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BOUNDARY RELATIONS AND THEIR WEYL FAMILIES

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Abstract. The concepts of boundary relations and the corresponding Weyl families are introduced. Let S be a closed symmetric linear operator or, more generally, a closed symmetric relation in a Hilbert space $\mathfrak{H}$, let $\mathcal{H}$ be an auxiliary Hilbert space, let $J_{\mathfrak{H}} = \begin{pmatrix} 0 & -iI_{\mathfrak{H}} \\ iI_{\mathfrak{H}} & 0 \end{pmatrix}$, and let $J_{\mathcal{H}}$ be defined analogously. A unitary relation Γ from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$ is called a *boundary relation* for the adjoint S^* if $\ker \Gamma = S$. The corresponding *Weyl family* $M(\lambda)$ is defined as the family of images of the defect subspaces $\widehat{\mathfrak{N}}_\lambda$ ($\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$) under Γ. Here Γ need not be surjective and is allowed to be even multivalued. While this leads to fruitful connections between certain classes of holomorphic families of linear relations on the complex Hilbert space $\mathcal{H}$ and the class of unitary relations $\Gamma : (\mathfrak{H}^2, J_{\mathfrak{H}}) \to (\mathcal{H}^2, J_{\mathcal{H}})$, it also generalizes the notion of so-called boundary value space and extends essentially the applicability of abstract boundary mappings in the connection of boundary value problems. Moreover, these new notions yield, for instance, the following realization theorem: every $\mathcal{H}$-valued maximal dissipative (for $\lambda \in \mathbb{C}_+$) holomorphic family of linear relations is the Weyl family of a boundary relation, which is unique up to unitary equivalence if certain minimality conditions are satisfied. Further connections between analytic properties of Weyl families and geometric properties of boundary relations are investigated and some applications are given.

1. Introduction

Up till the seventies most papers about the extension theory of symmetric operators in a Hilbert space were mainly based on von Neumann's formula or a simplified version of it when the symmetric operator has points of regular type on the real line. Later an alternative approach was proposed by V.M. Bruck and A.N. Kochubei (see [17] and the references therein), which is based on an abstract version of Green's identity. The basic object that arises here is the notion of a boundary triplet, also called a boundary value space, see [17, 12, 13, 9].

Definition 1.1. ([17]) Let S be a closed densely defined symmetric operator in a Hilbert space $\mathfrak{H}$. A triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is a Hilbert space and Γ_i, $i = 0, 1$, are operators from $\operatorname{dom} S^*$ to $\mathcal{H}$, is said to be an (ordinary) *boundary triplet* for S^*, if:

(BT1) the abstract Green's identity

$$(S^*f, g) - (f, S^*g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}} \tag{1.1}$$

holds for all $f, g \in \operatorname{dom} S^*$;

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(BT2) the closed linear mapping $\Gamma := \{\Gamma_0, \Gamma_1\} : \operatorname{dom} S^* \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

Here and in the following $[\mathfrak{H}, \mathfrak{K}]$ stands for the set of all bounded linear operators between the Banach spaces $\mathfrak{H}$ and $\mathfrak{K}$; when $\mathfrak{K} = \mathfrak{H}$ this is abbreviated to $[\mathfrak{H}]$. For the present paper it is useful to interpret the mapping Γ in a different manner. Identify the operator S^* with its graph in $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$ and provide the Hilbert spaces $\mathfrak{H}^2$ and $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$ with the inner products induced by the operators $J_{\mathfrak{H}}$ and $J_{\mathcal{H}}$ of the form

$$J = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}.$$

Then $(\mathfrak{H}^2, J_{\mathfrak{H}})$ and $(\mathcal{H}^2, J_{\mathcal{H}})$ are Kreĭn spaces and Definition 1.1 is equivalent to the fact that the mapping Γ is a partial isometry from the subspace S^* of the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ onto the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$.

In [11, 12] the concept of a Weyl function was associated to an ordinary boundary triplet as an abstract version of the m-function appearing in boundary value problems for differential operators.

Definition 1.2. ([11, 12]) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*. The operator-valued function $M(\lambda)$ defined by

$$\Gamma_1 f_\lambda = M(\lambda)\Gamma_0 f_\lambda, \quad f_\lambda \in \mathfrak{N}_\lambda := \ker(S^* - \lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{1.2}$$

is called the *Weyl function*, corresponding to the triplet Π.

The mappings Γ_0 and Γ_1 induce two selfadjoint extensions A_0 and A_1 of S, given by $\operatorname{dom} A_i := \ker \Gamma_i$, $i \in \{0, 1\}$. By definition the Weyl function $M(\cdot)$ is an operator-valued function with values in $[\mathcal{H}]$, which is holomorphic on $\rho(A_0)$, while the inverse $M(\cdot)^{-1}$ is holomorphic on $\rho(A_1)$.

The motivation for the introduction of (abstract) Weyl functions goes back to the theory of singular Sturm-Liouville operators. Let $-d^2/dx^2 + q$ be a Sturm-Liouville operator in the Hilbert space $L^2(0, \infty)$ with a real potential q, which is assumed to be in the limit-point case at ∞. The corresponding minimal operator S is densely defined, closed, and symmetric; its defect numbers are $(1, 1)$. For y in the domain of the corresponding maximal operator S^* one can define $\Gamma_0 y = y(0)$ and $\Gamma_1 y = y'(0)$. Then $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for S^* and the corresponding Weyl function $M(\cdot)$ coincides with the m-function introduced originally by H. Weyl [33] and E.C. Titchmarsh [32].

The (abstract) Weyl function $M(\cdot)$ plays an important role in the spectral theory of the selfadjoint extension A_0 (where $\operatorname{dom} A_0 = \ker \Gamma_0$) of S. The selfadjoint extension A_0 of S generates the so-called γ-field defined by $\gamma(\lambda) := (\Gamma_0 \upharpoonright \mathfrak{N}_\lambda(S^*))^{-1}$. Then $\gamma(\cdot)$ is an operator function with values in $[\mathcal{H}, \mathfrak{N}_\lambda]$, which is holomorphic on $\rho(A_0)$ and satisfies

$$\gamma(\lambda) = [I + (\lambda - \mu)(A_0 - \lambda)^{-1}]\gamma(\mu), \quad \lambda, \mu \in \rho(A_0), \tag{1.3}$$

cf. [22]. It was shown in [12], [24] that the Weyl function $M(\cdot)$ satisfies the identity

$$M(\lambda) - M(\mu)^* = (\lambda - \bar{\mu})\gamma(\mu)^*\gamma(\lambda), \quad \lambda, \mu \in \rho(A_0). \tag{1.4}$$

The condition in Definition 1.1 that the operator S is densely defined can be relaxed. However, if S is nondensely defined, then the adjoint S^* of S is a linear relation and the mappings Γ_i now belong to $[S^*, \mathcal{H}]$, where S^* is considered as a subspace of $\mathfrak{H}^2$ equipped with the graph norm. In this case the condition (BT1) is replaced by

$$(f', g) - (f, g') = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g})_{\mathcal{H}} - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g})_{\mathcal{H}}, \quad \widehat{f} := \{f, f'\}, \widehat{g} := \{g, g'\} \in S^*, \tag{1.5}$$

and the condition (BT2) requires the closed linear mapping $\Gamma := \{\Gamma_0, \Gamma_1\} : S^* \to \mathcal{H} \oplus \mathcal{H}$ to be surjective. Moreover, the definition of the Weyl function takes the form

$$\Gamma_1 \widehat{f}_\lambda = M(\lambda)\Gamma_0 \widehat{f}_\lambda, \quad \widehat{f}_\lambda := \{f_\lambda, \lambda f_\lambda\} \in S^*, \tag{1.6}$$

and, similarly, one modifies the definition of the γ-field, cf. [13], [24].

Recall that the class $R[\mathcal{H}]$ of *Nevanlinna functions* (also called Pick or Herglotz functions, see [15], [16]) is the set of all operator functions $M(\cdot)$ with values in $[\mathcal{H}]$ which are holomorphic on $\mathbb{C} \setminus \mathbb{R}$, and satisfy $M(\lambda) = M(\bar{\lambda})^*$ and $\operatorname{Im}\lambda \operatorname{Im} M(\lambda) \geq 0$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The subclass $R^s[\mathcal{H}]$ of strict Nevanlinna functions in $R[\mathcal{H}]$ is the set of all functions $M(\cdot) \in R[\mathcal{H}]$ for which $0 \notin \sigma_p(\operatorname{Im} M(i))$. The subclass $R^u[\mathcal{H}]$ of uniformly strict Nevanlinna functions in $R^s[\mathcal{H}]$ is the set of all functions $M(\cdot) \in R[\mathcal{H}]$ for which $0 \in \rho(\operatorname{Im} M(i))$. The identity (1.4) means that $M(\cdot)$ is a Q-function of the pair $\{S, A_0\}$ in the sense of M.G. Kreĭn and H. Langer, see [22, 23], and hence it belongs to the subclass $R^u[\mathcal{H}]$ (whether S is densely defined or not). As a Q-function it determines the pair $\{S, A_0\}$ up to a unitary equivalence. It was shown in [11, 13] that for each Nevanlinna function in $R^u[\mathcal{H}]$ there exists a boundary triplet in the above sense for which it is the Weyl function. In [13] the concept of boundary triplet in Definition 1.1 has been extended to the case where the corresponding Weyl function belongs to the subclass $R^s[\mathcal{H}]$ and the inverse result for this subclass has been established.

Now the natural problem arises, whether every Nevanlinna function in the class $R[\mathcal{H}]$ can be interpreted as a Weyl function of some generalized boundary triplet. In fact, the same question can be asked for the more general notion of an arbitrary Nevanlinna family (see the definition below). This last problem is also inspired by the Kreĭn-Naimark formula for generalized resolvents of a symmetric operator S:

$$P_{\mathfrak{H}}(\widetilde{A} - \lambda))^{-1}|_{\mathfrak{H}} = (A_0 - \lambda)^{-1} - \gamma(\lambda)(M(\lambda) + \tau(\lambda))^{-1}\gamma(\bar{\lambda})^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{1.7}$$

which establishes a bijective correspondence between the set of all selfadjoint (canonical and exit space) extensions $\widetilde{A}$ of A and the set of all Nevanlinna families $\tau(\cdot)$. Here $P_{\mathfrak{H}}$ is the orthogonal projection from the exit space onto $\mathfrak{H}$. While $M(\cdot)$ in (1.7) appears as a Weyl function of an (ordinary) boundary triplet (or as a Q-function of the pair $\{S, A_0\}$), there is in general no analogous interpretation for the family $\tau(\cdot)$ in (1.7).

The class of all *Nevanlinna families* $M(\cdot)$ in $\mathcal{H}$ is denoted by $\widetilde{R}(\mathcal{H})$; it is the set of holomorphic families of linear relations $M(\lambda) : \mathcal{H} \to \mathcal{H}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, (i.e. $M(\lambda)$ is a linear subspace of $\mathcal{H} \oplus \mathcal{H}$), which satisfy

(NF1) $M(\lambda)$ is dissipative for all $\lambda \in \mathbb{C}_+$;
(NF2) $M(\lambda) = M(\bar{\lambda})^*$ for all $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$;
(NF3) $0 \in \rho(M(\lambda) + i)$ for one (equivalently for all) $\lambda \in \mathbb{C}_+$.

Nevanlinna families are maximal dissipative in the upper halfplane by the properties (NF1) and (NF3), and maximal accumulative in the lower halfplane by the symmetry property (NF2). When $M(\cdot)$ is a Nevanlinna function in $R[\mathcal{H}]$ it is clear that the property (NF3) is automatically satisfied. In the present paper the new concepts of a *boundary relation* and the corresponding *Weyl family* are introduced. These new concepts make it possible to realize every Nevanlinna family (possibly unbounded and even multivalued) as the Weyl family of a boundary relation. In a forthcoming paper the usefulness of these new concepts will be demonstrated for the Kreĭn-Naimark theory of generalized resolvents. In particular, it will be shown that $\tau(\cdot)$ is the Weyl family of the symmetric relation S_2 which is given by $S_2 := \widetilde{A} \cap (\widetilde{\mathfrak{H}} \ominus \mathfrak{H})^2$, where $\widetilde{A}$ is associated to an appropriate boundary relation. In the special case when $\tau(\cdot) \in R^u[\mathcal{H}]$ this fact has been established by the authors in [9].

In order to explain these new notions assume for the moment that S is densely defined and rewrite Green's identity (1.1) in assumption (BT1) of Definition 1.1 as

$$(S^*f, g) - (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} = (f, S^*g) - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \operatorname{dom} S^*. \tag{1.8}$$

The interpretation of (1.8) is that the operator $\widetilde{A}$ defined by

$$\widetilde{A} : \begin{pmatrix} f \\ \Gamma_0 f \end{pmatrix} \mapsto \begin{pmatrix} S^*f \\ -\Gamma_1 f \end{pmatrix}, \quad f \in \operatorname{dom} S^*, \tag{1.9}$$

is symmetric in $\mathfrak{H} \oplus \mathcal{H}$. Moreover, the assumption (BT2) of Definition 1.1 guarantees that $\widetilde{A}$ is selfadjoint in $\mathfrak{H} \oplus \mathcal{H}$. If S is not densely defined, similar observations can be made when (1.9) is appropriately interpreted. The precise definition of a boundary relation will be given in Section 3, but in an equivalent form it can be reformulated as follows. A pair $\{\mathcal{H}, \Gamma\}$, where $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ is a closed linear relation (i.e. a linear subspace of $\mathfrak{H}^2 \oplus \mathcal{H}^2$) is said to be a boundary relation for S^* if $\operatorname{dom} \Gamma$ is dense in S^* and if the transform $\widetilde{A}$ of Γ determined by

$$\widetilde{A} = \left\{ \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' \\ -h' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \right\} \tag{1.10}$$

is a selfadjoint relation in $\mathfrak{H} \oplus \mathcal{H}$. The linear relation Γ from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$ turns out to be unitary in the sense of relations, cf. [27]. In this definition S is not necessarily densely defined and S is allowed to have infinite and unequal defect numbers. The corresponding Weyl family is now defined by

$$M(\lambda) = \Gamma\left(\{\{f, f'\} \in \operatorname{dom} \Gamma : f' = \lambda f\}\right) \tag{1.11}$$

as an extension of Definition 1.2. The given assumptions are enough to guarantee that the Weyl family $M(\cdot)$ is a Nevanlinna family in the sense of Definition (NF1)–(NF3). Moreover, one of the main results in this paper shows that every Nevanlinna family can be realized as a Weyl family of some boundary relation which is unique, up to unitary equivalence, when a certain minimality condition is satisfied; see Theorem 3.9. The proof is based on the generalized Naimark theorem and does not use any operator model as was done in the case of a uniformly strict Nevanlinna function (see [22], [23], [13]). Note in this connection that a simple proof of the Naimark dilation theorem is recently presented in [25]. Observe that the definition of boundary relation allows Γ to be multivalued in which case it may happen that Γ is indecomposable into the orthogonal sum $\Gamma_0 \oplus \Gamma_1$, where $\Gamma_j : \mathfrak{H}^2 \to \mathcal{H}$, $j = 0, 1$. When the decomposition $\Gamma = \Gamma_0 \oplus \Gamma_1$ makes sense, the new concept of the boundary relation reduces to a natural generalization of the notion of an ordinary boundary triplet in Definition 1.1 as well as of the notion of a generalized boundary triplet in [13]; in this case the notation "boundary triplet" will still be kept for Γ in the present paper.

The connection between the boundary relation Γ and the selfadjoint operator or, in general, relation $\widetilde{A}$ plays a fundamental role in the sequel. The interpretation of $\widetilde{A}$ is that of a selfadjoint exit space extension of S determined by the boundary relation Γ. The given procedure can be applied, for instance, in the linearization of boundary value problems with eigenvalue parameters in the boundary conditions; here arbitrary (finite or infinite and equal or unequal) defect numbers for the underlying operators are allowed. This will be further investigated in the forthcoming paper as well as the extension of the notions of boundary relations and the corresponding Weyl families to the case where S is defined on a space with an indefinite inner product. The appearance of unbounded Weyl functions is not excluded here either; this makes it unnecessary to find regularizations for boundary mappings for treating boundary value problems involving partial differential operators, cf. [17], [13]. The present paper establishes for the first time on a general level the link between the abstracts boundary triplets (here the mapping Γ) and the exit space extensions $\widetilde{A}$ (the transform $\mathcal{J}$ which connects Γ and $\widetilde{A}$). In what follows, this connection is effectively used in building up the general theory of boundary relations and their Weyl families and it plays a key role in proving some of the central theorems of the present paper. Some of the main results of the paper have been announced in [10].

In Section 2 some preparatory material is presented, including results on linear relations in Kreĭn spaces. Here the main transform $\mathcal{J}$ acting between two Kreĭn spaces is introduced and its properties are investigated. In Section 3 the concepts of boundary relation and the corresponding Weyl family are introduced. The main result of this section is the following inverse theorem: every Nevalinna family can be realized as the Weyl function of a boundary relation. In Section 4 the investigation of geometrical properties of boundary relations and the analytical properties of the corresponding Weyl families is continued. Several known results on Q-functions or equivalently Weyl functions of ordinary boundary triplets are extended to wider subclasses of Nevanlinna families. In particular, geometrical properties of boundary relations whose Weyl families $M(\lambda)$ are domain invariant are studied in detail. In Section 5 the connection between the new concepts and the earlier concepts of boundary triplets and the corresponding Weyl functions is investigated. Section 6 contains several examples, which demonstrate the applicability of the new concepts and sharpness of several statements in the earlier sections of the paper, as well as some new unexpectable effects.

2. Preliminaries

2.1. **Linear relations in Hilbert spaces.** Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be separable Hilbert spaces. A subspace $T \subset \mathfrak{H}_1 \oplus \mathfrak{H}_2$ is called a linear relation from $\mathfrak{H}_1$ to $\mathfrak{H}_2$. It is also convenient to write $\Gamma : \mathfrak{H}_1 \to \mathfrak{H}_2$ and interpret Γ as a multivalued linear mapping from $\mathfrak{H}_1$ into $\mathfrak{H}_2$. In the case where $\mathfrak{H}_1 = \mathfrak{H}_2 (= \mathfrak{H})$, Γ is called a linear relation in $\mathfrak{H}$. In what follows $[\mathfrak{H}_1, \mathfrak{H}_2]$ denotes the set of all bounded linear operators from $\mathfrak{H}_1$ to $\mathfrak{H}_2$; $[\mathfrak{H}]$ is the set of all bounded linear operators in $\mathfrak{H}$; $\operatorname{dom} T$, $\ker T$, $\operatorname{ran} T$, and $\operatorname{mul} T$ are the domain, kernel, range, and the multivalued part of the linear relation T, respectively. The inverse T^{-1} is a relation from $\mathfrak{H}_2$ to $\mathfrak{H}_1$ defined by $\{\, \{f', f\} : \{f, f'\} \in T \,\}$. The adjoint T^* is the closed linear relation from $\mathfrak{H}_2$ to $\mathfrak{H}_1$ defined by

$$T^* = \{\, \{h, k\} \in \mathfrak{H}_2 \oplus \mathfrak{H}_1 : (k, f)_{\mathfrak{H}_1} = (h, g)_{\mathfrak{H}_2},\ \{f, g\} \in T \,\}, \tag{2.1}$$

(see [4], [6]). Moreover, $\rho(T)$ ($\hat{\rho}(T)$) is the set of regular (regular type) points of T. Often a linear operator T will be identified with its graph. The sum $T_1 + T_2$ and the componentwise sum $T_1 \widehat{+} T_2$ of two linear relations T_1 and T_2 are defined by

$$\begin{aligned} T_1 + T_2 &= \{\, \{f, g + h\} : \{f, g\} \in T_1, \{f, h\} \in T_2 \,\}, \\ T_1 \widehat{+} T_2 &= \{\, \{f + h, g + k\} : \{f, g\} \in T_1, \{h, k\} \in T_2 \,\}. \end{aligned} \tag{2.2}$$

Clearly,

$$\ker(T_1 - T_2) = \operatorname{dom}(T_1 \cap T_2), \quad \ker(T_1^{-1} - T_2^{-1}) = \operatorname{ran}(T_1 \cap T_2). \tag{2.3}$$

The null spaces of $T - \lambda$, $\lambda \in \mathbb{C}$, are defined by

$$\mathfrak{N}_\lambda(T) = \ker(T - \lambda), \quad \widehat{\mathfrak{N}}_\lambda(T) = \{\, \{f, \lambda f\} \in T : \, f \in \mathfrak{N}_\lambda(T) \,\}. \tag{2.4}$$

Recall the following simple result (cf. [13, Lemma 2.1]), which will be used in the proof of the next proposition.

Lemma 2.1. *([13]) Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, let $\mathfrak{M}$ be a closed linear subspace of $\mathfrak{X}$, and let $P \in [\mathfrak{X}, \mathfrak{Y}]$ be surjective. Then the range $P\mathfrak{M}$ is closed in $\mathfrak{Y}$ if and only if the sum of the linear subspaces $\mathfrak{M} + \mathfrak{N}$ is closed in $\mathfrak{X}$, where $\mathfrak{N} := \ker P$.*

Proposition 2.2. *Let T be a closed linear relation from a Hilbert space $\mathfrak{H}_1$ to a Hilbert space $\mathfrak{H}_2$. Then:*

(i) $\operatorname{dom} T$ *is closed if and only if* $\operatorname{dom} T^*$ *is closed;*
(ii) $\operatorname{ran} T$ *is closed if and only if* $\operatorname{ran} T^*$ *is closed.*

Proof. The statement (i) is equivalent to the statement (ii) (by inversion of T). So it suffices to prove the last statement. Let P be the orthoprojection from $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ onto $\mathfrak{H}_2$, so that $\ker P = \mathfrak{H}_1 \oplus \{0\}$. Assume that $\operatorname{ran} T = PT$ is closed. Then by Lemma 2.1 also $T \,\widehat{+}\, (\mathfrak{H}_1 \oplus \{0\})$ is closed. By a theorem of Kato [19, Chapter 4, Theorem 4.8] the corresponding sum of the orthogonal complements in $\mathfrak{H}_1 \oplus \mathfrak{H}_2$

$$T^\perp \,\widehat{+}\, (\{0\} \oplus \mathfrak{H}_2) \tag{2.5}$$

is also closed. The operator $J : \mathfrak{H}_1 \oplus \mathfrak{H}_2 \to \mathfrak{H}_2 \oplus \mathfrak{H}_1$ given by $J\{h, h'\} = \{-h', h\}$ is unitary, and it follows from (2.1) that $T^* = JT^\perp$. Hence, (2.5) implies that

$$T^* \,\widehat{+}\, (\mathfrak{H}_2 \oplus \{0\}),$$

is also closed. In other words, $T^* \,\widehat{+}\, \ker Q$ is closed. Here Q is the orthoprojection from $\mathfrak{H}_2 \oplus \mathfrak{H}_1$ onto $\mathfrak{H}_1$, so that $\ker Q = \mathfrak{H}_2 \oplus \{0\}$. Another application of Lemma 2.1 shows that $QT^* = \operatorname{ran} T^*$ is closed. □

Recall that a linear relation T in $\mathfrak{H}$ is called *symmetric* (*dissipative* or *accumulative*) if $\operatorname{Im}(h', h) = 0$ (≥ 0 or ≤ 0, respectively) for all $\{h, h'\} \in T$. These properties remain invariant under closures. By polarization it follows that a linear relation T in $\mathfrak{H}$ is symmetric if and only if $T \subset T^*$. A linear relation T in $\mathfrak{H}$ is called *selfadjoint* if $T = T^*$, and it is called *essentially selfadjoint* if $\operatorname{clos} T = T^*$. A dissipative (accumulative) linear relation T in $\mathfrak{H}$ is called maximal dissipative (maximal accumulative) if it has no proper dissipative (accumulative) extensions. Clearly, a linear relation T is selfadjoint if and only if it is both maximal dissipative and maximal accumulative.

Assume that T is closed. If T is dissipative or accumulative, then $\operatorname{mul} T \subset \operatorname{mul} T^*$. In this case the orthogonal decomposition $\mathfrak{H} = (\operatorname{mul} T)^\perp \oplus \operatorname{mul} T$ induces an orthogonal decomposition of T as

$$T = T_s \oplus T_\infty, \quad T_\infty = \{0\} \oplus \operatorname{mul} T, \quad T_s = \{\, \{f, g\} \in T : \, g \perp \operatorname{mul} T \,\},$$

where T_∞ is a selfadjoint relation in $\operatorname{mul} T$ and T_s is an operator in $\mathfrak{H} \ominus \operatorname{mul} T$ with $\overline{\operatorname{dom} T_s} = \overline{\operatorname{dom} T} = (\operatorname{mul} T^*)^\perp$, which is dissipative or accumulative. Moreover, if the relation T is maximal dissipative or accumulative, then $\operatorname{mul} T = \operatorname{mul} T^*$. In this case the orthogonal decomposition $(\overline{\mathrm{dom}}\, T)^\perp = \mathrm{mul}\, T^*$ shows that T_s is a densely defined dissipative or accumulative operator in $(\mathrm{mul}\, T)^\perp$, which is maximal (as an operator). In particular, if T is a selfadjoint relation, then there is such a decomposition where T_s is a selfadjoint operator (densely defined in $(\mathrm{mul}\, T)^\perp$).

Let S be a closed symmetric linear relation in a Hilbert spaces $\mathfrak{H}$. Then the adjoint relation S^* can be decomposed via the von Neumann formula:

$$S^* = S \mathbin{\widehat{+}} \widehat{\mathfrak{N}}_\lambda(S^*) \mathbin{\widehat{+}} \widehat{\mathfrak{N}}_{\bar\lambda}(S^*), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad \text{direct sums,} \tag{2.6}$$

where $\widehat{\mathfrak{N}}_\lambda(S^*)$ is defined as in (2.4). When $\lambda = \pm i$ the decomposition (2.6) is orthogonal:

$$S^* = S \oplus \widehat{\mathfrak{N}}_i(S^*) \oplus \widehat{\mathfrak{N}}_{-i}(S^*), \tag{2.7}$$

where the orthogonality is with respect to the inner product topology in S^*, cf. [4], [6]. A symmetric linear relation S is called *simple* if there is no nontrivial orthogonal decomposition of the Hilbert space $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and no corresponding orthogonal decomposition $S = S_1 \oplus S_2$ with S_1 a symmetric relation in $\mathfrak{H}_1$ and S_2 a selfadjoint relation in $\mathfrak{H}_2$. The above decomposition $S = S_s \oplus S_\infty$ shows that a simple closed symmetric relation is necessarily an operator. Recall that (cf. [23]) a closed symmetric linear relation S in a Hilbert space $\mathfrak{H}$ is simple if and only if

$$\mathfrak{H} = \overline{\mathrm{span}}\, \{\, \mathfrak{N}_\lambda(S^*) : \lambda \in \mathbb{C} \setminus \mathbb{R} \,\}.$$

2.2. **Linear relations in Kreĭn spaces.** Now let $\mathfrak{H}$ and $\mathcal{H}$ be Hilbert spaces and let $j_\mathfrak{H}$ and $j_\mathcal{H}$ be signature operators in them. Recall that a bounded linear operator j in a Hilbert space is a signature operator, if $j = j^* = j^{-1}$. Interpret the spaces $\mathfrak{H}$ and $\mathcal{H}$ are Kreĭn space whose inner product is determined by the fundamental symmetries $j_\mathfrak{H}$ and $j_\mathcal{H}$. Then the adjoint of a linear relation T from the Kreĭn space $(\mathfrak{H}, j_\mathfrak{H})$ to the Kreĭn space $(\mathcal{H}, j_\mathcal{H})$ is given by $T^{[*]} = j_\mathfrak{H} T^* j_\mathcal{H}$. It satisfies the following equalities familiar from the Hilbert space case

$$(\mathrm{dom}\, T)^{[\perp]} = \mathrm{mul}\, T^{[*]}, \quad (\mathrm{ran}\, T)^{[\perp]} = \ker T^{[*]}. \tag{2.8}$$

Here the orthogonal complements, denoted by $[\perp]$, are with respect to the Kreĭn space structures. The inner products in $(\mathfrak{H}, j_\mathfrak{H})$, $(\mathcal{H}, j_\mathcal{H})$ will be denoted by

$$[\varphi, \psi]_\mathcal{H} = (j_\mathfrak{H}\varphi, \psi)_\mathcal{H}, \quad [\varphi', \psi']_\mathfrak{H} = (j_\mathfrak{H}\varphi', \psi')_\mathfrak{H} \quad \varphi, \psi \in \mathcal{H}, \;\; \varphi', \psi' \in \mathfrak{H}.$$

A linear relation T from the Kreĭn space $(\mathfrak{H}, j_\mathfrak{H})$ to the Kreĭn space $(\mathcal{H}, j_\mathcal{H})$ is called *isometric* if

$$[\varphi', \varphi']_\mathcal{H} = [\varphi, \varphi]_\mathfrak{H}, \quad \{\varphi, \varphi'\} \in T, \tag{2.9}$$

and *contractive* or *expanding* if equality in (2.9) is replaced by $\leq$ or by $\geq$, respectively. These properties are invariant under closures. By polarization it follows that a linear relation T is isometric if and only if $T^{-1} \subset T^{[*]}$. A linear relation T is called *unitary* if $T^{-1} = T^{[*]}$; it is called *essentially unitary* if $(\mathrm{clos}\, T)^{-1} = T^{[*]}$.

The first statement in the next proposition can be found in a paper of Yu.L. Shmul'jan n, see [27]. Namely he mentioned that it can be obtained by combining [27, Theorem 3] with one result of Spitkovskiĭ in [28]. A simple and essentially different proof of this statement is presented below. The second statement is proved in [27, Theorem 2] by using a result of R. Arens [2].

