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Singular perturbations as range perturbations in a Pontryagin space

Vladimir Derkach, Seppo Hassi, and Henk de Snoo

Abstract. When the singular finite rank perturbations of an unbounded selfadjoint operator A_0 in a Hilbert space $\mathfrak{H}_0$, formally defined by $A_{(\alpha)} = A_0 + G\alpha G^*$, are lifted to an exit Pontryagin space $\mathfrak{H}$ by means of an operator model, they become ordinary range perturbations of a selfadjoint operator H_∞ in $\mathfrak{H} \supset \mathfrak{H}_0$: $H_\tau = H_\infty - \Omega\tau^{-1}\Omega^*$. Here G is a mapping from $\mathbb{C}^d$ into some scale space $\mathfrak{H}_{-k}(A_0)$, $k \in \mathbb{N}$, of generalized elements associated with A_0, while Ω is a mapping from $\mathbb{C}^d$ into the extended space $\mathfrak{H}$, where H_τ is defined. The connection between these two perturbation formulas is studied.

1. Introduction

Let A_0 be an unbounded selfadjoint operator in a Hilbert space $\mathfrak{H}_0$ and let $\mathfrak{H}_{-k}(A_0)$, $k \in \mathbb{N}$, be the dual space of generalized elements corresponding to the space $\mathfrak{H}_{+k}(A_0) = \operatorname{dom}|A_0|^{k/2}$ equipped with the graph norm, cf. [5]. Singular finite rank perturbations of an unbounded selfadjoint operator A_0 in a Hilbert space $\mathfrak{H}_0$ are defined formally as

$$A_{(\alpha)} = A_0 + G\alpha G^*, \tag{1.1}$$

where G is an injective linear mapping from $\mathcal{H} = \mathbb{C}^d$ into $\mathfrak{H}_{-k}(A_0)$ and α is a selfadjoint operator in $\mathcal{H}$. In [13], [21] an operator model for the singular perturbations (1.1) was constructed by extending the space $\mathfrak{H}_0$ with a finite-dimensional exit space $\mathfrak{H}_Q$; see also [2] for the case of $\mathfrak{H}_{-2}$-perturbations and for further references about in this topic. The model given in [7], [9] uses a coupling method for identifying the singular perturbations $A_{(\alpha)}$ with the selfadjoint extensions H_τ of a symmetric operator in $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_Q$. It turns out that the extensions H_τ are in fact ordinary range perturbations of one of the extensions, namely of the selfadjoint operator H_∞ in $\mathfrak{H} \supset \mathfrak{H}_0$:

$$H_\tau = H_\infty - \Omega\tau^{-1}\Omega^*, \tag{1.2}$$

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where Ω is a mapping from $\mathcal{H}$ into $\mathfrak{H}$ and τ is a selfadjoint parameter in $\mathcal{H}$. The perturbations H_τ in (1.2) induce a symmetric restriction S of H_∞ in $\mathfrak{H}$ via

$$\operatorname{dom} S = \{\, F \in \operatorname{dom} H_\infty : \, \Omega^* F = 0 \,\},$$

which, due to the assumption $\operatorname{ran}\Omega \subset \mathfrak{H}$, is maximally nondensely defined in $\mathfrak{H}$. Therefore, among the selfadjoint extensions of S there are linear relations which are not operators. In particular, the generalized Friedrichs extension (see [15], [16]) of S is not an operator. A classification of the perturbations H_τ by decomposing the selfadjoint parameter τ into its operator and multivalued parts leads to intermediate symmetric extensions of S and their generalized Friedrichs extensions. These extensions of S turn out to be precisely those which are given by the so-called extremal boundary conditions and whose compressed resolvents to the original space $\mathfrak{H}_0$ are canonical, i.e., coincide with a resolvent of a selfadjoint relation in $\mathfrak{H}_0$.

The contents of this paper are now briefly described. Section 2 contains the necessary facts concerning boundary triplets and Weyl functions in a Pontryagin space. A concise introduction to finite rank singular perturbations of a selfadjoint operator in a Hilbert space is given in Section 3. Such finite rank singular perturbations are identified with selfadjoint relations in a larger Pontryagin space. They are interpreted as range perturbations in Section 4. A connection with so-called extremal boundary conditions can be found in Section 5.

2. Boundary triplets and abstract Weyl functions

Let $\mathfrak{H}$ be a Pontryagin space with negative index κ, cf. [4]. The set of all bounded everywhere defined linear operators acting on $\mathfrak{H}$ is denoted by $[\mathfrak{H}]$. If T is a linear relation in $\mathfrak{H}$, then $\operatorname{dom} T$, $\ker T$, $\operatorname{ran} T$, and $\operatorname{mul} T$ indicate the domain, kernel, range, and multivalued part of T, respectively; moreover, $\rho(T)$ denotes the set of regular points of the linear relation T. Let S be a not necessarily densely defined closed symmetric relation in $\mathfrak{H}$ with equal defect numbers $d_+(S) = d_-(S) < \infty$ and let S^* be the adjoint linear relation of S, so that $S \subset S^*$. Recall (see [14], [6]) that a triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of a Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} = n_\pm(S)$ and two linear mappings Γ_j, $j = 0, 1$, from S^* to $\mathcal{H}$ is called a *boundary triplet* for S^*, if the mapping $\Gamma = (\Gamma_0, \Gamma_1)^\top : \widehat{f} \to (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f})^\top$ from S^* into $\mathcal{H} \oplus \mathcal{H}$ is surjective and the following abstract Green's identity holds for every $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in S^*$:

$$(f', g) - (f, g') = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g})_{\mathcal{H}} - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g})_{\mathcal{H}} = i(\Gamma \widehat{g})^* J (\Gamma \widehat{f});$$

here J stands for the block operator

$$J = \begin{pmatrix} 0 & -iI_{\mathcal{H}} \\ iI_{\mathcal{H}} & 0 \end{pmatrix}.$$

The adjoint S^* of every closed symmetric relation S with equal defect numbers has a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$. All other boundary triplets $\widetilde{\Pi} = \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ are related to Π via a J-unitary transformation W: $\widetilde{\Gamma} = W\Gamma$. In particular, the *transposed boundary triplet* $\Pi^\top = \{\mathcal{H}, \Gamma_0^\top, \Gamma_1^\top\}$, is defined by $\Gamma^\top = iJ\Gamma$. When S

is densely defined, S^* can be identified with its domain $\operatorname{dom} S^*$ and the boundary mappings can be interpreted as mappings from $\operatorname{dom} S^*$ onto $\mathcal{H}$.

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*. The mapping $\Gamma^\top : \widehat{f} \to \{ \Gamma_1 \widehat{f}, -\Gamma_0 \widehat{f} \}$ from S^* onto $\mathcal{H} \oplus \mathcal{H}$ establishes a one-to-one correspondence between the set of all selfadjoint extensions of S and the set of all selfadjoint linear relations τ in $\mathcal{H}$ via

(2.1)
$$A_\tau := \ker (\Gamma_0 + \tau \Gamma_1) = \{ \widehat{f} \in S^* : \{\Gamma_1 \widehat{f}, -\Gamma_0 \widehat{f}\} \in \tau \} = \{ \widehat{f} \in S^* : \Gamma^\top \widehat{f} \in \tau \}.$$

When the parameter τ is an operator in $\mathcal{H}$ the equation (2.1) takes the form

$$\Gamma_0 \widehat{f} + \tau \Gamma_1 \widehat{f} = 0, \quad \widehat{f} \in S^*. \tag{2.2}$$

The identity $\tau = \infty$ is to be interpreted as $\tau^{-1} = 0$ or, more precisely, by using graph notation as $\tau = \{0, I_\mathcal{H}\}$; in this case the equation in (2.2) takes the form $\Gamma_1 \widehat{f} = 0$. More generally, there is a similar interpretation, when τ is decomposed orthogonally in terms of its operator part and multivalued part. To each boundary triplet Π one may naturally associate two selfadjoint extensions of S by $A_0 = \ker \Gamma_0$, $A_1 (= A_\infty) = \ker \Gamma_1$, corresponding to the linear relations $\tau = 0$ and $\tau = \infty$ via (2.1).

