimension and nondegeneracy of the root subspace $\mathsf{R}_\infty(S'_F)$; further equivalent conditions and descriptions (without a specific model space) for arbitrary generalized poles and zeros of $Q \in \mathbf{N}_\kappa$ with direct (nonconstructive) proofs can be found from [15].

Lemma 4.7. *Let $Q_\infty \in \mathbf{N}_\kappa$, let $\kappa_\infty(Q_\infty) = \deg q (= k) > 0$, and let S', S'_F, and $\omega \in \mathfrak{H}$, $[\omega,\omega] \le 0$, be as in Theorem 4.6. Then:*

(i) $\dim \mathsf{R}_\infty(S'_F) \ge \deg q$;

(ii) $\dim \mathsf{R}_\infty(S'_F)$ *is equal to ν if and only if* $\omega \in \operatorname{dom} {S'_F}^{\nu-1} \setminus \operatorname{dom} {S'_F}^{\nu}$;

(iii) $\mathsf{R}_\infty(S'_F)$ *is a regular subspace of dimension ν if and only if*

$$\omega \in \operatorname{dom} {S'_F}^{\nu-1} \setminus \operatorname{dom} {S'_F}^{\nu} \text{ and } [\omega,\omega_{\nu-1}] \ne 0 \text{ for } \{\omega,\omega_{\nu-1}\} \in {S'_F}^{\nu-1};$$

(iv) $\mathsf{R}_\infty(S'_F)$ *is a singular subspace of dimension ν if and only if*

$$\omega \in \operatorname{dom} {S'_F}^{\nu-1} \setminus \operatorname{dom} {S'_F}^{\nu} \text{ and } [\omega,\omega_{\nu-1}] = 0 \text{ for } \{\omega,\omega_{\nu-1}\} \in {S'_F}^{\nu-1}. \tag{4.16}$$

Proof. (i) Consider a proper factorization (4.7) for Q_∞ and the model operator $S(Q_\infty)$ constructed in Theorem 4.4 and denote by $A_0 = \ker \Gamma_0$ the selfadjoint extension $S_F(Q_\infty)$ in (4.11) of $S(Q_\infty)$ in (4.8). Clearly, the vectors

$$w_0 = \begin{pmatrix} 0 \\ e_k \\ 0 \end{pmatrix}, \dots, w_{k-2} = \begin{pmatrix} 0 \\ e_2 \\ 0 \end{pmatrix},\ w_{k-1} = \begin{pmatrix} 0 \\ e_1 \\ 0 \end{pmatrix} \tag{4.17}$$

form a Jordan chain in $\mathsf{R}_\infty(A_0)$, that is, $w_0 \in \operatorname{mul} A_0$ and $\{w_{j-1}, w_j\} \in A_0$ for $j = 1,\dots,k-1$. If $S(Q_\infty)$ is simple then $A_0 = S'_F$ and this proves (i). In the case when $S(Q_\infty)$ is not simple, but $\operatorname{mul} S(Q_\infty)$ is trivial the vectors $w_0,\dots,w_{k-1}$ still belong to $\mathsf{R}_\infty(S'_F)$ since for all $\alpha \in \sigma_p(A_0^0) \cap \sigma(q^\sharp)$ these vectors are orthogonal to the eigenspaces $\ker(S(Q_\infty) - \alpha)$ which were described in (4.14).

Assume that $\operatorname{mul} S(Q_\infty)$ is not trivial. Again it follows from Lemma 4.5 that the vectors $w_0,\dots,w_{k-1}$ are orthogonal to $\ker(S(Q_\infty) - \alpha)$ for all $\alpha \in \sigma_p(A_0^0) \cap \sigma(q^\sharp)$. Moreover, by using (4.13) in Lemma 4.5 it is seen that the vectors $w_0,\dots,w_{k-2}$ are orthogonal to $\operatorname{mul} S(Q_\infty)$. Since $\operatorname{mul} S(Q_\infty)[\perp]\ker(S(Q_\infty) - \alpha)$ for all $\alpha \in \sigma_p(A_0^0) \cap \sigma(q^\sharp)$, it is easy to check that the projections $w'_j = P' w_j$ of

w_j, $j = 1, \dots, k-1$, to the subspace $\mathfrak{H}' = \mathfrak{H} \ominus \mathfrak{H}''$, where $\mathfrak{H}''$ is given by (4.15), take the form

$$w_0' = w_0, \dots, w_{k-2}' = w_{k-2}, w_{k-1}' = \begin{pmatrix} -(\overline{\Gamma_0^0 \widehat{g}})g \\ e_1 \\ -|\Gamma_0^0 \widehat{g}|^2 e_k \end{pmatrix}, \tag{4.18}$$

where $g \in \operatorname{mul} A_1^0$, $[g, g] = 1$, $\widehat{g} = \{0, g\} \in A_1^0$, and $\Gamma_0^0 \widehat{g} \neq 0$. Therefore, the sequence $\{w_0', \dots, w_{k-1}'\}$ forms a Jordan chain in $\mathsf{R}_\infty(S_F')$ and hence again $\dim \mathsf{R}_\infty(S_F') \geq k$.

(ii) This statement holds true since S_F' has a simple spectrum.

(iii)&(iv) Let $\omega_0, \dots, \omega_{\nu-1}$ be a maximal chain in $\mathsf{R}_\infty(S_F')$. Since $\{0, \omega_0\} \in S_F'$ one obtains immediately $[\omega_0, \omega_j] = 0$ for all $j < \nu - 1$. Therefore, the subspace $\mathsf{R}_\infty(S_F')$ is regular if and only if $[\omega_0, \omega_{\nu-1}] \neq 0$. This proves (iii) and (iv). □

The explicit construction of a maximal Jordan chain $\{\omega_0, \dots, \omega_{\nu-1}\}$ spanning $\mathsf{R}_\infty(S_F')$ in the factorization model will be carried out by a continuation of the chains (4.17) and (4.18), which correspond to the cases $\operatorname{mul} S(Q_\infty) = \{0\}$ and $\operatorname{mul} S(Q_\infty) \neq \{0\}$, respectively. Observe, that by Lemma 4.5 $\operatorname{mul} S(Q_\infty)$ is nontrivial if and only if $\operatorname{mul} A_1^0$ is nontrivial. Moreover, since S_0 is simple at most one of the selfadjoint extensions A_0^0 or A_1^0 of S_0 can be multivalued.

Case I: $\operatorname{mul} S(Q_\infty) = \{0\}$. The proof of Lemma 4.7 shows that $\{\omega_0, \dots, \omega_{\nu-1}\}$ can be constructed as a continuation of the chain (4.17) if $\nu > k$ with $\omega_j = w_j$, $j = 0, \dots, k-1$. The formula (4.11) implies that the vector ω_k should be of the form $\omega_k = (g_0, f, \widetilde{f})^\top$, where

$$\widehat{g}_0 = \{0, g_0\} \in S_0^*, \quad \Gamma_1^0 \widehat{g}_0 = 1, \quad f = C_{q^\sharp} e_1 + \widetilde{\varphi} e_k, \quad \widetilde{f} = \Gamma_0^0 \widehat{g}_0 e_k.$$

By choosing $\widetilde{\varphi} e_k = -C_{q^\sharp} e_1$ one obtains

$$\omega_k = \begin{pmatrix} g_0 \\ 0 \\ \Gamma_0^0 \widehat{g}_0 e_k \end{pmatrix}, \quad \widehat{g}_0 = \{0, g_0\} \in S_0^*, \quad \Gamma_1^0 \widehat{g}_0 = 1. \tag{4.19}$$

Here $g_0 \neq 0$ if and only if $\omega_k \neq 0$. Therefore, one can continue the chain (4.17) if and only if $\operatorname{mul} S_0^*$ is nontrivial. This proves the following

Lemma 4.8. *If* $\operatorname{mul} S_0^* = \{0\}$*, then* $\mathsf{R}_\infty(S_F')$ *is isotropic and*

$$\dim \mathsf{R}_\infty(S_F') = \kappa_0(\mathsf{R}_\infty(S_F')) = k.$$

Now assume that $\operatorname{mul} S_0^*$ is nontrivial, so that one can continue the chain (4.17) by a vector $\omega_k \neq 0$ of the form (4.19). Here two further different cases can occur: either $\operatorname{mul} A_0^0$ is nontrivial, in which case $\operatorname{mul} A_0^0 = \operatorname{mul} S_0^*$, or $\operatorname{mul} A_0^0 = \{0\}$. These two cases give rise to two different types of maximal Jordan chains in $\mathsf{R}_\infty(S_F')$.

Lemma 4.9. *Let* $\operatorname{mul} A_0^0$ *be nontrivial. Then* $\mathsf{R}_\infty(S_F')$ *is regular if and only if* $\nu = 2k+1$. *If* $\mathsf{R}_\infty(S_F')$ *is singular then* $k+1 \le \nu \le 2k$ *and*

$$\kappa_0(\mathsf{R}_\infty(S_F')) = 2k+1-\nu, \quad \kappa_-(\mathsf{R}_\infty(S_F')) = \nu-k-1, \quad \kappa_+(\mathsf{R}_\infty(S_F')) = \nu-k. \tag{4.20}$$

Proof. Since $\operatorname{mul} A_0^0 \ne \{0\}$ one has $\Gamma_0^0 \widehat{g}_0 = 0$. It follows from (4.11) and (4.19) that the chain $\{\omega_k, \dots, \omega_{\nu-1}\}$ can be taken to be of the form

$$\omega_k = \begin{pmatrix} g_0 \\ 0 \\ 0 \end{pmatrix}, \omega_{k+1} = \begin{pmatrix} g_1 \\ 0 \\ \widetilde{f}_{(1)} \end{pmatrix}, \dots, \omega_{\nu-1} = \begin{pmatrix} g_{\nu-k-1} \\ 0 \\ \widetilde{f}_{(\nu-k-1)} \end{pmatrix}, \tag{4.21}$$

with $\{g_{j-1}, g_j\} \in A_1^0$ and

$$\widetilde{f}_{(j)} = a\widetilde{e}_{k-j+1} + \sum_{i=0}^{j-2} c_{j,i}\widetilde{e}_{k-i}, \quad c_{j,i} \in \mathbb{C}, \quad j = 1, \dots, \nu-k-1,$$

where

$$a := \Gamma_0^0\{g_0, g_1\} = [\omega_{k-1}, \omega_{k+1}] = [\omega_k, \omega_k] = [g_0, g_0] > 0,$$

since the vector $g_0 \in \operatorname{mul} A_0^0$ is positive in view of Lemma 4.5. There are two reasons for the chain to break: either $g_{\nu-k-1} \notin \operatorname{dom} A_1^0$, or one has $(\widetilde{f}_{(\nu-k-1)})_1 \ne 0$ in which case $\nu = 2k+1$. Since $[\omega_{\nu-1}, \omega_0] = (\widetilde{f}_{\nu-k-1})_1$, the subspace $\mathsf{R}_\infty(A_0)$ is regular if and only if $\nu = 2k+1$.