Proposition 2.3. *Let T be a unitary relation from the Kreĭn space $(\mathfrak{H}, j_\mathfrak{H})$ to the Kreĭn space $(\mathcal{H}, j_\mathcal{H})$. Then:*

(i) $\operatorname{dom} T$ *is closed if and only if* $\operatorname{ran} T$ *is closed;*
(ii) *the following equalities hold:*

$$\ker T = (\operatorname{dom} T)^{[\perp]}, \quad \operatorname{mul} T = (\operatorname{ran} T)^{[\perp]}. \tag{2.10}$$

Proof. By definition T satisfies the identity $T^{-1} = T^{[*]}$, where the Kreĭn space adjoint $T^{[*]}$ of T is connected to the Hilbert space adjoint T^* via $T^{[*]} = j_{\mathfrak{H}} T^* j_{\mathcal{H}}$. It is clear that $\operatorname{dom} T^{[*]}$ ($\operatorname{ran} T^{[*]}$) is closed if and only if $\operatorname{dom} T^*$ (resp. $\operatorname{ran} T^*$) is closed. Therefore, for a unitary relation T the equivalence $\operatorname{dom} T$ is closed if and only if $\operatorname{ran} T$ is closed follows now from Proposition 2.2.

To get the identities (2.10) it is enough to apply (2.8) and the equality $T^{-1} = T^{[*]}$. □

In its present generality it is useful to give criteria for a unitary relation $T : (\mathfrak{H}, j_{\mathfrak{H}}) \to (\mathcal{H}, j_{\mathcal{H}})$ to be an operator (not necessarily densely defined).

Corollary 2.4. *Let T be a unitary relation from the Kreĭn space $(\mathfrak{H}, j_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}, j_{\mathcal{H}})$. Then:*

(i) *T is single-valued if and only if $\overline{\operatorname{ran}}\, T = \mathcal{H}$;*
(ii) *T is a densely defined operator if and only if $\overline{\operatorname{ran}}\, T = \mathcal{H}$ and $\ker T = \{0\}$;*
(iii) *T is bounded and single-valued if and only if $\operatorname{ran} T = \mathcal{H}$;*
(iv) *$T \in [\mathfrak{H}, \mathcal{H}]$ if and only if $\operatorname{ran} T = \mathcal{H}$ and $\ker T = \{0\}$.*

Proof. By Proposition 2.3 $\operatorname{mul} T = (\operatorname{ran} T)^{[\perp]}$ and this gives (i). Moreover, according to Proposition 2.3 $\operatorname{ran} T$ is closed if and only if $\operatorname{dom} T$ is closed, and thus (iii) follows from the closed graph theorem. To get (ii) and (iv) is remains to apply the identity $\ker T = (\operatorname{dom} T)^{[\perp]}$ in Proposition 2.3. □

Using Kreĭn space terminology, Proposition 2.3 shows that for a unitary relation T, the isotropic part of $\operatorname{dom} T$ is equal to $\ker T$ and the isotropic part of $\operatorname{ran} T$ is equal to $\operatorname{mul} T$. For an isometric relation T from the Kreĭn space $(\mathfrak{H}, j_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}, j_{\mathcal{H}})$ the situation is different. It follows from $T^{-1} \subset T^{[*]}$ and the identities (2.8) that

$$\ker T \subset (\operatorname{dom} T)^{[\perp]}, \quad \operatorname{mul} T \subset (\operatorname{ran} T)^{[\perp]}, \tag{2.11}$$

so that $\ker T$ is contained in the isotropic part of $\operatorname{dom} T$ and $\operatorname{mul} T$ is contained in the isotropic part of $\operatorname{ran} T$. It turns out that isometric relations whose domain satisfies the additional property

$$(\operatorname{dom} T)^{[\perp]} \subset \operatorname{dom} T \tag{2.12}$$

play a central role in the construction of boundary mappings. The following results give sufficient conditions for such an isometric relation T to be unitary. The connection to ordinary boundary triplets becomes clear in Section 5.

Proposition 2.5. *Let T be an isometric linear relation from the Kreĭn space $(\mathfrak{H}, j_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}, j_{\mathcal{H}})$. If the conditions*

(i) *$(\operatorname{dom} T)^{[\perp]} \subset \operatorname{dom} T$;*
(ii) *$(\operatorname{ran} T)^{[\perp]} \subset \operatorname{mul} T$,*

are satisfied, then T also satisfies

$$\ker T = (\operatorname{dom} T)^{[\perp]}. \tag{2.13}$$

Moreover, if the condition (2.13) *and the condition*

$$\operatorname{dom} T^{[*]} \subset \operatorname{ran} T \tag{2.14}$$

are satisfied, then T is a unitary relation.

Proof. Assume that $f \in (\operatorname{dom} T)^{[\perp]}$ so that $[f,h]_{\mathfrak{H}} = 0$ for all $h \in \operatorname{dom} T$. By assumption (i) there exists an element $f' \in \mathcal{H}$ so that $\{f,f'\} \in T$. Hence for all $\{h,h'\} \in T$

$$[f',h']_{\mathcal{H}} = [f,h]_{\mathfrak{H}} = 0,$$

so that $f' \in (\operatorname{ran} T)^{[\perp]}$. Hence, assumption (ii) implies that $f' \in \operatorname{mul} T$. Therefore $f \in \ker T$, so that $(\operatorname{dom} T)^{[\perp]} \subset \ker T$. Hence, (2.11) implies that (2.13) is satisfied.

Now assume that (2.13) and (2.14) hold. Let $\{f,f'\} \in T^{[*]}$ so that $[f',h]_{\mathfrak{H}} = [f,h']_{\mathcal{H}}$ for all $\{h,h'\} \in T$. The condition (2.14) implies the existence of an element $\varphi \in \mathfrak{H}$ such that $\{\varphi, f\} \in T$. Since T is isometric it follows that $[f,h']_{\mathcal{H}} = [\varphi,h]_{\mathfrak{H}}$, so that $[f',h]_{\mathfrak{H}} = [\varphi,h]_{\mathfrak{H}}$ for all $h \in \operatorname{dom} T$. This implies that $f' = \varphi + \gamma$ with $\gamma \in (\operatorname{dom} T)^{[\perp]}$. The condition (2.13) shows that $\gamma \in \ker T$. Hence

$$\{f',f\} = \{\varphi+\gamma, f\} = \{\varphi, f\} + \{\gamma, 0\} \in T,$$

which implies that $T^{[*]} \subset T^{-1}$. □

Corollary 2.6. *Condition (ii) in Proposition 2.5 is automatically satisfied when* $\operatorname{ran} T$ *is dense in $\mathcal{H}$, in which case T is an operator. In particular, if (i) holds and* $\operatorname{ran} T = \mathcal{H}$*, then* $\operatorname{dom} T$ *is closed and T is a bounded unitary operator.*

Proof. Finally, assume that $\operatorname{ran} T$ is dense in $\mathcal{H}$. Since $(\operatorname{ran} T)^{[\perp]} = \{0\}$, clearly (ii) is satisfied. Since T is isometric it follows from the second inclusion in (2.11) that T is an operator.

Now assume $\operatorname{ran} T = \mathcal{H}$. Hence, $\operatorname{ran} T$ is dense in $\mathcal{H}$, so that (ii) follows and (2.13) automatically follows. Moreover, (2.14) is also automatically satisfied, so that (ii) implies that T is a unitary operator. It follows from Proposition 2.3 that $\operatorname{dom} T$ is closed. The boundedness of T follows from the closed graph theorem. □

Clearly, with T also the inverse T^{-1} is isometric. Hence, a formal inversion in Proposition 2.5 gives the following equivalent version.

Proposition 2.7. *Let T be an isometric linear relation from the Kreĭn space $(\mathfrak{H}, j_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}, j_{\mathcal{H}})$. If the conditions*

(i) $(\operatorname{ran} T)^{[\perp]} \subset \operatorname{ran} T$*;*
(ii) $(\operatorname{dom} T)^{[\perp]} \subset \operatorname{mul} T$*,*

are satisfied, then T also satisfies

$$\operatorname{mul} T = (\operatorname{ran} T)^{[\perp]}. \tag{2.15}$$

Moreover, if the condition (2.15) *and the condition*

$$\operatorname{ran} T^{[*]} \subset \operatorname{dom} T, \tag{2.16}$$

are satisfied, then T is a unitary relation.

Corollary 2.8. *Condition (ii) in Proposition 2.7 is automatically satisfied when* $\operatorname{dom} T$ *is dense in $\mathcal{H}$, in which case T^{-1} is an operator. In particular, if (i) holds and* $\operatorname{dom} T = \mathcal{H}$*, then* $\operatorname{ran} T$ *is closed and T^{-1} is a bounded unitary operator.*

Since with T also the closure of T is isometric, it is possible to replace in Propositions 2.5, 2.7, and their corollaries the relation T by its closure to conclude that T is an essentially unitary relation.

Let T be an isometric operator from the Kreĭn space $(\mathfrak{H}, j_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}, j_{\mathcal{H}})$ such that $\overline{\operatorname{dom}}\, T = \mathfrak{H}$ and $\overline{\operatorname{ran}}\, T = \mathcal{H}$. If either (2.14) or (2.16) holds, then T and T^{-1} are unitary relations, which are in general unbounded, see Example 6.4. In particular, if either $\overline{\operatorname{dom}}\, T = \mathfrak{H}$ and $\operatorname{ran} T = \mathcal{H}$, or $\operatorname{dom} T = \mathfrak{H}$ and $\overline{\operatorname{ran}}\, T = \mathcal{H}$, then by Corollary 2.6 or 2.8 both T and T^{-1} are unitary operators, which are bounded (cf. [3, Chapter 2,Definition 5.1 and Corollary 5.8]).

In what follows it is convenient to interpret the Hilbert space $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$ as a Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ whose inner product is determined by the fundamental symmetry $J_{\mathfrak{H}}$:

$$J_{\mathfrak{H}} := \begin{pmatrix} 0 & -iI_{\mathfrak{H}} \\ iI_{\mathfrak{H}} & 0 \end{pmatrix}. \tag{2.17}$$

The adjoint T^* in (2.1) of a linear relation T in the Hilbert space $\mathfrak{H}$ can be written in terms of $J_{\mathfrak{H}}$ as:

$$T^* = J_{\mathfrak{H}} T^{\perp} = (J_{\mathfrak{H}} T)^{\perp}. \tag{2.18}$$

The following connections between linear relations in the Hilbert space $\mathfrak{H}$ and subspaces in the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ will be useful.

Proposition 2.9 ([3]). *Let T be a linear relation in the Hilbert space $\mathfrak{H}$. Then*

- (i) *T is symmetric (selfadjoint) if and only if T is a neutral (hypermaximal neutral) subspace of $(\mathfrak{H}^2, J_{\mathfrak{H}})$;*
- (ii) *T is dissipative (maximal dissipative) if and only if T is a nonnegative (maximal nonnegative) subspace of $(\mathfrak{H}^2, J_{\mathfrak{H}})$;*
- (iii) *T is a accumulative (maximal accumulative) if and only if T is a nonpositive (maximal nonpositive) subspace of $(\mathfrak{H}^2, J_{\mathfrak{H}})$.*

2.3. **The main transform.** Let $\mathfrak{H}$ and $\mathcal{H}$ be Hilbert spaces and define their orthogonal sum as $\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathcal{H}$. In this section linear relations from $\mathfrak{H}^2$ to $\mathcal{H}^2$ will be related to linear relations in $\widetilde{\mathfrak{H}}$, i.e., from $\widetilde{\mathfrak{H}}$ to $\widetilde{\mathfrak{H}}$. For this purpose some interpretation of notation is needed. An element of a linear relation Γ from $\mathfrak{H}^2$ to $\mathcal{H}^2$ is usually denoted by $\{\widehat{f}, \widehat{h}\}$ with the understanding that $\widehat{f} = \{f, f'\} \in \mathfrak{H}^2$ and $\widehat{h} = \{h, h'\} \in \mathcal{H}^2$. However, it will also be convenient to think of such a general element as

$$\{\widehat{f}, \widehat{h}\} = \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \quad \text{with} \quad \widehat{f} = \begin{pmatrix} f \\ f' \end{pmatrix} \in \begin{pmatrix} \mathfrak{H} \\ \mathfrak{H} \end{pmatrix}, \quad \widehat{h} = \begin{pmatrix} h \\ h' \end{pmatrix} \in \begin{pmatrix} \mathcal{H} \\ \mathcal{H} \end{pmatrix}.$$

This interpretation will be assumed whenever needed without explicit mention. A similar interpretation applies to the notation of the elements of a linear relation $\widetilde{A}$ in $\widetilde{\mathfrak{H}}$. Explicitly, the linear relations Γ and $\widetilde{A}$ are interpreted to act as follows:

$$\Gamma : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{H} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{H} \end{pmatrix}, \quad \widetilde{A} : \begin{pmatrix} \mathfrak{H} \\ \mathcal{H} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathcal{H} \end{pmatrix}. \tag{2.19}$$

In this sense, the following linear transform from $\mathfrak{H}^2 \oplus \mathcal{H}^2$ to $(\mathfrak{H} \oplus \mathcal{H})^2$

$$\mathcal{J} : \Gamma \mapsto \widetilde{A} := \left\{ \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' \\ -h' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \right\} \tag{2.20}$$

defines a one-to-one correspondence between the (closed) linear relations $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ and the (closed) linear relations $\widetilde{A}$ in $\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathcal{H}$. The correspondence in (2.20) is denoted by $\widetilde{A} = \mathcal{J}\Gamma$.

Proposition 2.10. *Let the linear relation Γ from $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to $(\mathcal{H}^2, J_{\mathcal{H}})$ and the linear relation $\widetilde{A}$ in $\mathfrak{H} \oplus \mathcal{H}$ be connected by $\widetilde{A} = \mathcal{J}\Gamma$. Then*

$$\widetilde{A}^* = \mathcal{J}((\Gamma^{[*]})^{-1}). \tag{2.21}$$

Moreover, the transform $\mathcal{J}$ establishes a one-to-one correspondence between the isometric (contractive, expanding, unitary) relations Γ from $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to $(\mathcal{H}^2, J_{\mathcal{H}})$ and the symmetric (dissipative, accumulative, selfadjoint) relations in $\mathfrak{H} \oplus \mathcal{H}$.

Proof. It is straightforward to check that for all elements of the form

$$\left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\}, \left\{ \begin{pmatrix} g \\ g' \end{pmatrix}, \begin{pmatrix} k \\ k' \end{pmatrix} \right\} \in \begin{pmatrix} \mathfrak{H} \\ \mathfrak{H} \end{pmatrix} \oplus \begin{pmatrix} \mathcal{H} \\ \mathcal{H} \end{pmatrix},$$

the following identity is satisfied:

$$\begin{aligned} &\frac{1}{i} \left\{ \left(\begin{pmatrix} f' \\ -h' \end{pmatrix}, \begin{pmatrix} g \\ k \end{pmatrix} \right)_{\mathfrak{H}\oplus\mathcal{H}} - \left(\begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} g' \\ -k' \end{pmatrix} \right)_{\mathfrak{H}\oplus\mathcal{H}} \right\} \\ &= \left(J_{\mathfrak{H}} \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} g \\ g' \end{pmatrix} \right)_{\mathfrak{H}^2} - \left(J_{\mathcal{H}} \begin{pmatrix} h \\ h' \end{pmatrix}, \begin{pmatrix} k \\ k' \end{pmatrix} \right)_{\mathcal{H}^2}. \end{aligned} \tag{2.22}$$

This identity implies the equivalence:

$$\left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' \\ -h' \end{pmatrix} \right\} \in \widetilde{A}^* \Leftrightarrow \left\{ \begin{pmatrix} h \\ h' \end{pmatrix}, \begin{pmatrix} f \\ f' \end{pmatrix} \right\} \in \Gamma^{[*]},$$

which leads to the identity (2.21). Hence it follows that

$$\widetilde{A} \subset \widetilde{A}^* \Leftrightarrow \Gamma^{-1} \subset \Gamma^{[*]}, \quad \widetilde{A} = \widetilde{A}^* \Leftrightarrow \Gamma^{-1} = \Gamma^{[*]}.$$

Observe that (2.22) in particular leads to the following identity:

$$2\mathrm{Im} \left(\begin{pmatrix} f' \\ -h' \end{pmatrix}, \begin{pmatrix} f \\ h \end{pmatrix} \right)_{\mathfrak{H}\oplus\mathcal{H}} = \left(J_{\mathfrak{H}} \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} f \\ f' \end{pmatrix} \right)_{\mathfrak{H}^2} - \left(J_{\mathcal{H}} \begin{pmatrix} h \\ h' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right)_{\mathcal{H}^2}.$$

This implies the connection between the contractive (expanding) relations Γ from $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to $(\mathcal{H}^2, J_{\mathcal{H}})$ and the dissipative (accumulative) relations $\widetilde{A}$ in $\mathfrak{H} \oplus \mathcal{H}$. □

Remark 2.11. Let $\mathcal{C}$ be a Cayley transform of $\widetilde{A}$

$$\mathcal{C} : \widetilde{A} \mapsto U = \left\{ \{u' + iu, u' - iu\} : \{u, u'\} \in \widetilde{A} \right\}.$$

Then the transform $\mathcal{C} \circ \mathcal{J}$ is a kind of Potapov-Ginzburg transform (see [27]) which establishes a one-to-one correspondence between isometric (contractive, expanding, unitary) relations from $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to $(\mathcal{H}^2, J_{\mathcal{H}})$ and isometric (contractive, expanding, unitary) operators in $\mathfrak{H} \oplus \mathcal{H}$.

With the Hilbert spaces $\mathfrak{H}$ and $\mathcal{H}$ define the orthogonal sum $\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathcal{H}$. The following identifications will be used:

$$\mathfrak{H}_1 = \begin{pmatrix} \mathfrak{H} \\ 0 \end{pmatrix}, \quad \mathfrak{H}_2 = \begin{pmatrix} 0 \\ \mathcal{H} \end{pmatrix}, \quad \widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2 = \begin{pmatrix} \mathfrak{H} \\ \mathcal{H} \end{pmatrix}. \tag{2.23}$$

Let P_j be the orthogonal projection from $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}_j$, $j = 1, 2$.

Proposition 2.12. *Let the linear relations* Γ *and* $\widetilde{A}$ *be related by* (2.20), *i.e.* $\widetilde{A} = \mathcal{J}\Gamma$. *Then the linear relations*

$$S_1 = \ker \Gamma, \quad -S_2 = \operatorname{mul} \Gamma, \quad T_1 = \operatorname{dom} \Gamma, \quad -T_2 = \operatorname{ran} \Gamma, \tag{2.24}$$

are given by

$$S_j = \widetilde{A} \cap \mathfrak{H}_j^2, \quad T_j = \left\{ \{P_j\varphi, P_j\varphi'\} : \{\varphi, \varphi'\} \in \widetilde{A} \right\}. \tag{2.25}$$

Moreover, if $\widetilde{A}$ *is a symmetric linear relation in* $\widetilde{\mathfrak{H}}$, *then* $T_j \subset S_j^*$ *and in particular* S_j *is a symmetric linear relation in* $\mathfrak{H}_j$, $j = 1, 2$. *If, in addition,* $\widetilde{A}$ *is selfadjoint, then*

$$\operatorname{clos} T_j = S_j^* \quad j = 1, 2. \tag{2.26}$$

Proof. The equalities (2.25) are immediate from (2.20). The inclusions $T_j \subset S_j^*$ with $\widetilde{A}$ symmetric (Γ isometric) and the equalities (2.26) with $\widetilde{A}$ selfadjoint (Γ unitary) are implied by Proposition 2.3 in view of (2.24) and (2.18). □

If for $j = 1$ or $j = 2$, the relation S_j is densely defined, then it follows from (2.26) that $\operatorname{clos} T_j = S_j^*$ is an operator, and (2.25) shows that $P_j \operatorname{mul} \widetilde{A} = \{0\}$.

The next result gives some mapping properties of isometric relations in product spaces.

Proposition 2.13. *Let* Γ *be an isometric relation from* $(\mathfrak{H}^2, J_{\mathfrak{H}})$ *to* $(\mathcal{H}^2, J_{\mathcal{H}})$ *and let* $A \subset \operatorname{dom} \Gamma$ *be a linear relation in* $\mathfrak{H}^2$. *Then:*

(i) A *is symmetric (dissipative, accumulative) in* $\mathfrak{H}^2$ *if and only if* $\Gamma(A)$ *is symmetric (dissipative, accumulative) in* $\mathcal{H}^2$;
(ii) *if* $A^* \subset \operatorname{dom} \Gamma$ *then* $\Gamma(A^*) \subset \Gamma(A)^*$;
(iii) *if* $A^* \subset \operatorname{dom} \Gamma$ *and* $\Gamma(A)$ *is essentially selfadjoint in* $\mathcal{H}^2$, *then* A *is essentially selfadjoint in* $\mathfrak{H}^2$.

Proof. (i) By definition $\Gamma(A) = \{ \widehat{h} : \{\widehat{f}, \widehat{h}\} \in \Gamma \text{ for some } \widehat{f} \in A \}$ and the statement follows from

$$2\operatorname{Im}(f', f) = \left(J_{\mathfrak{H}} \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} f \\ f' \end{pmatrix} \right)_{\mathfrak{H}^2} = \left(J_{\mathcal{H}} \begin{pmatrix} h \\ h' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right)_{\mathcal{H}^2} = 2\operatorname{Im}(h', h).$$

(ii) Let $\widehat{g} \in A^*$ and let $\{\widehat{g}, \widehat{k}\} \in \Gamma$. Then for every $\widehat{h} \in \Gamma(A)$ one obtains

$$0 = \left(J_{\mathfrak{H}} \begin{pmatrix} g \\ g' \end{pmatrix}, \begin{pmatrix} f \\ f' \end{pmatrix} \right)_{\mathfrak{H}^2} = \left(J_{\mathcal{H}} \begin{pmatrix} k \\ k' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right)_{\mathcal{H}^2},$$

since here $\widehat{f} \in A$. This means that $\widehat{k} \in \Gamma(A)^*$ and hence $\Gamma(A^*) \subset \Gamma(A)^*$.

(iii) If $\Gamma(A)$ is essentially selfadjoint, then by part (i) A is symmetric. Now part (ii) shows that $\Gamma(A) \subset \Gamma(A^*) \subset \Gamma(A)^*$. Hence, $\operatorname{clos} \Gamma(A) = \operatorname{clos} \Gamma(A^*)$ and $\Gamma(A^*)$ is essentially selfadjoint. Therefore, A^* must be symmetric by part (i) and consequently $A^* = A^{**} =$ *closA*. □

2.4. **Orthogonal couplings.** Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be arbitrary Hilbert spaces (not necessarily the same as in (2.23)) and let $\widetilde{A}$ be a selfadjoint linear relation in the orthogonal sum $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Then the formula (2.25) defines closed symmetric linear relations S_1 and S_2, and not necessarily closed linear relations T_1 and T_2, in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. The relation $\widetilde{A}$ can be interpreted as a selfadjoint extension of the orthogonal sum $S_1 \oplus S_2$. It is called the *orthogonal coupling* of S_1 and T_2 (or of T_1 and S_2), see [30]. The selfadjoint relation $\widetilde{A}$ is said to be *minimal* with respect to the Hilbert space $\mathfrak{H}_j$ (j is fixed, j=1,2) if

$$\mathfrak{H}_1 \oplus \mathfrak{H}_2 = \overline{\operatorname{span}} \left\{ \mathfrak{H}_j + (\widetilde{A} - \lambda)^{-1} \mathfrak{H}_j : \lambda \in \rho(\widetilde{A}) \right\}. \tag{2.27}$$

The null spaces associated to T as in (2.4) are said to be "defect spaces" of the linear relations T_j, i.e.,

$$\mathfrak{N}_\lambda(T_j) = \ker (T_j - \lambda), \quad \widehat{\mathfrak{N}}_\lambda(T_j) = \{ \{f, \lambda f\} \in T_j : f \in \mathfrak{N}_\lambda(T_j) \}. \tag{2.28}$$

For the notational convenience the usual defect spaces of S_j are denoted here by $\mathfrak{N}_\lambda(S_j^*)$ and $\widehat{\mathfrak{N}}_\lambda(S_j^*)$.

Lemma 2.14. *Let $\widetilde{A}$ be a selfadjoint linear relation in $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, and let the linear relations S_j and T_j, $j = 1, 2$, be defined by* (2.25). *Then:*

(i) $\mathfrak{N}_\lambda(T_1) = P_1(\widetilde{A} - \lambda)^{-1} \mathfrak{H}_2$, $\mathfrak{N}_\lambda(T_2) = P_2(\widetilde{A} - \lambda)^{-1} \mathfrak{H}_1$;
(ii) *$\mathfrak{N}_\lambda(T_j)$ is dense in $\mathfrak{N}_\lambda(S_j^*)$ for all $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$, $j = 1, 2$;*
(iii) *The deficiency indices of S_1 and $-S_2$ coincide: $n_\pm(S_1) = n_\mp(S_2)$;*
(iv) *$\widetilde{A}$ is minimal with respect to $\mathfrak{H}_1$ (resp. $\mathfrak{H}_2$) if and only if S_2 (resp. S_1) is simple.*

Proof. First observe that

$$(\widetilde{A} - \lambda)^{-1} \begin{pmatrix} f' - \lambda f \\ -h' - \lambda h \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}, \quad \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' \\ -h' \end{pmatrix} \right\} \in \widetilde{A}. \tag{2.29}$$

(i) Note in (2.29) that $f \in \mathfrak{N}_\lambda(T_1)$ if and only if $f' = \lambda f$. This gives the first assertion. The proof of the second assertion is similar.

(ii) Since $\operatorname{ran}(S_1 - \lambda) = \{ f' - \lambda f : f \in \operatorname{dom} S_1 \}$ the following identities follow easily from (2.29):

$$\ker P_2(\widetilde{A} - \lambda)^{-1} \restriction \mathfrak{H}_1 = \operatorname{ran}(S_1 - \lambda), \quad \ker P_1(\widetilde{A} - \lambda)^{-1} \restriction \mathfrak{H}_2 = \operatorname{ran}(S_2 - \lambda). \tag{2.30}$$

Note that $\overline{\operatorname{ran}}\, X^* = (\ker X)^\perp$ for any bounded linear operator X. Thus the identities in (2.30) imply that the ranges of

$$P_1(\widetilde{A} - \lambda)^{-1} \restriction \mathfrak{H}_2 = \left(P_2(\widetilde{A} - \bar{\lambda})^{-1} \restriction \mathfrak{H}_1 \right)^*, \quad P_2(\widetilde{A} - \lambda)^{-1} \restriction \mathfrak{H}_1 = \left(P_1(\widetilde{A} - \bar{\lambda})^{-1} \restriction \mathfrak{H}_2 \right)^*,$$

are dense subsets of $\mathfrak{N}_\lambda(S_1^*)$ and $\mathfrak{N}_\lambda(S_2^*)$ for all $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$, respectively.

(iii) In view of (2.30) the statements (i) and (ii) can be rewritten in the form

$$\overline{\operatorname{span}}\, P_1(\widetilde{A} - \lambda)^{-1} \mathfrak{N}_{\bar{\lambda}}(S_2^*) = \mathfrak{N}_\lambda(S_1^*), \quad \overline{\operatorname{span}}\, P_2(\widetilde{A} - \lambda)^{-1} \mathfrak{N}_{\bar{\lambda}}(S_1^*) = \mathfrak{N}_\lambda(S_2^*), \tag{2.31}$$

respectively. These identities imply the equality of the defect numbers.

(iv) If $\widetilde{A}$ is minimal with respect to $\mathfrak{H}_1$, then it follows from (i), (ii), and (2.27) that

$$\mathfrak{H}_2 = \overline{\operatorname{span}} \{ P_2(\widetilde{A} - \lambda)^{-1} \mathfrak{H}_1 : \lambda \in \rho(\widetilde{A}) \} = \overline{\operatorname{span}} \{ \mathfrak{N}_\lambda(S_2^*) : \lambda \in \rho(\widetilde{A}) \},$$

so that S_2 is simple. Conversely, if S_2 is simple, then clearly (2.27) is satisfied and $\widetilde{A}$ is minimal with respect to $\mathfrak{H}_1$. □

Proposition 2.15. *Let $\widetilde{A}$ be a selfadjoint relation in $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, and let the linear relations T_1 and T_2 be given by* (2.25)*. Then T_1 is closed if and only if T_2 is closed.*

Proof. Let Γ be defined by $\widetilde{A} = \mathcal{J}\Gamma$, so that Γ is a unitary relation. By definition $T_1 = \operatorname{dom}\Gamma$ and $T_2 = \operatorname{ran}\Gamma$. Hence, the statement follows from Proposition 2.3. □

2.5. **Nevanlinna families.** A family of linear relations $M(\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, in a Hilbert space $\mathcal{H}$ is called a Nevanlinna family if

(i) for every $\lambda \in \mathbb{C}^+(\mathbb{C}^-)$ the relation $M(\lambda)$ is (maximal) dissipative (resp. accumulative);
(ii) $M(\lambda)^* = M(\bar{\lambda})$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$;
(iii) for some, and hence for all, $\mu \in \mathbb{C}^+(\mathbb{C}^-)$ the operator family $(M(\lambda)+\mu)^{-1} \in [\mathcal{H}]$ is holomorphic for all $\lambda \in \mathbb{C}^+(\mathbb{C}^-)$.

By the maximality condition, each relation $M(\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, is necessarily closed. The class of all Nevanlinna families in a Hilbert space is denoted by $\widetilde{R}(\mathcal{H})$. Nevanlinna families were considered in [21], [14], and [23], where the following orthogonal decomposition can be found.

Proposition 2.16. *If $M(\cdot) \in \widetilde{R}(\mathcal{H})$, then the multivalued part* $\operatorname{mul} M(\lambda)$ *is independent of $\lambda \in \mathbb{C}\setminus\mathbb{R}$, so that*

$$M(\lambda) = M_s(\lambda) \oplus M_\infty, \quad M_\infty = \{0\} \oplus \operatorname{mul} M(\lambda), \quad \lambda \in \mathbb{C}\setminus\mathbb{R}, \tag{2.32}$$

where $M_s(\lambda)$ is a Nevanlinna family of densely defined operators in $\mathcal{H} \ominus \operatorname{mul} M(\lambda)$.