Let $\mathfrak{N}_\lambda(S^*) = \ker (S^* - \lambda)$, $\lambda \in \widehat{\rho}(S)$, be the defect subspace of S and let $\widehat{\mathfrak{N}}_\lambda(S^*) := \{ \{f_\lambda, \lambda f_\lambda\} : f_\lambda \in \mathfrak{N}_\lambda(S^*) \}$; here the notations $\mathfrak{N}_\lambda$ and $\widehat{\mathfrak{N}}_\lambda$ are used when the context is clear. Every boundary triplet Π gives rise to two operator functions defined for $\lambda \in \rho(A_0) \,(\neq \emptyset)$ by the formulas

$$\gamma(\lambda) = p_1 (\Gamma_0 \restriction \widehat{\mathfrak{N}}_\lambda)^{-1} (\in [\mathcal{H}, \mathfrak{N}_\lambda]), \quad M(\lambda) = \Gamma_1 (\Gamma_0 \restriction \widehat{\mathfrak{N}}_\lambda)^{-1} \,(\in [\mathcal{H}]). \tag{2.3}$$

Here p_1 denotes the orthogonal projection onto the first component of $\mathcal{H} \oplus \mathcal{H}$. The functions γ and M in (2.3) are holomorphic on $\rho(A_0)$ and they are called the *γ-field* and the *Weyl function* of S corresponding to the boundary triplet Π, respectively; cf. [6], [11]. The function M is also the *Q-function* of the pair (S, A_0) in the sense of [19]). The γ-field $\gamma^\top$ and the abstract Weyl function $M^\top$ corresponding to the transposed boundary triplet $\Pi^\top$ are related to γ and M via

$$\gamma^\top(\lambda) = \gamma(\lambda) M(\lambda)^{-1}, \quad M(\lambda)^\top = -M(\lambda)^{-1}, \quad \lambda \in \rho(A_1) \,(\neq \emptyset).$$

When $\mathfrak{H}$ is a Hilbert space, a Weyl function M of S belongs to the class of Nevanlinna functions, that is, M is holomorphic in the upper halfplane $\mathbb{C}_+$, $\operatorname{Im} M(\lambda) \geq 0$ for all $\lambda \in \mathbb{C}_+$, and M satisfies the symmetry condition $M(\lambda)^* = M(\bar\lambda)$ for $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$. In the case where $\mathfrak{H}$ is a Pontryagin space of negative index κ, the Weyl function M of S belongs to the class $\mathbf{N}_k$, $k \leq \kappa$, of *generalized Nevanlinna functions* which are meromorphic on $\mathbb{C}_+ \cup \mathbb{C}_-$, satisfy $M(\lambda)^* = M(\bar\lambda)$, and for which the kernel

$$\mathsf{N}_M(\lambda, \mu) = \frac{M(\lambda) - M(\bar\mu)}{\lambda - \bar\mu}, \quad \mathsf{N}_M(\lambda, \bar\lambda) = \frac{d}{d\lambda} M(\lambda), \quad \lambda, \mu \in \mathbb{C}_+,$$

has k negative squares [19]. If S is *simple*, that is,

$$\mathfrak{H} = \overline{\operatorname{span}} \{ \mathfrak{N}_\lambda(S^*) : \lambda \in \rho(A_0) \,(\neq \emptyset) \},$$

then S is an operator without eigenvalues. In this case the Weyl function M belongs to the class $\mathbf{N}_\kappa$, i.e. $k=\kappa$, and the domain of holomorphy $\rho(M)$ of M coincides with the resolvent set $\rho(A_0)$.

The resolvent of the extension A_τ and its spectrum $\sigma(A_\tau)$ can be expressed in terms of τ and the Weyl function M via Kreĭn's formula. In the terminology of boundary triplets the result can be formulated as follows, see [10], [11], [6].

Proposition 2.1. *Let S be a closed symmetric relation in the Pontryagin space $\mathfrak{H}$ with equal defect numbers (d,d), $d<\infty$, let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triplet for S^* with the Weyl function M, let τ be a linear relation in $\mathcal{H}$ connected with A_τ via* (2.1). *Then the resolvent of A_τ is given by*

$$(A_\tau-\lambda)^{-1}=(A_0-\lambda)^{-1}-\gamma(\lambda)(\tau^{-1}+M(\lambda))^{-1}\gamma(\bar\lambda)^*,\quad \lambda\in\rho(A_\tau)\cap\rho(A_0).$$

Moreover, for every $\lambda\in\rho(A_0)$ the following equivalences hold:

(i) $\lambda\in\rho(A_\tau)$ *if and only if* $\tau^{-1}+M(\lambda)$ *is invertible;*
(ii) $\lambda\in\sigma_p(A_\tau)$ *if and only if* $\ker(\tau^{-1}+M(\lambda))$ *is nontrivial.*

Similarly, for a (generalized) Nevanlinna family $\widetilde{\tau}(\lambda)$ the function

$$(A_0-\lambda)^{-1}-\gamma(\lambda)(\widetilde{\tau}(\lambda)+M(\lambda))^{-1}\gamma(\bar\lambda)^*,$$

is the compressed resolvent of an exit space extension of S in a Hilbert (or a Pontryagin) space, cf. [19], [22], [10], [6].

3. A model for singular perturbations

In a number of papers singular rank one perturbations of A_0 generated by $\omega\in\mathfrak{H}_{-2n-2}$ have been studied by means of exit space extensions of a symmetric operator S connected with A_0, see [21], [12], [13], [20]. In this section the main ingredients for constructing a model for *finite rank singular perturbations* of A_0 generated by G with $\operatorname{ran} G\subset\mathfrak{H}_{-2n-j}$, $j=1,2$, are given. This model was established in [7] and further used in [9], see also [8] for a special case. The model uses a coupling of two symmetric operators and it is motivated by a perturbation result concerning the extending inner product space $\mathfrak{H}\supset\mathfrak{H}_0$: the resolvents associated with the perturbations of A_0 should be finite rank perturbations of the resolvent generated in $\mathfrak{H}$ by $(A_0-\lambda)^{-1}$ (see Theorem 3.1).

3.1. Some operators associated with matrix polynomials

Let q be a monic $d\times d$ matrix polynomial of the form

(3.1) $$q(\lambda)=I_{\mathcal{H}}\lambda^n+q_{n-1}\lambda^{n-1}+\cdots+q_1\lambda+q_0,$$

and let r be a selfadjoint $d\times d$ matrix polynomial of the form

(3.2) $$r(\lambda)=r_{2n-1}\lambda^{2n-1}+r_{2n-2}\lambda^{2n-2}+\cdots+r_1\lambda+r_0,\quad r_j=r_j^*,\quad j=0,\dots,2n-1.$$

Observe, that the function Q in

$$Q(\lambda)=\begin{pmatrix} 0 & q(\lambda) \\ q^\sharp(\lambda) & r(\lambda) \end{pmatrix}, \tag{3.3}$$

is a $2d\times 2d$ matrix polynomial whose leading coefficient is, in general, noninvertible. In fact, Q is a strict generalized matrix Nevanlinna function whose Nevanlinna kernel has dn negative (and dn positive) squares.

