VAASAN YLIOPISTO

HENDRIK LUIT WIETSMA

# On Unitary Relations between Kreĭn Spaces

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**Julkaisun nimike**

Kreĭn-avaruuksien välisistä unitaarisista relaatioista

**Tiivistelmä**

Operaattoriteorian alueella on kehitetty useita menetelmiä tavallisten ja osittaisdifferentiaaliyhtälöiden reuna-arvo-ongelmien indusoimien (itseadjungoitujen) reaalisaatioiden tutkimiseksi, kuten redusoivat operaattorit, reuna-arvoavaruudet, (yleistetyt) reunakolmikot ja splitatut Dirac-rakenteet. Hiljattain on osoitettu, että kaikille edellä mainituille menetelmille voidaan antaa tulkinta Kreĭn-avaruuksien moniarvoisten unitaaristen operaattoreiden eli unitaaristen relaatioiden avulla. Tämä sekä J.W. Calkinin varhainen julkaisu ovat muodostaneet lähtökohdan nyt käsillä olevalle tutkimukselle, jossa tarkastellaan Kreĭn-avaruuksien välisiä unitaarisia relaatioita.

Tutkimuksessa kehitetään kaksi geometrisluontoista menetelmää, jotka tuottavat aiempaa yksityiskohtaisempaa tietoa Kreĭn-avaruuksien välisistä unitaarisista relaatioista, niiden rakenteesta ja keskeisistä kuvausominaisuuksista sekä toisaalta isometristen ja unitaaristen relaatioiden välillä vallitsevista eroavuuksista. Lähtökohtana näille menetelmille tutkimuksessa tarkastellaan unitaaristen relaatioiden käyttäytymistä tiettyjä maksimaalisuusominaisuuksia omaavien aliavaruuksien suhteen ja johdetaan tämän jälkeen kumpaankin menetelmään liittyen unitaarisille relaatioille lohkomuotoiset esitykset, joita voidaan pitää tämän tutkimustyön keskeisinä päätuloksina. Nämä esitykset mahdollistavat unitaaristen relaatioiden hankalasti hallittavien ominaisuuksien – jotka johtuvat Kreĭn-avaruuksien unitaaristen relaatioiden epäjatkuvuudesta – aiempaa syvällisemmän ymmärtämyksen. Työssä osoitetaan muun muassa kuinka J.W. Calkinin edellä mainitun julkaisun päätulokset voidaan todistaa helposti mainittujen lohkoesitysten avulla. Työn tuloksia sovelletaan myös laajoihin epäjatkuvien isometristen ja unitaaristen operaattoreiden luokkiin, joita tyypillisesti esiintyy osittaisdifferentiaaliyhtälöiden alueella tehtävässä tutkimuksessa.

**Asiasanat**

Operaattoreiden laajennusteoria, isometrinen relaatio, unitaarinen relaatio, Kreĭn-avaruus, tavallinen ja osittaisdifferentiaaliyhtälö, reunakolmikko, kvasireunakolmikko, Weyl funktio.

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On unitary relations between Kreĭn spaces

**Abstract**

In order to study (selfadjoint) realizations of ordinary and partial differential equations, different operator-theoretical objects have been introduced; for instance, reduction operators, boundary value spaces, (generalized) boundary triplets and split Dirac structures. Recently it was shown that all those objects can be interpreted as unitary multi-valued operators (unitary relations) between certain Kreĭn spaces. Motivated thereby, and by an early paper of J.W. Calkin, the author has investigated unitary relations between arbitrary Kreĭn spaces.

In this dissertation two geometrical approaches to unitary relations between Kreĭn spaces are developed and used to obtain further information about the structure and the essential mapping properties of unitary relations, as well as to describe the difference between isometric and unitary relations. A starting point for both approaches is an investigation of the behavior of unitary relations with respect to special types of maximal subspaces. As a consequence, block representations for unitary relations are established. Those representations, which are the main contribution of this dissertation, provide a deeper understanding of the unbounded behavior of unitary relations between Kreĭn spaces. For example, the derived representations lead to simple proofs for the main statements in Calkin's above mentioned paper. The obtained results are also applied to a large class of unbounded unitary and isometric operators which naturally occur in the study of partial differential equation.

**Keywords**

Extension theory of operators, isometric relation, unitary relation, Kreĭn space, ordinary and partial differential equation, boundary triplet, quasi-boundary triplet, Weyl function.

# PREFACE

Lying before you is a part of the results obtained in almost five years of research in mathematics. In this long period, I have been able to learn many mathematical facts and, more importantly, I have come to understand more about the methods and structure of mathematical discovery which have fascinated me for a long time. Fortunately, during this period, it has also become apparent that there are still many things left to understand.

Although I'm, naturally, solely responsible for the contents of this dissertation, it would not have been possible to get this dissertation into its current form without the aid of several people. Therefore I would, first and foremost, like to thank my supervisor Seppo Hassi who has been extremely generous with his time not only in teaching me to better understand mathematics, and in discussing and considering my work, but also in helping me settle here in Vaasa. Secondly, I would like to thank Jussi Behrndt and Henk de Snoo with whom I had the pleasure to write a number of papers. In particular, their different approach to mathematics made me understand that proving a result and making it understandable to the reader are not trivially connected. I would also like to express my gratitude to the pre-examiners Vladimir Derkach and Harald Woracek for their feedback.

Furthermore, I would also like to express my gratitude to the department of Mathematics and Statistics of the University of Vaasa and the Finnish Academy for their financial assistance which made it possible to attend a number of conferences and to visit researchers abroad. In that connection I would like to mention that the support offered at the Technical University of Berlin and the Technical University of Graz was very much appreciated. Finally, I would like to thank our department of Mathematics and Statistics for the pleasant working environment.

Vaasa, June 2012.

# CONTENTS

# List of publications

This dissertation is based on the following three articles and one preprint:

(I) Wietsma, H. L. (2011). On unitary relations between Kreĭn spaces. *Preprint available at* http://www.uwasa.fi/materiaali/pdf/isbn_978-952-476-356-1.pdf;

(II) Wietsma, H.L. (2012). Representations of unitary relations between Kreĭn spaces. *Integral Equations Operator Theory* 72, 309–344;

(III) Wietsma, H.L. (2012). Block representations for classes of isometric operators between Kreĭn spaces. *Submitted to Operators and Matrices*;

(IV) Hassi, S. and Wietsma, H.L. (2012). On Calkin's abstract symmetric boundary conditions. *To appear in London Math. Soc. Lecture Note Ser.*.

Here (II)-(IV) are refereed articles and the preprint (I) is an extended version of (II).

In addition to the aforementioned articles the writer has coauthored the following refereed articles:

(i) Behrndt, J., Hassi, S., de Snoo, H.S.V. & Wietsma, H.L. (2010). Monotone convergence theorems for semi-bounded operators and forms with applications. *Proc. Roy. Soc. Edinburth* 140A, 927–951.

(ii) Behrndt, J., Hassi, S., de Snoo, H.S.V. & Wietsma, H.L. (2011). Square-integrable solutions and Weyl functions for singular canonical systems. *Math. Nachr.* 284: no. 11–12, 1334–1384.

(iii) Behrndt, J., Hassi, S., de Snoo, H.S.V. & Wietsma, H.L. (2011). Limit properties of monotone matrix functions. *Linear Algebra Appl.* 436: no. 5, 935–953.

(iv) Behrndt, J., Hassi, S., de Snoo, H.S.V., Wietsma, H.L. & Winkler, H. (2011). Linear fractional transformations of Nevanlinna functions associated with a nonnegative operator. *To appear in Complex Anal. Oper. Theory.*.

# 1 INTRODUCTION

The subject of this dissertation is unitary relations between Kreĭn spaces. As is well known, unitary operators between Hilbert spaces are bounded everywhere defined isometric operators with bounded everywhere defined inverses. I.e., if $\{\mathfrak{H}_1, (\cdot,\cdot)_1\}$ and $\{\mathfrak{H}_2, (\cdot,\cdot)_2\}$ are Hilbert spaces, then $U$ is a unitary operator from $\{\mathfrak{H}_1, (\cdot,\cdot)_1\}$ to $\{\mathfrak{H}_2, (\cdot,\cdot)_2\}$ if and only if $\operatorname{ran} U = \mathfrak{H}_2$ and

$$(f,g)_1 = (Uf,Ug)_2, \quad \forall f,g \in \operatorname{dom} U = \mathfrak{H}_1.$$

Unitary operators between Kreĭn spaces were initially introduced as everywhere defined isometric operators with everywhere defined inverse, see (Azizov & Iokhvidov 1989: Ch. II, §5 and the remarks to that section). I.e., if $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ are Kreĭn spaces, then $U$ is a unitary operator between $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ if and only if $\operatorname{ran} U = \mathfrak{K}_2$ and

$$[f,g]_1 = [Uf,Ug]_2, \quad \forall f,g \in \operatorname{dom} U = \mathfrak{K}_1.$$

Such unitary operators, which are here called standard unitary operators, are closely connected to unitary operators between Hilbert spaces. In particular, they behave geometrically essentially the same as those unitary operators. R. Arens (1961) introduced an alternative, very general, definition of unitary relations (multi-valued operators): a relation $U$ between Kreĭn spaces is *unitary* if

$$U^{-1} = U^{[*]},$$

where the adjoint is taken with respect to the underlying indefinite inner products, cf. Yu.L. Shmul'jan (1976) and P. Sorjonen (1980). Note that all standard unitary operators satisfy the above equality. With this definition unitary relations are closed, however, they need not be bounded nor densely defined and they can be multi-valued. Therefore their behavior differs essentially from Hilbert space unitary operators.

**Example 1.1.** Let $B$ be a closed relation in the Hilbert space $\{\mathfrak{H}, (\cdot,\cdot)\}$ and on $\mathfrak{H}^2$ define the indefinite inner product $<\cdot,\cdot>$ by

$$<\{f,f'\},\{g,g'\}>= i\left[(f,g') - (f',g)\right], \quad f,f',g,g' \in \mathfrak{H}.$$

Then $\{\mathfrak{H}^2, <\cdot,\cdot>\}$ is a Kreĭn space and $U$ defined on $\mathfrak{H}^2$ as

$$U\{f_1,f_2\} = \{Bf_1, (B^*)^{-1}f_2\}, \quad f_1 \in \operatorname{dom} B,\ f_2 \in \operatorname{ran} B^*$$

is a unitary relation in $\{\mathfrak{H}^2, < \cdot, \cdot >\}$ with $\ker U = \ker B \times \operatorname{mul} B^*$ and $\operatorname{mul} U = \operatorname{mul} B \times \ker B^*$. Clearly, $U$ has closed domain (and range) if and only if $B$ and $B^{-1}$ have closed domain. Moreover, $U$ is a unitary operator with a trivial kernel if and only if $\ker B = \{0\} = \operatorname{mul} B$ and $\overline{\operatorname{dom}}\, B = \mathfrak{H} = \overline{\operatorname{ran}}\, B$. In particular, if $B$ satisfies the preceding conditions, then $U$ has an operator block representation:

$$U = \begin{pmatrix} B & 0 \\ 0 & B^{-*} \end{pmatrix},$$

where the representation is with respect to the decomposition $\mathfrak{H} \oplus \mathfrak{H}$ of $\mathfrak{H}^2$.

# Motivation

The motivation for the present study of unitary relations between Kreĭn spaces comes from the extension theory of symmetric relations in Hilbert and Kreĭn spaces. Therein unitary relations naturally appear, although usually under a different name. In particular, this work was motivated by the rediscovery of J.W. Calkin's 1939 paper on extension theory by V. Derkach, the recent investigations of extension theory in connection with partial differential equations by J. Behrndt and M. Langer, see (Behrndt & Langer 2007), and by the recent papers of V. Derkach, S. Hassi, M. Malamud and H.S.V. de Snoo where unitary relations between Kreĭn spaces appeared in the setting of extension theory, see (Derkach et al. 2006; 2009). In order to make this motivation more concrete, a short overview of the extension theory of symmetric relations is presented. This overview at the same time shows how unitary relation appear/can be used in a more practical setting.

Maximal symmetric extensions of (unbounded) symmetric operators in (separable) Hilbert spaces have initially been studied by J. von Neumann in the late twenties. He used the Cayley transform to obtain a formula which expresses the domain of the adjoint $S^*$ of a symmetric operator $S$ in terms of the domain of the symmetric operator and its defect spaces:

$$\operatorname{dom} S^* = \operatorname{dom} S + \{f_i : (S^* - i)f_i = 0\} + \{f_{-i} : (S^* + i)f_{-i} = 0\},$$

see (von Neumann 1930: Satz 29). The above expression is now known as the von Neumann formula and has formed the basis for the early investigations of extensions of symmetric operators. In particular, J. von Neumann showed that the defect numbers of a symmetric operator, which can be defined by means of the von Neumann formula, characterize which type of maximal symmetric extensions an (unbounded) symmetric operator has.

Motivated by questions connected with selfadjoint realizations of partial differential equations, cf. Example 1.5 below and see (Calkin 1939b), the investigations of maximal extension of symmetric operators was continued by J.W. Calkin almost a decade later. As the main tool in his investigations J.W. Calkin introduced reduction operators for the adjoint of symmetric operators (in Hilbert spaces), see (Calkin 1939a); these operators can in fact be interpreted as unitary operators between Kreĭn spaces, see (Hassi & Wietsma 2012: Proposition 2.7). For instance, using bounded reduction operators an elegant and complete description was given for all the maximal symmetric extensions of a symmetric operator, see (Calkin 1939a: Theorem 4.1); that result would only later be rediscovered, see (Gorbachuk & Gorbachuk 1991). Moreover, using unbounded reduction operators J.W. Calkin studied maximal extensions of a symmetric operator whose graph is contained in a dense subspace of the graph of the adjoint of the symmetric operator; this is a problem which naturally occurs in connection with partial differential equations. As in the case of bounded reduction operators, he showed that there are two possibilities: Either each maximal symmetric extension of a symmetric operator has the same defect numbers or there exist maximal symmetric extensions with ”arbitrary” defect numbers. J.W. Calkin also investigated the structure and mapping properties of reduction operators. Of particular interest is his domain decomposition of such operators, see (Calkin 1939a: Theorem 3.5); that decomposition is the central result in the aforementioned paper.

The parametrization of selfadjoint extensions of symmetric operators resurfaced in the book of N. Dunford and J.T. Schwartz (1963). Recall therefore that one can associate to ordinary differential equation a symmetric operator, the so-called minimal operator, and that its adjoint is called the maximal operator. Wanting to apply the spectral theory of selfadjoint operators to this setting, they needed to describe the selfadjoint restrictions of the maximal operator. This they in fact did by means (of systems) of so-called boundary values for the maximal operator, see Example 1.2 below.

**Example 1.2.** In the Hilbert space $L^2(\imath)$, where $\imath = [0,1]$, consider the following differential expression

$$(\ell f)(x) = f''(x) + f(x), \quad x \in \imath.$$

To study this differential equation, a maximal and minimal operator, $T_{\text{max}}$ and $T_{\text{min}}$, are associated to it:

$$T_{\text{max}} f = \ell f, \quad \operatorname{dom} T_{\text{max}} = \{f \in L^2(\imath) : \ell f \in L^2(\imath),\ f, f' \in \text{AC}_{\text{loc}}(\imath)\}$$

and

$$T_{\text{min}} f = \ell f, \quad \operatorname{dom} T_{\text{min}} = \{f \in \operatorname{dom} T_{\text{max}} : f(0) = f'(0) = f(1) = f'(1) = 0\},$$

see e.g. (Behrndt et al. 2011b). Boundary values for this setting, in terms of Dunform and Schwartz (1963), would be for example

$$a_1 f = f(0) \quad \text{or} \quad a_2 f = f'(1), \quad \operatorname{dom} a_1 = \operatorname{dom} a_2 = \operatorname{dom} T_{\text{max}}.$$

Note that it can be shown that the operator $A$ defined as

$$Af = \ell f, \quad \operatorname{dom} A = \{ f \in \operatorname{dom} T_{\text{max}} : a_1(f) = 0 = a_2(f) \}.$$

is a selfadjoint restriction of $T_{\text{max}}$.

In the seventies V.M. Bruck and A.N. Kochubeĭ independently introduced so-called boundary value spaces (BVS's) to describe the selfadjoint extensions of densely defined symmetric operators in Hilbert spaces with equal defect numbers, see (Gorbachuk & Gorbachuk 1991) and the references therein. For a densely defined symmetric operator $S$, this BVS is a triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is a auxiliary Hilbert space, often called the boundary space, and $\Gamma_0$ and $\Gamma_1$ are mappings defined on the domain of $S^*$ and mapping onto $\mathcal{H}$. As a consequence of their structure, BVS's would later usually be called ordinary boundary triplets. By means of these objects the selfadjoint extensions of $S$ can be parameterized by selfadjoint relations in $\mathcal{H}$.

**Example 1.3.** For the situation in Example 1.2 a possible choice of a boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for $T_{\text{max}}$ is

$$\mathcal{H} = \mathbb{C}^2, \quad \Gamma_0\{f, T_{\text{max}} f\} = \begin{pmatrix} f(0) \\ f'(1) \end{pmatrix} \text{ and } \Gamma_1\{f, T_{\text{max}} f\} = -\begin{pmatrix} f'(0) \\ f(1) \end{pmatrix}. \quad (1.1)$$

Note that with this definition the selfadjoint extension $A$ of $T_{\text{min}}$ in Example 1.2 is the restriction of $T_{\text{max}}$ to $\ker \Gamma_0$ and that $\Gamma = \Gamma_0 \times \Gamma_1$ is a (bounded) reduction operator for $T_{\text{max}}$ in the terminology of J.W. Calkin.

Not only was the boundary triplet introduced to describe selfadjoint extensions of symmetric operators, it was also used to describe maximal dissipative and accumulative extensions of symmetric operators and to describe spectral properties of those extensions. In order to obtain the latter results the so-called characteristic function of a symmetric operator was introduced by A.N. Kochubeĭ, see (Gorbachuk et al. 1989) and the references therein. In the middle of the eighties V. Derkach and M.M. Malamud investigated the Cayley transform of this characteristic function, see (Derkach & Malamud 1985; 1991), and showed that this transform is a so-called Q-function for the symmetric operator; those $Q$-functions had been studied earlier by M.G. Kreĭn and H. Langer. In the literature of boundary triplets this transformed characteristic function is nowadays called the Weyl function (associated to a boundary triplet).

Later V. Derkach and M.M. Malamud generalized the concept of a boundary triplet to the concept of a generalized boundary triplet, see (Derkach & Malamud 1995). This generalization allows for the realization of a greater class of functions as Weyl functions and also allows for the applicability of boundary triplet methods to a larger class of problems (without regularizing). For instance, the closure of the triplet $\{L^2(\partial\Omega), \Gamma_1, -\Gamma_0\}$ from Example 1.5 below is a generalized boundary triplet which is not an ordinary boundary triplet, see (Behrndt & Langer 2007: Proposition 4.6). There is however a price to pay for using generalized boundary triplets instead of ordinary boundary triplets, the latter are bounded (with respect to the appropriate topologies) while the former are not.

In the present millennium the aforementioned two authors together with S. Hassi and H.S.V. de Snoo developed the boundary triplet approach by, among other things, incorporating Kreĭn space terminology and methods into it, see (Derkach et al. 2006; 2009). In particular, they showed that ordinary boundary triplets, and their various generalizations, can be seen as unitary relations between Kreĭn spaces whose inner products have a specific, fixed, structure.

**Example 1.4.** Recall that for a Hilbert space $\{\mathfrak{H}, (\cdot,\cdot)_{\mathfrak{H}}\}$, $\mathfrak{H}^2$ equipped with the indefinite inner product $<\cdot,\cdot>_{\mathfrak{H}}$, defined by

$$< \{f, f'\}, \{g, g'\} >_{\mathfrak{H}} = i\left[(f, g')_{\mathfrak{H}} - (f', g)_{\mathfrak{H}}\right], \quad f, f', g, g' \in \mathfrak{H},$$

becomes a Kreĭn space. With this notation consider the mapping $\Gamma = \Gamma_0 \times \Gamma_1$ from $L^2([0,1]) \times L^2([0,1])$ to $\mathbb{C}^4$, where $\Gamma_0$ and $\Gamma_1$ are as in (1.1). Then $\Gamma$ is a (bounded) unitary operator from the Kreĭn space $\{L^2([0,1]) \times L^2([0,1]), <\cdot,\cdot>_{L^2([0,1])}\}$ onto the Kreĭn space $\{\mathbb{C}^2 \times \mathbb{C}^2, <\cdot,\cdot>_{\mathbb{C}^2}\}$.

In order to apply boundary triplet type techniques to partial differential equations, J. Behrndt and M. Langer generalized the concept of a generalized boundary triplet to the concept of a quasi-boundary triplet in their 2007 paper. Quasi-boundary triplets can not be interpreted as unitary operators between Kreĭn spaces. However, they can be interpreted as a special type of isometric operators between Kreĭn spaces, which are closely related to unitary operators between Kreĭn spaces. Quasi-boundary triplets naturally appear in the setting of partial differential equations as the following example taken from (Behrndt & Langer 2007) shows.

**Example 1.5.** Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with $C^\infty$-boundary $\partial\Omega$ and define the differential expression $\ell$ as

$$(\ell f)(x, y) = \frac{\partial^2}{\partial x^2} f(x, y) + \frac{\partial^2}{\partial y^2} f(x, y), \quad (x, y) \in \Omega,$$

i.e., $\ell$ is the Laplacian in $\mathbb{R}^2$. With $\ell$ associate a maximal operator $T$ and a minimal operator $S$ in the Hilbert space $L^2(\Omega)$ via

$$Tf = \ell f, \quad \operatorname{dom} T = H^2(\Delta)$$

and

$$Sf = \ell f, \quad \operatorname{dom} S = \left\{ f \in H^2(\Delta) : f|_{\partial\Omega} = 0 = \frac{\partial f}{\partial \nu}\bigg|_{\partial\Omega} \right\},$$

where $H^2(\Delta)$ is the Sobolev space of order two. Define the mappings $\Gamma_0$ and $\Gamma_1$ from $L^2(\Omega) \times L^2(\Omega)$ to $L^2(\partial\Omega)$ via

$$\Gamma_0\{f, Tf\} = f|_{\partial\Omega} \quad \text{and} \quad \Gamma_1\{f, Tf\} = -\frac{\partial f}{\partial \nu}\bigg|_{\partial\Omega}, \quad f \in \operatorname{dom} T.$$

Then $\ker \Gamma_0 \cap \ker \Gamma_1 = \operatorname{gr} S$ and with these operators the Laplace (or Green's) identity takes the following form:

$$(Tf, g)_\Omega - (f, Tg)_\Omega = (\Gamma_1\{f, Tf\}, \Gamma_0\{g, Tg\})_{\partial\Omega} - (\Gamma_0\{f, Tf\}, \Gamma_1\{g, Tg\})_{\partial\Omega}.$$

The above equality is precisely saying that $\Gamma = \Gamma_0 \times \Gamma_1$ is an isometric operator from the Kreĭn space $\{L^2(\Omega) \times L^2(\Omega), < \cdot, \cdot >_{L^2(\Omega)}\}$ to the Kreĭn space $\{L^2(\partial\Omega) \times L^2(\partial\Omega), < \cdot, \cdot >_{L^2(\partial\Omega)}\}$, see Example 1.4 for the notation. Moreover, it can be shown that $\overline{\operatorname{ran}}\, \Gamma = L^2(\partial\Omega) \times L^2(\partial\Omega)$ and that $A_N$ defined via

$$A_N f = Tf, \quad \operatorname{dom} A_N = \{f \in \operatorname{dom} T : \Gamma_0\{f, Tf\} = f|_{\partial\Omega} = 0\}$$

is a selfadjoint extension of the symmetric operator $S$. As a consequence of these properties, $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ is a quasi-boundary triplet for the adjoint of $S$. Moreover, the closure of $\Gamma$ is a unitary operator between $\{L^2(\Omega) \times L^2(\Omega), < \cdot, \cdot >_{L^2(\Omega)}\}$ and $\{L^2(\partial\Omega) \times L^2(\partial\Omega), < \cdot, \cdot >_{L^2(\partial\Omega)}\}$, see (Behrndt & Langer 2007: Proposition 4.6).

Other extensions of the concept of a boundary triplet have been made by V.A. Derkach, who introduced boundary triplets in Kreĭn spaces so as to be able to study extension theory of symmetric operators in Kreĭn spaces, see (Derkach 1995; 1999), and by V. Mogilevskii, who introduced D-boundary triplets to investigate extensions of symmetric operators with unequal defect numbers, see (Mogilevskii 2006). Also those objects can be interpreted as unitary operators between Kreĭn spaces. Note also that for instance the notion of a (split) Dirac structure, which appears in system theory, can be interpreted as a unitary relation, see (Behrdnt et al. 2010: Proposition 4.6).

# Aims

The main aim of this dissertation is to obtain a better understanding of the structure and geometrical behavior of unitary relations between Kreĭn spaces; in particular, of unitary relations with an unbounded operator part or, equivalently, with a non-closed domain. More specifically, it is first of all attempted to understand how much (special types of) isometric relations differ from unitary relations (here a relation $V$ between Kreĭn spaces is isometric if $V \subseteq V^{-[*]}$) and how unitary relations with a closed domain differ from those with a non-closed domain. The second major aim of this dissertation is to investigate the essential mapping properties of unitary relations. In particular, an aim is to obtain conditions for the pre-image of a neutral subspace under a unitary relation (or, more generally, under an isometric relation) to be (hyper-)maximal neutral.

# Outline

Following is an outline of this dissertation which consists out of nine chapters, including this introduction, and an appendix.

The second chapter contains preparations for the later chapters. In particular, there the basics of Kreĭn spaces are recalled and the Kreĭn space notation that will be used in this dissertation is fixed. Thereafter a special class of maximal semi-definite subspaces is introduced and characterized. This is followed by a short section on decompositions of a subspace with respect to another subspace and a section on multi-valued operators. The final section of this chapter contains some representations of semi-definite subspaces by means of multi-valued operators.

In the third chapter the basic properties and characterizations of (maximal) isometric and unitary relations are given. In particular, it is shown that the behavior of unitary relations with respect to their kernel, multi-valued part and closed uniformly definite subspaces contained in their domain and range is of a simple nature.

Thereupon, in Chapter 4, special classes of unitary relations are investigated. More specifically, unitary relations with a closed domain and standard unitary operators are considered and, moreover, two types of isometric (unitary) relations having a simple block representation are introduced. Those latter isometric (unitary) relations, which will be called archetypical isometric (unitary) relations, will play a big role in the later chapters; they, and their composition, essentially show what kind of geometrical behavior unitary relations can exhibit.

In the fifth chapter it is shown how unitary relations are characterized by means of their behavior with respect to uniformly definite subspaces. In particular, it is there shown that unitary relations can essentially be characterized by one identity; the so-called Weyl identity. Using this approach a known quasi-block representation for unitary operator is obtained which is thereafter extended to a quasi-block representation for maximal isometric operators. Also some applications of this approach to unitary relations are presented there.