Observe, that the maximal chain $\{\omega_0, \dots, \omega_{\nu-1}\}$ in $\mathsf{R}_\infty(S_F)$ is also a maximal chain in $\mathsf{R}_\infty(S_F')$, because the vectors $\omega_j \in \mathsf{R}_\infty(S_F)$ $(j = 0, \dots, \nu-1)$ are orthogonal to the eigenspaces $\ker(S(Q_\infty) - \alpha)$ for $\alpha \in \sigma_p(A_0^0) \cap \sigma(q)$.

If $\mathsf{R}_\infty(S_F')$ is singular then $k+1 \le \nu \le 2k$. The isotropic subspace of $\mathsf{R}_\infty(S_F')$ is spanned by the vectors ω_j, $j = 0, \dots, 2k-\nu$, and

$$[\omega_{2k-\nu+1}, \omega_{\nu-1}] = [\omega_k, \omega_k] = [g_0, g_0] = a > 0.$$

Therefore, the Gram matrix of the chain $\omega_0, \dots, \omega_{\nu-1}$ is a Hankel matrix of the form

$$G = \begin{pmatrix} 0 & 0 \\ 0 & G_0 \end{pmatrix} \text{ with } G_0 = \begin{pmatrix} 0 & & a \\ & \iddots & \\ a & & * \end{pmatrix}, \tag{4.22}$$

where the left upper corner of G is the matrix $0_{2k+1-\nu}$. The other two equalities in (4.20) are implied by the structure of G_0 in (4.22), since $a > 0$. ☐

Lemma 4.10. *Let* $\operatorname{mul} A_0^0 = \operatorname{mul} A_1^0 = \{0\}$ *and let* $\operatorname{mul} S_0^* \ne \{0\}$. *Then* $\mathsf{R}_\infty(S_F')$ *is regular if and only if* $\nu = 2k$. *If* $\mathsf{R}_\infty(S_F')$ *is singular then* $k+1 \le \nu \le 2k-1$ *and*

$$\kappa_0(\mathsf{R}_\infty(S_F')) = 2k-\nu, \quad \kappa_-(\mathsf{R}_\infty(S_F')) = \kappa_+(\mathsf{R}_\infty(S_F')) = \nu-k. \tag{4.23}$$

Proof. By assumptions $\Gamma_0^0\widehat{g}_0 \neq 0$. In this case the chain $\{\omega_k, \dots, \omega_{\nu-1}\}$ takes the form

$$\omega_k = \begin{pmatrix} g_0 \\ 0 \\ a\widetilde{e}_k \end{pmatrix}, \omega_{k+1} = \begin{pmatrix} g_1 \\ 0 \\ \widetilde{f}_{(1)} \end{pmatrix}, \dots, \omega_{\nu-1} = \begin{pmatrix} g_{\nu-k-1} \\ 0 \\ \widetilde{f}_{(\nu-k-1)} \end{pmatrix}, \tag{4.24}$$

where $a := \Gamma_0^0\widehat{g}_0 \neq 0$, $\{g_{j-1}, g_j\} \in A_1^0$, and $\widetilde{f}_{(j)}$ are given by

$$\widetilde{f}_{(j)} = a\widetilde{e}_{k-j} + \sum_{i=0}^{j-1} c_{j,i}\widetilde{e}_{k-i}, \quad c_{j,i} \in \mathbb{C}, \quad j = 1, \dots, \nu - k - 1. \tag{4.25}$$

As in Lemma 4.9 it is seen from (4.24) that the subspace $\mathsf{R}_\infty(S_F')$ is regular if and only if $\nu = 2k$.

If $\mathsf{R}_\infty(S_F')$ is singular then $k+1 \leq \nu \leq 2k-1$. The isotropic subspace of $\mathsf{R}_\infty(S_F')$ is spanned by the vectors ω_j, $j = 0, \dots, 2k-1-\nu$, and

$$[\omega_{2k-\nu}, \omega_{\nu-1}] = [\omega_k, \omega_{k-1}] = a \neq 0.$$

Therefore, the Gram matrix of the chain $\omega_0, \dots, \omega_{\nu-1}$ takes the form (4.22), where the left upper corner of G is the matrix $0_{2k-\nu}$. The equalities in (4.23) are implied by the structure of G. □

Case II: $\operatorname{mul} S(Q_\infty) \neq \{0\}$. The proof of Lemma 4.7 shows that a maximal Jordan chain $\{\omega_0, \dots, \omega_{\nu-1}\}$ in $\mathsf{R}_\infty(S_F')$ can now be constructed as a continuation of the chain (4.18) with $\omega_j = w_j'$, $j = 0, \dots, k-1$.

Lemma 4.11. *Let* $\operatorname{mul} S(Q_\infty) \neq \{0\}$*. Then* $\mathsf{R}_\infty(S_F')$ *is regular if and only if* $\nu = 2k-1$*. If* $\mathsf{R}_\infty(S_F')$ *is singular then* $1 < k \leq \nu \leq 2k-2$ *and*

$$\kappa_0(\mathsf{R}_\infty(S_F')) = 2k-1-\nu, \quad \kappa_-(\mathsf{R}_\infty(S_F')) = \nu-k+1, \quad \kappa_+(\mathsf{R}_\infty(S_F')) = \nu-k. \tag{4.26}$$

Proof. If $k=1$ then $\omega_0 = (-(\overline{\Gamma_0^0\widehat{g}})g, 1, -|\Gamma_0^0\widehat{g}|^2)^\top$ and it follows from (4.11) that the chain (4.18) cannot be continued, in which case $\nu = k = 1$ and $[\omega_0, \omega_0] = -|\Gamma_0^0\widehat{g}|^2 < 0$.

Now let $k > 1$ and consider the continuation of the chain (4.18). According to (4.11) the condition $\{\omega_{k-1}, \omega_k\} \in A_0$ for some ω_k means that for some vector g_1 one has

$$\widehat{h} := \{-(\overline{\Gamma_0^0\widehat{g}})g, g_1\} \in S_0^*, \quad \Gamma_1^0\widehat{h} = 1, \tag{4.27}$$

where $\widehat{g} = \{0, g\} \in A_1^0$, $[g,g] = 1$, $\Gamma_0^0\widehat{g} \neq 0$. Observe, that here the conditions $\Gamma_1^0\widehat{h} = 1$ and $[g,g] = 1$ are equivalent, since by Green's identity (4.1)

$$-\Gamma_0^0\widehat{g}[g,g] = \Gamma_1^0\widehat{g}\overline{\Gamma_0^0\widehat{h}} - \Gamma_0^0\widehat{g}\overline{\Gamma_1^0\widehat{h}} = -\Gamma_0^0\widehat{g}\overline{\Gamma_1^0\widehat{h}}.$$

Observe also, that $g \notin \operatorname{dom} A_1^0$, since $g \in \operatorname{mul} A_1^0$ and the vector g is positive. Moreover, it follows from (4.11) that the continuation of the chain (4.18) can be taken to be of the form

$$\omega_k = \begin{pmatrix} g_1 \\ 0 \\ \widetilde{f}_{(1)} \end{pmatrix}, \dots, \omega_{\nu-1} = \begin{pmatrix} g_{\nu-k} \\ 0 \\ \widetilde{f}_{(\nu-k)} \end{pmatrix}, \tag{4.28}$$

where $\{g_{j-1}, g_j\} \in A_1^0$ for $j = 2, \dots, \nu - k$ and $\widetilde{f}_{(j)}$, $j = 1, \dots, \nu - k$, are given by (4.25), where

$$a := -|\Gamma_0^0 \widehat{g}|^2 = [\omega_{k-1}, \omega_{k-1}] < 0.$$

By (4.13) the vector ω_j is orthogonal to $\operatorname{mul} S(Q_\infty)$ for all $j = k, \dots, \nu - 2$. In addition, one can select $g_{\nu-k}$ in (4.28) such that $[g_{\nu-k}, g] = 0$ and then also $\omega_{\nu-1}$ is orthogonal to $\operatorname{mul} S(Q_\infty)$. Moreover, the vectors ω_j, $j = k, \dots, \nu - 1$, are orthogonal to the eigenspaces $\ker(S(Q_\infty) - \alpha)$, $\alpha \in \sigma_p(A_0^0) \cap \sigma(q)$. Hence (4.18) together with (4.28) forms a maximal Jordan chain in $\mathsf{R}_\infty(S'_F)$. Moreover, the subspace $\mathsf{R}_\infty(S'_F)$ is regular if and only if $\nu = 2k - 1$.

If $\mathsf{R}_\infty(S'_F)$ is singular then $k > 1$ and $k \le \nu \le 2k-2$, see (4.27). The isotropic subspace of $\mathsf{R}_\infty(S'_F)$ is spanned by the vectors ω_j, $j = 0, \dots, 2k - \nu - 2$, and

$$[\omega_{2k-\nu-1}, \omega_{\nu-1}] = [\omega_{k-1}, \omega_{k-1}] = a < 0.$$

Therefore, the Gram matrix of the chain $\omega_0, \dots, \omega_{\nu-1}$ is a Hankel matrix of the form (4.22), where the left upper corner of G is the matrix $0_{2k-1-\nu}$ and $a < 0$. This proves the equalities (4.26). □

The above considerations show that three different types of maximal Jordan chains in $\mathsf{R}_\infty(S'_F)$ can occur, cf. (4.21), (4.24), and (4.28). The longest Jordan chains appear in the first case, where $\operatorname{mul} A_0^0 \ne \{0\}$, see (4.21), while the shortest Jordan chains appear in the third case, where $\operatorname{mul} A_1^0 \ne \{0\}$, or equivalently, $\operatorname{mul} S(Q_\infty) \ne \{0\}$, see (4.28). The main properties associated with each of these maximal Jordan chains in Lemmas 4.8–4.11 are collected in the next theorem.

