Working Papers of the University of Vaasa,
Department of Mathematics and Statistics
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Preprint, September 2002

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Preprints are available at: http://www.uwasa.fi/julkaisu/sis.html

SINGULAR PERTURBATIONS OF SELFADJOINT OPERATORS

VLADIMIR DERKACH, SEPPO HASSI, AND HENK DE SNOO

Abstract. Singular finite rank perturbations of an unbounded selfadjoint operator A_0 in a Hilbert space $\mathfrak{H}_0$ are defined formally as $A_{(\alpha)} = A_0 + G\alpha G^*$, where G is an injective linear mapping from $\mathcal{H} = \mathbb{C}^d$ to the scale space $\mathfrak{H}_{-k}(A_0)$, $k \in \mathbb{N}$, of generalized elements associated with the selfadjoint operator A_0, and where α is a selfadjoint operator in $\mathcal{H}$. The cases $k = 1$ and $k = 2$ have been studied extensively in the literature with applications to problems involving point interactions or zero range potentials. The scalar case with $k = 2n > 1$ has been considered recently by various authors from a mathematical point of view. In this paper singular finite rank perturbations $A_{(\alpha)}$ in the general setting $\operatorname{ran} G \subset \mathfrak{H}_{-k}(A_0)$, $k \in \mathbb{N}$, are studied by means of a recent operator model induced by a class of matrix polynomials. As an application singular perturbations of the Dirac operator are considered.

1. Introduction

Let A_0 be an unbounded selfadjoint operator in a Hilbert space $\mathfrak{H}_0$ and let $\mathfrak{H}_{+2}(A_0) \subset \mathfrak{H}_0 \subset \mathfrak{H}_{-2}(A_0)$ be the triplet of Hilbert spaces, where $\mathfrak{H}_{+2}(A_0)$ is $\operatorname{dom} A_0$ equipped with the graph inner product and where $\mathfrak{H}_{-2}(A_0)$ is the corresponding dual space of generalized elements, cf. [6]. Let G be an injective linear mapping from $\mathcal{H} = \mathbb{C}^d$ to $\mathfrak{H}_{-2}(A_0)$. For each selfadjoint operator α in $\mathcal{H}$ there is a (singular) finite rank perturbation $A_{(\alpha)}$ of A_0 formally given by

$$A_{(\alpha)} = A_0 + G\alpha G^*. \tag{1.1}$$

Such perturbations can be found in many areas, especially in the theory of point interactions or zero range potentials, see [1], [3], [40]. In order to give a meaning to $A_{(\alpha)}$ in (1.1) introduce a restriction S_0 of A_0 via

$$\operatorname{dom} S_0 = \operatorname{dom} A_0 \cap \ker G^*. \tag{1.2}$$

Then S_0 is a closed symmetric operator with defect numbers (d, d). A natural interpretation for the perturbation $A_{(\alpha)}$ in (1.1) is now as the selfadjoint extension of S_0 corresponding to the selfadjoint operator α in $\mathcal{H}$ via Kreĭn's formula [28], [42]. If the operator A_0 is semibounded and $\operatorname{ran} G \subset \mathfrak{H}_{-1}(A_0)$, the (singular) perturbation (1.1) is said to be form-bounded and the operator $A_{(\alpha)}$ can be constructed directly via the first representation theorem [32], [36], [42]. For an extension of this approach to the case of a nonsemibounded operator A_0, cf. [2], [20], [24], [26]. More general singular finite rank perturbations of A_0 where $\operatorname{ran} G$ belongs to the scale space $\mathfrak{H}_{-k}(A_0)$, $k > 2$, of generalized elements have received a lot of attention

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1991 *Mathematics Subject Classification.* Primary: 47A55, 47B25, 47B50; Secondary 34L40, 81Q10, 81Q15.

Key words and phrases. Singular finite rank perturbations, extension theory, Kreĭn's formula, boundary triplet, Weyl function, generalized Nevanlinna function, operator model.

The first author was partially supported by the Academy of Finland (project 52528) and the Dutch Association for Mathematical Physics (MF00/34). The second author was supported by the Academy of Finland (project 40362).

recently. For an extensive list of references, see [3]. Here $\mathfrak{H}_{-k}(A_0)$, $k \in \mathbb{N}$, is the dual space corresponding to the space $\mathfrak{H}_{+k}(A_0) = \operatorname{dom}|A_0|^{k/2}$ equipped with the graph norm. Singular perturbations with $k > 2$ cannot be treated in terms of the extension theory of the operator S_0 in the original space $\mathfrak{H}_0$, since now the restriction of A_0 to $\operatorname{dom} A_0 \cap \ker G^*$ is in general essentially selfadjoint. However, there exists an interpretation for the singular perturbations $A_{(\alpha)}$ in (1.1), in the general setting where $k > 2$ and $d \geq 1$, as exit space extensions of an appropriate restriction of A_0. These extensions act in a space which is a finite-dimensional extension of $\mathfrak{H}_0$. They are nonselfadjoint with respect to the underlying Hilbert space inner product, but become selfadjoint when a suitable Pontryagin space scalar product is introduced.

Singular rank one perturbations ($d = 1$) in the case $k = 2n+2$, $n \geq 1$, have been recently studied in [17], [18], [41]. The approach in these papers is based on a construction involving the Hilbert space $\mathfrak{H}_0$, a sequence of vectors in the scale spaces $\mathfrak{H}_{-2k}(A_0)$, $k = 0, 1, \dots, n+1$, and some auxiliary set of parameters in $\mathbb{C}$. After certain restrictions on these parameters, a Pontryagin space Π_n is constructed and the operator A_0 is lifted (in the notation of the present paper) to a selfadjoint relation H_0 in Π_n. Then a one-dimensional restriction S of H_0 in Π_n is introduced. These constructions are related to the model for generalized Nevanlinna functions in [31]. The Q-function M of the pair (S, H_0) is a generalized Nevanlinna function, cf. [35], which characterizes this pair up to unitary equivalence; it has a representation of the form

$$M = r + q^\sharp M_0 q, \tag{1.3}$$

cf. [18, Proposition 3.1]. Here $q(\lambda) = (\lambda - i)^n$, $q^\sharp(\lambda) = q(\bar{\lambda})^*$, r is a polynomial with real coefficients of degree at most $2n-1$, and M_0 is the Q-function of A_0 and a one-dimensional (densely defined) restriction S_0 of A_0, so that M_0 is an ordinary Nevanlinna function.

In the present paper singular finite rank perturbations of the form (1.1) are considered with G an injective linear mapping from $\mathcal{H} = \mathbb{C}^d$ to the space $\mathfrak{H}_{-2n-2}(A_0)$ or $\mathfrak{H}_{-2n-1}(A_0)$ with $n \geq 1$. These perturbations are interpreted by means of a general operator model which was given for a class of matrix polynomials in [13], see also [8]. The construction is as follows. Select an n-th order monic $d \times d$ matrix polynomial q, and define $G_0 = q(A_0)^{-1} G$. Then G_0 maps $\mathcal{H} = \mathbb{C}^d$ into $\mathfrak{H}_{-2}(A_0)$ or $\mathfrak{H}_{-1}(A_0)$, respectively. Introduce the restriction S_0 of A_0 to $\operatorname{dom} A_0 \cap \ker G_0^*$, so that S_0 is a closed symmetric operator in $\mathfrak{H}_0$ with defect numbers (d, d). The polynomial q, together with a selfadjoint $d \times d$ matrix polynomial r of degree at most $2n-1$, determine a matrix polynomial Q of the form

$$Q = \begin{pmatrix} 0 & q \\ q^\sharp & r \end{pmatrix}. \tag{1.4}$$

The function Q gives rise to a model involving a reproducing kernel Pontryagin space $\mathfrak{H}_Q$ and a corresponding multiplication operator S_Q in it, cf. [13]. Via G_0 the polynomial q determines the operator S_0 in $\mathfrak{H}_0$ and the coefficients of the polynomials q and r serve as parameters for the model space $\mathfrak{H}_Q$. The orthogonal coupling of the symmetric operator S_0 in the Hilbert space $\mathfrak{H}_0$ and the symmetric operator S_Q in the Pontryagin space $\mathfrak{H}_Q$ leads to a symmetric extension S of $S_0 \oplus S_Q$ and its selfadjoint extension H_0 in the Pontryagin space $\mathfrak{H}_0 \oplus \mathfrak{H}_Q$, such that the corresponding Weyl function M is given by (1.3), cf. [13]. The symmetric operator S associated to M is maximally nondensely defined in the sense that the dimension of the multivalued part of S^* is maximal, and the extension H_0 is the generalized Friedrichs extension of S in the sense of [10]. The selfadjoint parameters τ in $\mathcal{H}$ generate selfadjoint extensions H_τ of S in $\mathfrak{H}_0 \oplus \mathfrak{H}_Q$ via Kreĭn's formula relative to S and H_0. The pair (S, H_0) in $\mathfrak{H}_0 \oplus \mathfrak{H}_Q$ is the lifting of (S_0, A_0) in $\mathfrak{H}_0$. The singular perturbations $A_{(\alpha)}$ of A_0 in (1.1) are now "identified" with those extensions H_τ of S for which the parameter τ is a selfadjoint operator in $\mathcal{H}$. The motivation for this identification is obtained from a perturbation result for the extended resolvent acting in the rigging of $\mathfrak{H}_0$ generated by A_0 (see Theorem 4.8). Now the singular perturbations $A_{(\alpha)}$ can be seen as exit space extensions of S_0, whose compressed resolvents are characterized by the exit space version of Kreĭn's formula. This gives the connection between the singular finite rank perturbations $A_{(\alpha)}$ and the selfadjoint extensions H_τ as perturbations in $\mathfrak{H}_0 \oplus \mathfrak{H}_Q$ that were studied in [10], [11], [12]. Since the extensions H_τ are described by means of abstract boundary conditions for the adjoint S^* in $\mathfrak{H}_0 \oplus \mathfrak{H}_Q$ as well as via interface conditions for the adjoint S_0^* in $\mathfrak{H}_0$, the results in this paper are directly applicable for studying singular finite rank perturbations of differential operators.

In the case of rank one perturbations ($d = 1$) with $q(\lambda) = (\lambda - i)^n$, the model in this paper with $k = 2n+2$ is unitarily equivalent to the model in [18] since the Weyl function M coincides with the Q-function in [18]. For $n = 1$ a similar description for the model operator S, based on abstract boundary conditions, was given in [12, Theorem 3.1]. Therefore the results in [12] can be used to analyse singular rank one perturbations with $\operatorname{ran} G \subset \mathfrak{H}_{-3}$ or $\operatorname{ran} G \subset \mathfrak{H}_{-4}$; see also [38] for a different approach.

The paper is organized as follows. Some preliminary results are given in Section 2. They include necessary facts concerning boundary triplets, Weyl functions, and generalized resolvents of symmetric operators. In addition, the model concerning a class of matrix polynomials from [13] is briefly recalled. In Section 3 the factorization model from [13] is presented and the selfadjoint extensions H_τ of the model operator S are defined via abstract boundary conditions. The compressed resolvents $P_{\mathfrak{H}_0}(H_\tau - \lambda)^{-1} \upharpoonright \mathfrak{H}_0$, and the corresponding Štraus extensions in $\mathfrak{H}_0$ are described in terms of "interface conditions" which in general are λ-depending. Singular finite rank perturbations (1.1) of a selfadjoint operator A_0 are considered in Section 4. In the case where $\operatorname{ran} G \subset \mathfrak{H}_{-1}(A_0)$ or $\operatorname{ran} G \subset \mathfrak{H}_{-2}(A_0)$ the boundary triplets for S_0^* are expressed in terms of G and A_0. The general case $\operatorname{ran} G \subset \mathfrak{H}_{-2n-j}(A_0)$, $j = 1, 2$, is reduced to the previous two by replacing G by $G_0 = q(A_0)^{-1}G$. In Section 5 certain two-dimensional perturbations $A_0 + G\alpha G^*$ of the Dirac operator $A_0 = D$ with

$$D = -ic\frac{d}{dx} \otimes \sigma_1 + (c^2/2) \otimes \sigma_3, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are considered. In the case, where $Gh = \delta \otimes h$, $h \in \mathbb{C}^2$, an application of Theorem 3.1 leads to a description of the perturbations $A_{(\alpha)}$ in $\mathfrak{H}_{-2}(A_0)$,

$$y \in \operatorname{dom} A_{(\alpha)} \Leftrightarrow y(0+) = \Lambda y(0-),$$

where Λ is a linear-fractional transformation of α given by

$$\Lambda = (2ic\sigma_1 - \alpha)^{-1}(2ic\sigma_1 + \alpha).$$

This coincides with the descriptions of $A_{(\alpha)}$ in [1], [2], [3], [5], [21]. The case $Gh = -ic\delta' \otimes \sigma_1 h + (c^2/2)\delta \otimes \sigma_3 h$, $h \in \mathbb{C}^2$, leads to perturbations in $\mathfrak{H}_{-4}(A_0)$. Then the function $y = P_{\mathfrak{H}_0}(H_\tau - \lambda)^{-1}z$ is shown to be a solution of a boundary value problem with the λ-depending interface conditions of the form

$$y(0+) = \Lambda(\lambda) y(0-), \quad \Lambda(\lambda) = (2ic\sigma_1 - \lambda^2\tau)^{-1}(2ic\sigma_1 + \lambda^2\tau).$$

Some further applications of the model for singular perturbations will be studied elsewhere.

2. Preliminaries

The necessary ingredients for the present paper are briefly reviewed in this section. They involve the extension theory of symmetric linear relations in Pontryagin spaces, and the construction of operator models for a class of polynomials.

2.1. **Boundary triplets and abstract Weyl functions.** Let $\mathfrak{H}$ be a Pontryagin space with negative index κ, cf. [4]. Let S be a not necessarily densely defined closed symmetric relation in $\mathfrak{H}$ with equal defect numbers $d_+(S) = d_-(S) < \infty$ and let S^* be the adjoint linear relation of S. The symmetry of S can be expressed by $S \subset S^*$. Here and later operators will be identified with their graphs. In the rest of this paper $[\mathfrak{H}]$ stands for the set of all bounded everywhere defined linear operators in $\mathfrak{H}$. If T is a closed linear relation in $\mathfrak{H}$, i.e. $T \in \widetilde{\mathcal{C}}(\mathfrak{H})$, then $\operatorname{dom} T$, $\ker T$, $\operatorname{ran} T$, and $\operatorname{mul} T$ indicate the domain, kernel, range, and multivalued part of T, respectively. Moreover, $\rho(T)$ denotes the set of regular points of the linear relation T. Recall (see [23], [7]) that a triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of a Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} = n_\pm(S)$ and two linear mappings Γ_j, $j = 0, 1$, from S^* to $\mathcal{H}$ is called a *boundary triplet* for S^*, if $\Gamma = (\Gamma_0, \Gamma_1)^\top : \widehat{f} \to (\Gamma_0\widehat{f}, \Gamma_1\widehat{f})^\top$ is a surjective linear mapping from S^* onto $\mathcal{H} \oplus \mathcal{H}$ and the abstract Green's identity

$$(2.1) \quad (f', g) - (f, g') = (\Gamma_1\widehat{f}, \Gamma_0\widehat{g})_{\mathcal{H}} - (\Gamma_0\widehat{f}, \Gamma_1\widehat{g})_{\mathcal{H}} = i(\Gamma\widehat{g})^* J(\Gamma\widehat{f}), \quad J = \begin{pmatrix} 0 & -iI_{\mathcal{H}} \\ iI_{\mathcal{H}} & 0 \end{pmatrix},$$

holds for all $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in S^*$. The adjoint S^* of any closed symmetric relation S with equal defect numbers has a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$. Every other boundary triplet $\widetilde{\Pi} = \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is related to Π via a J-unitary transformation W: $\widetilde{\Gamma} = W\Gamma$. In particular, the *transposed boundary triplet* $\Pi^\top = \{\mathcal{H}, \Gamma_0^\top, \Gamma_1^\top\}$, is defined by $\Gamma^\top = iJ\Gamma$. When S is densely defined, S^* can be identified with its domain $\operatorname{dom} S^*$, in which case the boundary mappings are interpreted as mappings from $\operatorname{dom} S^*$ to $\mathcal{H}$.

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*. The mapping $\Gamma^\top : \widehat{f} \to \{\Gamma_1\widehat{f}, -\Gamma_0\widehat{f}\}$ from S^* onto $\mathcal{H} \oplus \mathcal{H}$ establishes a one-to-one correspondence between the set of all selfadjoint extensions of S and the set of all selfadjoint linear relations τ in $\mathcal{H}$ via

$$(2.2) \qquad A_\tau := \ker(\Gamma_0 + \tau\Gamma_1) = \{\widehat{f} \in S^* : \{\Gamma_1\widehat{f}, -\Gamma_0\widehat{f}\} \in \tau\} = \{\widehat{f} \in S^* : \Gamma^\top\widehat{f} \in \tau\}.$$

When the parameter τ is an operator in $\mathcal{H}$ the equation (2.2) takes the form

$$(2.3) \qquad \Gamma_0\widehat{f} + \tau\Gamma_1\widehat{f} = 0.$$

For $\tau = \infty$, meaning that $\tau^{-1} = 0$ or $\tau = \{0, I_{\mathcal{H}}\}$, the equation in (2.2) reads as $\Gamma_1\widehat{f} = 0$. More generally, there is a similar interpretation, when τ is decomposed orthogonally in terms of an operator part and a multivalued part. To each boundary triplet Π one may naturally associate two selfadjoint extensions of S by $A_0 = \ker \Gamma_0$, $A_1 (= A_\infty) = \ker \Gamma_1$, corresponding to the linear relations $\tau = 0$ and $\tau = \infty$ via (2.2).