Clearly, if $M(\cdot) \in \widetilde{R}(\mathcal{H})$, then $M_\infty \subset M(\lambda) \cap M(\lambda)^*$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$. The following subclasses of the class $\widetilde{R}(\mathcal{H})$ will be useful:

$R(\mathcal{H})$ is the set of all $M(\cdot) \in \widetilde{R}(\mathcal{H})$ for which $\operatorname{mul} M(\lambda) = \{0\}$;
$R^s(\mathcal{H})$ is the set of all $M(\cdot) \in \widetilde{R}(\mathcal{H})$ for which $M(\lambda) \cap M(\lambda)^* = \{0\}$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$;
$R^u(\mathcal{H})$ is the set of all $M(\cdot) \in \widetilde{R}(\mathcal{H})$ for which $M(\lambda) \widehat{+} M(\lambda)^* = \mathcal{H}^2$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Hence, $M(\cdot) \in R^s(\mathcal{H})$ or $M(\cdot) \in R^u(\mathcal{H})$, if $M(\lambda)$ and $M(\lambda)^*$ are disjoint or transversal, respectively, for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$. With the classes $\widetilde{R}(\mathcal{H})$, $R(\mathcal{H})$, $R^s(\mathcal{H})$, and $R^u(\mathcal{H})$ correspond the classes $\widetilde{R}_{inv}(\mathcal{H})$, $R_{inv}(\mathcal{H})$, $R^s_{inv}(\mathcal{H})$, and $R^u_{inv}(\mathcal{H})$ of Nevanlinna families $M(\cdot)$ whose domain $\operatorname{dom} M(\lambda)$ does not depend on $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Furthermore, the following subclasses of $\widetilde{R}(\mathcal{H})$ will be important:

$\widetilde{R}[\mathcal{H}]$ is the set of all $M(\cdot) \in \widetilde{R}(\mathcal{H})$ for which $\operatorname{dom} M(\lambda)$ is closed for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$;
$R[\mathcal{H}]$ is the set of all $M(\cdot) \in \widetilde{R}[\mathcal{H}]$ for which $\operatorname{dom} M(\lambda) = \mathcal{H}$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$;
$R^s[\mathcal{H}]$ is the set of all $M(\cdot) \in R[\mathcal{H}]$ for which $\ker \operatorname{Im} M(\lambda) = \{0\}$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$;
$R^u[\mathcal{H}]$ is the set of all $M(\cdot) \in R^s[\mathcal{H}]$ for which $0 \in \rho(\operatorname{Im} M(\lambda))$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Remark 2.17. In Section 4 (cf. also the Appendix) it will be shown that various properties which were used above to define different subclasses of Nevanlinna families are independent from $\lambda \in \mathbb{C}\setminus\mathbb{R}$. This means that the corresponding subclasses of $\widetilde{R}(\mathcal{H})$ can be equivalently defined by assuming the corresponding property of $M(\lambda)$ only at a single point $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

For each $M(\cdot)$ in $\widetilde{R}[\mathcal{H}]$ or its subclasses, the operator $M_s(\lambda)$ is necessarily bounded. In what follows, the Nevanlinna functions in $R^s[\mathcal{H}]$ and $R^u[\mathcal{H}]$ will be called *strict* and *uniformly strict*, respectively.

Proposition 2.18. *Let $M(\cdot) \in \widetilde{R}(\mathcal{H})$. Then the following statements are equivalent:*

(i) *$M(\cdot) \in R^u(\mathcal{H})$;*

(ii) *$M(\lambda) \in [\mathcal{H}]$ and $0 \in \rho(\operatorname{Im} M(\lambda))$ for some, and hence for all, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

The result in Proposition 2.18 is a consequence of Propositions 4.5 and 5.3, see also Theorem 4.13 and Proposition 4.22; one further independent proof is given in the Appendix, Proposition A.8).

The definitions and Proposition 2.18 give rise to the inclusions and the equalities in the following array:

$$\begin{array}{ccccccc} R^u(\mathcal{H}) & \subset & R^s(\mathcal{H}) & \subset & R(\mathcal{H}) & \subset & \widetilde{R}(\mathcal{H}) \\ \| & & \cup & & \cup & & \cup \\ R^u_{inv}(\mathcal{H}) & \subset & R^s_{inv}(\mathcal{H}) & \subset & R_{inv}(\mathcal{H}) & \subset & \widetilde{R}_{inv}(\mathcal{H}) \\ \| & & \cup & & \cup & & \cup \\ R^u[\mathcal{H}] & \subset & R^s[\mathcal{H}] & \subset & R[\mathcal{H}] & \subset & \widetilde{R}[\mathcal{H}] \end{array} \tag{2.33}$$

In the infinite dimensional situation each of the inclusions is strict. However, in the finite-dimensional situation the vertical inclusions in this array reduce to equalities

$$R^s(\mathcal{H}) = R^s_{inv}(\mathcal{H}) = R^s[\mathcal{H}], \quad R(\mathcal{H}) = R_{inv}(\mathcal{H}) = R[\mathcal{H}], \quad \widetilde{R}(\mathcal{H}) = \widetilde{R}_{inv}(\mathcal{H}) = \widetilde{R}[\mathcal{H}]. \tag{2.34}$$

If $M(\cdot) \in R[\mathcal{H}]$, then it admits the following integral representation

$$M(\lambda) = A + B\lambda + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1} \right) d\Sigma(t), \quad \int_{\mathbb{R}} \frac{d\Sigma(t)}{t^2+1} \in [\mathcal{H}], \tag{2.35}$$

where $A = A^* \in [\mathcal{H}]$, $0 \le B = B^* \in [\mathcal{H}]$, the $[\mathcal{H}]$-valued family $\Sigma(\cdot)$ is nondecreasing, and the integral is uniformly convergent in the strong topology, cf. [5].

3. Boundary relations and Weyl families

3.1. **Definition of a boundary relation and its Weyl family.** Let S be a closed symmetric linear relation in the Hilbert space $\mathfrak{H}$. It is not assumed that the defect numbers of S are equal or finite. A boundary relation for S^* is defined as follows.

Definition 3.1. Let S be a closed symmetric linear relation in a Hilbert space $\mathfrak{H}$ and let $\mathcal{H}$ be an auxiliary Hilbert space. A linear relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is called a *boundary relation* for S^*, if:

(G1) $\operatorname{dom} \Gamma$ is dense in S^* and the identity

$$(f', g)_{\mathfrak{H}} - (f, g')_{\mathfrak{H}} = (h', k)_{\mathcal{H}} - (h, k')_{\mathcal{H}}, \tag{3.1}$$

holds for every $\{\widehat{f}, \widehat{h}\}$, $\{\widehat{g}, \widehat{k}\} \in \Gamma$;

(G2) Γ is maximal in the sense that if $\{\widehat{g}, \widehat{k}\} \in \mathfrak{H}^2 \oplus \mathcal{H}^2$ satisfies (3.1) for every $\{\widehat{f}, \widehat{h}\} \in \Gamma$, then $\{\widehat{g}, \widehat{k}\} \in \Gamma$.

Here $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in \operatorname{dom} \Gamma (\subset \mathfrak{H}^2)$, $\widehat{h} = \{h, h'\}$, $\widehat{k} = \{k, k'\} \in \operatorname{ran} \Gamma (\subset \mathcal{H}^2))$.

The condition (3.1) in (G1) can be interpreted as an abstract Green's identity. Using the terminology of Kreĭn spaces (3.1) means that Γ is an isometric relation from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$, since

$$(J_{\mathfrak{H}}\widehat{f}, \widehat{g})_{\mathfrak{H}^2} = (J_{\mathcal{H}}\widehat{h}, \widehat{k})_{\mathcal{H}^2}, \quad \{\widehat{f}, \widehat{h}\}, \quad \{\widehat{g}, \widehat{k}\} \in \Gamma. \tag{3.2}$$

The maximality condition (G2) and Proposition 2.3 now imply the following result.

Proposition 3.2. *Let $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ be a boundary relation for S^*. Then Γ is a unitary relation from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$. Moreover, $S = \ker \Gamma$.*

Proof. In view of (3.2) Γ is isometric, i.e., $\Gamma^{-1} \subset \Gamma^{[*]}$. Now assume that $\{\widehat{k}, \widehat{g}\} \in \Gamma^{[*]}$. Then

$$(J_{\mathfrak{H}}\widehat{g}, \widehat{f})_{\mathfrak{H}^2} - (J_{\mathcal{H}}\widehat{k}, \widehat{h})_{\mathcal{H}^2} = 0,$$

holds for every $\{\widehat{f}, \widehat{h}\} \in \Gamma$ and hence (3.1) is satisfied. By assumption (G2) one concludes that $\{\widehat{g}, \widehat{k}\} \in \Gamma$, or equivalently, that $\{\widehat{k}, \widehat{g}\} \in \Gamma^{-1}$. This proves the reverse inclusion $\Gamma^{[*]} \subset \Gamma^{-1}$.

Since $\overline{\operatorname{dom}}\,\Gamma = S^*$, the identity $S = \ker \Gamma$ is implied by Proposition 2.3 and (2.18):

$$\ker \Gamma = (\operatorname{dom} \Gamma)^{[\perp]} = (S^*)^{[\perp]} = S.$$

This completes the proof. □

Note that the boundary relation Γ is automatically closed and linear, since it is a unitary relation from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$. However, it can be multivalued, nondensely defined, or unbounded.

Let Γ be a boundary relation for S^* and let $T = \operatorname{dom} \Gamma$. According to Proposition 2.12 (see (2.26)) the linear relation T in $\mathfrak{H}$ satisfies

$$S \subset T \subset S^*, \quad \operatorname{clos} T = S^*. \tag{3.3}$$

The defect spaces $\mathfrak{N}_\lambda(T)$ and $\widehat{\mathfrak{N}}_\lambda(T)$ for T are defined as in (2.28). For all $\{\widehat{f}_\lambda, \widehat{h}\}, \{\widehat{g}_\mu, \widehat{k}\} \in \Gamma$ with $\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(T)$ and $\widehat{g}_\mu \in \widehat{\mathfrak{N}}_\mu(T)$ one has

$$(\lambda - \bar{\mu})(f_\lambda, g_\mu)_{\mathfrak{H}} = (h', k)_{\mathcal{H}} - (h, k')_{\mathcal{H}}, \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}, \tag{3.4}$$

which follows from the identity (3.1). Hence, the subspace $\widehat{\mathfrak{N}}_\lambda(T)$ is positive in the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ for $\lambda \in \mathbb{C}_+$ and negative for $\lambda \in \mathbb{C}_-$.

Definition 3.3. The *Weyl family* $M(\lambda)$ of S corresponding to the boundary relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is defined by

$$M(\lambda) := \Gamma(\widehat{\mathfrak{N}}_\lambda(T)) := \left\{ \widehat{h} \in \mathcal{H}^2 : \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma \text{ for some } \widehat{f}_\lambda = \{f, \lambda f\} \in \mathfrak{H}^2 \right\}. \tag{3.5}$$

In the case where $M(\lambda)$ is operator-valued it is called the *Weyl function* of S corresponding to the boundary relation Γ.

It will be shown that each Weyl family is a Nevanlinna family, and conversely, that each Nevanlinna family can be realized as the Weyl family of a minimal boundary relation.

Definition 3.4. The boundary relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is called *minimal*, if

$$\mathfrak{H} = \mathfrak{H}_{min} := \overline{\operatorname{span}}\, \{ \mathfrak{N}_\lambda(T) : \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \}.$$

Since $\mathfrak{N}_\lambda(T)$ is dense in $\mathfrak{N}_\lambda(S^*)$ (cf. Lemma 2.14) the boundary relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is minimal if and only if S is simple. In general, if S_{min} is the simple part of S the restriction $\Gamma_{min} : \mathfrak{H}^2_{min} \mapsto \mathcal{H}^2$ of the linear relation Γ to $\mathfrak{H}_{min}$ is a boundary relation for S^*_{min}. Clearly, the Weyl families corresponding to the linear relations Γ and Γ_{min} coincide.

Associate with Γ the following linear relations which are not necessarily closed:
(3.6)

$$\Gamma_0 = \left\{ \{\widehat{f}, h\} : \{\widehat{f}, \widehat{h}\} \in \Gamma, \, \widehat{h} = \{h, h'\} \right\}, \quad \Gamma_1 = \left\{ \{\widehat{f}, h'\} : \{\widehat{f}, \widehat{h}\} \in \Gamma, \, \widehat{h} = \{h, h'\} \right\}.$$

It is clear that

$$\operatorname{dom} M(\lambda) = \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) \subset \operatorname{ran} \Gamma_0, \quad \operatorname{ran} M(\lambda) = \Gamma_1(\widehat{\mathfrak{N}}_\lambda(T)) \subset \operatorname{ran} \Gamma_1. \tag{3.7}$$

If the boundary relation Γ is single-valued the triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ will be called a *boundary triplet* associated with the boundary relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$. In this case the Weyl family corresponding to the boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ can be also defined via the equality

$$\Gamma_1(\{f_\lambda, \lambda f_\lambda\}) = M(\lambda)\Gamma_0(\{f_\lambda, \lambda f_\lambda\}), \quad \{f_\lambda, \lambda f_\lambda\} \in T. \tag{3.8}$$

Finally, observe the following useful fact. Let $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ be a boundary relation for S^*. Then

$$\Gamma^\top = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Gamma \tag{3.9}$$

is a unitary relation from $\mathfrak{H}^2$ to $\mathcal{H}^2$. Clearly, $\Gamma^\top$ is also a boundary relation for S^* (so that, in particular, $\ker \Gamma^\top = S$). Consequently, if $M(\cdot)$ is the Weyl family for Γ, then $-M(\cdot)^{-1}$ is the Weyl family for $\Gamma^\top$.

3.2. **Orthogonal coupling associated with a boundary relation.** In this subsection the linear transform $\mathcal{J}$ introduced in Subsection 2.3 will be used in order to obtain some criteria for a linear relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ to be a boundary relation. For a boundary relation Γ from $\mathfrak{H}^2$ to $\mathcal{H}^2$ the relation $\widetilde{A} = \mathcal{J}\Gamma$ is defined by (2.20). In the following proposition some results of Subsection 2.3 are reformulated in terms of boundary relations.

Proposition 3.5. *Let Γ be a subspace in $\mathfrak{H}^2 \oplus \mathcal{H}^2$ and let $S = \ker \Gamma$. Then Γ is a boundary relation for S^* if and only if $\widetilde{A} = \mathcal{J}\Gamma$ is a selfadjoint linear relation in $\mathfrak{H} \oplus \mathcal{H}$. In this case the boundary relation Γ is minimal if and only if $\widetilde{A} = \mathcal{J}\Gamma$ is a minimal selfadjoint extension of $S_2 = \operatorname{mul} \Gamma$.*

Proof. The first statement is immediate from Propositions 2.10 and 2.12 and Definition 3.1. By Definition 3.4 the minimality of the linear relation Γ is equivalent to the simplicity of S which, in turn, is equivalent to the minimality of $\widetilde{A} = \mathcal{J}\Gamma$ as a selfadjoint extension of S_2 (see Lemma 2.14). □

Proposition 3.6. *The linear relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is a boundary relation for S^* if and only if*

(i) $\operatorname{dom} \Gamma$ *is dense in S^*;*
(ii) Γ *is closed and isometric from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$;*
(iii) $\operatorname{ran}(\Gamma(\widehat{\mathfrak{N}}_\lambda(T)) + \lambda) (= \operatorname{ran}(M(\lambda) + \lambda)) = \mathfrak{H}$ *for some (and, hence, for all) $\lambda \in \mathbb{C}_+$ and for some (and, hence, for all) $\lambda \in \mathbb{C}_-$.*

Proof. Let Γ be a boundary relation for S^*. Then (i) is satisfied by definition. Furthermore, by Proposition 3.2, Γ is a unitary relation from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$. Hence, (ii) is also satisfied. The transform $\widetilde{A} = \mathcal{J}\Gamma$ is selfadjoint, so that $\operatorname{ran}(\widetilde{A}-\lambda) = \mathfrak{H} \oplus \mathcal{H}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In particular,

$$\mathfrak{H}_2 := \begin{pmatrix} 0 \\ \mathcal{H} \end{pmatrix} \subset \operatorname{ran}(\widetilde{A} - \lambda). \tag{3.10}$$

It follows from (2.20) that

$$\widetilde{A} - \lambda = \left\{ \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' - \lambda f \\ -h' - \lambda h \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \right\}, \tag{3.11}$$

which together with (3.5) and (3.10) gives (iii).

Conversely, assume that the linear relation Γ satisfies (i), (ii), and (iii). By (ii) and Proposition 2.10 it follows that $\widetilde{A} = \mathcal{J}\Gamma$ is closed and symmetric. In order to prove that $\widetilde{A}$ is selfadjoint it suffices to show that $\operatorname{ran}(\widetilde{A}-\lambda)$ is dense for some $\lambda \in \mathbb{C}_+$ and for some $\lambda \in \mathbb{C}_-$. It follows from (iii) and (3.11) that $\mathfrak{H}_2 \subset \operatorname{ran}(\widetilde{A} - \lambda)$. To complete the argument, assume that $\varphi \in \mathfrak{H}_1$ is orthogonal to $\operatorname{ran}(\widetilde{A}-\lambda)$. This implies that $\{\varphi, \bar{\lambda}\varphi\} \in T^*$, where $T = \operatorname{dom}\Gamma$. By (i) T is dense in S^* and hence $S = T^*$ and $\{\varphi, \bar{\lambda}\varphi\} \in S$. Since S is symmetric this yields $\varphi = 0$. □

Next it will be shown that for every closed symmetric linear relation S there exists a boundary relation for S^*; in the case of equal defect numbers this fact is well known.

Proposition 3.7. *Let S be any closed symmetric linear relation with arbitrary defect numbers in a Hilbert space $\mathfrak{H}$. Then there exists a boundary relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ for S^*.*

Proof. Let $\widetilde{A}$ be any selfadjoint exit space extension of S with the property $\widetilde{A} \cap \mathfrak{H}^2 = S$. Then by Proposition 2.12 the transform $\Gamma = \mathcal{J}^{-1}\widetilde{A}$ of $\widetilde{A}$ satisfies $\ker\Gamma = S$ and hence by Proposition 3.5 Γ defines a boundary relation for S^*. A particular construction of such an extension $\widetilde{A}$ can be given as follows.

In the orthogonal sum $\mathfrak{H} \oplus \mathfrak{H}$ the relation $S \oplus (-S)$ is closed and symmetric with equal defect numbers. Define the relation $\widetilde{A}$ in $\mathfrak{H} \oplus \mathfrak{H}$ by

$$\widetilde{A} = \left\{ \widehat{f} = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1' \\ -f_2' \end{pmatrix} \right\} : \widehat{f_1} = \{f_1, f_1'\},\ \widehat{f_2} = \{f_2, f_2'\} \in S^*,\ P_{\mathfrak{N}}\widehat{f_1} = P_{\mathfrak{N}}\widehat{f_2} \right\}, \tag{3.12}$$

where $P_{\mathfrak{N}}$ is the orthogonal projection from S^* onto $\mathfrak{N} = \widehat{\mathfrak{N}}_i(S^*) \oplus \widehat{\mathfrak{N}}_{-i}(S^*)$, cf. (2.7). The elements $\widehat{f_j} \in S^*$, $j = 1, 2$, in (3.12) have the representations

$$\{f_j, f_j'\} = \{h_j, h_j'\} \widehat{+} \{\varphi_i, i\varphi_i\} \widehat{+} \{\varphi_{-i}, -i\varphi_{-i}\},$$

where $\{h_j, h_j'\} \in S$, $\{\varphi_i, i\varphi_i\} \in S^*$, $\{\varphi_{-i}, -i\varphi_{-i}\} \in S^*$. With this notation a typical element of the Cayley transform U of $\widetilde{A}$

$$U = \{ \{u' - iu, u' + iu\} : \{u, u'\} \in \widetilde{A} \},$$

is of the form

$$\left\{ \begin{pmatrix} h_1' - ih_1 - 2i\varphi_{-i} \\ -(h_2' + ih_2) - 2i\varphi_i \end{pmatrix}, \begin{pmatrix} h_1' + ih_1 + 2i\varphi_i \\ -(h_2' - ih_2) + 2i\varphi_{-i} \end{pmatrix} \right\}.$$

This shows immediately that U is isometric and that $\operatorname{dom} U = \operatorname{ran} U = \mathfrak{H} \oplus \mathcal{H}$, so that U is unitary. Hence $\widetilde{A}$ is a selfadjoint relation and it clearly extends $S \oplus (-S)$. Moreover, clearly $S \subset \widetilde{A} \cap \mathfrak{H}^2$. In order to prove the reverse inclusion, assume that

$$\widehat{f} = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1' \\ -f_2' \end{pmatrix} \right\} \in \widetilde{A} \cap \mathfrak{H}^2.$$

Then $f_2 = f_2' = 0$, and by the definition (3.12) of the relation $\widetilde{A}$, it follows that $P_{\mathfrak{N}} \widehat{f}_1 = 0$, so that $\{f_1, f_1'\} \in S$. Hence $\widetilde{A} \cap \mathfrak{H}^2 \subset S$, and consequently, $\widetilde{A} \cap \mathfrak{H}^2 = S$. □

Remark 3.8. One can simplify the construction of the extension $\widetilde{A}$ in the previous proposition when S is a closed symmetric operator with equal defect numbers. Let V be an isometric mapping from $\mathfrak{N}_{-i}(S^*)$ onto $\mathfrak{N}_i(S^*)$ and let $\mathcal{H} = \mathfrak{N}_i(S^*)$. Define the linear relation $\widetilde{A}$ by

$$\widetilde{A} = \left\{ \left\{ \begin{pmatrix} f \\ \varphi_i + V\varphi_{-i} \end{pmatrix}, \begin{pmatrix} f' \\ -i\varphi_i + iV\varphi_{-i} \end{pmatrix} \right\} : \widehat{f} = \{f, f'\} \in S^*, \ \varphi_{\pm i} = \pi_{\pm i} \widehat{f} \right\},$$

where $\pi_{\pm i}$ are the orthoprojections onto $\mathfrak{N}_{\pm i}(S^*)$ in the decomposition (2.7). Then $\widetilde{A}$ is a selfadjoint extention of S such that $\widetilde{A} \cap \mathfrak{H}^2 = S$.

The transform $\Gamma = \mathcal{J}^{-1}\widetilde{A}$ defines a boundary relation for S^* with the additional property $\operatorname{ran} \Gamma = \mathcal{H}^2$ (so that $\operatorname{dom} \Gamma = S^*$, $\operatorname{mul} \Gamma = \{0, 0\}$, which implies that Γ is a bounded linear operator)], cf. Corollary 2.6. It corresponds to an ordinary boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with the boundary operators Γ_0, Γ_1 given by

$$\Gamma_0 \widehat{f} = \pi_i \widehat{f} + V\pi_{-i}\widehat{f}, \quad \Gamma_1 \widehat{f} = i\pi_i \widehat{f} - iV\pi_{-i}\widehat{f}, \tag{3.13}$$

cf. [24]. In the case of a densely defined operator S with equal defect numbers, the statement of Proposition 3.7 and the formulas (3.13) go back to V. Bruck and A. Kochubeĭ (see [17]).

3.3. **A characterization of Weyl families: the main realization theorem.** As was shown in [13] for every Nevanlinna function $M(\cdot) \in R^u[\mathcal{H}]$ there exist a symmetric operator S in a Hilbert space $\mathfrak{H}$ and an ordinary boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ such that the corresponding Weyl function is $M(\cdot)$. Since every Weyl function is also a Q-function of the pair $\{S, A_0\}$ (see [13]) this gives a realization for every R-function as a Q-function of a pair $\{S, A_0\}$. The latter problem has been originally solved by M.G. Kreĭn and H. Langer in [22] for the case $\overline{\operatorname{dom} S} = \mathfrak{H}$ and extended to the case $\overline{\operatorname{dom} S} \neq \mathfrak{H}$ in [23]. That every function $M(\cdot)$ from $R^s[\mathcal{H}]$ can be realized as the Weyl function of an appropriate generalized boundary triplet was shown in [13]. In this subsection this realization theorem is extended to the class $\widetilde{R}(\mathcal{H})$ of all Nevanlinna families and arbitrary boundary relations. The present approach is based on the generalized Naimark theorem and hence it differs from those used in [22], [23], [13].

Two boundary relations $\Gamma^{(j)} : (\mathfrak{H}^{(j)})^2 \to \mathcal{H}^2$, $j = 1, 2$, are said to be *unitary equivalent* if there is a unitary operator $U : \mathfrak{H}^{(1)} \to \mathfrak{H}^{(2)}$ such that

$$\Gamma^{(2)} = \left\{ \left\{ \begin{pmatrix} Uf \\ Uf' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma^{(1)} \right\}. \tag{3.14}$$

If the boundary relations $\Gamma^{(1)}$ and $\Gamma^{(2)}$ satisfy (3.14) and $S_j = \ker \Gamma^{(j)}$, $T_j = \operatorname{dom} \Gamma^{(j)}$, $j = 1, 2$, then $S_2 = US_1U^{-1}$ and $T_2 = UT_1U^{-1}$.

Theorem 3.9. *Let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^*. Then the corresponding Weyl family $M(\cdot)$ belongs to the class $\widetilde{R}(\mathcal{H})$.*

Conversely, if $M(\cdot)$ belongs to the class $\widetilde{R}(\mathcal{H})$ then there exists (up to unitary equivalence) a unique minimal boundary relation whose Weyl function coincides with $M(\cdot)$.

Proof. Necessity. If $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ is a boundary relation, so that $\widetilde{A}$ is selfadjoint, it follows from the (2.20), (3.11) that

$$(\widetilde{A}-\lambda)^{-1}\begin{pmatrix} f'-\lambda f \\ -h'-\lambda h\end{pmatrix} = \begin{pmatrix} f \\ h\end{pmatrix}, \quad \left\{\begin{pmatrix} f \\ f'\end{pmatrix}, \begin{pmatrix} h \\ h'\end{pmatrix}\right\} \in \Gamma,$$

which implies

$$R(\lambda) := P_{\mathcal{H}}(\widetilde{A}-\lambda)^{-1}\restriction \mathcal{H} = -(M(\lambda)+\lambda)^{-1}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{3.15}$$

The latter equality can be rewritten as

$$M(\lambda) = \{\,\{R(\lambda)h, -(I+\lambda R(\lambda))h\} : \, h \in \mathcal{H}\,\}.$$

Since the kernel

$$K(\lambda,\mu) = \frac{R(\lambda)-R(\mu)^*}{\lambda-\bar{\mu}} - R(\mu)^*R(\lambda) = P_{\mathcal{H}}(\widetilde{A}-\bar{\mu})^{-1}(I-P_{\mathcal{H}})(\widetilde{A}-\lambda)^{-1}|_{\mathcal{H}}$$

is nonnegative (see [29]) it follows that $M(\lambda)$ is a Nevanlinna family. Indeed, for every

$$\{f,f'\} = \{\,\{R(\lambda)h, -(I+\lambda R(\lambda))h\} \in M(\lambda) \quad (h \in \mathcal{H})$$

one obtains

$$\frac{(f',f)-(f,f')}{\lambda-\bar{\lambda}} = (K(\lambda,\lambda)h,h) \geq 0.$$

Sufficiency. Assume that $M(\cdot)$ belongs to $\widetilde{R}(\mathcal{H})$. Then $M_1(\lambda) := -(M(\lambda)+\lambda)^{-1}$ belongs to $R[\mathcal{H}]$ and therefore it admits an integral representation of the form (2.35). The estimate $\|M_1(iy)\| \leq 1/y$ and the monotonicity of $y\mathrm{Im}\, M_1(iy)$ show that the strong limit $s-\lim_{y\to\infty} y\mathrm{Im}\, M_1(iy)$ exists and defines a bounded operator in $\mathcal{H}$ such that

$$0 \leq s - \lim_{y\to\infty} y\mathrm{Im}\, M_1(iy) \leq I_{\mathcal{H}}.$$

Moreover, $s-\lim_{y\to\infty} M_1(iy) = 0$. Hence, the integral representation of $M_1(\lambda)$ takes the form

$$M_1(\lambda) = \int_{\mathbb{R}} \frac{d\Sigma(t)}{t-\lambda}, \qquad 0 \leq \int_{\mathbb{R}} d\Sigma(t) = s - \lim_{y\to\infty} y\mathrm{Im}\, M_1(iy) \leq I_{\mathcal{H}}. \tag{3.16}$$

Without loss of generality one may assume that $\Sigma(-\infty)=0$, in which case $0 \leq \Sigma(+\infty) := s-\lim_{t\to\infty}\Sigma(t) \leq I_{\mathcal{H}}$. It follows from the generalized Naimark theorem (cf. [1, 29]) that there is an orthogonal dilation $E(t)$ of $\Sigma(t)$ (i.e. a spectral family of a selfadjoint linear relation $\widetilde{A}$ in some Hilbert space $\widetilde{\mathfrak{H}} \supset \mathcal{H}$ with $\Sigma(t) = P_{\mathcal{H}}E(t)\restriction_{\mathcal{H}}$ for all $t \in \mathbb{R}$. Note that $E(\infty)$ is an orthogonal projection in $\widetilde{\mathfrak{H}}$, which is equal to $I_{\widetilde{\mathcal{H}}}$ if and only if $\widetilde{A}$ is an operator. The linear relation $\widetilde{A}$ can be chosen minimal in the sence that

$$\widetilde{\mathfrak{H}} = \overline{\mathrm{span}}\,\{\,\mathcal{H}, E(t)\mathcal{H} : \, t \in \mathbb{R}\,\}, \tag{3.17}$$

which, of course, is equivalent to the minimality of the selfadjoint extension $\widetilde{A}$ with respect to $\mathcal{H}$. The multivalued part of a minimal selfadjoint extension $\widetilde{A}$ is trivial if and only if $\Sigma(+\infty) = I_{\mathcal{H}}$.

It follows from (3.16) that $M_1(\lambda)$ takes the form

$$M_1(\lambda) = P_{\mathcal{H}}(\widetilde{A} - \lambda)^{-1}\upharpoonright_{\mathcal{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{3.18}$$

where $\widetilde{A}$ is a minimal selfadjoint relation with the above properties. Let $\mathfrak{H} = \widetilde{\mathfrak{H}} \ominus \mathcal{H}$. Decompose the graph of $\widetilde{A}$ as in (2.20) according to the decomposition $\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathcal{H}$ and let $\Gamma = \mathcal{J}^{-1}\widetilde{A}$. Due to Proposition 3.5 Γ defines a boundary relation for S^* where $S := \ker \Gamma = \widetilde{A} \cap (\widetilde{\mathfrak{H}} \ominus \mathcal{H})^2$. By Lemma 2.14 Γ is minimal by the minimality of the selfadjoint extension $\widetilde{A}$ with respect to $\mathcal{H}$, see (3.17). Now the first part of the present proof shows that the Weyl family associated with Γ satisfies (3.15) with the compressed resolvent of $\widetilde{A}$ in $\mathcal{H}$ given by (3.18). Therefore, the Weyl family associated with Γ coincides with the given family $M(\cdot) \in \widetilde{R}_{\mathcal{H}}$.

