Working Papers of the University of Vaasa,
Department of Mathematics and Statistics
3

Shiga-Watanabe's time inversion property for self-similar diffusion processes

Juha Vuolle-Apiala

Preprint, December 2002

University of Vaasa
Department of Mathematics and Statistics
P.O. Box 700, FIN-65101 Vaasa, Finland
Preprints are available at: http://www.uwasa.fi/julkaisu/sis.html

SHIGA-WATANABE's TIME INVERSION PROPERTY FOR SELF-SIMILAR DIFFUSION PROCESSES

Juha Vuolle-Apiala
Department of Mathematics and Statistics, University of Vaasa, P. O. Box 700, 65101 Vaasa, Finland
e-mail: jmva@uwasa.fi

Summary: Assume (X_t, P^0) is 1/2-self-similar, rotation invariant diffusion on R^d, $d \geq 2$, starting at 0 and assume $\{0\}$ is a polar set. We will show, using the corresponding well-known result for the radial process, that Shiga-Watanabe's time inversion property holds for (X_t, P^0). The generalization for an α-self-similar, rotation invariant diffusion, $\alpha>0$, is also given.

Key words: time inversion, self-similar, diffusion, rotation invariant, skew product, radial process, spherical process

AMS Mathematics Subject Classification (2000): 60G18, 60J60, 60J25

0. Introduction. Theorem.

The following time inversion property is well known for Brownian motion in R^d, $d \geq 1$, and Bessel diffusions on $[0, \infty)$, starting at 0 (see Shiga, Watanabe (1973) and Watanabe (1975)):

(0.1) (X_t) has the same finite dimensional distributions as $(tX_{1/t})$ under P^0, for all $t>0$.

This was in Graversen, Vuolle-Apiala (2000) shown to be true also for symmetrized Bessel processes on R, starting at 0, in the case of the index $\nu \in (-1,0)$, that is, (X_t) can both hit 0 and can be started at 0. Symmetrized Bessel processes form the class of one dimensional rotation invariant 1/2-self similar diffusions (see the definition below). If the index $\nu \leq -1$ then 0 is an exit boundary point, that is, (X_t) can hit 0 but it cannot be started there. If $\nu \geq 0$ then 0 is an entrance boundary point, that is, (X_t) can be started there and it will never come back. Thus in this case (X_t) in fact lives either on $[0, \infty)$ or on $(-\infty, 0]$ and $\{0\}$ is a polar set. Obviously, (0.1) is valid. (0.1) has been generalized for α-self-similar diffusions on $[0, \infty)$ and for symmetric α-self-similar diffusions on R, $\alpha>0$, in Graversen, Vuolle-Apiala (2000). The corresponding generalization of (0.1) is then

(0.2) (X_t) has the same finite dimensional distributions as $(t^{2\alpha}X_{1/t})$ under P^0, for all $t>0$.

In this note we will show that (0.2) holds for all rotation invariant, d-dimensional, α- self-similar diffusions (that is, strong Markov processes with continuous paths), $d \geq 2$, $\alpha>0$, for which $\{0\}$ is a polar set. Our main tool is a skew product representation for rotation invariant diffusions starting at 0; see Itô, Mc Kean Jr., 1974, p. 274 - 276.

Let (X_t, P^x) be a rotation invariant (RI) α-self-similar (α-ss) diffusion on R^d, $d \geq 2$, $\alpha>0$, such that $\{0\}$ is a polar set. By α-self-similarity we mean that

(0.3) (X_t) under P^x has the same finite dimensional distributions as $(a^{-\alpha}X_{at})$ under $P^{a^{\alpha}x}$ for all $x \in R^d$, $a>0$

and by (RI) that

(0.4) (X_t) under P^x has the same finite dimensional distributions as $(T^{-1}(X_t))$ under $P^{T(x)}$ for all $T \in O(d)$.