Thereafter, complementing the fifth chapter, the behavior of unitary relations with respect to hyper-maximal semi-definite subspaces is investigated in the sixth chapter. In particular, there it is shown that unitary relations contain hyper-maximal semi-definite subspaces in their domain (and range).

Extending upon the results from Chapter 6, block representations for unitary relations, and also for certain types of isometric relations, are presented in the seventh chapter. Those block representations will be expressed in terms of the archetypical isometric operators introduced in the fourth chapter. In particular, it is shown that the obtained block representations for unitary operators are a useful tool by giving simple proofs for the most important statements from (Calkin 1939a).

In the eight chapter a classification from (Calkin 1939a) is considered; that classification was introduced by J.W. Calkin in order to describe the maximal neutral subspaces contained in the domain of an unbounded unitary operator (between Kreĭn spaces). In Chapter 8 that classification is extended, further implications of it are stated and new characterizations for it are given. In particular, a characterization of the classification in terms of a block representation for unitary operators is given.

Finally, Chapter 9 contains a summary of obtained results. In particular, there it is shown how the above formulated aims have been fulfilled. Furthermore, to indicate the applicability of the results the bibliography is followed by an appendix in which part of the obtained results are applied to different types of boundary triplets.

# 2 PRELIMINARIES

This chapter containing preliminary results consists out of five sections. In the first section some elementary facts about Kreĭn spaces are recalled from (Azizov & Iokhvidov 1989) and (Bognár 1974), and the Kreĭn space notation used in this dissertation is fixed. Thereupon in the second section the notion of hyper-maximality of a neutral subspace in a Kreĭn space is recalled from (Azizov & Iokhvidov 1989) and that notion is extended to all semi-definite subspaces of a Kreĭn spaces; such subspaces will be naturally encountered when unitary relations are considered, see Chapter 6. The most important property of hyper-maximal semi-definite subspaces is that they induce a orthogonal decomposition of the space. In the third section the abstract equivalents of the von Neumann formulas, used in the analysis of symmetric operators, are identified/stated. The fourth section contains a short introduction to multi-valued operators, which are also called linear relations. In the last section of this chapter representations of semi-definite subspaces by means of (Hilbert space) relations are presented. Two types of angular representation are given: The traditional representation with respect to a canonical decomposition of the space, see (Azizov & Iokhvidov 1989: Ch. 1, §8), and a second representation with respect to hyper-maximal neutral subspaces.

## 2.1 Basic properties of Kreĭn spaces

A vector space $\mathfrak{K}$ with an indefinite inner product $[\cdot,\cdot]$ is called a *Kreĭn space* if there exists a decomposition of $\mathfrak{K}$ into the direct sum of two subspaces (linear subsets) $\mathfrak{K}^+$ and $\mathfrak{K}^-$ of $\mathfrak{K}$ such that $\{\mathfrak{K}^+,[\cdot,\cdot]\}$ and $\{\mathfrak{K}^-,-[\cdot,\cdot]\}$ are Hilbert spaces and $[f^+,f^-]=0$, $f^+\in\mathfrak{K}^+$ and $f^-\in\mathfrak{K}^-$; a decomposition $\mathfrak{K}^+[+]\mathfrak{K}^-$ of $\mathfrak{K}$ is called a *canonical decomposition* of $\{\mathfrak{K},[\cdot,\cdot]\}$. (Here the sum of two subspaces $\mathfrak{M}$ and $\mathfrak{N}$ is said to be *direct* if $\mathfrak{M}\cap\mathfrak{N}=\{0\}$, in which case the sum is denoted by $\mathfrak{M}\dot{+}\mathfrak{N}$.) The dimensions of $\mathfrak{K}^+$ and $\mathfrak{K}^-$ are independent of the canonical decomposition of $\{\mathfrak{K},[\cdot,\cdot]\}$ and are denoted by $k^+$ and $k^-$, respectively.

For a Kreĭn space $\{\mathfrak{K},[\cdot,\cdot]\}$ there exists a linear operator $\mathfrak{j}$ in $\mathfrak{K}$ such that $\{\mathfrak{K},[\mathfrak{j}\cdot,\cdot]\}$ is a Hilbert space and with respect to its inner product $\mathfrak{j}^*=\mathfrak{j}^{-1}=\mathfrak{j}$. Any mapping $\mathfrak{j}$ satisfying the preceding properties is called *a fundamental symmetry* of $\{\mathfrak{K},[\cdot,\cdot]\}$. Conversely, if $\{\mathfrak{H},(\cdot,\cdot)\}$ is a Hilbert space and $\mathfrak{j}$ is a fundamental symmetry in $\{\mathfrak{H},(\cdot,\cdot)\}$, then $\{\mathfrak{H},(\mathfrak{j}\cdot,\cdot)\}$ is a Kreĭn space. Each fundamental symmetry induces a canonical decomposition and, conversely, each canonical decomposition induces a fundamental symmetry. However, all the norms generated by the different fundamental symmetries are equivalent. Hence a subspace of the Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$ is called *closed* if it is closed with respect to the definite inner product $[\mathfrak{j}\cdot,\cdot]$ for one (and hence for every) fundamental symmetry of $\{\mathfrak{K}, [\cdot,\cdot]\}$.

**Example 2.1.** Let $\{\mathfrak{H}, (\cdot,\cdot)\}$ be a Hilbert space and define $\mathfrak{j}$ on $\mathfrak{H}^2$ as

$$\mathfrak{j}\{f, f'\} = i\{-f', f\}, \quad f, f' \in \mathfrak{H}.$$

Then $\mathfrak{j}$ is a fundamental symmetry in the Hilbert space $\{\mathfrak{H}^2, (\cdot,\cdot)\}$, i.e. $\mathfrak{j} = \mathfrak{j}^{-1} = \mathfrak{j}^*$. Hence, with the sesqui-linear form $<\cdot,\cdot>$ defined on $\mathfrak{H}^2$ by

$$< \{f, f'\}, \{g, g'\} >= (\mathfrak{j}\{f, f'\}, \{g, g'\}) = i\left[(f, g') - (f', g)\right], \quad f, f', g, g' \in \mathfrak{H},$$

$\{\mathfrak{H}^2, <\cdot,\cdot>\}$ is a Kreĭn space for which $\mathfrak{j}$ is a fundamental symmetry. Note that if $\mathfrak{K}^+[+]\mathfrak{K}^-$ is the canonical decomposition of $\{\mathfrak{H}^2, <\cdot,\cdot>\}$ corresponding to the fundamental symmetry $\mathfrak{j}$, then

$$\mathfrak{K}^+ = \ker(\mathfrak{j} - I) = \{\{f, if\} : f \in \mathfrak{H}\};$$
$$\mathfrak{K}^- = \ker(\mathfrak{j} + I) = \{\{f, -if\} : f \in \mathfrak{H}\}.$$

For a subspace $\mathfrak{L}$ of the Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$ the *orthogonal complement* of $\mathfrak{L}$, denoted by $\mathfrak{L}^{[\perp]}$, is the closed subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ defined as

$$\mathfrak{L}^{[\perp]} = \{f \in \mathfrak{K} : [f, g] = 0, \ \forall g \in \mathfrak{L}\}.$$

If $\mathfrak{j}$ is a fixed fundamental symmetry of $\{\mathfrak{K}, [\cdot,\cdot]\}$, then the $\mathfrak{j}$-orthogonal complement of $\mathfrak{L}$, i.e. the orthogonal complement with respect to $[\mathfrak{j}\cdot,\cdot]$, is denoted by $\mathfrak{L}^\perp$. Clearly, $\mathfrak{L}^{[\perp]} = \mathfrak{j}\mathfrak{L}^\perp = (\mathfrak{j}\mathfrak{L})^\perp$. For subspaces $\mathfrak{M}$ and $\mathfrak{N}$ of the Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$ with a fixed fundamental symmetry $\mathfrak{j}$ the notation $\mathfrak{M}[+]\mathfrak{N}$ and $\mathfrak{M} \oplus \mathfrak{N}$ is used to indicate that the sum of $\mathfrak{M}$ and $\mathfrak{N}$ is orthogonal or $\mathfrak{j}$-orthogonal, respectively. Note that

$$\mathfrak{M}^{[\perp]} \cap \mathfrak{N}^{[\perp]} = (\mathfrak{M} + \mathfrak{N})^{[\perp]} \quad \text{and} \quad \mathfrak{M}^{[\perp]} + \mathfrak{N}^{[\perp]} \subseteq (\mathfrak{M} \cap \mathfrak{N})^{[\perp]}. \tag{2.1}$$

Lemma 2.2 below gives a condition for the inclusion in (2.1) to be an equality, see (Kato 1966: Ch. IV: Theorem 4.8).

**Lemma 2.2.** *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed subspaces of the Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then $\mathfrak{M} + \mathfrak{N}$ is closed if and only if $\mathfrak{M}^{[\perp]} + \mathfrak{N}^{[\perp]}$ is closed.*

*Moreover, if either of the above equivalent conditions holds, then*

$$\mathfrak{M}^{[\perp]} + \mathfrak{N}^{[\perp]} = (\mathfrak{M} \cap \mathfrak{N})^{[\perp]}.$$

A projection $P$ or $\mathcal{P}$ onto a closed subspace of the Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$ with fundamental symmetry $\mathfrak{j}$ is called *orthogonal* or $\mathfrak{j}$-*orthogonal* if

$$\mathfrak{K} = \ker P[+]\operatorname{ran} P \quad \text{or} \quad \mathfrak{K} = \ker \mathcal{P} \oplus \operatorname{ran} \mathcal{P},$$

respectively. Recall in this connection that $\{\ker P, [\cdot,\cdot]\}$ and $\{\operatorname{ran} P, [\cdot,\cdot]\}$ are Kreĭn spaces, see (Azizov & Iokhvidov 1989: Ch. 1, Theorem 7.16). Note that for a canonical decomposition $\mathfrak{K}^+[+]\mathfrak{K}^-$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$, with associated fundamental symmetry $\mathfrak{j}$, the projections $P^+$ and $P^-$ onto $\mathfrak{K}^+$ and $\mathfrak{K}^-$, respectively, are orthogonal and $\mathfrak{j}$-orthogonal projections. For a subspace $\mathfrak{L}$ those projections satisfy

$$\mathfrak{L}^{[\perp]} \cap \mathfrak{K}^+ = \mathfrak{K}^+ \ominus P^+\mathfrak{L} \quad \text{and} \quad \mathfrak{L}^{[\perp]} \cap \mathfrak{K}^- = \mathfrak{K}^- \ominus P^-\mathfrak{L}. \tag{2.2}$$

A subspace $\mathfrak{L}$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$ is called *positive, negative, nonnegative, nonpositive* or *neutral* if $[f,f] > 0$, $[f,f] < 0$, $[f,f] \geq 0$, $[f,f] \leq 0$ or $[f,f] = 0$ for every $f \in \mathfrak{L} \setminus \{0\}$, respectively. A positive or negative subspace $\mathfrak{L}$ is called *uniformly positive* or *negative* if there exists a constant $\alpha > 0$ such that $[\mathfrak{j}f,f] \leq \alpha[f,f]$ or $[\mathfrak{j}f,f] \leq -\alpha[f,f]$ for all $f \in \mathfrak{L} \setminus \{0\}$ and a fundamental symmetry $\mathfrak{j}$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$, respectively. Note that a subspace $\mathfrak{L}$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$ is neutral if and only if $\mathfrak{L} \subseteq \mathfrak{L}^{[\perp]}$. This observation together with (2.2) yields the following result.

**Proposition 2.3.** *(Azizov & Iokhvidov 1989: Ch. 1, Corollary 5.8) Let $\mathfrak{L}$ be a neutral subspace of the Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then $\{\mathfrak{L}^{[\perp]}/\operatorname{clos}(\mathfrak{L}), [\cdot,\cdot]\}$ is a Kreĭn space*[1].

Furthermore, a subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ having a certain property is said to be *maximal* with respect to that property, if there does not exist an extensions of it having the same property. A subspace is said to *essentially* have a certain property if its closure has the indicated property.

**Remark 2.4.** In this dissertation the notation $\{\mathfrak{H}, (\cdot,\cdot)\}$ and $\{\mathfrak{K}, [\cdot,\cdot]\}$ is always used to denote Hilbert and Kreĭn spaces, respectively. To distinguish different Hilbert and Kreĭn spaces subindexes are used: $\mathfrak{H}_1, \mathfrak{K}_1, \mathfrak{H}_2, \mathfrak{K}_2$, etc.. Closed subspaces of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$, which are themselves Kreĭn spaces with the inner product $[\cdot,\cdot]_i$, are denoted by $\widetilde{\mathfrak{K}}_i$ or $\widehat{\mathfrak{K}}_i$. A canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ is denoted by $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$, its associated fundamental symmetry is denoted by $\mathfrak{j}_i$, and $P_i^+$ and $P_i^-$ always denoted the orthogonal projection onto $\mathfrak{K}_i^+$ and $\mathfrak{K}_i^+$, respectively.

[1]The indefinite inner product on the quotient space, induced by the indefinite inner product on the original space, is always indicated by the same symbol.

## 2.2 Hyper-maximal semi-definite subspaces

Recall the following characterizations of maximal nonnegative and maximal nonpositive subspaces, see (Bognár 1974: Ch. V, Section 4).

**Proposition 2.5.** *Let $\mathfrak{L}$ be a nonnegative (nonpositive) subspace of $\{\mathfrak{K},[\cdot,\cdot]\}$ and let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K},[\cdot,\cdot]\}$ with associated projections $P^+$ and $P^-$. Then equivalent are*

(i) *$\mathfrak{L}$ is a maximal nonnegative (nonpositive) subspace of $\{\mathfrak{K},[\cdot,\cdot]\}$;*

(ii) *$P^+\mathfrak{L}=\mathfrak{K}^+$ ($P^-\mathfrak{L}=\mathfrak{K}^-$);*

(iii) *$\mathfrak{L}$ is closed and $\mathfrak{L}^{[\perp]}$ is a nonpositive (nonnegative) subspace of $\{\mathfrak{K},[\cdot,\cdot]\}$;*

(iv) *$\mathfrak{L}$ is closed and $\mathfrak{L}^{[\perp]}$ is a maximal nonpositive (nonnegative) subspace of $\{\mathfrak{K},[\cdot,\cdot]\}$.*

Next recall that a (neutral) subspace $\mathfrak{L}$ of $\{\mathfrak{K},[\cdot,\cdot]\}$ is called *hyper-maximal neutral* if it is simultaneously maximal nonnegative and maximal nonpositive, see (Azizov & Iokhvidov 1989: Ch. 1, Definition 4.15). Equivalently, $\mathfrak{L}$ is hyper-maximal neutral if and only if $\mathfrak{L}=\mathfrak{L}^{[\perp]}$, cf. Proposition 2.5. I.e., if $\mathfrak{j}$ is a fundamental symmetry for $\{\mathfrak{K},[\cdot,\cdot]\}$, then $\mathfrak{L}$ is hyper-maximal neutral if and only if $\mathfrak{K}$ has the following orthogonal decomposition:

$$\mathfrak{K}=\mathfrak{L}\oplus\mathfrak{j}\mathfrak{L}. \tag{2.3}$$

The following result gives additional characterizations of hyper-maximal neutrality by means of a canonical decomposition of the corresponding Kreĭn space, see (Azizov & Iokhvidov 1989: Ch. 1, Theorem 4.5 & Theorem 8.10).

**Proposition 2.6.** *Let $\mathfrak{L}$ be a neutral subspace of $\{\mathfrak{K},[\cdot,\cdot]\}$ and let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K},[\cdot,\cdot]\}$ with associated projections $P^+$ and $P^-$. Then equivalent are*

(i) *$\mathfrak{L}$ is hyper-maximal neutral;*

(ii) *$P^+\mathfrak{L}=\mathfrak{K}^+$ and $P^-\mathfrak{L}=\mathfrak{K}^-$;*

(iii) *$U_\mathfrak{L}$ defined via $\operatorname{gr}U_\mathfrak{L}=\{\{P^+f,P^-f\}\in\mathfrak{K}^+\times\mathfrak{K}^- : f\in\mathfrak{L}\}$ is a standard unitary operator from $\{\mathfrak{K}^+,[\cdot,\cdot]\}$ onto $\{\mathfrak{K}^-,-[\cdot,\cdot]\}$.*

Note that $U_{\mathfrak{L}}$ in Proposition 2.6 (iii) is called the angular operator w.r.t. $\mathfrak{K}^+$ of $\mathfrak{L}$, see Section 2.5 below. As a consequence of Proposition 2.6, $k^+ = k^-$ if there exists a hyper-maximal neutral subspace in $\{\mathfrak{K}, [\cdot,\cdot]\}$. The converse also holds: If $k^+ = k^-$, then there exist hyper-maximal neutral subspaces in $\{\mathfrak{K}, [\cdot,\cdot]\}$, see Example 2.7 below. By definition hyper-maximal neutral subspaces are maximal neutral subspaces, the converse does not in general hold as the next example shows.

**Example 2.7.** Let $\{\mathfrak{H}, (\cdot,\cdot)\}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n\geq 0}$, $e_n \in \mathfrak{H}$. Define the indefinite inner product $[\cdot,\cdot]$ on $\mathfrak{H}^2$ by

$$[\{f, f'\}, \{g, g'\}] = (f, g) - (f', g'), \quad f, f', g, g' \in \mathfrak{H}.$$

Then $\{\mathfrak{H}^2, [\cdot,\cdot]\}$ is a Kreĭn space. Now define the subspace $\mathfrak{L}_1$ and $\mathfrak{L}_2$ of $\mathfrak{K}$ as

$$\mathfrak{L}_1 = \overline{\operatorname{span}}\,\{\{e_n, e_n\} : n \in \mathbb{N}\} \quad \text{and} \quad \mathfrak{L}_2 = \overline{\operatorname{span}}\,\{\{e_n, e_{2n}\} : n \in \mathbb{N}\}.$$

Then $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are maximal neutral subspaces of $\{\mathfrak{H}^2, [\cdot,\cdot]\}$, but only $\mathfrak{L}_1$ is a hyper-maximal neutral subspace of $\{\mathfrak{H}^2, [\cdot,\cdot]\}$.

The above example can be modified to show that there also exist different types of maximal nonpositive and nonnegative subspaces of Kreĭn spaces. Hence the notion of hyper-maximality can meaningfully be extended to semi-definite subspaces.

**Definition 2.8.** Let $\mathfrak{L}$ be a nonnegative or nonpositive subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then $\mathfrak{L}$ is called *hyper-maximal nonnegative* or *hyper-maximal nonpositive* if $\mathfrak{L}$ is closed and $\mathfrak{L}^{[\perp]}$ is a neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$.

Some alternative characterizations for semi-definite subspaces to be hyper-maximal semi-definite are provided by the following proposition.

**Proposition 2.9.** *Let $\mathfrak{L}$ be a nonnegative (nonpositive) subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$ with associated fundamental symmetry $\mathfrak{j}$ and projections $P^+$ and $P^-$. Then equivalent are*

(i) *$\mathfrak{L}$ is hyper-maximal nonnegative (nonpositive);*

(ii) *$\mathfrak{L}$ is closed, $\mathfrak{L}^{[\perp]} \subseteq \mathfrak{L}$ and $P^-\mathfrak{L}^{[\perp]} = \mathfrak{K}^-$ ($P^+\mathfrak{L}^{[\perp]} = \mathfrak{K}^+$);*

(iii) *$\mathfrak{L}$ is closed and $\mathfrak{L} = \mathfrak{L}^{[\perp]} + \mathfrak{L} \cap \mathfrak{K}^+$ ($\mathfrak{L} = \mathfrak{L}^{[\perp]} + \mathfrak{L} \cap \mathfrak{K}^-$);*

(iv) *$\mathfrak{L}$ is closed and induces the following orthogonal decomposition of $\mathfrak{K}$:*

$$\mathfrak{K} = \mathfrak{L}^{[\perp]} \oplus (\mathfrak{L} \cap \mathfrak{j}\mathfrak{L}) \oplus \mathfrak{j}\mathfrak{L}^{[\perp]}.$$

*Proof.* The statement will only be proven in the case that $\mathfrak{L}$ is a nonnegative subspace, the other case can be proven by similar arguments.

(i) $\Rightarrow$ (ii): Since $\mathfrak{L}^{[\perp]}$ is neutral, $\mathfrak{L}^{[\perp]} \subseteq \mathfrak{L}^{[\perp][\perp]} = \operatorname{clos}\mathfrak{L} = \mathfrak{L}$. Next let $f^- \in \mathfrak{K}^- \ominus P^-\mathfrak{L}^{[\perp]} = \operatorname{clos}(\mathfrak{L}) \cap \mathfrak{K}^-$, see (2.2). Since $\mathfrak{L}$ is by assumption closed and nonnegative, it follows that $f^- = 0$, i.e., $P^-\mathfrak{L}^{[\perp]} = \mathfrak{K}^-$.

(ii) $\Rightarrow$ (iii): It suffices to prove the inclusion $\mathfrak{L} \subseteq \mathfrak{L}^{[\perp]} + \mathfrak{L} \cap \mathfrak{K}^+$. Hence, let $f \in \mathfrak{L}$ be decomposed as $f^+ + f^-$, where $f^\pm \in \mathfrak{K}^\pm$. Then the assumption $P^-\mathfrak{L}^{[\perp]} = \mathfrak{K}^-$ implies that there exists a $g^+ \in \mathfrak{K}^+$ such that $g^+ + f^- \in \mathfrak{L}^{[\perp]}$ and, hence, $f - (g^+ + f^-) = f^+ - g^+ \in \mathfrak{L} \cap \mathfrak{K}^+$, because by assumption $\mathfrak{L}^{[\perp]} \subseteq \mathfrak{L}$.

(iii) $\Rightarrow$ (iv): Since $\mathfrak{L}$ is closed, $\mathfrak{L} \cap \mathfrak{K}^+ = \mathfrak{L} \cap \mathrm{j}\mathfrak{L}$ is a closed subspace. Moreover, since $\mathfrak{L}$ is nonnegative, the second assumption in (iii) implies that $\mathfrak{L}$ is the orthogonal sum of $\mathfrak{L}^{[\perp]}$ and $\mathfrak{L} \cap \mathfrak{K}^+$. In other words, $\mathfrak{L}^{[\perp]}$ is a hyper-maximal neutral subspace of the Kreĭn space $\{\mathfrak{K} \ominus (\mathfrak{L} \cap \mathrm{j}\mathfrak{L}), [\cdot,\cdot]\}$. Hence, (2.3) implies (iv).

(iv) $\Rightarrow$ (i): The decomposition in (iv) implies that $\mathfrak{L} \cap \mathrm{j}\mathfrak{L}$ is closed and the assumption that $\mathfrak{L}$ is nonnegative implies that $\mathfrak{L} \cap \mathrm{j}\mathfrak{L} \subseteq \mathfrak{K}^+$. Consequently, the decomposition in (iv) implies that $\mathfrak{L}^{[\perp]}$ is a hyper-maximal neutral subspace of the Kreĭn space $\{\mathfrak{K} \ominus (\mathfrak{L} \cap \mathrm{j}\mathfrak{L}), [\cdot,\cdot]\}$, see (2.3). Hence, $\mathfrak{L}^{[\perp]}$ is a maximal neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and, consequently, (i) holds, because $\mathfrak{L}$ is by assumption closed. □

Recall that by definition $\mathfrak{L}^{[\perp]}$ is a maximal neutral subspace if $\mathfrak{L}$ is a hyper-maximal semi-definite subspace. Proposition 2.9 shows that the converse also holds: if $\mathfrak{L}$ is a maximal neutral subspace, then $\mathfrak{L}^{[\perp]}$ is a hyper-maximal semi-definite subspace. Corollary 2.10 below shows that hyper-maximal semi-definite subspaces can also be characterized by means of projections associated with canonical decompositions of the space. Note that different from the case of hyper-maximal neutral subspaces, see Proposition 2.6, here conditions on one pair of projections do not suffice.

**Corollary 2.10.** *Let $\mathfrak{L}$ be a semi-definite subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then $\mathfrak{L}$ is hyper-maximal semi-definite if and only if $P^+\mathfrak{L} = \mathfrak{K}^+$ and $P^-\mathfrak{L} = \mathfrak{K}^-$ for every canonical decomposition $\mathfrak{K}^+[+]\mathfrak{K}^-$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$ with associated projections $P^+$ and $P^-$.*

*Proof.* To prove the statement w.l.o.g. assume that $\mathfrak{L}$ is nonnegative.

Let $\mathfrak{L}$ be hyper-maximal nonnegative and let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$ with associated projections $P^+$ and $P^-$. Then Proposition 2.9 (iv) implies that $\mathfrak{L} \cap \mathrm{j}\mathfrak{L} = \mathfrak{L} \cap \mathfrak{K}^+$ is closed and that $\mathfrak{L}^{[\perp]}$ is a hyper-maximal neutral subspace of the Kreĭn space $\{\mathfrak{K} \ominus (\mathfrak{L} \cap \mathrm{j}\mathfrak{L}), [\cdot,\cdot]\}$. Hence, $P^\pm\mathfrak{L}^{[\perp]} = \mathfrak{K}^\pm \ominus (\mathfrak{L} \cap \mathrm{j}\mathfrak{L})$, see Proposition 2.6, and $\mathfrak{L}^{[\perp]} + \mathfrak{L} \cap \mathrm{j}\mathfrak{L} \subseteq \mathfrak{L}$, because $\mathfrak{L}$ is by assumption closed. These observations show that the stated characterization holds.

Conversely, if $P^+\mathfrak{L} = \mathfrak{K}^+$ and $P^-\mathfrak{L} = \mathfrak{K}^-$ for every projection $P^+$ and $P^+$ as in the statement. Then $P^+\mathfrak{L} = \mathfrak{K}^+$ implies that $\mathfrak{L}$ is maximal nonnegative, and hence closed, and that $\mathfrak{L}^{[\perp]}$ is a maximal nonpositive subspace, see Proposition 2.5. Suppose that $f \in \mathfrak{L}^{[\perp]}$ is such that $[f, f] < 0$, then there exists a canonical decomposition $\mathfrak{K}_a^+[+]\mathfrak{K}_a^-$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that $f \in \mathfrak{K}_a^-$, see (Bognár 1974: Ch. V, Theorem 5.6). I.e., $f \in \mathfrak{L}^{[\perp]} \cap \mathfrak{K}_a^- = \mathfrak{K}_a^- \ominus P_a^-\mathfrak{L}$, see (2.2), which is in contradiction with the assumption that $P_a^-\mathfrak{L} = \mathfrak{K}_a^-$. Consequently, $\mathfrak{L}^{[\perp]}$ is neutral and, hence, $\mathfrak{L}$ is a hyper-maximal nonnegative subspace. □

Corollary 2.10 shows that hyper-maximal nonnegative (nonpositive) subspaces are also maximal nonnegative (nonpositive), justifying the terminology. It also shows that in a Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$ with $k^+ > k^-$ or $k^+ < k^-$ every hyper-maximal semi-definite subspace is nonnegative or nonpositive, respectively. If $k^+ = k^-$, then a hyper-maximal semi-definite subspace can be neutral, nonnegative or nonpositive.