Theorem 4.12. *Let $Q_\infty \in \mathbf{N}_\kappa$, $\kappa_\infty(Q_\infty) = k > 0$, $\nu = \dim(\mathsf{R}_\infty(S'_F))$, and let S', S'_F be as in Theorem 4.6. Then one of the following three cases occurs:*

(i) *If $\operatorname{mul} A_0^0$ is nontrivial, then $k+1 \le \nu \le 2k+1$, $\kappa_-(\mathsf{R}_\infty(S'_F)) = \kappa_+(\mathsf{R}_\infty(S'_F)) - 1$, and $\kappa_0(\mathsf{R}_\infty(S'_F)) = 2k + 1 - \nu$. Moreover, $\mathsf{R}_\infty(S'_F)$ is singular if and only if $\nu \le 2k$.*

(ii) *If $\operatorname{mul} A_0^0 = \operatorname{mul} A_1^0 = \{0\}$, then $k \le \nu \le 2k$, $\kappa_-(\mathsf{R}_\infty(S'_F)) = \kappa_+(\mathsf{R}_\infty(S'_F))$, and $\kappa_0(\mathsf{R}_\infty(S'_F)) = 2k - \nu$. Moreover, $\mathsf{R}_\infty(S'_F)$ is singular if and only if $\nu \le 2k - 1$.*

(iii) *If $\operatorname{mul} A_1^0$ is nontrivial, then $k \le \nu \le 2k-1$, $\kappa_-(\mathsf{R}_\infty(S'_F)) = \kappa_+(\mathsf{R}_\infty(S'_F)) + 1$, and $\kappa_0(\mathsf{R}_\infty(S'_F)) = 2k - 1 - \nu$. Moreover, $\mathsf{R}_\infty(S'_F)$ is singular if and only if $\nu \le 2k - 2$ and $k > 1$.*

Proof. It is enough to prove the statement (ii). For this one combines the results in Lemma 4.8 and Lemma 4.10. Indeed, if $\operatorname{mul} S_0^* = \{0\}$ then by Lemma 4.8 one has $\nu = \dim \mathsf{R}_\infty(S'_F) = k \le 2k - 1$ and $\kappa_0(\mathsf{R}_\infty(S'_F)) = k = 2k - \nu$. □

5. Analytic characterizations of the root subspace at ∞

Let $Q_\infty \in \mathbf{N}_\kappa$ have a minimal operator representation (1.2) with $\gamma = 0$, where A is a selfadjoint operator in $\mathfrak{H}$, and let S and S_F be defined by (1.3) and (1.4), respectively. In Section 4 one such minimal operator representation was constructed in Theorem 4.6 by using the factorization (4.2) of Q_∞ in Theorem 4.4. By unitary equivalence general statements concerning S, A, and S_F in a minimal representation of the function Q_∞ can be obtained by considering the corresponding objects S', A', and S'_F in the model of Theorem 4.6. In this section the results concerning the root subspace of the generalized Friedrichs extension S_F in Section 4 are connected with the asymptotic expansions in Section 3. In particular, the connection between the classification of generalized poles and zeros introduced in Subsection 3.2 and the three different types of maximal Jordan chains in Section 4 is explained.

5.1. The root subspace of the generalized Friedrichs extension at ∞ and operator representations.

In order to establish the connection between the Jordan chains in the root subspace $\mathsf{R}_\infty(S_F)$ of the generalized Friedrichs extension S_F and the asymptotic expansions in Section 3 the following lemma will be useful.

Lemma 5.1. *Let A be a selfadjoint operator in a Pontryagin $\mathfrak{H}$, let $\omega \in \mathfrak{H}$, and let S and S_F be defined by* (1.3) *and* (1.4)*. Then:*

(i) $\operatorname{dom} S^n = \{\, f \in \operatorname{dom} A^n : [A^j f, \omega] = 0$ *for all* $j < n \,\}$, $n \in \mathbb{N}$*;*
(ii) $\omega \in \operatorname{dom} S^n$ *if and only if* $\omega \in \operatorname{dom} S_F^n$, $n \in \mathbb{N}$*;*
(iii) *if* $\omega_0, \omega_1, \dots, \omega_n$ *with* $\omega_0 = \omega$ *is a Jordan chain in* $\mathsf{R}_\infty(S_F)$ *such that*

$$[\omega_i, \omega_j] = 0 \quad \text{for all} \quad i + j < l \le 2n, \quad l \in \mathbb{Z}, \tag{5.1}$$

then

$$[S^i\omega, S^j\omega] = [\omega_i, \omega_j] \quad \text{for all} \quad i + j \le l \le 2n. \tag{5.2}$$

Proof. (i) If $f \in \operatorname{dom} S^n$, then the definition of S in (1.3) shows that $f \in \operatorname{dom} A^n$ and $[A^j f, \omega] = 0$ for all $j < n$, since $A^j f \in \operatorname{dom} S$ for all $j < n$ and ω is othogonal to $\operatorname{dom} S$. Conversely, assume that $f \in \operatorname{dom} A^n$ and $[A^j f, \omega] = 0$ for all $j < n$. Then the definition (1.3) shows that $A^j f \in \operatorname{dom} S$ for all $j < n$. Hence $f \in \operatorname{dom} S^n$.

(ii) Note that the condition $\omega \in \operatorname{dom} S_F^n$ means that there is a chain of vectors $\omega_0, \omega_1, \dots, \omega_n$, $\omega_0 = \omega$, such that $\{\omega_{j-1}, \omega_j\} \in S_F$ for $j \le n$. According to (1.4) this is equivalent to

$$\omega_j = S\omega_{j-1} + c_j\omega \quad \text{for some } c_j \in \mathbb{C}, \quad j = 1, \dots, n. \tag{5.3}$$

Hence, if $\omega \in \operatorname{dom} S_F^n$, then $\omega_j \in \operatorname{dom} S$ and $\omega_{j-1} \in \operatorname{dom} S^2$ for all $j < n$, and this leads to $\omega \in \operatorname{dom} S^n$. The reverse implication is also clear from (5.3).

(iii) It follows from (5.1) and (5.3) with $l \ge 1$ that

$$[\omega_0, \omega_1] = [\omega_0, S\omega_0].$$

Now, if $[S^i\omega_0, S^j\omega_0] = [\omega_i, \omega_j] = 0$ for all $i+j < m \le l$, then one obtains from (5.3) that

$$\begin{aligned}[\omega_{i+1}, \omega_j] &= [S^{i+1}\omega_0 + \sum_{\alpha=0}^{i} c_{i+1-\alpha} S^\alpha \omega_0, S^j\omega_0 + \sum_{\beta=0}^{j-1} c_{j-\beta} S^\beta \omega_0] \\ &= [S^{i+1}\omega_0, S^j\omega_0],\end{aligned}$$

which completes the proof. □

The statements (iii) and (iv) in Lemma 4.7 can now be reformulated in terms of the moments $s_j = [A^j\omega, \omega]$ of the operator A, cf. also [15].

Proposition 5.2. *Let S, S_F, $\omega \in \mathfrak{H}$, and $[\omega, \omega] \le 0$, be as in* (1.3) *and* (1.4)*, in a minimal representation of $Q_\infty \in \mathbf{N}_\kappa$ with $\kappa_\infty(Q_\infty) = k > 0$. Then:*

(i) $\mathsf{R}_\infty(S_F)$ *is a regular subspace of dimension ν if and only if*

$$\omega \in \operatorname{dom} A^{\nu-1}, \text{ and } s_{\nu-1} \neq 0,\ s_j = 0 \text{ for all } j < \nu - 1; \tag{5.4}$$

(ii) $\mathsf{R}_\infty(S_F)$ *is a singular subspace of dimension ν if and only if*

$$\omega \in \operatorname{dom} A^{\nu-1} \setminus \operatorname{dom} A^\nu, \text{ and } s_j = 0 \text{ for all } j \le \nu - 1. \tag{5.5}$$

Proof. (i) Assume that $\mathsf{R}_\infty(S_F)$ is a regular subspace of dimension ν. Then by Lemma 4.7 $\omega \in \operatorname{dom} S_F^{\nu-1} \setminus \operatorname{dom} S_F^\nu$ and by Lemma 5.1 $\omega \in \operatorname{dom} S^{\nu-1} \subset \operatorname{dom} A^{\nu-1}$. This implies that $s_j = [A^j\omega, \omega] = [S^j\omega, \omega] = 0$ for $j < \nu - 1$. Moreover, if $\{\omega, \omega_{\nu-1}\} \in S_F^{\nu-1}$ then

$$s_{\nu-1} = [A^{\nu-1}\omega, \omega] = [S^{\nu-1}\omega, \omega] = [\omega_{\nu-1}, \omega] \neq 0,$$

so that (5.4) follows.

Conversely, if (5.4) holds, then $\omega \in \operatorname{dom} S_F^{\nu-1}$ by Lemma 5.1. Moreover, for $\{\omega, \omega_{\nu-1}\} \in S_F^{\nu-1}$ one has

$$[\omega_{\nu-1}, \omega] = [A^{\nu-1}\omega, \omega] = s_{\nu-1} \neq 0,$$

so that $\omega \notin \operatorname{dom} S_F^\nu$. Hence, $\mathsf{R}_\infty(S_F)$ is a regular subspace of dimension ν.

(ii) Assume that $\mathsf{R}_\infty(S_F)$ is a singular subspace of dimension ν. Then it follows from Lemma 4.7 and Lemma 5.1 that $\omega \in \operatorname{dom} S^{\nu-1} \subset \operatorname{dom} A^{\nu-1}$ and $[A^j\omega, \omega] = [S^j\omega, \omega] = 0$ for all $j \le \nu - 1$. If $\omega \in \operatorname{dom} A^\nu$, then by Lemma 5.1 $\omega \in \operatorname{dom} S_F^\nu$, a contradiction to (4.16). Thus, $\omega \notin \operatorname{dom} A^\nu$ and (5.5) follows.