Self-similar diffusions on R or on $[0, \infty)$ are defined similarly. Brownian motion fullfills both (0.3) and (0.4). See more about processes fullfilling (0.3) and (0.4) in Graversen, Vuolle-Apiala (1986), Lamperti (1972) and Vuolle-Apiala, Graversen (1986). According to Graversen, Vuolle-Apiala (1986) and Vuolle-Apiala (2002), when $X_0 \neq 0$ the diffusion processes which fullfill (0.3) and (0.4) can be represented as skew products

(0.5) $[|X_t|, \theta_{A_t}]$,

where $A_t = \lambda \int_0^t |X_s|^{-1/\alpha} ds$ for some $\lambda>0$, the radial part $(|X_t|)$ is an α-ss diffusion on $(0, \infty)$, and (θ_t) is a spherical Brownian motion on S^{d-1} independent of $(|X_t|)$.

Remark: As showed in Graversen, Vuolle-Apiala (1986), (0.5) is valid for all strong Markov processes with cadlag paths fullfilling (0.3) and (0.4). However, as showed by a counterexample by J. Bertoin, W. Werner (1996), the independence between $(|X_t|)$ and (θ_t) is not necessarily true if the paths are only right continuous. There is an error in the proof of Proposition 2.4, p.19-20 in Graversen, Vuolle-Apiala (1986). It was showed in Vuolle-Apiala (2002), Lemma 2.1, that $(|X_t|)$ and (θ_t) are independent in the case of continuous paths.

We want to prove the following:

Theorem: Let (X_t, P^0) be an (RI) α-ss diffusion on R^d, $\alpha>0$, $d \geq 2$, starting at 0, having $\{0\}$ as a polar set. Then the time inversion property (0.2) is valid.

1. The Proof of the Theorem

The proof will be based on

Proposition: Let (r_t) be an α-ss diffusion on $[0,\infty)$, $\alpha>0$, such that 0 is an entrance, non-exit boundary point. Then the skew product

(0.6) $[r_t, \theta_{\lambda \int_0^t r_s^{-1/\alpha} ds}]$, $t>0$, $r_0>0$, where

(θ_t, Q^θ) is a spherical Brownian motion on S^{d-1} independent of (r_t), such that $Q^\theta(\theta_0=\theta) = 1$ $\forall \theta \in S^{d-1}$, can be completed to be an α-ss diffusion on R^d by defining

(0.7) $[r_t, \nu_{\lambda \int_1^t r_s^{-1/\alpha} ds}]$, $t>0$, when $r_0=0$.

where (ν_t, Q) is an independent, spherical Brownian motion defined for $-\infty<t<+\infty$ and the law of (ν_0) is the uniform spherical distribution $m(d\theta)$.

Remark 1: Because of a uniqueness result of RI measures on S^{d-1} there is at most one way to complete (0.6) to be RI on the whole R^d.

Remark 2: It is obvious that (ν_t, Q) in fact is a stationary process and ν_t is uniformly distributed for all $t \in R$ (see Kuznetsov, 1973).

We have

(0.8) $$Q\{\nu_{t_1} \in d\theta_1, \dots, \nu_{t_n} \in d\theta_n\} = m(d\theta_1)Q^{\theta_1}(\theta_{t_2-t_1} \in d\theta_2) \dots Q^{\theta_{n-1}}(\theta_{t_n-t_{n-1}} \in d\theta_n)$$

for $-\infty<t_1< \dots <t_n<+\infty$.

In order to prove Proposition we need

Lemma 1: $(\nu_{t_1}, \dots, \nu_{t_n})$ under Q has the same distribution as $(\nu_{-t_1}, \dots, \nu_{-t_n})$.

Proof: Follows immediately from (0.8) and the fact that (θ_t, Q^θ) has a symmetric density with respect to the uniform measure $m(d\theta)$ on S^{d-1}(see Vuolle-Apiala, Graversen, 1986, Lemma 3, p.329). □

In the proof of Proposition we will use the result of Itô-McKean (1974), p. 275, which says that the skew product (0.6) can be completed to be a diffusion (which obviously is RI) on the whole R^d having the skew product (0.7) when $r_0=0$ iff $A_{0+} = \infty$ a.s P^0. Here we need