Let $\mathfrak{N}_\lambda(S^*) = \ker(S^* - \lambda)$, $\lambda \in \widehat{\rho}(S)$, be the defect subspace of S and let $\widehat{\mathfrak{N}}_\lambda(S^*) := \{\{f_\lambda, \lambda f_\lambda\} : f_\lambda \in \mathfrak{N}_\lambda(S^*)\}$; here the notations $\mathfrak{N}_\lambda$ and $\widehat{\mathfrak{N}}_\lambda$ are used when the context is clear. Associated with the boundary triplet Π are two operator functions

$$(2.4)\quad \gamma(\lambda) = p_1(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda)^{-1} (\in [\mathcal{H}, \mathfrak{N}_\lambda]), \quad M(\lambda) = \Gamma_1(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda)^{-1} (\in [\mathcal{H}]), \quad \lambda \in \rho(A_0)\,(\neq \emptyset),$$

which are holomorphic on $\rho(A_0)$. Here p_1 denotes the orthogonal projection onto the first component of $\mathcal{H} \oplus \mathcal{H}$. The functions γ and M are called the *γ-field* and the *Weyl function* of S corresponding to the boundary triplet Π, cf. [7], [15], [16], [39] (or the Q-function corresponding to the pair (S, A_0), cf. [35]). The γ-field $\gamma^\top$ and the abstract Weyl function $M^\top$ corresponding to the transposed boundary triplet $\Pi^\top$ are related to γ and M via

$$\gamma^\top(\lambda) = \gamma(\lambda) M(\lambda)^{-1}, \quad M(\lambda)^\top = -M(\lambda)^{-1}, \quad \lambda \in \rho(A_1)\,(\neq \emptyset).$$

If $\mathfrak{H}$ is a Hilbert space, a Weyl function M of S is a so-called Nevanlinna function, that is, M is holomorphic in the upper halfplane $\mathbb{C}_+$, $\operatorname{Im} M(\lambda) \geq 0$ for all $\lambda \in \mathbb{C}_+$, and M satisfies the symmetry condition $M(\lambda)^* = M(\bar{\lambda})$ for $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$. In the case where $\mathfrak{H}$ is a Pontryagin space of negative index κ, the Weyl function M of S belongs to the class $\mathbf{N}_k$, $k \leq \kappa$, of *generalized Nevanlinna functions* which are meromorphic on $\mathbb{C}_+ \cup \mathbb{C}_-$, satisfy $M(\lambda)^* = M(\bar{\lambda})$, and for which the kernel

$$(2.5)\qquad \mathsf{N}_M(\lambda, \mu) = \frac{M(\lambda) - M(\bar{\mu})}{\lambda - \bar{\mu}}, \quad \mathsf{N}_M(\lambda, \bar{\lambda}) = \frac{d}{d\lambda} M(\lambda), \quad \lambda, \mu \in \mathbb{C}_+,$$

has k negative squares [35]. When S is *simple*, that is,

$$(2.6)\qquad \mathfrak{H} = \overline{\operatorname{span}}\, \{\, \mathfrak{N}_\lambda(S^*) : \lambda \in \rho(A_0)\,(\neq \emptyset) \,\},$$

then S is an operator without eigenvalues. Moreover, in this case the Weyl function M belongs to the class $\mathbf{N}_\kappa$, so that $k = \kappa$, and the domain of holomorphy $\rho(M)$ of M coincides with the resolvent set $\rho(A_0)$.

The resolvent of the extension A_τ and its spectrum $\sigma(A_\tau)$ can be expressed in terms of τ and the Weyl function M via Kreĭn's formula. In the terminology of boundary triplets the result can be formulated as follows, see [7], [15], [16].

Proposition 2.1. *Let S be a closed symmetric relation in the Pontryagin space $\mathfrak{H}$ with equal defect numbers (d, d), $d < \infty$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* with the Weyl function M, let τ be a linear relation in $\mathcal{H}$ connected with A_τ via* (2.2). *Then the resolvent of A_τ is given by*

$$(2.7)\qquad (A_\tau - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)(\tau^{-1} + M(\lambda))^{-1} \gamma(\bar{\lambda})^*, \quad \lambda \in \rho(A_\tau) \cap \rho(A_0).$$

Moreover, for every $\lambda \in \rho(A_0)$ the following equivalences hold:
(i) *$\lambda \in \rho(A_\tau)$ if and only if $\tau^{-1} + M(\lambda)$ is invertible;*
(ii) *$\lambda \in \sigma_p(A_\tau)$ if and only if $\ker(\tau^{-1} + M(\lambda))$ is nontrivial.*

In a similar way, for a (generalized) Nevanlinna family $\widetilde{\tau}(\lambda)$ the function

$$(A_0 - \lambda)^{-1} - \gamma(\lambda)(\widetilde{\tau}(\lambda) + M(\lambda))^{-1} \gamma(\bar{\lambda})^*,$$

is the compressed resolvent of an exit space extension of S in a Hilbert (or a Pontryagin) space, cf. [35], [43], [15], [39], [7], [9].

2.2. **A model for a class of matrix polynomials.** The construction of a model for a class of matrix polynomials as given in [13] is now briefly reviewed. Let q be a monic $d\times d$ matrix polynomial of the form

$$q(\lambda)=I_{\mathcal H}\lambda^n+q_{n-1}\lambda^{n-1}+\cdots+q_1\lambda+q_0, \tag{2.8}$$

and let r be a selfadjoint $d\times d$ matrix polynomial of the form

$$r(\lambda)=r_{2n-1}\lambda^{2n-1}+r_{2n-2}\lambda^{2n-2}+\cdots+r_1\lambda+r_0,\quad r_j=r_j^*,\quad j=0,\dots,2n-1. \tag{2.9}$$

Observe, that the function Q in

$$Q(\lambda)=\begin{pmatrix}0 & q(\lambda)\\ q^\sharp(\lambda) & r(\lambda)\end{pmatrix}, \tag{2.10}$$

is a $2d\times 2d$ matrix polynomial whose leading coefficient is, in general, noninvertible. Let the $n\times n$ block matrices $\mathcal B_q$ and $\mathcal C_q$ be defined by

$$\mathcal B_q=\begin{pmatrix} q_1 & q_2 & \dots & q_{n-1} & I_{\mathcal H}\\ q_2 & \dots & q_{n-1} & I_{\mathcal H} & 0\\ \vdots & \iddots & \iddots & 0 & 0\\ q_{n-1} & I_{\mathcal H} & \iddots & \iddots & \vdots\\ I_{\mathcal H} & 0 & 0 & \dots & 0\end{pmatrix},\quad \mathcal C_q=\begin{pmatrix} 0 & I_{\mathcal H} & 0 & \dots & 0\\ 0 & 0 & I_{\mathcal H} & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \dots & 0 & I_{\mathcal H}\\ -q_0 & -q_1 & \dots & -q_{n-2} & -q_{n-1}\end{pmatrix}. \tag{2.11}$$

Define the operators

$$\mathcal B=\begin{pmatrix}0 & \mathcal B_q\\ \mathcal B_{q^\sharp} & \mathcal B_r\end{pmatrix},\quad \mathcal C=\begin{pmatrix}\mathcal C_{q^\sharp} & \mathcal C_{12}\\ 0 & \mathcal C_q\end{pmatrix},\quad \mathcal B_r=(r_{j+k+1})_{j,k=0}^{n-1},\quad \mathcal C_{12}=\mathcal B_{q^\sharp}^{-1}\mathcal D, \tag{2.12}$$

where

$$\mathcal D=\begin{pmatrix} r_n\\ r_{n+1}\\ \vdots\\ r_{2n-1}\end{pmatrix}(q_0,q_1,\dots,q_{n-1})-\begin{pmatrix} I_{\mathcal H}\\ 0\\ \vdots\\ 0\end{pmatrix}(r_0,r_1,\dots,r_{n-1}). \tag{2.13}$$

Denote $\Lambda=(I_{\mathcal H},\lambda I_{\mathcal H},\dots,\lambda^{n-1}I_{\mathcal H})$, and define

$$\Lambda_1=\lambda^n\Lambda\widetilde{\mathcal B}_{(r)}\mathcal B_q^{-1},\quad \widetilde{\mathcal B}_{(r)}=\begin{pmatrix} r_{n+1} & \dots & r_{2n-1} & 0\\ \vdots & \iddots & 0 & 0\\ r_{2n-1} & \iddots & \iddots & \vdots\\ 0 & 0 & \dots & 0\end{pmatrix}. \tag{2.14}$$

In terms of these notions the kernel $\mathsf N_Q(\ell,\lambda)$ has the following factorization

$$\mathsf N_Q(\ell,\lambda)=\begin{pmatrix}L & 0\\ L_1 & L\end{pmatrix}\mathcal B\begin{pmatrix}\Lambda & 0\\ \Lambda_1 & \Lambda\end{pmatrix}^*, \tag{2.15}$$

where L and L_1 are defined similar to Λ and Λ_1. Hence, Q is a strict generalized matrix Nevanlinna function with dn negative (and dn positive) squares. The representation (2.15) leads to an explicit form for the reproducing kernel Pontryagin space $\mathfrak H(Q)$ associated with Q in (2.10) and the corresponding operator $S(Q)$ of multiplication by the independent variable in $\mathfrak H(Q)$, cf. [13].

Theorem 2.2. *Let the matrix polynomial Q be given by* (2.10) *with q and r as in* (2.8), (2.9)*. Let $\mathcal{B}$ and $\mathcal{C}$ be given by* (2.12)*. Then:*

(i) *The reproducing kernel Pontryagin space $\mathfrak{H}(Q)$ is isometrically isomorphic to the space $\mathfrak{H}_Q = \mathcal{H}^n \oplus \mathcal{H}^n$ $(= \mathbb{C}^{2dn})$ equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}_Q} = (\mathcal{B}\,\cdot, \cdot)$.*

(ii) *The operator $\mathcal{C}$ is selfadjoint in $\mathfrak{H}_Q$. Its restriction S_Q to the subspace*

$$\operatorname{dom} S_Q = \left\{ F = \begin{pmatrix} f \\ \widetilde{f} \end{pmatrix} \in \mathfrak{H}_Q : f_1 = \widetilde{f}_1 = 0 \right\}$$

is a closed simple symmetric operator in $\mathfrak{H}_Q$ with defect numbers $(2d, 2d)$, which is unitarily equivalent to $S(Q)$.

(iii) *The adjoint linear relation ${S_Q}^*$ of S_Q takes the form*

$${S_Q}^* = \left\{ \widehat{F} = \left\{ F, \mathcal{C}F + \mathcal{B}^{-1} \begin{pmatrix} \varphi \otimes e_1 \\ \widetilde{\varphi} \otimes e_1 \end{pmatrix} \right\} : F \in \mathfrak{H}_Q,\ \varphi, \widetilde{\varphi} \in \mathcal{H} \right\}.$$

(iv) *A boundary triplet $\Pi_Q = \{\mathcal{H} \oplus \mathcal{H}, \Gamma_0^Q, \Gamma_1^Q\}$ for ${S_Q}^*$ can be defined by*

$$\Gamma_0^Q \widehat{F} = \begin{pmatrix} f_1 \\ \widetilde{f}_1 \end{pmatrix}, \quad \Gamma_1^Q \widehat{F} = \begin{pmatrix} \varphi \\ \widetilde{\varphi} \end{pmatrix}, \quad \widehat{F} \in {S_Q}^*.$$

(v) *The Weyl function of S_Q associated with Π_Q coincides with Q and the corresponding γ-field is given by*

$$\gamma_Q(\lambda) h = \begin{pmatrix} \Lambda^\top & \Lambda_1^\top \\ 0 & \Lambda^\top \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad h_1, h_2 \in \mathcal{H}.$$

3. Construction of the factorization model

3.1. **The model for symmetric operator.** Let S_0 be a closed symmetric operator in a Hilbert space $\mathfrak{H}_0$ with defect numbers (d, d) whose Weyl function is M_0. Let S_Q be a symmetric operator in a Pontryagin space $\mathfrak{H}_Q$ with the Weyl function (2.10) where q and r are $d \times d$ matrix polynomials, q is monic and r is selfadjoint. In [13] a Pontryagin space symmetric linear relation S was constructed as a coupling of the operators S_0 and S_Q, such that the following function is a Weyl function for S:

$$M(\lambda) = r(\lambda) + q^\sharp(\lambda) M_0(\lambda) q(\lambda). \tag{3.1}$$

Theorem 3.1. ([13, Theorem 4.2]) *Let S_0 be a closed symmetric operator in the Hilbert space $\mathfrak{H}_0$ and let $\Pi^0 = \{\mathcal{H}, \Gamma_0^0, \Gamma_1^0\}$ be a boundary triplet for S_0^* with the Weyl function M_0 and the γ-field γ_0. Let S_Q be the symmetric operator in $\mathfrak{H}_Q$ as defined in Theorem 2.2 with the boundary triplet $\Pi_Q = \{\mathcal{H} \oplus \mathcal{H}, \Gamma_0^Q, \Gamma_1^Q\}$ and with q, r, and Q as in* (2.8), (2.9), *and* (2.10)*, respectively. Then:*

(i) *The linear relation*

$$S = \left\{ \left\{ f_0 \oplus \begin{pmatrix} f \\ \widetilde{f} \end{pmatrix}, f_0' \oplus \left(\mathcal{C} \begin{pmatrix} f \\ \widetilde{f} \end{pmatrix} + \mathcal{B}^{-1} \begin{pmatrix} \Gamma_0^0 \widehat{f}_0 \otimes e_1 \\ 0 \end{pmatrix} \right) \right\} \in S_0^* \oplus S_Q^* : \begin{matrix} f_1 = \Gamma_1^0 \widehat{f}_0 \\ \widetilde{f}_1 = 0 \end{matrix} \right\}$$

is closed and symmetric in $\mathfrak{H}_0 \oplus \mathfrak{H}_Q$ and has defect numbers (d, d).

(ii) *The adjoint S^* is given by*

$$S^* = \left\{ \left\{ f_0 \oplus \begin{pmatrix} f \\ \widetilde{f} \end{pmatrix}, f_0' \oplus \left(\mathcal{C} \begin{pmatrix} f \\ \widetilde{f} \end{pmatrix} + \mathcal{B}^{-1} \begin{pmatrix} \Gamma_0^0 \widehat{f}_0 \otimes e_1 \\ \widetilde{\varphi} \otimes e_1 \end{pmatrix} \right) \right\} \in S_0^* \oplus S_Q^* : \begin{array}{c} f_1 = \Gamma_1^0 \widehat{f}_0 \\ \widetilde{\varphi} \in \mathcal{H} \end{array} \right\}.$$

(iii) *A boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for S^* is determined by*

$$\Gamma_0(\widehat{f}_0 \oplus \widehat{F}) = \widetilde{f}_1, \quad \Gamma_1(\widehat{f}_0 \oplus \widehat{F}) = \widetilde{\varphi}, \quad \widehat{f}_0 \oplus \widehat{F} \in S^*.$$

(iv) *The corresponding Weyl function M is of the form* (3.1) *and the γ-field γ is given by*

$$\gamma(\lambda)h = \gamma_0(\lambda)q(\lambda)h \oplus ((\Lambda^\top M_0(\lambda)q(\lambda) + \Lambda_1^\top)h \dotplus \Lambda^\top h), \quad h \in \mathcal{H}. \tag{3.2}$$

If the operator S_0 is densely defined in $\mathfrak{H}_0$, then S is an operator. When $r = 0$ the formulas for S and S^* in Theorem 3.1 can be simplified and the Weyl function is factorized as

$$M(\lambda) = q^\sharp(\lambda) M_0(\lambda) q(\lambda). \tag{3.3}$$

Theorem 3.1 was obtained earlier in [12, Section 3] in the special case that $d = n = 1$ and $q(\lambda) = \lambda - \alpha$, $\alpha \in \mathbb{C}$. The problem of simplicity of the model operator S was investigated in [12, 13].

3.2. **Selfadjoint extensions of the model operator.** The model in Theorem 3.1 leads to an explicit form for the extension $H_\tau = \ker(\Gamma_0 + \tau\Gamma_1)$.