Conversely, assume that (5.5) holds. Then one has $\omega \in \operatorname{dom} S_F^{\nu-1} \setminus \operatorname{dom} S_F^\nu$ by Lemma 5.1. Moreover, by (5.2) one has

$$[\omega_{\nu-1}, \omega] = [A^{\nu-1}\omega, \omega] = s_{\nu-1} = 0.$$

Hence, (4.16) holds and $\mathsf{R}_\infty(S_F)$ is a singular subspace of dimension ν. □

5.2. Asymptotic expansions and the classification for the generalized zero (pole) ∞ in the model space.

The classification of generalized poles of Q_∞ or, equivalently, of the generalized zeros of $Q = -1/Q_\infty$ in Subsection 3.2 is now connected with the maximal Jordan chains constructed in Section 4. For this purpose observe that the assumptions of Theorem 4.12 can be expressed in terms of the Weyl function $Q_0(\lambda)$ of S_0 in the following equivalent form:

$$\operatorname{mul} A_0^0 = \{0\} \quad \Leftrightarrow \quad \lim_{z \widehat{\to} \infty} \frac{Q_0(z)}{z} = 0, \tag{5.6}$$

$$\operatorname{mul} A_1^0 = \{0\} \quad \Leftrightarrow \quad \lim_{z \widehat{\to} \infty} z Q_0(z) = \infty. \tag{5.7}$$

Now one can reformulate Theorem 4.12 in the form which makes clear the connection with the classification of generalized zeros and poles of nonpositive type introduced in Section 3.

Theorem 5.3. *Let $Q \in \mathbf{N}_\kappa$ have a minimal representation* (1.2) *and let S and S_F be defined by* (1.3) *and* (1.4). *Let ∞ be a generalized zero of negative type of Q with multiplicity $\pi_\infty(Q) = k > 0$ and let the root subspace $\mathsf{R}_\infty(S_F)$ be of dimension ν. Then $\mathsf{R}_\infty(S_F)$ is regular if and only if Q has an asymptotic expansion of the form*

$$Q(z) \sim -\frac{s_{\nu-1}}{z^\nu} - \cdots - \frac{s_{2\nu-2}}{z^{2\nu-1}} + o\left(\frac{1}{z^{2\nu-1}}\right), \quad z \widehat{\to} \infty, \quad s_{\nu-1} \neq 0. \tag{5.8}$$

Moreover, precisely one of the following three cases occurs:

(i) *If ∞ is a GZNT1, then $k+1 \le \nu \le 2k+1$ and Q has the asymptotic expansion*

$$Q(z) \sim -\frac{s_{2k}}{z^{2k+1}} - \cdots - \frac{s_{2\nu-2}}{z^{2\nu-1}} + o\left(\frac{1}{z^{2\nu-1}}\right), \quad z \widehat{\to} \infty, \tag{5.9}$$

where $s_{2k} > 0$. In this case $\kappa_-(\mathsf{R}_\infty(S_F)) = \kappa_+(\mathsf{R}_\infty(S_F)) - 1$, $\kappa_0(\mathsf{R}_\infty(S_F)) = 2k + 1 - \nu$, and $\mathsf{R}_\infty(S_F)$ is singular if and only if $\nu \le 2k$.

(ii) *If ∞ is a GZNT2, then $k \le \nu \le 2k$ and for $\nu \ge k+1$ Q has the asymptotic expansion*

$$Q(z) \sim -\frac{s_{2k-1}}{z^{2k}} - \cdots - \frac{s_{2\nu-2}}{z^{2\nu-1}} + o\left(\frac{1}{z^{2\nu-1}}\right), \quad z \widehat{\to} \infty, \tag{5.10}$$

where $s_{2k-1} \neq 0$, and for $\nu = k$ Q has the asymptotic expansion

$$Q(z) \sim o\left(\frac{1}{z^{2\nu-1}}\right), \quad z \widehat{\to} \infty. \tag{5.11}$$

In this case $\kappa_-(\mathsf{R}_\infty(S_F)) = \kappa_+(\mathsf{R}_\infty(S_F))$, $\kappa_0(\mathsf{R}_\infty(S_F)) = 2k-\nu$, and $\mathsf{R}_\infty(S_F)$ is singular if and only if $\nu \le 2k-1$.

(iii) *If ∞ is a GZNT3, then $k \le \nu \le 2k-1$ and Q has the asymptotic expansion*

$$Q(z) \sim -\frac{s_{2k-2}}{z^{2k-1}} - \cdots - \frac{s_{2\nu-2}}{z^{2\nu-1}} + o\left(\frac{1}{z^{2\nu-1}}\right), \quad z \widehat{\to} \infty, \tag{5.12}$$

where $s_{2k-2} < 0$. *In this case* $\kappa_-(\mathsf{R}_\infty(S_F)) = \kappa_+(\mathsf{R}_\infty(S_F))+1$, $\kappa_0(\mathsf{R}_\infty(S_F)) = 2k-1-\nu$, *and* $\mathsf{R}_\infty(S_F)$ *is singular if and only if* $\nu \leq 2k-2$ *and* $k>1$.

Proof. Since the root subspace $\mathsf{R}_\infty(S_F)$ is of dimension ν it follows from Proposition 5.2 that $\omega \in \operatorname{dom} A^{\nu-1}$. By Theorem 3.3 this means that $Q \in \mathbf{N}_{\kappa,-2(\nu-1)}$. Now Corollary 3.4 and (5.4) show that in the regular case the asymptotic expansion is of the form (5.8).

If the root subspace $\mathsf{R}_\infty(S_F)$ is singular then Proposition 5.2 and Theorem 3.3 yield $\omega \in \operatorname{dom} A^{\nu-1} \setminus \operatorname{dom} A^\nu$ and $Q \in \mathbf{N}_{\kappa,-2(\nu-1)} \setminus \mathbf{N}_{\kappa,-2\nu}$. Now consider the classification given in Subsection 3.1.

(i) Assume that ∞ is a GZNT1 of Q. Then by (T1) the following limit exists:

$$\lim_{z \widehat{\to} \infty} z^{2k+1} Q(z) < 0.$$

It follows from the factorization (4.7) that

$$\lim_{z \widehat{\to} \infty} \frac{Q_0(z)}{z} = \lim_{z \widehat{\to} \infty} \frac{Q_\infty(z)}{z^{2k+1}} = -\lim_{z \widehat{\to} \infty} \frac{1}{z^{2k+1}Q(z)} > 0.$$

According to (5.6) this means that $\operatorname{mul} A_0^0 \neq \{0\}$. The asymptotic expansion (5.9) is implied by (3.15) and Corollary 3.4. The remaining statements are obtained from part (i) of Theorem 4.12.

(ii) Assume that ∞ is a GZNT2 of Q. Then by (T2) one has

$$\lim_{z \widehat{\to} \infty} z^{2k+1} Q(z) = \infty, \quad \lim_{z \widehat{\to} \infty} z^{2k-1} Q(z) = 0.$$

It follows from the factorization (4.7) that

$$\lim_{z \widehat{\to} \infty} \frac{Q_0(z)}{z} = -\lim_{z \widehat{\to} \infty} \frac{1}{z^{2k+1}Q(z)} = 0, \quad \lim_{z \widehat{\to} \infty} zQ_0(z) = -\lim_{z \widehat{\to} \infty} \frac{1}{z^{2k-1}Q(z)} = \infty.$$

According to (5.6) and (5.7) this means that $\operatorname{mul} A_0^0 = \operatorname{mul} A_1^0 = \{0\}$. The asymptotic expansion (5.10) is implied by (3.17), (3.18), and Corollary 3.4, while the expansion (5.11) is obtained from (3.18), (3.19). The remaining statements are obtained from part (ii) of Theorem 4.12.

(iii) Finally, assume that ∞ is a GZNT3 of Q. Then by (T3) the following limit exists:

$$\lim_{z \widehat{\to} \infty} z^{2k-1} Q(z) > 0.$$

It follows from (4.7) that

$$\lim_{z \widehat{\to} \infty} zQ_0(z) = -\lim_{z \widehat{\to} \infty} \frac{1}{z^{2k-1}Q(z)} < 0,$$

and in view of (5.7) this means that $\operatorname{mul} A_1^0 \neq \{0\}$. The asymptotic expansion (5.12) is implied by (3.16) and Corollary 3.4, and the remaining statements are obtained from part (iii) of Theorem 4.12. □

The characterizations in Theorem 5.3 can be translated also for the generalized poles of nonpositive type of the function Q_∞ by means of the following theorem. In fact, this result is an extension of [6, Theorem 5.2] and can be seen to augment also the result stated in Theorem 3.3.

Theorem 5.4. *Let $Q \in \mathbf{N}_\kappa$, $Q \neq 0$, with $\lim_{z \widehat{\to} \infty} Q(z) = 0$ and let $Q_\infty = -1/Q$. Then the following statements are equivalent:*

(i) *$Q \in \mathbf{N}_{\kappa,-2n} \setminus \mathbf{N}_{\kappa,-2n-2}$ and $m \geq 0$ is the maximal integer such that $s_j = 0$ for all $j \leq m-1 (\leq 2n)$;*

(ii) *$Q(z) = [(A-z)^{-1}\omega, \omega]$, $z \in \rho(A)$, for some selfadjoint operator A in a Pontryagin space $\mathfrak{H}$ and a cyclic vector $\omega \in \operatorname{dom} A^n \setminus \operatorname{dom} A^{n+1}$ satisfying $[A^j\omega, A^i\omega] = 0$ for all $i + j \leq m - 1 (\leq 2n)$, $i, j \leq n$;*

(iii) *Q has an asymptotic expansion of the form*

$$Q(z) = -\frac{s_m}{z^{m+1}} - \cdots - \frac{s_{2n}}{z^{2n+1}} + o\left(\frac{1}{z^{2n+1}}\right), \quad z \widehat{\to} \infty, \tag{5.13}$$

where $m (\leq 2n+1)$ and n are maximal nonnegative integers, such that (5.13) holds;

(iv) *$Q_\infty = -1/Q$ has an asymptotic expansion of the form*

$$Q_\infty(z) = p_{m+1} z^{m+1} + \cdots + p_{2\ell+1} z^{2\ell+1} + o\left(z^{2\ell+1}\right), \quad z \widehat{\to} \infty, \tag{5.14}$$

where $p_{m+1} \neq 0$ if $m \geq 2\ell$ and $\ell \in \mathbb{Z}$ (with $2\ell \leq m+1$) is minimal such that (5.14) holds.