Lemma 2: Let (r_t) be an α-ss diffusion on $[0, \infty)$ such that 0 is an entrance, non-exit boundary point. Then

$$P^0\{\int_0^\epsilon r_s^{-1/\alpha} ds = \infty\} = 1 \quad \forall\epsilon>0.$$

Proof of Lemma 2: (0.3) implies that

$$P^0\{\int_0^\epsilon r_s^{-1/\alpha} ds = \infty\} = P^0\{\int_0^\epsilon (a^{-\alpha} r_{as})^{-1/\alpha} ds = \infty\} = P^0\{\int_0^{a\epsilon} r_s^{-1/\alpha} ds = \infty\}.$$

So it suffices to show that

$$P^0\{\int_0^\infty r_s^{-1/\alpha} ds = \infty\} = 1.$$

The Markov property gives

$$P^0\{\int_0^\infty r_s^{-1/\alpha} ds = \infty\} \geq P^0\{\int_t^\infty r_s^{-1/\alpha} ds = \infty\} = E^0\{ P^{r_t}\{\int_0^\infty r_s^{-1/\alpha} ds = \infty\}\} \quad \forall t>0.$$

Because 0 is an entrance, non-exit boundary point, $r_t>0$ a.s. (P^0). Now, according to Lamperti (1972),

$$P^r\{\int_0^\infty r_s^{-1/\alpha} ds = \infty\} = 1 \quad \text{for all } r>0$$

and thus

$$E^0\{P^{r_t}\{\int_0^\infty r_s^{-1/\alpha}ds = \infty\}\} = 1$$

which implies

$$P^0\{\int_0^\infty r_s^{-1/\alpha}ds = \infty\} = 1. \quad \square$$

Proof of Proposition: It only remains to prove that the skew product

(0.7) $$[r_t, \nu_{\lambda\int_1^t r_s^{-1/\alpha}ds}] \quad \text{when } r_0=0,$$

fullfills the α-self-similarity condition (0.3) under P^0. Let $I_1, \ldots, I_n$ be Borel subsets of $[0,\infty)$ and $J_1, \ldots, J_n$ Borel subsets of S^{d-1}. We will show

(*) $$P^0\{r_{t_1} \in I_1, \ldots, r_{t_n} \in I_n, \nu_{\lambda\int_1^{t_1} r_s^{-1/\alpha}ds} \in J_1, \ldots, \nu_{\lambda\int_1^{t_n} r_s^{-1/\alpha}ds} \in J_n\} =$$

$$P^0\{a^{-\alpha}r_{at_1} \in I_1, \ldots, a^{-\alpha}r_{at_n} \in I_n, \nu_{\lambda\int_1^{at_1} r_s^{-1/\alpha}ds} \in J_1, \ldots, \nu_{\lambda\int_1^{at_n} r_s^{-1/\alpha}ds} \in J_n\}$$

for all t>0.

For simplicity, assume n=2, the general case is analogeous.

Now the right hand side of (*) for n=2 is equal to

$$P^0\{a^{-\alpha}r_{at_1} \in I_1, a^{-\alpha}r_{at_2} \in I_2, \nu_{\lambda\int_{1/a}^{t_1} (a^{-\alpha}r_{as})^{-1/\alpha}ds} \in J_1, \nu_{\lambda\int_{1/a}^{t_2} (a^{-\alpha}r_s)^{-1/\alpha}ds} \in J_2\} =$$

$$P^0\{r_{t_1} \in I_1, r_{t_2} \in I_2, \nu_{\lambda\int_{1/a}^{t_1} r_s^{-1/\alpha}ds} \in J_1, \nu_{\lambda\int_{1/a}^{t_2} r_s^{-1/\alpha}ds} \in J_2\}$$

because (r_t) fullfills (0.3) and because of independence between $(r_.)$ and $(\nu_.)$.