Proposition 3.2. *Let the assumptions be as in Theorem 3.1, and let γ and M be given by* (3.2) *and* (3.1), *respectively. Then:*

(i) *The selfadjoint extensions H_τ of S in $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_Q$ are in a one-to-one correspondence with the selfadjoint relations τ in $\mathcal{H}$ via*

$$H_\tau = \left\{ \left\{ f_0 \oplus \begin{pmatrix} f \\ \widetilde{f} \end{pmatrix}, f_0' \oplus \left(\mathcal{C} \begin{pmatrix} f \\ \widetilde{f} \end{pmatrix} + \mathcal{B}^{-1} \begin{pmatrix} \Gamma_0^0 \widehat{f}_0 \otimes e_1 \\ \widetilde{\varphi} \otimes e_1 \end{pmatrix} \right) \right\} \in S_0^* \oplus S_Q^* : \begin{array}{c} f_1 = \Gamma_1^0 \widehat{f}_0 \\ \widetilde{f}_1 + \tau\widetilde{\varphi} = 0 \end{array} \right\}.$$

(ii) *The resolvent $(H_\tau - \lambda)^{-1}$ is given by*

$$(H_\tau - \lambda)^{-1} = (H_0 - \lambda)^{-1} - \gamma(\lambda)(\tau^{-1} + M(\lambda))^{-1}\gamma(\bar{\lambda})^*, \quad \lambda \in \rho(H_\tau) \cap \rho(H_0). \tag{3.4}$$

(iii) *For every $\lambda \in \rho(H_0)$ the following equivalences hold:*

$$\begin{aligned} \lambda \in \sigma_p(H_\tau) &\Leftrightarrow 0 \in \sigma_p(\tau^{-1} + M(\lambda)), \\ \lambda \in \rho(H_\tau) &\Leftrightarrow 0 \in \rho(\tau^{-1} + M(\lambda)). \end{aligned}$$

Proof. By part (iii) of Theorem 3.1 the condition $\widehat{f}_0 \oplus \widehat{F} \in \ker(\Gamma_0 + \tau\Gamma_1)$ is equivalent to $\{\widetilde{\varphi}, \widetilde{f}_1\} \in -\tau$, or to $\widetilde{f}_1 + \tau\widetilde{\varphi} = 0$, when correctly interpreted if τ is multivalued. The representation of H_τ now follows from (ii) of Theorem 3.1. This proves (i). The form of the resolvent of H_τ in (ii) is obtained from Proposition 2.1 and Theorem 3.1. The statement (iii) is immediate from Proposition 2.1. □

Define the block matrices

$$X_n = \begin{pmatrix} 0 & 0 & \dots & 0 \\ I & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \lambda^{n-2} & \dots & I & 0 \end{pmatrix}, \quad X_{n-1} = \begin{pmatrix} I & \dots & 0 \\ \vdots & \ddots & \vdots \\ \lambda^{n-2} & \dots & I \end{pmatrix}. \tag{3.5}$$

The following properties of the companion matrix $\mathcal{C}_q$ are useful and easily checked, e.g. the last one is a simple corollary of the Frobenius formula.

Lemma 3.3. *Let $\mathcal{C}_q$ be the companion matrix corresponding to the polynomial q of the form (2.8). Then:*

(i) $(\mathcal{C}_q-\lambda)\Lambda_{(n)}^\top h=(0,\dots,0,-q(\lambda)h)^\top$ *for all* $\lambda\in\mathbb{C}$, $h\in\mathcal{H}$;

(ii) $\sigma(\mathcal{C}_q)=\sigma(q)$ *and* $\ker(\mathcal{C}_q-\lambda)=\{\Lambda_{(n)}^\top h : h\in\ker q(\lambda)\}$;

(iii) $(\mathcal{C}_q-\lambda)X_n=I_{\mathcal{H}^n}-\begin{pmatrix}0 & 0\\ \widetilde{q}_\lambda X_{n-1} & I\end{pmatrix}$, *where* $\widetilde{q}_\lambda=(q_1,\dots,q_{n-2},q_{n-1}+\lambda)$;

(iv) *For every* $\lambda\in\mathbb{C}\setminus\sigma(q)$, $g\in\mathcal{H}^n$,

$$(\mathcal{C}_q-\lambda)^{-1}g=X_ng-\frac{1}{q(\lambda)}\Lambda_{(n)}^\top(g_n+\widetilde{q}_\lambda X_{n-1}(g_1,\dots,g_{n-1})^\top). \tag{3.6}$$

Proposition 3.4. *Let the assumptions be as in Theorem 3.1 and let $H_0=\ker\Gamma_0$ be as in Proposition 3.2 (with $\tau=0$). Then:*

(i) $\rho(H_0)=\rho(A_0)$;

(ii) *the compression of the resolvent of H_0 to the subspace $\mathfrak{H}_0$ is given by*

$$P_{\mathfrak{H}_0}(H_0-\lambda)^{-1}\restriction\mathfrak{H}_0=(A_0-\lambda)^{-1},\quad \lambda\in\rho(H_0); \tag{3.7}$$

(iii) *the subspace $\mathcal{L}=\{0\}\oplus\mathcal{H}^n\oplus\{0\}$ of $\mathfrak{H}=\mathfrak{H}_0\oplus\mathfrak{H}_Q$ is maximal neutral and invariant under the resolvent $(H_0-\lambda)^{-1}$. It satisfies $(H_0-\lambda)^{-n}\mathcal{L}=\{0\}$, $\lambda\in\rho(H_0)$.*

Proof. (i) Let $G=(g_0,g,\widetilde{g})^\top\in\mathfrak{H}$ and let $\widehat{f}_0=\{f_0,f_0'\}\in S_0^*$. By Proposition 3.2 the relation $G\in\operatorname{ran}(H_0-\lambda)$ can be rewritten as a system of equalities

$$\begin{cases} f_0'-\lambda f_0=g_0,\\ (\mathcal{C}_{q^\sharp}-\lambda)f+\mathcal{C}_{12}\widetilde{f}+\widetilde{\varphi}\otimes e_n-\mathcal{B}_{q^\sharp}^{-1}\mathcal{B}_r(\Gamma_0^0\widehat{f}_0\otimes e_n)=g,\\ (\mathcal{C}_q-\lambda)\widetilde{f}+\Gamma_0^0\widehat{f}_0\otimes e_n=\widetilde{g},\quad f_1=\Gamma_1^0\widehat{f}_0,\quad \widetilde{f}_1=0.\end{cases} \tag{3.8}$$

Since $\widetilde{f}_1=0$ the third identity in (3.8) and Lemma 3.3 (iii) yield

$$(\widetilde{f}_2,\dots,\widetilde{f}_n)^\top=X_{n-1}(\widetilde{g}_1,\dots,\widetilde{g}_{n-1})^\top, \tag{3.9}$$

$$\Gamma_0^0\widehat{f}_0=\widetilde{g}_n+\sum_{j=1}^{n-1}q_j\widetilde{f}_{j+1}+\lambda\widetilde{f}_n. \tag{3.10}$$

Clearly, $\widehat{h}_0=\widehat{f}_0-\widehat{\gamma}_0(\lambda)\Gamma_0^0\widehat{f}_0\in A_0$. The first equality in (3.8) implies $h_0'-\lambda h_0=f_0'-\lambda f_0=g_0$. This means that $\{h_0,g_0\}\in A_0-\lambda$, or equivalently, that $\{g_0,h_0\}\in(A_0-\lambda)^{-1}$. Now assume that $\lambda\in\rho(A_0)$. Then $h_0=(A_0-\lambda)^{-1}g_0$ and

$$f_0=(A_0-\lambda)^{-1}g_0+\gamma_0(\lambda)\Gamma_0^0\widehat{f}_0,\quad f_0'=\lambda f_0+g_0. \tag{3.11}$$

The second equality in (3.8) can be rewritten as

$$(\mathcal{C}_{q^\sharp}-\lambda)f+\widetilde{\varphi}\otimes e_n=k, \tag{3.12}$$

where $k=g-\mathcal{C}_{12}\widetilde{f}+\mathcal{B}_{q^\sharp}^{-1}\mathcal{B}_r(\Gamma_0^0\widehat{f}_0\otimes e_n)$. Using $f_1=\Gamma_1^0\widehat{f}_0$ and applying Lemma 3.3 (i), (iii) to (3.12) one obtains

$$(f_2,\dots,f_n)^\top=X_{n-1}(k_1,\dots,k_{n-1})^\top+\lambda\Lambda_{(n-1)}^\top\Gamma_1^0\widehat{f}_0, \tag{3.13}$$

$$\widetilde{\varphi} = k_n + \sum_{j=0}^{n-1} q_j^* f_{j+1} + \lambda f_n. \tag{3.14}$$

This shows that $\lambda \in \rho(H_0)$, and thus $\rho(A_0) \subset \rho(H_0)$.

Conversely, assume that $\lambda \in \rho(H_0)$. Then with $G = (g_0, 0, 0)^\top$ one obtains from the third identity in (3.8) and Lemma 3.3 (iii) that $\widetilde{f} = 0$ and $\Gamma_0^0 \widehat{f}_0 = 0$. Now the first identity in (3.8) gives $\operatorname{ran}(A_0 - \lambda) = \mathfrak{H}_0$ and, therefore, $\lambda \in \rho(A_0)$. In fact, Lemma 3.3 (i) yields

$$(H_0 - \lambda)^{-1}(g_0, 0, 0)^\top = ((A_0 - \lambda)^{-1} g_0, \Lambda_{(n)}^\top \Gamma_1^0 \widehat{f}_0, 0)^\top. \tag{3.15}$$

(ii) The equality (3.7) follows immediately from (3.15).

(iii) Clearly, $\mathcal{L}$ is a neutral subspace of $\mathfrak{H}_0 \oplus \mathfrak{H}_Q$ and has dimension dn, so that it is maximal neutral, cf. [4]. Moreover, again using Lemma 3.3 (iii) one obtains from (3.8) that for $G = (0, g, 0)^\top \in \mathcal{L}$,

$$(H_0 - \lambda)^{-1}(0, g, 0)^\top = (0, X_n g, 0)^\top, \quad (H_0 - \lambda)^{-n}(0, g, 0)^\top = (0, X_n^n g, 0)^\top = 0.$$

□

A more complete description of the structure of root subspaces in the scalar case can be found in [12]. The selfadjoint extensions $H_\tau = \ker(\Gamma_0 + \tau\Gamma_1)$ of S described in Proposition 3.2 can be interpreted as standard *range perturbations* of the selfadjoint extension $H_\infty = \ker \Gamma_1$ in the Pontryagin space $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_Q$, see [14]; cf. also [27], [29] for the Hilbert space case. These perturbations can be seen as liftings of the singular perturbations $A_{(\alpha)}$ of A_0 from $\mathfrak{H}_0$ to the extended space $\mathfrak{H}$, cf. Corollary 3.6. Various properties of range perturbations in a Pontryagin space setting were considered in [10], [11], [12]. A more detailed study of this connection leads to intermediate symmetric extensions of S and their generalized Friedrichs extensions which can be described by means of so-called extremal boundary conditions, cf. [14].

3.3. **Štraus extensions.** Let S_0 be a closed symmetric operator in $\mathfrak{H}_0$ and let H be a selfadjoint extension of S_0 in an exit space $\mathfrak{H}$ ($\supset \mathfrak{H}_0$). A family $\{ T(\lambda) : \lambda \in \mathbb{C} \}$ of extensions of S_0 in the original space $\mathfrak{H}_0$ defined by

$$T(\lambda) = \{ \{P_{\mathfrak{H}_0} f, P_{\mathfrak{H}_0} f'\} : \{f, f'\} \in H,\ f' - \lambda f \in \mathfrak{H}_0 \} \tag{3.16}$$

is called the family of Štraus extensions of S_0 corresponding to the selfadjoint extension H, cf. [43], [19], [9]. Recall that $S_0 \subset T(\lambda) \subset S_0^*$ for all $\lambda \in \mathbb{C}$. It follows from (3.16) that the compressed resolvent of H can be expressed by means of the family $T(\lambda)$ as follows:

$$P_{\mathfrak{H}_0}(H - \lambda)^{-1} {\upharpoonright} \mathfrak{H}_0 = (T(\lambda) - \lambda)^{-1}, \quad \lambda \in \rho(H). \tag{3.17}$$

In fact, the family of Štraus extensions can be characterized in terms of boundary operators. Let $\{\mathcal{H}, \Gamma_0^0, \Gamma_1^0\}$ be a boundary triplet for S_0^* and let the extension H of S_0 be related to a generalized Nevanlinna family $\widetilde{\tau}$ via Kreĭn's formula

(3.18)
$$P_{\mathfrak{H}_0}(H - \lambda)^{-1} {\upharpoonright} \mathfrak{H}_0 = (A_0 - \lambda)^{-1} - \gamma_0(\lambda)(\widetilde{\tau}(\lambda) + M_0(\lambda))^{-1}\gamma_0(\bar{\lambda})^*, \quad \lambda \in \rho(H) \cap \rho(A_0).$$

Then the family $T(\lambda)$ of Štraus extensions is given by the equality

$$\Gamma^0 T(\lambda) = \left\{ \{\Gamma_0^0 \widehat{f}_0, \Gamma_1^0 \widehat{f}_0 \} : \widehat{f}_0 \in T(\lambda) \right\} = -\widetilde{\tau}(\lambda), \tag{3.19}$$

see [16], [7].

Theorem 3.5. *Let the assumptions be as in Theorem 3.1. Then the compressed resolvent and the Štraus family* $T_\tau(\lambda)$ *of the extension* H_τ *in Proposition 3.2 are given by*

$$P_{\mathfrak{H}_0}(H_\tau-\lambda)^{-1}\upharpoonright\mathfrak{H}_0=(A_0-\lambda)^{-1}-\gamma_0(\lambda)\left(\widetilde{\tau}(\lambda)^{-1}+M_0(\lambda)\right)^{-1}\gamma_0(\bar{\lambda})^*, \tag{3.20}$$

and

$$T_\tau(\lambda)=\left\{\,\widehat{f}_0=\{f_0,f_0'\}\in S_0^*:\,(\Gamma_0^0+\widetilde{\tau}(\lambda)\Gamma_1^0)\widehat{f}_0=0\,\right\},\quad \lambda\in\rho(H_\tau)\cap\rho(A_0), \tag{3.21}$$

where $\widetilde{\tau}(\lambda)=q(\lambda)(\tau^{-1}+r(\lambda))^{-1}q^\sharp(\lambda)$.

Proof. The resolvent of H_τ is given by (3.4) in Proposition 3.2. In view of the identity (3.7) and the form of the γ-field in (3.2) the compression of this formula to $\mathfrak{H}_0$ gives

$$\begin{aligned}P_{\mathfrak{H}_0}(H_\tau-\lambda)^{-1}\upharpoonright\mathfrak{H}_0&=P_{\mathfrak{H}_0}(H_0-\lambda)^{-1}\upharpoonright\mathfrak{H}_0-P_{\mathfrak{H}_0}\gamma(\lambda)\left(\tau^{-1}+M(\lambda)\right)^{-1}\gamma(\bar{\lambda})^*\upharpoonright\mathfrak{H}_0\\&=(A_0-\lambda)^{-1}-\gamma_0(\lambda)q(\lambda)\left(\tau^{-1}+M(\lambda)\right)^{-1}q^\sharp(\lambda)\gamma_0(\bar{\lambda})^*.\end{aligned}$$

Taking into account (3.1) this leads to (3.20) with $\widetilde{\tau}=q(\tau^{-1}+r)^{-1}q^\sharp$. The second statement follows now from (3.19), since

$$T_\tau(\lambda)=\left\{\,\widehat{f}_0\in S_0^*:\,\{\,\Gamma_0^0\widehat{f}_0,\Gamma_1^0\widehat{f}_0\,\}\in-\widetilde{\tau}(\lambda)^{-1}\,\right\},$$

and this coincides with (3.21). □

The next result gives a connection between the selfadjoint extensions of S in $\mathfrak{H}_0\oplus\mathfrak{H}_Q$ and the selfadjoint extensions of S_0 in $\mathfrak{H}_0$. A similar result was obtained in [29, Theorem 3.2] in a simpler situation.

Corollary 3.6. *The selfadjoint extensions* H_τ *of* S *in* $\mathfrak{H}_0\oplus\mathfrak{H}_Q$ *and the selfadjoint extensions* $A_{\widetilde{\tau}}$ *of* S_0 *in* $\mathfrak{H}_0$ *are connected by*

$$A_{\widetilde{\tau}}=\ker(\Gamma_0^0+\widetilde{\tau}\Gamma_1^0)=\{\,\{P_{\mathfrak{H}_0}F,G\}:\,\{F,G\}\in H_\tau,\ G\in\mathfrak{H}_0\,\},$$

where $\widetilde{\tau}=q_0(\tau^{-1}+r_0)^{-1}q_0^*$ *and this product is understood in the sense of relations.*

Proof. When $0\in\rho(H_\tau)\cap\rho(A_0)$ this result follows directly from Theorem 3.5. To prove it in the general case one can proceed as in the proof of Proposition 3.4. Consider the first three equalities in (3.8) with $\lambda=0$ and $g=\widetilde{g}=0$. Then it follows from the third equality in (3.8) that $\Gamma_0^0\widehat{f}_0=q_0\widetilde{f}_1$ and $\widetilde{f}_2=\cdots=\widetilde{f}_n=0$. Next a simple calculation using (2.12), (2.13) shows that

$$k=-\mathcal{C}_{12}\widetilde{f}+\mathcal{B}_{q^\sharp}^{-1}\mathcal{B}_r(\Gamma_0^0\widehat{f}_0\otimes e_n)=r_0\widetilde{f}_1\otimes e_n.$$

Now the second equality in (3.8), or equivalently (3.12), implies that $\mathcal{C}_{q^\sharp}f=(r_0\widetilde{f}_1-\widetilde{\varphi})\otimes e_n$. This gives $q_0^*f_1=\widetilde{\varphi}-r_0\widetilde{f}_1$ and $f_2=\cdots=f_n=0$. Hence, together with the description of H_τ in Proposition 3.2, one arrives at the following conditions for $\widehat{f}_0$:

$$\Gamma_0^0\widehat{f}_0=q_0\widetilde{f}_1,\quad q_0^*\Gamma_1^0\widehat{f}_0=\widetilde{\varphi}-r_0\widetilde{f}_1,\quad \{\widetilde{\varphi},\widetilde{f}_1\}\in-\tau.$$

It can be checked that these three conditions are equivalent to

$$\{\Gamma_1^0\widehat{f}_0,\Gamma_0^0\widehat{f}_0\}\in-q_0(\tau^{-1}+r_0)^{-1}q_0^*.$$

The linear relation $\widetilde{\tau} = q_0(\tau^{-1} + r_0)^{-1}q_0^*$ (where the products and inverses are to be understood in the sense of relations) is selfadjoint. Therefore, $A_{\widetilde{\tau}}$ is a selfadjoint extension of S_0 and the claim follows. □

Of course, when $r_0 = 0$ and $q_0 = I$ the "inverse compression" of H_τ in Corollary 3.6 gives the extension A_τ with precisely the same parameter $\widetilde{\tau} = \tau$. In this sense the selfadjoint extensions H_τ of S can be seen as *liftings* of the selfadjoint extensions of S_0.