In this case the integers $m, n \geq 0$ and ℓ are connected by $\ell = m - n$. Moreover, in (5.13) $s_m \neq 0$ if and only if $p_{m+1} \neq 0$ in (5.14), in which case $p_{m+1} = 1/s_m$ and $m \leq 2n$ or, equivalently, $2\ell \leq m$.

Proof. (i) $\Leftrightarrow$ (ii) This equivalence follows from Theorem 3.3 and the formulas (3.10) for the moments s_j, $j \leq 2n$.

(i) $\Leftrightarrow$ (iii) If (i) holds then by Corollary 3.4 Q has an asymptotic expansion of the form (5.13) and maximality of n in this expansion follows from the assumption $Q \notin \mathbf{N}_{\kappa,-2n+2}$. The converse statement is also clear.

(iii) $\Leftrightarrow$ (iv) Assume that Q satisfies (5.13). If $m \leq 2n$ then $s_m \neq 0$. Otherwise $m = 2n+1$ and the expansion (5.13) reduces to

$$Q(z) = o\left(\frac{1}{z^{2n+1}}\right), \quad z \widehat{\to} \infty. \tag{5.15}$$

In the case that $m \leq 2n$, $s_m \neq 0$ and the expansion (5.13) can be rewritten in the form

$$z^{m+1} Q(z) = -s_m - \cdots - \frac{s_{2n}}{z^{2n-m}} + o\left(\frac{1}{z^{2n-m}}\right), \quad z \widehat{\to} \infty. \tag{5.16}$$

Since $Q_\infty = -1/Q$, by inverting the expansion (5.16) one concludes that the expansion (5.13) for Q with $s_m \neq 0$ is equivalent for Q_∞ to admit an expansion of the form

$$Q_\infty(z) = p_{m+1} z^{m+1} + \cdots + p_{2(m-n)+1} z^{2(m-n)+1} + o\left(z^{2(m-n)+1}\right), \quad z \widehat{\to} \infty,$$

where $p_{m+1} = 1/s_m \neq 0$. This means that Q_∞ has an asymptotic expansion of the form (5.14), where the integer $\ell = m - n$, $2\ell < m+1$, is minimal if n is maximal, and conversely. Hence the equivalence of (iii) and (iv) is shown in the case that $m \leq 2n$.

Next consider the case that $m = 2n+1$. Then Q satisfies (5.15) and this is an expansion of the form (5.13) with the maximal integers $n \geq 0$ and $m = 2n+1$. Since $Q_\infty = -1/Q$, one concludes that

$$Q_\infty(z) = o(z^{2n+3}), \quad z \widehat{\rightarrow} \infty,$$

so that Q_∞ has an expansion of the form (5.14) with $\ell := n+1 > 0$ and $m+1 = 2\ell$. Moreover, here $\ell = n+1$ is the minimal integer, such that Q_∞ admits an expansion of the form (5.14). Observe, that since $\ell = n+1$ is minimal, the maximal m in (5.13) is equal to $m = 2n+1$, so that the equality $\ell = m - n$ holds also in this case. In particular, (5.13) and (5.14) are still equivalent if $m = 2n+1$, in which case one can take $m + 1 = 2\ell$. □

To establish the classification of the asymptotic expansions for the function Q_∞ along the lines of Theorem 5.3 it is enough to consider expansions of the form (5.14) with $\ell = m - n \geq 0$ (so that in (5.13) $n \leq m$) and then apply the last statement of Theorem 5.3. In this case (5.14) takes the form

$$Q_\infty(z) = P(z) + o\left(z^{2\ell+1}\right), \quad z \widehat{\rightarrow} \infty,$$

where

$$P(z) := p_{m+1} z^{m+1} + \cdots + p_{2\ell+1} z^{2\ell+1}, \quad \ell \geq 0,$$

is a real polynomial of degree $\deg P = m+1$, whose leading coefficient is given by $p_{m+1} = 1/s_m$ if $\deg P > 2\ell$. If $m + 1 = 2\ell$, then one can take $P = 0$.

5.3. Asymptotic expansions and the index of singularity.

As another consequence of Theorem 5.4 some characterizations for the index of singularity of the generalized pole (zero) of nonpositive type of Q_∞ (of $Q = -1/Q_\infty$) at ∞ are given. The motivation for this notion is given in the end of this section.

Definition 5.5. Let ∞ be a generalized pole of Q_∞ (zero of Q) of nonpositive type of order $\nu\,(= \dim \mathsf{R}_\infty(S_F))$ and let $H_\nu = (s_{i+j})_{i,j=0}^{\nu-1}$ be the $\nu \times \nu$ Hankel matrix which is determined by the finite moments s_j, $0 \leq j \leq 2\nu - 2$, of Q. Then $\kappa_\infty^0 = \dim(\ker H_\nu)$ is called the *index of singularity* of Q_∞ at ∞.

Theorem 5.6. *Let n and m be maximal nonnegative integers with $n \leq m$, such that $Q \in \mathbf{N}_{\kappa,-2n}$ and $s_j = 0$ for all $j \leq m-1 (\leq 2n)$, and let S, A, S_F, ω be from the minimal operator representation* (1.2) *of Q. Then the following assertions are equivalent:*

- (i) *∞ is a generalized zero of nonpositive type of Q with the index of singularity equal to κ_∞^0;*
- (ii) *$\kappa_\infty^0 = m - n (\geq 0)$;*

(iii) $Q_\infty = -1/Q$ *admits an asymptotic expansion of the form*

$$Q_\infty(z) = P_{n+1}(z) + o(z^{2\kappa_\infty^0+1}), \quad z\widehat{\rightarrow}\infty, \tag{5.17}$$

where P_{n+1} *is a polynomial of degree* $n+1 \geq 2\kappa_\infty^0$ *and* $\kappa_\infty^0 \geq 0$ *is the minimal integer such that* (5.17) *holds;*

(iv) κ_∞^0 *is equal to the dimension of the isotropic subspace of* $\mathsf{R}_\infty(S_F)$.

Proof. (i) ⇔ (ii) It follows from Theorem 5.3 and the maximality of n and m, $n \leq m$, that $n = \nu - 1$, where $\nu = \dim \mathsf{R}_\infty(S_F)$, cf. [15, Theorem 5.2]. Since clearly $\dim(\ker H_\nu) = m - n (\geq 0)$ the equivalence of (i) and (ii) is shown.

(ii) ⇔ (iii) This follows immediately from Theorem 5.4 with $\ell = m - n = \kappa_\infty^0 \geq 0$. Here minimality of $\kappa_\infty^0 \geq 0$ in (5.17) is equivalent to the maximality of $n\,(= m - \kappa_\infty^0)$ in (5.13).

(ii) ⇔ (iv) To prove this equivalence the indices m and n are calculated in each of the cases (i)–(iii) in Theorem 5.3. In the case (i) one obtains from (5.9) $m = 2k$, $n = \nu - 1$. Hence

$$\kappa_\infty^0 = m - n = 2k - \nu + 1 = \kappa_0(\mathsf{R}_\infty(S_F)).$$

Similarly in the case (ii) for both expansions (5.10) and (5.11) one has $m = 2k - 1$, $n = \nu - 1$. Thus

$$\kappa_\infty^0 = m - n = 2k - \nu = \kappa_0(\mathsf{R}_\infty(S_F)).$$

Finally in the case (iii) $m = 2k - 2$, $n = \nu - 1$, and

$$\kappa_\infty^0 = m - n = 2k - 1 - \nu = \kappa_0(\mathsf{R}_\infty(S_F)).$$

This completes the proof. □

The equivalence of (i) and (iv) in Theorem 5.6 is also a direct consequence of [15, Corollary 4.4]. From Theorem 5.6 one obtains immediately the following characterization for the regularity of a critical point.

Corollary 5.7. ([6, Theorem 4.1], [17, Proposition 1.6]) *The root* $\mathsf{R}_\infty(S_F)$ *is nondegenerate (equivalently* ∞ *is a regular critical point of* S_F*) if and only if* Q_∞ *admits the representation*

$$Q_\infty(z) = P(z) + Q_0(z), \quad \text{where } Q_0(z) = o(z), \quad z\widehat{\rightarrow}\infty.$$

Proof. This is immediate from the equivalence of (iii) and (iv) in Theorem 5.6. □

The index κ_∞^0 measures the degree of singularity of the singular critical point ∞ of S_F. The characterization of κ_∞^0 via the asymptotic expansion (5.17) in Theorem 5.6 is particularly appealing: it extends the result stated in Corollary 5.7 to the case of *singular critical points* in an explicit manner.

6. Spectral characterizations via the underlying Weyl functions

In this section the structure of the underlying root subspace corresponding to the three different types of maximal Jordan chains constructed in Section 4 is studied by means of the factorized integral representations of the underlying Weyl functions. First detailed results are presented for the point ∞. Then the case of finite generalized zeros and poles of nonpositive type is treated briefly. Furthermore, it is shown how the classification of all generalized zeros and poles of $Q \in \mathbf{N}_\kappa$ belonging to $\mathbb{R} \cup \{\infty\}$ can be applied in establishing analytic criteria for the minimality of (not necessarily canonical) factorization models of $\mathbf{N}_\kappa$-functions.

6.1. The classification of generalized zeros and poles at ∞.

The canonical factorization of generalized Nevanlinna functions in (1.6) implies that $Q(z) = -1/Q_\infty(z)$ has the following integral representation:

$$Q(z) = -1/Q_\infty(z) = \frac{\widetilde{p}(z)\widetilde{p}^\sharp(z)}{\widetilde{q}(z)\widetilde{q}^\sharp(z)} \left(a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\rho(t) \right), \tag{6.1}$$

where $\widetilde{p}$ and $\widetilde{q}$ are as in (2.5), $a \in \mathbb{R}$, $b \geq 0$, and $\rho(t)$ satisfies

$$\int_{\mathbb{R}} \frac{d\rho(t)}{t^2+1} < \infty. \tag{6.2}$$

In the case that $\int_{\mathbb{R}} d\rho(t) < \infty$, denote $a_0 = a - \int_{\mathbb{R}} t/(t^2+1)\, d\rho(t) \in \mathbb{R}$.