Associated with the matrix polynomial q there are $n\times n$ block matrices $\mathcal{B}_q$ and $\mathcal{C}_q$ defined by

$$\mathcal{B}_q=\begin{pmatrix} q_1 & q_2 & \dots & q_{n-1} & I_{\mathcal{H}} \\ q_2 & \dots & q_{n-1} & I_{\mathcal{H}} & 0 \\ \vdots & \iddots & \iddots & 0 & 0 \\ q_{n-1} & I_{\mathcal{H}} & \iddots & \iddots & \vdots \\ I_{\mathcal{H}} & 0 & 0 & \dots & 0 \end{pmatrix}$$

and

$$\mathcal{C}_q=\begin{pmatrix} 0 & I_{\mathcal{H}} & 0 & \dots & 0 \\ 0 & 0 & I_{\mathcal{H}} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & I_{\mathcal{H}} \\ -q_0 & -q_1 & \dots & -q_{n-2} & -q_{n-1} \end{pmatrix}.$$

Moreover, the following block matrices are needed

$$\mathcal{B}=\begin{pmatrix} 0 & \mathcal{B}_q \\ \mathcal{B}_{q^\sharp} & \mathcal{B}_r \end{pmatrix},\quad \mathcal{C}=\begin{pmatrix} \mathcal{C}_{q^\sharp} & \mathcal{C}_{12} \\ 0 & \mathcal{C}_q \end{pmatrix},\quad \mathcal{B}_r=(r_{j+k+1})_{j,k=0}^{n-1},\quad \mathcal{C}_{12}=\mathcal{B}_{q^\sharp}^{-1}\mathcal{D}, \tag{3.4}$$

where

$$\mathcal{D}=\begin{pmatrix} r_n \\ r_{n+1} \\ \vdots \\ r_{2n-1} \end{pmatrix}(q_0,q_1,\dots,q_{n-1})-\begin{pmatrix} I_{\mathcal{H}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}(r_0,r_1,\dots,r_{n-1}).$$

In addition, the following vectors depending on $\lambda\in\mathbb{C}$ are used:

$$\Lambda=(I_{\mathcal{H}},\lambda I_{\mathcal{H}},\dots,\lambda^{n-1}I_{\mathcal{H}}),$$

$$\Lambda_1=\lambda^n\Lambda\widetilde{\mathcal{B}}_{(r)}\mathcal{B}_q^{-1},\quad \widetilde{\mathcal{B}}_{(r)}=\begin{pmatrix} r_{n+1} & \dots & r_{2n-1} & 0 \\ \vdots & \iddots & 0 & 0 \\ r_{2n-1} & \iddots & \iddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

The main objective here is the matrix polynomial Q defined in (3.3). It determines the structure of the exit space $\mathfrak{H}_Q=\mathcal{H}^n\oplus\mathcal{H}^n$ $(=\mathbb{C}^{2dn})$ used for constructing the model for singular perturbations. The inner product in $\mathfrak{H}_Q$ is defined by the block matrix $\mathcal{B}$ via $\langle\cdot,\cdot\rangle_{\mathfrak{H}_Q} = (\mathcal{B}\,\cdot,\cdot)$ in which case *the companion type operator* $\mathcal{C}$ in (3.4) becomes selfadjoint in $\mathfrak{H}_Q$. The restriction of $\mathcal{C}$ to the subspace

$$\operatorname{dom} S_Q = \left\{ F = \begin{pmatrix} f \\ \widetilde{f} \end{pmatrix} \in \mathfrak{H}_Q : \; f_1 = \widetilde{f}_1 = 0 \right\} \tag{3.5}$$

defines a closed simple symmetric operator S_Q in $\mathfrak{H}_Q$ with defect numbers $(2d, 2d)$. It is maximally nondensely defined and a straighforward calculation shows that its adjoint S_Q^* (a linear relation in $\mathfrak{H}_Q$) is given by

$${S_Q}^* = \left\{ \widehat{F} = \left\{ F, \mathcal{C}F + \mathcal{B}^{-1} \begin{pmatrix} \varphi \otimes e_1 \\ \widetilde{\varphi} \otimes e_1 \end{pmatrix} \right\} : \; F \in \mathfrak{H}_Q, \; \varphi, \widetilde{\varphi} \in \mathcal{H} \right\}.$$

It is possible to associate a boundary triplet $\Pi_Q = \{\mathcal{H} \oplus \mathcal{H}, \Gamma_0^Q, \Gamma_1^Q\}$ with ${S_Q}^*$ by defining the boundary mappings on S_Q^* via

$$\Gamma_0^Q \widehat{F} = \begin{pmatrix} f_1 \\ \widetilde{f}_1 \end{pmatrix}, \quad \Gamma_1^Q \widehat{F} = \begin{pmatrix} \varphi \\ \widetilde{\varphi} \end{pmatrix}, \quad \widehat{F} \in {S_Q}^*.$$

In this case the Weyl function of S_Q associated with the boundary triplet Π_Q coincides with the matrix polynomial Q, cf. [7].

3.2. A perturbation result for the resolvents

Let G be a linear mapping from $\mathcal{H} = \mathbb{C}^d$ into the scale space $\mathfrak{H}_{-2n-1}$ generated by the selfadjoint operator A_0 and let $\widetilde{A}_0$ be the $[\mathfrak{H}_{-2n+1}, \mathfrak{H}_{-2n-1}]$-continuation of A_0. The adjoint operator G^* maps $\mathfrak{H}_{2n+1}$ into $\mathcal{H}$. The case where G is a mapping into $\mathfrak{H}_{-2n}$ is similar to the present case; it can be found in [9]. Observe, that if $\operatorname{ran} G \cap \mathfrak{H}_{-2} = \{0\}$, then the restriction of A_0 to

$$\operatorname{dom} S_0 = \operatorname{dom} A_0 \cap \ker G^*$$

gives rise to an essentially selfadjoint operator S_0 whose closure coincides with A_0. Moreover, the vector $\widetilde{R}_\lambda G h = (\widetilde{A}_0 - \lambda)^{-1} G h$, $h \in \mathcal{H}$, $\lambda \in \rho(A_0)$, does not belong to the space $\mathfrak{H}_0$. However, one can give a sense to the vector $\widetilde{R}_\lambda G h$ by extending the space $\mathfrak{H}_0$ suitably. For instance, if $0 \in \rho(A_0)$, then the vector

$$\gamma(\lambda) h := \widetilde{R}_\lambda G h = \widetilde{A}_0^{-1} G h + \cdots + \lambda^{n-1} \widetilde{A}_0^{-n} G h + \lambda^n \widetilde{R}_\lambda \widetilde{A}_0^{-n} G h$$

can be considered as a vector from an extended inner product space $\mathfrak{H}$ satisfying the condition

$$\mathfrak{H} \supset \operatorname{span}\{\, \mathfrak{H}_0, \widetilde{A}_0^{-j} \operatorname{ran} G : \; j = 1, \dots, n \,\}. \tag{3.6}$$

In this space the continuation $\widetilde{A}_0$ of A_0 generates an operator, say H_0, for which the operator function $\gamma(\lambda)$, $\lambda \in \rho(A_0)$, can be interpreted to form its γ-field in the sense that

$$\frac{\gamma(\lambda) - \gamma(\mu)}{\lambda - \mu} = (H_0 - \lambda)^{-1} \gamma(\mu), \quad \lambda, \mu \in \rho(A_0).$$

This identity implies that

$$\frac{d}{d\lambda} \gamma(\lambda) = (H_0 - \lambda)^{-1} \gamma(\lambda), \quad \lambda \in \rho(A_0).$$

The inner product $\langle u,\varphi\rangle_{\mathfrak{H}}$ in $\mathfrak{H}$ should coincide with the form (u,φ) generated by the inner product in $\mathfrak{H}_0$ if the vectors u, φ are in duality, say, $u \in \mathfrak{H}_{2(n-j)+1}$, $\varphi \in \widetilde{A}_0^{-j}\operatorname{ran} G$. Now, for the other vectors in (3.6) it will be supposed that the conditions

$$\left\langle \widetilde{A}_0^{-j}Gh, \widetilde{A}_0^{-k}Gf \right\rangle_{\mathfrak{H}} = (t_{j+k-1}h, f)_{\mathcal{H}}, \quad j,k = 1,\dots,n;\; h,f \in \mathcal{H}, \tag{3.7}$$

are satisfied for some operators $t_j = t_j^* \in [\mathcal{H}]$, $j = 1,\dots,2n-1$. The next result shows that under such mild conditions on the extending space the structure of perturbed resolvents becomes completely fixed under some basic assumptions on H_0. This fact gives rise to the model presented in [7] for singular finite rank perturbations of A_0.