According to (3.20) the exit space for S_0 is determined by the $d \times d$ matrix function $\widetilde{\tau}^{-1} = q^{-\sharp}(\tau^{-1} + r)q^{-1}$. This observation yields another construction of the model space associated with M. Namely, one may use the coupling methods as presented in [9], [25] of the model spaces corresponding to the sum of two Nevanlinna functions M_0 and $\widetilde{\tau}^{-1}$. Here the degree of the rational matrix function $\widetilde{\tau}^{-1}$ is equal to $2n$ and therefore the corresponding exit space will have the dimension $2nd$. However, it is not clear if the exit spaces $\mathfrak{H}_{\widetilde{\tau}^{-1}}$ can be taken to be equal for different values of $\tau \in \widetilde{\mathcal{C}}(\mathcal{H})$. In the present approach the situation is different. To see this observe that $\widetilde{\tau}^{-1} = q^{-\sharp}(\tau^{-1} + r)q^{-1}$ is obtained from Q given in (2.10) by using a Schur complement and a transposed boundary triplet via the following steps,

$$Q \to \begin{pmatrix} 0 & q \\ q^\sharp & r + \tau^{-1} \end{pmatrix} \to -q(r + \tau^{-1})^{-1}q^\sharp \to \widetilde{\tau}^{-1} = q^{-\sharp}(r + \tau^{-1})q^{-1}.$$

This shows that for each τ the exit space determined by $\widetilde{\tau}^{-1}$ can be taken to be $\mathfrak{H}_Q$. Moreover, the model operator $S_{\widetilde{\tau}^{-1}}$ for $\widetilde{\tau}^{-1}$ is a closed symmetric extension of the model operator S_Q in Theorem 2.2 with smaller defect numbers (nd, nd) in $\mathfrak{H}_Q$.

4. Singular finite rank perturbations

Let A_0 be a selfadjoint operator in the Hilbert space $\mathfrak{H}_0$ and let G be a linear injective mapping from $\mathcal{H} = \mathbb{C}^d$ into $\mathfrak{H}_0$. For a $d \times d$ matrix $\alpha = \alpha^*$ define the operator $A_{(\alpha)}$ by (1.1) so that $A_{(\alpha)}$ is a finite rank perturbation of A_0, cf. e.g. [32]. Let S_0 be the restriction of A_0 defined by (1.2). Then S_0 is a closed, symmetric, and nondensely defined operator with defect numbers (d, d). Its adjoint S_0^* is a closed linear relation, given by

$$S_0^* = \{\, \widehat{f} = \{f, A_0 f - Gh\} : \, f \in \operatorname{dom} A_0, \, h \in \mathcal{H} \,\}. \tag{4.1}$$

A boundary triplet for S_0^* can be defined by

$$\mathcal{H} = \mathbb{C}^d, \quad \Gamma_0^0 \widehat{f} = h, \quad \Gamma_1^0 \widehat{f} = G^* f, \quad \widehat{f} \in S_0^*, \tag{4.2}$$

where Γ_0^0 is well defined, since $\ker G = \{0\}$. The corresponding γ-field and the Weyl function are given by

$$\gamma_0(\lambda) = (A_0 - \lambda)^{-1}G, \quad M_0(\lambda) = G^*(A_0 - \lambda)^{-1}G, \quad \lambda \in \rho(A_0). \tag{4.3}$$

The perturbations $A_{(\alpha)}$ in (1.1) are now selfadjoint operator extensions of S_0. The Weyl function characterizes A_0 and S_0, up to unitary equivalence, cf. [35]. It also can be used to describe the spectrum of each perturbation $A_{(\alpha)}$, cf. Proposition 2.1. Such and more general perturbations have been considered in several recent papers, see e.g. [3], [22], [27], [33], [34], [42].

The perturbations $A_{(\alpha)}$ in (1.1) with $\operatorname{ran} G \subset \mathfrak{H}_0$ are ordinary (range) perturbations of the selfadjoint operator A_0. To introduce perturbations of A_0 of a more general type consider a rigging of the Hilbert space $\mathfrak{H}_0$, generated by the operator $|A_0|$:

$$\mathfrak{H}_{+k} \subset \cdots \subset \mathfrak{H}_{+2} \subset \mathfrak{H}_{+1} \subset \mathfrak{H}_0 \subset \mathfrak{H}_{-1} \subset \mathfrak{H}_{-2} \subset \cdots \subset \mathfrak{H}_{-k}, \tag{4.4}$$

where $\mathfrak{H}_{+k} = \operatorname{dom}|A_0|^{k/2}$, $k \in \mathbb{N}$, equipped with the graph inner product and $\mathfrak{H}_{-k}$ is the corresponding dual space, cf. [6]. Here the notation $\mathfrak{H}_{\pm k}$ for $\mathfrak{H}_{\pm k}(|A_0|)$ is used for simplicity. If $\operatorname{ran} G \subset \mathfrak{H}_{-k} \backslash \mathfrak{H}_0$, the perturbing term $G\alpha G^*$ becomes unbounded in $\mathfrak{H}_0$, and the expression in (1.1) needs an interpretation. In the sequel, interpretations for such (singular) perturbations will be presented for each of the following cases, respectively:

$$\operatorname{ran} G \subset \mathfrak{H}_{-1}, \quad \operatorname{ran} G \subset \mathfrak{H}_{-2}, \quad \operatorname{ran} G \subset \mathfrak{H}_{-k}, \quad k > 2.$$

4.1. **Perturbations in $\mathfrak{H}_{-1}$.** Let G be an injective linear mapping from $\mathcal{H} = \mathbb{C}^d$ into $\mathfrak{H}_{-1}$ and denote by G^* its adjoint operator from $\mathfrak{H}_{+1}$ into $\mathcal{H}$. The identity (1.2) gives again rise to a symmetric operator S_0 in $\mathfrak{H}_0$. Let $\widetilde{A}_0$ be the $[\mathfrak{H}_{+1}, \mathfrak{H}_{-1}]$-continuation of A_0 to all of $\mathfrak{H}_{+1}$. Then the expressions for the γ-field γ_0 and the Weyl function M_0 in (4.3) are still well defined, after A_0 is replaced by $\widetilde{A}_0$. The connection of the finite rank perturbations $A_{(\alpha)}$ to the extension theory in this case can be given in terms of boundary triplets as follows, cf. [10, Theorem 6.2] for the scalar case.

Theorem 4.1. *Let A_0 be a selfadjoint operator in the Hilbert space $\mathfrak{H}_0$ and let $\widetilde{A}_0$ be its $[\mathfrak{H}_{+1}, \mathfrak{H}_{-1}]$-continuation. Let G be an injective linear mapping from $\mathcal{H} = \mathbb{C}^d$ into $\mathfrak{H}_{-1}$ and define the restriction S_0 of A_0 by (1.2). Then:*

(i) *The operator S_0 is closed and symmetric in $\mathfrak{H}_0$ and has defect numbers (d, d).*
(ii) *The adjoint linear relation S_0^* of S_0 is given by*

$$S_0^* = \{\, \widehat{f} = \{f, \widetilde{A}_0 f - Gh\} : \, f \in \mathfrak{H}_{+1}, \, \widetilde{A}_0 f - Gh \in \mathfrak{H}_0, \, h \in \mathcal{H} \,\}. \tag{4.5}$$

(iii) *A boundary triplet for S_0^* can be defined by (4.2).*
(iv) *The corresponding γ-field and Weyl function are given by*

$$\gamma_0(\lambda) = (\widetilde{A}_0 - \lambda)^{-1} G, \quad M_0(\lambda) = G^* (\widetilde{A}_0 - \lambda)^{-1} G. \tag{4.6}$$

(v) *The perturbation*

$$A_{(\alpha)} = \{\, \{f, (\widetilde{A}_0 + G\alpha G^*) f\} : \, f \in \mathfrak{H}_{+1}, \, (\widetilde{A}_0 + G\alpha G^*) f \in \mathfrak{H}_0 \,\} \tag{4.7}$$

coincides with the selfadjoint extension $A_\tau = \ker(\Gamma_0^0 + \tau \Gamma_1^0)$ of S_0 with $\alpha = \tau = \tau^* \in [\mathcal{H}]$ and the resolvent of $A_{(\alpha)}$ is given by (2.7).

Proof. As a restriction of A_0, S_0 is symmetric and its closedness follows from the closedness of $\ker G^*$ in $\mathfrak{H}_{+1}\,(\supset \mathfrak{H}_{+2})$. The defect numbers are equal and they cannot be greater than (d, d), since $\ker G^*$ has co-dimension d in $\mathfrak{H}_{+1}$. The continuation $\widetilde{A}_0$ is a selfadjoint operator from $\mathfrak{H}_{+1}$ into $\mathfrak{H}_{-1}$ and, in particular, in the sense of the duality between these spaces, the equality $(\widetilde{A}_0 f, g) = (f, \widetilde{A}_0 g)$ holds for all $f, g \in \mathfrak{H}_{+1}$, cf. [24]. The resolvent $\widetilde{R}_\lambda = (\widetilde{A}_0 - \lambda)^{-1}$ of $\widetilde{A}_0$ is a $[\mathfrak{H}_{-1}, \mathfrak{H}_{+1}]$-continuous operator for $\lambda \in \rho(A_0)$. Therefore, it follows from the definition (1.2) that for all $f \in \operatorname{dom} S_0$ and all $\lambda \in \rho(A_0)$:

$$((\widetilde{A}_0 - \lambda)^{-1} Gh, (S_0 - \bar{\lambda}) f)_{\mathfrak{H}_0} = (Gh, f)_{\mathfrak{H}_0} = (h, G^* f)_{\mathcal{H}} = 0. \tag{4.8}$$

Hence, $\widetilde{R}_\lambda(\operatorname{ran} G) \subset \mathfrak{N}_\lambda(S_0^*)$ and a dimension argument shows that

$$\widetilde{R}_\lambda(\operatorname{ran} G) = \mathfrak{N}_\lambda(S_0^*), \quad \lambda \in \rho(A_0). \tag{4.9}$$

In particular, the defect numbers of S_0 are (d,d), and hence (i) has been proved.

To see (ii), recall the decomposition

$$S_0^* = A_0 \hat{+} \widehat{\mathfrak{N}}_\lambda(S_0^*), \quad \lambda \in \rho(A_0). \tag{4.10}$$

It follows from (4.9) and (4.10) that every $\{f,f'\} \in S_0^*$ admits the representation

$$\{f,f'\} = \{f_0 + \widetilde{R}_i Gh, A_0 f_0 + i\widetilde{R}_i Gh\} = \{f, \widetilde{A}_0 f - Gh\},$$

where $f_0 \in \operatorname{dom} A_0$, $h \in \mathcal{H}$ and, hence,

$$f = f_0 + \widetilde{R}_i Gh \in \mathfrak{H}_{+1}, \quad \widetilde{A}_0 f - Gh \in \mathfrak{H}_0.$$

This gives (4.5).

As to (iii), it is clear from (4.5) that the mapping $\Gamma^0 : S_0^* \to \mathcal{H}\oplus\mathcal{H}$ determined by (4.2) is surjective. With the vectors $\{f,f'\} = \{f, \widetilde{A}_0 f - Gh\} \in S_0^*$ and $\{g,g'\} = \{g, \widetilde{A}_0 g - Gk\} \in S_0^*$ one obtains

$$(f',g) - (f,g') = (\widetilde{A}_0 f - Gh, g) - (f, \widetilde{A}_0 g - Gk) = (G^* f, k)_{\mathcal{H}} - (h, G^* g)_{\mathcal{H}},$$

so that the abstract Green's identity holds.

Each vector $\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(S_0^*)$ admits the representation

$$\{f_\lambda, \lambda f_\lambda\} = \{\widetilde{R}_\lambda Gh, \lambda \widetilde{R}_\lambda Gh\} = \{\widetilde{R}_\lambda Gh, \widetilde{A}_0 \widetilde{R}_\lambda Gh - Gh\}.$$

This implies

$$\Gamma_0^0 \widehat{f}_\lambda = h, \quad \Gamma_1^0 \widehat{f}_\lambda = G^*(\widetilde{A}_0 - \lambda)^{-1} G,$$

which gives (iv) in view of (2.4).

Finally to prove (v), observe that with $\widehat{f} \in S_0^*$,

$$\Gamma_0^0 \widehat{f} + \tau \Gamma_1^0 \widehat{f} = h + \tau G^* f.$$

Thus, $\widehat{f} \in \ker(\Gamma_0^0 + \tau\Gamma_1^0)$ precisely when $h = -\tau G^* f$. Substituting this into (4.5) gives the representation (4.7) for the extension $\ker(\Gamma_0^0 + \tau\Gamma_1^0)$ with $\alpha = \tau$. □

If $\operatorname{ran} G \subset \mathfrak{H}_0$, then the statements in Theorem 4.1 clearly reduce to the facts presented in the introduction of the present section. When $\operatorname{ran} G \subset \mathfrak{H}_{-1}$, the operator in the righthand side of (4.7) will be written shortly as

$$A_{(\alpha)} = \widetilde{A}_0 + G\alpha G^*, \quad \alpha \in [\mathcal{H}].$$

Observe, that if $\operatorname{ran} G \subset \mathfrak{H}_{-1}\backslash\mathfrak{H}_0$, then the operator S_0 in Theorem 4.1 is densely defined and its adjoint S_0^* in (4.5) is an operator. In the case where $A_0 \ge 0$, the operator $A_{(\alpha)}$ is a *form-bounded* perturbation of A_0 in the sense of [2]. When the operator A_0 is not semibounded, but $\operatorname{ran} G \subset \mathfrak{H}_{-1}\backslash\mathfrak{H}_0$, the lifting of the extensions $A_\tau = \ker(\Gamma_0^0 + \tau\Gamma_1^0)$ to the space triplet $\mathfrak{H}_{+1} \subset \mathfrak{H}_0 \subset \mathfrak{H}_{-1}$ gives rise to a situation where the lifted extensions $\widetilde{A}_\tau$ behave like usual finite rank perturbations of A_0 in $\mathfrak{H}_0$ and they give rise to a generalized Friedrichs extension of S_0 in the original space $\mathfrak{H}_0$. Such results, involving so-called Kac subclasses of Nevanlinna functions, have been obtained in [24], [26], and then extended in [10] to Pontryagin spaces. The next results shows that perturbations in $\mathfrak{H}_{-1}$ as described in Theorem 4.1 are additive with respect to the parameter $\alpha \in [\mathcal{H}]$.

Proposition 4.2. *For each selfadjoint $\tau \in [\mathcal{H}]$ the space triplets $\mathfrak{H}_{+1}(A_\tau) \subset \mathfrak{H}_0 \subset \mathfrak{H}_{-1}(A_\tau)$ are (topologically) independent of τ and A_τ is a representation of the additive group $[\mathcal{H}]$, i.e.*

$$A_{\tau_1+\tau_2} = (A_{\tau_1})_{\tau_2}, \quad \tau_j = \tau_j^* \in [\mathcal{H}], \quad j = 1, 2. \tag{4.11}$$

Proof. The equality of the domains $\operatorname{dom}|A_\tau|^{\frac{1}{2}}$ for the extensions $A_\tau = \ker(\Gamma_0^0 + \tau\Gamma_1^0)$ in (4.7) corresponding to the selfadjoint (operator) parameters $\tau \in [\mathcal{H}]$ can be proved along the lines of [24], [29]. It follows then from the closed graph theorem that the norms on the spaces $\mathfrak{H}_{\pm 1}(A_\tau)$ are equivalent, and therefore the space triplets $\mathfrak{H}_{+1}(A_\tau) \subset \mathfrak{H}_0 \subset \mathfrak{H}_{-1}(A_\tau)$ for $\tau \in [\mathcal{H}]$ coincide, up to equivalent inner products.