Uniqueness. To prove the uniqueness assume that $\Gamma^{(j)} : (\mathfrak{H}^{(j)})^2 \to \mathcal{H}^2$, $j = 1, 2$, are two minimal boundary relations with the same Weyl family $M(\lambda)$. Then $\widetilde{A}^{(j)} = \mathcal{J}\Gamma^{(j)}$, $j = 1, 2$, are two selfadjoint linear relations in Hilbert spaces $\widetilde{\mathfrak{H}}^{(j)}(\supset \mathcal{H})$ minimal with respect to $\mathcal{H}$ and such that

$$P_{\mathcal{H}}(\widetilde{A}^{(j)} - \lambda)^{-1}\upharpoonright \mathcal{H} = -(M(\lambda) + \lambda)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.19}$$

Then the corresponding resolutions of identities $E^{(j)}(t)$ also have the minimality properties

$$\widetilde{\mathfrak{H}}^{(j)} = \operatorname{clos} \mathcal{L}^{(j)}, \quad \mathcal{L}^{(j)} = \operatorname{span} \{ \mathcal{H}, E^{(j)}(t)\mathcal{H} : t \in \mathbb{R} \}$$

and by the Stieltjes inversion formula they satisfy the equality $P_{\mathcal{H}}E^{(1)}(t)\upharpoonright_{\mathcal{H}} = P_{\mathcal{H}}E^{(2)}(t)\upharpoonright_{\mathcal{H}}$, for all $t \in \mathbb{R}$. Define the mapping $U_0 : \mathcal{L}^{(1)} \to \mathcal{L}^{(2)}$ by the equalities

$$U_0 h = h, \quad U_0 E^{(1)}(t)h = E^{(2)}(t)h, \quad h \in \mathcal{H}, \ t \in \mathbb{R}. \tag{3.20}$$

It follows from

$$\|E^{(1)}(t)h\|^2_{\widetilde{\mathfrak{H}}^{(1)}} = \int_{(-\infty,t]} d(E^{(1)}(s)h, h)_{\widetilde{\mathfrak{H}}^{(1)}} = \int_{(-\infty,t]} d(E^{(2)}(s)h, h)_{\widetilde{\mathfrak{H}}^{(2)}} = \|E^{(2)}(t)h\|^2_{\widetilde{\mathfrak{H}}^{(2)}}, \ h \in \mathcal{H}$$

that U_0 is a well-defined isometric mapping from $\mathcal{L}^{(1)}$ onto $\mathcal{L}^{(2)}$. Its closure $\widetilde{U}$ is a unitary operator from $\widetilde{\mathfrak{H}}^{(1)}$ onto $\widetilde{\mathfrak{H}}^{(2)}$ and according to the decompositions $\widetilde{\mathfrak{H}}^{(j)} = \mathfrak{H}^{(j)} \oplus \mathcal{H}$ it can be represented as $\widetilde{U} = U \oplus I_{\mathcal{H}}$, where $U : \mathfrak{H}^{(1)} \to \mathfrak{H}^{(2)}$ is unitary. It follows from (3.20) that $\widetilde{U}E_1(t) = E_2(t)\widetilde{U}$ for all $t \in \mathbb{R}$ and, therefore, the selfadjoint linear relations $\widetilde{A}^{(j)}$ are unitary equivalent

$$\widetilde{A}^{(2)} = \left\{ \left\{ \begin{pmatrix} Uf \\ h \end{pmatrix}, \begin{pmatrix} Uf' \\ -h' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' \\ -h' \end{pmatrix} \right\} \in \widetilde{A}^{(1)} \right\}.$$

This leads to the unitary equivalence (3.14) of the boundary relations $\Gamma^{(1)}$ and $\Gamma^{(2)}$. □

4. Weyl families of symmetric operators

4.1. **Subclasses of Weyl families.** The main theorem in the previous section gives a one-to-one correspondence between Nevanlinna families and boundary relations. In this subsection geometric characterizations of subclasses of Nevanlinna families or functions are given in terms of the boundary relation. The following preliminary result is important.

Lemma 4.1. *Let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^* with the Weyl family $M(\lambda) = \Gamma(\widehat{\mathfrak{N}}_\lambda(T))$. Then the following equalities hold for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$:*

(i) $M(\lambda) \cap M(\lambda)^* = \operatorname{mul} \Gamma$ *and* $\operatorname{clos}(M(\lambda) \widehat{+} M(\lambda)^*) = \overline{\operatorname{ran}}\, \Gamma$;

(ii) $\{\ker M(\lambda), 0\} = \operatorname{mul} \Gamma \cap (\mathcal{H} \oplus \{0\})$ *and* $\{0, \operatorname{mul} M(\lambda)\} = \operatorname{mul} \Gamma \cap (\{0\} \oplus \mathcal{H})$;

(iii) $\ker(M(\lambda) - M(\lambda)^*) = \operatorname{mul} \Gamma_0$ *and* $\ker(M(\lambda)^{-1} - M(\lambda)^{-*}) = \operatorname{mul} \Gamma_1$.

Proof. (i) Let $\{0, \widehat{h}\} \in \Gamma$, $\widehat{h} = \{h, h'\}$. Then (3.1) and (3.4) show that for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $\{k, k'\} \in \Gamma(\widehat{\mathfrak{N}}_\lambda(T))$ the identity $(h', k) - (h, k') = 0$ holds. Hence, $\widehat{h} \in M(\lambda)^*$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, which proves that $\operatorname{mul} \Gamma \subset M(\lambda) \cap M(\lambda)^*$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Conversely, if $\widehat{h} \in M(\lambda) \cap M(\lambda)^*$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $\{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma$ for some $\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(T)$ and, moreover, according to (3.4) $(\lambda - \bar{\lambda})\|f_\lambda\|^2 = 0$, which implies that $\widehat{f}_\lambda = \{0, 0\}$. Therefore, $\{0, \widehat{h}\} \in \Gamma$ and this proves the reverse inclusion $M(\lambda) \cap M(\lambda)^* \subset \operatorname{mul} \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Hence the first statement of (i) has been shown.

The second statement in (i) follows from the first one by taking adjoints. Then use the symmetry property $M(\lambda)^* = M(\bar{\lambda})$ and apply Proposition 2.12.

(ii) Let $\widehat{h} = \{h, 0\} \in \operatorname{mul} \Gamma$, $h \in \mathcal{H}$. Then $\widehat{h} \in M(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, so that $h \in \ker M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Conversely, assume that $h \in \ker M(\lambda)$, i.e., that $\widehat{h} = \{h, 0\} \in M(\lambda)$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then $\{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma$ for some $\widehat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} \in \widehat{\mathfrak{N}}_\lambda(T)$ and (3.4) gives $(\lambda - \bar{\lambda})\|f_\lambda\|^2 = (0, h) - (h, 0) = 0$. Hence, $f_\lambda = 0$ and $\widehat{f}_\lambda = 0$, which shows that $\widehat{h} \in \operatorname{mul} \Gamma \cap (\mathcal{H} \oplus \{0\})$. This proves the first equality in (i). The proof of the second equality is similar.

(iii) These identities follow immediately from (i) and (2.3), and the definition in (3.6). □

Remark 4.2. Lemma 4.1 combined with the realization theorem proved in the previous section (see Theorem 3.9) yields immediately the following invariance results for an arbitrary Nevanlinna family $M(\cdot) \in \widetilde{R}(\mathcal{H})$:

$$M(\lambda) \cap M(\lambda)^*, \quad \ker M(\lambda), \quad \operatorname{mul} M(\lambda), \quad \ker(M(\lambda) - M(\lambda)^*), \quad \ker(M(\lambda)^{-1} - M(\lambda)^{-*}) \tag{4.1}$$

do not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

This indicates that via Theorem 3.9 boundary relations in fact offer a new method for studying function and spectral theoretical properties of Nevanlinna families by means of geometric properties of boundary relations, and vice versa. Observe, that direct function theoretical proofs for the invariance properties of Nevanlinna families formulated in (4.1) maybe based e.g. on an application of the maximality principle. By using so-called Nevanlinna pairs some operator theoretical proofs for the corresponding invariance properties are presented in the Appendix (see Proposition A.6).

Lemma 4.1 gives also the following result.

Corollary 4.3. *Let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^* with the Weyl family $M(\lambda) = \Gamma(\widehat{\mathfrak{N}}_\lambda(T))$. Then the following equalities hold for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$:*

$$\overline{\operatorname{dom}}\, M(\lambda) = \overline{\operatorname{ran}}\, \Gamma_0, \quad \overline{\operatorname{ran}}\, M(\lambda) = \overline{\operatorname{ran}}\, \Gamma_1. \tag{4.2}$$

Proof. The definition of the Weyl family $M(\lambda)$, the symmetry property $M(\lambda)^* = M(\bar{\lambda})$, and part (i) of Lemma 4.1 imply that

$$M(\lambda) \widehat{+} M(\bar{\lambda}) \subset \operatorname{ran} \Gamma = \operatorname{clos}(M(\lambda) \widehat{+} M(\bar{\lambda})), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.3}$$

Since $\operatorname{mul} M(\lambda)$ is independent from $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the equality $\overline{\operatorname{dom}}\, M(\lambda) = \overline{\operatorname{dom}}\, M(\bar\lambda)$ holds. Now the identities (4.2) follow from (4.3). □

In general, the first inclusion in (4.3) need not be an equality, cf. Example 6.4. However, sufficient conditions for the equality to hold can be found in the next lemma.

Lemma 4.4. *Let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^* with corresponding Weyl family $M(\lambda) = \Gamma(\widehat{\mathfrak{N}}_\lambda(T))$. Then the following statements are equivalent:*

(i) $\operatorname{ran}\Gamma$ *is closed;*
(ii) $M(\lambda) \,\widehat{+}\, M(\lambda)^*$ *is closed for some (equivalently for every)* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*;*
(iii) $T = S^*$.

If any one of these conditions is satisfied, then

$$M(\lambda) \,\widehat{+}\, M(\lambda)^* = \operatorname{ran}\Gamma, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.4}$$

Proof. (i) ⇔ (iii) This is clear from Proposition 2.3 since $T = \operatorname{dom}\Gamma$ dense in S^*.

(ii) ⇒ (i) If (ii) holds for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then part (i) of Lemma 4.1 and the inclusion (4.3) yield

$$\operatorname{ran}\Gamma \subset \overline{\operatorname{ran}}\,\Gamma = M(\lambda) \,\widehat{+}\, M(\lambda)^* \subset \operatorname{ran}\Gamma.$$

(iii) ⇒ (ii) Von Neumann's formula (2.6) implies that for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$\Gamma(\widehat{\mathfrak{N}}_\lambda(T)) \,\widehat{+}\, \Gamma(\widehat{\mathfrak{N}}_{\bar\lambda}(T)) = \operatorname{ran}\Gamma,$$

which is closed since $\operatorname{ran}\Gamma$ is closed by the equivalence of (i) and (iii).

If one of the conditions (i), (ii), or (iii) is satisfied, then the identity (4.4) is clear from the above arguments. □

Proposition 4.5. *Let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^* with corresponding Weyl family $M(\lambda) = \Gamma(\widehat{\mathfrak{N}}_\lambda(T))$. Then:*

(i) $M(\cdot) \in R(\mathcal{H})$ *if and only if* $\operatorname{mul}\Gamma \cap (\{0\} \oplus \mathcal{H}) = \{0\}$*;*
(ii) $M(\cdot) \in R^s(\mathcal{H})$ *if and only if* $\operatorname{ran}\Gamma$ *is dense in* $\mathcal{H}^2$*;*
(iii) $M(\cdot) \in R^u(\mathcal{H})$ *if and only if* $\operatorname{ran}\Gamma = \mathcal{H}^2$.

Proof. (i) Observe that $M(\cdot) \in R(\mathcal{H})$ if and only if $\operatorname{mul} M(\lambda) = \{0\}$. Hence, the statement follows from part (ii) of Lemma 4.1.

(ii) By definition $M(\cdot) \in R^s(\mathcal{H})$ if and only if $M(\lambda) \cap M(\lambda)^* = \{0\}$. The statement now follows from part (i) of Lemma 4.1 and Proposition 2.3.

(iii) By definition $M(\cdot) \in R^u(\mathcal{H})$ if and only if $M(\lambda) \,\widehat{+}\, M(\lambda)^* = \mathcal{H}^2$. Hence, if $M(\cdot) \in R^u(\mathcal{H})$, then clearly $\operatorname{ran}\Gamma = \mathcal{H}^2$, cf. the inclusion (4.3). Conversely, if $\operatorname{ran}\Gamma = \mathcal{H}^2$, then $\operatorname{ran}\Gamma$ is closed, so that by Lemma 4.1 and Lemma 4.4, $M(\lambda) \,\widehat{+}\, M(\lambda)^* = \operatorname{ran}\Gamma = \mathcal{H}^2$, and thus $M(\cdot) \in R^u(\mathcal{H})$. □

The class $\widetilde{R}_{inv}(\mathcal{H})$ is the set of all $M(\cdot) \in \widetilde{R}(\mathcal{H})$ such that $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}_0$ for some linear subspace $\mathcal{H}_0 \subset \mathcal{H}$ with $\operatorname{clos}\mathcal{H}_0 = (\operatorname{mul} M(\lambda))^\perp$.

Corollary 4.6. *The invariant subclasses $R_{inv}(\mathcal{H})$ and $R^s_{inv}(\mathcal{H})$ are characterized by:*

(i) $M(\cdot) \in R_{inv}(\mathcal{H})$ *if and only if* $\operatorname{mul}\Gamma \cap (\{0\} \oplus \mathcal{H}) = \{0\}$ *and* $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}_0$ *for some dense linear subspace* $\mathcal{H}_0 \subset \mathcal{H}$*;*
(ii) $M(\cdot) \in R^s_{inv}(\mathcal{H})$ *if and only if* $\operatorname{mul}\Gamma = \{0\}$ *and* $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}_0$ *for some dense linear subspace* $\mathcal{H}_0 \subset \mathcal{H}$.

Proposition 4.7. *Let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^* with corresponding Weyl family $M(\lambda) = \Gamma(\widehat{\mathfrak{N}}_\lambda(T))$. Then:*

(i) *$M(\cdot) \in \widetilde{R}[\mathcal{H}]$ if and only if $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}_0$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, where $\mathcal{H}_0 \subset \mathcal{H}$ is a closed linear subspace;*

(ii) *$M(\cdot) \in R[\mathcal{H}]$ if and only if $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;*

(iii) *$M(\cdot) \in R^s[\mathcal{H}]$ if and only if $\operatorname{mul}\Gamma_0 = \{0\}$ and $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

Proof. For any $M(\cdot) \in \widetilde{R}(\mathcal{H})$ the following orthogonal decomposition holds:

$$\mathcal{H} = \overline{\operatorname{dom}} \, M(\lambda) \oplus \operatorname{mul} M(\lambda),$$

and the multivalued part $\operatorname{mul} M(\lambda)$ does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Define $\mathcal{H}_0 = \mathcal{H} \ominus \operatorname{mul} M(\lambda)$. Since $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \operatorname{dom} M(\lambda)$, parts (i) and (ii) follow from the definitions of $\widetilde{R}[\mathcal{H}]$ and $\widetilde{R}[\mathcal{H}]$. As to (iii): $M(\cdot) \in R^s[\mathcal{H}]$ if and only if $M(\lambda) \in R[\mathcal{H}]$ and $\ker \operatorname{Im} M(\lambda) = \{0\}$. Hence, the assertion follows from Lemma 4.1. □

4.2. **Boundary relations and their γ-fields.** The identity (3.4) implies that for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$,

$$\ker(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(T)) = \{0\}, \quad \ker(\Gamma_1 \upharpoonright \widehat{\mathfrak{N}}_\lambda(T)) = \{0\}, \quad \ker(\Gamma \upharpoonright \widehat{\mathfrak{N}}_\lambda(T)) = \{0\}.$$

In particular, the inverse of $\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(T)$ is a single-valued linear mapping from $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \operatorname{dom} M(\lambda)$ onto $\widehat{\mathfrak{N}}_\lambda(T)$; it is denoted by $\widehat{\gamma}(\lambda) := (\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(T))^{-1}$. The γ-field $\gamma(\cdot)$ associated with the boundary relation $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ is defined by

$$\gamma(\lambda) = \{ \{h, f_\lambda\} : \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma, \widehat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} \}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{4.5}$$

so that $\gamma(\lambda)$ corresponds to the first component of the mapping $\widehat{\gamma}(\lambda)$. It maps $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T))$ onto $\mathfrak{N}_\lambda(T)$ and satisfies $\gamma(\lambda)\Gamma_0\widehat{f}_\lambda = f_\lambda$ for all $\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(T)$. With $\gamma(\lambda)$ the relation $\Gamma \upharpoonright \widehat{\mathfrak{N}}_\lambda(T)$ can be rewritten as follows

$$\Gamma \upharpoonright \widehat{\mathfrak{N}}_\lambda(T) := \{ \{\{\gamma(\lambda)h, \lambda\gamma(\lambda)h\}, \{h, h'\}\} : \{h, h'\} \in M(\lambda) \}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.6}$$

In the case that Γ is single-valued one can decompose $\Gamma = \Gamma_0 \oplus \Gamma_1$. Then by part (ii) of Lemma 4.1 the corresponding Weyl family $M(\cdot)$ is operator-valued. In this case the identity (4.6) takes the form

$$\Gamma_0\widehat{\gamma}(\lambda)h = h, \quad \Gamma_1\widehat{\gamma}(\lambda)h = M(\lambda)h, \quad h \in \operatorname{dom} M(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.7}$$

These formulas are typically used in the case of ordinary boundary triplets for defining the corresponding Weyl function.

Proposition 4.8. *Let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^* with corresponding Weyl family $M(\lambda) = \Gamma(\widehat{\mathfrak{N}}_\lambda(T))$ and let $\widetilde{A} = \mathcal{J}\Gamma$ be as in (2.20). Then the corresponding γ-field $\gamma(\cdot)$ in (4.5) and the Weyl family $M(\cdot)$ are connected by*

$$(\widetilde{A} - \lambda)^{-1} \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = - \begin{pmatrix} \gamma(\lambda)(M(\lambda) + \lambda)^{-1}\varphi \\ (M(\lambda) + \lambda)^{-1}\varphi \end{pmatrix}, \quad \varphi \in \mathcal{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.8}$$

Furthermore, the γ-field $\gamma(\cdot)$ satisfies with $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$ the identity

$$\frac{(M_s(\lambda)h, k)_{\mathcal{H}} - (h, M_s(\mu)k)_{\mathcal{H}}}{\lambda - \bar{\mu}} = (\gamma(\lambda)h, \gamma(\mu)k)_{\mathfrak{H}}, \quad h \in \operatorname{dom} M(\lambda), \quad k \in \operatorname{dom} M(\mu), \tag{4.9}$$

and, in particular,

$$\ker \gamma(\lambda) = \operatorname{mul} \Gamma_0 = \ker (M(\lambda) - M(\lambda)^*). \tag{4.10}$$

Proof. It follows from (2.20) and (4.5) that

$$(\widetilde{A} - \lambda)^{-1} \begin{pmatrix} 0 \\ -h' - \lambda h \end{pmatrix} = \begin{pmatrix} \gamma(\lambda)h \\ h \end{pmatrix}, \quad h \in \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)),$$

which gives (4.8) immediately. The identity (4.9) follows from (3.4) and the description (4.6). Finally, the identities in (4.10) are obtained from the definition (4.5) and Lemma 4.1. □

The identity (4.8) shows the sense in which the mapping $\gamma(\lambda) : \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) \to \mathfrak{N}_\lambda(T)$ can be seen to be holomorphic. In general, the closure of the mapping $\gamma(\lambda)$ is not single-valued, cf. Example 6.6. However, there is a useful sufficient condition which guarantees that the closure of $\gamma(\lambda)$ is single-valued.

Proposition 4.9. *Assume that for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the inclusion*

$$\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) \subset \Gamma_0(\widehat{\mathfrak{N}}_{\bar\lambda}(T)) \tag{4.11}$$

is satisfied. Then $\gamma(\lambda)$ admits a single-valued closure.

Proof. It follows from (4.9) that for all $h \in \operatorname{dom} M(\lambda)$

$$(\lambda - \bar\lambda)(\gamma(\lambda)h, \gamma(\lambda)h)_{\mathfrak{H}} = (M_s(\lambda)h, h)_{\mathcal{H}} - (h, M_s(\lambda)h)_{\mathcal{H}},$$

and the assumption $\operatorname{dom} M(\lambda) \subset \operatorname{dom} M(\lambda)^*$ implies then that

$$(\gamma(\lambda)h, \gamma(\lambda)h)_{\mathfrak{H}} = \left(\frac{M_s(\lambda) - M_s(\lambda)^*}{\lambda - \bar\lambda} h, h \right)_{\mathcal{H}}. \tag{4.12}$$

Now, for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the operator

$$N(\lambda) := \frac{M_s(\lambda) - M_s(\lambda)^*}{\lambda - \bar\lambda}$$

is a nonnegative densely defined operator in $\mathcal{H} \ominus \operatorname{mul} M(\lambda)$. Therefore, both quadratic forms in (4.12) are closable (see [19]). Hence, $\gamma(\lambda)$ admits a single-valued closure. □

It follows from the identity $(N(\lambda)h, h) = (\gamma(\lambda)h, \gamma(\lambda)h)$, $h \in \operatorname{dom} M(\lambda)$, that the operator $N(\lambda)$ is nonnegative. Hence, it has a Friedrichs extension $N_F(\lambda)$. According to the Second Representation Theorem (see [19]) the single-valued closure of $\gamma(\lambda)$, denoted by $\gamma^{**}(\lambda)$, satisfies $\operatorname{dom} \gamma^{**}(\lambda) = \operatorname{dom} N_F(\lambda)^{1/2}$. Observe that the original mapping is onto $\mathfrak{N}_\lambda(T)$ and its closure is into $\mathfrak{N}_\lambda(S^*)$. In general the closure does not map onto $\mathfrak{N}_\lambda(S^*)$, cf. Example 6.7. In the case when (4.11) fails to hold it may happen that the closure of $\gamma(\lambda)$ is multivalued, see Example 6.6.

4.3. **Characterization of domain invariance.** The boundary relation for S^* and the associated mappings Γ_0 and Γ_1 in (3.6) induce two linear relations:

$$A_0 = \ker \Gamma_0, \quad A_1 = \ker \Gamma_1, \tag{4.13}$$

in the Hilbert space $\mathfrak{H}$. Clearly, these relations are symmetric and satisfy

$$S \subset A_0 \subset T, \quad S \subset A_1 \subset T.$$

The relations A_0 and A_1 need not be closed and their defect numbers may be unequal, cf. Example 6.3. The following lemma is useful in the further study of these relations.

Lemma 4.10. *Let $\Gamma : \mathcal{H}^2 \to \mathfrak{H}^2$ be a boundary relation for S^*. Then for $j = 1, 2$:*

(i) $\Gamma_j(\widehat{\mathfrak{N}}_\lambda(T)) \subset \operatorname{ran}\Gamma_j \subset \operatorname{clos}(\Gamma_j(\widehat{\mathfrak{N}}_\lambda(T)))$ *for all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(ii) $\operatorname{ran}\Gamma_j^{[*]} = \ker\Gamma_j$, *where* Γ_j *is understood as a linear subspace of* $\mathfrak{H}^2 \oplus \mathcal{H}^2$;

(iii) *if* $\operatorname{ran}\Gamma_j$ *is closed, then* A_j *is closed.*

The condition in (iii) is satisfied if $\Gamma_j(\widehat{\mathfrak{N}}_\lambda(T))$ *is closed for some* $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. (i) The first inclusion is obvious. Since by definition $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \operatorname{dom} M(\lambda)$ and $\Gamma_1(\widehat{\mathfrak{N}}_\lambda(T)) = \operatorname{ran} M(\lambda)$, the second inclusion is immediate from Corollary 4.3.

(ii) Identifying Γ_0 as a linear subspace of $\Gamma \subset \mathfrak{H}^2 \oplus \mathcal{H}^2$ it takes the form

$$\Gamma_0 = \{\, \{\widehat{f}, \{h, 0\}\} : \{\widehat{f}, \{h, h'\}\} \in \Gamma \,\}.$$

Assume that $\widehat{g} \in \operatorname{ran}\Gamma_0^{[*]}$, i.e. $\{\widehat{k}, \widehat{g}\} \in \Gamma_0^{[*]}$ for some $\widehat{k} = \{k, k'\}$. Then for all $\{\widehat{f}, \{h, 0\}\} \in \Gamma_0$,

$$0 = \left(J_{\mathfrak{H}}\widehat{g}, \widehat{f}\right) - (J_{\mathcal{H}}\{k, k'\}, \{h, 0\}) = \left(J_{\mathfrak{H}}\widehat{g}, \widehat{f}\right) - (J_{\mathcal{H}}\{0, k'\}, \{h, h'\}).$$

This means that $\{\{0, k'\}, \widehat{g}\} \in \Gamma^{[*]} = \Gamma^{-1}$ or equivalently that $\{\widehat{g}, \{0, k'\}\} \in \Gamma$, i.e., $\widehat{g} \in A_0 = \ker\Gamma_0$. Therefore, $\operatorname{ran}\Gamma_0^{[*]} = \ker\Gamma_0$. Similarly one proves the identity $\operatorname{ran}\Gamma_1^{[*]} = \ker\Gamma_1$.

(iii) Let $\operatorname{ran}\Gamma_j$ be closed. Then also $\operatorname{ran}(\Gamma_j)^{**}$ is closed. By Proposition 2.2, more precisely by its Kreĭn space version, equivalently then $\operatorname{ran}\Gamma_j^{[*]}$ is closed. Now the assertion follows from the equalities in (ii).

The last statement is clear from the inclusions in (i). □

The condition in (iii) is sufficient, but not necessary. In fact, in Example 6.5 $\operatorname{ran}\Gamma_0$ is not closed, while A_0 is selfadjoint.

Proposition 4.11. *The following statements are equivalent:*

(i) $\mathcal{H}_0 := \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T))$ *is independent from* $\lambda \in \mathbb{C}^+$ *(resp. from* $\lambda \in \mathbb{C}^-$*)*;

(ii) $\mathfrak{N}_\mu(T) \subset \operatorname{ran}(A_0 - \lambda)$ *for all* $\lambda, \mu \in \mathbb{C}^+$ *(resp. for all* $\lambda, \mu \in \mathbb{C}^-$*),* $\lambda \neq \mu$.

If one of these conditions is satisfied, then the γ*-field* $\gamma(\cdot)$ *satisfies*

$$\gamma(\lambda) = [I + (\lambda - \mu)(A_0 - \lambda)^{-1}]\gamma(\mu), \quad \lambda, \mu \in \mathbb{C}^+(\mathbb{C}^-). \tag{4.14}$$

Proof. (i) ⇒ (ii) Let $\mathcal{H}_0 = \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T))$ for all $\lambda \in \mathbb{C}^+$. It follows from (4.6) that for every $h \in \mathcal{H}_0$ there exist $h', h'' \in \mathcal{H}$ such that

(4.15)
$$\{\{\gamma(\lambda)h, \lambda\gamma(\lambda)h\}, \{h, h'\}\} \in \Gamma \upharpoonright \widehat{\mathfrak{N}}_\lambda(T) \subset \Gamma, \quad \{\{\gamma(\mu)h, \mu\gamma(\mu)h\}, \{h, h''\}\} \in \Gamma \upharpoonright \widehat{\mathfrak{N}}_\mu(T) \subset \Gamma.$$

Hence,

$$\{\{(\gamma(\lambda) - \gamma(\mu))h, (\lambda\gamma(\lambda) - \mu\gamma(\mu))h\}, \{0, h' - h''\}\} \in \Gamma,$$

and therefore

$$\{(\gamma(\lambda) - \gamma(\mu))h, (\lambda\gamma(\lambda) - \mu\gamma(\mu))h\} \in A_0. \tag{4.16}$$

It follows from (4.16) that

$$\{(\gamma(\lambda) - \gamma(\mu))h, (\lambda - \mu)\gamma(\mu)h\} \in A_0 - \lambda,$$

so that $\gamma(\mu)h \in \operatorname{ran}(A_0 - \lambda)$ for every $h \in \mathcal{H}_0$. Therefore (ii) follows.

(ii) ⇒ (i) Let $h \in \Gamma_0(\widehat{\mathfrak{N}}_\mu(T))$. By definition there is an element $h' \in \mathcal{H}$ so that

$$\{\, \{\gamma(\mu)h, \mu\gamma(\mu)h\}, \{h, h'\} \,\} \in \Gamma. \tag{4.17}$$

The assumption in (ii) shows that $\gamma(\mu)h \in \operatorname{ran}(A_0-\lambda)$, so that there is an element $k \in \mathfrak{H}$ such that $\{k, \gamma(\mu)h+\lambda k\} \in A_0$. Hence, there exists $\varphi \in \mathcal{H}$ such that

$$\{\,\{(\lambda-\mu)k, (\lambda-\mu)\gamma(\mu)h+\lambda(\lambda-\mu)k\}, \{0,\varphi\}\,\} \in \Gamma. \tag{4.18}$$

It follows from (4.17) and (4.18) that

$$\{\,\{\gamma(\mu)h+(\lambda-\mu)k, \lambda(\gamma(\mu)h+(\lambda-\mu)k)\}, \{h, h'+\varphi\}\,\} \in \Gamma.$$

In other words, $h \in \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T))$. Hence $\Gamma_0(\widehat{\mathfrak{N}}_\mu(T)) \subset \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T))$, and equality follows by symmetry.