**Example 2.11.** With the notation as in Example 2.7, $\{\mathfrak{H}^2, [\cdot,\cdot]\}$ is a Kreĭn space with $k^+ = k^-$. In this Kreĭn space $\mathfrak{L}_1$ is a hyper-maximal neutral subspace, whilst $\mathfrak{L}_2^{[\perp]}$ is a hyper-maximal nonpositive subspace.

## 2.3 Abstract von Neumann formulas

Let $\mathfrak{L}$ be a neutral subspace of the Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$ with a canonical decomposition $\mathfrak{K}^+[+]\mathfrak{K}^-$. Then the *(abstract) first von Neumann formula holds:*

$$\mathfrak{L}^{[\perp]} = \operatorname{clos}(\mathfrak{L})[\oplus](\mathfrak{L}^{[\perp]} \cap \mathfrak{K}^+)[\oplus](\mathfrak{L}^{[\perp]} \cap \mathfrak{K}^-), \tag{2.4}$$

see (Azizov & Iokhvidov 1989: Ch. 1, 4.20) and (2.2). Note that (2.4) is nothing else than the canonical decomposition for the Kreĭn space $\{\mathfrak{L}^{[\perp]} \ominus \mathfrak{L}, (\mathrm{j}\cdot,\cdot)\}$ induced by the canonical decomposition $\mathfrak{K}^+[+]\mathfrak{K}^-$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$. As a consequence of the first von Neumann formula and Lemma 2.3, the notion of defect numbers for neutral subspaces of Kreĭn spaces as introduced below is well-defined, see (Azizov & Iokhvidov 1989: Ch. 1, Theorem 6.7). This definition extends the usual definition of defect numbers for symmetric relations, see Appendix A.

**Definition 2.12.** Let $\mathfrak{L}$ be a neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then the defect numbers $n_+(\mathfrak{L})$ and $n_-(\mathfrak{L})$ of $\mathfrak{L}$ are defined as

$$n_+(\mathfrak{L}) = \dim(\mathfrak{L}^{[\perp]} \cap \mathfrak{K}^-) \quad \text{and} \quad n_-(\mathfrak{L}) = \dim(\mathfrak{L}^{[\perp]} \cap \mathfrak{K}^+).$$

The following generalization of the *second von Neumann formula* will be useful in the analysis of unitary relations.

**Proposition 2.13.** *Let $\mathfrak{L}$ and $\mathfrak{M}$ be subspaces of $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that $\mathfrak{M} \subseteq \mathfrak{L}$ and let $P$ be an orthogonal projection in $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then*

$$P\mathfrak{L} = P\mathfrak{M} \quad \text{if and only if} \quad \mathfrak{L} = \mathfrak{M} + \mathfrak{L} \cap \ker P. \tag{2.5}$$

*Furthermore, if $\mathfrak{M}$ is closed, $P\mathfrak{L} = P\mathfrak{M}$ and $(I-P)\mathfrak{M}^{[\perp]} + (I-P)(\mathfrak{L} \cap \ker P)^{[\perp]}$ is closed, then*

(i) $\mathfrak{L} \cap \ker P$ *is closed if and only if* $\mathfrak{L}$ *is closed;*

(ii) $\operatorname{clos}(\mathfrak{L} \cap \ker P) = (\operatorname{clos} \mathfrak{L}) \cap \ker P$.

*Proof.* Clearly, if $\mathfrak{L} = \mathfrak{M} + \mathfrak{L} \cap \ker P$, then $P\mathfrak{L} = P\mathfrak{M}$. To prove the converse let $f \in \mathfrak{L}$, then the assumption that $P\mathfrak{L} = P\mathfrak{M}$ implies that there exists a $g \in \mathfrak{M}$ such that $Pf = Pg$, i.e. $f - g \in \ker P$. Since by assumption $\mathfrak{M} \subseteq \mathfrak{L}$, $f - g$ is also contained in $\mathfrak{L}$, i.e. $f - g \in \mathfrak{L} \cap \ker P$. These arguments show that $\mathfrak{L} \subseteq \mathfrak{M} + \mathfrak{L} \cap \ker P$. Since the reverse inclusion clearly holds, this completes the proof of (2.5).

(i): If $\mathfrak{L}$ is closed, then $\mathfrak{L} \cap \ker P$ is clearly closed. To prove the converse note first that $\operatorname{ran} P \subseteq (\mathfrak{L} \cap \ker P)^{[\perp]}$. Therefore the assumption that $(I-P)\mathfrak{M}^{[\perp]} + (I-P)(\mathfrak{L} \cap \ker P)^{[\perp]}$ is closed implies that $\mathfrak{M}^{[\perp]} + (\mathfrak{L} \cap \ker P)^{[\perp]}$ is closed. This fact together with the assumptions that $\mathfrak{M}$ and $\mathfrak{L} \cap \ker P$ are closed implies that $\mathfrak{M} + \mathfrak{L} \cap \ker P$ is closed, see Lemma 2.2. Consequently, the closedness of $\mathfrak{L}$ now follows from (2.5).

(ii): The assumptions $P\mathfrak{L} = P\mathfrak{M}$ and $\mathfrak{M} \subseteq \mathfrak{L}$ yield by (2.5) that

$$\mathfrak{L} = \mathfrak{M} + (\mathfrak{L} \cap \ker P) \subseteq \mathfrak{M} + \operatorname{clos}(\mathfrak{L} \cap \ker P) \subseteq \operatorname{clos}(\mathfrak{L}).$$

Since $\mathfrak{M} + \operatorname{clos}(\mathfrak{L} \cap \ker P)$ is closed (see the proof of (i)), taking closures in the above equation yields that $\operatorname{clos}(\mathfrak{L}) = \mathfrak{M} + \operatorname{clos}(\mathfrak{L} \cap \ker P)$ and therefore $P(\operatorname{clos} \mathfrak{L}) \subseteq P\mathfrak{M}$. Now, (2.5) implies that $\operatorname{clos}(\mathfrak{L}) = \mathfrak{M} + (\operatorname{clos} \mathfrak{L}) \cap \ker P$, i.e.,

$$\mathfrak{M} + \operatorname{clos}(\mathfrak{L} \cap \ker P) = \operatorname{clos}(\mathfrak{L}) = \mathfrak{M} + (\operatorname{clos} \mathfrak{L}) \cap \ker P.$$

From this it follows that (ii) holds. □

Let $\mathfrak{j}$ be a fundamental symmetry of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then observe that $(I-P)\mathfrak{M}^{[\perp]} + (I-P)(\mathfrak{L} \cap \ker P)^{[\perp]}$ is closed, if the following inclusion holds

$$\begin{aligned}(I-P)\mathfrak{M}^{[\perp]} &\supseteq ((I-P)(\mathfrak{L} \cap \ker P)^{[\perp]})^{\perp} \cap \ker P \\ &= \mathfrak{j}\operatorname{clos}(\mathfrak{L} \cap \ker P) \cap \ker P + (\mathfrak{j}\operatorname{ran} P) \cap \ker P.\end{aligned}$$

**Corollary 2.14.** *Let $\mathfrak{L}$ and $\mathfrak{M}$ be subspaces of $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that $\mathfrak{M} \subseteq \mathfrak{L}$ and let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$ with associated projections $P^+$ and $P^-$. Then*

$$P^-\mathfrak{L} = P^-\mathfrak{M} \quad \textit{if and only if} \quad \mathfrak{L} = \mathfrak{M} + \mathfrak{L} \cap \mathfrak{K}^+;$$
$$P^+\mathfrak{L} = P^+\mathfrak{M} \quad \textit{if and only if} \quad \mathfrak{L} = \mathfrak{M} + \mathfrak{L} \cap \mathfrak{K}^-.$$

*Furthermore, if $\mathfrak{M}$ is closed, $P^-\mathfrak{L} = P^-\mathfrak{M}$ and $\operatorname{clos}(\mathfrak{L} \cap \mathfrak{K}^+) \subseteq P^+\mathfrak{M}^{[\perp]}$, then*

*(i)* $\mathfrak{L} \cap \mathfrak{K}^+$ *is closed if and only if* $\mathfrak{L}$ *is closed;*

*(ii)* $\operatorname{clos}(\mathfrak{L} \cap \mathfrak{K}^+) = (\operatorname{clos} \mathfrak{L}) \cap \mathfrak{K}^+$*;*

*and if $\mathfrak{M}$ is closed, $P^+\mathfrak{L} = P^+\mathfrak{M}$ and $\operatorname{clos}(\mathfrak{L} \cap \mathfrak{K}^-) \subseteq P^-\mathfrak{M}^{[\perp]}$, then*

*(i')* $\mathfrak{L} \cap \mathfrak{K}^-$ *is closed if and only if* $\mathfrak{L}$ *is closed;*

*(ii')* $\operatorname{clos}(\mathfrak{L} \cap \mathfrak{K}^-) = (\operatorname{clos} \mathfrak{L}) \cap \mathfrak{K}^-$.

*Proof.* The observation preceding this statement shows that the condition that the subspace $(I - P)\mathfrak{M}^{[\perp]} + (I - P)(\mathfrak{L} \cap \ker P)^{[\perp]}$ is closed for $P = P^-$ or $P = P^+$, if $\operatorname{clos}(\mathfrak{L} \cap \mathfrak{K}^+) \subseteq P^+\mathfrak{M}^{[\perp]}$ or $\operatorname{clos}(\mathfrak{L} \cap \mathfrak{K}^-) \subseteq P^-\mathfrak{M}^{[\perp]}$, respectively. Hence, this statement follows from Proposition 2.13 by taking $P$ to be $P^-$ and $P^+$. □

Note that if $\mathfrak{L}$ is a subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$, then the conditions $P^-\mathfrak{L} = P^-\mathfrak{M}$ and $\operatorname{clos}(\mathfrak{L}\cap\mathfrak{K}^+) \subseteq P^+\mathfrak{M}^{[\perp]}$ are satisfied for any hyper-maximal nonpositive subspace $\mathfrak{M} \subseteq \mathfrak{L}$, and the conditions $P^+\mathfrak{L} = P^+\mathfrak{M}$ and $\operatorname{clos}(\mathfrak{L} \cap \mathfrak{K}^-) \subseteq P^-\mathfrak{M}^{[\perp]}$ are satisfied for any hyper-maximal nonnegative subspace $\mathfrak{M} \subseteq \mathfrak{L}$.

## 2.4 Multi-valued operators in Kreĭn spaces

Recall that a mapping $H$ from a set $X$ to set $Y$ is called a multi-valued mapping if $Hx := H(x)$ is a subset of $Y$ for every $x \in X$. Using this concept $H$ is called a *(linear) multi-valued operator* from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ if $H$ is a linear multi-valued mapping from a subspace of $\mathfrak{K}_1$, called the domain of $H$ or $\operatorname{dom} H$ for short, to $\mathfrak{K}_2$ such that

$$H(f + cg) = Hf + cHg, \quad f, g \in \operatorname{dom} H,\ c \in \mathbb{C},$$

see (Cross 1998). Here $Hf + cHg$ is the sum of subspaces of $\mathfrak{K}_2$, i.e., $Hf + cHg = \{f' + cg' : f' \in Hf \text{ and } g' \in Hg\}$. For a (linear) multi-valued operator $H$ and a subspace $\mathfrak{L} \subseteq \operatorname{dom} H$, the subspace $H(\mathfrak{L})$ of $\mathfrak{K}_2$ is defined as

$$H(\mathfrak{L}) = \{f' \in \mathfrak{K}_2 : \exists f \in \mathfrak{L} \text{ s.t. } f' \in Hf\}.$$

Using this definition, the range, the multi-valued part and the kernel of a multi-valued operator $H$ are defined as follows:

$$\operatorname{ran} H = H(\operatorname{dom} H), \quad \operatorname{mul} H = H0, \quad \ker H = \{f \in \operatorname{dom} H : Hf = \operatorname{mul} H\}.$$

Since a multi-valued operator is linear, for a fixed fundamental symmetry $\mathfrak{j}_2$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ there exists for every $f \in \operatorname{dom} H$ a unique $f' \in (\operatorname{mul} H)^{\perp_2}$ such that $Hf = f' + \operatorname{mul} H$. A (single-valued) linear operator which on the basis of the preceding observation can be associated to a multi-valued operator, will be called *an operator part* of a multi-valued operator $H$ (w.r.t. $\mathfrak{j}_2$) and is denoted by $H_o$. In particular, $Hf = H_o f + \operatorname{mul} H$, $f \in \operatorname{dom} H$, and, hence, $H = H_o$ if and only if $\operatorname{mul} H = \{0\}$. In that case the multi-valued operator is an ordinary (linear) operator and the above definitions of the domain and kernel reduce to their usual form. A multi-valued operator is called *closed* if an operator part is a closed operator and $\operatorname{mul} H$ is closed subspace (of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$). The graph of a multi-valued operator $H$ is the subspace $\operatorname{gr} H$ of $\mathfrak{K}_1 \times \mathfrak{K}_2$ defined as

$$\operatorname{gr} H = \{\{f, f'\} \in \mathfrak{K}_1 \times \mathfrak{K}_2 : f \in \operatorname{dom} H \text{ and } f' \in Hf\}.$$

Conversely, with each subspace of $\mathfrak{K}_1 \times \mathfrak{K}_2$ a multi-valued operator can be associated. Recall that subspaces of $\mathfrak{K}_1 \times \mathfrak{K}_2$ are called (linear) relations, see e.g. (Arens 1961). Here, following Cross (1998), the term relation will be used as a synonym for a multi-valued operator[2].

The inverse of a relation $H$ is the relation $H^{-1}$ defined as

$$H^{-1}f' = \{f \in \mathfrak{K}_1 : f' \in Hf\}, \quad f' \in \operatorname{dom} H^{-1} := \operatorname{ran} H.$$

Clearly, $(H^{-1})^{-1} = H$, $\ker H = \operatorname{mul} H^{-1}$, $\operatorname{mul} H = \ker H^{-1}$ and

$$H^{-1}Hf = f + \ker H, \quad f \in \operatorname{dom} H, \quad HH^{-1}f' = f' + \operatorname{mul} H, \quad f' \in \operatorname{ran} H.$$

The adjoint of a relation $H$ is the relation $H^{[*]}$ whose graph is given by

$$\operatorname{gr} H^{[*]} = \{\{f, f'\} \in \mathfrak{K}_2 \times \mathfrak{K}_1 : [f', g]_1 = [f, g']_2, \;\; \forall \{g, g'\} \in \operatorname{gr} H\}.$$

[2]It is emphasized that here the multi-valued operator (relation) and its graph will not be identified; a multi-valued operator (relation) is always to be understood as a multi-valued mapping between two spaces and its graph is used to describe the geometrical properties of this mapping.

If $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ are Hilbert spaces, then the usual notation $H^*$ is used for the adjoint of a relation $H$. From the above definition of the adjoint of a relation it follows immediately that

$$(\operatorname{dom} H)^{[\perp]_1} = \operatorname{mul} H^{[*]} \quad \text{and} \quad (\operatorname{ran} H)^{[\perp]_2} = \ker H^{[*]}. \tag{2.6}$$

For relations $G$ and $H$ from $\mathfrak{K}_1$ to $\mathfrak{K}_2$ the notation $G + H$ is used to denote the sum of relations:

$$(G + H)f = Gf + Hf, \quad f \in \operatorname{dom} G \cap \operatorname{dom} H.$$

Moreover, the notation $G \subseteq H$ is used to denote that $H$ is an extension of $G$, i.e. $\operatorname{gr} G \subseteq \operatorname{gr} H$. In particular, with this notation

$$G = H \quad \text{if and only if} \quad G \subseteq H, \ \operatorname{dom} H \subseteq \operatorname{dom} G, \ \operatorname{mul} H \subseteq \operatorname{mul} G \tag{2.7}$$

or, by passing to the inverses,

$$G = H \quad \text{if and only if} \quad G \subseteq H, \ \operatorname{ran} H \subseteq \operatorname{ran} G, \ \ker H \subseteq \ker G. \tag{2.8}$$

If $G$ is a relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and $H$ is a relation from $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ to $\{\mathfrak{K}_3, [\cdot,\cdot]_3\}$, then their composition is the relation $HG$ defined as

$$(HG)f = \{f' \in \mathfrak{K}_2 : \exists g \in Gf \text{ s.t. } f' \in Hg\}, \quad f \in G^{-1}(\operatorname{ran} G \cap \operatorname{dom} H).$$

The following basic facts about relations can essentially be found in e.g. (Arens 1961); for the last statement in Lemma 2.15 below see also (Derkach et al. 2009: Lemma 2.9).

**Lemma 2.15.** *Let $G$ be a relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $H$ be a relation from $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ to $\{\mathfrak{K}_3, [\cdot,\cdot]_3\}$. Then*

$$\left(H^{[*]}\right)^{-1} = \left(H^{-1}\right)^{[*]}, \quad (HG)^{-1} = G^{-1}H^{-1} \quad \textit{and} \quad G^{[*]}H^{[*]} \subseteq (HG)^{[*]}.$$

*Moreover, if $G$ is closed, $\operatorname{ran} G$ is closed and $\operatorname{dom} H \subseteq \operatorname{ran} G$ or $H$ is closed, $\operatorname{dom} H$ is closed and $\operatorname{ran} G \subseteq \operatorname{dom} H$, then $G^{[*]}H^{[*]} = (HG)^{[*]}$.*

In particular, as a consequence of Lemma 2.15, the notation $H^{-[*]}$ is used as a shorthand notation for $\left(H^{[*]}\right)^{-1} = (H^{-1})^{[*]}$.

Let $P_i$ be a projection in $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$, for $i = 1, 2$, then the projection $P_1 \times P_2$ in $\mathfrak{K}_1 \times \mathfrak{K}_2$ is defined as $(P_1 \times P_2)(f_1 \times f_2) = P_1 f_1 \times P_2 f_2$, $f_1 \in \mathfrak{K}_1$ and $f_2 \in \mathfrak{K}_2$. From $\operatorname{gr} H \subseteq \operatorname{dom} H \times \operatorname{ran} H$ for a relation $H$ from $\mathfrak{K}_1$ to $\mathfrak{K}_2$, it follows immediately that $(P_1 \times P_2)\operatorname{gr} H \subseteq (P_1 \times P_2)(\operatorname{dom} H \times \operatorname{ran} H)$. Characterizations for when the inverse inclusion holds are provided by the following statement.

**Proposition 2.16.** *Let $H$ be a relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $P_i$ be an orthogonal projection in $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$, for $i = 1, 2$. Then equivalent are*

(i) $(P_1 \times P_2)(\mathrm{dom}\, H \times \mathrm{ran}\, H) \subseteq (P_1 \times P_2)\mathrm{gr}\, H$;

(ii) $P_1\mathrm{dom}\, H = P_1 H^{-1}(\mathrm{ran}\, H \cap \ker P_2)$;

(iii) $P_2\mathrm{ran}\, H = P_2 H(\mathrm{dom}\, H \cap \ker P_1)$;

(iv) $\mathrm{dom}\, H = H^{-1}(\mathrm{ran}\, H \cap \ker P_2) + (\mathrm{dom}\, H \cap \ker P_1)$;

(v) $\mathrm{ran}\, H = H(\mathrm{dom}\, H \cap \ker P_1) + (\mathrm{ran}\, H \cap \ker P_2)$.

*Proof.* (i) $\Rightarrow$ (ii): If (i) holds, then for every $f_1 \in P_1\mathrm{dom}\, H$ there exists $\{f, f'\} \in \mathrm{gr}\, H$ such that $P_1 f = f_1$ and $P_2 f' = 0$. Therefore $f' \in \mathrm{ran}\, H \cap \ker P_2$ and hence $P_1\mathrm{dom}\, H \subseteq P_1 H^{-1}(\mathrm{ran}\, H \cap \ker P_2)$. Since the inverse inclusion clearly holds, this shows that (ii) holds.

(ii) $\Leftrightarrow$ (iv): Let $f \in \mathrm{dom}\, H$, then by (ii) there exists $\{g, g'\} \in \mathrm{gr}\, H$ such that $P_1 g = P_1 f$ and $g' \in \mathrm{ran}\, H \cap \ker P_2$. Hence $f = g + h$, where $h = f - g \in \mathrm{dom}\, H$ and $P_1 h = P_1(f - g) = 0$, i.e., $\mathrm{dom}\, H \subseteq H^{-1}(\mathrm{ran}\, H \cap \ker P_2) + (\mathrm{dom}\, H \cap \ker P_1)$. Since the inverse inclusion clearly holds, this proves the implication from (ii) to (iv). The reverse implication is direct.

(iii) $\Leftrightarrow$ (v): This is similar to the equivalence of (ii) and (iv).

(iv) $\Leftrightarrow$ (v): This follows by applying $H$ and $H^{-1}$.

(ii) & (iii) $\Rightarrow$ (i): If $f \in P_1\mathrm{dom}\, H$ and $f' \in P_2\mathrm{ran}\, H$, then by (ii) there exists $\{g, g'\} \in \mathrm{gr}\, H$ such that $P_1 g = f$, $P_2 g' = 0$ and by (iii) there exists $\{h, h'\} \in \mathrm{gr}\, H$ such that $P_1 h = 0$ and $P_2 h' = f'$. Hence, $\{g + h, g' + h'\} \in \mathrm{gr}\, H$, $P_1(g + h) = P_1 g = f$ and $P_2(g' + h') = P_2 h' = f'$. □

This section is concluded by stating a several properties of operator ranges which will be used throughout the dissertation. Therefore recall that a subspace of a Hilbert space is called an *operator range* if it is the range of a bounded (or, equivalently, of a closed) operator in that space. Most of the below stated operator range results can be found from (Fillmore & Williams 1971); it is however worth mentioning that statements (iii) and (vi) (in the separable case) of Proposition 2.17 go back to Calkin (1939a: Lemma 3.1 & Lemma 4.2). For the proof of (vi) Calkin used the well-known part of Proposition 2.17 (v), which goes back to von Neumann (1929).

**Proposition 2.17.** *Let $\{\mathfrak{H}, (\cdot,\cdot)\}$ be a Hilbert space. Then the following statements hold:*

*(i) if $\mathscr{R}_1 \subseteq \mathfrak{H}$ and $\mathscr{R}_2 \subseteq \mathfrak{H}$ are operator ranges, then $\mathscr{R}_1 + \mathscr{R}_2$ and $\mathscr{R}_2 \cap \mathscr{R}_2$ are also operator ranges;*

*(ii) if $\mathscr{R} \subseteq \mathfrak{H}$ is an operator range such that $\operatorname{clos} \mathscr{R} = \mathfrak{H}$, then there exists a nonnegative selfadjoint operator $B$ in $\{\mathfrak{H}, (\cdot,\cdot)\}$ such that $\operatorname{dom} B = \mathfrak{H}$, $\operatorname{ran} B = \mathscr{R}$ and $\ker B = \{0\}$;*

*(iii) if $\mathscr{R} \subseteq \mathfrak{H}$ is an operator range which does not contain an infinite-dimensional closed subspaces, then $\mathscr{R}$ is a ”compact operator range”: If $B$ is an operator in $\{\mathfrak{H}, (\cdot,\cdot)\}$ such that $\operatorname{ran} B = \mathscr{R}$, then $B$ is a compact operator;*

*(iv) if $\mathscr{R}_1 \subseteq \mathfrak{H}$ is the operator range of a compact operator and $\mathscr{R}_2 \subseteq \mathfrak{H}$ is an operator range such that $\operatorname{clos}(\mathscr{R}_1 + \mathscr{R}_2) = \mathfrak{H}$, where $\mathscr{R}_2$ is nonclosed or the co-dimension of $\mathscr{R}_2$ is infinite, then $\mathscr{R}_1 + \mathscr{R}_2 \neq \mathfrak{H}$;*

*(v) if $\mathscr{R}_1 \subseteq \mathfrak{H}$ is a nonclosed operator range and $\{\mathfrak{H}, (\cdot,\cdot)\}$ is a separable Hilbert space, then there exists an operator range $\mathscr{R}_2$ of a noncompact operator with $\operatorname{clos}(\mathscr{R}_2) = \mathfrak{H}$ such that*

$$\mathscr{R}_1 \cap \mathscr{R}_2 = \{0\} \quad \textit{and} \quad \operatorname{clos}(\mathscr{R}_1 + \mathscr{R}_2) \neq \mathfrak{H};$$

*(vi) if $\mathscr{R} \subseteq \mathfrak{H}$ is a nonclosed operator range such that $\operatorname{clos}(\mathscr{R}) = \mathfrak{H}$, then there exists an infinite-dimensional closed subspace $\mathfrak{L} \subseteq \mathfrak{H}$ such that $\mathscr{R} \cap \mathfrak{L} = \{0\}$.*

*Proof.* (i): These two statements can be found in (Fillmore & Williams 1971: Theorem 2.2 & Corollary 2 on p. 260).

(ii): If $\mathscr{R}$ is an operator range, then the polar decomposition of closed operators implies that there exists a bounded nonnegative selfadjoint operator $B$ in $\{\mathfrak{H}, (\cdot,\cdot)\}$ such that $\mathscr{R} = \operatorname{ran} B$. Hence, the fact that (ii) holds, follows now from the assumption that $\operatorname{clos}(\mathscr{R}) = \overline{\operatorname{ran}}\, B = \mathfrak{H}$, see (2.6).

(iii): For this statement see (Calkin 1939a: Lemma 3.1) or (Fillmore & Williams 1971: Theorem 2.5).

(iv): To prove (iv) assume the converse, i.e. that $\mathscr{R}_1 + \mathscr{R}_2 = \mathfrak{H}$. Then by (Fillmore & Williams 1971: Theorem 2.4) there exist closed disjoint subspaces $\mathfrak{M}_1 \subseteq \mathscr{R}_1$ and $\mathfrak{M}_2 \subseteq \mathscr{R}_2$ such that $\mathfrak{M}_1 + \mathfrak{M}_2 = \mathfrak{H}$. Now by either of the assumptions on $\mathscr{R}_2$ in (iv) it follows that $\mathfrak{M}_1$ must be an infinite-dimensional subspace. Since $\mathfrak{M}_1$ is contained in the range of a compact operator, that is not possible, see (iii). This contradiction shows that $\mathscr{R}_1 + \mathscr{R}_2 \neq \mathfrak{H}$.

(v): The first part of this statement is the contents of (Fillmore & Williams 1971: Theorem 3.6) and the second part of it follows from an inspection of that proof.