Theorem 3.1. ([9, Theorem 4.8]) *Let $0 \in \rho(A_0)$, let $\operatorname{ran} G \subset \mathfrak{H}_{-2n-1}\setminus\mathfrak{H}_{-2n}$, and let $G_0 = \widetilde{A}_0^{-n}G$. Moreover, assume that $\mathfrak{H} \supset \mathfrak{H}_0$ is (an isometric image of) an inner product space satisfying* (3.6), (3.7), *and let H and H_0 be selfadjoint linear relations in $\mathfrak{H}$ such that*

(i) *$\rho(H_0) = \rho(A_0)$;*
(ii) *$\gamma(\lambda)' = (H_0-\lambda)^{-1}\gamma(\lambda)$ holds for (an isometric image of) $\gamma(\lambda) = (\widetilde{A}_0 - \lambda)^{-1}G$, $\lambda \in \rho(A_0)$;*
(iii) *$(H-\lambda)^{-1} - (H_0-\lambda)^{-1} = -\gamma(\lambda)\sigma(\lambda)\gamma(\bar\lambda)^*$, $\lambda \in \rho(H)\cap\rho(H_0)$;*

for some matrix function $\sigma(\lambda)$ holomorphic and invertible for $\lambda \in \rho(H_0)\cap\rho(H)$. Then $\sigma(\lambda)^{-1}$ can be represented in the form

$$\sigma^{-1}(\lambda) = \beta + t(\lambda) + \lambda^{2n}M_0(\lambda), \tag{3.8}$$

where $\beta = \beta^* \in [\mathcal{H}]$, $t(\lambda) = t_1\lambda + \cdots + t_{2n-1}\lambda^{2n-1}$, and $M_0(\lambda) = G_0^*\widetilde{R}_\lambda G_0$ is a Nevanlinna function in $\mathcal{H}$.

In Theorem 3.1 the function σ^{-1} can be seen as a Weyl function (or a Q-function) of an underlying symmetric operator S. The formula for σ^{-1} in (3.8) shows that it is a generalized Nevanlinna function and therefore in general the operator S cannot be symmetric in some Hilbert space. The model constructed in [7] for S uses a coupling method resulting in a Pontryagin space $\mathfrak{H}$ such that S becomes symmetric in $\mathfrak{H}$. The construction of the model space via the coupling method is briefly recalled in the next subsection.

Note that the condition $0 \in \rho(A_0)$, which was assumed for simplicity, leads to the particular form of $\sigma(\lambda)^{-1}$ in (3.8). Other invertibility conditions on A_0 lead to the more general form of $\sigma(\lambda)^{-1}$ in (3.9).

3.3. The model

Let S_0 be a closed symmetric operator in a Hilbert space $\mathfrak{H}_0$ with defect numbers (d,d) and the Weyl function M_0. Let S_Q be the symmetric operator in the Pontryagin space $\mathfrak{H}_Q = \mathbb{C}^{dn} \oplus \mathbb{C}^{dn}$ defined as the restriction of $\mathcal{C}$ to (3.5). The next theorem (cf. [7]) gives a symmetric linear relation S in the Pontryagin space

$\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_Q = \mathfrak{H} \oplus (\mathbb{C}^{dn} \oplus \mathbb{C}^{dn})$ as a coupling of the operators S_0 and S_Q, such that the following function is a Weyl function for S:

$$M(\lambda) = r(\lambda) + q^\sharp(\lambda) M_0(\lambda) q(\lambda). \tag{3.9}$$

Here the matrix polynomials q and r are as in (3.1) and (3.2).

Theorem 3.2. ([7, Theorem 4.2]) *Let S_0 be a closed symmetric operator in the Hilbert space $\mathfrak{H}_0$ and let $\Pi^0 = \{\mathcal{H}, \Gamma_0^0, \Gamma_1^0\}$ be a boundary triplet for S_0^* with the Weyl function M_0 and the γ-field γ_0. Let q, r, and Q are the same as in* (3.1), (3.2), *and* (3.3), *respectively. Then:*

(i) *The linear relation*

$$S = \left\{ \left\{ \begin{pmatrix} f_0 \\ f \\ \widetilde{f} \end{pmatrix}, \begin{pmatrix} f_0' \\ \mathcal{C}\begin{pmatrix} f \\ \widetilde{f} \end{pmatrix} + \mathcal{B}^{-1}\begin{pmatrix} \Gamma_0^0 \widehat{f}_0 \otimes e_1 \\ 0 \end{pmatrix} \end{pmatrix} \right\} : \begin{array}{c} \widehat{f}_0 = \{f_0, f_0'\} \in S_0^* \\ f, \widetilde{f} \in \mathbb{C}^{dn} \\ f_1 = \Gamma_1^0 \widehat{f}_0, \widetilde{f}_1 = 0 \end{array} \right\}$$

is closed and symmetric in $\mathfrak{H}_0 \oplus \mathfrak{H}_Q$ and has defect numbers (d, d).

(ii) *The adjoint S^* is given by*

$$S^* = \left\{ \left\{ \begin{pmatrix} f_0 \\ f \\ \widetilde{f} \end{pmatrix}, \begin{pmatrix} f_0' \\ \mathcal{C}\begin{pmatrix} f \\ \widetilde{f} \end{pmatrix} + \mathcal{B}^{-1}\begin{pmatrix} \Gamma_0^0 \widehat{f}_0 \otimes e_1 \\ \widetilde{\varphi} \otimes e_1 \end{pmatrix} \end{pmatrix} \right\} : \begin{array}{c} \widehat{f}_0 = \{f_0, f_0'\} \in S_0^* \\ f, \widetilde{f} \in \mathbb{C}^{dn}, \widetilde{\varphi} \in \mathbb{C}^d \\ f_1 = \Gamma_1^0 \widehat{f}_0 \end{array} \right\}.$$

(iii) *A boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for S^* is determined by*

$$\Gamma_0(\widehat{f}_0 \oplus \widehat{F}) = \widetilde{f}_1, \quad \Gamma_1(\widehat{f}_0 \oplus \widehat{F}) = \widetilde{\varphi}, \quad \widehat{f}_0 \oplus \widehat{F} \in S^*.$$

(iv) *The corresponding Weyl function M is of the form* (3.9) *and the γ-field γ is given by*

$$\gamma(\lambda)h = \gamma_0(\lambda)q(\lambda)h \oplus ((\Lambda^\top M_0(\lambda)q(\lambda) + \Lambda_1^\top)h \dotplus \Lambda^\top h), \quad h \in \mathcal{H}. \tag{3.10}$$

If the operator S_0 is densely defined in $\mathfrak{H}_0$, then S is an operator. When $r = 0$ the formulas for S and S^* in Theorem 3.2 can be simplified and the Weyl function takes the factorized form

$$M(\lambda) = q^\sharp(\lambda) M_0(\lambda) q(\lambda).$$

3.4. Selfadjoint extensions of the model operator

The selfadjoint extensions of the model operator S can be parametrized by the selfadjoint relations τ in the parameter space $\mathcal{H}$ via $H_\tau = \ker(\Gamma_0 + \tau\Gamma_1)$. ¿From Theorem 3.2 one obtains the following explicit expressions for H_τ, cf. [9].