Now assume that one of the equivalent conditions (i) or (ii) is satisfied. The assumption (i) implies the identity (4.16), which may be rewritten as

$$\{\gamma(\mu)h, \gamma(\lambda)h\} \in I + (\lambda-\mu)(A-\lambda)^{-1},$$

where $(A_0-\lambda)^{-1}$ is a bounded linear operator on $\operatorname{ran}(A_0-\lambda)$, since A_0 is symmetric. Hence (4.14) is valid. □

Corollary 4.12. *If (i) or equivalently (ii) in Proposition 4.11 holds for all $\lambda, \mu \in \mathbb{C}\setminus\mathbb{R}$, then A_0 is essentially selfadjoint and (4.14) holds for all $\lambda, \mu \in \mathbb{C}\setminus\mathbb{R}$.*

Proof. Assume first that S is simple. Then part (ii) of Proposition 4.11 implies that $\overline{\operatorname{ran}}\,(A_0-\lambda) = \mathfrak{H}$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, so that A_0 is essentially selfadjoint. If S is not simple, decompose $S = S' \oplus S''$ where S'' is selfadjoint. The symmetric extension A_0 of S decomposes accordingly: $A_0 = A_0' \oplus S''$. The earlier argument shows that A_0' is essentially selfadjoint, so that A_0 itself is also essentially selfadjoint. □

The case of equality in the first inclusion of (i) in Lemma 4.10 is characterized in the following theorem.

Theorem 4.13. *The following statements are equivalent for every fixed $\lambda \in \mathbb{C}^+(\mathbb{C}^-)$:*

(i) $\operatorname{ran}(A_0-\lambda) = \mathfrak{H}$ *(i.e. A_0 is maximal symmetric);*
(ii) $T = A_0 \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(T)$*;*
(iii) $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \operatorname{ran}\Gamma_0$.

If (i), (ii), or (iii) holds for some $\lambda \in \mathbb{C}^+(\mathbb{C}^-)$, then (i), (ii), and (iii) hold for every $\lambda \in \mathbb{C}^+(\mathbb{C}^-)$ and, moreover, S satisfies

$$S = \{\,\{f,g\} \in A_0^* : \ (g-\bar{\lambda}f, \gamma(\lambda)h)_{\mathfrak{H}} = 0, \ h \in \mathcal{H}_0 := \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T))\,\}, \quad \lambda \in \mathbb{C}^+(\mathbb{C}^-). \tag{4.19}$$

Proof. (i) ⇒ (ii) If $\operatorname{ran}(A_0-\lambda) = \mathfrak{H}$ for $\lambda \in \mathbb{C}^+$, it follows that

$$S^* = A_0 \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(S^*).$$

Now observe that $T \subset S^*$, $A_0 \subset T$, and $T \cap \widehat{\mathfrak{N}}_\lambda(S^*) = \widehat{\mathfrak{N}}_\lambda(T)$, which gives (ii).

(ii) ⇒ (i) The identity in (ii) shows that

$$\operatorname{ran}(T-\lambda) = \operatorname{ran}(A_0-\lambda), \quad \lambda \in \mathbb{C}^+.$$

Since $\widetilde{A}$ is selfadjoint, it follows from (2.20) and (2.25) that $\operatorname{ran}(T-\lambda) = \mathfrak{H}$ for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Hence, in particular $\operatorname{ran}(A_0-\lambda) = \mathfrak{H}$ for $\lambda \in \mathbb{C}^+$, which gives (i).

(ii) ⇒ (iii) Let $h \in \operatorname{ran}\Gamma_0$, so that $\{\widehat{f}, \{h,h'\}\} \in \Gamma$ for some $h' \in \mathcal{H}$, $\widehat{f} \in T$. Decompose $\widehat{f} = \widehat{f}_0 + \widehat{f}_\lambda$, where $\{\widehat{f}_0, \{0,k\}\} \in \Gamma$ and $\{\widehat{f}_\lambda, \{h, h'-k\}\} \in \widehat{\mathfrak{N}}_\lambda(T)$. This shows that $h \in \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T))$, and (iii) follows.

(iii) $\Rightarrow$ (ii) Since clearly $A_0 \widehat{+} \widehat{\mathfrak{N}}_\lambda(T) \subset T$, it suffices to prove the reverse inclusion. Let $\widehat{f} \in T$, so that $\{\widehat{f}, \{h, h'\}\} \in \Gamma$ for some $h, h' \in \mathcal{H}$. According to (iii) there exists $\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(T)$, such that $\{\widehat{f}_\lambda, h\} \in \Gamma_0$, i.e., $\{\widehat{f}_\lambda, \{h, h''\}\} \in \Gamma$ for some $h'' \in \mathcal{H}$. This implies that $\{\widehat{f} - \widehat{f}_\lambda, \{0, h' - h''\}\} \in \Gamma$, and therefore $\widehat{f}_0 := \widehat{f} - \widehat{f}_\lambda \in A_0$, which shows that $\widehat{f} = \widehat{f}_0 + \widehat{f}_\lambda \in A_0 \widehat{+} \widehat{\mathfrak{N}}_\lambda(T)$. Hence, (ii) follows.

If any of the equivalent statements (i), (ii), or (iii) holds for some $\lambda \in \mathbb{C}^+(\mathbb{C}^-)$ then symmetry of A_0 forces that these statements hold for every $\lambda \in \mathbb{C}^+(\mathbb{C}^-)$. Moreover, it follows from (ii) that

$$T^* = A_0^* \cap \left(\widehat{\mathfrak{N}}_\lambda(T)\right)^*,$$

which leads to the identity (4.19). □

As a consequence of Theorem 4.13 one obtains criteria for A_0 to be selfadjoint.

Corollary 4.14. *The relation A_0 is selfadjoint if and only if one (and hence all) of the statements (i), (ii), or (iii) in Theorem 4.13 holds for some $\lambda \in \mathbb{C}^+$ and some $\lambda \in \mathbb{C}^-$. Moreover, in this case*

$$S = \{\, \{f, g\} \in A_0 : \ (g - \bar{\lambda} f, \gamma(\lambda) h)_{\mathfrak{H}} = 0, \ h \in \mathcal{H}_0 := \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) \,\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.20}$$

The case of equality in the second inclusion of (i) in Lemma 4.10 is characterized in the next proposition.

Proposition 4.15. *The following statements are equivalent for every fixed $\lambda \in \mathbb{C}^+(\mathbb{C}^-)$:*

(i) $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T))$ *is closed;*
(ii) $\overline{\operatorname{ran}}\,(A_0 - \lambda) = \mathfrak{H}$ *and* $\operatorname{ran}\Gamma_0$ *is closed.*

In this case A_0 is closed and $\operatorname{ran}(A_0 - \lambda) = \mathfrak{H}$ *for every $\lambda \in \mathbb{C}^+(\mathbb{C}^-)$, so that A_0 is maximal symmetric.*

Proof. (i) $\Rightarrow$ (ii) Assume that $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T))$ is closed for some $\lambda \in \mathbb{C}^+(\mathbb{C}^-)$. Then by Lemma 4.10 $\operatorname{ran}\Gamma_0$ is closed and Theorem 4.13 implies that $\operatorname{ran}(A_0 - \lambda) = \mathfrak{H}$ for $\lambda \in \mathbb{C}^+(\mathbb{C}^-)$.

(ii) $\Rightarrow$ (i) If $\operatorname{ran}\Gamma_0$ is closed, then A_0 is closed by Lemma 4.10. Therefore the assumption $\overline{\operatorname{ran}}\,(A_0 - \lambda) = \mathfrak{H}$ for $\lambda \in \mathbb{C}^+(\mathbb{C}^-)$ leads to $\operatorname{ran}(A_0 - \lambda) = \mathfrak{H}$, $\lambda \in \mathbb{C}^+(\mathbb{C}^-)$. Then by Theorem 4.13 $\operatorname{ran}\Gamma_0 = \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T))$, and this subspace is closed by the second assumption in (ii).

The last statement follows from Lemma 4.10. □

As a consequence of Proposition 4.15 one obtains some further invariance results known for an arbitrary Nevanlinna family $M(\cdot) \in \widetilde{R}(\mathcal{H})$. This in turn leads to a more precise statement concerning A_0 in the previous proposition.

Proposition 4.16. *Let the Nevanlinna family $M(\cdot) \in \widetilde{R}(\mathcal{H})$ be the Weyl family associated to the boundary relation $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ via Theorem 3.9. Then:*

(i) *if* $\operatorname{dom} M(\lambda_0)$ ($\operatorname{ran} M(\lambda_0)$) *is closed for some $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$, then* $\operatorname{ran}\Gamma_0$ *(resp.* $\operatorname{ran}\Gamma_1$*) is closed and the operator part of $M(\lambda)$ (of $M(\lambda)^{-1}$) is bounded for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$;*
(ii) *if* $\operatorname{ran}\Gamma_0$ ($\operatorname{ran}\Gamma_1$) *is not closed, then* $\operatorname{dom} M(\lambda)$ *(resp.* $\operatorname{ran} M(\lambda)$*) is nonclosed and the operator part of $M(\lambda)$ (of $M(\lambda)^{-1}$) is unbounded for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

Proof. It is enough to prove (i). Since $M(\lambda_0)^* = M(\bar\lambda_0)$ Proposition 2.2 shows that $\operatorname{dom} M(\lambda_0)$ is closed if and only if $\operatorname{dom} M(\bar\lambda_0)$ is closed. In this case $\operatorname{dom} M(\lambda_0) = \operatorname{dom} M(\bar\lambda_0) = \operatorname{ran}\Gamma_0$ by Corollary 4.3. Now it is clear that the properties in part (ii) of Proposition 4.15 are satisfied for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and consequently $\operatorname{dom} M(\lambda) = \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T))$ is closed for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$. By the closed graph theorem this means that the operator part of $M(\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, is bounded (see (2.32)). Similar one proves the assertion for $\operatorname{ran} M(\lambda)$. □

Corollary 4.17. *If one of the equivalent statements (i) or (ii) in Proposition 4.15 holds at a single point $\lambda_0 \in \mathbb{C}\setminus\mathbb{R}$, then A_0 is selfadjoint.*

Proof. It was shown in the proof of Proposition 4.16 that $\operatorname{dom} M(\lambda_0) = \operatorname{dom} M(\bar\lambda_0) = \operatorname{ran}\Gamma_0$ and consequently the equality $\operatorname{ran}(A_0-\lambda) = \mathfrak{H}$ holds in fact for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$. □

Further invariance results concerning the spectra of an arbitrary Nevanlinna family $M(\cdot) \in \widetilde{R}(\mathcal{H})$ are now easily established.

Proposition 4.18. *Let $M(\cdot) \in \widetilde{R}(\mathcal{H})$ be a Nevanlinna family and let $\lambda_0 \in \mathbb{C}\setminus\mathbb{R}$. Then:*

(i) *if $M(\lambda_0) \in [\mathcal{H}]$ then $M(\lambda) \in [\mathcal{H}]$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$;*
(ii) *if $\alpha = \bar\alpha \in \rho(M(\lambda_0))$ then $\alpha = \bar\alpha \in \rho(M(\lambda))$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$;*
(iii) *if $\alpha = \bar\alpha \in \sigma_p(M(\lambda_0))$ then $\alpha = \bar\alpha \in \sigma_p(M(\lambda))$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$;*
(iv) *if $\alpha = \bar\alpha \in \sigma_c(M(\lambda_0))$ then $\alpha = \bar\alpha \in \sigma_c(M(\lambda))$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$.*

Proof. To prove the statements let $M(\cdot) \in \widetilde{R}(\mathcal{H})$ be the Weyl family associated to the boundary relation $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ via Theorem 3.9.

(i) By Lemma 4.1 $\overline{\operatorname{dom}}\, M(\lambda) = (\operatorname{mul} M(\lambda))^\perp$ does not depend on $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and hence the statement follows from part (i) of Proposition 4.16.

(ii) The condition $\alpha \in \rho(M(\lambda))$ means that $(M(\lambda)-\alpha I_{\mathcal{H}})^{-1} \in [\mathcal{H}]$. Since $-(M(\lambda)-\alpha I_{\mathcal{H}})^{-1} \in \widetilde{R}(\mathcal{H})$ the assertion is obtained immediately from (i).

(iii) For every $\alpha \in \mathbb{R}$ is clear that $M(\lambda) - \alpha I_{\mathcal{H}} \in \widetilde{R}(\mathcal{H})$. Therefore, by Lemma 4.1 $\ker(M(\lambda)-\alpha I_{\mathcal{H}})$ does not depend on $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

(iv) Observe that $\sigma(M(\lambda)) = \sigma_p(M(\lambda)) \cup \sigma_c(M(\lambda))$. Hence the statement follows by combining (ii) and (iii). □

Remark 4.19. The proof of Proposition 4.18 shows that the eigenspaces $\ker(M(\lambda)-\alpha I_{\mathcal{H}})$, $\alpha \in \mathbb{R}$, actually do not depend on $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Independent operator theoretical proofs for the invariance results in Proposition 4.18 are presented in the Appendix (see Proposition A.6).

According to Corollary 4.14 the assumption that A_0 is selfadjoint is equivalent to the decomposition $T = A_0 \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(T)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Clearly, this decomposition is direct: $\widehat{\mathfrak{N}}_\lambda(T) \cap A_0 = \{0,0\}$. In this case the intersection $\mathfrak{N}_\lambda(T) \cap \operatorname{dom} A_0$ can be described as follows.

Proposition 4.20. *Assume that A_0 is selfadjoint. Then for $\lambda \in \mathbb{C}\setminus\mathbb{R}$,*

$$\mathfrak{N}_\lambda(T) \cap \operatorname{dom} A_0 = (A_0-\lambda)^{-1}(\operatorname{mul} T). \tag{4.21}$$

Moreover, $h \in \Gamma_0\{0,-\omega\}$, $\omega \in \operatorname{mul} T$, if and only if $\gamma(\lambda)h = (A_0-\lambda)^{-1}\omega$ for $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Proof. The identity in (4.21) is equivalent to

$$\{\,\{f,\lambda f\} \in T : f \in \operatorname{dom} A_0\,\} = \{\,\{(A_0-\lambda)^{-1}\omega, \lambda(A_0-\lambda)^{-1}\omega\} : \omega \in \operatorname{mul} T\,\}. \tag{4.22}$$

Let $\{f, \lambda f\} \in T$ have the property that $f \in \operatorname{dom} A_0$. Then there is an element $\omega \in \mathfrak{H}$ for which $f = (A_0 - \lambda)^{-1}\omega$. Hence

$$\{f, \lambda f\} - \{(A_0 - \lambda)^{-1}\omega, (I + \lambda(A_0 - \lambda)^{-1})\omega\} = \{0, -\omega\}. \tag{4.23}$$

Since the elements in the lefthand side belong to T, it follows that $\omega \in \operatorname{mul} T$. Hence the lefthand side of (4.22) is contained in the righthand side.

Conversely, observe that for $\omega \in \mathfrak{H}$

$$\{(A_0 - \lambda)^{-1}\omega, \lambda(A_0 - \lambda)^{-1}\omega\} = \{(A_0 - \lambda)^{-1}\omega, (I + \lambda(A_0 - \lambda)^{-1})\omega\} \mathbin{\widehat{+}} \{0, -\omega\}.$$

Thus, if $\omega \in \operatorname{mul} T$, then the elements in the righthand side belong to $A_0 \mathbin{\widehat{+}} (\{0\} \oplus \operatorname{mul} T) \subset T$. Hence the righthand side of (4.22) is contained in the lefthand side.

It follows from (4.23) that $h \in \Gamma_0\{0, -\omega\}$ if and only if $h \in \Gamma_0\{f, \lambda f\}$ and, by definition of $\gamma(\lambda)$ this is equivalent to $f = \gamma(\lambda)h$. □

Remark 4.21. The assumption that A_0 is selfadjoint is essential for the inclusion $\operatorname{mul} T \subset \operatorname{ran}(A_0 - \lambda)$. If A_0 is not selfadjoint, the identity (4.21) need not be valid, cf. Example 6.4.

The next result is a strengthening of Lemma 4.10.

Proposition 4.22. *Let Γ be a boundary relation for S^* and let $\operatorname{ran}\Gamma_0$ be closed. Then:*

(i) $\Gamma(A_0)$ *is essentially selfadjoint in $\mathcal{H}$ and the defect numbers of S are equal;*
(ii) *if $A_0^* \subset \operatorname{dom}\Gamma$ then $A_0 = \ker\Gamma_0$ is selfadjoint and the linear relation Γ_0 is closed;*
(iii) *if $\operatorname{ran}\Gamma$ is closed, and in particular if the defect numbers of S are finite, then $A_0 = \ker\Gamma_0$ is selfadjoint and the linear relation Γ_0 is closed.*

Proof. (i) By assumption $\operatorname{dom} T_2 = \operatorname{ran}\Gamma_0$ is closed. Then also $\operatorname{dom}(\operatorname{clos} T_2) = \operatorname{dom} S_2^*$ is closed and hence by Proposition 2.2 $\operatorname{dom} S_2$ is closed. Decompose $S_2 = S_0 \oplus (\{0\} \oplus \operatorname{mul} S_2)$, where S_0 is the operator part of S_2 in $\mathcal{H} \ominus (\operatorname{mul} S_2) = \overline{\operatorname{dom} S_2^*} = \operatorname{dom} T_2\, (\supset \operatorname{dom} S_2)$ and $\{0\} \oplus \operatorname{mul} S_2$ is a selfadjoint relation in $\operatorname{mul} S_2$. Since $\operatorname{dom} S_0 = \operatorname{dom} S_2$ is closed, S_0 is a bounded symmetric operator in $\operatorname{dom} T_2$. It is well known that S_0 has bounded selfadjoint extensions in $\operatorname{dom} T_2$, cf. e.g. [1]. Let B_0 be a bounded selfadjoint extension of S_0 in $\operatorname{dom} T_2$. Then $S_0^* = B_0 \mathbin{\widehat{+}} (\{0\} \oplus \operatorname{mul} S_0^*)$, since clearly $(B_0 \mathbin{\widehat{+}} (\{0\} \oplus \operatorname{mul} S_0^*))^* = S_0$ and moreover, $B_0 \mathbin{\widehat{+}} (\{0\} \oplus \operatorname{mul} S_0^*)$ is closed, which follows from the fact that B_0 is a closed bounded operator in $\operatorname{dom} T_2$. Consequently,

$$\operatorname{clos} T_2 = S_2^* = S_0^* \oplus (\{0\} \oplus \operatorname{mul} S_2) = B_0 \mathbin{\widehat{+}} (\{0\} \oplus \operatorname{mul} S_2^*). \tag{4.24}$$

To prove essential selfadjointness of $\Gamma(A_0)$ first observe that

$$\Gamma(A_0) = \operatorname{mul}\Gamma \mathbin{\widehat{+}} (\{0\} \oplus \operatorname{mul} T_2) = S_2 \mathbin{\widehat{+}} (\{0\} \oplus \operatorname{mul} T_2).$$

Since A_0 is symmetric, $\Gamma(A_0)$ is symmetric by Proposition 2.13. Moreover, $\operatorname{mul} T$ is dense in $\operatorname{mul} S_2^*$, which follows from (4.24) and the boundedness of B_0. Since $\operatorname{dom} S_2 = \operatorname{dom} S_0$ is closed this together with (4.24) implies that

$$\begin{aligned}\Gamma(A_0)^* &= S_2^* \cap ((\operatorname{mul} T_2)^\perp \oplus \mathcal{H}) = S_2^* \cap (\operatorname{dom} S_2 \oplus \mathcal{H}) \\ &= S_0 \mathbin{\widehat{+}} (\{0\} \oplus \operatorname{mul} S_2^*) \subset \operatorname{clos}\Gamma(A_0) \subset \Gamma(A_0)^*.\end{aligned}$$

Hence, $\Gamma(A_0)$ is essentially selfadjoint and the defect numbers of S_2, and therefore also of $S = S_1$, are equal.

(ii) According to Proposition 2.13 A_0 is essentially selfadjoint and by Lemma 4.10 it is closed. Thus, A_0 is selfadjoint. The closedness of Γ_0 follows from the general implication

$$A_0 = A_0^*, \quad \operatorname{ran}\Gamma_0 \text{ is closed } \Rightarrow \quad \Gamma_0 \text{ is closed }.$$

(iii) If $\operatorname{ran}\Gamma$ is closed then equivalently $T = \operatorname{dom}\Gamma$ is closed; i.e., $T = S^*$. Consequently, $A_0^* \subset \operatorname{dom}\Gamma$ and the statement is obtained from part (ii). Observe, that if the defect numbers of S are finite then T is closed as a finite-dimensional extension of S. □

Remark 4.23. (i) $\operatorname{mul} M(\lambda) = \operatorname{mul} S_2$ and the operator part $M_s(\lambda)$ of $M(\lambda)$ acts on $\overline{\operatorname{dom}}\, T_2$;
(ii) if $\operatorname{ran}\Gamma_0$ is closed, then $\Gamma(A_0)$ is selfadjoint if and only if $\operatorname{mul} T_2$ is closed;
(iii) if $\operatorname{ran}\Gamma_1$ is closed, then $\Gamma(A_1)$ is selfadjoint if and only if $\ker T_2$ is closed;
(iv) if $\operatorname{ran}\Gamma_0$ and $\operatorname{ran}\Gamma_1$ both are closed and $\Gamma(A_0)$ or $\Gamma(A_1)$ is selfadjoint, then A_0 and A_1 both are selfadjoint. Moreover, $A_0 \widehat{+} A_1 = T$, so that A_0 and A_1 are disjoint w.r.t. S.
(v) Part (iii) of Proposition 4.22 applies in particular to the case $\operatorname{ran}\Gamma = \mathcal{H}^2$; this implies for instance the equality $R^u(\mathcal{H}) = R^u[\mathcal{H}]$, cf. Theorem 4.13.

4.4. **Domain invariance and operator representations of Weyl families.** A Weyl family $M(\cdot) \in \widetilde{R}(\mathcal{H})$ belongs to the subclass $\widetilde{R}_{inv}(\mathcal{H})$ when there exists a linear (not necessarily closed) subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that

$$\operatorname{dom} M(\lambda) = \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}_0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.25}$$

In this case Proposition 4.9 may be applied to show that the closure of the γ-field $\gamma(\cdot)$ is single-valued. Note that in this proposition the notations $[\mathfrak{H}_0, \mathfrak{N}_\lambda(T)]$ and $[\mathcal{H}_0]$ refer to the bounded linear operators in the respective spaces, even when $\mathcal{H}_0$ is not complete.

Proposition 4.24. *Let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation with the corresponding Weyl family $M(\cdot) \in \widetilde{R}(\mathcal{H})$. Assume that $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}_0$ holds for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then:*
(i) *$\gamma(\lambda)$ admits a single-valued closure for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*
(ii) *$\mathcal{H}_0 \subset \operatorname{dom}\gamma(\lambda)^{**} = \operatorname{dom}\gamma(\mu)^{**} \subset \operatorname{clos}\mathcal{H}_0, \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$;*
(iii) *$\mathcal{H}_0 \subset \operatorname{dom}\gamma(\lambda)^*\gamma(\mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$;*
(iv) *$\gamma(\lambda) \in [\mathfrak{H}_0, \mathfrak{N}_\lambda(T)]$ if and only if $\operatorname{Im} M_s(\lambda) \in [\mathcal{H}_0]$.*

Proof. (i) This is a direct consequence of Proposition 4.9.
(ii) It follows from (4.14) that there exists $c > 0$, such that

$$\frac{1}{c}\|\gamma(\mu)\|_{\mathfrak{H}} \le \|\gamma(\lambda)\|_{\mathfrak{H}} \le c\|\gamma(\mu)\|_{\mathfrak{H}},$$

which shows that the topology induced on $\mathcal{H}_0$ by the form $(\gamma(\lambda)h, \gamma(\lambda)k)_{\mathfrak{H}}$ does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Therefore, the domain of the closure of this form is also independent of $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and clearly the closure is given by $(\gamma(\lambda)^{**}h, \gamma(\lambda)^{**}k)_{\mathfrak{H}}$, $h, k \in \operatorname{dom}\gamma(\lambda)^{**}$, where $\gamma(\lambda)^{**} : \mathcal{H} \to \mathfrak{H}$ is the closure of $\gamma(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.
(iii) It follows from (4.9) that for all $h, k \in \mathcal{H}_0$

$$(\lambda - \bar{\mu})(\gamma(\lambda)h, \gamma(\mu)k)_{\mathfrak{H}} = (h, (M_s(\lambda)^* - M_s(\mu))k)_{\mathcal{H}}. \tag{4.26}$$

Hence, for $k \in \operatorname{dom}\gamma(\mu) = \mathcal{H}_0$ and $\lambda \neq \bar{\mu}$,
(4.27)

$$\sup_{h \in \mathcal{H}_0} \frac{|(\gamma(\lambda)h, \gamma(\mu)k)_{\mathfrak{H}}|}{\|h\|} = \sup_{h \in \mathcal{H}_0} \frac{|(h, (M_s(\lambda)^* - M_s(\mu))k)_{\mathcal{H}}|}{|\lambda - \bar{\mu}|\,\|h\|} \le \frac{\|(M_s(\lambda)^* - M_s(\mu))k\|_{\mathcal{H}}}{|\lambda - \bar{\mu}|} < \infty,$$

which means that $\gamma(\mu)k \in \operatorname{dom}\gamma(\lambda)^*$. Hence, $\mathcal{H}_0 \subset \operatorname{dom}\gamma(\lambda)^*\gamma(\mu)$.

(iv) This follows immediately from the identity

$$\frac{(\operatorname{Im} M_s(\lambda)h,h)_{\mathfrak{H}}}{\operatorname{Im}\lambda} = \frac{(M_s(\lambda)h,h)_{\mathcal{H}} - (M_s(\lambda)^*h,h)_{\mathcal{H}}}{\lambda-\bar\lambda} = (\gamma(\lambda)h,\gamma(\lambda)h)_{\mathfrak{H}}, \quad h \in \mathcal{H}_0. \tag{4.28}$$

□

Proposition 4.25. *Assume that the Weyl family $M(\cdot)$ belongs to the subclass $\widetilde{R}_{inv}(\mathcal{H})$, so that* (4.25) *holds with a linear subspace $\mathcal{H}_0 \subset \mathcal{H}$. Then the operator part of the Weyl family $M(\lambda) = M_s(\lambda) \oplus M_\infty$ has the operator representation*

$$M_s(\lambda) = M_s(\mu)^* + (\lambda-\bar\mu)\gamma(\mu)^*[I + (\lambda-\mu)(A_0-\lambda)^{-1}]\gamma(\mu), \quad \lambda,\mu \in \mathbb{C}\setminus\mathbb{R}, \tag{4.29}$$

where $\gamma(\mu)^* : \mathfrak{H} \to \operatorname{clos}\mathcal{H}_0$.

Proof. This follows immediately from the identity

$$\frac{(M_s(\lambda)h,k)_{\mathcal{H}} - (M_s(\mu)^*h,k)_{\mathcal{H}}}{\lambda-\bar\mu} = (\gamma(\lambda)h,\gamma(\mu)k)_{\mathfrak{H}}, \quad h,k \in \mathcal{H}_0,$$

the identity (4.14), and part (iii) of Proposition 4.24. □

Under the assumption of domain invariance the symmetric relation $A_0 = \ker\Gamma_0$ is essentially selfadjoint (cf. Corollary 4.12), so that its closure $A_0^{**} = A_0^*$ is a selfadjoint extension of S. Recall that under these circumstances the identity (4.14) now holds for all $\lambda,\mu \in \mathbb{C}\setminus\mathbb{R}$:

$$\gamma(\lambda)h = [I + (\lambda-\mu)(A_0-\lambda)^{-1}]\gamma(\mu)h, \quad h \in \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)), \quad \lambda,\mu \in \mathbb{C}\setminus\mathbb{R}. \tag{4.30}$$

Denote by A_s the (orthogonal) operator part of A_0^{**}, so that A_0^{**} admits the decomposition

$$A_0^{**} = A_s \oplus A_\infty, \quad A_\infty = \{0, \operatorname{mul} A_0^{**}\}.$$

Let P be the orthogonal projection onto $\mathfrak{H}_s = \overline{\operatorname{dom} A_0^{**}}$, so that $I-P$ is the orthogonal projection onto $\operatorname{mul} A_0^{**}$. Let $E(t)$ be the spectral family (of orthogonal projections) of A_0^{**}, so that in particular $\ker E(\infty) = \operatorname{ran}(I-P)$. Then the operator A_s in $\mathfrak{H}_s$ satisfies

$$I_{\mathfrak{H}_s} + (\lambda-\mu)(A_s-\lambda)^{-1} = \int_{\mathbb{R}} \frac{t-\mu}{t-\lambda}\, dE(t). \tag{4.31}$$

In view of (4.31) the identity (4.30) may now be rewritten as follows (with $\lambda,\mu \in \mathbb{C}\setminus\mathbb{R}$):

$$\gamma(\lambda)h = (I-P)\gamma(\mu)h + [I_{\mathfrak{H}_s} + (\lambda-\mu)(A_s-\lambda)^{-1}]P\gamma(\mu)h, \quad h \in \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)), \tag{4.32}$$

which is an orthogonal decomposition.

The next result is an extension of the integral representation (2.35) from the subclass $R[\mathcal{H}]$ to the subclass $\widetilde{R}_{inv}(\mathcal{H})$ of Nevanlinna families.

Proposition 4.26. *Assume that the Weyl family $M(\cdot)$ belongs to the subclass $\widetilde{R}_{inv}(\mathcal{H})$, so that* (4.25) *holds with a linear subspace $\mathcal{H}_0 \subset \mathcal{H}$. Then the operator part of the Weyl family $M(\lambda) = M_s(\lambda) \oplus M_\infty$ has the integral representation*

$$(M_s(\lambda)h,h)_{\mathcal{H}} = a_h + b_h\lambda + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\right) d\sigma_h(t), \quad h \in \mathcal{H}_0\ (= \operatorname{dom} M(\lambda)), \tag{4.33}$$

where $a_h = (\operatorname{Re} M_s(i)h,h)_{\mathcal{H}}$, $d\sigma_h(t) = (t^2+1)d(E(t)P\gamma(i)h, P\gamma(i)h)_{\mathfrak{H}}$,

$$b_h = \|(I-P)\gamma(i)h\|^2_{\mathfrak{H}}, \tag{4.34}$$

P is the orthogonal projection onto $\overline{\operatorname{dom}}\, A_0^{**}$, and $E(t)$ is the spectral family of A_0^{**}.