In view of Theorem 4.1 the extension $A_{\tau_1} \subset S_0^*$, $\tau_1 \in [\mathcal{H}]$, is given by

$$\widetilde{A}_{\tau_1} f = \widetilde{A}_0 f + G\tau_1 G^* f, \quad f \in \mathfrak{H}_{+1}, \tag{4.12}$$

with $f \in \operatorname{dom} A_{\tau_1}$ if and only if $\widetilde{A}_0 f + G\tau_1 G^* f \in \mathfrak{H}_0$. Now, applying (4.12) again with A_{τ_1} and $\tau_2 \in [\mathcal{H}]$ yields

$$(\widetilde{A}_{\tau_1})_{\tau_2} = \widetilde{A}_{\tau_1} + G\tau_2 G^* = \widetilde{A}_0 + G(\tau_1 + \tau_2)G^* = \widetilde{A}_{\tau_1+\tau_2},$$

and clearly $f \in \operatorname{dom}(A_{\tau_1})_{\tau_2}$ if and only if $f \in \operatorname{dom} A_{\tau_1+\tau_2}$. This proves (4.11). □

4.2. **Perturbations in $\mathfrak{H}_{-2}$.** Let G be an injective linear mapping from $\mathcal{H} = \mathbb{C}^d$ into $\mathfrak{H}_{-2}$ and let G^* be its adjoint operator from $\mathfrak{H}_{+2}$ into $\mathcal{H}$. The identity (1.2) still gives rise to a symmetric operator S_0 in $\mathfrak{H}_0$. However, when $\operatorname{ran} G \subset \mathfrak{H}_{-2}$ the operator Γ_1^0 in (4.2) is not well defined anymore, and it has to be regularized. Let $\widetilde{A}_0$ be the $[\mathfrak{H}_0, \mathfrak{H}_{-2}]$-continuation of A_0 to all of $\mathfrak{H}_0$. The resolvent $\widetilde{R}_\lambda = (\widetilde{A}_0 - \lambda)^{-1}$ of $\widetilde{A}_0$ is an $[\mathfrak{H}_{-2}, \mathfrak{H}_0]$-continuous operator for $\lambda \in \rho(A_0)$, see [44]. Then the expression for the γ-field γ_0 in (4.3) is well defined, after A_0 is replaced by $\widetilde{A}_0$, but a regularization of the Weyl function M_0 is needed.

An operator $\mathcal{R} \in [\mathfrak{H}_{-2}, \mathfrak{H}_0]$ is said to be a *regularizing operator* of $\widetilde{R}_\lambda$ if $\widetilde{R}_\lambda - \mathcal{R} \in [\mathfrak{H}_{-2}, \mathfrak{H}_{+2}]$, and $(\widetilde{R}_\lambda - \mathcal{R})^* = \widetilde{R}_{\bar\lambda} - \mathcal{R}$ for $\lambda \in \rho(A_0)$. For example, one can take $\mathcal{R} = \frac{1}{2}(\widetilde{R}_i + \widetilde{R}_{-i})$ as a regularizing operator of $\widetilde{R}_\lambda$, cf. [44]. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are two regularizing operators of $\widetilde{R}_\lambda$ then clearly $\mathcal{R}_2 - \mathcal{R}_1 \in [\mathfrak{H}_{-2}, \mathfrak{H}_{+2}]$.

Theorem 4.3. *Let A_0 be a selfadjoint operator in the Hilbert space $\mathfrak{H}_0$, let $\widetilde{A}_0$ be the $[\mathfrak{H}_0, \mathfrak{H}_{-2}]$-continuation of A_0, and let $\mathcal{R}$ be a regularizing operator of $\widetilde{R}_\lambda = (\widetilde{A}_0 - \lambda)^{-1}$. Let G be a linear injective mapping from $\mathcal{H} = \mathbb{C}^d$ into $\mathfrak{H}_{-2}$ and define the restriction S_0 of A_0 by (1.2). Then:*

(i) *The operator S_0 is closed and symmetric in $\mathfrak{H}_0$ and has defect numbers (d, d).*

(ii) *The adjoint linear relation S_0^* of S_0 is given by*

$$S_0^* = \{\, \widehat{f} = \{f, \widetilde{A}_0 f - Gh\} : f \in \mathfrak{H}_0,\ \widetilde{A}_0 f - Gh \in \mathfrak{H}_0,\ h \in \mathcal{H} \,\}. \tag{4.13}$$

(iii) *A boundary triplet for S_0^* can be defined by*

$$\mathcal{H} = \mathbb{C}^d, \quad \Gamma_0^0 \widehat{f} = h, \quad \Gamma_1^0 \widehat{f} = G^*(f - \mathcal{R}Gh) + Bh, \quad \widehat{f} \in S_0^*, \tag{4.14}$$

where B is a selfadjoint operator in $\mathcal{H}$.

(iv) *The corresponding γ-field and the Weyl function are given by*

$$\gamma_0(\lambda) = (\widetilde{A}_0 - \lambda)^{-1} G, \quad M_0(\lambda) = G^*((\widetilde{A}_0 - \lambda)^{-1} - \mathcal{R})G + B. \tag{4.15}$$

(v) *The resolvent of the extension* $\widetilde{A}_\tau = \ker(\Gamma_0 + \tau\Gamma_1)$, $\tau = \tau^* \in \widetilde{\mathcal{C}}(\mathcal{H})$, *is given by*

$$(A_\tau - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma_0(\lambda)(\tau^{-1} + G^*((\widetilde{A}_0 - \lambda)^{-1} - \mathcal{R})G + B)^{-1}\gamma_0(\bar{\lambda})^*, \quad \lambda \in \rho(A_\tau) \cap \rho(A_0).$$

Proof. (i) Observe, that $\ker G^*$ is closed in $\mathfrak{H}_{+2}$. This implies that S_0 is a closed symmetric operator in $\mathfrak{H}_0$ with equal defect numbers which cannot be greater than (d,d), since $\ker G^*$ has co-dimension d in $\mathfrak{H}_{+2}$. The continuation $\widetilde{A}_0$ admits the following symmetry property

$$(f, \widetilde{A}_0 g) = (A_0 f, g)_{\mathfrak{H}_0}, \quad f \in \operatorname{dom} A_0,\ g \in \mathfrak{H}_0,$$

where $(\cdot,\cdot)$ stands for the duality between $\mathfrak{H}_{+2}$ and $\mathfrak{H}_{-2}$. Hence, the equality (4.9) follows from (4.8). In particular, the defect numbers of S_0 are (d,d).

(ii) It follows from (4.9) and (4.10) that every $\{f,f'\} \in S_0^*$ admits the representation

$$\{f,f'\} = \{f_0 + \widetilde{R}_\lambda Gh, A_0 f_0 + \lambda \widetilde{R}_\lambda Gh\}, \quad f_0 \in \operatorname{dom} A_0, \quad h \in \mathcal{H}, \quad \lambda \in \rho(A_0), \tag{4.16}$$

which, due to the relation $\widetilde{R}_\lambda Gh - \mathcal{R}Gh \in \operatorname{dom} A_0$, can be rewritten as

$$\{f,f'\} = \{f_0' + \mathcal{R}Gh, \widetilde{A}_0(f_0' + \mathcal{R}Gh) - Gh\}, \quad f_0' \in \operatorname{dom} A_0.$$

Hence, S_0^* belongs to the left side of (4.13). Conversely, if $\widetilde{A}_0 f - Gh \in \mathfrak{H}_0$ for some $f \in \mathfrak{H}_0$ and $h \in \mathcal{H}$, then equivalently $f - \widetilde{R}_\lambda Gh \in \operatorname{dom} A_0$. In this case, $f = f_0 + \widetilde{R}_\lambda Gh$ for some $f_0 \in \operatorname{dom} A_0$ and $f' = \widetilde{A}_0 f - Gh = A_0 f_0 + \lambda \widetilde{R}_\lambda Gh$, so that $\{f,f'\} \in S_0^*$ by (4.16).

(iii) It is clear from (4.13) that the mapping $\Gamma^0 : S_0^* \to \mathcal{H} \oplus \mathcal{H}$ determined by (4.14) is surjective. Moreover, for every $\{f,f'\} \in S_0^*$ of the form (4.16) and

$$\{g,g'\} = \{g_0 + \widetilde{R}_{\bar{\lambda}} Gk, A_0 g_0 + \bar{\lambda} \widetilde{R}_{\bar{\lambda}} Gk\} \in S_0^*, \quad g_0 \in \operatorname{dom} A_0, \quad k \in \mathcal{H},$$

one obtains

$$\begin{aligned}(f',g) - (f,g') &= ((A_0 - \lambda)f_0, \widetilde{R}_{\bar{\lambda}} Gk) - (\widetilde{R}_\lambda Gh, (A_0 - \bar{\lambda})g_0)\\ &= (f_0, Gk) - (Gh, g_0)\\ &= (G^* f_0 + Bh, k)_{\mathcal{H}} - (h, G^* g_0 + Bk)_{\mathcal{H}}\\ &= (G^*(f - \mathcal{R}Gh) + Bh, k)_{\mathcal{H}} - (h, G^*(g - \mathcal{R}Gk) + Bk)_{\mathcal{H}}.\end{aligned}$$

(iv) Decompose the defect vectors $f_\lambda \in \mathfrak{N}_\lambda(S_0^*)$ as follows:

$$\{f_\lambda, \lambda f_\lambda\} = \{\widetilde{R}_\lambda Gh, \lambda \widetilde{R}_\lambda Gh\} = \{\widetilde{R}_\lambda Gh, \widetilde{A}_0 \widetilde{R}_\lambda Gh - Gh\}.$$

Then according to (4.14),

$$\Gamma_0^0 \widehat{f}_\lambda = h, \quad \Gamma_1^0 \widehat{f}_\lambda = G^* \left(\widetilde{R}_\lambda - \mathcal{R}\right) Gh + Bh,$$

which in view of (2.4) leads to (4.15).

(v) The statement follows from Proposition 2.1. □

The boundary operator Γ_1^0 in Theorem 4.3 depends on a free parameter $B \in [\mathcal{H}]$. When this parameter B is fixed, the family of perturbations $A_{(\alpha)}$ of A_0 can be defined, in analogy with Theorem 4.1, as the family of selfadjoint extensions

$$A_\tau = \ker(\Gamma_0^0 + \tau\Gamma_1^0), \quad \tau = \tau^* \in \widetilde{\mathcal{C}}(\mathcal{H}). \tag{4.17}$$

Some other, but equivalent, forms for defining these perturbations have been given in [22], [33], [2], [29]. Observe that the resulting family $A_{(\alpha)}$ is not additive in the sense of (4.11).

Remark 4.4. Comparing the statement (v) in Theorems 4.1 and 4.3 in the case where $\operatorname{ran} G \subset \mathfrak{H}_{-1}$ it is seen that $A_{(\alpha)} = A_\tau$ if and only if

$$\tau^{-1} - G^*\mathcal{R}G + B = \alpha^{-1}. \tag{4.18}$$

It follows from (4.18) that the operator α should be equal to 0 for G satisfying $\operatorname{ran} G \subset \mathfrak{H}_{-2} \setminus \mathfrak{H}_{-1}$ and $\tau = \tau^*$ invertible. Such "perturbations" A_τ of A_0 were called in [33] *infinitesimal*; they can be interpreted as selfadjoint extensions of the symmetric operator S_0 in Theorem 4.3 and hence they can be parametrized by the resolvent formula (v) in Theorem 4.3 with $\tau = \tau^* \in \widetilde{\mathcal{C}}(\mathcal{H})$.

Remark 4.5. If $0 \in \rho(A_0)$, one can take $\widetilde{A}_0^{-1} \in [\mathfrak{H}_{-2}, \mathfrak{H}_0]$ as a regularizing operator for the resolvent $\widetilde{R}_\lambda$. Then the corresponding Weyl function M_0 takes the form

$$M_0(\lambda) = G^*\left((\widetilde{A}_0 - \lambda)^{-1} - \widetilde{A}_0^{-1}\right) G + B. \tag{4.19}$$

The case of perturbations in $\mathfrak{H}_{-2}$ is general in the sense that every closed symmetric operator S with defect numbers (d, d) can be obtained as a restriction of a selfadjoint operator A_0 via (1.2) with some linear injective mapping from $\mathcal{H} = \mathbb{C}^d$ into $\mathfrak{H}_{-2}$, cf. e.g. [27].

The following lemma gives some formulas which will be useful in Subsection 4.3 in order to describe the renormalization procedure for the resolvent $\widetilde{R}_\lambda = (\widetilde{A}_0 - \lambda)^{-1}$ and the Weyl function generated by singular perturbations of A_0 with $\operatorname{ran} G \subset \mathfrak{H}_{-k}$, $k > 2$.

Lemma 4.6. *Let G be a linear injective mapping from $\mathcal{H} = \mathbb{C}^d$ into $\mathfrak{H}_{-1}$, let q be a scalar polynomial of degree $n \in \mathbb{N}$, such that $\sigma(q) \cap \sigma(A_0) = \emptyset$, and let $G_0 = q(\widetilde{A}_0)^{-1}G$. Define the block matrix T by*

$$T = \operatorname{col}(t_{2n-1}, \dots, t_1, t_0), \quad t_j = G_0^*\widetilde{A}_0^{2n-1-j}G_0, \quad j = 0, 1, \dots, 2n-1. \tag{4.20}$$

Then the following identities hold

$$G^*(\widetilde{A}_0 - \lambda)^{-1}G = r(\lambda) + q^\sharp(\lambda)G_0^*(\widetilde{A}_0 - \lambda)^{-1}G_0 q(\lambda), \tag{4.21}$$

$$G^*(\widetilde{A}_0 - \lambda)^{-2}G = \frac{d}{d\lambda}\{r(\lambda) + q^\sharp(\lambda)G_0^*(\widetilde{A}_0 - \lambda)^{-1}G_0 q(\lambda)\}, \tag{4.22}$$

where

$$r(\lambda) = \Lambda_{(2n)}(\mathcal{B}_{q^\sharp q} \otimes I)T, \quad \Lambda_{(2n)} = (I_\mathcal{H}, \lambda I_\mathcal{H}, \dots, \lambda^{2n-1}I_\mathcal{H}) \tag{4.23}$$

and $\mathcal{B}_{q^\sharp q}$ is the matrix associated with the polynomial $q^\sharp q$ via (2.11).

Proof. It follows from

$$p(\lambda, z) = \frac{q(\lambda)^\sharp q(\lambda) - q(z)^\sharp q(z)}{\lambda - z} \tag{4.24}$$

that the corresponding matrix polynomial admits the representation

$$p(\lambda, z)I_\mathcal{H} = \Lambda_{(2n)}(\mathcal{B}_{q^\sharp q} \otimes I)Z_{(2n)}^\top = \sum_{j,k=0}^{2n-1} b_{jk}\lambda^j z^k I_\mathcal{H}, \tag{4.25}$$

where $Z_{(2n)}$ is defined similar to $\Lambda_{(2n)}$. Moreover, (4.24) implies that

$$G^*(\widetilde{A}_0 - \lambda)^{-1}G = G_0^* p(\lambda, \widetilde{A}_0)G_0 + q^\sharp(\lambda)G_0^*(\widetilde{A}_0 - \lambda)^{-1}G_0 q(\lambda).$$

To prove (4.21), it remains to notice that due to (4.25),

$$G_0^* p(\lambda, \widetilde{A}_0) G_0 = \sum_{j,k=0}^{2n-1} b_{jk} \lambda^j G_0^* \widetilde{A}_0^k G_0 = \Lambda_{(2n)}(\mathcal{B}_{q^\sharp q} \otimes I_{\mathcal{H}})T.$$

The identity (4.22) is obtained from (4.21) by differentiation. □

Remark 4.7. Statements similar to those in Lemma 4.6 are still valid if $\operatorname{ran} G \subset \mathfrak{H}_{-2}$. In this case $t_0 = G_0^* \widetilde{A}_0^{2n-1} G_0$ is not well defined and the resolvent needs a regularizing term. Let $\mathcal{R}$ be a regularizing operator of $\widetilde{R}_\lambda$ commuting with $\widetilde{R}_\lambda$. Then the difference

$$t_0 - G^* \mathcal{R} G = G^*(q^\sharp(\widetilde{A}_0)^{-1} \widetilde{A}_0^{2n-1} q(\widetilde{A}_0)^{-1} - \mathcal{R})G \tag{4.26}$$

makes sense and by incorporating the regularizing terms $G^* \mathcal{R} G$ and $q^\sharp G_0^* \mathcal{R} G_0 q$ in the formula (4.21) one arrives at the following identity

$$G^*(\widetilde{R}_\lambda - \mathcal{R})G = r(\lambda) + q^\sharp(\lambda) q(\lambda)[G_0^*(\widetilde{R}_\lambda - \mathcal{R})G_0 + B], \tag{4.27}$$

where $B = G_0^* \mathcal{R} G_0$ and the matrix polynomial $r(\lambda)$ is now given by

$$r(\lambda) = \Lambda_{(2n)}(\mathcal{B}_{q^\sharp q} \otimes I)T - G^* \mathcal{R} G. \tag{4.28}$$

In (4.28) $r(\lambda)$ is well defined in view of (4.26). Differentiation of (4.27) gives again an expression for $G^*(\widetilde{A}_0 - \lambda)^{-2} G$ analogous to (4.22). Notice also that the selection of a regularizing operator $\mathcal{R}$ or even two different regularizing operators $\mathcal{R}_1$, $\mathcal{R}_2$ in (4.27) results in a difference for r and B only by some constant well-defined selfadjoint operators in $\mathcal{H}$, since $\mathcal{R} - \mathcal{R}_j \in [\mathfrak{H}_{-2}, \mathfrak{H}_{+2}]$, $j = 1, 2$.