This is further equal to

$$P^0\{r_{t_1} \in I_1, r_{t_2} \in I_2, \nu_{\lambda\int_{1/a}^{1} r_s^{-1/\alpha}ds + \lambda\int_1^{t_1} r_s^{-1/\alpha}ds} \in J_1, \nu_{\lambda\int_{1/a}^{1} r_s^{-1/\alpha}ds + \lambda\int_1^{t_2} r_s^{-1/\alpha}ds} \in J_2\} =$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} P^0\{r_{t_1} \in I_1, r_{t_2} \in I_2, \lambda\int_{1/a}^{1} r_s^{-1/\alpha}ds \in du, \lambda\int_1^{t_1} r_s^{-1/\alpha}ds \in dv, \lambda\int_1^{t_2} r_s^{-1/\alpha}ds \in dw,$$

$$\nu_{u+v} \in J_1, \nu_{u+w} \in J_2\} =$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} P^0\{r_{t_1} \in I_1, r_{t_2} \in I_2, \lambda\int_{1/a}^{1} r_s^{-1/\alpha} ds \in du, \lambda\int_{1}^{t_1} r_s^{-1/\alpha} ds \in dv, \lambda\int_{1}^{t_2} r_s^{-1/\alpha} ds \in dw\}$$

$$Q(\nu_{u+v} \in J_1, \nu_{u+w} \in J_2)$$

because of independence between $(r_.)$ and $(\nu_.)$. Now $(\nu_.)$ is a stationary process and thus this is equal to

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} P^0\{r_{t_1} \in I_1, r_{t_2} \in I_2, \lambda\int_{1/a}^{1} r_s^{-1/\alpha} ds \in du, \lambda\int_{1}^{t_1} r_s^{-1/\alpha} ds \in dv, \lambda\int_{1}^{t_2} r_s^{-1/\alpha} ds \in dw\}$$

$$Q(\nu_v \in J_1, \nu_w \in J_2) =$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} P^0\{r_{t_1} \in I_1, r_{t_2} \in I_2, \lambda\int_{1}^{t_1} r_s^{-1/\alpha} ds \in dv, \lambda\int_{1}^{t_2} r_s^{-1/\alpha} ds \in dw\} Q(\nu_v \in J_1, \nu_w \in J_2) =$$

$$P^0\ \{r_{t_1} \in I_1, r_{t_2} \in I_2, \nu_{\lambda\int_1^{t_1} r_s^{-1/\alpha}ds} \in J_1, \nu_{\lambda\int_1^{t_2} r_s^{-1/\alpha}ds} \in J_2\}. \qquad \square$$

Now we can prove Theorem:

<u>Proof of Theorem:</u> (X_t) has according to Graversen, Vuolle-Apiala (1986), Vuolle-Apiala (2002) and Proposition a skew product representation

$$[r_t, \theta_{\lambda\int_0^{t} r_s^{-1/\alpha}ds}] \quad \text{as } X_0 \neq 0 \tag{0.6}$$

and

$$[r_t, \nu_{\lambda\int_1^{t} r_s^{-1/\alpha}ds}] \quad \text{as } X_0 = 0, \tag{0.7}$$

where (r_t) is the radial process, (θ_t, Q^θ) is an independent spherical Brownian motion such that $Q^\theta(\theta_0=\theta)=1$ and (ν_h, Q) is an independent, stationary, spherical Brownian motion defined for $-\infty<h<+\infty$ and the law of (ν_h) is the uniform spherical distribution for all $h \in R$. To show (0.2) let us consider the distribution of $\{t_1^{2\alpha}X_{1/t_1}, \ldots, t_n^{2\alpha}X_{1/t_n}\}$ under P^0. Assume for simplicity n=2, α=1/2, the general case is analogeous. Let I_i and J_i, i=1,2, be Borel subsets of $(0, \infty)$ and S^{d-1}, respectively. Now