Proposition 3.3. *Let the assumptions be as in Theorem 3.2, and let γ and M be given by* (3.10) *and* (3.9), *respectively. Then:*

(i) *The selfadjoint extensions H_τ of S in $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_Q$ are in a one-to-one correspondence with the selfadjoint relations τ in $\mathcal{H}$ via*

$$H_\tau = \left\{ \left\{ \begin{pmatrix} f_0 \\ f \\ \widetilde{f} \end{pmatrix}, \begin{pmatrix} f_0' \\ \mathcal{C}\begin{pmatrix} f \\ \widetilde{f} \end{pmatrix} + \mathcal{B}^{-1}\begin{pmatrix} \Gamma_0^0 \widehat{f}_0 \otimes e_1 \\ \widetilde{\varphi} \otimes e_1 \end{pmatrix} \end{pmatrix} \right\} : \begin{array}{c} \widehat{f}_0 = \{f_0, f_0'\} \in S_0^* \\ f, \widetilde{f} \in \mathbb{C}^{dn} \\ f_1 = \Gamma_1^0 \widehat{f}_0, \widetilde{f}_1 + \tau \widetilde{\varphi} = 0 \end{array} \right\}.$$

(ii) *For every $\lambda \in \rho(H_\tau) \cap \rho(H_0)$ the resolvent $(H_\tau - \lambda)^{-1}$ satisfies the relation*

$$(H_\tau - \lambda)^{-1} = (H_0 - \lambda)^{-1} - \gamma(\lambda)(\tau^{-1} + M(\lambda))^{-1}\gamma(\bar{\lambda})^*. \tag{3.11}$$

(iii) *For every $\lambda \in \rho(H_0)$ the following equivalences hold:*

$$\lambda \in \sigma_p(H_\tau) \Leftrightarrow 0 \in \sigma_p(\tau^{-1} + M(\lambda)),$$
$$\lambda \in \rho(H_\tau) \Leftrightarrow 0 \in \rho(\tau^{-1} + M(\lambda)).$$

Proof. (i) The condition $\widehat{f}_0 \oplus \widehat{F} \in \ker(\Gamma_0 + \tau\Gamma_1)$ means that $\{\widetilde{\varphi}, \widetilde{f}_1\} \in -\tau$, or equivalently, that $\widetilde{f}_1 + \tau\widetilde{\varphi} = 0$, see (iii) of Theorem 3.2. The representation of H_τ is now obtained from the formula for S^* in Theorem 3.2.

(ii) The form of the resolvent of H_τ is obtained by applying Proposition 2.1 to the data in Theorem 3.2.

(iii) This statement is immediate from Proposition 2.1. □

The operator S_0 in Theorem 3.2 is allowed to be nondensely defined in the original Hilbert space $\mathfrak{H}_0$. If S_0 is densely defined in $\mathfrak{H}_0$ then S is an operator in the model Pontryagin space $\mathfrak{H}_0 \oplus \mathfrak{H}_Q$. However, even in this case the model operator S is not densely defined in $\mathfrak{H}_0 \oplus \mathfrak{H}_Q$. Therefore, among the selfadjoint extensions of S there are linear relations which are not operators. In fact, the following result holds.

Proposition 3.4. *The multivalued parts of S and H_τ are given by*

$$\operatorname{mul} S = \left\{ \begin{pmatrix} f_0' \\ \mathcal{B}^{-1}\begin{pmatrix} \Gamma_0^0 \widehat{f}_0 \otimes e_1 \\ 0 \end{pmatrix} \end{pmatrix} : \widehat{f}_0 = \{0, f_0'\} \in A_1 \right\}, \tag{3.12}$$

$$\operatorname{mul} H_\tau = \left\{ \begin{pmatrix} f_0' \\ \mathcal{B}^{-1}\begin{pmatrix} \Gamma_0^0 \widehat{f}_0 \otimes e_1 \\ \widetilde{\varphi} \otimes e_1 \end{pmatrix} \end{pmatrix} : \widehat{f}_0 = \{0, f_0'\} \in A_1, \widetilde{\varphi} \in \ker\tau \right\}. \tag{3.13}$$

and the equalities

$$\dim \operatorname{mul} S = \dim \operatorname{mul} A_1, \quad \dim \operatorname{mul} H_\tau = \dim \operatorname{mul} A_1 + \dim \ker \tau \tag{3.14}$$

hold. In particular, H_τ is an operator in $\mathfrak{H}$ if and only if $A_1 = \ker \Gamma_1^0$ is an operator in $\mathfrak{H}_0$ and $\ker \tau = \{0\}$. Moreover, H_0 is the unique selfadjoint extension H_τ of S for which the equality $\operatorname{mul} H_\tau = \operatorname{mul} S^*$ holds, and it has the representation

$$H_0 = S \hat{+} (\{0\} \oplus \operatorname{mul} S^*), \tag{3.15}$$

where $\hat{+}$ stands for the componentwise sum of the graphs.

Proof. The form of $\operatorname{mul} H_\tau$ is straighforward to check by using the formulas for the selfadjoint extensions H_τ in Proposition 3.3. By letting $\widetilde{\varphi} = 0$ in (3.13) one obtains the description (3.12) for $\operatorname{mul} S$. The equalities (3.14) follow from (3.12) and (3.13).

Moreover, by comparing $\operatorname{mul} H_\tau$ with the multivalued part of the adjoint relation S^* (see [7]) one concludes that the condition $\operatorname{mul} H_\tau = \operatorname{mul} S^*$ is equivalent to $\dim \ker \tau = d$. This means that $\tau = 0$, i.e., the only selfadjoint extension with the maximal multivalued part $\operatorname{mul} S^*$ is H_0.

The representation (3.15) of H_0 is now obvious. □

If the selfadjoint extension $A_1 = \ker \Gamma_1^0$ of S_0 is an operator in $\mathfrak{H}_0$, then $\operatorname{mul} S = \{0\}$ and H_τ is an operator in $\mathfrak{H}$ if and only if $\ker \tau = \{0\}$. In view of (3.15) the extension H_0 is always multivalued, since S is nondensely defined in $\mathfrak{H}$. In fact, H_0 has a natural interpretation as a *generalized Friedrichs extension* of S, see [15], [16]. The representation (3.15) shows that, together with S, H_0 is maximally nondensely defined in $\mathfrak{H}$. In fact, H_0 has a nontrivial root subspace $\mathfrak{L}$ at ∞ and, moreover, the following results shows that the finite spectrum of H_0 coincides with the spectrum of the selfadjoint extension A_0 of S_0 in the original Hilbert space $\mathfrak{H}_0$. Hence, in particular the assumption (i) in Theorem 3.1 is satisfied.

Proposition 3.5. ([9, Proposition 3.4]) *Let the assumptions be as in Theorem 3.2 and let $H_0 = \ker \Gamma_0$ be as in Proposition 3.3 (with $\tau = 0$). Then:*

(i) $\rho(H_0) = \rho(A_0)$*;*

(ii) *the compression of the resolvent of H_0 to the subspace $\mathfrak{H}_0$ is given by*

$$P_{\mathfrak{H}_0}(H_0 - \lambda)^{-1} \upharpoonright \mathfrak{H}_0 = (A_0 - \lambda)^{-1}, \quad \lambda \in \rho(H_0);$$

(iii) *the subspace $\mathcal{L} = \{0\} \oplus \mathcal{H}^n \oplus \{0\}$ of $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_Q$ is maximal neutral and invariant under the resolvent $(H_0 - \lambda)^{-1}$. It satisfies $(H_0 - \lambda)^{-n}\mathcal{L} = \{0\}$, $\lambda \in \rho(H_0)$.*

This result will be extended in Section 5 to a certain subclass of selfadjoint extensions of the model operator S in $\mathfrak{H}$ (i.e. for certain singular perturbations of A_0).

4. Singular perturbations as range perturbations

Let $H_\infty = \ker \Gamma_1$ be the selfadjoint extension of S corresponding to $\tau^{-1} = 0$ in Proposition 3.3. The selfadjoint extensions H_τ of S in Proposition 3.3 can be seen as "range perturbations" of H_∞ in the Pontryagin space $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_Q$, cf. [17], [18] for the Hilbert space case. For simplicity the results in this section are stated when $A_1 = \ker \Gamma_1^0$ is an operator in $\mathfrak{H}_0$, which is always the case when S_0 is densely defined in $\mathfrak{H}_0$. In this case H_∞ is also an operator by Proposition 3.4. Introduce

$\Omega : \mathcal{H} \to \operatorname{mul} S^* \subset \mathfrak{H}_0 \oplus \mathfrak{H}_Q$ by

$$\Omega h = \begin{pmatrix} 0 \\ h \otimes e_n \\ 0 \end{pmatrix}, \quad h \in \mathcal{H}. \tag{4.1}$$

In the rest of this paper the following notations will be used

$$\mathbf{F} = \begin{pmatrix} f_0 \\ f \\ \widetilde{f} \end{pmatrix}, \; \mathbf{G} = \begin{pmatrix} g_0 \\ g \\ \widetilde{g} \end{pmatrix} \in \mathfrak{H}_0 \oplus \mathfrak{H}_Q.$$

Proposition 4.1. *Let the assumptions be as in Theorem 3.2 and assume that* $A_1 = \ker \Gamma_1^0$ *is an operator. Then* S *is a domain restriction of* H_∞ *given by*

$$\operatorname{dom} S = \{ F \in \operatorname{dom} H_\infty : \Omega^* F = 0 \}, \tag{4.2}$$

and the selfadjoint extensions H_τ *and* H_∞ *of* S *in Proposition 3.3 are connected by*

$$H_\tau = H_\infty - \Omega \tau^{-1} \Omega^*, \tag{4.3}$$

where the difference is understood in the sense of relations.