4.3. **Perturbations in $\mathfrak{H}_{-2n-1}$ and $\mathfrak{H}_{-2n-2}$.** In a number of papers singular rank one perturbations of A_0 generated by $\omega \in \mathfrak{H}_{-2n-2}$ have been studied by means of exit space extensions of a symmetric operator S connected with A_0, see [41], [17], [18], [38], [37]. In this subsection a model for *finite rank singular perturbations* of A_0 generated by G with $\operatorname{ran} G \subset \mathfrak{H}_{-2n-j}$, $j = 1, 2$, is established also in terms of exit space extensions of A_0. Here the model for such perturbations is derived from a basic assumption that in an extending inner product space $\mathfrak{H} \supset \mathfrak{H}_0$ the resolvents associated with the perturbations of A_0 should be finite rank perturbations of the resolvent generated in $\mathfrak{H}$ by $(A_0 - \lambda)^{-1}$ (see Theorem 4.8).

First consider the case, where G is a linear mapping from $\mathcal{H} = \mathbb{C}^d$ into $\mathfrak{H}_{-2n-1}$ and let $\widetilde{A}_0$ be the $[\mathfrak{H}_{-2n+1}, \mathfrak{H}_{-2n-1}]$-continuation of A_0. The adjoint operator G^* maps $\mathfrak{H}_{2n+1}$ into $\mathcal{H}$. Observe, that if $\operatorname{ran} G \cap \mathfrak{H}_{-2} = \{0\}$, then the identity (1.2) gives rise to an essentially selfadjoint operator whose closure is equal to A_0. Moreover, the vector $\widetilde{R}_\lambda G h = (\widetilde{A}_0 - \lambda)^{-1} G h$, $h \in \mathcal{H}$, $\lambda \in \rho(A_0)$, does not belong to the space $\mathfrak{H}_0$. To give a sense to the vector $\widetilde{R}_\lambda G h$ and to the resolvent formula (2.7) one needs to extend the space $\mathfrak{H}_0$ by adding the subspaces

$$\widetilde{A}_0^{-1} \operatorname{ran} G, \dots, \widetilde{A}_0^{-n} \operatorname{ran} G, \tag{4.29}$$

assuming, for simplicity, that $0 \in \rho(A_0)$. Then the vector

$$\gamma(\lambda) h := \widetilde{R}_\lambda G h = \widetilde{A}_0^{-1} G h + \cdots + \lambda^{n-1} \widetilde{A}_0^{-n} G h + \lambda^n \widetilde{R}_\lambda \widetilde{A}_0^{-n} G h \tag{4.30}$$

can be considered as a vector from an extended inner product space $\mathfrak{H}$ which contains both $\mathfrak{H}_0$ and the subspaces (4.29):

$$\mathfrak{H} \supset \operatorname{span}\{\, \mathfrak{H}_0, \widetilde{A}_0^{-j} \operatorname{ran} G : \; j = 1, \dots, n \,\}. \tag{4.31}$$

In this space the continuation $\widetilde{A}_0$ of A_0 generates an operator, say H_0, for which the operator function $\gamma(\lambda)$, $\lambda \in \rho(A_0)$, can be interpreted to form its γ-field in the sense that

$$\frac{\gamma(\lambda) - \gamma(\mu)}{\lambda - \mu} = (H_0 - \lambda)^{-1} \gamma(\mu), \quad \lambda, \mu \in \rho(A_0). \tag{4.32}$$

This identity implies that

$$\frac{d}{d\lambda} \gamma(\lambda) = (H_0 - \lambda)^{-1} \gamma(\lambda), \quad \lambda \in \rho(A_0). \tag{4.33}$$

The inner product $\langle u, \varphi \rangle_{\mathfrak{H}}$ in $\mathfrak{H}$ should coincide with the form (u, φ) generated by the inner product in $\mathfrak{H}_0$ if the vectors u, φ are in duality, say, $u \in \mathfrak{H}_{2(n-j)+1}$, $\varphi \in \widetilde{A}_0^{-j} \operatorname{ran} G$. Now, for the other vectors in (4.31) it will be supposed that the conditions

$$\left\langle \widetilde{A}_0^{-j} G h, \widetilde{A}_0^{-k} G f \right\rangle_{\mathfrak{H}} = (t_{j+k-1} h, f)_{\mathcal{H}}, \quad j, k = 1, \dots, n; \; h, f \in \mathcal{H}, \tag{4.34}$$

are satisfied for some operators $t_j = t_j^* \in [\mathcal{H}]$, $j = 1, \dots, 2n-1$. The next result shows that under such weak conditions on the extending space the structure of perturbed resolvents becomes already completely fixed even under some mild assumptions on H_0. This fact yields an interpretation and a model for singular finite rank perturbations of A_0.

Theorem 4.8. *Assume that* $0 \in \rho(A_0)$ *and let* $\operatorname{ran} G \subset \mathfrak{H}_{-2n-1} \backslash \mathfrak{H}_{-2n}$, *let* $G_0 = \widetilde{A}_0^{-n} G$, *let* $\mathfrak{H} \supset \mathfrak{H}_0$ *be (an isometric image of) an inner product space satisfying* (4.31), (4.34), *and let* H *and* H_0 *be selfadjoint linear relations in* $\mathfrak{H}$ *such that*

(i) $\rho(H_0) = \rho(A_0)$;
(ii) $\gamma(\lambda)' = (H_0 - \lambda)^{-1} \gamma(\lambda)$ *holds for (an isometric image of)* $\gamma(\lambda) = (\widetilde{A}_0 - \lambda)^{-1} G$, $\lambda \in \rho(A_0)$;
(iii) $(H - \lambda)^{-1} - (H_0 - \lambda)^{-1} = -\gamma(\lambda) \sigma(\lambda) \gamma(\bar{\lambda})^*$, $\lambda \in \rho(H) \cap \rho(H_0)$;

for some matrix function $\sigma(\lambda)$ *holomorphic and invertible for* $\lambda \in \rho(H_0) \cap \rho(H)$. *Then* $\sigma(\lambda)^{-1}$ *can be represented in the form*

$$\sigma^{-1}(\lambda) = \beta + t(\lambda) + \lambda^{2n} M_0(\lambda), \tag{4.35}$$

where $\beta = \beta^* \in [\mathcal{H}]$, $t(\lambda) = t_1 \lambda + \cdots + t_{2n-1} \lambda^{2n-1}$, *and* $M_0(\lambda) = G_0^* \widetilde{R}_\lambda G_0$ *is a Nevanlinna function in* $\mathcal{H}$.

Proof. Denote $R_\lambda = (H_0 - \lambda)^{-1}$, $\widetilde{R}_\lambda = (\widetilde{A}_0 - \lambda)^{-1}$. By assumption (ii)

$$\gamma(\lambda)' = R_\lambda \gamma(\lambda), \quad [\gamma(\bar{\lambda})^*]' = \gamma(\bar{\lambda})^* R_\lambda. \tag{4.36}$$

Now, differentiation of (iii) yields

$$(H - \lambda)^{-2} - R_\lambda^2 = -\gamma(\lambda) \sigma(\lambda)' \gamma(\bar{\lambda})^* - R_\lambda \gamma(\lambda) \sigma(\lambda) \gamma(\bar{\lambda})^* - \gamma(\lambda) \sigma(\lambda) \gamma(\bar{\lambda})^* R_\lambda, \tag{4.37}$$

which together with (iii) implies that

$$\gamma(\lambda) \sigma(\lambda)' \gamma(\bar{\lambda})^* = -\gamma(\lambda) \sigma(\lambda) \gamma(\bar{\lambda})^* \gamma(\lambda) \sigma(\lambda) \gamma(\bar{\lambda})^*. \tag{4.38}$$

The identity (4.38) can be rewritten (by the assumption of isometry in (ii)) as

$$\frac{d\sigma^{-1}}{d\lambda} = \gamma(\bar{\lambda})^*\gamma(\lambda) = (\widetilde{R}_{\bar{\lambda}}G)^*(\widetilde{R}_\lambda G). \tag{4.39}$$

It follows from (4.30) and (4.34) that for every $h, f \in \mathcal{H}$, $j = 1, \dots, n$,

$$\left\langle \widetilde{A}_0^{-j}\widetilde{R}_\lambda Gh, \widetilde{A}_0^{-1}Gf \right\rangle_{\mathfrak{H}} = \left\langle \widetilde{R}_\lambda Gh, \widetilde{A}_0^{-j-1}Gf \right\rangle_{\mathfrak{H}},$$

$$\left\langle \widetilde{R}_\lambda Gh, \widetilde{A}_0^{-1}\widetilde{R}_{\bar{\lambda}}Gf \right\rangle_{\mathfrak{H}} = \left\langle \widetilde{R}_\lambda^2 Gh, \widetilde{A}_0^{-1}Gf \right\rangle_{\mathfrak{H}}.$$

Therefore,

$$\left\langle \widetilde{R}_\lambda Gh, \widetilde{R}_{\bar{\lambda}}Gf \right\rangle_{\mathfrak{H}} = \left\langle \widetilde{R}_\lambda Gh, (I+\bar{\lambda}\widetilde{R}_{\bar{\lambda}})\widetilde{A}_0^{-1}Gf \right\rangle_{\mathfrak{H}} = \frac{d}{d\lambda}\left\langle \lambda\widetilde{R}_\lambda Gh, \widetilde{A}_0^{-1}Gf \right\rangle_{\mathfrak{H}}, \tag{4.40}$$

and by applying (4.30), with $2n-1$ instead of n, and (4.34) one obtains

$$\begin{aligned}\left\langle \lambda\widetilde{R}_\lambda Gh, \widetilde{A}_0^{-1}Gf \right\rangle_{\mathfrak{H}} &= \lambda\left\langle \widetilde{A}_0^{-1}Gh, \widetilde{A}_0^{-1}Gf \right\rangle_{\mathfrak{H}} + \dots + \lambda^{2n-1}\left\langle \widetilde{A}_0^{-2n+1}Gh, \widetilde{A}_0^{-1}Gf \right\rangle_{\mathfrak{H}} \\ &\quad + \lambda^{2n}\left\langle \widetilde{R}_\lambda\widetilde{A}_0^{-2n+1}Gh, \widetilde{A}_0^{-1}Gf \right\rangle_{\mathfrak{H}} \\ &= (t(\lambda)h, f)_{\mathcal{H}} + \lambda^{2n}(G_0^*\widetilde{R}_\lambda G_0 h, f)_{\mathcal{H}}.\end{aligned} \tag{4.41}$$

It follows from (4.39), (4.40), and (4.41) that $\sigma^{-1}(\lambda)$ in (iii) takes the form (4.35), where β is a selfadjoint operator in $\mathcal{H}$ and $M_0(\lambda) = G_0^*\widetilde{R}_\lambda G_0$ is a Nevanlinna function since $\operatorname{ran} G_0 \subset \mathfrak{H}_{-1}$. □

Remark 4.9. Observe that one arrives at the same formula (4.35) for $\sigma^{-1}(\lambda)$ by comparing (4.39) with (4.22) in Lemma 4.6. Similarly, in the case $\sigma(q) \cap \sigma(A_0) = \emptyset$ one can derive from Lemma 4.6 the following representation of $\sigma^{-1}(\lambda)$:

$$\sigma^{-1}(\lambda) = r(\lambda) + q^\sharp(\lambda)q(\lambda)M_0(\lambda), \tag{4.42}$$

where r is given by (4.23). One can extend Theorem 4.8 also to the case where $\operatorname{ran} G \subset \mathfrak{H}_{-2n-2}$. Then the function $\sigma^{-1}(\lambda)$ in (iii) still has the same form (4.42), but the function $M_0(\lambda)$ in (4.42) takes the form $G_0^*(\widetilde{R}_\lambda - \mathcal{R})G_0 + B$, where $\mathcal{R}$ is a regularizing operator of $\widetilde{R}_\lambda$, $B = B^*$, and r with $\deg r \le 2n-1$ is given by (4.28); see Remark 4.7.

Remark 4.10. The function $r(\lambda) + q^\sharp(\lambda)q(\lambda)M_0(\lambda)$ is the Weyl function of the model Pontryagin space symmetric operator S considered in [13, Theorem 4.2] and does not belong to the class of Nevanlinna functions. In fact, substituting the formula (4.42) with $\beta = \tau^{-1}$, $\tau = \tau^* \in \widetilde{\mathcal{C}}(\mathcal{H})$, for σ in (iii) one obtains the resolvent formula (2.7) in Proposition 2.1 with the Weyl function

$$M = r + q^\sharp M_0 q. \tag{4.43}$$

The formulas (4.21) and (4.27) can now be seen as a *renormalization procedure* for the Weyl function associated with the γ-field $\gamma(\lambda) = (\widetilde{A}_0 - \lambda)^{-1}G$. In view of Theorem 4.8 it is natural to identify the family of finite rank singular perturbations $A_0 + G\alpha G^*$ with the family of selfadjoint extensions H_τ of the symmetric operator S in a Pontryagin space; see Theorem 4.12. The formula (4.42) implies also that one cannot find a Hilbert space selfadjoint family satisfying the properties (ii) and (iii) in Theorem 4.8. Of course, one can still give a description in purely Hilbert space terminology, but then the extensions will not be selfadjoint anymore (cf. [38], [37]).

A model space $\mathfrak{H}$ and a selfadjoint relation H_0 in $\mathfrak{H}$ which satisfy the assumptions in Theorem 4.8 can be constructed directly from A_0 and G without using riggings of the Hilbert space $\mathfrak{H}_0$ as follows. Let G be an injective linear mapping from $\mathcal{H} = \mathbb{C}^d$ into $\mathfrak{H}_{-2n-j}$, $j = 1, 2$. Let q be an n-th order monic $d \times d$ matrix polynomial such that $\sigma(q) \cap \sigma(A_0) = \emptyset$ and define $G_0 = q(A_0)^{-1}G$, so that G_0 maps $\mathcal{H} = \mathbb{C}^d$ into $\mathfrak{H}_{-1}(A_0)$ or $\mathfrak{H}_{-2}(A_0)$, respectively. The restriction S_0 of A_0 to $\operatorname{dom} A_0 \cap \ker G_0^*$, is a closed symmetric operator in $\mathfrak{H}_0$ with defect numbers (d, d). The corresponding Weyl function M_0 is as in (4.6), (4.15), or as in (4.19) if in addition $0 \in \rho(A_0)$, with G replaced by G_0. Let $t_0, \dots, t_{2n-1}$ be arbitrary selfadjoint $d \times d$ matrices and define the matrix polynomial r, $\deg r \le 2n - 1$, by (4.23) or (4.28). Parallel to (4.21) or (4.27) depending on $\operatorname{ran} G \subset \mathfrak{H}_{-2n-1}$ or $\operatorname{ran} G \subset \mathfrak{H}_{-2n-2}$, respectively, the generalized Nevanlinna function M is defined by (4.43). The matrix polynomial Q of the form (2.10) gives rise to a model involving a reproducing kernel Pontryagin space $\mathfrak{H}_Q$ and a corresponding multiplication operator S_Q in it, see Theorem 2.2. The model for M in (4.43) is now obtained by applying Theorem 3.1. For simplicity the result is formulated for the case $0 \in \rho(A_0)$ and $\operatorname{ran} G \subset \mathfrak{H}_{-2n-1} \backslash \mathfrak{H}_{-2n}$.

Theorem 4.11. *Let A_0 be a selfadjoint operator in the Hilbert space $\mathfrak{H}_0$ such that $0 \in \rho(A_0)$ and let $G : \mathcal{H} = \mathbb{C}^d \to \mathfrak{H}_{-2n-1} \backslash \mathfrak{H}_{-2n}$ be injective. Let $t_0, \dots, t_{2n-1}$ be arbitrary selfadjoint $d \times d$ matrices and let the matrix polynomial r, $\deg r \le 2n - 1$, be given by* (4.23)*, let $G_0 = (\widetilde{A}_0)^{-n} G$, and let S_0 be a symmetric restriction of A_0 defined by*

$$\operatorname{dom} S_0 = \{\, f \in \operatorname{dom} A_0 : \; G_0^* f = 0 \,\}.$$

Let $\{\mathcal{H}, \Gamma_0^0, \Gamma_1^0\}$ be a boundary triplet for S_0^* with the Weyl function M_0 and moreover, let the symmetric operator S_Q in $\mathfrak{H}_Q$ and the boundary triplet for S_Q^* be as in Theorem 2.2. Then the operator S_0 is densely defined in $\mathfrak{H}_0$ and, moreover, the following statements hold:

(i) *The linear relation S defined in Theorem 3.1 is a closed simple symmetric operator in $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_Q$ with defect numbers (d, d).*
(ii) *The adjoint linear relation S^* and the boundary triplet for S^* are as given in parts (ii) and (iii) of Theorem 3.1.*
(iii) *The corresponding Weyl function has the form* (4.43) *where $q(\lambda) = \lambda^n$ and M_0 is given by* (4.6).
(iv) *The linear relations $H_0 = \ker \Gamma_0$ and $H_\tau = \ker(\Gamma_0 + \tau\Gamma_1)$, $\tau \in [\mathcal{H}]$, are selfadjoint extensions of S in $\mathfrak{H}_0 \oplus \mathfrak{H}_Q$.*
(v) *The resolvent set $\rho(H_\tau)$ of H_τ, $\tau \in [\mathcal{H}]$, is nonempty, the spectrum $\sigma(H_\tau)$ in $\rho(A_0)$ coincides with*

$$\{\, \lambda \in \rho(A_0) : \; \det(I + (r(\lambda) + \lambda^{2n} M_0(\lambda))\tau) = 0 \,\},$$

and the compressed resolvent of H_0 and H_τ are of the form (3.7) *and* (3.20).
(vi) *The corresponding Štraus extensions T_τ, $\tau \in [\mathcal{H}]$, are given by the "interface conditions" in* (3.21).