In the next theorem the classification of the Jordan chains is characterized via the spectral properties of the function Q using the factorized integral representation (6.1).

Theorem 6.1. *Let $Q_\infty \in \mathbf{N}_\kappa$, let $k = \kappa_\infty(Q_\infty) > 0$, and let $Q(z) = -1/Q_\infty(z)$ have the factorized integral representation* (6.1)*. Then:*

(i) *the GZNT ∞ of Q is regular and of type (T1) if and only if*

$$b = a_0 = 0, \quad \int_{\mathbb{R}} d\rho(t) > 0, \ \text{and} \ \int_{\mathbb{R}} (1+|t|)^{2k} d\rho(t) < \infty; \tag{6.3}$$

(ii) *the GZNT ∞ of Q is regular and of type (T2) if and only if*

$$b = 0, \ a_0 \neq 0, \ \text{and} \ \int_{\mathbb{R}} (1+|t|)^{2(k-1)} d\rho(t) < \infty; \tag{6.4}$$

(iii) *the GZNT ∞ of Q is regular and of type (T3) if and only if*

$$b > 0 \ \text{and} \ \int_{\mathbb{R}} (1+|t|)^{2(k-2)} d\rho(t) < \infty; \tag{6.5}$$

(iv) *the GZNT ∞ of Q is singular and of type (T1) with the index of singularity $\kappa_\infty^0 (> 0)$ if and only if $b = a_0 = 0$ and*

$$\int_{\mathbb{R}} (1+|t|)^{2(k-\kappa_\infty^0)} d\rho(t) < \infty, \quad \int_{\mathbb{R}} (1+|t|)^{2(k-\kappa_\infty^0+1)} d\rho(t) = \infty; \tag{6.6}$$

(v) *the GZNT* ∞ *of* Q *is singular of type (T2) with the index of singularity* $(0<)\kappa_\infty^0<k$ *if and only if* $b=0$, $a_0\neq 0$, *and*

$$\int_{\mathbb{R}}(1+|t|)^{2(k-1-\kappa_\infty^0)}d\rho(t)<\infty,\quad \int_{\mathbb{R}}(1+|t|)^{2(k-\kappa_\infty^0)}d\rho(t)=\infty, \tag{6.7}$$

and with the index of singularity $\kappa_\infty^0=k(>0)$ *if and only if* $b=0$ *and*

$$\int_{\mathbb{R}}d\rho(t)=\infty; \tag{6.8}$$

(vi) *the GZNT* ∞ *of* Q *is singular and of type (T3) with the index of singularity* $\kappa_\infty^0(>0)$ *if and only if* $b>0$ *and*

$$\int_{\mathbb{R}}(1+|t|)^{2(k-2-\kappa_\infty^0)}d\rho(t)<\infty,\quad \int_{\mathbb{R}}(1+|t|)^{2(k-1-\kappa_\infty^0)}d\rho(t)=\infty. \tag{6.9}$$

Proof. By Proposition 5.2 the root subspace $\mathsf{R}_\infty(S_F)$ is regular (singular) of dimension ν if and only if (5.4) holds (respectively (5.5) holds). According to Theorem 3.3 the condition $\omega\in\operatorname{dom}A^{\nu-1}$ is equivalent to the condition

$$\int_{\mathbb{R}}(1+|t|)^{2(\nu-k-1)}d\rho(t)<\infty. \tag{6.10}$$

This leads to the integrability conditions in (6.3)–(6.5) in the regular case and to the integrability conditions (6.7)–(6.9) in the singular case, see Theorem 4.12.

It follows from the expansions (5.9)–(5.12) of Q in Theorem 5.3 and the factorized integral representation of Q in (6.1) that

$$s_j=0 \text{ for } j<2k-2,\quad s_{2k-2}(Q)=-b, \tag{6.11}$$

and moreover, that

$$\text{if } \int_{\mathbb{R}}d\rho(t)<\infty \text{ and } b=0,\qquad \text{then } s_{2k-1}=-a_0, \tag{6.12}$$

and

$$\text{if } \int_{\mathbb{R}}d\rho(t)<\infty \text{ and } b=a_0=0,\qquad \text{then } s_{2k}=\int_{\mathbb{R}}d\rho(t). \tag{6.13}$$

All the statements of the theorem can now be obtained from Theorem 5.3 by combining Proposition 5.2 with (6.10)–(6.13). $\square$

Observe that the regular cases (i)–(iii) are obtained from the singular cases (iv)–(vi) by taking $\kappa_\infty^0=0$ and excluding the second condition in (6.6), (6.7), and (6.9), respectively. All of the conditions in Theorem 6.1 are based on the canonical factorization of $Q=rr^\sharp Q_{00}$ in (6.1) involving the ordinary Nevanlinna function $Q_{00}\in\mathbf{N}_0$ in (2.5). In the factorization model of Theorem 4.4 the factor Q_0 belongs to the class $\mathbf{N}_{\kappa-k}$, where $k=\kappa_\infty(Q_\infty)$. The classification of Jordan chains was described in Theorem 4.12 by means of the selfadjoint extensions A_0^0 and A_1^0 of S_0 in the Pontryagin space $\mathfrak{H}_0$ whose negative index is equal to $\kappa(Q_0)=\kappa(Q_\infty)-\kappa_\infty(Q_\infty)$. Analogous descriptions remain true also for the canonical factorization of Q_∞.

Proposition 6.2. *Let $Q = rr^\sharp Q_{00}$ be the canonical factorization of $Q \in \mathbf{N}_\kappa$ in* (6.1) *with $k = \pi_\infty(Q) > 0$, let S_{00} be a simple symmetric operator in a Hilbert space $\mathfrak{H}_{00}$ with a boundary triplet $\Pi_{00} = \{\mathbb{C}, \Gamma_0^{00}, \Gamma_1^{00}\}$ whose Weyl function is equal to Q_{00}, and let $A_0^{00} = \ker \Gamma_0^{00}$ and $A_1^{00} = \ker \Gamma_1^{00}$ be the corresponding selfadjoint extensions of S_{00} in $\mathfrak{H}_{00}$. Then:*

(i) *the GZNT ∞ of Q is of type* (T1) *if and only if* $\operatorname{mul} A_1^{00} \neq \{0\}$*;*
(ii) *the GZNT ∞ of Q is of type* (T2) *if and only if* $\operatorname{mul} A_0^{00} = \operatorname{mul} A_1^{00} = \{0\}$*;*
(iii) *the GZNT ∞ of Q is of type* (T3) *if and only if* $\operatorname{mul} A_0^{00} \neq \{0\}$.

Proof. The identities (6.11) and (6.12) concern the function Q_{00}. The conditions which describe GZNT ∞ of type (T1) are $b = a_0 = 0$ and $\int_{\mathbb{R}} d\rho(t) < \infty$. These conditions are equivalent to $-\lim_{z \widehat{\to} \infty} zQ_{00}(z) < \infty$, which holds if and only if

$$-\lim_{z \widehat{\to} \infty} \frac{1}{zQ_{00}(z)} > 0 \quad \Leftrightarrow \quad \operatorname{mul} A_1^{00} \neq \{0\},$$

where the last equivalence follows from the simplicity of the operator S_{00} in $\mathfrak{H}_{00}$. Similarly, for type (T2) one has the conditions $b = 0$ and $a_0 \neq 0$ if $\int_{\mathbb{R}} d\rho(t) < \infty$, or the conditions $b = 0$ and $\int_{\mathbb{R}} d\rho(t) = \infty$. These conditions are equivalent to

$$\lim_{z \widehat{\to} \infty} \frac{Q_{00}(z)}{z} = 0, \quad \lim_{z \widehat{\to} \infty} zQ_{00} = \infty \Leftrightarrow \quad \operatorname{mul} A_0^{00} = \operatorname{mul} A_1^{00} = \{0\}.$$

Finally, for type (T3) one has the condition $b > 0$ which is equivalent to

$$\lim_{z \widehat{\to} \infty} \frac{Q_{00}(z)}{z} > 0 \quad \Leftrightarrow \quad \operatorname{mul} A_0^{00} \neq \{0\},$$

which completes the proof. □

The spectral theoretic characterization in Theorem 6.1 was based on the canonical factorization of the function $Q = -1/Q_\infty$. Since ∞ is a GPNT of the function Q_∞ it is natural to translate this result for the canonical factorization of Q_∞:

$$Q_\infty(z) = \frac{\widetilde{q}(z)\widetilde{q}^\sharp(z)}{\widetilde{p}(z)\widetilde{p}^\sharp(z)} \left(a_\infty + b_\infty z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\sigma_\infty(t) \right), \tag{6.14}$$

where $\widetilde{p}$, $\widetilde{q}$ are as in (6.1), $a_\infty \in \mathbb{R}$, $b_\infty \geq 0$, and $\sigma_\infty(t)$ satisfies the analog of (6.2). In the case that $\int_{\mathbb{R}} d\sigma_\infty(t) < \infty$, denote $\gamma_\infty = a_\infty - \int_{\mathbb{R}} t/(t^2+1)\, d\sigma_\infty(t) \in \mathbb{R}$.