$$P^0\{t_1X_{1/t_1} \in (I_1, J_1), t_2X_{1/t_2} \in (I_2, J_2)\} =$$

$$P^0\ \{t_1r_{1/t_1} \in I_1, t_2r_{1/t_2} \in I_2, \nu_{\lambda\int_1^{1/t_1} r_s^2 ds} \in J_1, \nu_{\lambda\int_1^{1/t_2} r_s^2 ds} \in J_2\} =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P^0\,(t_1 r_{1/t_1} \in I_1,\, t_2 r_{1/t_2} \in I_2,\, \nu_u \in J_1,\, \nu_v \in J_2,\, \lambda\int_1^{1/t_1} r_s^{-2} ds \in du,\, \lambda\int_1^{1/t_2} r_s^{-2} ds \in dv) =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P^0\,(t_1 r_{1/t_1} \in I_1,\, t_2 r_{1/t_2} \in I_2,\, \lambda\int_1^{1/t_1} r_s^{-2} ds \in du,\, \lambda\int_1^{1/t_2} r_s^{-2} ds \in dv)\; Q(\nu_u \in J_1,\, \nu_v \in J_2)=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P^0(t_1 r_{1/t_1} \in I_1, t_2 r_{1/t_2} \in I_2, -\lambda\int_1^{t_1} (s r_{1/s})^{-2} ds \in du, -\lambda\int_1^{t_2} (s r_{1/s})^{-2} ds \in dv) Q(\nu_u \in J_1,\, \nu_v \in J_2).$$

Because (0.1) is true for (r_t) this is equal to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P^0\,(r_{t_1} \in I_1,\, r_{t_2} \in I_2,\, -\lambda\int_1^{t_1} r_s^{-2} ds \in du,\, -\lambda\int_1^{t_2} r_s^{-2} ds \in dv)\; Q(\nu_u \in J_1,\, \nu_v \in J_2) =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P^0\,(r_{t_1} \in I_1,\, r_{t_2} \in I_2,\, \lambda\int_1^{t_1} r_s^{-2} ds \in du,\, \lambda\int_1^{t_2} r_s^{-2} ds \in dv)\; Q(\nu_{-u} \in J_1,\, \nu_{-v} \in J_2).$$

Using Lemma 1 we get this equal to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P^0\,(r_{t_1} \in I_1,\, r_{t_2} \in I_2,\, \lambda\int_1^{t_1} r_s^{-2} ds \in du,\, \lambda\int_1^{t_2} r_s^{-2} ds \in dv)\; Q(\nu_u \in J_1,\, \nu_v \in J_2) =$$

$$P^0\,\{r_{t_1} \in I_1,\, r_{t_2} \in I_2,\, \nu_{\lambda\int_1^{t_1} r_s^{-2} ds} \in J_1,\, \nu_{\lambda\int_1^{t_2} r_s^{-2} ds} \in J_2\} = P^0\{X_{t_1} \in (I_1, J_1)\,,\, X_{t_2} \in (I_2, J_2)\}.$$

□

Remark: It would be interesting to know if the result still is true when {0} is not polar.

REFERENCES

Bertoin, J., Werner, W. (1996), Stable windings, Ann. Probab. 24, n:o 3, 1269-1279

Graversen, S.E., Vuolle-Apiala, J (1986), α-self-similar Markov processes, Probab. Th. Related Fields 71, n:o 1, 149-158

Graversen, S.E., Vuolle-Apiala, J (2000), On Paul Lévy's arc sine law and Shiga-Watanabe's time inversion result, Probab. Math. Statist. 20, n:o 1, 63-73

Itô, K., McKean Jr., H.P. (1974), Diffusion Processes and Their Sample Paths (Springer-Verlag Berlin Heidelberg New York)

Kuznetsov, S.E. (1973), Construction of Markov processes with random birth and death times, Theor. Probability Appl. 18, 571-575

Lamperti, J.W. (1972), Semi-stable Markov processes I, Z. Wahrschein. verw. Gebiete 22, 205-225

Shiga, T., Watanabe, S. (1973), Bessel diffusions as one parameter family of diffusion processes, Z. Wahrschein. verw. Gebiete 27, 37-46

Vuolle-Apiala, J., Graversen, S.E. (1986), Duality theory for self-similar processes, Ann. Inst. H. Poincaré Probab. Statist. 22, n:o 3, 323-332

Vuolle-Apiala, J. (2002), On certain extensions of a rotation invariant Markov process, submitted for publication

Watanabe, S. (1975), On time inversion of one-dimensional diffusion process, Z. Wahrscein. verw. Gebiete 31, 115-124