The statements of the above theorem follow easily from Theorems 3.1, 3.5, Proposition 3.2, and [13, Theorem 5.3]. One can state a similar result for the case when $\operatorname{ran} G \subset \mathfrak{H}_{-2n-2} \backslash \mathfrak{H}_{-2n}$ and with q an arbitrary polynomial by using the formula (4.28) for the polynomial r and the formula (4.15) for M_0.

4.4. **Completion theorem.** Another construction of a model for singular rank one perturbations generated by $\omega \in \mathfrak{H}_{-2n}$, was given in [18] via a *completion procedure* which resulted in a symmetric operator in a Pontryagin space whose Q-function was of the form (4.43). Although the model in [18] differs from the one given in Theorem 3.1, the corresponding Weyl functions coincide and, hence, the underlying selfadjoint extensions are unitarily equivalent. In the next theorem the spaces $\mathfrak{H}_0$, $\mathfrak{H}$, as well as A_0, and its lifting H_0 in $\mathfrak{H}$ are connected to each others after such a completion procedure, when applied to the model in the present paper.

Theorem 4.12. *Let* $\operatorname{ran} G \subset \mathfrak{H}_{-2n-1} \backslash \mathfrak{H}_{-2n}$ *and assume that* $0 \in \rho(A_0)$. *Let* $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet defined by the equality* (4.2). *Let* $G_0 = \widetilde{A}_0^{-n} G$, *let* S_0 *be defined by* $\operatorname{dom} S_0 = \operatorname{dom} A_0 \cap \ker G_0^*$, *let* t_j *be selfadjoint operators in* $[\mathcal{H}]$ $(j = 1, 2, \ldots, 2n-1)$, *let* Q *be as in* (2.10), *where* $q(\lambda) = \lambda^n$ *and* r *is of the form* (4.23), *and let* $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_Q$ *and* $H_0 = \ker \Gamma_0$ *be constructed as in Theorem 3.1. Consider the linear space*

$$\mathcal{P}_n = \operatorname{span}\{\mathfrak{H}_{2n}, \widetilde{A}_0^{-1}(\operatorname{ran} G), \ldots, \widetilde{A}_0^{-2n}(\operatorname{ran} G)\},$$

and define the inner product of the vectors $\widetilde{A}_0^{-j} G h_j$, $h_j \in \mathcal{H}$, $j = 1, \ldots, 2n$, *by the identities* (4.34) *with* $j, k = 1, \ldots, 2n$, *where* $t_j = G_0^* \widetilde{A}_0^{2n-j-1} G_0$ *if* $j \geq 2n$. *Then:*

(i) *the mapping* V *from* $\mathcal{P}_n$ *to* $\mathfrak{H}_0 \oplus \mathfrak{H}_Q$ *defined by*

$$V : \varphi + \sum_{j=1}^{2n} \widetilde{A}_0^{-j} G h_j \mapsto \left(\varphi + \sum_{j=n+1}^{2n} \widetilde{A}_0^{-j} G h_j \right) \oplus \begin{pmatrix} \sum_{k=1}^n f_k \otimes e_k \\ \sum_{k=1}^n h_k \otimes \widetilde{e}_k \end{pmatrix},$$

where

$$f_k = G_0^* A_0^{k-1} \varphi + \sum_{i=n+1}^{2n} t_{n-k+i} h_i, \quad 1 \leq k \leq n,$$

is isometric and $\operatorname{ran} V$ *is dense in* $\mathfrak{H}_0 \oplus \mathfrak{H}_Q$;

(ii) *the closure of the graph of the operator*

$$\widehat{A}_0 : \varphi + \sum_{j=2}^{2n} \widetilde{A}_0^{-j} G h_j \mapsto A_0 \varphi + \sum_{j=2}^{2n} \widetilde{A}_0^{-(j-1)} G h_j, \quad \varphi \in \mathfrak{H}_{2n+2},$$

coincides with the linear relation H_0 *in* $\mathfrak{H}_0 \oplus \mathfrak{H}_Q$ *under the isometry* V.

(iii) V *maps* $(\widetilde{A}_0 - \lambda)^{-1} G$, $\lambda \in \rho(A_0)$, *to the* γ*-field* $\gamma(\lambda)$ *in* (3.2).

Proof. (i) Let $\varepsilon_j = \widetilde{A}_0^{-j} G h_j$, $h_j \in \mathcal{H}$, $j = 1, \ldots, 2n$. Since for $i \leq n$ the vector $V\varepsilon_i$ takes the form

$$V\varepsilon_i = 0 \oplus \begin{pmatrix} 0 \\ h_i \otimes \widetilde{e}_i \end{pmatrix}, \tag{4.44}$$

one obtains for $i, j \leq n$

$$[V\varepsilon_i, V\varepsilon_j] = (\mathcal{B}_r h_i \otimes \widetilde{e}_i, h_j \otimes \widetilde{e}_j)_{\mathfrak{H}(Q)} = (t_{i+j-1} h_i, h_j) = \langle \varepsilon_i, \varepsilon_j \rangle_{\mathcal{P}_n}.$$

For $j > n$ the vector $V\varepsilon_j$ takes the form

$$V\varepsilon_j = \varepsilon_j \oplus \begin{pmatrix} \sum_{k=1}^n t_{n-k+j} h_j \otimes e_k \\ 0 \end{pmatrix}, \tag{4.45}$$

and hence one obtains for $i \le n < j \le 2n$

$$\begin{aligned}[V\varepsilon_i, V\varepsilon_j] &= \left(\begin{pmatrix} h_i \otimes e_{n-i+1} \\ \sum_{l=1}^n t_{i+l-1} h_i \otimes \widetilde{e}_i \end{pmatrix}, \begin{pmatrix} \sum_{k=1}^n t_{n-k+j} h_j \otimes e_k \\ 0 \end{pmatrix} \right) \\ &= (t_{i+j-1} h_i, h_j) = \langle \varepsilon_i, \varepsilon_j \rangle_{\mathcal{P}_n} .\end{aligned}$$

It follows from (4.45) that

$$[V\varepsilon_i, V\varepsilon_j] = \langle \varepsilon_i, \varepsilon_j \rangle_{\mathcal{P}_n} , \quad i, j > n.$$

Finally, the equality

$$V\varphi = \varphi \oplus \begin{pmatrix} \sum_{k=1}^n G_0^* A_0^{k-1} \varphi \otimes e_k \\ 0 \end{pmatrix} \tag{4.46}$$

yields

$$[V\varphi, V\varepsilon_j] = (\varphi, \varepsilon_j)_{\mathfrak{H}_0} = \langle \varphi, \varepsilon_j \rangle_{\mathcal{P}_n} , \quad j > n,$$

and

$$[V\varphi, V\varepsilon_j] = (G_0^* A_0^{n-j} \varphi, h_j)_{\mathcal{H}} = \left(\varphi, \widetilde{A}_0^{-j} G^* h_j \right) = \langle \varphi, \varepsilon_j \rangle_{\mathcal{P}_n} , \quad j \le n.$$

Thus, the mapping V is isometric. It follows from (4.44), (4.45), (4.46) also that $\operatorname{ran} V$ is dense in $\mathfrak{H}_0 \oplus \mathfrak{H}_Q$, since $\mathfrak{H}_{2n}$ is dense in $\mathfrak{H}$.

(ii) It follows from (3.9)–(3.13) that

$$H_0^{-1} V A_0 f_0 = f_0 \oplus \begin{pmatrix} \sum_{j=1}^n G_0^* A_0^j f_0 \otimes e_j \\ 0 \end{pmatrix} = V f_0, \quad f_0 \in \mathfrak{H}_{2n+2}.$$

Therefore, to prove the second statement, it remains to show that

$$V \widetilde{A}_0^{-(j+1)} G h = H_0^{-1} V \widetilde{A}_0^{-j} G h, \quad j = 1, \dots, 2n-1. \tag{4.47}$$

For $j < n$ it follows from (4.44) and (3.9)–(3.13) that

$$H_0^{-1} V \widetilde{A}_0^{-j} G h = 0 \oplus \begin{pmatrix} 0 \\ h_j \otimes e_{j+1} \end{pmatrix} = V \widetilde{A}_0^{-(j+1)} G h.$$

If $j = n$ one obtains from (3.9)–(3.13)

$$H_0^{-1} V \widetilde{A}_0^{-n} G h = \left(\widetilde{A}_0^{-1} G_0 h \right) \oplus \begin{pmatrix} \sum_{i=1}^n t_{2n-i+1} h \otimes e_i \\ 0 \end{pmatrix} = V \widetilde{A}_0^{-(n+1)} G h.$$

Similarly, for $j > n$ one obtains from (3.9)–(3.13) that

$$H_0^{-1} V \widetilde{A}_0^{-j} G h = \left(\widetilde{A}_0^{-(j+1)} G h \right) \oplus \begin{pmatrix} \sum_{i=1}^n t_{n+j-i+1} h \otimes e_i \\ 0 \end{pmatrix} = V \widetilde{A}_0^{-(j+1)} G h.$$

To see (iii) decompose $\widetilde{R}_\lambda G = (\widetilde{A}_0 - \lambda)^{-1} G$ as in (4.30) with n replaced by $2n$. Then

$$(\widetilde{A}_0 - \lambda)^{-1} G h = \varepsilon_1 + \dots + \lambda^{2n-1} \varepsilon_{2n} + \lambda^{2n} \widetilde{R}_\lambda \varepsilon_{2n}.$$

It is easy to see that V maps the sum $u_1 = \varepsilon_1 + \dots + \lambda^{n-1} \varepsilon_n$ to the vector $(0, 0, \Lambda^\top h)$. Moreover, the image of $u_2 = \lambda^n (\varepsilon_{n+1} + \dots + \lambda^{n-1} \varepsilon_{2n}) + \lambda^n \widetilde{R}_\lambda \varepsilon_{2n}$ is of the form $V(u_2) = (u_2, v, 0)$. In view of (4.30)

$$u_2 = \lambda^n \widetilde{R}_\lambda \widetilde{A}_0^{-n} G h = q(\lambda) (\widetilde{A}_0 - \lambda)^{-1} G_0 h = q(\lambda) \gamma_0(\lambda) h.$$

The expression for the components of v is obtained after several applications of (4.30) with different values of n and by taking into account the definition of Λ_1 in (2.14). The results is

$$v = \lambda^n \Lambda^\top (G_0^*(\widetilde{A}_0 - \lambda)^{-1} G_0)h + \Lambda_1^\top h = q(\lambda)\Lambda^\top M_0(\lambda)h + \Lambda_1^\top h.$$

Therefore, $V(\widetilde{R}_\lambda G) = \gamma(\lambda)$, $\lambda \in \rho(A_0)$. □

Remark 4.13. An analog of Theorem 4.12 is still true when $\operatorname{ran} G \subset \mathfrak{H}_{-2n-2} \backslash \mathfrak{H}_{-2n}$. Then t_{2n}, the resolvent $\widetilde{R}_\lambda$ on the left side of (4.30) and the constant term $\widetilde{A}_0^{-1}$ on the right side of (4.30) should be regularized. If t_{2n} is replaced by $t_{2n} = G_0^*(\widetilde{A}_0^{-1} - \mathcal{R})G_0 + B$ then $V(\widetilde{R}_\lambda G) = \gamma(\lambda)$ is still given by (3.2) with $M_0(\lambda) = G_0^*(\widetilde{R}_\lambda - \mathcal{R})G_0 + B$; here also the selection $\mathcal{R} = \widetilde{A}_0^{-1}$ is allowed since $0 \in \rho(A_0)$.

Finally, it is emphasized that the model constructed above for singular perturbations admits all the properties in Theorem 4.8. The property (i) in Theorem 4.8 was shown in Proposition 3.4. Part (iii) of Theorem 4.12 shows that the isometric image $\gamma(\lambda)$ of $(\widetilde{A}_0 - \lambda)^{-1}G$ is the γ-field associated with H_0, so that it satisfies (4.32) and hence also (4.33): $\gamma(\lambda)' = (H_0 - \lambda)^{-1}\gamma(\lambda)$. (One can check this last identity also directly by applying the formulas given for $(H_0 - \lambda)^{-1}$ in the proof of Proposition 3.4 with $G = \gamma(\lambda)$.) Moreover, the property (iii) in Theorem 4.8 was proved in Proposition 3.2).

5. The Dirac operator

As an application of the model constructed in Section 3 some singular perturbations of the *Dirac operator* are studied.

5.1. **Perturbations in $\mathfrak{H}_{-2}$.** Let A_0 be the free Dirac operator in $\mathfrak{H}_0 = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ given on the domain $W_2^1(\mathbb{R}) \oplus W_2^1(\mathbb{R})$ by the expression

$$D = -ic\frac{d}{dx} \otimes \sigma_1 + (c^2/2) \otimes \sigma_3 = \begin{pmatrix} \frac{c^2}{2} & -ic\frac{d}{dx} \\ -ic\frac{d}{dx} & -\frac{c^2}{2} \end{pmatrix}, \tag{5.1}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are Pauli matrices in $\mathbb{C}^2$ and $c > 0$ is the velocity of light. The spectrum of A_0 coincides with the set $(-\infty, -c^2/2] \cup [c^2/2, \infty)$, and the resolvent operator $(A_0 - \lambda)^{-1}$ is given by the integral kernel

$$R_k(\lambda, x - x') = \frac{i}{2c} \begin{pmatrix} \zeta(\lambda) & \operatorname{sgn}(x - x') \\ \operatorname{sgn}(x - x') & \zeta(\lambda)^{-1} \end{pmatrix} e^{ik(\lambda)|x - x'|}, \tag{5.2}$$

where

$$k(\lambda) = \frac{1}{c}\sqrt{\lambda^2 - c^4/4}, \quad \operatorname{Im} k(\lambda) \ge 0, \quad \zeta(\lambda) = \frac{\lambda + c^2/2}{ck(\lambda)}.$$

Define $\gamma_0 : \mathcal{H} = \mathbb{C}^2 \to \mathfrak{H}_0$ by $\gamma_0(\lambda) = R_k(\lambda, x)$, so that in particular

$$\gamma_0(0) = \frac{1}{2c} \begin{pmatrix} 1 & i \operatorname{sgn} x \\ i \operatorname{sgn} x & -1 \end{pmatrix} e^{-c/2|x|}. \tag{5.3}$$

Consider the two-dimensional perturbations of A_0,

$$A_{(\alpha)} = A_0 + G_0 \alpha G_0^*, \quad \alpha \in [\mathcal{H}], \tag{5.4}$$

with the operator $G_0 : \mathcal{H} \to \mathfrak{H}_{-2}$ given by $G_0 h = \delta \otimes h$, $h \in \mathcal{H}$. Then $G_0^* : \mathfrak{H}_{+2} \,(= \operatorname{dom} D) \to \mathcal{H}$ is given by $G_0^* f = f(0)$. Let S_0 be the domain restriction of A_0 given by

$$\operatorname{dom} S_0 = \{\, y = (y_1, y_2)^\top \in W_2^1(\mathbb{R}) \oplus W_2^1(\mathbb{R}) : \, y(0) = 0 \,\},$$

and let $\widetilde{A}_0$ be the $[\mathfrak{H}_0, \mathfrak{H}_{-2}]$-continuation of A_0. Then $\gamma_0(\lambda) = (\widetilde{A}_0 - \lambda)^{-1} G_0$ and according to Theorem 4.3,

$$S_0^* = \{\, \{y_0 + \gamma_0(0)h, \widetilde{A}_0 y - \delta \otimes h\} : \, y_0 \in W_2^1(\mathbb{R}) \oplus W_2^1(\mathbb{R}), \, h \in \mathcal{H} \,\}. \tag{5.5}$$

The boundary operators Γ_0^0 and Γ_1^0 for S_0^* can be given by (4.14). It follows from (5.3) and (5.5) that

$$\Gamma_0^0 y = h = -ic\sigma_1 (y(0+) - y(0-)), \tag{5.6}$$

and, for the special choice of $B = \frac{1}{2c}\sigma_3$, that

$$\Gamma_1^0 y = y_0(0) + \frac{1}{2c}\sigma_3 h = \frac{y(0+) + y(0-)}{2}. \tag{5.7}$$

Due to (4.17) the perturbations $A_{(\alpha)}$ are determined by the selfadjoint extensions $A_\tau = \ker(\Gamma_0 + \tau \Gamma_1)$ with τ a selfadjoint 2×2 matrix in $\mathcal{H}$. Now $\widehat{f} \in A_\tau$ can be rewritten as

$$y(0+) = \Lambda y(0-), \tag{5.8}$$

where Λ is a σ_1-unitary matrix given by

$$\Lambda = \left(i\sigma_1 - \frac{1}{2c}\tau \right)^{-1} \left(i\sigma_1 + \frac{1}{2c}\tau \right). \tag{5.9}$$

In view of (5.7) the corresponding Weyl function is given by

$$M_0(\lambda) = \frac{\gamma_{0+}(\lambda) + \gamma_{0-}(\lambda)}{2} = \frac{i}{2c} \begin{pmatrix} \zeta(\lambda) & 0 \\ 0 & \zeta(\lambda)^{-1} \end{pmatrix}.$$

Clearly, $\lim_{y\to\infty} M_0(iy) = (i/2c) I_{\mathcal{H}}$ is not selfadjoint and therefore, by [26, Section 2] or [10, Theorem 4.4], $\operatorname{ran} G_0 \subset \mathfrak{H}_{-2} \backslash \mathfrak{H}_{-1}$. The description (5.8), (5.9) of selfadjoint extensions A_τ of the operator S_0 was given in [5]. In [3] it was shown that the extensions A_τ can be considered as perturbations of A_0. In fact, the definition of $A_{(\alpha)}$ depends on the choice of a free parameter B. In the present paper the choice $B = \frac{1}{2c}\sigma_3$ is made in order to obtain the same family A_τ as in [5] and [3], cf. also [30]. For the special cases

$$\tau = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \tau = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, \quad a, b \in \mathbb{R},$$

one obtains the boundary conditions

$$y_2(0+) - y_2(0-) = -\frac{i}{c} a y_1(0) \ \text{ or } \ y_1(0+) - y_1(0-) = -\frac{i}{c} b y_2(0), \tag{5.10}$$

which characterize the one-parameter families D_a and T_b of perturbations of A_0,

$$D_a = A_0 + a\delta \otimes e_1(\,\cdot\,, \delta \otimes e_1) \ \text{ or } \ T_b = A_0 + b\delta \otimes e_2(\,\cdot\,, \delta \otimes e_2)$$

respectively, cf. [1] and [21]. Here $(f, \delta \otimes e_1) = f_1(0)$ and $(f, \delta \otimes e_2) = f_2(0)$ for all $f \in \operatorname{dom} D$.