That proof consists out of two parts: First a dense operator range $\mathscr{L}$ is constructed such that $\mathscr{L} \cap V\mathscr{L} = \{0\}$ for a unitary operator $V$ in $\{\mathfrak{H}, (\cdot,\cdot)\}$ and secondly it is shown that there exists a unitary operator $W$ in $\{\mathfrak{H}, (\cdot,\cdot)\}$ such that $W\mathscr{R}_1 \subseteq \mathscr{L}$. From these two fact it follows that the first assertion in (v) holds by taking $\mathscr{R}_2$ to be $W^{-1}V\mathscr{L}$. Hence, to show that the second assertion in (v) holds it suffices to show that $\mathscr{L} + V(\mathscr{L}) \neq \mathfrak{H}$ and that $\operatorname{clos}\mathscr{L} = \mathfrak{H}$. Therefore note that $\mathscr{L}$ and $V$ are in (Fillmore & Williams 1971: Theorem 3.6) constructed as countably infinite sums: $\mathscr{L} = \bigoplus_{i=1}^{\infty} \mathscr{L}_i$ and $V = \bigoplus_{i=1}^{\infty} V_i$, where $\mathscr{L}_i$ is the operator range of a compact operator in an infinite-dimensional Hilbert space $\{\mathfrak{H}_i, (\cdot,\cdot)\}$ with $\operatorname{clos}\mathscr{L}_i = \mathfrak{H}_i$ and $V_i$ is a unitary operator in $\{\mathfrak{H}_i, (\cdot,\cdot)\}$. Clearly, from $\operatorname{clos}\mathscr{L}_i = \mathfrak{H}_i$, it follows that $\operatorname{clos}\mathscr{L} = \mathfrak{H}$. Moreover, since $\mathscr{L}_i$ is the operator range of a compact operator, (iv) implies that $\mathscr{L}_i + V\mathscr{L}_i \neq \mathfrak{H}_i$ and, hence, $\mathscr{L} + V\mathscr{L} \neq \mathfrak{H}$. Note also that the above construction shows that $\mathscr{R}_2 = W^{-1}V\mathscr{L}$ is not the operator range of a compact operator, because $\mathscr{L}$ contains by construction infinite-dimensional closed subspaces, cf. (iii).

(vi): Let $B$ be a bounded nonnegative selfadjoint operator in $\{\mathfrak{H}, (\cdot,\cdot)\}$ such that $\mathscr{R} = \operatorname{ran} B$, see (ii). Moreover, let $\{E_t\}_{t\in\mathbb{R}}$ be its spectral family and define $F_n = E_{2^{-n}||B||} - E_{2^{-n-1}||B||}$, $n \in \mathbb{N}$, then

$$\operatorname{ran} B = \left\{ \sum_{n=0}^{\infty} \phi_n : \phi_n \in \operatorname{ran} F_n \text{ and } \sum_{n=0}^{\infty} 4^n(\phi_n, \phi_n) < \infty \right\}, \tag{2.9}$$

see (Fillmore & Williams 1971: proof of Theorem 1.1). To prove that (vi) holds two disjoint cases are considered.

*Case 1:* There exist infinitely many Hilbert spaces $\{\operatorname{ran} F_n, (\cdot,\cdot)\}$ which are infinite-dimensional. Then let $\{n_k\}_{k\in\mathbb{N}}$ be a subsequence of $\mathbb{N}$ such that $\{\operatorname{ran} F_{n_k}, (\cdot,\cdot)\}$ is an infinite-dimensional Hilbert space, and let $\{\phi^i_{n_k}\}_{i\in\mathbb{N}}$ be an infinite orthonormal sequence in $\{\operatorname{ran} F_{n_k}, (\cdot,\cdot)\}$. Define $\psi_i = \sum_{k=0}^{\infty} 2^{-n_k}\phi^i_{n_k}$, $i \in \mathbb{N}$, then $\psi_i \notin \operatorname{ran} H$ by (2.9) whilst $\psi_i \in \mathfrak{H}$, because

$$(\psi_i, \psi_i) = \sum_{k=0}^{\infty} (2^{-n_k}\phi^i_{n_k}, 2^{-n_k}\phi^i_{n_k}) = \sum_{k=0}^{\infty} 4^{-n_k} \leq \sum_{k=0}^{\infty} 4^{-k} < \infty.$$

Since the $\phi^i_{n_k}$ are orthogonal by construction, $\mathfrak{L} := \overline{\operatorname{span}}\,\{\psi_1, \psi_2, \ldots\}$ is an infinite-dimensional closed subspace such that $\mathfrak{L} \cap \operatorname{ran} B = \{0\}$.

*Case 2:* There do not exist infinitely many Hilbert spaces $\{\operatorname{ran} F_n, (\cdot,\cdot)\}$ which are infinite-dimensional. Then $B$ is the orthogonal sum of a compact operator and a bounded and boundedly invertible operator. W.l.o.g. assume that $B$ is an (everywhere defined) compact operator in $\{\mathfrak{H}, (\cdot,\cdot)\}$. Since $\overline{\operatorname{ran}}\, B = \mathfrak{H}$ and $\operatorname{ran} B \neq \mathfrak{H}$, $\{\mathfrak{H}, (\cdot,\cdot)\}$ must be an (infinite-dimensional) separable Hilbert space and, hence, the existence of an infinite-dimensional closed subspace $\mathfrak{L}$ such that $\mathfrak{L} \cap \operatorname{ran} B = \{0\}$ follows from (v) and (iii). □

**Corollary 2.18.** *Let $H$ be a closed unbounded operator in $\{\mathfrak{H}, (\cdot,\cdot)\}$ with $\overline{\operatorname{dom}} H = \mathfrak{H}$, $\operatorname{ran} H = \mathfrak{H}$ and $\ker H = \{0\}$. Then for every $0 \leq m \leq \aleph_0$, there exists an $m$-dimensional closed subspace $\mathfrak{L}_m$ of $\mathfrak{H}$ such that*

$$\mathfrak{H} = \operatorname{clos}(H^{-*}(\mathfrak{L}_m^{\perp})).$$

*Proof.* Note first that the assumptions on $H$ imply that $H^*$ is an operator with $\operatorname{ran} H^* = \mathfrak{H}$ and that by Proposition 2.17 (vi) (applied to the operator range $\operatorname{dom} H$) there exists for $m$ as in this statement a closed subspace $\mathfrak{L}_m \subseteq \mathfrak{H}$ such that $\mathfrak{L}_m \cap \operatorname{dom} H = \{0\}$. Now let $g \in \mathfrak{H}$ be orthogonal to $H^{-*}(\mathfrak{L}_m^{\perp})$, then

$$0 = (g, H^{-*}f) = (H^{-1}g, f), \quad \forall f \in \mathfrak{L}_m^{\perp}.$$

This implies that $H^{-1}g \in \operatorname{dom} H \cap \mathfrak{L}_m = \{0\}$. Consequently, $g = 0$ and, hence, $\mathfrak{H} = \operatorname{clos}(H^{-*}(\mathfrak{L}^{\perp_2}))$. □

## 2.5 Angular and quasi-angular operators

In this section first the concept of an angular operator for (semi-definite) subspaces of a Kreĭn space is shortly recalled; for details see (Azizov & Iokhvidov 1989: Ch. 1, §8). That overview is followed by an other manner of characterizing semi-definite subspace of a Kreĭn space by means of operators.

Let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$, then the angular operators $K^+$ and $K^-$ of a subspace $\mathfrak{L}$ of $\mathfrak{K}$ w.r.t. $\mathfrak{K}^+$ and $\mathfrak{K}^-$ are the relations from $\mathfrak{K}^+$ to $\mathfrak{K}^-$ and from $\mathfrak{K}^-$ to $\mathfrak{K}^+$ defined via

$$\operatorname{gr} K^+ := \{\{P^+f, P^-f\} : f \in \mathfrak{L}\} \quad \text{and} \quad \operatorname{gr} K^- := \{\{P^-f, P^+f\} : f \in \mathfrak{L}\},$$

respectively. In other words, $K^+$ and $K^-$ are such that

$$\mathfrak{L} = \operatorname{gr} K^+ = \{f^+ + (K^+)_o f^+ : f^+ \in \operatorname{dom} K^+\} + \operatorname{mul} K^+;$$
$$\mathfrak{L} = \operatorname{gr} K^- = \{f^- + (K^-)_o f^- : f^- \in \operatorname{dom} K^-\} + \operatorname{mul} K^-,$$

where $\operatorname{mul} K^+ = \mathfrak{L} \cap \mathfrak{K}^-$ and $\operatorname{mul} K^- = \mathfrak{L} \cap \mathfrak{K}^+$. Proposition 2.19 below contains the characterizations of semi-definite subspaces by means of angular operators.

**Proposition 2.19.** *Let $\mathfrak{L}$ be a subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and let $K^+$ and $K^-$ be the angular operators of $\mathfrak{L}$ w.r.t. $\mathfrak{K}_1^+$ and $\mathfrak{K}_1^-$, respectively, for a canonical decomposition $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then the following equivalences hold:*

(i) *the subspace $\mathfrak{L}$ is a (closed, maximal) neutral or hyper-maximal neutral subspace of $\{\mathfrak{K},[\cdot,\cdot]\}$ if and only if $K^+$ or $(K^-)^{-1}$ is a (closed, maximal) isometric or unitary operator from (the Hilbert space) $\{\mathfrak{K}^+,[\cdot,\cdot]\}$ to (the Hilbert space) $\{\mathfrak{K}^-,-[\cdot,\cdot]\}$, respectively;*

(ii) *the subspace $\mathfrak{L}$ is a (closed, maximal) (uniform) nonnegative subspace of $\{\mathfrak{K},[\cdot,\cdot]\}$ if and only if $K^+$ is a (closed, everywhere defined) (uniform) contraction from $\{\mathfrak{K}^+,[\cdot,\cdot]\}$ to $\{\mathfrak{K}^-,-[\cdot,\cdot]\}$;*

(iii) *the subspace $\mathfrak{L}$ is a (closed, maximal) (uniform) nonpositive subspace of $\{\mathfrak{K},[\cdot,\cdot]\}$ if and only if $K^-$ is a (closed, everywhere defined) (uniform) contraction from $\{\mathfrak{K}^-,-[\cdot,\cdot]\}$ to $\{\mathfrak{K}^+,[\cdot,\cdot]\}$.*

*Moreover, the angular operator for $\mathfrak{L}^{[\perp]}$ w.r.t. $\mathfrak{K}^-$ or $\mathfrak{K}^+$ is $(K^+)^*$ or $(K^-)^*$, respectively. Here the adjoint is taken as a relation from $\{\mathfrak{K}^+,[\cdot,\cdot]\}$ to $\{\mathfrak{K}^-,-[\cdot,\cdot]\}$ or as a relation from $\{\mathfrak{K}^-,-[\cdot,\cdot]\}$ to $\{\mathfrak{K}^+,[\cdot,\cdot]\}$, respectively.*

*Proof.* The equivalences in (i)-(ii) can all be found in (Azizov & Iokhvidov 1989: Ch. 1, §8) and there also special cases of the last statement can be found. For completeness, a proof for the general case of the last statement is included here.

Let $K^+$ be the angular operator for the subspace $\mathfrak{L}$ w.r.t. $\mathfrak{K}^+$. If $g = g^+ + g^- \in \mathfrak{L}^{[\perp]}$, $g^+ \in \mathfrak{K}^+$ and $g^- \in \mathfrak{K}^-$, then

$$0 = [f + K^+ f, g^+ + g^-] = [f, g^+] + [K^+ f, g^-], \quad \forall f \in P_1^+ \mathfrak{L}.$$

This shows that $g^+ = (K^+)^* g^-$. Conversely, if $g^+ = (K^+)^* g^-$, then reversing the above arguments shows that $g^+ + g^- \in \mathfrak{L}^{[\perp]}$. Consequently, $(K^+)^*$ is the angular operator for $\mathfrak{L}^{[\perp]}$ w.r.t. $\mathfrak{K}^-$. □

Above semi-definite subspaces have been characterized by means of a canonical decomposition of the space. Next a characterization of semi-definite subspaces by means of a neutral decomposition of the space is presented. More precisely, let $\{\mathfrak{K},[\cdot,\cdot]\}$ be a Kreĭn space for which $\mathfrak{j}$ is a fundamental symmetry and assume that there exists a hyper-maximal neutral subspace $\mathfrak{M}$ in $\{\mathfrak{K},[\cdot,\cdot]\}$. Then recall that $\mathfrak{M}$ induces an orthogonal decomposition of the space into hyper-maximal neutral subspaces: $\mathfrak{K} = \mathfrak{M} \oplus \mathfrak{j}\mathfrak{M}$. If $\mathfrak{L}$ is now a subspace of $\mathfrak{K}$, then its *quasi-angular operator w.r.t.* $\mathfrak{M}$ is the relation $A$ in the Hilbert space $\{\mathfrak{M},[\mathfrak{j}\cdot,\cdot]\}$ defined via

$$\operatorname{gr} A = \{\{\mathcal{P}_{\mathfrak{M}} f, i\mathcal{P}_{\mathfrak{M}}\mathfrak{j} f\} : f \in \mathfrak{L}\}, \tag{2.10}$$

where $\mathcal{P}_{\mathfrak{M}}$ is the orthogonal projection onto $\mathfrak{M}$ w.r.t. $[\mathfrak{j}\cdot,\cdot]$. I.e., $A$ is such that

$$\mathfrak{L} = \{f - \mathfrak{j}iAf : f \in \operatorname{dom} A\} = \{f - \mathfrak{j}iA_o f : f \in \operatorname{dom} A\} + \mathfrak{j}(\operatorname{mul} A),$$

where $\operatorname{mul} A = \mathfrak{L} \cap \mathfrak{j}\mathfrak{M}$. Proposition 2.20 below contains a characterization of the different types of semi-definite subspaces $\mathfrak{L}$ in terms of their associated quasi-angular operators. Therefore recall that a relation $H$ in $\{\mathfrak{K}, [\cdot,\cdot]\}$ is called *dissipative* or *accumulative* if

$$\operatorname{Im}[f', f] \geq 0 \quad \text{or} \quad \operatorname{Im}[f', f] \leq 0, \quad \forall \{f, f'\} \in \operatorname{gr} H,$$

respectively. A dissipative or accumulative relation $H$ is called *maximal dissipative* or *maximal accumulative* if it has no proper dissipative or accumulative extensions, respectively. In particular, a dissipative or accumulative relation $H$ in a Hilbert space $\{\mathfrak{H}, (\cdot,\cdot)\}$ is maximal dissipative or maximal accumulative if and only if $\lambda \in \rho(A)$ for a (and hence for every) $\lambda \in \mathbb{C}_-$ or $\lambda \in \mathbb{C}_+$, respectively.

**Proposition 2.20.** *Let* $\mathfrak{j}$ *be a fundamental symmetry of* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *and assume that there exists a hyper-maximal neutral subspace* $\mathfrak{M}$ *in* $\{\mathfrak{K}, [\cdot,\cdot]\}$. *Moreover, let* $\mathfrak{L}$ *be a subspace of* $\mathfrak{K}$ *with quasi-angular operator* $A$ *w.r.t.* $\mathfrak{M}$. *Then the following equivalences hold:*

(i) *the subspace* $\mathfrak{L}$ *is a (closed, maximal) neutral or hyper-maximal neutral subspace of* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *if and only if* $A$ *is a (closed, maximal) symmetric or self-adjoint relation in the Hilbert space* $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$, *respectively;*

(ii) *the subspace* $\mathfrak{L}$ *is a (closed, maximal) nonnegative or nonpositive subspace of* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *if and only if* $A$ *is a (closed, maximal) dissipative or accumulative operator in* $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$, *respectively.*

*Moreover, the quasi-angular operator of* $\mathfrak{L}^{[\perp]}$ *w.r.t.* $\mathfrak{M}$ *is* $A^*$.

*Proof.* First the final assertion is shown to hold. Therefore observe that by definition $g - \mathfrak{j}ig' \in \mathfrak{L}^{[\perp]}$, $g, g' \in \mathfrak{M}$, if and only if

$$0 = [f - \mathfrak{j}if', g - \mathfrak{j}ig'] = [f, -\mathfrak{j}ig'] + [-\mathfrak{j}if', g] = -i([\mathfrak{j}f', g] - [\mathfrak{j}f, g'])$$

for all $\{f, f'\} \in \operatorname{gr} A$. This shows that $g - \mathfrak{j}ig' \in \mathfrak{L}^{[\perp]}$ if and only if $\{g, g'\} \in \operatorname{gr} A^*$. I.e., the final assertion holds. Clearly, (i) follows immediately from the proven assertion. Next assume that $\mathfrak{L}$ is a nonnegative subspace, then

$$0 \leq [f - \mathfrak{j}if', f - \mathfrak{j}if'] = i[-\mathfrak{j}f', f] - i[f, -\mathfrak{j}f'] = 2\operatorname{Im}([\mathfrak{j}f', f])$$

for all $\{f, f'\} \in \operatorname{gr} A$. I.e., $A$ is a dissipative relation in the Hilbert space $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$. The converse is proven by reserving the above arguments. Furthermore, it is clear that $\mathfrak{L}$ and $A$ are closed simultaneously, see Section 2.4, and that $\mathfrak{L}$ has a nonnegative extension if and only if $A$ has a dissipative extension. $\square$

# 3 BASIC PROPERTIES OF UNITARY RELATIONS

Basic properties of isometric and unitary relations are presented here. More precisely, in the first of the five sections of this chapter the definition of isometric and unitary relations are stated and some basic characterizations of them are recalled. In the second and third section, the behavior of isometric and unitary relations with respect to special subspaces is investigated. More specifically, first the kernel and multi-valued part of isometric relations are investigated and, secondly, the behavior of isometric relations with respect to the closure of subspaces is investigated. In that connection it is shown that basically only for uniformly definite subspaces one can say something in general about the closedness of their image after mapping them by a (closed) isometric relation. In the fourth section it is shown how from isometric relations the kernel, multi-valued part and closed uniformly definite subspaces contained in their domain, which were studied in the preceding sections, can be removed. Thereby one remains with the more involved part of isometric relations. Finally, in the fifth section maximal isometric relations are shortly considered.

## 3.1 Isometric and unitary relations

A relation $U$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ is called *isometric* or *unitary* if

$$U^{-1} \subseteq U^{[*]} \quad \text{or} \quad U^{-1} = U^{[*]}, \tag{3.1}$$

respectively, see (Arens 1961). An isometric relation is called *maximal* isometric, if it has no proper isometric extension. The above definition says that a relation $V$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ is isometric if and only if

$$[f,g]_1 = [f',g']_2, \quad \forall \{f,f'\}, \{g,g'\} \in \operatorname{gr} V.$$

Hence, polarization yields that $V$ is isometric if and only if $[f,f]_1 = [f',f']_2$, for all $\{f,f'\} \in \operatorname{gr} V$. Furthermore, (3.1) implies that unitary relations are maximal isometric relations in a special sense.

**Proposition 3.1.** *Let $U$ be a relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then $U$ is unitary if and only if $U$ is isometric and if $\{f,f'\} \in \mathfrak{K}_1 \times \mathfrak{K}_2$ is such that*

$$[f,g]_1 = [f',g']_2, \quad \forall \{g,g'\} \in \operatorname{gr} U,$$

*then $\{f,f'\} \in \operatorname{gr} U$.*

*Proof.* If $\{f, f'\} \in \mathfrak{K}_1 \times \mathfrak{K}_2$ satisfies the stated condition, then $\{f', f\} \in \operatorname{gr} U^{[*]}$ by the definition of $U^{[*]}$. Hence the equivalence follows directly from the definition of unitary relations, see (3.1). □

Since $\left(U^{[*]}\right)^{-1} = (U^{-1})^{[*]}$, the definitions of isometric and unitary relations in (3.1) imply that a relation is isometric or unitary if and only if its inverse is isometric or unitary, respectively. In particular, the action of an isometric or a unitary relation and their inverse are of the same type and, hence, the structure of their domain and range is of the same type. Since the adjoint of a relation is automatically closed, (3.1) also implies that every unitary relation is closed and that a relation is isometric if and only if its closure is isometric.

For Kreĭn spaces $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ the notation $[\cdot,\cdot]_{1,-2}$ is used to denote the indefinite inner product on $\mathfrak{K}_1 \times \mathfrak{K}_2$ defined by

$$[f_1 \times f_2, g_1 \times g_2]_{1,-2} = [f_1, g_1]_1 - [f_2, g_2]_2, \quad f_1, g_1 \in \mathfrak{K}_1, f_2, g_2 \in \mathfrak{K}_2. \tag{3.2}$$

With this inner product, $\{\mathfrak{K}_1 \times \mathfrak{K}_2, [\cdot,\cdot]_{1,-2}\}$ is a Kreĭn space and for a relation $H$ from $\mathfrak{K}_1$ to $\mathfrak{K}_2$ one has that $(\operatorname{gr} H)^{[\perp]_{1,-2}} = \operatorname{gr} H^{-[*]}$. The preceding observation yields the following result which can be partly found in (Shmul'jan 1976).

**Proposition 3.2.** *Let $U$ be a relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then $U$ is a (closed, maximal) isometric or unitary relation if and only if $\operatorname{gr} U$ is a (closed, maximal) neutral or hyper-maximal neutral subspace of $\{\mathfrak{K}_1 \times \mathfrak{K}_2, [\cdot,\cdot]_{1,-2}\}$, respectively.*

In light of Proposition 2.6 and the discussion following tit, Proposition 3.2 implies that if $U$ is a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, then

$$k_1^+ + k_2^- = k_1^- + k_2^+.$$

The following statement, which generalizes an equivalence in Proposition 2.6, can be interpreted as an inverse to Proposition 3.2; it shows how hyper-maximal neutral subspaces can be interpreted (nonuniquely) as unitary relations.

**Proposition 3.3.** *Let $\mathfrak{L}$ be a subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and let $P$ be an orthogonal projection in $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then $\mathfrak{L}$ is a (closed, maximal) neutral or hyper-maximal neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ if and only if the relation $U_{\mathfrak{L}}$ defined via*

$$\operatorname{gr} U_{\mathfrak{L}} := \{\{Pf, (I-P)f\} : f \in \mathfrak{L}\}$$

*is a (closed, maximal) isometric or unitary relation, respectively, from the Kreĭn space $\{\operatorname{ran} P, [\cdot,\cdot]\}$ to the Kreĭn space $\{\ker P, -[\cdot,\cdot]\}$,* *Proof.* If $\mathfrak{L}$ is neutral, then for $f, g \in \mathfrak{L}$

$$0 = [f, g] = [(I-P)f + Pf, (I-P)g + Pg] = [Pf, Pg] + [(I-P)f, (I-P)g],$$

i.e. $U_{\mathfrak{L}}$ is an isometric relation. Reversing the above argument shows that $\mathfrak{L}$ is neutral if $U_{\mathfrak{L}}$ is an isometric relation. From the fact that a relation is closed if and only if its graph is closed it follows immediately that $\mathfrak{L}$ and $U_{\mathfrak{L}}$ are simultaneously closed. Furthermore, by the proven equivalence it is also clear that $\mathfrak{L}$ can be extended neutrally if and only if $U_{\mathfrak{L}}$ can be extended isometrically.

Hence it only remains to prove that $\mathfrak{L}$ is hyper-maximal neutral if and only if $U_{\mathfrak{L}}$ is a unitary relation between the indicated Kreĭn spaces. Therefore assume that $\mathfrak{L}$ is hyper-maximal neutral and let $\{g, g'\} \in \operatorname{ran} P \times \ker P$ be such that $[Pf, g] = -[(I-P)f, g']$ for all $f \in \mathfrak{L}$. Then $[f, g+g'] = 0$ for all $f \in \mathfrak{L}$, i.e. $g + g' \in \mathfrak{L}$, because $\mathfrak{L}$ is hyper-maximal neutral. Hence $U_{\mathfrak{L}}$ is a unitary relation, see Proposition 3.1. Conversely, if $U_{\mathfrak{L}}$ is a unitary relation, then $\mathfrak{L}$ is hyper-maximal neutral by Proposition 3.2. □

Combing the preceding two propositions with Proposition 2.6 shows that with each unitary operator between Kreĭn spaces one can associate a unitary relation between Hilbert spaces; that association is a so-called Potapov-Ginzburg transformation, see (Azizov & Iokhvidov 1989: Ch. 5, §1) or Proposition 4.14 below.

## 3.2 Kernels and multi-valued parts of isometric relations

For an isometric relation $V$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ (2.6) becomes

$$\ker V \subseteq (\operatorname{dom} V)^{[\perp]_1} \quad \text{and} \quad \operatorname{mul} V \subseteq (\operatorname{ran} V)^{[\perp]_2}. \tag{3.3}$$

Hence, in particular, $\ker V$ and $\operatorname{mul} V$ are neutral subspaces of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, respectively. For a unitary relation $U$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ the inequalities in (3.3) become equalities:

$$\ker U = (\operatorname{dom} U)^{[\perp]_1} \quad \text{and} \quad \operatorname{mul} U = (\operatorname{ran} U)^{[\perp]_2}. \tag{3.4}$$

Lemma 3.4 below contains a useful consequence for an isometric relation if equality holds in (3.3) for one of the inclusions therein.

**Lemma 3.4.** *Let $V$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\ker V = (\operatorname{dom} V)^{[\perp]_1}$ and let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ with associated orthogonal projections $P_1^+$ and $P_1^-$. Then*

$$P_1^+\overline{\operatorname{dom}}\, V = \mathfrak{K}_1^+ \quad \textit{and} \quad P_1^-\overline{\operatorname{dom}}\, V = \mathfrak{K}_1^-.$$

*Proof.* The assumption $\ker V = (\operatorname{dom} V)^{[\perp]_1}$ together with (2.2) implies that

$$\overline{\operatorname{dom}}\, V \cap \mathfrak{K}_1^+ = \mathfrak{K}_1^+ \ominus_1 P_1^+ \ker V \quad \text{and} \quad \overline{\operatorname{dom}}\, V \cap \mathfrak{K}_1^- = \mathfrak{K}_1^- \ominus_1 P_1^- \ker V.$$

Since $\ker V \subseteq \operatorname{dom} V$, the conclusion follows from the preceding equalities. □

Next a condition is given under which the inequalities in (3.3) become equalities given that equality holds in either of the two inclusions, cf. (Derkach et al. 2006: Section 2.3).

**Lemma 3.5.** *Let $V$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then*

$$\begin{cases} \ker V = (\operatorname{dom} V)^{[\perp]_1} \\ (\operatorname{ran} V)^{[\perp]_2} \subseteq \operatorname{ran} V \end{cases} \quad \textit{if and only if} \quad \begin{cases} \operatorname{mul} V = (\operatorname{ran} V)^{[\perp]_2} \\ (\operatorname{dom} V)^{[\perp]_1} \subseteq \operatorname{dom} V \end{cases}$$

*Proof.* Since $V^{-1}$ is an isometric relation if and only if $V$ is an isometric relation, it suffices to prove only one implication. Therefore assume that $\ker V = (\operatorname{dom} V)^{[\perp]_1}$ and that $(\operatorname{ran} V)^{[\perp]_2} \subseteq \operatorname{ran} V$. Then, clearly, $(\operatorname{dom} V)^{[\perp]_1} \subseteq \operatorname{dom} V$. Moreover, the assumption $\ker V = (\operatorname{dom} V)^{[\perp]_1}$ and an application of Lemma 3.8 below yield

$$\begin{aligned} \operatorname{mul} V &= V(\ker V) = V((\operatorname{dom} V)^{[\perp]_1}) = (V(\operatorname{dom} V))^{[\perp]_2} \cap \operatorname{ran} V \\ &= (\operatorname{ran} V)^{[\perp]_2} \cap \operatorname{ran} V. \end{aligned}$$

Hence, the assumption $(\operatorname{ran} V)^{[\perp]_2} \subseteq \operatorname{ran} V$ yields $\operatorname{mul} V = (\operatorname{ran} V)^{[\perp]_2}$. □

A further condition for the equality $\ker V = (\operatorname{dom} V)^{[\perp]_1}$ to hold is contained in the following statement.