Proof. It follows from (4.29), (4.31), and (4.32) that for all $h \in \mathcal{H}_0$ and $\mu = i$

$$\begin{aligned}(M_s(\lambda)h, h)_{\mathcal{H}} =& (M_s(i)^* h, h)_{\mathcal{H}} + (\lambda + i)\|(I - P)\gamma(i)\|_{\mathfrak{H}}^2 \\ &+ (\lambda + i)([I + (\lambda - i)(A_s - \lambda)^{-1}]P\gamma(i)h, \gamma(i)h)_{\mathfrak{H}} \\ =& (M_s(i)^* h, h)_{\mathcal{H}} + (\lambda + i)\left(b_h + \int_{\mathbb{R}} \frac{t - i}{t - \lambda} d(E(t)P\gamma(i)h, P\gamma(i)h)_{\mathfrak{H}}\right).\end{aligned}$$

Since

$$\operatorname{Im}(M_s(i)h, h)_{\mathcal{H}} = b_h + \int_{\mathbb{R}} d(E(t)P\gamma(i)h, P\gamma(i)h)_{\mathfrak{H}} = b_h + \int_{\mathbb{R}} \frac{d\sigma_h(t)}{t^2 + 1}$$

one obtains

$$(M_s(\lambda)h, h)_{\mathcal{H}} = a_h + b_h\lambda + \int_{\mathbb{R}} \left(\frac{1}{t - \lambda} - \frac{1}{t + i} - \frac{i}{t^2 + 1}\right) d\sigma_h(t),$$

and this leads to (4.33). □

The coefficient (4.34) of the linear term in the integral representation (4.33) may be obtained by a limiting procedure.

Proposition 4.27. *Assume that the Weyl family $M(\cdot)$ belongs to the subclass $\widetilde{R}_{inv}(\mathcal{H})$, so that (4.25) holds with a linear subspace $\mathcal{H}_0 \subset \mathcal{H}$. Then for all $\mu \in \mathbb{C} \setminus \mathbb{R}$ and $h \in \mathcal{H}_0$,*

$$\lim_{y \to \infty} \frac{(M_s(iy)h, h)_{\mathcal{H}}}{iy} = \|(I - P)\gamma(i)h\|_{\mathfrak{H}}^2 = \|(I - P)\gamma(\mu)h\|_{\mathfrak{H}}^2. \tag{4.35}$$

In particular, with $h \in \mathcal{H}_0$,

$$\lim_{y \to \infty} \frac{(M_s(iy)h, h)_{\mathcal{H}}}{iy} = 0 \quad \textit{if and only if} \quad \gamma(\mu)h \in \overline{\operatorname{dom}}\, A_0^{**}. \tag{4.36}$$

Proof. The first equality in (4.35) is implied by (4.33) and (4.34). To obtain the second equality in (4.35) first observe that as a consequence of (4.32) the following limiting result holds with $\lambda = iy$, $y \to \infty$:

$$\lim_{\lambda = iy \to i\infty} (\gamma(\lambda)h, \gamma(\mu)h) = \|(I - P)\gamma(\mu)h\|_{\mathfrak{H}}^2.$$

Now apply (4.29). □

Corollary 4.28. *Assume that the Weyl family $M(\cdot)$ belongs to the subclass $\widetilde{R}_{inv}(\mathcal{H})$ and let S be an operator, i.e., $\operatorname{mul} S = \{0\}$. Then A_0^{**} is an operator if and only if*

$$\lim_{y \to \infty} \frac{(M_s(iy)h, h)_{\mathcal{H}}}{iy} = 0 \quad \textit{for every } h \in \mathcal{H}_0 = \operatorname{dom} M_s(\lambda). \tag{4.37}$$

Proof. Assume that A_0^{**} is an operator. Then it follows from Proposition 4.27 that (4.37) holds.

Conversely, assume that (4.37) holds. If S is simple, it follows from Proposition 4.27 that $\mathfrak{N}_\lambda(T)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, are orthogonal to $\operatorname{mul} A_0^{**}$. Since $\mathfrak{H} = \overline{\operatorname{span}}\, \{ \mathfrak{N}_\lambda(T) : \lambda \in \mathbb{C} \setminus \mathbb{R} \}$, this implies that $\operatorname{mul} A_0^{**} = \{0\}$. In the general case, when S is not necessarily simple, decompose the operator S as an orthogonal sum $S_0 \oplus S_1$ with a simple symmetric operator S_0 and a selfadjoint operator S_1. Then the result follows from the fact that $M(\cdot)$ is the Weyl family of the simple part S_1 of S. □

As a direct consequence of (4.31) and (4.32) one obtains

$$\|\gamma(\lambda)h\|^2_{\mathfrak{H}} = \|(I-P)\gamma(\mu)h\|^2_{\mathfrak{H}} + \int_{\mathbb{R}} \left|\frac{t-\mu}{t-\lambda}\right|^2 \|dE(t)P\gamma(\mu)h\|^2_{\mathfrak{H}}. \tag{4.38}$$

It also follows from (4.32) that if $\gamma(\lambda)h \in \operatorname{dom} A_0^{**}$ for some $\lambda \in \mathbb{C}\setminus\mathbb{R}$, then the same is true for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Proposition 4.29. *Assume that the Weyl family $M(\cdot)$ belongs to the subclass $\widetilde{R}_{inv}(\mathcal{H})$ and let $h \in \mathcal{H}_0 = \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T))$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Then*

$$\sup_{y>0} y\big(\operatorname{Im} M_s(iy)h,h\big)_{\mathcal{H}} < \infty \Leftrightarrow \gamma(\mu)h \in \operatorname{dom} A_0^{**}, \quad \mu \in \mathbb{C}\setminus\mathbb{R}. \tag{4.39}$$

Proof. It follows from (4.9) that

$$y\left(\frac{M_s(iy)-M_s(iy)^*}{2i}h,h\right)_{\mathcal{H}} = y^2\|\gamma(iy)h\|^2_{\mathfrak{H}}. \tag{4.40}$$

Combining (4.40) with (4.38) leads to

$$y\operatorname{Im}\big(M_s(iy)h,h\big)_{\mathcal{H}} = y^2\|(I-P)\gamma(i)h\|^2_{\mathfrak{H}} + y^2\int_{\mathbb{R}} \frac{t^2+1}{t^2+y^2} d\|E_tP\gamma(i)h\|^2_{\mathfrak{H}}. \tag{4.41}$$

An application of Lebesgue's monotone convergence theorem gives

$$\sup_{y>0} y^2\int_{\mathbb{R}} \frac{t^2+1}{t^2+y^2} d\|E_tP\gamma(i)h\|^2_{\mathfrak{H}} = \int_{\mathbb{R}} (t^2+1) d\|E_tP\gamma(i)h\|^2_{\mathfrak{H}}. \tag{4.42}$$

Therefore, the righthand side of (4.41) is finite if and only if $(I-P)\gamma(i)h = 0$ and $P\gamma(i)h \in \operatorname{dom} A_0^{**}$. □

Define *the subclass* $\widetilde{R}^s_{inv}(\mathcal{H})$ as the set of all Nevanlinna families $M(\cdot) \in \widetilde{R}_{inv}(\mathcal{H})$ for which $\ker(\operatorname{Im} M_s(\lambda)) = \{0\}$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Corollary 4.30. *Assume that the Weyl family $M(\cdot)$ belongs to the subclass $\widetilde{R}^s_{inv}(\mathcal{H})$. Moreover, assume that A_0 is selfadjoint and* $\operatorname{mul} S = \{0\}$. *Then* $\operatorname{mul} T = \{0\}$ *if and only if*

$$\lim_{y\to\infty} \frac{(M_s(iy)h,h)_{\mathcal{H}}}{iy} = 0, \quad \textit{and} \quad \lim_{y\to\infty} y(\operatorname{Im} M_s(iy)h,h)_{\mathcal{H}} = \infty \quad \textit{for every } h \in \mathcal{H}_0. \tag{4.43}$$

Proof. If (4.43) holds then A_0 is an operator and $\mathfrak{N}_\lambda(T) \cap \operatorname{dom} A_0 = \{0\}$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Now Proposition 4.20 shows that $\operatorname{mul} T = \{0\}$.

Conversely, assume that $\operatorname{mul} T = \{0\}$. Then by Proposition 4.20 $\mathfrak{N}_\lambda(T) \cap \operatorname{dom} A_0 = \{0\}$ for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and this implies (4.43), since $\gamma(\lambda)$ is injective (see (4.28)). □

Remark 4.31. The results in this subsection are generalizations of similar statements which are well known for Q-functions of symmetric operators and Weyl functions of ordinary boundary triplets, i.e., for the subclass $R^u[\mathcal{H}]$ of Nevanlinna functions, see e.g. [23], [24].

4.5. **Forbidden lineals.** The concept of a forbidden (isometric) operator V in the framework of von Neumann's theory has originally been introduced by M.A. Krasnosel'skiĭ in [20]. The connection between the operator V and limit values of the characteristic function was discovered by A.V. Straus [31]. In the case of ordinary boundary triplets the so-called forbidden lineal has been introduced and studied in [24], cf. also [13].

Definition 4.32. Let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary linear relation and let $T = \operatorname{dom}\Gamma$. The *forbidden lineal* of Γ is defined by

$$\mathcal{F}_\Gamma = \Gamma(\{0\} \oplus \operatorname{mul} T).$$

In this subsection the forbidden lineal of Γ will be characterized by using the asymptotic properties of the corresponding Weyl family $M(\cdot)$, under the assumptions that A_0 is a self-adjoint and the operator part $M_s(\cdot)$ of $M(\cdot)$ has a bounded imaginary part for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The approach given below is rather straightforward and the proof of the main statement in Proposition 4.34 is essentially simpler than the one used earlier in the case of ordinary boundary triplets.

Proposition 4.33. *Let $A_0 = A_0^*$ and let P be the orthogonal projection onto $\overline{\operatorname{dom}} A_0$. Let $h \in \Gamma_0\{0, -\omega\}$ for some $\omega \in \operatorname{mul} T$ and let $\operatorname{Im} M(\lambda)$ be a bounded operator for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then:*

(i) $\gamma(\lambda)h \to 0$ *strongly in $\mathcal{H}$ as $\lambda = iy \to \infty$;*
(ii) $\lambda\gamma(\lambda)h \to -P\omega$ *strongly in $\mathfrak{H}$ as $\lambda = iy \to \infty$;*
(iii) *the following strong limit exists*

$$M_s(i\infty)h := \lim_{y\to\infty} M_s(iy)h = M_s(\bar{\mu})h - \gamma(\mu)P\omega. \tag{4.44}$$

Proof. It follows from Proposition 4.20 that $h \in \operatorname{dom}\gamma(\lambda)$ and

$$\gamma(\lambda)h = (A_0 - \lambda)^{-1}\omega, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.45}$$

The statements (i) and (ii) are immediate from the representation (4.45). Moreover, (iii) is implied by (4.29) since $\gamma^*(\mu)$ is bounded due to Proposition 4.24. □

Proposition 4.34. *Let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary linear relation such that $A_0 = A_0^*$ and $\operatorname{Im} M_s(\lambda)$ is a bounded operator for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\operatorname{dom}\mathcal{F}_\Gamma = \{\, h \in \mathcal{H} : \sup_{y>0} y(\operatorname{Im} M_s(iy)h, h) < \infty \,\}, \tag{4.46}$$

and the forbidden lineal $\mathcal{F}_\Gamma$ admits the representation

$$\mathcal{F}_\Gamma = \{\{h, M_s(i\infty)h\} : h \in \operatorname{dom}\mathcal{F}_\Gamma\} \,\widehat{+}\, \Gamma(\{0\} \oplus \operatorname{mul} A_0). \tag{4.47}$$

Proof. The following characterization concerning $\operatorname{dom}\mathcal{F}_\Gamma$ is implied by Proposition 4.20

$$h \in \operatorname{dom}\mathcal{F}_\Gamma \text{ if and only if } \gamma(\lambda)h \in \operatorname{dom} A_0. \tag{4.48}$$

Combining (4.48) with the equivalence (4.39) in Proposition 4.29 gives the description of $\operatorname{dom}\mathcal{F}$ in (4.46).

Next assume that $\{h, h'\} \in \Gamma\{0, -\omega\}$, where $\omega \in \operatorname{mul} T$. Then

$$\left\{ \begin{pmatrix} 0 \\ h \end{pmatrix}, \begin{pmatrix} -\omega \\ -h' \end{pmatrix} \right\} \in \widetilde{A}. \tag{4.49}$$

Consider also the elements

$$\left\{ \begin{pmatrix} \gamma(\lambda)h \\ h \end{pmatrix}, \begin{pmatrix} \lambda\gamma(\lambda)h \\ -M_s(\lambda)h \end{pmatrix} \right\} \in \widetilde{A}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.50}$$

It follows from Proposition 4.33 that with $\lambda = iy \to \infty$ these elements converge to

$$\left\{ \begin{pmatrix} 0 \\ h \end{pmatrix}, \begin{pmatrix} -P\omega \\ -M_s(i\infty)h \end{pmatrix} \right\} \in \widetilde{A}. \tag{4.51}$$

Now subtract from this element the element in (4.49), so that

$$\left\{ \begin{pmatrix} 0 \\ h \end{pmatrix}, \begin{pmatrix} -P\omega \\ -M_s(i\infty)h \end{pmatrix} \right\} - \left\{ \begin{pmatrix} 0 \\ h \end{pmatrix}, \begin{pmatrix} -\omega \\ -h' \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} (I-P)\omega \\ h' - M_s(i\infty)h \end{pmatrix} \right\} \in \widetilde{A}.$$

This shows that $\{0, h' - M_s(i\infty)h\} \in \Gamma(\{0\} \oplus \operatorname{mul} A_0)$. Hence,

$$\{h, h'\} = \{h, M_s(i\infty)h\} \,\widehat{+}\, \{0, h' - M_s(i\infty)h\},$$

proving one inclusion in (4.47). The reverse inclusion is obvious from (4.46), (4.51), and $\operatorname{mul} A_0 \subset \operatorname{mul} T$. □

Corollary 4.35. *Let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation, which is single-valued and satisfies the assumptions of Proposition 4.34. Then*

$$\operatorname{mul} \mathcal{F}_\Gamma = \Gamma(\{0\} \oplus \operatorname{mul} A_0).$$

5. Special boundary relations and their Weyl families

5.1. **Ordinary boundary triplets.** A combination of Definition 1.1 of an ordinary boundary triplet for the case of densely defined symmetric operator from [17] (see also [12], [24]) and the adaptation for the case of a nondensely defined symmetric operator leads to the following definition. The adjoint S^* of a symmetric operator S in $\mathfrak{H}$ is a closed linear relation in $\mathfrak{H}$; it can be considered as a Hilbert space with the graph norm.

Definition 5.1. ([17]) Let S be a closed symmetric operator in a Hilbert space $\mathfrak{H}$ with equal defect numbers. A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is a Hilbert space with $\dim \mathcal{H} = n_\pm(S)$ and $\Gamma_i \in [S^*, \mathcal{H}]$, is said to be an *ordinary boundary triplet* for S^*, if:

(A1) the abstract Green's identity (1.5) holds;
(A2) the mapping $\Gamma := \{\Gamma_0, \Gamma_1\} : S^* \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

Lemma 5.2. *Let $\Gamma : (\mathfrak{H}^2, J_\mathfrak{H}) \to (\mathcal{H}^2, J_\mathcal{H})$ be isometric and let $T = \operatorname{dom} \Gamma$ satisfy $S \subset T \subset S^*$. If T is dense in S^* and $\operatorname{ran} \Gamma = \mathcal{H}^2$, then $S = \ker \Gamma$ has equal defect numbers, $T = S^*$, and Γ is a bounded single-valued unitary operator.*

Proof. By assumptions $(\operatorname{dom} \Gamma)^{[\perp]} = T^* = S \subset \operatorname{dom} \Gamma$. Hence, Corollary 2.6 shows that Γ is a bounded unitary operator with $T = \operatorname{dom} \Gamma = S^*$. In particular, $\ker \Gamma = T^* = S$ and in view of Lemma 2.14 $\operatorname{mul} \Gamma = \{0\}$ implies that the defect numbers of S are equal. □

Proposition 5.3. *A triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triplet for S^* if and only if $\Gamma = \{\Gamma_0, \Gamma_1\} : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is a boundary relation for S^* such that*

$$\operatorname{ran} \Gamma = \mathcal{H}^2. \tag{5.1}$$

Proof. Let $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be an ordinary boundary triplet for S^*. Note that (A1) and (A2) mean that $\Gamma:\mathfrak{H}^2\mapsto\mathcal{H}^2$ is an isometric operator from the Kreĭn space $(\mathfrak{H}^2,J_{\mathfrak{H}})$ into the Kreĭn space $(\mathcal{H}^2,J_{\mathcal{H}})$ with $\operatorname{dom}\Gamma=S^*$ and $\operatorname{ran}\Gamma=\mathcal{H}^2$. By Lemma 5.2 Γ is unitary and (G1) and (G2) are satisfied.

Conversely, if $\Gamma:\mathfrak{H}^2\to\mathcal{H}^2$ is a boundary relation for S^* such that $\operatorname{ran}\Gamma=\mathcal{H}^2$, then Γ is unitary and $\operatorname{dom}\Gamma=S^*$. It remains to apply Lemma 5.2 to see that $\Gamma=\Gamma_0\oplus\Gamma_1$ satisfies all the assumptions in Definition 5.1. □

The kernels $A_i:=\ker\Gamma_i$, $i=0,1$, define two selfadjoint extensions of S. Associated with Π are two functions, which are holomorphic on $\rho(A_0)$.

Definition 5.4. ([12], [13]) Let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be an ordinary boundary triplet for S^*. Then the γ-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ corresponding to Π are defined by

$$\widehat{\gamma}(\lambda):=(\Gamma_0\upharpoonright\widehat{\mathfrak{N}}_\lambda)^{-1},\quad \gamma(\lambda):=\pi_1(\Gamma_0\upharpoonright\widehat{\mathfrak{N}}_\lambda)^{-1},\quad M(\lambda)=\Gamma_1\widehat{\gamma}(\lambda),\quad \lambda\in\rho(A_0). \tag{5.2}$$

Here $\widehat{\mathfrak{N}}_\lambda:=\widehat{\mathfrak{N}}_\lambda(S^*)$ and π_1 stands for the orthogonal projection onto the first component of $\mathcal{H}\oplus\mathcal{H}$.

In this case the corresponding Weyl function $M(\cdot)$ belongs to the class $R^u[\mathcal{H}]$. In fact, $M(\cdot)$ determines the pair $\{S,A_0\}$ up to unitary equivalence, cf. [23], and, conversely, for every such function $M(\cdot)\in R^u[\mathcal{H}]$ there exist a symmetric operator S and an ordinary boundary triplet for S^* whose Weyl function is equal to $M(\cdot)$.

It was shown in [12], [24] that $\gamma(\cdot)$ and $M(\cdot)$ satisfy (4.14) and the following identity

$$M(\lambda)-M(\mu)^*=(\lambda-\bar{\mu})\gamma(\mu)^*\gamma(\lambda),\quad \lambda,\mu\in\rho(A_0). \tag{5.3}$$

This means that the γ-field $\gamma(\cdot)$ of S in Definition 5.4 is generated by A_0 and that $M(\cdot)$ is a Q-function of the pair $\{S,A_0\}$.

5.2. **Generalized boundary triplets.** A more general definition of a boundary triplet for a symmetric operator S with equal defect numbers was given in [13]. This motivates the statement in the next lemma.

Lemma 5.5. *Let $\mathcal{H}$ and $\mathfrak{H}$ be Hilbert spaces and let Γ be an isometric relation from $(\mathfrak{H}^2,J_{\mathfrak{H}})$ to $(\mathcal{H}^2,J_{\mathcal{H}})$ with the properties*

(i) $\operatorname{ran}\Gamma_0=\mathcal{H}$;

(ii) $A_0:=\ker\Gamma_0$ *is essentially selfadjoint in* $\mathfrak{H}$,

where Γ_0 is as defined in (3.6). *Then Γ is a unitary relation from $(\mathfrak{H}^2,J_{\mathfrak{H}})$ to $(\mathcal{H}^2,J_{\mathcal{H}})$.*

Proof. Let $\{\widehat{k},\widehat{g}\}\in\Gamma^{[*]}$ with $\widehat{k}=\{k,k'\}$ and $\widehat{g}=\{g,g'\}$. By assumption (i) $\{\widehat{s},\{k,t'\}\}\in\Gamma$ for some $\widehat{s}=\{s,s'\}\in\mathcal{H}^2$ and $t'\in\mathfrak{H}$. Since $\Gamma^{-1}\subset\Gamma^{[*]}$, one concludes that

$$\{\{0,k'-t'\},\widehat{g}-\widehat{s}\}=\{\widehat{k},\widehat{g}\}-\{\{k,t'\},\widehat{s}\}\in\Gamma^{[*]}. \tag{5.4}$$

Moreover, the assumption (i) and Lemma 4.10 imply that A_0 is closed. Thus, by the assumption (ii) A_0 is a selfadjoint relation in $\mathfrak{H}$. The condition $\widehat{f}\in A_0$ means that $\{\widehat{f},\{0,h'\}\}\in\Gamma$ for some $h'\in\mathcal{H}$. Now it follows from (5.4) that for all $\widehat{f}\in A_0$,

$$(J_{\mathfrak{H}}\widehat{f},\widehat{g}-\widehat{s})=(J_{\mathcal{H}}\{0,h'\},\{0,k'-t'\})=0.$$

Therefore, $\widehat{g} - \widehat{s} \in A_0^* = A_0$ by assumption (ii). Hence, $\{\widehat{g} - \widehat{s}, \{0, v'\}\} \in \Gamma$ for some $v' \in \mathcal{H}$ and (5.4) implies that $\{\{0, k' - t' - v'\}, \{0, 0\}\} \in \Gamma^{[*]}$. This means that for all $\{\widehat{f}, \widehat{h}\} \in \Gamma$,

$$0 = (J_{\mathfrak{H}}\widehat{f}, \{0, 0\}) = (J_{\mathcal{H}}\widehat{h}, \{0, k' - t' - v'\}) = i(h, k' - t' - v')_{\mathcal{H}},$$

and now the assumption (i) yields $k' - t' - v' = 0$. This shows that $\{\widehat{g}, \widehat{k}\} = \{\widehat{g} - \widehat{s}, \{0, v'\}\} + \{\widehat{s}, \{k, t'\}\} \in \Gamma$. Thus, $\Gamma^{[*]} \subset \Gamma^{-1}$. □

Now recall the definition of a generalized boundary triplet as given in [13].

Definition 5.6. ([13]) Let S be a closed symmetric operator in a Hilbert space $\mathfrak{H}$ with equal defect numbers and let T be a linear relation in $\mathfrak{H}$ such that $S \subset T \subset \operatorname{clos} T = S^*$. Then the triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is a Hilbert space and $\Gamma = (\Gamma_0, \Gamma_1)^\top$ is a single-valued linear mapping from T to $\mathcal{H}^2$, is said to be a *generalized boundary triplet* for S^*, if:

(S1) the abstract Green's identity (1.5) holds for all $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in T$;
(S2) $\operatorname{ran} \Gamma_0 = \mathcal{H}$;
(S3) $A_0 := \ker \Gamma_0$ is a selfadjoint linear relation in $\mathfrak{H}$.

By definition $A_0 \subset \operatorname{dom} \Gamma = T$, which implies that A_0 is a selfadjoint extension of S.

Proposition 5.7. *A triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a generalized boundary triplet for S^* if and only if $\Gamma = \{\Gamma_0, \Gamma_1\} : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is a boundary relation for S^* such that*

$$\operatorname{mul} \Gamma = \{0\}, \quad \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{5.5}$$

In this case the corresponding Weyl family $M(\lambda)$ belongs to the class $R^s[\mathcal{H}]$. Conversely, every $R^s[\mathcal{H}]$-function is the Weyl function of some generalized boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$.

Proof. Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for S^*. Then $\Gamma = \{\Gamma_0, \Gamma_1\}$ is unitary by Lemma 5.5 and hence it is a boundary relation for S^* with $\operatorname{mul} \Gamma = \{0\}$. It follows from the assumption (S3) that $S^* = A_0 \widehat{+} \widehat{\mathfrak{N}}_\lambda(S^*)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and since $A_0 \subset T$, also the equality

$$T = A_0 \widehat{+} \widehat{\mathfrak{N}}_\lambda(T), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{5.6}$$

holds. This together with (S2) gives

$$\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \Gamma_0(T) = \mathcal{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{5.7}$$

The statement $M(\lambda) \in R^s[\mathcal{H}]$ is obtained from Proposition 4.7.

Conversely, let Γ be a boundary relation for S^* with the properties (5.5). Then $\mathcal{H} = \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) \subset \operatorname{ran} \Gamma_0$, so that $\operatorname{ran} \Gamma_0 = \mathcal{H}$, i.e., (S2) is satisfied. Also the property (S3) is obtained from $\Gamma_0(\mathfrak{N}_\lambda(T)) = \mathcal{H}$ by using Proposition 4.15, cf. Corollary 4.17. The remaining conditions for generalized boundary triplets are clearly satisfied.

The fact that every $R^s[\mathcal{H}]$-function is the Weyl function of some generalized boundary triplet (see [13]) is implied in the present context by Theorem 3.9 and Propositions 4.5, 4.7. □

The next result collects some further properties of generalized boundary triplets.

Proposition 5.8. ([13]) *Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for S^*. Then:*

(i) $T = A_0 \widehat{+} \widehat{\mathfrak{N}}_\lambda(T)$ *for every* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*;*
(ii) $\operatorname{clos} \Gamma_1(A_0) = \mathcal{H}$ *and* $\overline{\operatorname{ran}}\, \Gamma = \mathcal{H} \oplus \mathcal{H}$*;*
(iii) *the restriction* $\Gamma_0 : \widehat{\mathfrak{N}}_\lambda(T) \to \mathcal{H}$ *is a closed mapping for every* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*;*

(iv) *the equalities* (5.2) *define an* $[\mathcal{H}, \widehat{\mathfrak{N}}_\lambda]$*-valued function* $\widehat{\gamma}(\cdot)$*, an* $[\mathcal{H}, \mathfrak{N}_\lambda]$*-valued function* $\gamma(\cdot)$*, and an* $[\mathcal{H}]$*-valued function* $M(\cdot)$*, which are holomorphic on* $\mathbb{C} \setminus \mathbb{R}$ *and satisfy the identities* (4.14) *and* (5.3).

Proof. (i) This was shown in the proof of Proposition 5.7, cf. (5.6).

(ii) By Proposition 2.3 $\operatorname{ran}\Gamma$ is dense in $\mathcal{H}^2$. Now assume that $h \perp \Gamma_1(A_0)$. Then $\{k, h\} \perp \Gamma(A_0)$ in $\mathcal{H}^2$ for every $k \in \mathcal{H}$. Since $\{-M(\lambda)^*h, h\} \perp \Gamma(\widehat{\mathfrak{N}}_\lambda)$ in $\mathcal{H}^2$, it follows from (i) that $\{-M(\lambda)^*h, h\} \perp \operatorname{ran}\Gamma$ and, therefore, $h = 0$.

(iii) By Proposition 4.9 the mapping $\Gamma_0 : \widehat{\mathfrak{N}}_\lambda(T) \to \mathcal{H}$ $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is closable. Moreover, it is closed since $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}$.

(iv) The operators $\widehat{\gamma}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, are bounded by the closed graph theorem and thus also the operators $\gamma(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, are bounded. The corresponding statement for $M(\lambda)$ was shown in Proposition 5.9. The remaining statements are immediate from Proposition 4.11 and the operator representation (4.29). □

5.3. **Boundary relations with Weyl functions in** $R[\mathfrak{H}]$**.** In this subsection the class of boundary relations whose Weyl functions belong to the class $R[\mathcal{H}]$ is considered.

Proposition 5.9. *Let* $\mathcal{H}$ *and* $\mathfrak{H}$ *be Hilbert spaces and let* $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ *be a (possibly multivalued) linear relation such that:*

(B1) *Green's identity* (3.1) *holds;*
(B2) $\operatorname{ran}\Gamma_0 = \mathcal{H}$*;*
(B3) $A_0 := \ker\Gamma_0$ *is a selfadjoint linear relation in* $\mathfrak{H}$*.*

Then $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ *is a boundary relation for* $S^* := (\ker\Gamma)^*$ *such that*

$$\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{5.8}$$

Conversely, every boundary relation $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ *satisfying* (5.8) *satisfies also the conditions (B1)-(B3). In this case the corresponding Weyl function belongs to the class* $R[\mathcal{H}]$*, and moreover, every* $R[\mathcal{H}]$*-function is the Weyl function of some boundary relation* $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ *with the properties (B1)-(B3).*

Proof. The proof is completely analogous to that of Proposition 5.7 and hence it will be omitted. □

Proposition 5.10. *Let* $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ *be a boundary relation satisfying the conditions (B1)-(B3). Then:*

(i) $T = S_0 \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(T)$ *for every* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*;*
(ii) $(\Gamma_0 \restriction \widehat{\mathfrak{N}}_\lambda(T))^{-1} : \mathcal{H} \to \widehat{\mathfrak{N}}_\lambda(T)$ *is closed bounded and single-valued for every* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*;*
(iii) *the equalities* (5.2) *define* $[\mathcal{H}, \widehat{\mathfrak{N}}_\lambda]$*-valued,* $[\mathcal{H}, \mathfrak{N}_\lambda]$*-valued, and* $[\mathcal{H}]$*-valued functions* $\widehat{\gamma}(\cdot)$*,* $\gamma(\cdot)$*, and* $M(\cdot)$*, respectively, which are holomorphic on* $\mathbb{C} \setminus \mathbb{R}$ *and satisfy the identities* (4.14) *and* (5.3).

Proof. The proof is analogous to that of Proposition 5.8. The main difference is that the mapping $\Gamma_0 \restriction \widehat{\mathfrak{N}}_\lambda(T)) : \widehat{\mathfrak{N}}_\lambda(T) \to \mathcal{H}$, may be multivalued, but it has still a trivial kernel and its inverse is a closed bounded single-valued mapping for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ by Proposition 4.9. The identity (5.3) is implied by (4.29). □

The next proposition gives a reduction of a multivalued boundary relation $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ with the properties (B1)–(B3) to a generalized boundary triplet satisfying (S1)–(S3).