Proof. Since A_1 is assumed to be an operator, Proposition 3.4 shows that H_∞ is an operator. The adjoint $\Omega^* : \mathfrak{H}_0 \oplus \mathfrak{H}_Q \to \mathcal{H}$ of Ω in (4.1) is given by

$$\Omega^* \mathbf{F} = \widetilde{f}_1. \tag{4.4}$$

The equality (4.2) is now clear from the formulas for H_∞ in Proposition 3.3 and for S in Theorem 3.2.

Let $\mathbf{F} = (f_0, f, \widetilde{f})^\top, \mathbf{G} = (g_0, g, \widetilde{g})^\top \in \mathfrak{H}$. By definition, $\{\mathbf{F}, \mathbf{G}\} \in \Omega \tau^{-1} \Omega^*$ if and only if $\{\Omega^* \mathbf{F}, \widetilde{\varphi}\} = \{\widetilde{f}_1, \widetilde{\varphi}\} \in \tau^{-1}$ and $\mathbf{G} = \Omega \widetilde{\varphi}$ for some $\widetilde{\varphi} \in \mathcal{H}$. Consequently, $\{\mathbf{F}, \mathbf{G}\} \in H_\infty - \Omega \tau^{-1} \Omega^*$ if and only if

$$\{\mathbf{F}, \mathbf{G}\} = \{\mathbf{F}, H_\infty \mathbf{F} + \Omega \widetilde{\varphi}\}, \quad \mathbf{F} \in \operatorname{dom} H_\infty, \quad \{\widetilde{\varphi}, \widetilde{f}_1\} \in -\tau. \tag{4.5}$$

Now using (4.1) and comparing (4.5) with the expression for H_τ in Proposition 3.3 the equality (4.3) follows. □

The above result depends on the fact that the model operator S in Theorem 3.2 is maximally nondensely defined in $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_Q$. In the case of defect numbers $(1,1)$ the extension H_0 is the only selfadjoint extension of S which is not an operator and the other extensions H_τ, $\tau \neq 0$, are given by (4.3), cf. [8]. In the special case of defect numbers $(1,1)$ a result similar to Proposition 4.1 has been obtained in [18, Theorem 3.2] for the model concerning perturbations in $\mathfrak{H}_{-2}$.

The perturbation formula (4.3) in Proposition 4.1 gives an explicit expression for the selfadjoint extensions H_τ of S. Moreover, the resolvent formula (3.11) can be obtained by a straighforward calculation from (4.3), cf. e.g. [15]. It is also clear from (4.3) that H_τ is an operator if and only if the inverse τ^{-1} is an operator in $\mathcal{H}$, in which case H_τ, $\ker \tau = \{0\}$, is an ordinary range perturbation of the operator H_∞ in the Pontryagin space $\mathfrak{H}$. An opposite extreme case is $\tau = 0$. Then the condition $\{\widetilde{\varphi}, \widetilde{f}_1\} \in -\tau$ in (4.5) is equivalent to $\mathbf{F} \in \ker \Omega^*$ which together with $\mathbf{F} \in \operatorname{dom} H_\infty$ implies that $\mathbf{F} \in \operatorname{dom} S$, while $\operatorname{mul} \Omega\tau^{-1}\Omega^* = \operatorname{ran} \Omega$. Hence, the perturbation (4.3) for $\tau = 0$ coincides with the form of H_0 given in (3.15), i.e. with the generalized Friedrichs extension of S in $\mathfrak{H}$. A more specific classification associated with the perturbation formula (4.3) is obtained by decomposing the selfadjoint parameter τ into its operator and multivalued parts,

$$\tau = \tau_s \oplus \tau_\infty, \quad \tau_s = \{\, \{h,k\} \in \tau : \, k \perp \operatorname{mul} \tau \,\}, \quad \tau_\infty = \{0\} \oplus \operatorname{mul} \tau. \tag{4.6}$$

Here τ_s is a selfadjoint operator in $\mathcal{H}_s = \operatorname{dom} \tau$, τ_∞ is a selfadjoint relation in $\mathcal{H}_\infty = \operatorname{mul} \tau$, and $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_\infty$.

Proposition 4.2. *Let the assumptions be as in Theorem 3.2 and assume that* $A_1 = \ker \Gamma_1^0$ *is an operator. Let* P *be an orthogonal projection in* $\mathcal{H}$ *and define* $\Omega_P = \Omega{\upharpoonright} \operatorname{ran} P$*, where* Ω *is given by* (4.1)*. Then:*

(i) *The domain restriction* S_P *of* H_∞*, given by*

$$\operatorname{dom} S_P = \{\, \mathbf{F} \in \operatorname{dom} H_\infty : \, \Omega_P^* \mathbf{F} = 0 \,\},$$

is a closed symmetric operator in $\mathfrak{H}$ *with defect numbers are given by* (s, s)*,* $s = \dim \operatorname{ran} P$*.*

(ii) *The adjoint of* S_P *is given by*

$$S_P^* = \{\, \{\mathbf{F}, \mathbf{G}\} \in S^* : \, (I-P)\widetilde{\varphi} = 0 \,\}.$$

(iii) *The selfadjoint extensions* H_τ *of* S *with the property* $H_\tau \cap H_\infty = S_P$ *are in one-to-one correspondence with the parameters* τ *for which* $\operatorname{mul} \tau = \ker P$ *and they are given by*

$$H_\tau = H_\infty - \Omega_P \tau_s^{-1} \Omega_P^*, \tag{4.7}$$

where $\tau = \tau_s \oplus \tau_\infty$ *is decomposed as in* (4.6) *and the difference is understood in the sense of relations.*

(iv) *The generalized Friedrichs extension of* S_P *corresponds to* $\tau_s = 0$ *in* (4.7) *and is given by*

$$S_P \,\hat{+}\, (\{0\} \oplus \operatorname{mul} S_P^*) = \{\, \{\mathbf{F}, \mathbf{G}\} \in S^* : \, \Omega_P^* \mathbf{F} \, (= P\widetilde{f}_1) = 0, \, (I-P)\widetilde{\varphi} = 0 \,\}.$$

Proof. Let $\mathbf{F} = (f_0, f, \widetilde{f})^\top$, $\mathbf{G} = (g_0, g, \widetilde{g})^\top \in \mathfrak{H}$. In view of (4.4), $\Omega_P^* \mathbf{F} = P\Omega^* \mathbf{F} = P\widetilde{f}_1$. Hence, $S_P = \{\, \{\mathbf{F}, \mathbf{G}\} \in S^* : \, \Omega_P^* \mathbf{F} = P\widetilde{f}_1 = 0, \, \widetilde{\varphi} = 0 \,\}$ from which the statements (i) and (ii) easily follow.