**Lemma 3.6.** *Let $V$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and assume that there exists a hyper-maximal semi-definite subspace $\mathfrak{L}$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $\mathfrak{L} \subseteq \operatorname{dom} V$. Then $\ker V = (\operatorname{dom} V)^{[\perp]_1}$ if and only if $\mathfrak{j}_1\mathfrak{L} \cap \operatorname{dom} V + \ker V$ is an essentially hyper-maximal semi-definite subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ for a (and hence for every) fundamental symmetry $\mathfrak{j}_1$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$.*

*Proof.* Note first that $\ker V \subseteq \mathfrak{L}$, because the hyper-maximality of $\mathfrak{L}$ implies that $\mathfrak{L}^{[\perp]} \subseteq \mathfrak{L}$ and (3.3) implies that $\ker V \subseteq (\operatorname{dom} V)^{[\perp]_1}$. Since $\mathfrak{L}$ being hyper-maximal semi-definite is closed, the inclusion $\ker V \subseteq \mathfrak{L}$ implies that $\ker V$ is closed. Using this observation and the hyper-maximality of $\mathfrak{L}$, it follows that $\mathfrak{K}_1$ has the following $\mathfrak{j}_1$-orthogonal decomposition:

$$\mathfrak{K}_1 = \ker V \oplus_1 (\mathfrak{L}^{[\perp]_1} \ominus \ker V) \oplus_1 (\mathfrak{L} \cap \mathfrak{j}_1\mathfrak{L}) \oplus_1 \mathfrak{j}_1(\mathfrak{L}^{[\perp]_1} \ominus_1 \ker V) \oplus_1 \mathfrak{j}_1 \ker V, \quad (3.5)$$

cf. Proposition 2.9 (iv). Hence, $\operatorname{dom} V \subseteq (\ker V)^{[\perp]_1}$ and $(\ker V)^{[\perp]_1}$ have the decompositions:

$$\begin{aligned}\operatorname{dom} V &= \ker V \oplus_1 (\mathfrak{L}^{[\perp]_1} \ominus_1 \ker V) \oplus_1 (\mathfrak{L} \cap \mathfrak{j}_1\mathfrak{L}) \oplus_1 (\mathfrak{j}_1\mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} V),\\ (\ker V)^{[\perp]_1} &= \ker V \oplus_1 (\mathfrak{L}^{[\perp]_1} \ominus_1 \ker V) \oplus_1 (\mathfrak{L} \cap \mathfrak{j}_1\mathfrak{L}) \oplus_1 \mathfrak{j}_1(\mathfrak{L}^{[\perp]_1} \ominus_1 \ker V).\end{aligned}$$

Since $\ker V$ is closed, $\ker V = (\operatorname{dom} V)^{[\perp]_1}$ if and only if $(\ker V)^{[\perp]_1} = \overline{\operatorname{dom}} V$. Hence, the above two formula lines show that $\ker V = (\operatorname{dom} V)^{[\perp]_1}$ if and only if $\operatorname{clos}(\mathfrak{j}_1\mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} V) = \mathfrak{j}_1\mathfrak{L}^{[\perp]_1} \ominus_1 \mathfrak{j}_1\ker V$. Since $\mathfrak{j}_1\mathfrak{L} \ominus_1 \mathfrak{j}_1\ker V = (\mathfrak{L} \cap \mathfrak{j}_1\mathfrak{L}) \oplus \mathfrak{j}_1(\mathfrak{L}^{[\perp]_1} \ominus_1 \ker V)$, it follows from (3.5) together with Proposition 2.9 that the statement holds. □

This section is concluded with necessary and sufficient conditions for an isometric relation to be unitary which can be found in (Sorjonen 1980: Proposition 2.3.1).

**Proposition 3.7.** *Let $U$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then $U$ is a unitary if and only if $\operatorname{dom} U^{[*]} \subseteq \operatorname{ran} U$ and $(\operatorname{dom} U)^{[\perp]_1} \subseteq \ker U$ or, equivalently, $\operatorname{ran} U^{[*]} \subseteq \operatorname{dom} U$ and $(\operatorname{ran} U)^{[\perp]_2} \subseteq \operatorname{mul} U$.*

*Proof.* If $U$ is a unitary relation, then (3.1) and (3.4) imply that $\operatorname{dom} U^{[*]} = \operatorname{ran} U$ and $(\operatorname{dom} U)^{[\perp]_1} = \ker U$, respectively. Conversely, since $U$ is isometric $U^{-1} \subseteq U^{[*]}$, see (3.1). Moreover, the assumptions imply that $\operatorname{dom} U^{[*]} \subseteq \operatorname{dom} U^{-1}$ and that $\operatorname{mul} U^{[*]} = (\operatorname{dom} U)^{[\perp]_1} \subseteq \ker U = \operatorname{mul} U^{-1}$, see (2.6). Hence, the equality $U^{-1} = U^{[*]}$ holds, see (2.7). I.e., $U$ is unitary.

The second equivalence is obtained from the first by passing to the inverses. □

## 3.3 Isometric relations and closures of subspaces

A standard unitary operator $U$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ satisfies for every subspace $\mathfrak{L}$ of $\mathfrak{K}_1$ the following equality:

$$U(\mathfrak{L}^{[\perp]_1}) = (U(\mathfrak{L}))^{[\perp]_2}. \tag{3.6}$$

Since a unitary relation between Kreĭn spaces need not be everywhere defined, (3.6) does not in general hold for unitary relations between Kreĭn spaces. Instead a weaker form of (3.6) holds for all isometric relations.

**Lemma 3.8.** *Let $V$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{L} \subseteq \operatorname{dom} V$. Then*

$$V(\mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} V) = (V(\mathfrak{L}))^{[\perp]_2} \cap \operatorname{ran} V.$$

*Proof.* If $f' \in V\left(\mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} V\right)$, then there exists a $f \in \mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} V$ such that $f' \in Vf$. In particular, $[f, h]_1 = 0$ for all $h \in \mathfrak{L}$. Since $V$ is isometric, this implies that $[f', h']_2 = 0$ for all $h' \in V(\mathfrak{L})$, i.e., $f' \in (V(\mathfrak{L}))^{[\perp]_2} \cap \operatorname{ran} V$. This shows that $V\left(\mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} V\right) \subseteq (V(\mathfrak{L}))^{[\perp]_2} \cap \operatorname{ran} V$. The inverse inclusion follows from the proven inclusion by applying it to $V^{-1}$ and $V(\mathfrak{L})$. □

If $U$ is a standard unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, then (3.6) implies that $U(\operatorname{clos} \mathfrak{L}) = \operatorname{clos}(U(\mathfrak{L}))$ for any subspace $\mathfrak{L}$ of $\mathfrak{K}_1$. This equality does not in general hold for unitary relations, and a similar result only holds for certain subspaces. For instance, if $V$ is an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and $\ker V \subseteq \mathfrak{L} \subseteq \operatorname{dom} V$ is such that

$$\operatorname{clos} \mathfrak{L} = (\mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} V)^{[\perp]_1} \quad \text{and} \quad \operatorname{clos}(V(\mathfrak{L})) = (V(\mathfrak{L})^{[\perp]_2} \cap \operatorname{ran} V)^{[\perp]_2}.$$

Then applying Lemma 3.8 twice yields

$$V(\operatorname{clos}(\mathfrak{L}) \cap \operatorname{dom} V) = (\operatorname{clos} V(\mathfrak{L})) \cap \operatorname{ran} V.$$

The above example indicates that the behavior of isometric relations with respect to the closure of subspaces is in general not easy to describe. However, for uniformly definite subspaces this behavior is specific.

**Proposition 3.9.** *Let $V$ be a closed isometric relation between $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{D} \subseteq \operatorname{dom} V$ be a uniformly definite subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then the following statements hold:*

(i) *if $\mathfrak{D} = \operatorname{clos}(\mathfrak{D}) \cap \operatorname{dom} V$, then $V(\mathfrak{D})$ is closed;*

(ii) *$\mathfrak{D}$ is closed if and only if $V(\mathfrak{D}) \hat{+} [\operatorname{mul} V]$ is a closed uniformly definite subspace of $\{(\operatorname{mul} V)^{[\perp]_2}/\operatorname{mul} V, [\cdot,\cdot]_2\}$.*

*Proof.* To prove the statements w.l.o.g. assume $\mathfrak{D}$ to be uniformly positive and let $\mathfrak{j}_1$ and $\mathfrak{j}_2$ be fundamental symmetries of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, respectively.

(i) : Let $f' \in \operatorname{clos}(V(\mathfrak{D}))$, then there exists a sequence $\{\{f_n, f_n'\}\}_{n \geq 0}$, where $f_n \in \mathfrak{D}$ and $f_n' \in Vf_n$, such that $f' = \lim_{n\to\infty} f_n'$ in the Hilbert space $\{\mathfrak{K}_2, [\mathfrak{j}_2\cdot,\cdot]_2\}$. By the isometry of $V$

$$[\mathfrak{j}_2(f_m' - f_n'), f_m' - f_n']_2 \geq [f_m' - f_n', f_m' - f_n']_2 = [f_m - f_n, f_m - f_n]_1.$$

Since $\mathfrak{D}$ is uniformly positive, there exists a constant $\alpha > 0$ such that $\alpha[\mathfrak{j}_1 g, g]_1 \leq [g, g]_1$ for all $g \in \mathfrak{D}$. Combining this with the above inequality yields

$$[\mathfrak{j}_2(f_m' - f_n'), f_m' - f_n']_2 \geq \alpha[\mathfrak{j}_1(f_m - f_n), f_m - f_n]_1.$$

Since $\{f'_n\}_{n\geq 0}$ converges by assumption in $\{\mathfrak{K}_2, [\mathfrak{j}_2\cdot,\cdot]_2\}$, the preceding inequality shows that $\{f_n\}_{n\geq 0}$ is a Cauchy-sequence in the Hilbert space $\{\mathfrak{K}_1, [\mathfrak{j}_1\cdot,\cdot]_1\}$ and, hence, converges to an $f \in \operatorname{clos}(\mathfrak{D})$. Consequently, $\{\{f_n, f'_n\}\}_{n\geq 0}$ converges (in the graph norm) to $\{f, f'\} \in \mathfrak{K}_1 \times \mathfrak{K}_2$ and, hence, $\{f, f'\} \in \operatorname{gr} V$ by the closedness of $V$. Therefore $f \in \operatorname{clos}(\mathfrak{D}) \cap \operatorname{dom} V = \mathfrak{D}$ and, hence, $f' \in V(\mathfrak{D})$.

(ii) : For simplicity assume that $\operatorname{mul} V = \{0\}$. Let $\mathfrak{D} \subseteq \operatorname{dom} V$ be closed, then $V \upharpoonright_{\mathfrak{D}}$ is an everywhere defined closed (isometric) operator from the Hilbert space $\{\mathfrak{D}, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. I.e., there exists a $M > 0$ such that $[\mathfrak{j}_2 Vf, Vf]_2 \leq M[f,f]_1$, for all $f \in \mathfrak{D}$. Hence, using the fact that $V$ is isometric, it follows that

$$[\mathfrak{j}_2 Vf, Vf]_2 \leq M[f,f]_1 = M[Vf, Vf]_2, \quad f \in \mathfrak{D}.$$

I.e., $V(\mathfrak{D})$ is a uniformly definite subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. The converse implication is obtained by applying (i) to $V^{-1}$ and $V(\mathfrak{D})$. □

Note that (ii) essentially also holds for non-closed isometric relations. This follows by considering instead of non-closed isometric relations their closure.

## 3.4 Reduction of isometric relations

Here unitary relations are reduced in two different ways: By means of neutral subspaces contained in their domain (or range) and by splitting them. These reductions allow us to remove from unitary relations that part of their behavior which is well understood. In order to obtain the mentioned results the following composition results for isometric relations are used, see (Derkach et al. 2009: Section 2.2).

**Lemma 3.10.** *Let $S$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $T$ be an isometric relation from $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ to $\{\mathfrak{K}_3, [\cdot,\cdot]\}$. Then*

(i) *$TS$ is an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_3, [\cdot,\cdot]_3\}$;*

(ii) *if $S$ and $T$ are unitary, $\operatorname{ran} S \subseteq \operatorname{dom} T$ and $\operatorname{dom} T$ is closed or $\operatorname{dom} T \subseteq \operatorname{ran} S$ and $\operatorname{ran} S$ is closed, then $TS$ is a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_3, [\cdot,\cdot]_3\}$.*

*Proof.* Combine Lemma 2.15 with (3.1). □

Lemma 3.11 below associates with each neutral subspace a unitary operator which can be used to reduce unitary relations, see Corollary 3.12 below.

**Lemma 3.11.** *Let $\mathfrak{L}$ be a closed neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then $U_{\mathfrak{L}}$ defined as*

$$U_{\mathfrak{L}} f = f + [\mathfrak{L}], \quad f \in \operatorname{dom} U_{\mathfrak{L}} = \mathfrak{L}^{[\perp]}$$

*is a bounded unitary operator from $\{\mathfrak{K}, [\cdot,\cdot]\}$ onto the Kreĭn space $\{\mathfrak{L}^{[\perp]}/\mathfrak{L}, [\cdot,\cdot]\}$.*

*Proof.* Recall that the fact that $\{\mathfrak{L}^{[\perp]}/\mathfrak{L}, [\cdot,\cdot]\}$ is a Kreĭn space is the contents of Proposition 2.3 and note that the isometry of $U_{\mathfrak{L}}$ is a direct consequence of the neutrality of $\mathfrak{L}$. Next let $h \in \mathfrak{K}$ and $k \in \mathfrak{L}^{[\perp]}/\mathfrak{L}$ be such that $[f, h] = [U_{\mathfrak{L}} f, k]$ for all $f \in \mathfrak{L}^{[\perp]} = \operatorname{dom} U_{\mathfrak{L}}$. Since $U_{\mathfrak{L}}$ maps onto $\mathfrak{L}^{[\perp]}/\mathfrak{L}$ by its definition, there exists a $g \in \operatorname{dom} U_{\mathfrak{L}}$ such that $U_{\mathfrak{L}} g = k$ and, hence, $[f, h - g] = 0$ for all $f \in \mathfrak{L}^{[\perp]}$. This shows that $h - g \in \operatorname{clos}(\mathfrak{L}) = \mathfrak{L} \subseteq \ker U_{\mathfrak{L}}$. Consequently, $\{h, k\} = \{g, U_{\mathfrak{L}} g\} + \{h - g, 0\} \in \operatorname{gr} U_{\mathfrak{L}}$ and, hence, Proposition 3.1 implies that $U_{\mathfrak{L}}$ is unitary. □

Since $\ker V$ and $\operatorname{mul} V$ are neutral subspaces for an isometric relation $V$, see (3.3), composing isometric relations with unitary operators provided by Lemma 3.11 yields isometric operators without kernel and multi-valued part. In other words the interesting behavior of isometric relations takes place on the quotient spaces $(\ker V)^{[\perp]_1}/\overline{\ker}\, V$ and $(\operatorname{mul} V)^{[\perp]_2}/\overline{\operatorname{mul}}\, V$. Therefore Corollary 3.12 below can be for instance used to simplify proofs for statements concerning isometric relations to the case of isometric operators.

**Corollary 3.12.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then $U_r$ and $U_d$ defined via*

$$\operatorname{gr} U_r = \{\{f, f' + [\operatorname{mul} U]\} \in \mathfrak{K}_1 \times \overline{\operatorname{ran}}\, U/\operatorname{mul} U : \{f, f'\} \in U\};$$
$$\operatorname{gr} U_d = \{\{f + [\ker U], f'\} \in \overline{\operatorname{dom}}\, U/\ker U \times \mathfrak{K}_2 : \{f, f'\} \in U\},$$

*are a unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to the Kreĭn space $\{\overline{\operatorname{ran}}\, U/\operatorname{mul} U, [\cdot,\cdot]_2\}$ with dense range and a unitary relation from the Kreĭn space $\{\overline{\operatorname{dom}}\, U/\ker U, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with dense domain.*

*In particular, $(U_r)_d = (U_d)_r$ is a unitary operator from $\{\overline{\operatorname{dom}}\, U/\ker U, [\cdot,\cdot]_1\}$ to $\{\overline{\operatorname{ran}}\, U/\operatorname{mul} U, [\cdot,\cdot]_2\}$ with dense domain and dense range.*

*Proof.* Since $\ker U$ and $\operatorname{mul} U$ are closed neutral subspaces by (3.4), $U_{\ker U}$ and $U_{\operatorname{mul} U}$ are unitary operators by Lemma 3.11 with closed domain and closed range, respectively. Consequently, Lemma 3.10 implies that $U_r := U_{\operatorname{mul} U} U$ and $U_d := U (U_{\ker U})^{-1}$ are a unitary operator and relation, respectively, and a direct calculation shows that they have the stated form. □

Lemma 3.13 below is a statement about splitting a unitary relation into two unitary relations.

**Lemma 3.13.** *Let* $U$ *be an isometric relation from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *and let* $\{\widetilde{\mathfrak{K}}_i, [\cdot,\cdot]_i\}[+]\{\widehat{\mathfrak{K}}_i, [\cdot,\cdot]_i\}$ *be an orthogonal decomposition of* $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ *into two Kreĭn spaces, for* $i = 1, 2$*, such that* $\operatorname{gr} U = \operatorname{gr} \widetilde{U} + \operatorname{gr} \widehat{U}$*, where the isometric relations* $\widetilde{U}$ *and* $\widehat{U}$ *are defined via*

$$\operatorname{gr} \widetilde{U} := \operatorname{gr} U \cap (\widetilde{\mathfrak{K}}_1 \times \widetilde{\mathfrak{K}}_2) \quad \textit{and} \quad \operatorname{gr} \widehat{U} := \operatorname{gr} U \cap (\widehat{\mathfrak{K}}_1 \times \widehat{\mathfrak{K}}_2).$$

*Then* $U$ *is unitary if and only if* $\widetilde{U}$ *and* $\widehat{U}$ *are unitary.*

*Proof.* This follows from the definition of unitary relations ($U^{[*]} = U^{-1}$) and the orthogonal decomposition of $U$. □

Recall from Proposition 3.9 that if a unitary relation contains a closed uniformly definite subspace in its domain, then the unitary relation behaves like a Hilbert space unitary operator on that part of the space. Hence, using Lemma 3.13, one can reduce a unitary relation by taking out such parts.

**Corollary 3.14.** *Let* $U$ *be a closed and isometric relation between* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *and* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$*, let* $\mathfrak{D}_1 \subseteq \operatorname{dom} U$ *be a closed uniformly definite subspace of* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *and let* $\mathfrak{D}_2$ *be a closed uniformly definite subspace of* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *such that* $U(\mathfrak{D}_1) = \mathfrak{D}_2 + \operatorname{mul} U$*. Then* $U$ *is a unitary relation if and only if* $\widetilde{U}$ *defined via*

$$\operatorname{gr} \widetilde{U} = \operatorname{gr} U \cap (\mathfrak{D}_1^{[\perp]_1} \times \mathfrak{D}_2^{[\perp]_2})$$

*is a unitary relation from the Kreĭn space* $\{\mathfrak{K}_1 \cap \mathfrak{D}_1^{[\perp]_1}, [\cdot,\cdot]_1\}$ *to the Kreĭn space* $\{\mathfrak{K}_2 \cap \mathfrak{D}_2^{[\perp]_2}, [\cdot,\cdot]_2\}$*.*

*Proof.* Note first that the existence of $\mathfrak{D}_2$ as stated follows from Proposition 3.9 and that

$$\widehat{U} f = (Uf) \cap \mathfrak{D}_2, \quad f \in \operatorname{dom} \widehat{U} = \mathfrak{D}_1$$

is an everywhere defined isometric operator from the Hilbert space $\{\mathfrak{D}_1, [\cdot,\cdot]_1\}$ onto the Hilbert space $\{\mathfrak{D}_2, [\cdot,\cdot]_2\}$ and, hence, unitary. Since $\operatorname{gr} U = \operatorname{gr} \widetilde{U} + \operatorname{gr} \widehat{U}$, the statement follows now from Lemma 3.13. □

## 3.5 Maximal isometric and unitary relations

Recall that an isometric relation is called *maximal isometric* if there does not exist a proper isometric extension of it. In particular, unitary relation are maximal isometric. As a consequence of the graph characterizations in Proposition 3.2, the following characterizations of maximal isometric and unitary relations hold.

**Corollary 3.15.** *Let $V$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ with associated projections $P_i^+$ and $P_i^-$, for $i = 1, 2$. Then $V$ is maximal isometric if and only if*

$$(P_1^+ \times P_2^-)\,\mathrm{gr}\,V = \mathfrak{K}_1^+ \times \mathfrak{K}_2^- \quad or \quad (P_1^- \times P_2^+)\,\mathrm{gr}\,V = \mathfrak{K}_1^- \times \mathfrak{K}_2^+.$$

*Moreover, $V$ is unitary if and only if both the above equalities hold.*

*Proof.* Clearly, $(\mathfrak{K}_1^+ \times \mathfrak{K}_2^-)[+](\mathfrak{K}_1^- \times \mathfrak{K}_2^+)$ is a canonical decomposition of (the Kreĭn space) $\{\mathfrak{K}_1 \times \mathfrak{K}_2, [\cdot,\cdot]_{1,-2}\}$, see (3.2). Hence, the statement is a direct consequence of Proposition 3.2, Proposition 2.5 and Proposition 2.6. □

Using Proposition 2.16 alternative characterizations for the conditions in Corollary 3.15 can be obtained.

**Proposition 3.16.** *Let $V$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ with associated projections $P_i^+$ and $P_i^-$, for $i = 1, 2$. Then equivalent are:*

*(i)* $(P_1^+ \times P_2^-)\,\mathrm{gr}\,V = \mathfrak{K}_1^+ \times \mathfrak{K}_2^-$;

*(ii)* $P_1^+ V^{-1}(\mathrm{ran}\,V \cap \mathfrak{K}_2^+) = \mathfrak{K}_1^+$ *and* $P_2^- V(\mathrm{dom}\,V \cap \mathfrak{K}_1^-) = \mathfrak{K}_2^-$;

*(iii)* $P_1^+\mathrm{dom}\,V = \mathfrak{K}_1^+$, $P_2^-\mathrm{ran}\,V = \mathfrak{K}_2^-$ *and*

$$\mathrm{dom}\,V = \mathrm{dom}\,V \cap \mathfrak{K}_1^- + V^{-1}(\mathrm{ran}\,V \cap \mathfrak{K}_2^+),$$

*Similarly, equivalent are:*

*(i)* $(P_1^- \times P_2^+)\,\mathrm{gr}\,V = \mathfrak{K}_1^- \times \mathfrak{K}_2^+$;

*(ii)* $P_1^- V^{-1}(\mathrm{ran}\,V \cap \mathfrak{K}_2^-) = \mathfrak{K}_1^-$ *and* $P_2^+ V(\mathrm{dom}\,V \cap \mathfrak{K}_1^+) = \mathfrak{K}_2^+$;

*(iii)* $P_1^-\mathrm{dom}\,V = \mathfrak{K}_1^-$, $P_2^+\mathrm{ran}\,V = \mathfrak{K}_2^+$ *and*

$$\mathrm{dom}\,V = \mathrm{dom}\,V \cap \mathfrak{K}_1^+ + V^{-1}(\mathrm{ran}\,V \cap \mathfrak{K}_2^-).$$

*Proof.* Clearly, it suffices to prove only the first equivalences. By Proposition 2.16, the assumption $(P_1^+ \times P_2^-)\,\mathrm{gr}\,V = \mathfrak{K}_1^+ \times \mathfrak{K}_2^-$ yields that (ii) holds. If (ii) holds, then, clearly, $P_1^+\mathrm{dom}\,V = \mathfrak{K}_1^+$ and $P_2^-\mathrm{ran}\,V = \mathfrak{K}_2^-$, and the domain decomposition in (iii) holds by Proposition 2.16. Finally, if (iii) holds, then the domain decomposition therein implies that $(P_1^+\mathrm{dom}\,V) \times (P_2^-\mathrm{ran}\,V) \subseteq (P_1^+ \times P_2^-)\mathrm{gr}\,V$, see Proposition 2.16. Hence, the assumptions $P_1^+\mathrm{dom}\,V = \mathfrak{K}_1^+$ and $P_2^-\mathrm{ran}\,V = \mathfrak{K}_2^-$ imply that (i) holds. □

In particular, Proposition 3.16 implies that if $U$ is a unitary relation, then

$$P_1^+ \operatorname{dom} U = \mathfrak{K}_1^+,\ P_1^- \operatorname{dom} U = \mathfrak{K}_1^-,\ P_2^+ \operatorname{ran} U = \mathfrak{K}_2^+,\ P_2^- \operatorname{ran} U = \mathfrak{K}_2^-. \tag{3.7}$$

Combining Proposition 3.16 with Corollary 3.15 yields necessary and sufficient conditions for an isometric relation to be unitary.

**Corollary 3.17.** *Let $U$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ with associated projections $P_i^+$ and $P_i^-$, for $i = 1, 2$. Then $U$ is a unitary relation if and only* (3.7) *holds and the domain of $U$ has the following decompositions:*

$$\operatorname{dom} U \cap \mathfrak{K}_1^+ + U^{-1}(\operatorname{ran} U \cap \mathfrak{K}_2^-) = \operatorname{dom} U = \operatorname{dom} U \cap \mathfrak{K}_1^- + U^{-1}(\operatorname{ran} U \cap \mathfrak{K}_2^+).$$

Note that the domain decomposition conditions in Corollary 3.17 are equivalent to the graph of $U$ having the following decompositions:

$$\begin{aligned} \operatorname{gr} U &= \{\{f, f'\} \in \operatorname{gr} U : f \in \mathfrak{K}_1^+\} + \{\{f, f'\} \in \operatorname{gr} U : f' \in \mathfrak{K}_2^-\}; \\ \operatorname{gr} U &= \{\{f, f'\} \in \operatorname{gr} U : f \in \mathfrak{K}_1^-\} + \{\{f, f'\} \in \operatorname{gr} U : f' \in \mathfrak{K}_2^+\}, \end{aligned} \tag{3.8}$$

cf. (Calkin 1939a: Theorem 3.9). For an isometric relation (3.7) can be satisfied while neither of the domain decompositions in Corollary 3.17 holds; consider for instance the identity mapping on a hyper-maximal semi-definite subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Conversely, if both the equalities in (3.8) are satisfies for an isometric relation, then the relation is already very close to being unitary.

**Proposition 3.18.** *Let $U$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ with associated projections $P_i^+$ and $P_i^-$, for $i = 1, 2$. Then $U$ is unitary if and only if*

*(i)* *$U$ is closed;*

*(ii)* $\ker U = (\operatorname{dom} U)^{[\perp]_1}$ *and* $\operatorname{mul} U = (\operatorname{ran} U)^{[\perp]_2}$*;*

*(iii)* *the domain of $U$ has the following decompositions:*

$$\operatorname{dom} U \cap \mathfrak{K}_1^+ + U^{-1}(\operatorname{ran} U \cap \mathfrak{K}_2^-) = \operatorname{dom} U = \operatorname{dom} U \cap \mathfrak{K}_1^- + U^{-1}(\operatorname{ran} U \cap \mathfrak{K}_2^+).$$

*Proof.* If $U$ is unitary, then the closedness of $U$ follows from $U^{-1} = U^{[*]}$ and (ii) holds by (3.4). Moreover, Corollary 3.17 shows that (iii) holds.