Theorem 6.3. *Let $Q_\infty \in \mathbf{N}_\kappa$ with $k = \kappa_\infty(Q_\infty) > 0$ have the factorized integral representation* (6.14)*. Then:*

(i) *the GPNT ∞ of Q_∞ is regular and of type (T1) if and only if $b_\infty > 0$ and*
$$\int_{\mathbb{R}} (1+|t|)^{2(k-1)} d\sigma_\infty(t) < \infty; \tag{6.15}$$
(ii) *the GPNT ∞ of Q_∞ is regular and of type (T2) if and only if $b_\infty = 0$, $\gamma_\infty \neq 0$, and the integrability condition* (6.15) *is satisfied;*
(iii) *the GPNT ∞ of Q_∞ is regular and of type (T3) if and only if $b_\infty = \gamma_\infty = 0$, $\int_{\mathbb{R}} d\rho(t) > 0$, and the integrability condition* (6.15) *is satisfied;*

(iv) *the GPNT* ∞ *of* Q_∞ *is singular and of type (T1) with the index of singularity* $\kappa_\infty^0(>0)$ *if and only if* $b_\infty > 0$ *and*

$$\int_{\mathbb{R}} (1+|t|)^{2(k-1-\kappa_\infty^0)} d\sigma_\infty(t) < \infty, \quad \int_{\mathbb{R}} (1+|t|)^{2(k-\kappa_\infty^0)} d\sigma_\infty(t) = \infty; \tag{6.16}$$

(v) *the GPNT* ∞ *of* Q_∞ *is singular and of type (T2) with the index of singularity* $(0<)\kappa_\infty^0 < k$ *if and only if* $b_\infty = 0$, $\gamma_\infty \neq 0$, *and the integrability conditions* (6.16) *are satisfied, and with the index of singularity* $\kappa_\infty^0 = k(>0)$ *if and only if* $b_\infty = 0$ *and* $\int_{\mathbb{R}} d\sigma_\infty(t) = \infty$;

(vi) *the GPNT* ∞ *of* Q_∞ *is singular and of type (T3) with the index of singularity* $\kappa_\infty^0(>0)$ *if and only if* $b_\infty = \gamma_\infty = 0$, *and the integrability conditions* (6.16) *are satisfied.*

Proof. In each case the conditions concerning the parameters b_∞, a_∞, and γ_∞ are immediate from Proposition 6.2. It remains to establish the integrability conditions for $\sigma_\infty(t)$.

First observe that the integrability conditions for $\rho(t)$ in Theorem 6.1 concern the function Q_{00} in (2.5), while the integrability conditions for $\sigma_\infty(t)$ concern the function $-1/Q_{00}$. Now if the GPNT ∞ is of type (T1) then due to the conditions $b = a_0 = 0$ the measure $d\rho(t)$ has two more finite moments than the measure $d\sigma_\infty(t)$. Therefore, the integrability conditions in (6.3) and (6.6) are equivalent to those in (6.15) and (6.16), respectively; cf. [16, Theorem 4.2]. Similarly, due to $b_\infty = \gamma_\infty = 0$, the integrability conditions in (6.5) and (6.9) are equivalent to those in (6.15) and (6.16), respectively. Moreover, by [16, Theorem 4.2] the integrability conditions in (6.4) and (6.7) are equivalent to those in (6.15) and (6.16), respectively, while the conditions $b = 0$ and (6.8) are clearly equivalent to the conditions $b_\infty = 0$ and $\int_{\mathbb{R}} d\sigma_\infty(t) = \infty$. □

In the case that $\kappa(Q_\infty) = \kappa_\infty(Q_\infty) = 1$ the result in Theorem 6.3 simplifies to [8, Theorem 4.1]. Observe also that if the function $\sigma_\infty(t)$ has a compact support in $\mathbb{R}$, then (6.16) shows that ∞ cannot be a singular critical point of S_F. Moreover, if $\int_{\mathbb{R}} d\sigma_\infty(t) = 0$ and $\kappa_\infty(Q_\infty) > 0$, then only the cases (i) and (ii) in Theorem 6.3 can occur.

6.2. The classification of generalized zeros and poles in $\mathbb{R}$.

The classification of generalized zeros and poles of nonpositive type has been studied in detail at the point $z = \infty$. Similar results hold true also for finite generalized zeros and poles of nonpositive type of $Q \in \mathbf{N}_\kappa$ which belong to $\mathbb{R}$. Here some main characterizations for the classification of finite generalized zeros $\beta \in \mathbb{R}$ and finite generalized poles $\alpha \in \mathbb{R}$ of Q are presented.

First the classification of zeros and poles is characterized by means of the canonical factorization of $Q \in \mathbf{N}_\kappa$.

Lemma 6.4. *Let* $Q \in \mathbf{N}_\kappa$, $\kappa > 0$, *be factorized as in* (6.1). *Then the types (T1)–(T3) of a generalized zero* $\beta \in \mathbb{R}$ *of* Q *are characterized as follows:*

(i) *the GZNT* $\beta \in \mathbb{R}$ *of* Q *is of type (T1) if and only if* $\beta \in \sigma_p(A_1^{00})$;

(ii) *the GZNT* $\beta \in \mathbb{R}$ *of* Q *is of type (T2) if and only if* $\beta \notin \sigma_p(A_0^{00}) \cup \sigma_p(A_1^{00})$;
(iii) *the GZNT* $\beta \in \mathbb{R}$ *of* Q *is of type (T3) if and only if* $\beta \in \sigma_p(A_0^{00})$.

Moreover, the types (T1)–(T3) of a generalized pole $\alpha \in \mathbb{R}$ *of* Q *are characterized as follows:*

(iv) *the GPNT* $\alpha \in \mathbb{R}$ *of* Q *is of type (T1) if and only if* $\alpha \in \sigma_p(A_0^{00})$;
(v) *the GPNT* $\alpha \in \mathbb{R}$ *of* Q *is of type (T2) if and only if* $\alpha \notin \sigma_p(A_0^{00}) \cup \sigma_p(A_1^{00})$;
(vi) *the GPNT* $\alpha \in \mathbb{R}$ *of* Q *is of type (T3) if and only if* $\alpha \in \sigma_p(A_1^{00})$.

Proof. (i)–(iii) In view of the canonical factorization of Q in (6.1) one obtains the following representation for the limits in (2.3):

$$\lim_{z \hat{\to} \beta} \frac{Q(z)}{(z-\beta)^{2\pi_\beta - 1}} = \lim_{z \hat{\to} \beta} (z-\beta) Q_{00}(z) = -(\rho(\beta+) - \rho(\beta-)) \, (\leq 0) \tag{6.17}$$

and if $\int_{\mathbb{R}} \frac{d\rho(t)}{(t-\beta)^2} < \infty$ and $\lim_{z \hat{\to} \beta} Q_{00}(z) = 0$, then

$$\lim_{z \hat{\to} \beta} \frac{Q(z)}{(z-\beta)^{2\pi_\beta + 1}} = \lim_{z \hat{\to} \beta} \frac{Q_{00}(z)}{(z-\beta)} = b + \int_{\mathbb{R}} \frac{d\rho(t)}{(t-\beta)^2} \, (> 0), \tag{6.18}$$

and otherwise the limit in (6.18) is not finite. Observe that the limit in (6.17) is negative if and only if $\beta \in \sigma_p(A_0^{00})$. The limit in (6.18) is finite if and only if for the function $-1/Q_{00}$ the limit in (6.17) is negative, which is equivalent to $\beta \in \sigma_p(A_1^{00})$. The statements (i)–(iii) are now obvious from the defining properties of the classifications given in Subsection 3.2.

(iv)–(vi) Apply the characterizations of generalized zeros in the first part of the lemma to the function $-1/Q$. □

Next some characteristic properties of the underlying root subspaces associated with the classification of generalized zeros $\beta \in \mathbb{R}$ (generalized poles $\alpha \in \mathbb{R}$) are established in a minimal representation of Q (of $-1/Q$, respectively).

Let $S(Q)$ be a simple symmetric operator in a Pontryagin space $\mathfrak{H}$, let $\{\Gamma_0, \Gamma_1, \mathcal{H}\}$ be a boundary triplet for S^* such that the corresponding Weyl function is the given $\mathbf{N}_\kappa$-function Q, and let $A(Q) = \ker \Gamma_0$ and $A(-1/Q) = \ker \Gamma_1$. Moreover, let $\mathsf{R}_\alpha(A(Q))$ and $\mathsf{R}_\beta(A(-1/Q))$ be the root subspaces of the selfadjoint extension $A(Q)$ and $A(-1/Q)$ of $S(Q)$ associated with the generalized pole α and the generalized zero $\beta \in \mathbb{R}$ of Q, respectively. The following result is analogous to Theorem 4.12. For simplicity, the classification of a GPNT $\alpha \in \mathbb{R}$ and a GZNT $\beta \in \mathbb{R}$ of Q is characterized here by using the signature of the corresponding root subspace. Here the following notations will be used:

$$\begin{aligned}
&\nu_\alpha := \dim \mathsf{R}_\alpha(A(Q)), \quad \nu_\beta := \dim \mathsf{R}_\beta(A(-1/Q)), \\
&\kappa_\pm^\alpha := \kappa_\pm(\mathsf{R}_\alpha(A(Q))), \quad \kappa_\pm^\beta := \kappa_\pm(\mathsf{R}_\beta(A(-1/Q))), \\
&\kappa_0^\alpha := \kappa_0(\mathsf{R}_\alpha(A(Q))), \quad \kappa_0^\beta := \kappa_\pm(\mathsf{R}_\beta(A(-1/Q))).
\end{aligned}$$

Proposition 6.5. *With the notations given above the following assertions hold for a GPNT* $\alpha \in \mathbb{R}$ *and a GZNT* $\beta \in \mathbb{R}$ *of* $Q \in \mathbf{N}_\kappa$*:*

(i) *α is of type (T1) if and only if $\kappa_-^\alpha = \kappa_+^\alpha - 1$, in this case $\kappa_\alpha + 1 \le \nu_\alpha \le 2\kappa_\alpha + 1$ and $\kappa_0^\alpha = 2\kappa_\alpha + 1 - \nu_\alpha$;*
(ii) *α is of type (T2) if and only if $\kappa_-^\alpha = \kappa_+^\alpha$, in this case $\kappa_\alpha \le \nu_\alpha \le 2\kappa_\alpha$ and $\kappa_0^\alpha = 2\kappa_\alpha - \nu_\alpha$;*
(iii) *α is of type (T3) if and only if $\kappa_-^\alpha = \kappa_+^\alpha + 1$, in this case $\kappa_\alpha \le \nu_\alpha \le 2\kappa_\alpha - 1$ and $\kappa_0^\alpha = 2\kappa_\alpha - 1 - \nu_\alpha$;*
(iv) *β is of type (T1) if and only if $\kappa_-^\beta = \kappa_+^\beta - 1$, in this case $\pi_\beta + 1 \le \nu_\beta \le 2\pi_\beta + 1$ and $\nu_0^\beta = 2\pi_\beta + 1 - \nu_\beta$;*
(v) *β is of type (T2) if and only if $\kappa_-^\beta = \kappa_+^\beta$, in this case $\pi_\beta \le \nu_\beta \le 2\pi_\beta$ and $\nu_0^\beta = 2\pi_\beta - \nu_\beta$;*
(vi) *β is of type (T3) if and only if $\kappa_-^\beta = \kappa_+^\beta + 1$, in this case $\pi_\beta \le \nu_\beta \le 2\pi_\beta - 1$ and $\nu_0^\beta = 2\pi_\beta - 1 - \nu_\beta$.*

Proof. The statements (i)–(iii) are obtained from Theorem 4.12 by considering the transform $Q_\infty(z) := -Q(\alpha + 1/z)$. Namely, the root subspace $\mathsf{R}_\alpha(A(Q))$ associated with the generalized pole α of Q coincides with the root subspace $\mathsf{R}_\infty(S'_F)$ associated with the generalized pole ∞ of the function Q_∞.