(iii) Clearly, $\{\mathbf{F}, \mathbf{G}\} \in H_\infty \cap H_\tau$ if and only if $\{\mathbf{F}, \mathbf{G}\} \in S^*$, and the conditions $\widetilde{\varphi} = 0$ and $\{\widetilde{\varphi}, \widetilde{f}_1\} \in -\tau$ are satisfied. Equivalently, $\mathbf{F} \in \operatorname{dom} H_\infty$ and $\widetilde{f}_1 \in \operatorname{mul} \tau$. Comparing this with the condition $\Omega_P^* \mathbf{F} = P\widetilde{f}_1 = 0$ for S_P in (i), one concludes that $\operatorname{mul} \tau = \ker P$. It is easy to check that for such τ, the equality $\Omega\tau^{-1}\Omega^* = \Omega_P \tau_s^{-1} \Omega_P^*$ holds (cf. the proof of Proposition 4.1). Hence, (4.7) follows from (4.3).

(iv) The discussion concerning H_0 above shows that the generalized Friedrichs extensions of S_P corresponds to $\tau_s = 0$ in (4.7), in which case $\{\widetilde{\varphi}, \widetilde{f}_1\} \in -\tau$ is equivalent to $(I-P)\widetilde{\varphi} = 0$ and $P\widetilde{f}_1 = 0$. □

Proposition 4.2 shows that S_P is maximally nondensely defined: $\dim \operatorname{mul} S_P^* = s$. Clearly, the perturbation formula (4.7) is an analog of (4.3). The characterization of operator extensions in (4.7) agrees with the one in (4.3), since $\ker \tau_s = \ker \tau$.

5. The class of selfadjoint extensions with extremal boundary conditions

According to Proposition 3.5 the compressed resolvent of H_0 from $\mathfrak{H}$ to the Hilbert space $\mathfrak{H}_0 \subset \mathfrak{H}$ coincides with the resolvent of the (unperturbed) operator A_0 in $\mathfrak{H}_0$. In this section the corresponding property will be proved for a certain subclass of selfadjoint extensions H_τ of the model operator S. A compressed resolvent of S in $\mathfrak{H}_0$ is said to be canonical if it coincides with the resolvent of some selfadjoint extension $\widetilde{A}$ of S_0 in the Hilbert space $\mathfrak{H}_0$.

Proposition 4.2 shows that the generalized Friedrichs extension of the intermediate symmetric extension $S_P \subset H_\infty$ is determined by the (abstract) boundary conditions

$$P\Gamma_0\widehat{\mathbf{F}} = (I-P)\Gamma_1\widehat{\mathbf{F}} = 0, \quad \widehat{\mathbf{F}} \in S^*, \quad P = P^* = P^2, \tag{5.1}$$

where $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is the boundary triplet associated with S^* in Theorem 3.2. In what follows boundary conditions of the form (5.1) are called *extremal boundary conditions* associated with S^*, since they have an interpretation as extreme points in the parameter space, see e.g. [3]. When in the model of Theorem 3.2 the matrix polynomial $r = 0$ and the matrix polynomial q is of the form $q = I_{\mathcal{H}} \otimes \widetilde{q}$, where $\widetilde{q}$ is a monic scalar polynomial, a different description for this class of extensions of S can be obtained by means of the compressed resolvents in $\mathfrak{H}_0$. In fact, for these extensions of S the following analog of Proposition 3.5 can be proved.

Theorem 5.1. *Let the assumptions be as in Theorem 3.2. Let $r = 0$ in (3.2), let $q = I_{\mathcal{H}} \otimes \widetilde{q}$, where $\widetilde{q}$ is a monic scalar polynomial, and let $\mathcal{C}$ be given by (3.4). Then the compressed resolvent of H_τ to $\mathfrak{H}_0$ is canonical if and only if H_τ is given by the extremal boundary conditions of the form (5.1). In this case $\tau = \{\{Ph, (I-P)h\} : h \in \mathcal{H}\}$ and for the corresponding H_τ the following assertions hold:*

- (i) *$\rho(H_\tau) = \rho(A_\tau) \cap \rho(\mathcal{C})$ $(= \rho(A_\tau)\backslash\sigma(q^\sharp q))$, where $A_\tau = \ker(\Gamma_0^0 + \tau\Gamma_1^0)$ and $\tau \neq 0$.*
- (ii) *$P_{\mathfrak{H}_0}(H_\tau - \lambda)^{-1}\restriction \mathfrak{H}_0 = (A_\tau - \lambda)^{-1}, \quad \lambda \in \rho(H_\tau)$.*
- (iii) *The subspace $\mathcal{L}_\tau = \mathcal{L}_1 \dotplus \mathcal{L}_2$ with*

$$\mathcal{L}_1 = \{0\} \oplus (\operatorname{ran} P)^n \oplus \{0\}, \quad \mathcal{L}_2 = \{0\} \oplus \{0\} \oplus (\ker P)^n$$

is maximal neutral and invariant under the resolvent $(H_\tau - \lambda)^{-1}$. Moreover,

$$(H_\tau - \lambda)^{-n}\mathcal{L}_1 = \{0\}, \quad (H_\tau - \lambda)^{-1}(0,0,\widetilde{g})^\top = (0,0,(\mathcal{C}_q - \lambda)^{-1}\widetilde{g})^\top, \tag{5.2}$$

where $(0,0,\widetilde{g})^\top \in \mathcal{L}_2$ and $\lambda \in \rho(H_\tau)$.

(iv) *The Weyl functions* $\widetilde{M}_\tau$ *of* (S,H_τ) *and* $\widetilde{M}_{0,\tau}$ *of* (S_0,A_τ) *are given by*

$$\widetilde{M}_\tau=\begin{pmatrix}\widetilde{q}_1^\sharp & 0\\ 0 & \widetilde{q}_2^{-1}\end{pmatrix}\widetilde{M}_{0,\tau}\begin{pmatrix}\widetilde{q}_1 & 0\\ 0 & \widetilde{q}_2^{-\sharp}\end{pmatrix}$$

and

$$\widetilde{M}_{0,\tau}=\begin{pmatrix}M_{11}-M_{12}M_{22}^{-1}M_{21} & -M_{12}M_{22}^{-1}\\ -M_{22}^{-1}M_{21} & -M_{22}^{-1}\end{pmatrix},$$

where $\widetilde{q}_1=I_{\mathcal{H}_s}\otimes\widetilde{q}$, $\widetilde{q}_2=I_{\mathcal{H}_\infty}\otimes\widetilde{q}$, *and the decomposition of the Weyl function* $M_0=(M_{ij})_{i,j=1}^2$ *of* (S_0,A_0) *is according to* $\mathcal{H}=\mathcal{H}_s\oplus\mathcal{H}_\infty=\operatorname{ran}P\oplus\ker P$.

Proof. It follows from Proposition 3.3 that the compressed resolvent of H_τ is given by

$$P_{\mathfrak{H}_0}(H_\tau-\lambda)^{-1}\upharpoonright\mathfrak{H}_0=(A_0-\lambda)^{-1}-\gamma_0(\lambda)(\widetilde{\tau}(\lambda)^{-1}+M_0(\lambda))^{-1}\gamma_0(\bar{\lambda})^*, \tag{5.3}$$

where $\widetilde{\tau}(\lambda)=q(\lambda)\tau q^\sharp(\lambda)$ due to assumption $r=0$. The formula (5.3) coincides with a canonical resolvent of S_0 if and only if the function $\widetilde{\tau}$ does not depend on λ. Clearly, this condition is satisfied if and only if $\tau_s=0$ in (4.6), i.e., H_τ is given by the extremal boundary conditions (5.1) for some $P=P^*=P^2$.

To prove (i)–(iv) introduce the following boundary mappings

$$\begin{cases}\widetilde{\Gamma}_0=P\Gamma_0-(I-P)\Gamma_1,\\ \widetilde{\Gamma}_1=(I-P)\Gamma_0+P\Gamma_1,\end{cases}\qquad\begin{cases}\widetilde{\Gamma}_0^0=P\Gamma_0^0-(I-P)\Gamma_1^0,\\ \widetilde{\Gamma}_1^0=(I-P)\Gamma_0^0+P\Gamma_1^0,\end{cases} \tag{5.4}$$

so that $H_\tau=\ker\widetilde{\Gamma}_0$ and $A_\tau=\ker\widetilde{\Gamma}_0^0$.