Conversely, if (iii) holds, then by Proposition 2.16

$$\begin{aligned} (P_1^+ \times P_2^-)\operatorname{gr} U &= P_1^+ \operatorname{dom} U \times P_2^- \operatorname{ran} U; \\ (P_1^- \times P_2^+)\operatorname{gr} U &= P_1^- \operatorname{dom} U \times P_2^+ \operatorname{ran} U. \end{aligned} \tag{3.9}$$

Moreover, by condition (i) and Proposition 3.2 $\mathrm{gr}\, U$ is a closed neutral subspace of the Kreĭn space $\{\mathfrak{K}_1 \times \mathfrak{K}_2, [\cdot,\cdot]_{1,-2}\}$ and thus $(P_1^+ \times P_2^-)\mathrm{gr}\, U$ and $(P_1^- \times P_2^+)\mathrm{gr}\, U$ are closed subspaces, see (Azizov & Iokhvidov 1989: Ch. 1, §4). In view of (3.9), this implies that $P_1^+\mathrm{dom}\, U$, $P_1^-\mathrm{dom}\, U$, $P_2^+\mathrm{ran}\, U$ and $P_2^-\mathrm{ran}\, U$ are closed subspaces. Now the assumption (ii) implies by Lemma 3.4 that $P_1^+\mathrm{dom}\, U = \mathfrak{K}_1^+$, $P_1^-\mathrm{dom}\, U = \mathfrak{K}_1^-$, $P_2^+\mathrm{ran}\, U = \mathfrak{K}_2^+$ and $P_2^-\mathrm{ran}\, U = \mathfrak{K}_2^-$. Consequently, Corollary 3.17 yields that $U$ is unitary. □

Using Corollary 2.14 the following properties of maximal isometric relations are obtained.

**Lemma 3.19.** *Let $V$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ with associated projections $P_i^+$ and $P_i^-$, for $i = 1, 2$. If $(P_1^+ \times P_2^-)\,\mathrm{gr}\, V = \mathfrak{K}_1^+ \times \mathfrak{K}_2^-$, then*

$$\mathrm{clos}\,(\mathrm{dom}\, V \cap \mathfrak{K}_1^-) = \overline{\mathrm{dom}}\, V \cap \mathfrak{K}_1^- \quad \textit{and} \quad \mathrm{clos}\,(\mathrm{ran}\, V \cap \mathfrak{K}_2^+) = \overline{\mathrm{ran}}\, V \cap \mathfrak{K}_2^+.$$

*Similarly, if $(P_1^- \times P_2^+)\,\mathrm{gr}\, V = \mathfrak{K}_1^- \times \mathfrak{K}_2^+$, then*

$$\mathrm{clos}\,(\mathrm{dom}\, V \cap \mathfrak{K}_1^+) = \overline{\mathrm{dom}}\, V \cap \mathfrak{K}_1^+ \quad \textit{and} \quad \mathrm{clos}\,(\mathrm{ran}\, V \cap \mathfrak{K}_2^-) = \overline{\mathrm{ran}}\, V \cap \mathfrak{K}_2^-.$$

*Proof.* The statement follows from Corollary 2.14 applied to $\mathfrak{M} = \mathrm{gr}\, V$, $\mathfrak{L} = \mathrm{dom}\, V \times \mathrm{ran}\, V$, $P^+ = P_1^+ \times P_2^-$ and $P^- = P_1^- \times P_2^+$. □

**Corollary 3.20.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ with associated projections $P_i^+$ and $P_i^-$, for $i = 1, 2$. Then*

$$\begin{aligned} \mathrm{clos}\,(\mathrm{dom}\, U \cap \mathfrak{K}_1^+) &= \overline{\mathrm{dom}}\, U \cap \mathfrak{K}_1^+, \quad \mathrm{clos}\,(\mathrm{ran}\, U \cap \mathfrak{K}_2^+) = \overline{\mathrm{ran}}\, U \cap \mathfrak{K}_2^+; \\ \mathrm{clos}\,(\mathrm{dom}\, U \cap \mathfrak{K}_1^-) &= \overline{\mathrm{dom}}\, U \cap \mathfrak{K}_1^-, \quad \mathrm{clos}\,(\mathrm{ran}\, U \cap \mathfrak{K}_2^-) = \overline{\mathrm{ran}}\, U \cap \mathfrak{K}_2^-. \end{aligned}$$

*Proof.* Combine Lemma 3.19 with Corollary 3.15. □

Combining Corollary 3.20 with the first von Neumann formula (2.4) (applied to $\mathfrak{L} = \ker U = (\mathrm{dom}\, U)^{[\perp]_1}$) yields that for a unitary relation $U$

$$\overline{\mathrm{dom}}\, U = \ker U \dotplus \mathrm{clos}\,(\mathrm{dom}\, U \cap \mathfrak{K}_1^+) \dotplus \mathrm{clos}\,(\mathrm{dom}\, U \cap \mathfrak{K}_1^-), \tag{3.10}$$

see also (Derkach et al. 2006: Lemma 2.14 (ii)). Combining the above equality with (3.7) yields the following useful equalities:

$$\begin{aligned} \mathfrak{K}_1^+ &= P_1^+\ker U \dotplus \mathrm{clos}\,(\mathrm{dom}\, U \cap \mathfrak{K}_1^+); \\ \mathfrak{K}_1^- &= P_1^-\ker U \dotplus \mathrm{clos}\,(\mathrm{dom}\, U \cap \mathfrak{K}_1^-). \end{aligned} \tag{3.11}$$

# 4 SPECIAL CLASSES OF UNITARY RELATIONS

In this chapter some special classes of unitary relations are introduced and investigated. More specifically, in the first section unitary relations with a closed domain, or equivalently with a closed range, are considered. They are shown to be almost completely characterized by their behavior on uniformly definite subspaces and they are also shown to have essentially the same behavior as standard unitary operators. In the second section two types of unitary relations with a simple structure are introduced, which will be called *archetypical unitary relations*. Later results, see e.g. Section 7.3, show that essentially all the mapping properties of (unbounded) unitary relations can be understood by considering only (unbounded) archetypical unitary operators. Finally, in the third section standard unitary operators are shortly considered. In particular, it is shown how they can be written in terms of the introduced archetypical unitary operators.

## 4.1 Unitary relations with closed domain

As a starting point for investigating unitary relations with closed domain, consider the following characterization of such relations. Note that the following statement is a generalization of (Bognár 1974: Ch. VI, Theorem 3.5).

**Proposition 4.1.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then $U$ has closed domain if and only if $U$ maps every uniformly positive (negative) subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ contained in the domain of $U$ onto the sum a uniformly positive (negative) subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and the multi-valued part of $U$.*

*Proof.* Let $(U_r)_d$ be the unitary operator with dense domain associated with $U$ as in Corollary 3.12. Then one can easily see that $U$ has closed domain if and only if $(U_r)_d$ has closed domain. In other words, it suffices to prove the statement for a densely defined unitary operator.

If $U$ has closed domain and $\mathfrak{D} \subseteq \operatorname{dom} U$ is a uniformly positive (negative) subspace, then $\operatorname{clos}(\mathfrak{D}) \subseteq \overline{\operatorname{dom}}\, U = \operatorname{dom} U$ is a closed uniformly positive (negative) subspace which is mapped by $U$ onto a uniformly positive (negative) subspace, see Proposition 3.9. Hence $\mathfrak{D}$ itself is also mapped onto a uniformly positive subspace. To prove the converse implication let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then $\mathfrak{K}_1 = \overline{\operatorname{dom}}\, U = \operatorname{clos}(\operatorname{dom} U \cap \mathfrak{K}_1^+) + \operatorname{clos}(\operatorname{dom} U \cap \mathfrak{K}_1^-)$, see (3.10). By the assumption together with Proposition 3.9 $\operatorname{dom} U \cap \mathfrak{K}_1^+$ and $\operatorname{dom} U \cap \mathfrak{K}_1^-$ are closed. Hence, $\mathfrak{K}_1 = \operatorname{dom} U \cap \mathfrak{K}_1^+ + \operatorname{dom} U \cap \mathfrak{K}_1^- \subseteq \operatorname{dom} U$ shows that $\operatorname{dom} U$ is closed. □

The proof of Proposition 4.1 shows that if $U$ is a unitary operator with closed domain, then $\operatorname{dom} U = \ker U[+]\operatorname{dom} U \cap \mathfrak{K}_1^+[+]\operatorname{dom} U \cap \mathfrak{K}_1^-$. Hence, in that case $\operatorname{ran} U = U(\operatorname{dom} U \cap \mathfrak{K}_1^+)[+]U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$, where $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ and $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ are closed uniformly definite subspaces of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, see Proposition 3.9. Since the orthogonal sum of closed uniformly definite subspaces is a closed subspace, see e.g. (Bognár 1974: Ch. V, Theorem 3.4 & Theorem 5.3), an elementary proof for the following statement has been obtained, see (Shmul'jan 1976; Sorjonen 1978/1979).

**Proposition 4.2.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then $\operatorname{dom} U$ is closed if and only if $\operatorname{ran} U$ is closed.*

*Proof.* Since $U$ is unitary if and only if $U^{-1}$ is unitary, it suffices to prove that if $\operatorname{dom} U$ is closed, then $\operatorname{ran} U$ is closed. If $\operatorname{mul} U \neq \{0\}$, then $\operatorname{mul}(U_{\operatorname{mul} U} U) = \{0\}$, see Lemma 3.11. Since $U_{\operatorname{mul} U} U$ has closed range if and only if $U$ has closed range, the statement follows now from the discussion preceding this statement. □

**Corollary 4.3.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then the following statements hold:*

(i) *if $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ is a Hilbert space, then $\operatorname{dom} U = \mathfrak{K}_1$;*

(ii) *if $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ are Hilbert spaces, then $U$ is a standard unitary operator.*

*Proof.* Clearly, (ii) follows from (i). If the assumption in (i) holds, then by Proposition 4.1 (applied to $U^{-1}$) $U$ has closed range and, hence, closed domain, see Proposition 4.2. Since $\ker U = (\operatorname{dom} U)^{[\perp]_1}$ is a neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, the assumption also implies that $\ker U = \{0\}$ and, hence, $\operatorname{dom} U = \overline{\operatorname{dom}} U = \mathfrak{K}_1$. □

Proposition 4.2 can be extended to the case of isometric relations: If equalities hold in (3.3) for an isometric relation and, additionally, its domain or range is closed, then the isometry relation must be a unitary relation with closed domain and range.

**Corollary 4.4.** *Let $U$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ which has closed domain or closed range and satisfies*

$$\ker U = (\operatorname{dom} U)^{[\perp]_1} \quad \textit{and} \quad \operatorname{mul} U = (\operatorname{ran} U)^{[\perp]_2}.$$

*Then $U$ is a unitary relation with closed domain and range.*

*Proof.* Assume that $U$ has closed range, then the assumptions together with (2.6) yield

$$\operatorname{dom} U^{[*]} \subseteq \overline{\operatorname{dom}} U^{[*]} = (\operatorname{mul} \operatorname{clos}(U))^{[\perp]_2} \subseteq (\operatorname{mul} U)^{[\perp]_2} = \operatorname{ran} U.$$

Therefore $U$ is unitary by Proposition 3.7 and $U$ has closed domain by Proposition 4.2. The case that $U$ has closed domain follows by passing to the inverse. □

Note that the assumptions $\ker U = (\operatorname{dom} U)^{[\perp]_1}$ and $\operatorname{mul} U = (\operatorname{ran} U)^{[\perp]_2}$ in Corollary 4.4 can by Lemma 3.5 be weakened to $\ker U = (\operatorname{dom} U)^{[\perp]_1}$ and $(\operatorname{ran} V)^{[\perp]_2} \subseteq \operatorname{ran} V$ or $\operatorname{mul} U = (\operatorname{ran} U)^{[\perp]_2}$ and $(\operatorname{dom} V)^{[\perp]_1} \subseteq \operatorname{dom} V$, cf. (Derkach et al. 2006: Section 2.3).

Proposition 4.5 below shows that unitary relations with closed domain and range have almost the same properties as standard unitary operators (everywhere defined unitary operators with everywhere defined inverse, see (Derkach et al. 2009: Definition 2.5)).

**Proposition 4.5.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with closed domain. If $\mathfrak{L}$, $\ker U \subseteq \mathfrak{L} \subseteq \operatorname{dom} U$, is a subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, then*

$$U(\mathfrak{L}^{[\perp]_1}) = (U(\mathfrak{L}))^{[\perp]_2} \quad \textit{and} \quad U(\operatorname{clos}(\mathfrak{L})) = \operatorname{clos}(U(\mathfrak{L})), \tag{4.1}$$

*Moreover, if $\mathfrak{L}$, $\ker U \subseteq \mathfrak{L} \subseteq \operatorname{dom} U$, is a neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, then*

$$n_+(\mathfrak{L}) = n_+(U(\mathfrak{L})) \quad \textit{and} \quad n_-(\mathfrak{L}) = n_-(U(\mathfrak{L})). \tag{4.2}$$

*In particular, $\mathfrak{L}$ is an (essentially, closed) (hyper-maximal, maximal) nonnegative, nonpositive or neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ if and only if $U(\mathfrak{L})$ is an (essentially, closed) (hyper-maximal, maximal) nonnegative, nonpositive or neutral of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, respectively.*

*Proof.* Let $\ker U \subseteq \mathfrak{L} \subseteq \operatorname{dom} U$, then $(\operatorname{dom} U)^{[\perp]_1} \subseteq \mathfrak{L}^{[\perp]_1} \subseteq (\ker U)^{[\perp]_1}$. Hence, using (3.4) and the closedness of the domain (and range) of $U$, it follows that $\ker U \subseteq \mathfrak{L}^{[\perp]_1} \subseteq \operatorname{dom} U$. Similar arguments show that $\operatorname{mul} U \subseteq (U(\mathfrak{L}))^{[\perp]_2} \subseteq \operatorname{ran} U$. Consequently, the equality $U(\mathfrak{L}^{[\perp]_1}) = (U(\mathfrak{L}))^{[\perp]_2}$ follows directly from Lemma 3.8. The second equality in (4.1) follows by applying the first equality therein twice to a subspace $\mathfrak{L}$.

As a consequence (4.1), (4.2) needs only to be proven for the case that $\mathfrak{L}$ and $U(\mathfrak{L})$ are closed. Now let $U_{\mathfrak{L}}$ and $U_{U(\mathfrak{L})}$ be the bounded unitary operators associated to $\mathfrak{L}$ and $U(\mathfrak{L})$ as in Lemma 3.11, then $U_a := U_{U(\mathfrak{L})} U (U_{\mathfrak{L}})^{-1}$ is an everywhere defined isometric operator from the Kreĭn space $\{\mathfrak{L}^{[\perp]_1}/\mathfrak{L}, [\cdot,\cdot]_1\}$ to the Kreĭn space $\{(U(\mathfrak{L}))^{[\perp]_2}/U(\mathfrak{L}), [\cdot,\cdot]_2\}$. I.e., $U_a$ is a standard unitary operator and, hence, $n_\pm(0_d) = n_\pm(0_r)$, where $0_d$ and $0_r$ are the trivial subspaces in $\{\mathfrak{L}^{[\perp]_1}/\mathfrak{L}, [\cdot,\cdot]_1\}$ and $\{(U(\mathfrak{L}))^{[\perp]_2}/U(\mathfrak{L}), [\cdot,\cdot]_2\}$, respectively. This, together with the first von Neumann formula (2.4), shows that $n_\pm(\mathfrak{L}) = n_\pm(U(\mathfrak{L}))$. □

Next further characterizations of the closedness of the domain of a unitary relation are given; they are closely related to results on Weyl families of boundary relations stated in (Derkach et al. 2006). Note that the equivalence of (i), (ii) and (iii) in Proposition 4.6 goes back to Calkin (1939a: Theorem 3.10) and that the characterization (vii) is an inverse to a statement in (Derkach et al. 2006: Lemma 4.4).

**Proposition 4.6.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then equivalent are*

(i) $\operatorname{dom} U$ *is closed;*

(ii) $\operatorname{dom} U \cap \mathfrak{K}_1^+$ *is closed;*

(iii) $\operatorname{dom} U \cap \mathfrak{K}_1^-$ *is closed;*

(iv) $U(\operatorname{dom} U \cap \mathfrak{K}_1^+) + [\operatorname{mul} U]$ *is a uniformly positive subspace of the Kreĭn space* $\{\overline{\operatorname{ran}}\, U/\operatorname{mul} U, [\cdot,\cdot]_2\}$*;*

(v) $U(\operatorname{dom} U \cap \mathfrak{K}_1^-) + [\operatorname{mul} U]$ *is a uniformly negative subspace of the Kreĭn space* $\{\overline{\operatorname{ran}}\, U/\operatorname{mul} U, [\cdot,\cdot]_2\}$*;*

(vi) $\operatorname{dom} U = \ker U + \operatorname{dom} U \cap \mathfrak{K}_1^+ + \operatorname{dom} U \cap \mathfrak{K}_1^-$*;*

(vii) $\operatorname{ran} U = U(\operatorname{dom} U \cap \mathfrak{K}_1^+) + U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$*;*

*Proof.* (i)-(v): The implication from (i) to (ii) and (iii) is clear, the equivalences of (ii) and (iv), and (iii) and (v) follows from Proposition 3.9. Furthermore, the equivalence of (iv) and (v) follows from Proposition 2.5, (Bognár 1974: Ch. V, Corollary 7.4), and Proposition 5.1 below, and (3.10) shows that (ii) and (iii) imply (i).

(i)-(v) $\Leftrightarrow$ (vi) : By (3.10) the conditions (i)-(iii) imply (vi). If (vi) holds, then $\mathfrak{K}_1^+ = P_1^+ \operatorname{dom} U = P_1^+ \ker U + \operatorname{dom} U \cap \mathfrak{K}_1^+$, where $P_1^+$ is the orthogonal projection onto $\mathfrak{K}_1^+$, see (3.7). Comparing this with (3.11) shows that (ii) holds.

(vi) $\Leftrightarrow$ (vii) : This follows by applying $U$ and $U^{-1}$. □

Observe that the characterizations (ii) and (iii) in Proposition 4.6 in particular imply that a unitary relation has closed domain (and range) if either of the defect numbers of the kernel of $U$ is finite.

## 4.2 Archetypical unitary relations

Two types of unitary operators having a simple block structure are here introduced; they will be called *archetypical* unitary operator. Recall that, in the bounded case, archetypical unitary operators appear as so-called transformers in (Shmul'jan 1980). They also appear naturally in the framework of boundary relations; there they are used to normalize the Weyl family associated with a boundary relation, see (Derkach et al. 2009). Here archetypical unitary operators are considered in the general case.

Let $\mathfrak{j}$ be a fundamental symmetry of $\{\mathfrak{K},[\cdot,\cdot]\}$ and let $\mathfrak{M}$ be a hyper-maximal semi-definite subspace of $\{\mathfrak{K},[\cdot,\cdot]\}$. Then recall that $\mathfrak{M}$ induces an orthogonal decomposition of $\mathfrak{K}$: $\mathfrak{K}=\mathfrak{M}^{[\perp]}\oplus(\mathfrak{M}\cap\mathfrak{j}\mathfrak{M})\oplus\mathfrak{j}\mathfrak{M}^{[\perp]}$, see Proposition 2.9. Clearly, $\mathfrak{M}\cap\mathfrak{j}\mathfrak{M}$ is a closed uniformly definite subspace of $\{\mathfrak{K},[\cdot,\cdot]\}$ and the behavior of isometric operators on this subspace is essentially like a Hilbert space unitary operator, see Proposition 3.9. Hence, assume that $\mathfrak{M}$ is hyper-maximal neutral and introduce for a relation $S$ in (the Hilbert space) $\{\mathfrak{M},[\mathfrak{j}\cdot,\cdot]\}$, the relation $\Upsilon_1(S)$ in $\{\mathfrak{K},[\cdot,\cdot]\}$ as

$$\Upsilon_1(S)(f+\mathfrak{j}g)=f+\mathfrak{j}(iSf+g),\quad f\in\operatorname{dom}S,\ g\in\mathfrak{M}.$$

Note that $\Upsilon_1(S)$ is a relation or, equivalently, has a kernel if and only if $S$ is a relation, and that $(\Upsilon_1(S))^{-1}=\Upsilon_1(-S)$. If $S$ is an operator, then $\Upsilon_1(S)$ is an operator (without kernel) which has the following block representation:

$$\Upsilon_1(S)=\begin{pmatrix} I & 0\\ \mathfrak{j}iS & I\end{pmatrix},$$

where the righthand side block decomposition is w.r.t. the decomposition $\mathfrak{M}\oplus\mathfrak{j}\mathfrak{M}$ of $\mathfrak{K}$. As a consequence of its definition, $\Upsilon_1(S)$ is an isometric operator or relation if and only if $S$ is a symmetric operator or relation, respectively. Since $\operatorname{clos}(\Upsilon_1(S))=\Upsilon_1(\operatorname{clos}(S))$, $\Upsilon_1(S)$ can be an operator whilst its closure is a relation. Proposition 4.8 below summarizes the above discussion and provides a characterization for $\Upsilon_1(S)$ to be unitary, see (Derkach et al. 2009: Example 2.11). Here a short proof for the characterization of $\Upsilon_1(S)$ to be unitary is included; it is based on the following lemma, which yields in fact a characterization for unitary relations, see Theorem 6.8 below.

**Lemma 4.7.** *Let $U$ be an isometric relation from $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ and assume that there exist hyper-maximal neutral subspaces $\mathfrak{M}_1$ and $\mathfrak{M}_2$ in $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$, respectively, such that $\mathfrak{M}_1\subseteq\operatorname{dom}U$ and $U(\mathfrak{j}_1\mathfrak{M}\cap\operatorname{dom}U)=\mathfrak{M}_2$ for a fundamental symmetry $\mathfrak{j}_1$ of $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$. Then $U$ is a unitary relation.*

*Proof.* Let $\mathfrak{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $k \in \mathfrak{K}_1$ and $k' \in \mathfrak{K}_2$ be such that $[f,k]_1 = [f',k']_2$ for all $\{f,f'\} \in \operatorname{gr} U$. Then by the assumptions there exists $\{h,h'\} \in \operatorname{gr} U$ such that $k - h \in \mathfrak{j}_1\mathfrak{M}_1$ and $k' - h' \in \mathfrak{j}_2\mathfrak{M}_2$. Clearly,

$$[f, k-h]_1 = [f', k'-h']_2, \quad \forall\{f,f'\} \in \operatorname{gr} U. \tag{4.3}$$

By the assumption that $U(\mathfrak{j}_1\mathfrak{M}_1 \cap \operatorname{dom} U) = \mathfrak{M}_2$, there exists a $g \in \mathfrak{j}_1\mathfrak{M}_1 \cap \operatorname{dom} U$ such that $\{g, \mathfrak{j}_2(k'-h')\} \in \operatorname{gr} U$. Therefore (4.3) implies that

$$0 = [g, k-h]_1 = [\mathfrak{j}_2(k'-h'), (k'-h')]_2.$$

This shows that $k'-h' = 0$ and, hence, $[f,k-h]_1 = 0$ for all $f \in \operatorname{dom} U$ by (4.3), i.e. $k - h \in (\operatorname{dom} U)^{[\perp]_1} \subseteq \mathfrak{M}_1^{[\perp]_1} = \mathfrak{M}_1$. Since $k-h \in \mathfrak{j}_1\mathfrak{M}_1$, this implies that $k-h = 0$, i.e. $\{k,k'\} = \{h,h'\} \in \operatorname{gr} U$. Consequently, Proposition 3.1 implies that $U$ is a unitary relation. □

**Proposition 4.8.** *Let* $\mathfrak{j}$ *be a fundamental symmetry of* $\{\mathfrak{K}, [\cdot,\cdot]\}$*, assume that there exists a hyper-maximal neutral subspace* $\mathfrak{M}$ *in* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *and let* $S$ *be a relation in* $\mathfrak{M}$*. Then* $\Upsilon_1(S)$ *is a (closed) isometric relation or (extendable to) a unitary relation in* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *if and only if* $S$ *is a (closed) symmetric relation or (extendable to) a selfadjoint relation in the Hilbert space* $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$*, respectively. Moreover,* $\Upsilon_1(S)$ *is an isometric operator without kernel if and only if* $S$ *is an operator and* $\Upsilon_1(S)$ *is a standard unitary operator if and only if* $S$ *is a bounded selfadjoint operator.*

*Proof.* Only the first equivalence is proven, the remaining statements follow directly from it and the definition of $\Upsilon_1(S)$. To prove that equivalence first note that if $T$ is a symmetric extension of $S$, then $\Upsilon_1(T)$ is an isometric extension of $\Upsilon_1(S)$. Hence, it suffices to prove that $\Upsilon_1(S)$ is unitary if and only if $S$ is selfadjoint.

If $S$ is selfadjoint, then $\mathfrak{j}\mathfrak{M} \subseteq \operatorname{dom}(\Upsilon_1(S))$ and $\Upsilon_1(S)(\mathfrak{M} \cap \operatorname{dom}(\Upsilon_1(S))) = \{f + \mathfrak{j}iSf : f \in \operatorname{dom} S\}$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$, see Proposition 2.20. Hence, Lemma 4.7 implies that $\Upsilon_1(S)$ is unitary. To prove the converse assume that $S$ is a maximal symmetric relation which is not selfadjoint, and that $\Upsilon_1(S)$ is unitary. Then there exists $\{f,f'\} \in \operatorname{gr} S^*$ such that $\operatorname{Im}[\mathfrak{j}f, f'] \neq 0$, and a direct calculation shows that $[f,g] = [f + \mathfrak{j}if', g']$ for all $\{g,g'\} \in \operatorname{gr}(\Upsilon_1(S))$, i.e., $\{f, f + \mathfrak{j}if'\} \in \operatorname{gr}(\Upsilon_1(S))$ by Proposition 3.1. On the other hand, $[f,f] = 0$ and, by assumption $[f + \mathfrak{j}if', f + \mathfrak{j}if'] = i([\mathfrak{j}f', f] - [f, \mathfrak{j}f']) \neq 0$. Therefore $\{f, f + \mathfrak{j}if'\}$ cannot belong to the graph of an isometric relation. This contradiction completes the proof. □

Observe that Proposition 4.8 yields elementary examples of isometric operators which can not be extended to unitary operators (or relations); namely $\Upsilon_1(S)$ for symmetric operators in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ with unequal defect numbers.