Proposition 5.11. *Let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation satisfying the conditions (B1)-(B3) and let $\mathcal{H}_0 = \pi_0 \mathrm{mul}\, \Gamma$, $\mathcal{H}_1 = \mathcal{H} \ominus \mathcal{H}_0$. Then there is a selfadjoint operator $K \in [\mathcal{H}]$, such that the linear relation*

$$\Gamma^s := \left\{ \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} P_{\mathcal{H}_1} h \\ h' - Kh \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \right\} \tag{5.9}$$

is a single-valued boundary relation for S^* and the corresponding boundary triplet $\{\mathcal{H}, \Gamma_0^s, \Gamma_1^s\}$ satisfies the assumptions (S1)-(S3). The Weyl functions $M(\lambda)$, $M_s(\lambda)$ corresponding to the boundary relations Γ, Γ^s are connected by

$$M(\lambda) = K + 0_{\mathcal{H}_0} \oplus M_1(\lambda), \quad (\lambda \in \mathbb{C}_+). \tag{5.10}$$

Proof. Since $\mathrm{ran}\, \Gamma_0 = \mathcal{H}$ one derives from Lemma 2.1 that $\mathrm{ran}\, \Gamma \widehat{+} \{0, \mathcal{H}\}$ is closed. By a theorem of Kato [19, Theorem 4.8] and Proposition 2.3

$$\mathrm{mul}\, \Gamma \widehat{+} \{0, \mathcal{H}\} \quad \text{is closed.} \tag{5.11}$$

Using Lemma 2.1 again one obtains that $\mathcal{H}_0 = \pi_0 \mathrm{mul}\, \Gamma$ is a closed subspace of $\mathcal{H}$ and it follows from (5.11) that $\mathrm{mul}\, \Gamma$ is the graph of a bounded operator $K_0 : \mathcal{H}_0 \to \mathcal{H}$. Due to Proposition 2.9 K_0 is a symmetric operator since $\mathrm{mul}\, \Gamma$ is a neutral subspace in $\{\mathcal{H}^2, J_{\mathcal{H}}\}$. Let K be a bounded selfadjoint operator extension of K_0. Since $\mathrm{mul}\, \Gamma = \mathrm{ran}\, \Gamma^{[\perp]}$ one obtains from

$$0 = (h', h_0) - (h, K_0 h_0) = (h' - Kh, h_0) \quad (h_0 \in \mathcal{H}_0,\ \{h, h'\} \in \mathrm{ran}\, \Gamma).$$

that $h' - Kh$ is orthogonal to $\mathcal{H}_0$. This proves that $\mathrm{ran}\, \Gamma^s \subset \mathcal{H}_1^2$. The mapping Γ is single-valued since for $\widehat{h} = \{h, h'\} \in \mathrm{mul}\, \Gamma$ one has $P_{\mathcal{H}_1} h = h' - Kh = 0$. Now, the properties (S1)-(S3) for Γ^s are implied by the properties (B1)-(B3) for Γ. The equality 5.10 follows from (5.9). □

The following corollary is immediate from Proposition 4.30.

Corollary 5.12. *Let S be a symmetric operator in $\mathfrak{H}$ and let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^* satisfying the conditions (B1)-(B3), $\mathcal{H}_0 = \pi_0 \mathrm{mul}\, \Gamma$, $A_0 = \ker \Gamma_0$, $T = \mathrm{dom}\, \Gamma$ and let $M(\lambda)$ be the corresponding Weyl function. Then:*

(i) $\mathrm{mul}\, A_0 = \{0\}$, *if and only if*

$$\lim_{y \to \infty} \frac{(M(iy)h, h)_{\mathcal{H}}}{iy} = 0, \quad h \in \mathcal{H}; \tag{5.12}$$

(ii) $\mathrm{mul}\, T = \{0\}$ *if and only if M satisfies the condition* (5.12) *and*

$$\lim_{y \uparrow \infty} y \mathrm{Im}\, \big(M(iy)h, h\big) = \infty \quad \forall h \in \mathcal{H} \ominus \mathcal{H}_0,\ \ h \neq 0.$$

6. Examples

In this section a number of illustrative examples are presented. Each example by itself shows some characteristic behaviour of boundary relations.

Example 6.1. Let A be a selfadjoint relation in a Hilbert space $\mathfrak{H}$ and let B be a selfadjoint relation in a Hilbert space $\mathcal{H}$. Define the linear relation $\widetilde{A}$ in $\mathfrak{H} \oplus \mathcal{H}$ by $\widetilde{A} = A \oplus (-B)$, i.e.,

$$\widetilde{A} = \left\{ \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' \\ -h' \end{pmatrix} \right\} : \{f, f'\} \in A,\ \{h, h'\} \in B \right\}.$$

It is clear that $\widetilde{A}$ is selfadjoint. The transform

$$\Gamma = \mathcal{J}\widetilde{A} = \left\{ \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} : \{f, f'\} \in A,\ \{h, h'\} \in B \right\}$$

defines a boundary relation $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ for $S^* = A$ with

$$T = \operatorname{dom} \Gamma = A, \quad S = \ker \Gamma = A, \quad \operatorname{ran} \Gamma = B, \quad \operatorname{mul} \Gamma = B.$$

In particular, this implies that $A_0 = A_1 = S = S^*$ are selfadjoint. The corresponding Weyl family $M(\lambda) = \Gamma(\widehat{\mathfrak{N}}_\lambda(T))$ is given by $M(\lambda) \equiv B$. Since $\operatorname{ran} \Gamma_0 = \operatorname{dom} B$ it is seen that $M(\lambda) \in \widetilde{R}[\mathcal{H}]$ if and only if $\operatorname{dom} B$ is closed (in which case B is the orthogonal sum of a bounded selfadjoint operator and a closed multivalued part). Furthermore, note that $\mathfrak{N}_\lambda(S^*) = \mathfrak{N}_\lambda(A) = \{0, 0\}$, so that the γ-field satisfies $\gamma(\lambda)h = 0$ for all $h \in \operatorname{dom} B$ and hence its closure is single-valued.

Example 6.1 gives a realization for a constant Nevanlinna family $M(\lambda) \equiv B$ as a Weyl family of a boundary relation Γ. Observe, that the symmetry property $M(\lambda)^* = M(\bar{\lambda})$ forces that $B = B^*$. In Example 6.1 the boundary relation Γ is not minimal. The following result expresses this situation more explicitly.

Corollary 6.2. *Let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be an arbitrary boundary relation whose Weyl function satisfies $M(\lambda) \equiv B$. If Γ is minimal then $\mathfrak{H} = \{0\}$ and in particular $S = T = S^*$.*

Proof. Consider $\{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma$ with $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The identity (3.4) with $\lambda = \mu$ shows that

$$(\lambda - \bar{\lambda})\|f_\lambda\|^2 = (h', h) - (h, h') = 0.$$

since B is selfadjoint. This implies that $\widehat{f}_\lambda = \{0, 0\}$. Now the assumption that $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ is minimal yields $\mathfrak{H} = \overline{\operatorname{span}}\, \{\, \mathfrak{N}_\lambda(T) :\ \lambda \in \mathbb{C} \setminus \mathbb{R} \,\} = \{0\}$. In this case $S = T = S^* = \{0, 0\}$. □

Example 6.3. Let S be a closed symmetric relation in a Hilbert space $\mathfrak{H}$. The defect numbers of S need not be equal and may be infinite. Let $\widetilde{A}$ be a linear relation in $\mathfrak{H} \oplus \mathfrak{H}$ given by

$$\widetilde{A} = \left\{ \widehat{f} = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1' \\ -f_2' \end{pmatrix} \right\} :\ \widehat{f}_1 = \{f_1, f_1'\},\ \widehat{f}_2 = \{f_2, f_2'\} \in S^*,\ P_{\mathfrak{N}}\widehat{f}_1 = P_{\mathfrak{N}}\widehat{f}_2 \right\}, \tag{6.1}$$

where $P_{\mathfrak{N}}$ is the orthogonal projection from S^* onto $\mathfrak{N} = \widehat{\mathfrak{N}}_i(S^*) \oplus \widehat{\mathfrak{N}}_{-i}(S^*)$, see (3.12). As it was shown in the proof of Proposition 3.7 $\widetilde{A}$ is a selfadjoint extension of S such that $\widetilde{A} \cap \mathfrak{H}^2 = S$. The transform $\Gamma = \mathcal{J}^{-1}\widetilde{A}$ is a boundary relation for S^* with the following properties

$$\ker \Gamma = \operatorname{mul} \Gamma = S, \quad \operatorname{dom} \Gamma = \operatorname{ran} \Gamma = S^*.$$

Furthermore, the Weyl family $M(\lambda)$ corresponding to the boundary relation Γ is given by

$$M(\lambda) = S \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(S^*), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{6.2}$$

To see this assume $\{f_2, f_2'\} \in M(\lambda)$. Then it follows from (3.5) that

$$\{\,\{f_\lambda, \lambda f_\lambda\}, \{f_2, f_2'\}\,\} \in \Gamma \quad \text{for some} \quad \{f_\lambda, \lambda f_\lambda\} \in S^*.$$

By (6.1) this shows $P_{\mathfrak{N}}(\{f_2, f_2'\} - \{f_\lambda, \lambda f_\lambda\}) = 0$, which implies that $\{f_2, f_2'\} - \{f_\lambda, \lambda f_\lambda\} \in S$, so that $\{f_2, f_2'\} \in S \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(S^*)$. Hence $M(\lambda) \subset S \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(S^*)$, and the reverse inclusion can be seen immediately. Furthermore, it is clear that

$$\operatorname{mul} M(\lambda) = \operatorname{mul} S, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{6.3}$$

To see this, consider an arbitrary element in $M(\lambda)$ with $\lambda \in \mathbb{C} \setminus \mathbb{R}$:

$$\{h, h'\} \,\widehat{+}\, \{f_\lambda, \lambda f_\lambda\} \in M(\lambda) \quad \text{with} \quad \{h, h'\} \in S, \quad \{f_\lambda, \lambda f_\lambda\} \in \widehat{\mathfrak{N}}_\lambda(S^*),$$

so that $(h', f_\lambda) = \bar{\lambda}(f_\lambda, f_\lambda)$. If $h + f_\lambda = 0$, then $\{-f_\lambda, h'\} \in S$ and $(h', f_\lambda) \in \mathbb{R}$. Hence, it follows that $f_\lambda = 0$, and consequently, $h = 0$. Thus $\operatorname{mul} M(\lambda) \subset \operatorname{mul} S$ and the reverse inclusion is obvious.

In this example $M(\cdot) \in \widetilde{R}(\mathfrak{H})$. According to Proposition 3.5 the boundary relation Γ is minimal if and only if the selfadjoint relation $\widetilde{A} = \mathcal{J}\Gamma$ in $\mathfrak{H} \oplus \mathfrak{H}$ is minimal with respect to $\mathfrak{H}$, which is equivalent to S being simple. Hence, if S is simple, in which case S is an operator, this model provides a minimal realization for the Weyl family $M(\lambda)$ in (6.2). Moreover, its multivalued part is trivial due to (6.3). Therefore, if S is simple, $M(\cdot) \in R(\mathfrak{H})$.

Recall that the symmetric relations A_0 and A_1 in (4.13) are given by

$$A_0 = \{\,\{f_1, f_1'\} \in S^* : P_{\mathfrak{N}}\widehat{f}_1 = P_{\mathfrak{N}}\{0, f_2'\},\ \{0, f_2'\} \in S^*\,\},$$

and

$$A_1 = \{\,\{f_1, f_1'\} \in S^* : P_{\mathfrak{N}}\widehat{f}_1 = P_{\mathfrak{N}}\{f_2, 0\},\ \{f_2, 0\} \in S^*\,\}.$$

This implies, in a similar way as above, that

$$A_0 = S \,\widehat{+}\, \{0, \operatorname{mul} S^*\}, \quad A_1 = S \,\widehat{+}\, \{\ker S^*, 0\}.$$

Moreover, $\operatorname{ran}\Gamma_0 = \operatorname{dom} S^*$ and $\operatorname{ran}\Gamma_1 = \operatorname{ran} S^*$. The extension A_0 of S is symmetric and, in general, not selfadjoint. The defect numbers of A_0 are called semi-defect numbers of S, see [20].

Example 6.4. Let B be a closed densely defined linear operator in $\mathfrak{H}$ and let $\widetilde{A}$ be the operator determined by the block form

$$\widetilde{A} = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}.$$

By definition $\widetilde{A}$ is a selfadjoint operator in $\mathfrak{H} \oplus \mathfrak{H}$ whose graph is given by

$$\widetilde{A} = \left\{ \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} Bh \\ B^* f \end{pmatrix} \right\} : h \in \operatorname{dom} B,\ f \in \operatorname{dom} B^* \right\}.$$

Moreover,

$$(\widetilde{A} - \lambda)^{-1} \begin{pmatrix} \varphi \\ 0 \end{pmatrix} = \begin{pmatrix} (BB^* - \lambda^2)^{-1}\lambda\varphi \\ B^*(BB^* - \lambda^2)^{-1}\varphi \end{pmatrix}$$

and

$$(\widetilde{A} - \lambda)^{-1} \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} B(B^*B - \lambda^2)^{-1}\psi \\ (B^*B - \lambda^2)^{-1}\lambda\varphi \end{pmatrix}.$$

The transform

$$\Gamma = \mathcal{J}\widetilde{A} = \left\{ \left\{ \begin{pmatrix} f \\ Bh \end{pmatrix}, \begin{pmatrix} h \\ -B^*f \end{pmatrix} \right\} : h \in \operatorname{dom} B, \ f \in \operatorname{dom} B^* \right\}$$

defines a boundary relation for $S^* = \mathfrak{H} \oplus \overline{\operatorname{ran}}\, B$ in $\mathfrak{H} \oplus \mathfrak{H}$ with

$$T = \operatorname{dom}\Gamma = \operatorname{dom} B^* \oplus \operatorname{ran} B, \quad S = \ker\Gamma = \ker B^* \oplus \{0\},$$
$$\operatorname{ran}\Gamma = \operatorname{dom} B \oplus \operatorname{ran} B^*, \quad \operatorname{mul}\Gamma = \ker B \oplus \{0\}.$$

Hence, Γ is single-valued if and only if $\ker B = \{0\}$ and since $\operatorname{mul}\Gamma \cap \{0, \mathfrak{H}\} = \{0\}$ the corresponding Weyl family $M(\lambda)$ is operator-valued. Moreover,

$$\operatorname{ran}\Gamma_0 = \operatorname{dom} B, \quad \operatorname{ran}\Gamma_1 = \operatorname{ran} B^*.$$

Observe, that $\{f, \lambda f\} \in T$ if and only if

$$f = \frac{1}{\lambda} Bh, \quad h \in \operatorname{dom} B^*B.$$

Hence,

$$\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \operatorname{dom} B^*B, \quad \Gamma_1(\widehat{\mathfrak{N}}_\lambda(T)) = \operatorname{ran} B^*B,$$

and the corresponding γ-field and the Weyl family are given by

$$\gamma(\lambda) = \frac{1}{\lambda} B, \quad M(\lambda) = -\frac{1}{\lambda} B^*B$$

on $\operatorname{dom} B^*B$. In particular, $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T))$ and $\Gamma_1(\widehat{\mathfrak{N}}_\lambda(T))$ are independent of $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and in general they do not coincide with $\operatorname{ran}\Gamma_0$ and $\operatorname{ran}\Gamma_1$, respectively. The formulas

$$A_0 = \ker\Gamma_0 = \operatorname{dom} B^* \oplus \{0\}, \quad A_1 = \ker\Gamma_0 = \ker B^* \oplus \operatorname{ran} B,$$

show that A_0 and A_1 are essentially selfadjoint in $\mathfrak{H}$.

In this case also the equality $M(\lambda) - M(\mu)^* = (\lambda - \bar{\mu})\gamma(\mu)^*\gamma(\lambda)$ is satisfied.

Example 6.5. *(Weyl function for bounded perturbations.)* Let B be bounded and let D and E be selfadjoint operators in $\mathfrak{H}$. Define $\widetilde{A}$ by the block form

$$\widetilde{A} = \begin{pmatrix} D & B \\ B^* & -E \end{pmatrix}.$$

Then $\widetilde{A}$ is a selfadjoint operator in $\mathfrak{H} \oplus \mathfrak{H}$ whose graph is given by

$$\widetilde{A} = \left\{ \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} Df + Bh \\ B^*f - Eh \end{pmatrix} \right\} : f \in \operatorname{dom} D, \ h \in \operatorname{dom} E \right\}.$$

Note that

$$(\widetilde{A} - \lambda)^{-1} \begin{pmatrix} \varphi \\ 0 \end{pmatrix} = \begin{pmatrix} [B(E+\lambda)^{-1}B^* + D - \lambda]^{-1}\varphi \\ (E+\lambda)^{-1}B^*[B(E+\lambda)^{-1}B^* + D - \lambda]^{-1}\varphi \end{pmatrix}$$

and

$$(\widetilde{A} - \lambda)^{-1} \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} (D-\lambda)^{-1}B[B^*(D-\lambda)^{-1}B + E + \lambda]^{-1}\psi \\ -[B^*(D-\lambda)^{-1}B + E + \lambda]^{-1}\psi \end{pmatrix}.$$

Here the Schur complements $B(E+\lambda)^{-1}B^* + D - \lambda$ and $B^*(D-\lambda)^{-1}B + E + \lambda$ are invertible for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$ since $\widetilde{A}$ is selfadjoint. The transform

$$\Gamma = \mathcal{J}\widetilde{A} = \left\{ \left\{ \begin{pmatrix} f \\ Df + Bh \end{pmatrix}, \begin{pmatrix} h \\ Eh - B^*f \end{pmatrix} \right\} : f \in \operatorname{dom} D, \ h \in \operatorname{dom} E \right\}$$

defines a boundary relation in $\mathfrak{H} \oplus \mathfrak{H}$ with

$$T = \operatorname{dom}\Gamma = D \,\widehat{+}\, (\{0\} \oplus B(\operatorname{dom} E)), \quad S = \ker\Gamma = D{\upharpoonright}\ker B^*,$$
$$\operatorname{ran}\Gamma = E \,\widehat{+}\, (\{0\} \oplus B^*(\operatorname{dom} D)), \quad \operatorname{mul}\Gamma = E{\upharpoonright}\ker B,$$

so that $S^* = \operatorname{clos}(D \,\widehat{+}\, (\{0\} \oplus \overline{\operatorname{ran}} B))$. Since $\operatorname{mul}\Gamma \cap \{0, \mathfrak{H}\} = \{0\}$, the corresponding Weyl family $M(\lambda)$ is operator-valued. Moreover,

$$\operatorname{ran}\Gamma_0 = \operatorname{dom} E, \quad \operatorname{ran}\Gamma_1 = B^*(\operatorname{dom} D) + \operatorname{ran} E. \tag{6.4}$$

Observe, that $\{f, \lambda f\} \in T$ if and only if

$$f = -(D-\lambda)^{-1}Bh.$$

Hence,

$$\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \operatorname{dom} E, \quad \Gamma_1(\widehat{\mathfrak{N}}_\lambda(T)) = \operatorname{ran}\left(B^*(D-\lambda)^{-1}B + E\right),$$

and the corresponding γ-field and the Weyl family are given by

$$\gamma(\lambda) = -(D-\lambda)^{-1}B, \quad M(\lambda) = B^*(D-\lambda)^{-1}B + E,$$

on $\operatorname{dom} M(\lambda)$. In particular, $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \operatorname{ran}\Gamma_0$ for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$, while $\Gamma_1(\widehat{\mathfrak{N}}_\lambda(T))$ in general depends on $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and need not coincide with $\operatorname{ran}\Gamma_1$. The formulas

$$A_0 = \ker\Gamma_0 = D, \quad A_1 = \ker\Gamma_1 = \{\,\{f, Df + Bh\} : \, f \in \operatorname{dom} D,\, h \in \operatorname{dom} E,\, B^*f = Eh\,\}$$

show that A_0 is selfadjoint while A_1 need not be essentially selfadjoint in $\mathfrak{H}$.

However, if for instance $E = 0$ then A_1 takes the form

$$A_1 = \{\,\{f, f'\} \in T : \, B^*f = 0\,\} = S \,\widehat{+}\, (\{0\} \oplus \operatorname{ran} B).$$

By Theorem 4.13 the equality $\Gamma_1(\widehat{\mathfrak{N}}_\lambda(T)) = \operatorname{ran}\Gamma_1$ holds for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$ if and only if A_1 is selfadjoint: this holds if, for instance, $E = 0$ and $\operatorname{ran} B$ is finite-dimensional, in which case A_1 is in fact the generalized Friedrichs extension of S. Since A_0 is a selfadjoint extension of S, S has equal defect numbers. Observe, that $A_0 = S$ if and only if $B = 0$, so that when $B \neq 0$ then S is not selfadjoint. On the other hand, $A_1 = S$ if and only if

$$B^*f = Eh \quad \Rightarrow \quad h = 0 \text{ and } B^*f = 0,$$

or equivalently,

$$\ker E = 0 \quad \text{and} \quad \operatorname{ran} B^* \cap \operatorname{ran} E = \{0\}. \tag{6.5}$$

In this case A_1 cannot be essentially selfadjoint (unless $B = 0$) and thus $\Gamma_1(\widehat{\mathfrak{N}}_\lambda(T))$ must depend on $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Example 6.6. Let S_+ and S_- be the minimal differential operator generated in $\mathfrak{H} = L_2(0,\infty)$ and $\mathcal{H} = L_2(-\infty,0)$, respectively by the expression $-iD$. Let $\widetilde{A}$ be the selfadjoint operator in $\mathfrak{H} \oplus \mathcal{H}$ given by

$$\widetilde{A} = \left\{ \left\{ \begin{pmatrix} y_+ \\ y_- \end{pmatrix}, \begin{pmatrix} -iy_+' \\ -iy_-' \end{pmatrix} \right\} : \, y_+ \in W_2^1(0,\infty), \; y_- \in W_2^1(-\infty,0), \; y_+(0+) = y_-(0-) \right\}.$$

Clearly, the operator $\widetilde{A}$ is minimal with respect to the space $\mathcal{H}$ in the sense of (2.27) since the operator S_+ is completely nonselfadjoint, see Lemma 2.14. Then the linear relation $\Gamma = \mathcal{J}\widetilde{A} : \mathfrak{H}^2 \to \mathcal{H}^2$ is a minimal boundary relation for $S_+^* = T$, with

$$\operatorname{dom}\Gamma = S_+^*, \quad \ker\Gamma = S_+, \quad \operatorname{ran}\Gamma = S_-^*, \quad \operatorname{mul}\Gamma = S_-.$$

The symmetric extensions $A_0 = \ker \Gamma_0$ and $A_1 = \ker \Gamma_1$ associated to the boundary relation $\Gamma = \mathcal{J}\widetilde{A} : \mathfrak{H}^2 \to \mathcal{H}^2$ are equal and coincide with the minimal operator S_+, which is a maximal symmetric operator, cf. Proposition 4.11. Furthermore, observe that

$$\operatorname{ran} \Gamma_0 = \operatorname{dom} S_-^*, \quad \operatorname{ran} \Gamma_1 = \operatorname{ran} S_-^*.$$

The defect subspaces $\mathfrak{N}_\lambda(S_+^*)$ are given by

$$\mathfrak{N}_\lambda(S^*) = \operatorname{span}\{e^{i\lambda x}\}, \quad \lambda \in \mathbb{C}_+, \quad \mathfrak{N}_\lambda(S^*) = \{0\}, \quad \lambda \in \mathbb{C}_-,$$

and the corresponding Weyl family $M(\lambda) = \Gamma(\widehat{\mathfrak{N}}_\lambda(S^*))$ has the form

$$M(\lambda) = S_-^*, \quad \lambda \in \mathbb{C}_+, \quad M(\lambda) = S_-, \quad \lambda \in \mathbb{C}_-,$$

where S_- and S_-^* are the minimal and the maximal differential operators generated in $\mathcal{H} = L_2(-\infty, 0)$ by the expression $-iD$. Clearly, $M(\lambda)$ belongs to the class $R(\mathcal{H})$ and has the domain invariance property in each halfplane, but not on $\mathbb{C} \setminus \mathbb{R}$. In fact, each halfplane gives different behaviour. For $\lambda \in \mathbb{C}^+$ the corresponding γ-field is given by

$$\gamma(\lambda)h = h(0)\, e^{i\lambda x}, \quad h \in \operatorname{dom} S_-^*,$$

so that

$$\|\gamma(\lambda)h\|^2 = \frac{|h(0)|^2}{\operatorname{Im} \lambda} = \frac{(S_-^* h, h) - (h, S_-^* h)}{\lambda - \bar{\lambda}}, \quad h \in \operatorname{dom} S_-^*.$$

Clearly (4.11) is not satisfied for $\lambda \in \mathbb{C}^+$ since $S_-^* \not\subset S_-$. In fact, the closure of the γ-field $\gamma(\lambda)$ is not single-valued for $\lambda \in \mathbb{C}^+$. To see this, define a sequence of smooth functions $h_n \in \mathfrak{H}$ such that $h_n(0) = 1$ and $h_n \to 0$ in $\mathfrak{H} = L^2(0, \infty)$, in which case $\gamma(\lambda)h_n = e^{i\lambda x} \neq 0$. For $\lambda \in \mathbb{C}^-$ the corresponding γ-field is given by

$$\gamma(\lambda)h = 0, \quad h \in \operatorname{dom} S_-,$$

and

$$\|\gamma(\lambda)h\|^2 = \frac{(S_- h, h) - (h, S_- h)}{\lambda - \bar{\lambda}}, \quad h \in \operatorname{dom} S_-.$$

Clearly (4.11) is satisfied for $\lambda \in \mathbb{C}^-$ since $S_- \subset S_-^*$. Therefore, the closure of $\gamma(\lambda)$ is single-valued and, in fact, $\gamma(\lambda)h = 0$ for all $h \in \mathcal{H}$.

Example 6.7. Let $A = A^*(\geq I)$ be a semibounded operator in a Hilbert space $\mathcal{H}$ and let S be a minimal selfadjoint operator in $L_2((0, \infty), \mathcal{H})$ associated with the differential expression

$$l[y] = -y'' + A^2 y.$$

It is known that the domain of the maximal operator S^* consists of the vectors

$$y = e^{-\widehat{A}t} f_1 + \frac{1}{2} \int_0^\infty e^{-A|t-s|} A^{-1} h(s) ds,$$

where $f_1 \in \mathfrak{H}_{-1/2}$, $h = Sy \in L^2([0, \infty), \mathcal{H})$, cf. [17]. Here $\widehat{A}$ denotes the continuation of the operator A acting from $\mathcal{H}_\alpha(A)$ to $\mathcal{H}_{\alpha+1}(A)$, where $\{\mathcal{H}_\alpha(A)\}$ is the scale of Hilbert spaces generated by the operator A ($\mathcal{H}_\alpha(A) = \operatorname{dom} A^\alpha$), see [17]. The boundary values $y(0)$ and $y'(0)$ of $y \in \operatorname{dom} S^*$ belong to $\mathcal{H}_{-1/2}(A)$ and $\mathcal{H}_{-3/2}(A)$, respectively. The regularized boundary values $y(0)$ and $y'(0) + \widehat{A}y(0)$ of $y \in \operatorname{dom} S^*$ belong to $\mathcal{H}_{-1/2}(A)$ and $\mathcal{H}_{1/2}(A)$, respectively. Moreover, the mapping $y \mapsto \{y(0), y'(0) + \widehat{A}y(0)\}$ from $\operatorname{dom} S^*$ to $\mathcal{H}_{-1/2}(A)$ and $\mathcal{H}_{1/2}(A)$ is surjective (see [17, Chapter 4]).

Define the operator T as a restriction of S^* to the domain $\operatorname{dom} T = \{y \in \operatorname{dom} S^* : y(0), y'(0) \in \mathcal{H}\}$ and two mappings $\widetilde{\Gamma} : \operatorname{dom} S^* \to \mathcal{H}^2$ and $\widetilde{\Gamma} : \operatorname{dom} T \to \mathcal{H}^2$ by

$$\widetilde{\Gamma} y = \begin{pmatrix} \Gamma_0 y \\ \Gamma_1 y \end{pmatrix} = \begin{pmatrix} \widehat{A}^{-1/2} y(0) \\ A^{1/2}(y'(0) + \widehat{A} y(0)) \end{pmatrix}, \quad \Gamma y = \begin{pmatrix} \Gamma_0 y \\ \Gamma_1 y \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}.$$

It is shown in [17] (see also [12]) that the triplet $\{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is an ordinary boundary triplet for S^* and the following equality

$$(S^* y, z) - (y, S^* z) = (\widetilde{\Gamma}_1 y, \widetilde{\Gamma}_0 z)_{\mathcal{H}} - (\widetilde{\Gamma}_0 y, \widetilde{\Gamma}_1 z)_{\mathcal{H}} \tag{6.6}$$

holds for all $y, z \in \operatorname{dom} S^*$. Since $y(0), y'(0) \in \mathcal{H}$ and $y'(0) + Ay(0) \in \mathcal{H}_{1/2}$ for all $y \in \operatorname{dom} T$ one obtains $y(0) \in \mathcal{H}_1$ for all $y \in \operatorname{dom} T$. Therefore, for all $y, z \in \operatorname{dom} T$ the equality 6.6 can be rewritten as

$$(S^* y, z) - (y, S^* z) = (\Gamma_1 y, \Gamma_0 z)_{\mathcal{H}} - (\Gamma_0 y, \Gamma_1 z)_{\mathcal{H}} \tag{6.7}$$

To show that $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for S^* let us suppose that for some $z, \widetilde{z} \in \mathcal{H}$, $h, \widetilde{h} \in \mathcal{H}$ the following equality

$$(S^* y, z) - (y, \widetilde{z}) = (\Gamma_1 y, h)_{\mathcal{H}} - (\Gamma_0 y, \widetilde{h})_{\mathcal{H}} \tag{6.8}$$

holds for all $y \in \operatorname{dom} S^*$. With the choice $y \in \operatorname{dom} S$ it follows that $z \in \operatorname{dom} S^*$ and $\widetilde{z} = S^* z$. Next, the choice $y \in \ker \Gamma_0$ give

$$(\Gamma_1 y, h)_{\mathcal{H}} = (\Gamma_1 y, \Gamma_0 z)_{\mathcal{H}}$$

for all $y \in \operatorname{dom} S^*$. Since $\Gamma_1 \operatorname{dom} A_0 = \mathcal{H}_{1/2}$ is dense in $\mathcal{H}$ one obtains from (6.7), (6.8) $h = \Gamma_0 z$. Similarly, the equality $\widetilde{h} = \Gamma_1 z$ is derived from (6.7), (6.8) by choosing $y = \ker \Gamma_1$, since $\Gamma_0 \operatorname{dom} A_1 = \mathcal{H}_{3/2}$.