(i) Let $\mathbf{G}=(g_0,g,\widetilde{g})^\top\in\mathfrak{H}$, $\widehat{f}_0=\{f_0,f_0'\}\in S_0^*$, and let $\lambda\in\rho(A_\tau)\cap\rho(\mathcal{C})$. Then by Proposition 3.3 the relation $\mathbf{G}\in\operatorname{ran}(H_\tau-\lambda)$ can be rewritten as a system of equalities

$$\begin{cases}f_0'-\lambda f_0=g_0,\\ (\mathcal{C}_{q^\sharp}-\lambda)f+\widetilde{\varphi}\otimes e_n=g,\\ (\mathcal{C}_q-\lambda)\widetilde{f}+\Gamma_0^0\widehat{f}_0\otimes e_n=\widetilde{g},\quad f_1=\Gamma_1^0\widehat{f}_0,\quad P\widetilde{f}_1=0,\quad (I-P)\widetilde{\varphi}=0.\end{cases} \tag{5.5}$$

As in the proof of Proposition 3.5 Now, one can solve $P\Gamma_0^0\widehat{f}_0$ from the third equality in (5.5) by means of the companion operator $\mathcal{C}_q$ (cf. [9]). Since $\lambda\in\rho(\mathcal{C}_{q^\sharp})$ the second equality in (5.5) gives $(I-P)\otimes f=(\mathcal{C}_{q^\sharp}-\lambda)^{-1}(I-P)\otimes g$. In particular, $(I-P)\Gamma_1^0\widehat{f}_0=(I-P)f_1$ and consequently $\widetilde{\Gamma}_0^0\widehat{f}_0$ has been solved. Let $\widetilde{\gamma}_\tau$ be the γ-field for (S_0,A_τ) associated with the boundary triplet $\{\mathcal{H},\widetilde{\Gamma}_0^0,\widetilde{\Gamma}_1^0\}$ in (5.4). Then one can write f_0 and f_0' in the form

$$f_0=(A_\tau-\lambda)^{-1}g_0+\widetilde{\gamma}_\tau(\lambda)\widetilde{\Gamma}_0^0\widehat{f}_0,\quad f_0'=\lambda f_0+g_0.$$

Now $\widetilde{f}$ can be solved from the third equation in (5.5) and, since $f_1=\Gamma_1^0\widehat{f}_0$, the vectors $(f_2,\dots,f_n)$ and $\widetilde{\varphi}$ can be solved from the second equality in (5.5). This proves $\rho(A_\tau)\cap\rho(\mathcal{C})\subset\rho(H_\tau)$.

To prove the reverse inclusion it is first shown that $\sigma(\mathcal{C}) \subset \sigma_p(H_\tau)$ holds for every $\tau \neq 0$. In view of

$$(\mathcal{C}_q - \lambda)\Lambda^\top h = (0, \dots, 0, -q(\lambda)h)^\top, \quad \lambda \in \mathbb{C},\ h \in \mathcal{H},$$

the eigenspace of $\mathcal{C}_q$ at λ is given by

$$\ker(\mathcal{C}_q - \lambda) = \{\, \Lambda^\top h :\ h \in \ker q(\lambda)\,\}. \tag{5.6}$$

Assume that $\lambda \in \sigma(\mathcal{C}_q)$. Since $\tau \neq 0$, one has $P \neq I$ and hence in view of (5.6) and the assumption $q = I_{\mathcal{H}} \otimes \widetilde{q}$ one can find $\widetilde{f} \neq 0$ such that $P\widetilde{f}_1 = 0$ and $(\mathcal{C}_q - \lambda)\widetilde{f} = 0$. It is easy to check that $(0, 0, \widetilde{f})^\top \in \ker(H_\tau - \lambda)$. Hence, $\sigma(\mathcal{C}_q) \subset \sigma_p(H_\tau)$ and by the symmetry of spectra $\sigma(\mathcal{C}_{q^\sharp}) \subset \sigma_p(H_\tau)$, so that $\sigma(\mathcal{C}) \subset \sigma_p(H_\tau)$.

Now, let $\lambda \in \rho(H_\tau)$ and let $g = \widetilde{g} = 0$. Then $\lambda \in \rho(\mathcal{C})$ and it follows from the second and the third equalities in (5.5) that

$$P\Gamma_0^0 \widehat{f}_0 = (I - P)\Gamma_1^0 \widehat{f}_0 = 0.$$

Therefore, $\widehat{f}_0 \in A_\tau$ and the first equality in (5.5) means that

$$\{f_0, g_0\} \in A_\tau - \lambda.$$

By assumption $\lambda \in \rho(H_\tau)$ and since $g_0 \in \mathfrak{H}_0$ is arbitrary it follows that $\lambda \in \rho(A_\tau)$. Therefore, $\rho(H_\tau) \subset \rho(A_\tau) \cap \rho(\mathcal{C})$.

(ii) The statement follows from the identity (with $\lambda \in \rho(H_\tau)$)

$$(H_\tau - \lambda)^{-1}(g_0, 0, 0)^\top = \Big((A_\tau - \lambda)^{-1} g_0, \Lambda^\top \Gamma_1^0 \widehat{f}_0, -(\mathcal{C}_q - \lambda)^{-1}(\Gamma_0^0 \widehat{f}_0 \otimes e_n)\Big)^\top.$$

(iii) Clearly, $\mathcal{L}$ is a neutral subspace of $\mathfrak{H}_0 \oplus \mathfrak{H}_Q$ with dimension dn, so that it is maximal neutral, cf. [4]. Moreover,

$$(H_\tau - \lambda)^{-1}(0, Pg, (I - P)\widetilde{g})^\top = (0, X_n P g, (\mathcal{C}_q - \lambda)^{-1}(I - P)\widetilde{g}),$$

where X_n stands for

$$X_n = \begin{pmatrix} 0 & 0 & \dots & 0 \\ I & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \lambda^{n-2} & \dots & I & 0 \end{pmatrix}.$$

This implies (5.2).

(iv) The transform of the boundary mappings in (5.4) corresponds to the following transform of the Weyl function $M = q^\sharp M_0 q$ (cf. [11]):

$$\begin{aligned} \widetilde{M}_\tau &= [(I - P) + PM][P - (I - P)M]^{-1} \\ &= \begin{pmatrix} \widetilde{q}^\sharp M_{11} \widetilde{q} & \widetilde{q}^\sharp M_{12} \widetilde{q} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -\widetilde{q}^{-1} M_{22}^{-1} M_{21} \widetilde{q} & -\widetilde{q}^{-1} M_{22}^{-1} \widetilde{q}^{-\sharp} \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{q}^\sharp (M_{11} - M_{12} M_{22}^{-1} M_{21}) \widetilde{q} & -\widetilde{q}^\sharp M_{12} M_{22}^{-1} \widetilde{q}^{-\sharp} \\ -\widetilde{q}^{-1} M_{22}^{-1} M_{21} \widetilde{q} & -\widetilde{q}^{-1} M_{22}^{-1} \widetilde{q}^{-\sharp} \end{pmatrix}, \end{aligned}$$

from which the statement follows. □

Recall that $\rho(A_\tau) \subset \rho(\widetilde{M}_{0,\tau})$ and $\rho(H_\tau) \subset \rho(\widetilde{M}_\tau)$, and that the inclusions are equalities if S_0 and S are simple. These properties are also reflected in (i) and (iv) of Theorem 5.1.

The proof of Theorem 5.1 gives also the following result, which shows the difference between the cases $\tau = 0$ and $\tau \neq 0$, cf. Proposition 3.5.

Corollary 5.2. *Let the assumptions be as in Theorem 5.1 and let τ be given by $\tau = \{\{Ph, (I-P)h\} : h \in \mathcal{H}\}$ for some orthogonal projection P in $\mathcal{H}$. If $\tau \neq 0$ (i.e. $P \neq I$) then $\sigma(q^\sharp q) \subset \sigma_p(H_\tau)$ and $\sigma(H_\tau) = \sigma(A_\tau) \cup \sigma(q^\sharp q)$.*

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