Next define for a relation $B$ in the Hilbert space $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$, with adjoint $B^*$, the relation $\Upsilon_2(B)$ as

$$\Upsilon_2(B)(f+\mathfrak{j}g) = Bf + \mathfrak{j}B^{-*}g, \quad f \in \operatorname{dom} B,\ g \in \operatorname{dom} B^{-*}.$$

A direct calculation shows that $\Upsilon_2(B)$ is an isometric relation in $\{\mathfrak{K}, [\cdot,\cdot]\}$, which is an operator if and only if $\operatorname{mul} B = \{0\}$ and $\ker B^* = (\operatorname{ran} B)^{\perp} = \{0\}$, and that $\operatorname{clos}(\Upsilon_2(B)) = \Upsilon_2(\operatorname{clos}(B))$. Hence, if $B$ is a non-closable operator with $\overline{\operatorname{ran}}\, B = \mathfrak{M}$, then $\Upsilon_2(B)$ is an isometric operator whilst $\operatorname{clos}(\Upsilon_2(B))$ is an isometric relation. If $\Upsilon_2(B)$ is an operator, then it has the following block representation w.r.t the decomposition $\mathfrak{M} \oplus \mathfrak{j}\mathfrak{M}$ of $\mathfrak{K}$:

$$\Upsilon_2(B) = \begin{pmatrix} B & 0 \\ 0 & \mathfrak{j}B^{-*}\mathfrak{j} \end{pmatrix}.$$

Note that $\Upsilon_2(B)$ is an isometric operator without kernel if and only if $B$ satisfies

$$\ker B = \{0\}, \quad \overline{\operatorname{dom}}\, B = \mathfrak{M}, \quad \operatorname{mul} B = \{0\} \quad \text{and} \quad \overline{\operatorname{ran}}\, B = \mathfrak{M}. \tag{4.4}$$

Furthermore, using (2.6), it follows that $\Upsilon_2(B)$ and $\operatorname{clos}(\Upsilon_2(B))$ are both isometric operators without kernel if and only if $B$ satisfies

$$\overline{\operatorname{dom}}\, B^* = \mathfrak{M}, \quad \overline{\operatorname{dom}}\, B = \mathfrak{M}, \quad \overline{\operatorname{ran}}\, B^* = \mathfrak{M} \quad \text{and} \quad \overline{\operatorname{ran}}\, B = \mathfrak{M}. \tag{4.5}$$

Clearly, the conditions in (4.5) are equivalent to those in (4.4) if $B$ is a closed operator. Proposition 4.9 below summarizes the above discussion and provides a characterization for $\Upsilon_2(B)$ to be unitary.

**Proposition 4.9.** *Let* $\mathfrak{j}$ *be a fundamental symmetry of* $\{\mathfrak{K}, [\cdot,\cdot]\}$*, assume that there exists a hyper-maximal neutral subspace* $\mathfrak{M}$ *in* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *and let* $B$ *be a relation in* $\mathfrak{M}$*. Then* $\Upsilon_2(B)$ *and* $\Upsilon_2(\operatorname{clos}(B)) = \operatorname{clos}(\Upsilon_2(B))$ *are an isometric and a unitary relation in* $\{\mathfrak{K}, [\cdot,\cdot]\}$*, respectively. Moreover,* $\Upsilon_2(B)$ *or* $\Upsilon_2(\operatorname{clos}(B))$ *is an isometric or unitary operator without kernel if and only if* $B$ *satisfies* (4.4) *or* (4.5)*, respectively, and* $\Upsilon_2(B)$ *is a standard unitary operator if and only if* $B$ *and* $B^{-1}$ *are everywhere defined operators.*

*Proof.* It suffices to prove that $\Upsilon_2(\operatorname{clos}(B))$ is unitary. Let $h, h', k, k' \in \mathfrak{M}$ be such that $[h+\mathfrak{j}h', f+\mathfrak{j}g] = [k+\mathfrak{j}k', f'+\mathfrak{j}g']$ for all $\{f,f'\} \in \operatorname{gr}(\operatorname{clos} B)$ and $\{g,g'\} \in \operatorname{gr} B^{-*}$. Then $[\mathfrak{j}h', f] = [\mathfrak{j}k', f']$ for all $\{f,f'\} \in \operatorname{gr}(\operatorname{clos} B)$ and $[h, \mathfrak{j}g] = [k, \mathfrak{j}g']$ for all $\{g,g'\} \in \operatorname{gr} B^{-*}$, i.e., $\{h',k'\} \in \operatorname{gr} B^{-*}$ and $\{h,k\} \in \operatorname{gr}(\operatorname{clos} B)$. Consequently, $\{h+\mathfrak{j}h', k+\mathfrak{j}k'\} \in \operatorname{gr}(\Upsilon_2(\operatorname{clos}(B)))$ and, hence, Proposition 3.1 implies that $\Upsilon_2(\operatorname{clos}(B))$ is unitary. □

Henceforth, the introduced isometric (unitary) relations $\Upsilon_1(S)$ and $\Upsilon_2(B)$ will be called *archetypical* isometric (unitary) relations.

Next it is shown that unitary operators of the type $\Upsilon_2(B)$ can map hyper-maximal neutral subspaces onto closed neutral subspaces with equal, but nonzero, defect numbers. In light of Theorem 7.16 below, this provides a simple proof for (Calkin 1939a: Lemma 4.4), see Corollary 7.25 below.

**Proposition 4.10.** *Let* $\mathfrak{j}$ *be a fundamental symmetry of* $\{\mathfrak{K},[\cdot,\cdot]\}$*, assume that there exists a hyper-maximal neutral subspace* $\mathfrak{M}$ *in* $\{\mathfrak{K},[\cdot,\cdot]\}$ *and let* $U:=\Upsilon_2(B)$*, where* $B$ *is a closed unbounded operator in the Hilbert space* $\{\mathfrak{M},[\mathfrak{j}\cdot,\cdot]\}$ *with* $\overline{\operatorname{dom}}\,B=\mathfrak{M}=\operatorname{ran}B$ *and* $\ker B=\{0\}$*. Then for every* $0\leq m\leq\aleph_0$ *there exists a hyper-maximal neutral subspace* $\mathfrak{L}\subseteq\operatorname{dom}U$ *of* $\{\mathfrak{K},[\cdot,\cdot]\}$ *such that* $U(\mathfrak{L})$ *is a closed neutral subspace of* $\{\mathfrak{K},[\cdot,\cdot]\}$ *with*

$$n_+(U(\mathfrak{L}))=m \quad \textit{and} \quad n_-(U(\mathfrak{L}))=m.$$

*Proof.* Since $B^*$ is a densely defined unbounded operator with $\operatorname{ran}B^*=\mathfrak{M}$ and $\ker B=\{0\}$, there exists an $m$-dimensional closed subspace $\mathfrak{N}_m$ of $\{\mathfrak{M},[\mathfrak{j}\cdot,\cdot]\}$ such that $\operatorname{dom}B^*\cap\mathfrak{N}_m=\{0\}$ and $\mathfrak{M}=\operatorname{clos}(B^{-1}(\mathfrak{M}\ominus\mathfrak{N}_m))$, see Corollary 2.18. Hence,

$$Cf=Bf,\quad f\in\operatorname{dom}C=B^{-1}(\mathfrak{M}\ominus\mathfrak{N}_m),$$

considered as an operator from $\mathfrak{M}$ to $\mathfrak{M}\ominus\mathfrak{N}_m$ is a closed operator which satisfies $\overline{\operatorname{dom}}\,C=\mathfrak{M}$, $\operatorname{ran}C=\mathfrak{M}\ominus\mathfrak{N}_m$ and $\ker C=\{0\}$. Now define the isometric operator $U_a$ from $\{\mathfrak{K},[\cdot,\cdot]\}$ to $\{\mathfrak{K}\ominus(\mathfrak{N}_m+\mathfrak{j}\mathfrak{N}_m),[\cdot,\cdot]\}$ as

$$U_a(f+\mathfrak{j}f')=Cf+\mathfrak{j}C^{-*}f',\quad f\in\operatorname{dom}C,\ f'\in\mathfrak{M}.$$

Then by definition $\operatorname{dom}U_a\subseteq\operatorname{dom}U$ and arguments as in Proposition 4.9 show that $U_a$ is a unitary operator from $\{\mathfrak{K},[\cdot,\cdot]\}$ to $\{\mathfrak{K}\ominus(\mathfrak{N}_m+\mathfrak{j}\mathfrak{N}_m),[\cdot,\cdot]\}$. Let $WK$ be the polar decomposition of $C$, then $K$ is a (nonnegative) selfadjoint operator in $\{\mathfrak{M},[\mathfrak{j}\cdot,\cdot]\}$ with $\operatorname{dom}K=\operatorname{dom}C$ and, hence, $\mathfrak{L}:=\{f+\mathfrak{j}iKf: f\in\operatorname{dom}K\}$ is a hyper-maximal neutral subspace of $\{\mathfrak{K},[\cdot,\cdot]\}$ contained in the domain of $U_a$.

By definition of $K$, $KC^{-1}$ is a closed operator from $\{\mathfrak{M}\ominus\mathfrak{N}_m,[\mathfrak{j}\cdot,\cdot]\}$ to $\{\mathfrak{M},[\mathfrak{j}\cdot,\cdot]\}$ with domain $\mathfrak{M}\ominus\mathfrak{N}_m$. Moreover, $KB^{-1}$ coincides with $KC^{-1}$ when the latter is considered as a mapping in $\{\mathfrak{M},[\mathfrak{j}\cdot,\cdot]\}$, because $\operatorname{dom}K=\operatorname{dom}C$ and $C\subseteq B$. Therefore $S:=B^{-*}KB^{-1}$ is a closed symmetry operator with domain $\mathfrak{M}\ominus\mathfrak{N}_m$, i.e. $S$ is a bounded symmetric operator with $n_\pm(S)=m$. Now the proof is completed by observing that $\mathfrak{L}\subseteq\operatorname{dom}U$ and that $U(\mathfrak{L})=\{f+\mathfrak{j}iSf: f\in\operatorname{dom}S\}$. □

## 4.3 Standard unitary operators

Let $\mathfrak{j}$ be a fundamental symmetry of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and assume that in $\{\mathfrak{K}, [\cdot,\cdot]\}$ there exists a hyper-maximal neutral subspace $\mathfrak{M}$, i.e., $\mathfrak{K} = \mathfrak{M} \oplus \mathfrak{j}\mathfrak{M}$. Let $\mathfrak{L}$ be a hyper-maximal neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$, then by Proposition 2.17 (ii) and Proposition 2.20 there exists a selfadjoint relation $K$ in (the Hilbert space) $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ and a closed operator $B$ in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ with $\operatorname{dom} B = \mathfrak{M}$, $\operatorname{ran} B = \operatorname{dom} K \oplus \operatorname{mul} K$ and $\ker B = \{0\}$, respectively, such that

$$\mathfrak{L} = \{\mathcal{P}_K Bf + \mathfrak{j}(i\mathcal{P}_K KBf + (I - \mathcal{P}_K)Bf) : f \in \mathfrak{M}\}.$$

Here $\mathcal{P}_K$ is the orthogonal projection onto $\overline{\operatorname{dom}} K = (\operatorname{mul} K)^{\perp}$ in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$. Using this observation, standard unitary operators can almost be decomposed in terms of the, in general unbounded, archetypical unitary operators introduced in the previous section. In particular, Theorem 4.11 below together with Theorem 7.16 below shows that to investigate compositions of unitary operators, it suffices to consider compositions of archetypical unitary operators.

**Theorem 4.11.** *Let $U$ be an isometric operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$ with fundamental symmetry $\mathfrak{j}$ and assume that there exists a hyper-maximal neutral subspace $\mathfrak{M}$ in $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then $U$ is a standard unitary operator if and only if there exists a closed subspace $\mathfrak{N}$ of $\mathfrak{M}$, selfadjoint operators $K_1$ and $K_2$ in the Hilbert space $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ with $\operatorname{dom} K_2 = \mathfrak{M}$ and $\operatorname{clos}(K_2^{-1} - K_1)$ being a selfadjoint relation in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$, a closed operator $B$ in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ satisfying $\operatorname{dom} B = \mathfrak{M}$, $\operatorname{ran} B = \operatorname{dom} K_1$, $\ker B = \{0\}$, $\operatorname{dom}\operatorname{clos}(K_2B^{-*}) = \mathfrak{M}$, $\operatorname{mul}\operatorname{clos}(K_2B^{-*}) = \{0\}$ and $\operatorname{ran}\operatorname{clos}(K_2B^{-*}) = \operatorname{dom}\operatorname{clos}(K_1^{-1} - K_2)$ such that*

$$U_{\mathfrak{N}}^{-1}U = \operatorname{clos}\left(\Upsilon_1(K_1)\mathfrak{j}\Upsilon_1(K_2)\mathfrak{j}\Upsilon_2(B)\right). \tag{4.6}$$

*Here, with $\mathcal{P}_{\mathfrak{N}}$ the orthogonal projection onto $\mathfrak{N}$ in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$, $U_{\mathfrak{N}}$ is the standard unitary operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$ defined as*

$$U_{\mathfrak{N}}(f + \mathfrak{j}f') = \mathcal{P}_{\mathfrak{N}}f + (I - \mathcal{P}_{\mathfrak{N}})f' + \mathfrak{j}((I - \mathcal{P}_{\mathfrak{N}})f + \mathcal{P}_{\mathfrak{N}}f'), \quad f, f' \in \mathfrak{M}.$$

*Proof.* If $U$ is a standard unitary operator, then $U(\mathfrak{M})$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$, see Proposition 4.5. Hence, by the discussion preceding this statement, there exists a selfadjoint relation $K$ in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ and a closed operator $B$ in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ with $\operatorname{dom} B = \mathfrak{M}$, $\operatorname{ran} B = \operatorname{dom} K \oplus \operatorname{mul} B$ and $\ker B = \{0\}$ such that with $\mathcal{P}_K$ the orthogonal projection onto $\overline{\operatorname{dom}} K$ in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$

$$U\restriction_{\mathfrak{M}} = \begin{pmatrix} \mathcal{P}_K B \\ \mathfrak{j}(i\mathcal{P}_K KB + (I - \mathcal{P}_K)B) \end{pmatrix} = U_{\mathfrak{N}} \begin{pmatrix} B \\ \mathfrak{j}i\mathcal{P}_K KB \end{pmatrix},$$

where $\mathfrak{N} = \overline{\mathrm{dom}}\, K$ and the block decomposition on the range is w.r.t. the decomposition $\mathfrak{M} \oplus \mathfrak{j}\mathfrak{M}$ of $\mathfrak{K}$. Note that $K_1 := \mathcal{P}_K K \oplus 0_{\mathrm{mul}\, K}$ is a selfadjoint operator in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$. These observations show that there exist operators $C$ and $D$ in $\mathfrak{M}$ with $\mathrm{dom}\, C = \mathfrak{M} = \mathrm{dom}\, D$ such that w.r.t. decomposition $\mathfrak{M} \oplus \mathfrak{j}\mathfrak{M}$ of $\mathfrak{K}$:

$$U_{\mathfrak{N}}^{-1} U = \begin{pmatrix} B & iC\mathfrak{j} \\ \mathfrak{j}iK_1B & \mathfrak{j}D\mathfrak{j} \end{pmatrix}. \tag{4.7}$$

Note that $U_{\mathfrak{N}}^{-1} U$ being a standard unitary operator is bounded. Hence, (4.7) implies that $C$ and $D$ are also bounded. Since $\mathrm{dom}\, C = \mathfrak{M} = \mathrm{dom}\, D$, this implies that $C$ and $D$ are closed operators. Since $(\Upsilon_1(K_1))^{-1} = \Upsilon_1(-K_1)$, it follows that

$$(\Upsilon_1(K_1))^{-1} U_{\mathfrak{N}}^{-1} U = \Upsilon_1(-K_1) \begin{pmatrix} B & iC\mathfrak{j} \\ \mathfrak{j}iK_1B & \mathfrak{j}D\mathfrak{j} \end{pmatrix} = \begin{pmatrix} B & iC\mathfrak{j} \\ 0 & \mathfrak{j}\,(D + K_1C)\,\mathfrak{j} \end{pmatrix}. \tag{4.8}$$

Since $U_{\mathfrak{N}}$ and $U$ are both standard unitary operators and $(\Upsilon_1(K_1))^{-1}$ is a unitary operator, the righthand side of (4.8) is also a unitary operator, see Lemma 3.10. The isometry of that operator implies that $(D + K_1C) \subseteq B^{-*}$ and the fact that $\mathfrak{j}\mathfrak{M} \subseteq \mathrm{ran}\,((\Upsilon_1(K_1))^{-1} U_{\mathfrak{N}}^{-1} U) = \mathrm{dom}\, \Upsilon_1(K_1)$ implies that $\mathrm{ran}\,(D + K_1C) = \mathfrak{M}$. Since $\ker B^{-*} = (\mathrm{dom}\, B)^{\perp} = \{0\}$, the preceding observations imply that $(D + K_1C) = B^{-*}$, see (2.8). Hence, $\mathrm{dom}\, B^* = \mathrm{ran}\, B^{-*} = \mathfrak{M}$ and

$$\begin{pmatrix} B & iC\mathfrak{j} \\ 0 & \mathfrak{j}\,(D + K_1C)\,\mathfrak{j} \end{pmatrix} = \begin{pmatrix} I & iCB^*\mathfrak{j} \\ 0 & I \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & \mathfrak{j}B^{-*}\mathfrak{j} \end{pmatrix} = \mathfrak{j}\Upsilon_1(CB^*)\mathfrak{j}\Upsilon_2(B). \tag{4.9}$$

Since $B$ is a closed operator satisfying $\mathrm{dom}\, B = \mathfrak{M} = \overline{\mathrm{ran}}\, B$ and $\ker B = \{0\}$, $\Upsilon_2(B)$ is a unitary operator without kernel. Consequently, (4.9) implies that $\Upsilon_1(CB^{-*})$ is isometric and, hence, $K_2 := CB^*$ is a symmetric operator, see Proposition 4.8. Since $\mathrm{dom}\, B^* = \mathfrak{M} = \mathrm{dom}\, C$, $K_2$ is in fact an everywhere defined symmetric operator, i.e., $K_2$ is a (bounded) selfadjoint operator in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$. Combining (4.8) and (4.9) yields that $U_t := I_{\mathrm{ran}\, \Upsilon_1(K_1)} U_{\mathfrak{N}}^{-1} U$ can be decomposed as follows:

$$U_t = \Upsilon_1(K_1)\mathfrak{j}\Upsilon_1(K_2)\mathfrak{j}\Upsilon_2(B) = \begin{pmatrix} B & iK_2B^{-*}\mathfrak{j} \\ \mathfrak{j}iK_1B\mathfrak{j} & (I - K_1K_2)B^{-*}\mathfrak{j} \end{pmatrix}. \tag{4.10}$$

Since $\overline{\mathrm{ran}}\,(\Upsilon_1(K_1)) = \mathfrak{K}$, the closure of $U_t$ coincides with $U_{\mathfrak{N}}^{-1} U$, i.e., (4.6) holds. As a consequence of (4.7), (4.10) and the proven closedness of $C$, $\mathrm{clos}\,(K_2B^{-*}) = C$ which yields $\mathrm{dom}\,\mathrm{clos}\,(K_2B^{-*}) = \mathfrak{M}$ and $\mathrm{mul}\,\mathrm{clos}\,(K_2B^{-*}) = \{0\}$. Moreover, since $\mathrm{clos}\,(U_t)$ is a standard unitary operator and $\mathrm{clos}\,(\mathrm{dom}\, U_t \cap \mathfrak{j}\mathfrak{M}) = \mathfrak{j}\mathfrak{M}$, $\mathrm{clos}\,(U_t(\mathrm{dom}\, U_t \cap \mathfrak{j}\mathfrak{M})) = \mathrm{clos}\,(\{if + \mathfrak{j}(K_2^{-1} - K_1)f : f \in \mathrm{ran}\,(K_2B^{-*})\})$ is a hyper-maximal neutral subspace. Consequently, Proposition 2.20 implies that

$\operatorname{clos}(K_2^{-1} - K_1)$ is a selfadjoint operator and also that $\operatorname{dom}\operatorname{clos}(K_2^{-1} - K_1) \subseteq \operatorname{ran}\operatorname{clos}((K_2B^{-*}))$. Finally, (4.10), $\overline{\operatorname{dom}}\, U_t = \mathfrak{K}$ and $\operatorname{dom}(\operatorname{clos}(K_2B^{-*})) = \mathfrak{M}$ imply that $\operatorname{dom}\operatorname{clos}(K_2^{-1} - K_1) = \operatorname{ran}(\operatorname{clos}(K_2B^{-*}))$.

Conversely, the assumptions imply that the closure of the righthand side of (4.10) is an everywhere defined isometric operator with dense range, i.e. $U_{\mathfrak{N}}^{-1}U$ and, hence, also $U$ is a standard unitary operator. □

Next some properties of standard unitary operators are presented. Recall that if $U$ is a standard unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ is a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, then the discussion preceding Proposition 4.2 shows that $U(\mathfrak{K}_1^+)[+]U(\mathfrak{K}_1^-)$ is a canonical decomposition of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Consequently, standard unitary operators in Kreĭn spaces are the orthogonal sum of two Hilbert space unitary operators. This implies that standard unitary operators give a one-to-one correspondence between fundamental symmetries.

**Lemma 4.12.** *Let $U$ be a standard unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then $\mathfrak{j}_1 \mapsto U\mathfrak{j}_1U^{-1}$ is a bijective mapping from the set of all fundamental symmetries of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ onto the set of all fundamental symmetries of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$.*

*Proof.* Let $\mathfrak{j}_1$ be a fundamental symmetry of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and let $\mathfrak{j}_2 := U\mathfrak{j}_1U^{-1}$. Then $\mathfrak{j}_2^{-1} = \mathfrak{j}_2$ and, clearly, $\{\mathfrak{K}_2, [\mathfrak{j}_2\cdot,\cdot]_2\}$ is a Hilbert space. Hence, $\mathfrak{j}_2$ is a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Since for any fundamental symmetry $\mathfrak{j}_2$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ one has that $\mathfrak{j}_2 = UU^{-1}\mathfrak{j}_2UU^{-1}$ and similar arguments as above show that $U^{-1}\mathfrak{j}_2U$ is a fundamental symmetry of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, the bijectivity of the mapping is evident. □

Analogues of Lemma 4.12 hold for unitary relations with closed domain and range. For instance, if $\operatorname{ran} U = \mathfrak{K}_2$, then the mapping in Lemma 4.12 is surjective.

For technical purposes the following property of standard unitary operators will be useful later on.

**Lemma 4.13.** *Let $\mathfrak{j}$ and $\mathfrak{j}'$ be fundamental symmetries of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and let $\mathfrak{M}$ and $\mathfrak{M}'$ be hyper-maximal neutral subspaces in $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then there exists a standard unitary operator $U$ in $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that*

$$U(\mathfrak{M}) = \mathfrak{M}' \quad \textit{and} \quad U(\mathfrak{j}\mathfrak{M}) = \mathfrak{j}'\mathfrak{M}'.$$

*Proof.* If the assumptions hold, then $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ and $\{\mathfrak{M}', [\mathfrak{j}'\cdot,\cdot]\}$ are Hilbert spaces of equal dimension. Let $U_t$ be a (standard) unitary operator between these Hilbert spaces, then $U$ defined by $U(f_0 + \mathfrak{j}f_1) = U_tf_0 + \mathfrak{j}'U_tf_1$, where $f_0, f_1 \in \mathfrak{M}$, is a standard unitary operator which has the stated properties. □

As a conclusion of this section it is shown that the Potapov-Ginzburg transformation, see (Azizov & Iokhvidov 1989: Ch. 5, §1), can be interpreted as a standard unitary operator. This transformation, which yields a one-to-one correspondence between unitary relations between Kreĭn spaces and Hilbert spaces (for fixed canonical decompositions of the spaces), can in turn be used to obtain conditions for when an isometric relation is unitary, see Lemma 5.2 below. To formulate the following statement introduce for Kreĭn spaces $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with canonical decompositions $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ and $\mathfrak{K}_2^+[+]\mathfrak{K}_2^-$, respectively, the Hilbert spaces $\{\mathfrak{H}_1, (\cdot,\cdot)_1\} := \{\mathfrak{K}_1^+ \times \mathfrak{K}_2^-, (\cdot,\cdot)_1\}$ and $\{\mathfrak{H}_2, (\cdot,\cdot)_2\} := \{\mathfrak{K}_2^+ \times \mathfrak{K}_1^-, (\cdot,\cdot)_2\}$, where

$$\begin{aligned} (f \times f', g \times g')_1 &= [f,g]_1 - [f',g']_2, \quad f,g \in \mathfrak{K}_1^+, f',g' \in \mathfrak{K}_2^-; \\ (f \times f', g \times g')_2 &= [f,g]_2 - [f',g']_1, \quad f,g \in \mathfrak{K}_2^+, f',g' \in \mathfrak{K}_1^-. \end{aligned} \tag{4.11}$$

**Proposition 4.14.** *Let $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ be Kreĭn spaces with associated Hilbert spaces $\{\mathfrak{H}_1, (\cdot,\cdot)_1\}$ and $\{\mathfrak{H}_2, (\cdot,\cdot)_2\}$ as defined above for the fundamental symmetries $\mathfrak{j}_1$ and $\mathfrak{j}_2$. Then the Potapov-Ginzburg transformation $\mathfrak{P}_{\mathfrak{j}_1,\mathfrak{j}_2}$ defined by*

$$\mathfrak{P}_{\mathfrak{j}_1,\mathfrak{j}_2}\{f,g\} = \{P_1^+ f \times P_2^- g, P_2^+ g \times P_1^- f\}$$

*is a standard unitary operator from the Kreĭn space $\{\mathfrak{K}_1, [\cdot,\cdot]_1\} \times \{\mathfrak{K}_2, -[\cdot,\cdot]_2\}$ to the Kreĭn space $\{\mathfrak{H}_1, (\cdot,\cdot)_1\} \times \{\mathfrak{H}_2, -(\cdot,\cdot)_2\}$. For a relation $H$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ denote its Potapov-Ginzburg transformation by $H_{PG}$, i.e. $\operatorname{gr} H_{PG} = \mathfrak{P}_{\mathfrak{j}_1,\mathfrak{j}_2}(\operatorname{gr} H)$. Then*

$$(H^{-[*]})_{PG} = (H_{PG})^{-*}.$$

*In particular, $\mathfrak{P}_{\mathfrak{j}_1,\mathfrak{j}_2}$ maps the graphs of (closed, maximal) isometric and unitary relations from the Kreĭn space $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to the Kreĭn space $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ onto the graphs of (closed, maximal) isometric and unitary relations from the Hilbert space $\{\mathfrak{H}_1, (\cdot,\cdot)_1\}$ to the Hilbert space $\{\mathfrak{H}_2, (\cdot,\cdot)_2\}$, respectively.*

*Proof.* Let $f,g \in \mathfrak{K}_1$ and $f',g' \in \mathfrak{K}_2$, then with the introduced inner products

$$\begin{aligned} &[f,g]_1 - [f',g']_2 = \\ &\quad = [P_1^+ f, P_1^+ g]_1 + [P_1^- f, P_1^- g]_1 - [P_2^+ f', P_2^+ g']_2 - [P_2^- f', P_2^- g']_2 \\ &\quad = (P_1^+ f \times P_2^- f', P_1^+ g \times P_2^- g')_1 - (P_2^+ f' \times P_1^- f, P_2^+ g' \times P_1^- g)_2. \end{aligned}$$

Hence the Potapov-Ginzburg transformation $\mathfrak{P}_{\mathfrak{j}_1,\mathfrak{j}_2}$ is an everywhere defined isometric operator from the Kreĭn space $\{\mathfrak{K}_1, [\cdot,\cdot]_1\} \times \{\mathfrak{K}_2, -[\cdot,\cdot]_2\}$ onto the Kreĭn space $\{\mathfrak{H}_1, (\cdot,\cdot)_1\} \times \{\mathfrak{H}_2, -(\cdot,\cdot)_2\}$, i.e., it is a standard unitary operator. Finally, the equality $\mathfrak{P}_{\mathfrak{j}_1,\mathfrak{j}_2}(H^{-[*]}) = (\mathfrak{P}_{\mathfrak{j}_1,\mathfrak{j}_2}(H))^{-*}$ follows from Proposition 4.5 combined with an interpretation of the orthogonal complement, cf. the arguments preceding Proposition 3.2. □

# 5 THE WEYL IDENTITY APPROACH

Proposition 4.1 showed that unitary relations with closed domain are essentially completely characterized by their behavior with respect to uniformly definite subspaces contained in their domain. Here it is shown how unitary relations can in general be distinguished from isometric relations by looking at their behavior with respect to uniformly definite subspaces contained in their domain. This approach to unitary relations will be called *the Weyl identity approach* to unitary relations. Therefore, continuing from the results obtained in Section 3.5, in the first section of this chapter it is shown that unitary relations satisfy the so-called Weyl identity and, moreover, that identity is also shown to characterize unitary relations. By means of the Weyl identity it is shown in the second section that unitary operators possess a quasi-block representation. That representation in particular shows that unitary relations in Kreĭn spaces are closely connected to nonnegative selfadjoint relations in Hilbert spaces; that connection will be used in the Chapter 6. In the third section it is shown that the obtained quasi-block decomposition for unitary operators can be generalized to a quasi-block representation for maximal isometric operators. There it is also shown that the Weyl identity approach can not be used to investigate general isometric relations. Finally, in the fourth section the Weyl identity approach to unitary relations is applied to obtain two types of results on unitary relations: First it is shown that this approach can be used to split unitary relations and, secondly, that it can be used to indicate how the defect numbers of neutral subspaces change under mapping by a unitary relation. In particular, in this last section of this chapter necessary and sufficient conditions are presented for the pre-image of a neutral subspace under a unitary relation to be a (hyper-)maximal neutral subspace.