The statements (iv)–(vi) follow by applying the results in (i)–(iii) to the function $-1/Q$. □

The classification for a GZNT $\beta \in \mathbb{R}$ and a GPNT α of Q can be characterized also via asymptotic expansions of the function Q in a neighborhood of these points. The defining properties of the types (T1)–(T3) are reflected in these expansions in a similar manner as in Theorems 5.3 and 5.4 above. Such expansions have been studied in [15] and, for instance, the analog of Theorem 5.3 for a GZNT $\beta \in \mathbb{R}$ of Q can be easily derived from [15, Theorem 5.2] and for a GPNT α of Q form [15, Theorem 5.4]. Moreover, these expansions can be characterized also via the canonical factorization of Q along the lines of Theorems 6.1 and 6.3 by using Lemma 6.4; see also [15, Section 6]. A detailed formulation of these results is left for the reader.

6.3. Analytic criteria for the minimality of (localized) factorization models.

The classification of generalized zeros and poles of nonpositive type of $Q \in \mathbf{N}_\kappa$ can be used to give analytic criteria for general (localized) factorizations models based on (proper) factorizations of Q of the form

$$Q(z) = \widehat{r}(z)\widehat{r}^\sharp(z)\widehat{Q}_0, \quad \widehat{r} = \frac{\widehat{p}}{\widehat{q}}, \tag{6.19}$$

where $\widehat{p}$ and $\widehat{q}$ are divisors of the polynomials $\widetilde{p}$ and $\widetilde{q}$ in the canonical factorization of Q in (6.1), with multiplicity of a zero equal to its original multiplicity. Such factorization models can be used, for instance, in studying *local spectral properties* of the function $Q \in \mathbf{N}_\kappa$, along the lines carried out in the previous sections of the present paper at ∞ with the aid of the model in Theorem 4.6 for proper factorizations of Q at ∞.

The basic observation here is that according to Lemma 4.11 the occurrence of a maximal Jordan chain of type (T3) means that $\operatorname{mul} S(Q_\infty) \neq \{0\}$. Therefore, the existence of such a Jordan chain is connected with the non-minimality of the factorization model of Q. The next result is formulated for the canonical factorization model as constructed in [3, Theorem 3.3]; by unitary equivalence the result holds for all other canonical factorization models, too; cf. [12].

Proposition 6.6. *The canonical factorization model constructed for $Q \in \mathbf{N}_\kappa$ in* [3, Theorem 3.3] *is simple if and only if all generalized zeros $\beta \in \mathbb{R} \cup \{\infty\}$ and generalized poles $\alpha \in \mathbb{R} \cup \{\infty\}$ of Q of nonpositive type are either of type (T1) or of type (T2).*

Proof. By Lemma 6.4 a real GZNT β (a real GPNT α) of Q is not of type (T3) if and only if $\beta \notin \sigma_p(A_0^{00})$ (respectively $\alpha \notin \sigma_p(A_1^{00})$). Moreover, by Proposition 6.2 the GZNT $\beta = \infty$ (the GPNT $\alpha = \infty$) is not of type (T3) if and only if $\operatorname{mul} A_0^{00} = \{0\}$ (respectively, $\operatorname{mul} A_1^{00} = \{0\}$). Now, according to [3, Theorem 4.1], these conditions characterize the simplicity of the corresponding factorization model for Q. □

These observations lead to the following analytic characterization for the simplicity of the factorization model in Theorem 4.4 ([9, Theorem 4.2]) which is based on a proper factorization of Q_∞ at ∞ (see Definition 4.2).

Proposition 6.7. *Let $Q_\infty \in \mathbf{N}_\kappa$ with $k = \kappa_\infty(Q_\infty) > 0$ and let $Q_\infty = qq^\sharp Q_0$ be a proper factorization of Q_∞ with some monic polynomial q, $\deg q = k$. Moreover, let the symmetric operator S_0 be simple in the Pontryagin space $\mathfrak{H}_0$ (see Lemma 4.5). Then the factorization model for Q_∞ in Theorem 4.4 is minimal if and only if the following two conditions are satisfied:*

(1) *$\alpha = \infty$ is not a GPNT of Q_∞ of type (T3);*
(2) *all the real zeros of q as GZNT of Q_∞ are either of the type (T1) or (T2).*

Proof. By Theorem 4.6 the symmetric relation $S(Q_\infty)$ is simple if and only if it has an empty point spectrum, or equivalently, the subspace $\mathfrak{H}''$ in (4.15) is trivial. In view of Lemma 4.5 the last condition is equivalent to

$$\operatorname{mul} A_1^0 = \{0\} \quad \text{and} \quad \sigma_p(A_0^0) \cap \sigma(q) = \emptyset. \tag{6.20}$$

By Theorem 4.12 the first condition in (6.20) is equivalent to the property formulated in part (1). The condition $\alpha \in \sigma_p(A_0^0) \cap \sigma(q)$ means that $q(\alpha) = 0$ and

$$\lim_{z \widehat{\to} \alpha} (z - \alpha) Q_0(z) < 0, \tag{6.21}$$

since by Lemma 4.5 $\ker (A_0^0 - \alpha)$ is spanned by a positive vector, cf. [9, Lemma 2.3]. Hence, if the multiplicity of α as a root of q is κ_α, then

$$\lim_{z \widehat{\to} \alpha} \frac{Q_\infty(z)}{(z - \alpha)^{2\kappa_\alpha - 1}} < 0, \tag{6.22}$$

and thus $\pi_\alpha(Q_\infty) = \kappa_\alpha$ and α is of type (T3). Conversely, (6.22) implies (6.21) with $\kappa_\alpha = \pi_\alpha(Q_\infty)$. This completes the proof. □

Since the polynomial q can be selected to be an arbitrary divisor of degree $k = \kappa_\infty(Q_\infty)$ of the polynomial $\widetilde{q}$ in the canonical factorization of Q_∞, the condition (2) in Proposition 6.7 can be satisfied if, for instance, the total number (counting multiplicities) of all generalized zeros in $\mathbb{C}_+$ and all generalized zeros of nonpositive type in $\mathbb{R}$ of Q_∞ which are of type (T1) or (T2) is at least equal to $\kappa_\infty(Q_\infty)$. Therefore, the factorization model in Theorem 4.4 can be minimal, while the canonical factorization model of Q_∞ need not be minimal.

Remark 6.8. Similar facts can be derived for other (localized) factorization models which are build on some (proper) factorization of $Q \in \mathbf{N}_\kappa$, cf. (6.19). The construction of such (localized) factorization models can be based on the (orthogonal) coupling of two minimal models (a method studied in another context in [4]), one of which is a finite-dimensional Pontryagin space model for the rational 2×2-matrix function

$$\widehat{R}(z) = \begin{pmatrix} 0 & \widehat{r}(z) \\ \widehat{r}^\sharp(z) & 0 \end{pmatrix}, \tag{6.23}$$

whose reproducing kernel space model has been identified in matrix terms with Bezoutians and companion operators in [3, Proposition 3.3]. The other is a minimal Pontryagin space model for the factor $\widehat{Q}_0$. The model for the product $Q = \widehat{r}\widehat{r}^\sharp\widehat{Q}_0$ in (6.19) is obtained from the orthogonal sum of the models for $\widehat{R}$ and $\widehat{Q}_0$ as a straighforward extension of the model constructed in [3, Theorem 3.3], simply by allowing the factor $\widehat{Q}_0$ to be a generalized Nevanlinna function, too. This procedure can also be described in pure function theoretic terms: perform suitable "block transforms" to the orthogonal sum $\widehat{R} \oplus \widehat{Q}_0$ to get the product $Q = \widehat{r}\widehat{r}^\sharp\widehat{Q}_0$, cf. [3, Section 3].

Such factorization models are not minimal in general. Non-minimality of such models reflects the fact that some poles and zeros of $\widehat{r}\widehat{r}^\sharp$ and $\widehat{Q}_0$ may cancel each others when these functions are multiplied. The simplest example here is the function $Q(z) = -z$, which belongs to $\mathbf{N}_1$ and whose canonical factorization is given by

$$Q(z) = -z = z^2 \left(-\frac{1}{z}\right),$$

where $Q_0(z) = -\frac{1}{z} \in \mathbf{N}_0$, cf. [9, Section 3]. Hence a model which is built on the canonical factorization of $Q(z)$ does not produce a minimal model for Q directly. The general characterization for minimality of canonical factorization models was established in [3, Theorem 4.1]. This result is equivalent to the analytic criterion given in Proposition 6.6: Q does not have any generalized zeros or poles in $\mathbb{R} \cup \{\infty\}$ which are of type (T3); cf. also [12, Theorem 4.4].

Finally, it is noted that the construction of a minimal model in the case of the canonical factorization of $Q \in \mathbf{N}_\kappa$ has been recently studied in [12] by using reproducing kernel Pontryagin spaces. In particular, in that paper a detailed analysis concerning the reproducing kernel space model for the matrix function $\widehat{R}$ in (6.23) has been carried out.

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