The defect subspace $\mathfrak{N}_\lambda(T)$ consists of the vectors $f_\lambda := e^{-\sqrt{A^2 - \lambda} t} f$, $f \in \mathcal{H}_1$. Since $\Gamma_0 f_\lambda = f$ the γ-field and the Weyl function $M(\lambda)$ of the boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ are given by

$$\gamma(\lambda) f = f_\lambda, \quad M(\lambda) f = -\sqrt{A^2 - \lambda} f, \quad f \in \mathcal{H}_1.$$

Clearly, $M \in R_{inv}(\mathcal{H}) \setminus R_{[\mathcal{H}]}$, since the function $-\sqrt{s^2 - \lambda}$ is unbounded on $[1, \infty)$ for every $\lambda \in \mathbb{C} \setminus [1, \infty)$ and $\operatorname{dom} M(\lambda) = \mathcal{H}_1$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Appendix A. Nevanlinna pairs

In abstract eigenvalue depending boundary value problems Nevanlinna family is often represented via its counterpart - Nevanlinna pair, see e.g. [13], [8], [9]. In this Appendix connections between Nevanlinna families and Nevanlinna pairs are investigated in the general Hilbert space setting.

Every closed linear relation T in a separable Hilbert space $\mathcal{H}$ can be represented as

$$T = \{ \{\Phi h, \Psi h\} : h \in \mathcal{L} \}, \tag{A.1}$$

where $\mathcal{L}$ is a parameter Hilbert space and the operators Φ, Ψ belong to $[\mathcal{L}, \mathcal{H}]$. To show this it is enough to take T as $\mathcal{L}$ and the projections π_1, π_2 onto the first and the second components of $T \subset \mathcal{H} \oplus \mathcal{H}$ as Φ and Ψ. Clearly, each pair $\{\Phi, \Psi\}$ of operators in $[\mathcal{L}, \mathcal{H}]$ gives rise to a linear relation T in $\mathcal{H}$ via (A.1). In the infinite-dimensional case ($\dim \mathcal{H} = \infty$)

the parameter Hilbert space $\mathcal{L}$ can be taken to be equal to $\mathcal{H}$. Note that when $\rho(T)$ is not empty and $\lambda_0 \in \rho(T)$ then

$$T = \{ \{(T-\lambda_0)^{-1}h, (I+\lambda_0(T-\lambda_0)^{-1})h\} : h \in \mathcal{H} \},$$

so that $\mathcal{L} = \mathcal{H}$ and there is a natural choice for the pair $\{\Phi, \Psi\}$ in $[\mathcal{H}]$. For linear relations given by the equation (A.1) the properties stated in Proposition 2.9 can be characterized in terms of the pair $\{\Phi, \Psi\}$.

Proposition A.1 ([9]). *Let T be a linear relation T in $\mathcal{H}$, defined by* (A.1). *Then:*

(i) *the adjoint T^* is a linear relation given by*

$$T^* = \{ \{h, h'\} \in \mathcal{H}^2 : \Psi^* h - \Phi^* h' = 0 \}. \tag{A.2}$$

(ii) *T is a dissipative (accumulative) relation if and only if*

$$-i(\Phi^*\Psi - \Psi^*\Phi) \geq 0, \quad (\leq 0); \tag{A.3}$$

(iii) *T is symmetric if and only if*

$$\Phi^*\Psi - \Psi^*\Phi = 0; \tag{A.4}$$

If, additionally, $\ker\Phi \cap \ker\Psi = \{0\}$, *than*

(iv) *$\lambda \in \rho(T)$ if and only if the operator $\Psi - \lambda\Phi$ has a bounded inverse;*

(v) *T is maximal dissipative (accumulative) if and only if* (A.3) *holds and the operator $\Psi + i\Phi$ ($\Psi - i\Phi$) has a bounded inverse;*

(vi) *T is selfadjoint if and only if* (A.4) *holds and the operators $\Psi \pm i\Phi$ have bounded inverses.*

Proof. (i) For $\{g, g'\} \in T^*$ and arbitrary $h \in \mathcal{H}$ one has the equality

$$0 = (g', \Phi h) - (g, \Psi h) = (\Phi^* g' - \Psi^* g, h),$$

which implies $\Psi^* g - \Phi^* g' = 0$.

(ii), (iii) If T is symmetric then for $\{\Phi h, \Psi h\} \in T$ one obtains

$$0 = -i[(\Psi h, \Phi h) - (\Phi h, \Psi h)] = -i((\Phi^*\Psi - \Psi^*\Phi)h, h), \quad \forall h \in \mathcal{H},$$

and conversely. Similarly, one obtains the conditions (A.3) for dissipative and accumulative linear relations.

(iv) It follows from(A.1) that

$$T - \lambda = \{ \{\Phi h, (\Psi - \lambda\Phi)h\} : h \in \mathcal{L} \}, \tag{A.5}$$

Assume that $\lambda \in \rho(T)$ and $(\Psi - \lambda\Phi)h = 0$. Then $\Psi h = \Phi h = 0$ and, hence, $h = 0$ (by the assumption $\ker\Phi \cap \ker\Psi = \{0\}$). Since $\operatorname{ran}(\Psi - \lambda\Phi) = \operatorname{ran}(T - \lambda) = \mathcal{H}$, it follows $0 \in \rho(\Psi - \lambda\Phi)$. Similarly, if $0 \in \rho(\Psi - \lambda\Phi)$ one obtains from (A.5) that $\lambda \in \rho(T)$ and $(T-\lambda)^{-1} = \Phi(\Psi - \lambda\Phi)^{-1}$.

(v), (vi) are immediate from (ii), (iii) and (iv). □

Let now a family of linear relations $\tau(\lambda)$ is represented in the form

$$\tau(\lambda) = \{\Phi(\lambda), \Psi(\lambda)\} := \{ \{\Phi(\lambda)h, \Psi(\lambda)h\} : h \in \mathcal{H} \} \tag{A.6}$$

where $\Phi(\lambda)$, $\Psi(\lambda)$ is a pair of holomorphic operator functions on $\mathbb{C}_+ \cup \mathbb{C}_-$. In the case when $\tau(\lambda)$ is a Nevanlinna family the corresponding pair $\{\Phi(\lambda), \Psi(\lambda)\}$ in the representation (A.6) is called the Nevanlinna pair.

Definition A.2. A pair $\{\Phi, \Psi\}$ of $[\mathcal{H}]$-valued functions $\Phi(\lambda)$, $\Psi(\lambda)$ holomorphic on $\mathbb{C}_+ \cup \mathbb{C}_-$ is said to be a Nevanlinna pair if:

(NP1) $\operatorname{Im} \Phi(\lambda)^* \Psi(\lambda)/\operatorname{Im} \lambda \geq 0$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$;
(NP2) $\Psi(\bar{\lambda})^* \Phi(\lambda) - \Phi(\bar{\lambda})^* \Psi(\lambda) = 0$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$;
(NP3) $0 \in \rho(\Psi(\lambda) \pm i\Phi(\lambda))$, $\lambda \in \mathbb{C}_\pm$.

Two Nevanlinna pairs $\{\Phi, \Psi\}$ and $\{\Phi_1, \Psi_1\}$ are said to be equivalent, if $\Phi_1(\lambda) = \Phi(\lambda)\chi(\lambda)$ and $\Psi_1(\lambda) = \Psi(\lambda)\chi(\lambda)$ for some operator function $\chi(\lambda) \in [\mathcal{H}]$, which is holomorphic and invertible on $\mathbb{C}_+ \cup \mathbb{C}_-$, cf. (iii) in Proposition A.5.

As follows from Proposition A.1 the property (NF1) for the family $\tau(\lambda)$ corresponding to the Nevanlinna pair $\{\Phi, \Psi\}$ is equivalent to the property (NP1) for the pair $\{\Phi, \Psi\}$. Moreover, the formula (A.6) establishes a one-to-one correspondence $\{\Phi, \Psi\} \mapsto M$ between the equivalence classes of Nevanlinna pairs and Nevanlinna families $\tau(\lambda) \in \widetilde{R}_{\mathcal{H}}$.

Proposition A.3 ([9])**.** *Let $\{\Phi, \Psi\}$ be a Nevanlinna pair of $[\mathcal{H}]$-valued functions on $\mathbb{C}_+ \cup \mathbb{C}_-$, and let $\tau(\lambda)$ be defined by* (A.6). *Then $\tau(\cdot)$ is a Nevanlinna family. Conversely, if $\tau(\cdot) \in \widetilde{R}(\mathcal{H})$ then there exists a Nevanlinna pair $\{\Phi, \Psi\}$ of $[\mathcal{H}]$-valued functions on $\mathbb{C}_+ \cup \mathbb{C}_-$, such that* (A.6) *holds.*

Every two Nevanlinna pairs $\{\Phi, \Psi\}$, $\{\Phi_1, \Psi_1\}$ satisfying (A.6) *are equivalent.*

Proof. Let $\{\Phi, \Psi\}$ be a Nevanlinna pair. Then it follows from (NP1), (NP3) and Proposition A.1 that the linear relation $\tau(\lambda)$ is maximal dissipative (maximal accumulative) for all $\lambda \in \mathbb{C}_+$ ($\lambda \in \mathbb{C}_-$). The assumption (NP2) concerning $\{\Phi, \Psi\}$ means that $\tau(\bar{\lambda}) \subset \tau(\lambda)^*$. Now it follows from the (A.2) that

$$\tau(\lambda)^* = \{\, \{h, h'\} \in \mathcal{H}^2 : \Psi(\lambda)^* h - \Phi(\lambda)^* h' = 0 \,\}. \tag{A.7}$$

and, hence

$$\tau(\lambda)^* - i = \{\{h, g\} : (\Psi(\lambda)^* - i\Phi(\lambda)^*)h = \Phi(\lambda)^* g\}$$

Using the hypothesis (NP3) one obtains $\ker(\tau(\lambda)^* - i) = \{0\}$, $\operatorname{ran}(\tau(\lambda)^* - i) = \mathcal{H}$ for $\lambda \in \mathbb{C}_+$. Since $\tau(\bar{\lambda})$ satisfies the same relations and $\tau(\bar{\lambda}) \subset \tau(\lambda)^*$ this implies $\tau(\bar{\lambda}) = \tau(\lambda)^*$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Conversely, assume that $\tau(\cdot) \in \widetilde{R}(\mathcal{H})$. Define $\Phi(\lambda)$ and $\Psi(\lambda)$ by

$$\Phi(\lambda) = (\tau(\lambda) \pm i)^{-1}, \quad \Psi(\lambda) = I \mp i(\tau(\lambda) \pm i)^{-1}, \quad \lambda \in \mathbb{C}_\pm. \tag{A.8}$$

Then $\tau(\lambda)$ has the representation (A.6). The property (NF3) implies that $\Phi(\cdot)$, $\Psi(\cdot)$ are holomorphic on $\mathbb{C}_+ \cup \mathbb{C}_-$ with the values in $[\mathcal{H}]$. Clearly, $\Psi(\lambda) \pm i\Phi(\lambda) = I$ and hence (NP3) holds. Moreover, the symmetry condition (NP2) is obvious and the positivity condition (NP1) follows from (NF1) in view of

$$\frac{((\Phi(\lambda)^*\Psi(\lambda) - \Psi(\lambda)^*\Phi(\lambda))h, h)}{\operatorname{Im} \lambda} = \frac{\operatorname{Im}(\Psi(\lambda)h, \Phi(\lambda)h)}{\operatorname{Im} \lambda} \geq 0.$$

□

Proposition A.4. *Let $\{\Phi, \Psi\}$ be a Nevanlinna pair. Then the following kernel is nonnegative on $\mathbb{C}_+ \cup \mathbb{C}_-$:*

$$\mathsf{N}_{\Phi,\Psi}(\lambda, \mu) = \frac{\Phi(\mu)^*\Psi(\lambda) - \Psi(\mu)^*\Phi(\lambda)}{\lambda - \bar{\mu}}, \quad \lambda, \mu \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{A.9}$$

Proof. Let $\tau(\cdot)$ be a family of linear relations associated to the Nevanlinna pair $\{\Phi,\Psi\}$. The Cayley transform $\mathcal{C}(\lambda)$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, of $\tau(\lambda)$ is given by

$$\mathcal{C}(\lambda)=(\Psi(\lambda)-i\Phi(\lambda))(\Psi(\lambda)+i\Phi(\lambda))^{-1}. \tag{A.10}$$

It is a holomorphic on $\mathbb{C}_+\cup\mathbb{C}_-$ and contractive for $\lambda\in\mathbb{C}_+$. It follows from the equality

$$\frac{I-\mathcal{C}(\mu)^*\mathcal{C}(\lambda)}{-i(\lambda-\bar{\mu})}=2(\Psi(\mu)+i\Phi(\mu))^{-*}\mathsf{N}_{\Phi,\Psi}(\lambda,\mu)(\Psi(\lambda)+i\Phi(\lambda))^{-1}$$

that the kernel $\mathsf{N}_{\Phi,\Psi}(\lambda,\mu)$ is nonnegative on $\mathbb{C}_+$, cf. [29]. Moreover, the nonnegativity of the kernel $\mathsf{N}_{\Phi,\Psi}(\lambda,\mu)$ on $\mathbb{C}_+\cup\mathbb{C}_-$ is implied by Ginzburg inequality (see [5, Section 1.6]). □

As follows from Proposition A.5 the conditions (NP3) and (NF3) can be replaced, for instance, by

$$0\in\rho(\Psi(\lambda)+\mu\Phi(\lambda))\quad\text{and}\quad 0\in\rho(\tau(\lambda)+\mu I),$$

respectively, for some (equivalently for every) $\mu\in\mathbb{C}_\pm$ and for all λ in the same halfplane as μ. Moreover, the following more general statement holds.

Proposition A.5. *Let $\{\Phi(\cdot),\Psi(\cdot)\}$ be a Nevanlinna pair, let W be a unitary operator in the Kreĭn space $(\mathcal{H}^2,J_{\mathcal{H}})$. and let $\begin{pmatrix}\widetilde{\Phi}(\lambda)\\ \widetilde{\Psi}(\lambda)\end{pmatrix}=W\begin{pmatrix}\Phi(\lambda)\\ \Psi(\lambda)\end{pmatrix}$. Then $\{\widetilde{\Phi}(\cdot),\widetilde{\Psi}(\cdot)\}$ is also a Nevanlinna pair. In particular, if $X=X^*\in[\mathcal{H}]$, Y is an invertible operator in $[\mathcal{H}]$ and $M(\cdot)\in R^u[\mathcal{H}]$, each of the following pairs is also a Nevanlinna pair:*

(A.11)

$$\{\Phi(\cdot),\Psi(\cdot)+X\Phi(\cdot)\},\quad \{Y^{-1}\Phi(\cdot),Y^*\Psi(\cdot)\},\quad \{-\Psi(\cdot),\Phi(\cdot)\},\quad \{\Phi(\cdot),\Psi(\cdot)+M(\cdot)\Phi(\cdot)\}.$$

Proof. Consider $\tau(\lambda)$ and $\widetilde{\tau}(\lambda)$ as the ranges of the block operators $T(\lambda)=\begin{pmatrix}\Phi(\lambda)\\ \Psi(\lambda)\end{pmatrix}$ and $\widetilde{T}(\lambda)=\begin{pmatrix}\widetilde{\Phi}(\lambda)\\ \widetilde{\Psi}(\lambda)\end{pmatrix}$. Then the kernel $\mathsf{N}_{\Phi,\Psi}(\lambda,\mu)$ can be represented as follows:

$$\mathsf{N}_{\Phi,\Psi}(\lambda,\mu)=\frac{T(\mu)^*J_{\mathcal{H}}T(\lambda)}{-i(\lambda-\bar{\mu})}. \tag{A.12}$$

The properties (NP1), (NP2) for $\{\widetilde{\Phi}(\cdot),\widetilde{\Psi}(\cdot)\}$ are implied by the equalities

$$\mathsf{N}_{\widetilde{\Phi},\widetilde{\Psi}}(\lambda,\mu)=\frac{\widetilde{T}(\mu)^*J_{\mathcal{H}}\widetilde{T}(\lambda)}{-i(\lambda-\bar{\mu})}=\frac{T(\mu)^*J_{\mathcal{H}}T(\lambda)}{-i(\lambda-\bar{\mu})}=\mathsf{N}_{\Phi,\Psi}(\lambda,\mu). \tag{A.13}$$

Due to Proposition 2.9 $\tau(\lambda)$ is a maximal nonnegative subspace of the Kreĭn space $(\mathcal{H}^2,J_{\mathcal{H}})$ for $\lambda\in\mathbb{C}_+$. Since $\widetilde{\tau}(\lambda)$ is the range of $\widetilde{T}(\lambda)$ it has the same property and, therefore, $\widetilde{\tau}(\cdot)\in\widetilde{R}_{\mathcal{H}}$. By Proposition (A.3) $\{\widetilde{\Phi},\widetilde{\Psi}\}$ is a Nevanlinna pair.

Applying this statement to the pair $\{\Phi,\Psi\}$ and the matrices

$$W=\begin{pmatrix}I&0\\X&I\end{pmatrix},\quad W=\begin{pmatrix}Y^{-1}&0\\0&Y^*\end{pmatrix},\quad W=\begin{pmatrix}0&-I\\I&0\end{pmatrix}$$

one shows that the first three pair in (A.11) are Nevanlinna pairs. The properties (NP1), (NP2) for the pair $\{\widetilde{\Phi}(\cdot),\widetilde{\Psi}(\cdot)\}=\{\Phi(\cdot),\Psi(\cdot)+M(\cdot)\Phi(\cdot)\}$ are implied by the identity

$$\mathsf{N}_{\widetilde{\Phi},\widetilde{\Psi}}(\lambda,\mu)=\mathsf{N}_{\widetilde{\Phi},\widetilde{\Psi}}(\lambda,\mu)+\Phi(\mu)\frac{M(\lambda)-M(\mu)^*}{\lambda-\bar{\mu}}\Phi(\lambda).$$

To show that the operator $\widetilde{\Psi}(\lambda)+i\widetilde{\Phi}(\lambda)$ is invertible for some $\lambda\in\mathbb{C}_+$ let us set $X=\mathrm{Re}\,M(\lambda)$, $Y=\mathrm{Im}\,M(\lambda)$ and apply the previous statement to the pairs:

$$\{\Phi_1(\lambda),\Psi_1(\lambda)\}=\{(Y+I)^{1/2}\Phi(\lambda),(Y+I)^{-1/2}\Psi(\lambda)\},$$
$$\{\Phi_2(\lambda),\Psi_2(\lambda)\}=\{\Phi_1(\lambda),\Psi_1(\lambda)+(Y+I)^{-1/2}X(Y+I)^{-1/2}\Phi_1(\lambda)\}.$$

Since these pairs are maximal dissipative it follows that the operator

$$\widetilde{\Psi}(\lambda)+i\widetilde{\Phi}(\lambda)=(Y+I)^{1/2}(\Psi_2(\lambda)+i\Phi_2(\lambda))$$

is also invertible. □

Proposition A.6. *Let $\{\Phi(\cdot),\Psi(\cdot)\}$ be a Nevanlinna pair and let $\tau(\cdot)\in\widetilde{R}_{\mathcal{H}}$ be the corresponding Nevanlinna family, $\lambda_0\in\mathbb{C}_+$. Then the following statements hold:*

(i) *if $0\in\sigma_p(\mathsf{N}_{\Phi,\Psi}(\lambda_0,\lambda_0))$ then $0\in\sigma_p(\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda))$ for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$;*
(ii) *if $0\in\rho(\mathsf{N}_{\Phi,\Psi}(\lambda_0,\lambda_0))$ then $0\in\rho(\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda))$ for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$;*
(iii) *$\tau(\lambda)\cap\tau(\bar{\lambda})$ does not depend on $\lambda\in\mathbb{C}\setminus\mathbb{R}$;*
(iv) *if $\alpha=\bar{\alpha}\in\sigma_p(\tau(\lambda_0))$ then $\alpha=\bar{\alpha}\in\sigma_p(\tau(\lambda))$ for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$;*
(v) *if $\alpha=\bar{\alpha}\in\sigma_c(\tau(\lambda_0))$ then $\alpha=\bar{\alpha}\in\sigma_c(\tau(\lambda))$ for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$;*
(vi) *if $\alpha=\bar{\alpha}\in\rho(\tau(\lambda_0))$ then $\alpha=\bar{\alpha}\in\rho(\tau(\lambda))$ for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$;*
(vii) $\mathrm{mul}\,(\tau(\lambda))$ *does not depend on $\lambda\in\mathbb{C}\setminus\mathbb{R}$.*
(viii) *if $\tau(\lambda_0)\in[\mathcal{H}]$ then $\tau(\lambda)\in[\mathcal{H}]$ for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$;*

Proof. (i) Due to Proposition 2.9 $\tau(\lambda_0)$ is a nonnegative subspace of the Kreĭn space $(\mathcal{H}^2,J_{\mathcal{H}})$. Let $0\in\sigma_p(\mathsf{N}_{\Phi,\Psi}(\lambda_0,\lambda_0))$ and denote $\mathcal{H}_0=\ker\mathsf{N}_{\Phi,\Psi}(\lambda_0,\lambda_0)$, $\lambda_0\in\mathbb{C}_+$. In view of (A.12), $T(\lambda_0)\mathcal{H}_0$ is equal to the isotropic part $\tau(\lambda_0)\cap\tau(\lambda_0)^*$ of $\tau(\lambda_0)$ in $(\mathcal{H}^2,J_{\mathcal{H}})$. By Proposition A.4 the matrix

$$\begin{pmatrix}(\mathsf{N}_{\Phi,\Psi}(\lambda_0,\lambda_0)h_0,h_0) & (\mathsf{N}_{\Phi,\Psi}(\lambda_0,\lambda)h_0,h)\\ (\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda_0)h,h_0) & (\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda)h,h)\end{pmatrix},$$

is nonnegative, which implies that $(\mathsf{N}_{\Phi,\Psi}(\lambda_0,\lambda)h_0,h)=0$ for all $h\in\mathcal{H}$, $h_0\in\mathcal{H}_0$, and $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Now (A.12) shows that

$$(JT(\lambda_0)\mathcal{H}_0,T(\lambda)h)=0$$

for all $h\in\mathcal{H}$ and $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Therefore, the subspace $T(\lambda_0)\mathcal{H}_0$ is orthogonal to the maximal nonnegative subspace $\tau(\lambda)$ in $(\mathcal{H}^2,J_{\mathcal{H}})$ and consequently $T(\lambda_0)\mathcal{H}_0$ is contained in the isotropic part of $\tau(\lambda)$. This shows that $0\in\sigma_p(\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda))$.

(ii) Let now $0\in\rho(\mathsf{N}_{\Phi,\Psi}(\lambda_0,\lambda_0))$. Then it follows from (i) that $\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda)$ has a trivial kernel for every $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Assume that $0\in\sigma_c(\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda))$ and let $\{h_n\}_{n=1}^\infty$ be a sequence of $\mathcal{H}$ such that $\|h_n\|=1$ and $\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda)h_n\to 0$ as $n\to\infty$. Since $T(\lambda_0)\mathcal{H}$ is a maximal positive subspace of the Kreĭn space $(\mathcal{H}^2,J_{\mathcal{H}})$ its orthogonal complement $(T(\lambda_0)\mathcal{H})^\perp$ is a negative subspace in $(\mathcal{H}^2,J_{\mathcal{H}})$. Decompose $T(\lambda)h_n$ as follows

$$T(\lambda)h_n=T(\lambda_0)h_n'+g_n,\quad (h_n'\in\mathcal{H},g_n\in(T(\lambda_0)\mathcal{H})^\perp).\tag{A.14}$$

Since the matrix

$$\begin{pmatrix}(\mathsf{N}_{\Phi,\Psi}(\lambda_0,\lambda_0)h_n',h_n') & (\mathsf{N}_{\Phi,\Psi}(\lambda_0,\lambda)h_n',h_n)\\ (\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda_0)h_n,h_n') & (\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda)h_n,h_n)\end{pmatrix}$$

is nonnegative and $(\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda)h_n,h_n)\to 0$ one obtains

$$(\mathsf{N}_{\Phi,\Psi}(\lambda_0,\lambda)h_n',h_n)=(J_{\mathcal{H}}T(\lambda_0)h_n',T(\lambda)h_n)=(J_{\mathcal{H}}T(\lambda_0)h_n',T(\lambda_0)h_n')\to 0\quad(n\to\infty)$$

and, therefore, $T(\lambda_0)h_n' \to 0$ as $n \to \infty$. Next, it follows from

$$(\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda)h_n, h_n) = (J_{\mathcal{H}}T(\lambda_0)h_n', T(\lambda_0)h_n') + (J_{\mathcal{H}}g_n, g_n)$$

that $(J_{\mathcal{H}}g_n, g_n) \to 0$. Since $(T(\lambda_0)\mathcal{H})^{\perp}$ is a negative subspace this implies $g_n \to 0$ and, hence, $T(\lambda)h_n \to 0$. This contradicts to (NP3) and, therefore, $0 \in \rho(\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda))$.

(iii) The arguments in (i) show that $\tau(\lambda_0) \cap \tau(\lambda_0)^* \subset \tau(\lambda) \cap \tau(\lambda)^*$. The roles of λ and λ_0 can be interchanged and, hence the equality in (iii) follows.

(iv) Let $\alpha = \bar{\alpha} \in \sigma_p(\tau(\lambda_0))$ and $\Psi(\lambda_0)h_0 = \alpha\Phi(\lambda_0)h_0$ for some $h_0 \neq 0$. This implies that $(\mathsf{N}_{\Phi,\Psi}(\lambda_0,\lambda_0)h_0, h_0) = 0$ and the vector $T(\lambda_0)h_0$ is neutral and belongs to the isotropic part $\tau(\lambda_0) \cap \tau(\lambda_0)^*$ of $\tau(\lambda_0)$. Now, it follows from (iii) that $\{\Phi(\lambda)h_0, \alpha\Phi(\lambda_0)h_0\} \in \tau(\lambda)$, and hence $\alpha \in \sigma_p(\tau(\lambda))$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

The proofs of (v), (vi), (vii), (viii) are analogous, but the further details will be omitted here. □

Proposition A.7. *Let $\{\Phi, \Psi\}$ be a Nevanlinna pair and let $\tau(\cdot) \in \widetilde{R}_{\mathcal{H}}$ be the corresponding Nevanlinna family, $\lambda_0 \in \mathbb{C}_+$. Then:*

(i) *$\tau(\cdot) \in R^s_{\mathcal{H}}$ if and only if $0 \notin \sigma_p(\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda))$;*
(ii) *$\tau(\cdot) \in R^u_{\mathcal{H}}$ if and only if $0 \in \rho(\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda))$.*

Proof. (i) Let $h \in \ker \mathsf{N}_{\Phi,\Psi}(\lambda,\lambda)$, that is $(\Phi(\lambda)^*\Psi(\lambda) - \Psi(\lambda)^*\Phi(\lambda))h = 0$. Than it follows from (A.7) that $\{\Phi(\lambda)h, \Psi(\lambda)h\} \in \tau(\lambda) \cap \tau(\lambda)^*$ and therefore $h = 0$ if and only if $\tau(\lambda) \cap \tau(\lambda)^* = \{0\}$.

(ii) Let f, $f' \in \mathcal{H}$ and let h, g satisfy the equations

$$\Phi(\lambda)h + \Phi(\bar{\lambda})g = f, \quad \Psi(\lambda)h + \Psi(\bar{\lambda})g = f', \tag{A.15}$$

Then it follows from (NP2) that

$$\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda)h = \Psi(\lambda)^*f - \Phi(\lambda)^*f', \quad \mathsf{N}_{\Phi,\Psi}(\bar{\lambda},\bar{\lambda})g = \Psi(\bar{\lambda})^*f - \Phi(\bar{\lambda})^*f'. \tag{A.16}$$

Assume that $0 \in \rho(\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda))$. Then it follows from (A.16) and Proposition A.6 that the system (A.15) has a unique solution for all f, $f' \in \mathcal{H}$ and therefore $\tau(\lambda) \widehat{+} \tau(\lambda)^* = \mathcal{H}^2$.

Conversely, let $\tau(\cdot) \in R^u_{\mathcal{H}}$ and thus the system (A.15) has a unique solution for all f, $f' \in \mathcal{H}$. Then it follows from the first equation in (A.16) and the hypothesis (NP3) that $\operatorname{ran} \mathsf{N}_{\Phi,\Psi}(\lambda,\lambda) = \mathcal{H}^2$. This implies that $0 \in \rho(\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda))$. □

Proposition A.8. *Let $\tau(\lambda) \in R^u_{\mathcal{H}}$. Then $\tau(\lambda) \in [\mathcal{H}]$ and $M^{-1}(\lambda) \in [\mathcal{H}]$ for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In particular, the following equality holds $R^u_{\mathcal{H}} = R^u_{[\mathcal{H}]}$.*

Proof. Let $\{\Phi, \Psi\}$ be a Nevanlinna pair associated to M. It is enough to prove that $\Phi(\lambda)$ and $\Psi(\lambda)$ are invertible. Now assume, for instance, that $\Phi(\lambda)h_n \to 0$ for some sequence $h_n \in \mathcal{H}$, $\|h_n\| = 1$. This together with $0 \in \rho(\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda))$ shows that for some $\alpha > 0$ one has

$$\alpha \le (\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda)h_n, h_n)_{\mathcal{H}} \to 0,$$

a contradiction. Since $\operatorname{ran} \Phi(\lambda) = \operatorname{dom} \tau(\lambda)$ ($\operatorname{ran} \Psi(\lambda) = \operatorname{ran} \tau(\lambda)$) is dense in $\mathcal{H}$, one concludes that $\Phi(\lambda)$ must be invertible. A similar argument shows that $\Psi(\lambda)$ is invertible. □

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