5.2. **Perturbations in $\mathfrak{H}_{-4}$.** Now assume that G maps $\mathcal{H} = \mathbb{C}^2$ into $\mathfrak{H}_{-4}$. For the sake of simplicity let

$$Gh = \widetilde{A}_0 G_0 h = -ic\delta' \otimes \sigma_1 h + (c^2/2)\delta \otimes \sigma_3 h, \quad h \in \mathcal{H} = \mathbb{C}^2.$$

Then $G^* : \mathfrak{H}_{+4} (= \operatorname{dom} D^2) \to \mathcal{H}$ is given by $G^* f = (Df)(0)$. Setting $q(\lambda) = I_{\mathcal{H}}\lambda$ one obtains in the model in Theorem 2.2:

$$\mathcal{C}_q = \mathcal{C}_{q^\sharp} = 0, \quad \mathcal{B} = \sigma_1 \otimes I_{\mathcal{H}}.$$

According to Theorem 3.1 and [13, Theorem 5.3], the operator

$$S = \left\{ \left\{ \begin{pmatrix} y \\ \Gamma_1^0 y \\ 0 \end{pmatrix}, \begin{pmatrix} S_0^* y \\ 0 \\ \Gamma_0^0 y \end{pmatrix} \right\} : y \in \operatorname{dom} S_0^* \right\}, \tag{5.11}$$

is a simple symmetric operator in the Pontryagin space $\mathfrak{H}_0 \oplus \mathbb{C}^4$ whose inner product is determined by $I_{\mathfrak{H}_0} \oplus \mathcal{B}$. The finite rank perturbations $A_0 + G\alpha G^*$ are identified with the selfadjoint extensions H_τ, $\tau \in [\mathcal{H}]$, of the operator S, as specified in the following theorem. It is obtained by applying Theorem 3.1 with the data given above.

Theorem 5.1. *Let the operator S in $\mathfrak{H}_0 \oplus \mathbb{C}^4$ be defined by* (5.11). *Then:*

(i) *The adjoint linear relation S^* takes the form*

$$S^* = \left\{ \widehat{F} = \left\{ \begin{pmatrix} y \\ \Gamma_1^0 y \\ \widetilde{f} \end{pmatrix}, \begin{pmatrix} S_0^* y \\ \widetilde{\varphi} \\ \Gamma_0^0 y \end{pmatrix} \right\} : y \in \operatorname{dom} S_0^*, \ \widetilde{f}, \widetilde{\varphi} \in \mathcal{H} \right\}.$$

(ii) *The boundary triplet $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ for S^* and the corresponding Weyl function M, which has two negative squares, are given by*

$$\Gamma_0 \widehat{F} = \widetilde{f}, \quad \Gamma_1 \widehat{F} = \widetilde{\varphi}, \quad M(\lambda) = \frac{i\lambda^2}{2c} \begin{pmatrix} \zeta(\lambda) & 0 \\ 0 & \zeta(\lambda)^{-1} \end{pmatrix}, \quad |\lambda| > c^2/2. \tag{5.12}$$

(iii) *The selfadjoint extensions $H_\tau = \ker(\Gamma_0 + \tau\Gamma_1)$ are given by*

$$H_\tau = \left\{ \left\{ \begin{pmatrix} y \\ \Gamma_1^0 y \\ \widetilde{f} \end{pmatrix}, \begin{pmatrix} S_0^* y \\ \widetilde{\varphi} \\ \Gamma_0^0 y \end{pmatrix} \right\} : y \in \operatorname{dom} S_0^*, \ \widetilde{f} + \tau\widetilde{\varphi} = 0 \right\}. \tag{5.13}$$

(iv) *The spectrum of H_τ in $\mathbb{C} \setminus ((-\infty, -c^2/2] \cup [c^2/2, \infty))$ is characterized by the equivalence*

$$\lambda \in \sigma_p(H_\tau) \Leftrightarrow \det \left(I + \frac{i\lambda^2}{2c} \begin{pmatrix} \zeta(\lambda) & 0 \\ 0 & \zeta(\lambda)^{-1} \end{pmatrix} \tau \right) = 0. \tag{5.14}$$

(v) *The compression of the resolvent $(H_\tau - \lambda)^{-1}$ to $\mathfrak{H}_0$ takes the form*

$$P_{\mathfrak{H}_0}(H_\tau - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \lambda^2 \gamma_0(\lambda)\tau \left(I + \frac{i\lambda^2}{2c} \begin{pmatrix} \zeta(\lambda) & 0 \\ 0 & \zeta(\lambda)^{-1} \end{pmatrix} \tau \right)^{-1} \gamma_0(\bar{\lambda})^*. \tag{5.15}$$

(vi) *The function $y = P_{\mathfrak{H}_0}(H_\tau - \lambda)^{-1} z$ is a solution of the boundary value problem with the λ-depending interface condition*

$$(S_0^* - \lambda) y = z, \quad y(0+) = \Lambda(\lambda) y(0-), \tag{5.16}$$

where $\Lambda(\lambda)$ *is given by the formula*

$$\Lambda(\lambda) = (2ic\sigma_1 - \lambda^2\tau)^{-1}(2ic\sigma_1 + \lambda^2\tau). \tag{5.17}$$

By special choices of $\tau \in [\mathcal{H}]$ it is possible to generate frequently occuring cases. For instance, if

$$\tau = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \tau = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, \quad a, b \in \mathbb{R},$$

then one obtains the one-parameter families of "perturbations" $D_a^{(1)}$ or $T_b^{(1)}$ of A_0,

$$D_a^{(1)} = A_0 + a\omega_1(\cdot, \omega_1), \quad \omega_1 = \begin{pmatrix} (c^2/2)\delta \\ -ic\delta' \end{pmatrix},$$
$$T_b^{(1)} = A_0 + b\omega_2(\cdot, \omega_2), \quad \omega_2 = \begin{pmatrix} ic\delta' \\ (c^2/2)\delta \end{pmatrix},$$

respectively. Here $(f, \omega_1) = (c^2/2)f_1(0) - icf_2'(0)$ and $(f, \omega_2) = icf_1'(0) + (c^2/2)f_2(0)$ for all $f \in \operatorname{dom} D^2$. Their compressed resolvents are characterized by the following interface conditions

$$y_2(0+) - y_2(0-) = -\frac{i}{c}\lambda^2 a y_1(0) \quad \text{or} \quad y_1(0+) - y_1(0-) = -\frac{i}{c}\lambda^2 b y_2(0), \tag{5.18}$$

respectively. As is known [1], [5], [21], the perturbations $A_{(\tau)}$ in (5.4) are related to the corresponding nonrelativistic interactions of the Schrödinger operator via the nonrelativistic limit. For perturbations in $\mathfrak{H}_{-4}$ the situation is different. The nonrelativistic limit does not distinguish the perturbations $D_a^{(1)}$. Namely,

$$\lim_{c\to\infty} P_{\mathfrak{H}_0}\left(D_a^{(1)} - \left(\lambda + \frac{c^2}{2}\right)\right)^{-1} \upharpoonright \mathfrak{H}_0 = (-A_{a,\infty} - \lambda)^{-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $A_{a,\infty}$ stands for

$$A_{a,\infty} = \{\, \{y, -D^2 y\} : y \in W_2^2(\mathbb{R}\backslash\{0\}), \, y(0) = 0 \,\}.$$

Acknowledgements. We would like to thank Yury Arlinskiĭ for discussions and valuable comments which led to some improvements in the paper.

References

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable models in quantum mechanics*, Springer-Verlag, New York, 1988.

[2] S. Albeverio and P. Kurasov, "Rank one perturbations of not semibounded operators", Integral Equations Operator Theory, 27 (1997), 379–400.

[3] S. Albeverio and P. Kurasov, *Singular perturbations of differential operators*, London Math. Soc., Lecture Notes Series 271, Cambridge University Press, 1999.

[4] T.Ya. Azizov and I.S. Iokhvidov, *Foundations of the theory of linear operators in spaces with an indefinite metric*, Nauka, Moscow, 1986 (Russian) (English translation: Wiley, New York, 1989).

[5] S. Benvegnu and L. Dabrowski, "Relativistic point interaction", Letters in Mathematical Physics, 30 (1994), 159–167.

[6] Ju.M. Berezanski, *Expansions in eigenfunctions of selfadjoint operators*, Naukova Dymka, Kiev, 1965 (Russian) (English translation: Translations of Mathematical Monographs, Volume 17, American Mathematical Society, 1968).

[7] V.A. Derkach, "On generalized resolvents of Hermitian relations", J. Math. Sciences, 97 (1999), 4420–4460.

[8] V.A. Derkach and S. Hassi, "A reproducing kernel space model for $\mathbf{N}_\kappa$-functions", preprint 2001.

[9] V.A. Derkach, S. Hassi, M.M. Malamud, and H.S.V. de Snoo, "Generalized resolvents of symmetric operators and admissibility", Methods of Functional Analysis and Topology, 6 (2000), 24–55.

[10] V.A. Derkach, S. Hassi, and H.S.V. de Snoo, "Operator models associated with Kac subclasses of generalized Nevanlinna functions", Methods of Functional Analysis and Topology, 5 (1999), 65–87.

[11] V.A. Derkach, S. Hassi, and H.S.V. de Snoo, "Generalized Nevanlinna functions with polynomial asymptotic behaviour and regular perturbations", Oper. Theory Adv. Appl., 122 (2001), 169–189.

[12] V.A. Derkach, S. Hassi, and H.S.V. de Snoo, "Rank one perturbations in a Pontryagin space with one negative square", J. Funct. Anal., 188 (2002), 317–349.

[13] V.A. Derkach, S. Hassi, and H.S.V. de Snoo, "Operator models associated with singular perturbations", Methods of Functional Analysis and Topology, 7 (2001), 1–21.

[14] V.A. Derkach, S. Hassi, and H.S.V. de Snoo, "Singular perturbations as range perturbations in a Pontryagin space", (in preparation).

[15] V.A. Derkach and M.M. Malamud, "Generalized resolvents and the boundary value problems for hermitian operators with gaps", J. Funct. Anal., 95 (1991), 1–95.

[16] V.A. Derkach and M.M. Malamud, "The extension theory of hermitian operators and the moment problem", J. Math. Sciences, 73 (1995), 141–242.

[17] J.F. van Diejen and A. Tip, "Scattering from generalized point interaction using selfadjoint extensions in Pontryagin spaces", J. Math. Phys., 32 (3), (1991), 631–641.

[18] A. Dijksma, H. Langer, Yu.G. Shondin, and C. Zeinstra, "Self-adjoint operators with inner singularities and Pontryagin spaces", Oper. Theory Adv. Appl., 117 (2000), 105–176.

[19] A. Dijksma, H. Langer, and H.S.V. de Snoo, "Unitary colligations in Π_κ-spaces, characteristic functions and Štraus extensions", Pacific J. Math., 125 (1986), 347-362.

[20] A. Fleige, S. Hassi, and H.S.V. de Snoo, "A Kreĭn space approach to representation theorems and generalized Friedrichs extensions", Acta Sci. Math. (Szeged), 66 (2000), 595–612.

[21] F. Gesztesy and P. Şeba, "New analytically solvable models of relativistic point interactions", Letters in Mathematical Physics, 13 (1987), 345–358.

[22] F. Gesztesy and B. Simon, "Rank one perturbations at infinite coupling", J. Funct. Anal., 128 (1995), 245–252.

[23] V.I. Gorbachuk and M.L. Gorbachuk, *Boundary value problems for operator differential equations*, Naukova Dumka, Kiev, 1984 (Russian) (English translation: Kluwer Academic Publishers, Dordrecht, Boston, London, 1990).

[24] S. Hassi, M. Kaltenbäck, and H.S.V. de Snoo, "Triplets of Hilbert spaces and Friedrichs extensions associated with the subclass $\mathbf{N}_1$ of Nevanlinna functions", J. Operator Theory, 37 (1997), 155–181.

[25] S. Hassi, M. Kaltenbäck, and H.S.V. de Snoo, "The sum of matrix Nevanlinna functions and selfadjoint extensions in exit spaces", Oper. Theory Adv. Appl., 103 (1998), 137–154.

[26] S. Hassi, H. Langer, and H.S.V. de Snoo, "Selfadjoint extensions for a class of symmetric operators with defect numbers $(1,1)$", 15th OT Conference Proceedings, (1995), 115–145.

[27] S. Hassi and H.S.V. de Snoo, "One-dimensional graph perturbations of selfadjoint relations", Ann. Acad. Sci. Fenn. A.I. Math., 22 (1997), 123–164.

[28] S. Hassi and H.S.V. de Snoo, "On rank one perturbations of selfadjoint operators", Integral Equations Operator Theory, 29 (1997), 288–300.

[29] S. Hassi and H.S.V. de Snoo, "Nevanlinna functions, perturbation formulas and triplets of Hilbert spaces", Math. Nachr., 195 (1998), 115–138.

[30] R.J. Hughes, "Finite-rank perturbations of the Dirac operator", J. Math. Anal. Appl., 238 (1999), 67–81.

[31] P. Jonas, H. Langer, and B. Textorius, "Models and unitary equivalence of cyclic selfadjoint operators in Pontrjagin space", Oper. Theory Adv. Appl., 59 (1992), 252–284.

[32] T. Kato, *Perturbation theory for linear operators*, Springer Verlag, New York, 1966.

[33] A. Kiselev and B. Simon, "Rank one perturbations with infinitesimal coupling", J. Funct. Anal., 130 (1995), 345–356.

[34] V. Koshmanenko, Singular operator as a parameter of self-adjoint extensions", Oper. Theory Adv. Appl., 118 (2000), 205–223.

[35] M.G. Kreĭn and H. Langer, "Über die Q-function eines π-hermiteschen Operators in Raume Π_κ", Acta Sci. Math. (Szeged), 34 (1973), 191–230.

[36] M.G. Kreĭn and V.A. Yavryan, "Spectral shift functions that arise in perturbations of a positive operator", J. Operator Theory, 6 (1981), 155–191 (Russian).

[37] P. Kurasov, "$\mathfrak{H}_{-n}$-perturbations of self-adjoint operators and Krein's resolvent formula", preprint 2001.

[38] P. Kurasov and K. Watanabe, "On rank one $\mathfrak{H}_{-3}$-perturbations of positive self-adjoint operators", preprint 2000.

[39] M.M. Malamud, "On a formula for the generalized resolvents of a nondensely defined Hermitian operator", Ukrain. Mat. Zh., 44 (1992), 1658–1688.

[40] B.S. Pavlov, "The theory of extensions and explicitly solvable models", Uspekhi Mat. Nauk, 42 (1987), 99–131.

[41] Yu.G. Shondin, "Quantum-mechanical models in $\mathbb{R}_n$ associated with extensions of the energy operator in Pontryagin space", Teor. Mat. Fiz., 74 (1988), 331–344) (Russian) (English translation: Theor. Math. Phys., 74 (1988), 220-230).

[42] B. Simon, "Spectral analysis of rank one perturbations and applications", in J. Feldman, R. Froese and L.M. Rosen (editors), Proceedings on Mathematical Quantum Theory II: Schrödinger operators, CRM Proceedings and Lecture Notes, Vol. 8, Amer. Math. Soc., Providence, R.I., 1995.

[43] A.V. Štraus, "Extensions and generalized resolvents of a symmetric operator which is not densely defined", Izv. Akad. Nauk SSSR, Ser. Mat., 34 (1970), 175-202 (Russian) (English translation: Math. USSR-Izvestija, 4 (1970), 179-208).

[44] E.R. Tsekanovskiĭ and Yu. L. Shmulyan, "The theory of bi-extensions of operators on rigged Hilbert spaces. Unbounded operator colligations and characteristic functions", Uspekhi Mat. Nauk, 32:5 (1977), 69–124 (Russian). (English translation: Russian Math. Surveys, 32:5 (1977), 73–131.)

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