## 5.1 The Weyl identity

Here it is shown that a unitary relation satisfies an identity which will be called *the Weyl identity*. The reason for this name is that in the case of boundary relations, which can be interpreted as unitary relations, see Section A.2, this identity is an identity for the Weyl family associated with the boundary relation, see Section A.3.

**Proposition 5.1.** *Let* $U$ *be a unitary relation from* $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ *and let* $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ *be a canonical decomposition of* $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$*. Then the Weyl identity holds:*

$$U(\operatorname{dom} U\cap\mathfrak{K}_1^+)=\big(U(\operatorname{dom} U\cap\mathfrak{K}_1^-)\big)^{[\perp]_2}.$$

*In particular,* $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ *and* $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ *are a maximal nonnegative and maximal nonpositive subspace of* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$*, respectively.*

*Proof.* Let $\mathfrak{K}_2^+[+]\mathfrak{K}_2^-$ be a canonical decomposition of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $P_i^+$ and $P_i^-$ be the projections associated to $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$, $i = 1, 2$. Then Proposition 3.16 together with Corollary 3.15 implies that $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ and $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ are a maximal nonnegative and a maximal nonpositive subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, respectively. Since, evidently, $\operatorname{dom} U \cap \mathfrak{K}_1^+ \subseteq (\operatorname{dom} U \cap \mathfrak{K}_1^-)^{[\perp]_1}$, applying $U$ and using Lemma 3.8 yields

$$U(\operatorname{dom} U \cap \mathfrak{K}_1^+) \subseteq (U(\operatorname{dom} U \cap \mathfrak{K}_1^-))^{[\perp]_2} \cap \operatorname{ran} U.$$

Since $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ and $(U(\operatorname{dom} U \cap \mathfrak{K}_1^-))^{[\perp]_2}$ are both maximal nonnegative, see Proposition 2.5, the Weyl identity follows from the previous inclusion. □

Note also that the equality $(\operatorname{dom} U \cap \mathfrak{K}_1^+) \cap (\operatorname{dom} U \cap \mathfrak{K}_1^-) = \{0\}$ yields

$$U(\operatorname{dom} U \cap \mathfrak{K}_1^+) \cap U(\operatorname{dom} U \cap \mathfrak{K}_1^-) = \operatorname{mul} U. \tag{5.1}$$

Using the Potapov-Ginzburg transformation, see Proposition 4.14, the following necessary and sufficient conditions for an isometric relation to be unitary are obtained, cf. Proposition 3.18. Those conditions are subsequently used to prove that the Weyl identity characterizes unitary relations almost completely.

**Lemma 5.2.** *Let* $U$ *be an isometric relation from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *and let* $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ *be a canonical decomposition of* $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ *with associated projections* $P_i^+$ *and* $P_i^-$*, for* $i = 1, 2$*. Then* $U$ *is unitary if and only if*

(i) $U$ *is closed,* $\ker U = (\operatorname{dom} U)^{[\perp]_1}$ *and* $\operatorname{mul} U = (\operatorname{ran} U)^{[\perp]_1}$*;*

(ii) *there exists a subspace* $\mathfrak{M}^+ \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^+$ *with* $P_2^+ U(\mathfrak{M}^+) = \mathfrak{K}_2^+$*;*

(iii) *there exists a subspace* $\mathfrak{M}^- \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-$ *with* $P_2^- U(\mathfrak{M}^-) = \mathfrak{K}_2^-$*.*

*Proof.* Necessity of (i) is clear by (3.1) and (3.4). Since $P_2^\pm U(\operatorname{dom} U \cap \mathfrak{K}_1^\pm) = \mathfrak{K}_2^\pm$ by Proposition 3.16 and Corollary 3.15, (ii) and (iii) hold for $\mathfrak{M}^\pm = \operatorname{dom} U \cap \mathfrak{K}_1^\pm$.

Conversely, assume that (i)-(iii) hold and let $U_{PG}$ be the Potapov-Ginzburg transformation of $U$, i.e.,

$$\operatorname{gr} U_{PG} = \{\{P_1^+ f \times P_2^- f', P_2^+ f' \times P_1^- f\} : \{f, f'\} \in \operatorname{gr} U\},$$

see Proposition 4.14. Since $U$ is by assumption a closed isometric relation, $U_{PG}$ is a closed isometric operator from the Hilbert space $\{\mathfrak{K}_1^+ \times \mathfrak{K}_2^-, (\cdot,\cdot)_1\}$ to the Hilbert space $\{\mathfrak{K}_2^+ \times \mathfrak{K}_1^-, (\cdot,\cdot)_2\}$, see (4.11). Now observe that the assumption (ii) implies that $\mathfrak{K}_2^+ \times \{0\} \subseteq \operatorname{ran} U_{PG}$. Moreover, the assumption $\ker U = (\operatorname{dom} U)^{[\perp]_1}$ implies that $P_1^- \overline{\operatorname{dom}} U = \mathfrak{K}_1^-$, see Lemma 3.4, and, hence, there exists a subspace $\mathfrak{N}_1^- \subseteq \mathfrak{K}_1^-$ satisfying $\operatorname{clos} \mathfrak{N}_1^- = \mathfrak{K}_1^-$, such that $P_1^- \operatorname{ran} U_{PG} = \mathfrak{N}_1^-$. Combining the preceding observations shows that $\mathfrak{K}_2^+ \times \mathfrak{N}_1^- \subseteq \operatorname{ran} U_{PG}$ and, hence, $\overline{\operatorname{ran}}\, U_{PG} = \mathfrak{K}_2^+ \times \mathfrak{K}_1^-$. Similar arguments show that $\overline{\operatorname{dom}}\, U_{PG} = \mathfrak{K}_1^+ \times \mathfrak{K}_2^-$. Consequently, $\operatorname{clos}(U_{PG}) = U_{PG}$ is a (Hilbert space) unitary operator and therefore, using the inverse Potapov-Ginzburg transformation, $U$ is a unitary relation. □

**Theorem 5.3.** *Let $U$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then $U$ is unitary if and only if*

(i) *$U$ is closed;*

(ii) $\ker U = (\operatorname{dom} U)^{[\perp]_1}$*;*

(iii) $U(\operatorname{dom} U \cap \mathfrak{K}_1^+) = \big(U(\operatorname{dom} U \cap \mathfrak{K}_1^-)\big)^{[\perp]_2}$.

*Proof.* Necessity of the conditions (i)-(iii) follows from (3.1), (3.4) and Proposition 5.1. Conversely, if (iii) holds, then

$$(\operatorname{ran} U)^{[\perp]_2} \subseteq \big(U(\operatorname{dom} U \cap \mathfrak{K}_1^-)\big)^{[\perp]_2} = U(\operatorname{dom} U \cap \mathfrak{K}_1^+) \subseteq \operatorname{ran} U.$$

By Lemma 3.5 the above inclusion combined with the assumption (ii) implies that $\operatorname{mul} U = (\operatorname{ran} U)^{[\perp]_2}$. Moreover, Proposition 3.9 yields that $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ and $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ are closed and, hence, assumption (iii) combined with Proposition 2.5 implies that $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ and $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ are a maximal nonnegative and nonpositive subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, respectively. Hence the sufficiency of the conditions (i)-(iii) follows now from Lemma 5.2. □

Geometrically Theorem 5.3 says that closed isometric relation are unitary precisely when they map certain uniformly definite subspaces onto maximal definite subspaces. It can be seen as an abstract extension of (Derkach et al. 2006: Proposition 3.6).

## 5.2 A quasi-block representation for unitary operators

Here a quasi-block representation for unitary operators and a consequence of it from (Nakagami 1988) are presented; see also (Gheondea 1988). For completeness here a proof based on the Weyl identity is included. As a preparation for the proof two lemmas will be stated. The first lemma shows that unitary relations possess a core which is connected to the Weyl identity. Note that the same subspace is also a core for certain maximal isometric relations, see Corollary 6.3 below; cf. also Example 5.10 below.

**Lemma 5.4.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then the subspace $\mathfrak{L} := \ker U + \operatorname{dom} U \cap \mathfrak{K}_1^+ + \operatorname{dom} U \cap \mathfrak{K}_1^-$ is a core for $U$, i.e., $\operatorname{clos}(U \upharpoonright_{\mathfrak{L}}) = U$.*

*Proof.* By definition $U_r := U \upharpoonright_{\mathfrak{L}}$ is an isometric relation such that

$$U_r(\operatorname{dom} U_r \cap \mathfrak{K}_1^\pm) = U(\operatorname{dom} U \cap \mathfrak{K}_1^\pm).$$

Hence, Proposition 5.1 implies that $U_r(\operatorname{dom} U_r \cap \mathfrak{K}_1^+) = \left(U_r(\operatorname{dom} U_r \cap \mathfrak{K}_1^-)\right)^{[\perp]_2}$. Furthermore, since $\overline{\operatorname{dom}}\, U_r = \overline{\operatorname{dom}}\, U$, see (3.10), it follows from (3.4) that

$$\ker U_r = \ker U = (\operatorname{dom} U)^{[\perp]_1} = (\operatorname{dom} U_r)^{[\perp]_1}.$$

Consequently, $\operatorname{clos} U_r$ is a closed isometric relation satisfying the conditions of Theorem 5.3, i.e., $\operatorname{clos} U_r$ is a unitary relation. Since $U_r \subseteq U$, this completes the proof. □

In Lemma 5.5 below certain unitary operators in a Kreĭn space with a trivial kernel are considered which are additionally nonnegative selfadjoint operators in an associated Hilbert space. Theorem 5.6 below shows that this class of unitary operators essentially explains the structure of unitary operators between Kreĭn spaces. As a preparation for Lemma 5.5, recall that for an everywhere defined contraction $K$ from the Hilbert space $\{\mathfrak{H}_1, (\cdot,\cdot)_1\}$ to the Hilbert space $\{\mathfrak{H}_2, (\cdot,\cdot)_2\}$ the following equivalence holds:

$$\ker(I - K^*K) = \{0\} \quad \text{if and only if} \quad \ker(I - KK^*) = \{0\}. \tag{5.2}$$

**Lemma 5.5.** *Let $\mathfrak{j}$ be a fundamental symmetry of $\{\mathfrak{K}, [\cdot,\cdot]\}$, let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be the associated canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and let $K$ be an everywhere defined contractive operator from $\{\mathfrak{K}^+, [\cdot,\cdot]\}$ to $\{\mathfrak{K}^-, -[\cdot,\cdot]\}$ with $\ker(I - K^*K) = \{0\}$. Then $U_K$ defined as*

$$U_K = \operatorname{clos}\left(\begin{pmatrix} I & K^* \\ K & I \end{pmatrix}\begin{pmatrix} (I - K^*K)^{-1/2} & 0 \\ 0 & (I - KK^*)^{-1/2} \end{pmatrix}\right) \tag{5.3}$$

*is a unitary operator in* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *with* $\ker U_K = \{0\}$ *and*

$$U_K(\operatorname{dom} U_K \cap \mathfrak{K}^+) = \{f^+ + Kf^+ : f^+ \in \mathfrak{K}^+\};$$
$$U_K(\operatorname{dom} U_K \cap \mathfrak{K}^-) = \{f^- + K^*f^- : f^- \in \mathfrak{K}^-\}.$$

*Moreover,* $U_K$ *is a nonnegative selfadjoint operator in the Hilbert space* $\{\mathfrak{K}, [\mathfrak{j}\cdot,\cdot]\}$.

*Proof.* In this proof the following notation is used

$$D_K = (I - K^*K)^{1/2} \quad \text{and} \quad D_{K^*} = (I - KK^*)^{1/2},$$

cf. (Sz.-Nagy & Foiaş 1970: Ch. I, Section 3). Note that the assumption $\ker(I - K^*K) = \{0\}$ implies that $D_K^{-1} := (D_K)^{-1}$ and $D_{K^*}^{-1} := (D_{K^*})^{-1}$ are operators, see (5.2).

*Step 1:* W.r.t. the decomposition $\mathfrak{K}^+ \times \mathfrak{K}^-$ of $\mathfrak{K}$, define $S$ and $T$ as

$$S = \begin{pmatrix} I & K^* \\ K & I \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} D_K^{-1} & 0 \\ 0 & D_{K^*}^{-1} \end{pmatrix}. \tag{5.4}$$

Then $S$ is an everywhere defined closed operator and, hence, by Lemma 2.15

$$(ST)^{[*]} = T^{[*]}S^{[*]} = \begin{pmatrix} D_K^{-1} & 0 \\ 0 & D_{K^*}^{-1} \end{pmatrix} \begin{pmatrix} I & -K^* \\ -K & I \end{pmatrix}.$$

Consequently, $V := ST$ satisfies

$$\begin{aligned} V^{[*]}V &= \begin{pmatrix} D_K^{-1} & 0 \\ 0 & D_{K^*}^{-1} \end{pmatrix} \begin{pmatrix} I & -K^* \\ -K & I \end{pmatrix} \begin{pmatrix} I & K^* \\ K & I \end{pmatrix} \begin{pmatrix} D_K^{-1} & 0 \\ 0 & D_{K^*}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} D_K^{-1} & 0 \\ 0 & D_{K^*}^{-1} \end{pmatrix} \begin{pmatrix} I - K^*K & 0 \\ 0 & I - KK^* \end{pmatrix} \begin{pmatrix} D_K^{-1} & 0 \\ 0 & D_{K^*}^{-1} \end{pmatrix} \\ &= I_{\operatorname{dom} V}. \end{aligned}$$

This shows that $V$ is an isometric operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$. Furthermore, the condition $\ker(I - K^*K) = \{0\}$ implies that $V$ has dense domain, see (5.2) and (2.6). Consequently, (3.3) implies that $\ker V = \{0\}$. Moreover, evidently, $\operatorname{dom} V \cap \mathfrak{K}^+ = \operatorname{dom} D_K^{-1}$, $\operatorname{dom} V \cap \mathfrak{K}^- = \operatorname{dom} D_{K^*}^{-1}$, $\operatorname{dom} D_K = \mathfrak{K}^+$ and $\operatorname{dom} D_{K^*} = \mathfrak{K}^-$. Hence

$$V(\operatorname{dom} V \cap \mathfrak{K}^+) = \{f^+ + Kf^+ : f^+ \in \mathfrak{K}^+\};$$
$$V(\operatorname{dom} V \cap \mathfrak{K}^-) = \{f^- + K^*f^- : f^- \in \mathfrak{K}^-\}.$$

These equalities show that $V(\operatorname{dom} V \cap \mathfrak{K}^+)$ and $V(\operatorname{dom} V \cap \mathfrak{K}^-)$ are a maximal nonnegative and a maximal nonpositive subspace, respectively, and that

$$V(\operatorname{dom} V \cap \mathfrak{K}^+) = (V(\operatorname{dom} V \cap \mathfrak{K}^-))^{[\perp]_2},$$

see Proposition 2.19. Consequently, $U_K = \mathrm{clos}\,(V)$ is unitary by Theorem 5.3. Finally, since $V(\mathrm{dom}\,V \cap \mathfrak{K}^+) \cap V(\mathrm{dom}\,V \cap \mathfrak{K}^-) = \{0\}$, because by assumption $\ker\,(I - K^*K) = \{0\}$, it follows from (5.1) that $\mathrm{mul}\,U_K = \{0\}$.

*Step 2:* Recall that

$$KD_K^{-1} \subseteq D_{K^*}^{-1}K \quad \text{and} \quad K^*D_{K^*}^{-1} \subseteq D_K^{-1}K^*,$$

see (Sz.-Nagy & Foiaş 1970: Ch. I, Section 3). Applying the above inclusions to $V$ yields:

$$V = \begin{pmatrix} D_K^{-1} & K^*D_{K^*}^{-1} \\ KD_K^{-1} & D_{K^*}^{-1} \end{pmatrix} \subseteq \begin{pmatrix} D_K^{-1} & 0 \\ 0 & D_{K^*}^{-1} \end{pmatrix} \begin{pmatrix} I & K^* \\ K & I \end{pmatrix} = \mathfrak{j}T^{[*]}S^{[*]}\mathfrak{j} = \mathfrak{j}U_K^{[*]}\mathfrak{j}.$$

Since $\mathfrak{j}U_K^{[*]}\mathfrak{j}$ is a unitary operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$, see Lemma 3.10, and $U_K = \mathrm{clos}\,(V)$ is also a unitary operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$, the above inclusion implies that $U_K = \mathfrak{j}U_K^{[*]}\mathfrak{j}$. I.e., $U_K$ is a selfadjoint operator in the Hilbert space $\{\mathfrak{K}, [\mathfrak{j}\cdot,\cdot]\}$.

*Step 3:* The arguments in (Sz.-Nagy & Foiaş 1970: Ch. I, Section 3) can also be used to show that

$$KD_K^{-1/2} \subseteq D_{K^*}^{-1/2}K \quad \text{and} \quad K^*D_{K^*}^{-1/2} \subseteq D_K^{-1/2}K^*.$$

Applying these inclusions to $V$ $(= ST)$ yields:

$$\begin{aligned} V &= \begin{pmatrix} I & K^* \\ K & I \end{pmatrix} \begin{pmatrix} D_K^{-1} & 0 \\ 0 & D_{K^*}^{-1} \end{pmatrix} \\ &\subseteq \begin{pmatrix} D_K^{-1/2} & 0 \\ 0 & D_{K^*}^{-1/2} \end{pmatrix} \begin{pmatrix} I & K^* \\ K & I \end{pmatrix} \begin{pmatrix} D_K^{-1/2} & 0 \\ 0 & D_{K^*}^{-1/2} \end{pmatrix} \end{aligned}$$

Since $K$ is a contraction, $S$ (in (5.4)) is a nonnegative operator in $\{\mathfrak{K}, [\mathfrak{j}\cdot,\cdot]\}$. Consequently, the above calculation shows that $V$ is a nonnegative operator in $\{\mathfrak{K}, [\mathfrak{j}\cdot,\cdot]\}$ and, hence, also $U_K = \mathrm{clos}\,(V)$ is a nonnegative operator in $\{\mathfrak{K}, [\mathfrak{j}\cdot,\cdot]\}$. □

Note that the condition $\ker\,(I - K^*K) = \{0\}$ in Lemma 5.5 can be dropped by allowing $U_K$ to have a kernel and a multi-valued part. In that case the block representation for $U_K$ needs to be interpreted in a specific manner.

Following is the announced representation for unitary operators w.r.t. uniformly definite subspaces, see (Nakagami 1988) and (Gheondea 1988); see also (Azizov & Iokhvidov 1989: Ch. 2, Theorem 5.10).

**Theorem 5.6.** *Let $U$ be a unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_2^+[+]\mathfrak{K}_2^-$ be a canonical decompositions of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then there exists a* *bounded unitary operator* $U_t$ *from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *onto* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *and an everywhere defined contraction* $K$ *from* $\{\mathfrak{K}_2^+, [\cdot,\cdot]_2\}$ *to* $\{\mathfrak{K}_2^-, -[\cdot,\cdot]_2\}$ *with* $\ker(I - K^*K) = \{0\}$ *such that*

$$U = U_K U_t,$$

*where* $U_K$ *is as in Lemma 5.5. Conversely, if* $U_t$ *and* $K$ *are as above, then* $U_K U_t$ *is a unitary operator from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$.

*Proof.* Since $U_K$ is a unitary operator in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and $U_t$ is a bounded unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ onto $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, $U_K U_t$ is a unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ by Lemma 3.10.

To prove the converse note that if $\ker U \neq \{0\}$, then $U_o := U(U_{\ker U})^{-1}$, where $U_{\ker U}$ is as in Lemma 3.11, is a unitary operator without kernel. Hence, it suffices to prove that for the unitary operator $U_o$ with $\ker U_o = \{0\}$ there exists a representation $U_o = U_K U_t$, where $U_t$ is a standard unitary operator. Namely, in that case, $U$ has the representation $U = U_K(U_t U_{\ker U})$, where $U_t U_{\ker U}$ is a bounded unitary operator.

Hence, let $U$ be a unitary operator with $\ker U = \{0\}$ and let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then by Proposition 5.1 and 2.19 there exists a (unique) contractive operator $K$ from $\{\mathfrak{K}_2^+, [\cdot,\cdot]_2\}$ to $\{\mathfrak{K}_2^-, -[\cdot,\cdot]_2\}$ such that

$$U(\operatorname{dom} U \cap \mathfrak{K}_1^+) = \{f_2^+ + K f_2^+ : f_2^+ \in \mathfrak{K}_2^+\};$$
$$U(\operatorname{dom} U \cap \mathfrak{K}_1^-) = \{f_2^- + K^* f_2^- : f_2^- \in \mathfrak{K}_2^-\}.$$

Here $\ker(I - K^*K) = \{0\}$, because $\operatorname{mul} U = U(\operatorname{dom} U \cap \mathfrak{K}_1^+) \cap U(\operatorname{dom} U \cap \mathfrak{K}_1^-) = \{0\}$, see (5.1). With this $K$, let $U_K$ be the unitary operator in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\operatorname{mul} U_K = \{0\}$ given by Lemma 5.5 and define $\mathfrak{L}$ to be $\operatorname{dom} U \cap \mathfrak{K}_1^+ + \operatorname{dom} U \cap \mathfrak{K}_1^-$. Then $\operatorname{ran}(U \upharpoonright_{\mathfrak{L}}) \subseteq \operatorname{ran} U_K$ and, hence, $U_K^{-1} U \upharpoonright_{\mathfrak{L}}$ is an isometric operator, see Lemma 3.10, which satisfies

$$\left(U_K^{-1} U \upharpoonright_{\mathfrak{L}}\right)(\operatorname{dom} U \cap \mathfrak{K}_1^+) = \operatorname{dom}(I - K^*K)^{-1/2} \times \{0\} \subseteq \mathfrak{K}_2^+;$$
$$\left(U_K^{-1} U \upharpoonright_{\mathfrak{L}}\right)(\operatorname{dom} U \cap \mathfrak{K}_1^-) = \{0\} \times \operatorname{dom}(I - KK^*)^{-1/2} \subseteq \mathfrak{K}_2^-.$$

Now observe that $\operatorname{dom} U \cap \mathfrak{K}_1^+$ and $\operatorname{dom}(I - K^*K)^{-1/2}$ are dense in the Hilbert spaces $\{\mathfrak{K}_1^+, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2^+, [\cdot,\cdot]_2\}$, respectively, and that $\operatorname{dom} U \cap \mathfrak{K}_1^-$ and $\operatorname{dom}(I - KK^*)^{-1/2}$ are dense in the Hilbert spaces $\{\mathfrak{K}_1^-, -[\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2^-, -[\cdot,\cdot]_2\}$, respectively, see Corollary 3.20. Hence, there exist standard unitary operators $U_t^+$ and $U_t^-$ from $\{\mathfrak{K}_1^+, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2^+, [\cdot,\cdot]_2\}$ and from $\{\mathfrak{K}_1^-, -[\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2^-, -[\cdot,\cdot]_2\}$, respectively, such that with respect to the decompositions $\mathfrak{K}_1^+ \times \mathfrak{K}_1^-$ and $\mathfrak{K}_2^+ \times \mathfrak{K}_2^-$ of $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively, $U_t := \operatorname{clos}(U_K^{-1} U \upharpoonright_{\mathfrak{L}}) = U_t^+ \times U_t^-$, i.e., $U \upharpoonright_{\mathfrak{L}} \subseteq U_K U_t$.

Since $U_K U_t$ is unitary by the proven part of the statement and $U$ is by assumption a unitary operator, the preceding inclusion implies that $U = U_K U_t$. $\square$

If $\mathfrak{j}_2$ is the fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ corresponding to the canonical decomposition $\mathfrak{K}_2^+[+]\mathfrak{K}_2^-$ in the statement of Theorem 5.6 and $\mathfrak{j}_1$ is the fundamental symmetry of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ corresponding to the canonical decomposition $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ in the proof of Theorem 5.6, then $K$ in Theorem 5.6 is the operator such that $U(\operatorname{dom} U \cap \mathfrak{K}_1^+) = \{f_2^+ + Kf_2^+ : f_2^+ \in \mathfrak{K}_2^+\}$. Furthermore, if $\ker U = \{0\}$, then, with the above notation, $U_t$ in Theorem 5.6 is a standard unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $U_t\mathfrak{j}_1 = \mathfrak{j}_2 U_t$. Therefore, in that case, $U_t$ is also an everywhere defined unitary operator from (the Hilbert space) $\{\mathfrak{K}_1, [\mathfrak{j}_1\cdot,\cdot]_1\}$ onto (the Hilbert space) $\{\mathfrak{K}_2, [\mathfrak{j}_2\cdot,\cdot]_2\}$. Consequently, in light of Lemma 5.5, the decomposition $U_K U_t$ of $U$ in Theorem 5.6 is in fact a polar decomposition of $U$ as an operator from the Hilbert space $\{\mathfrak{K}_1, [\mathfrak{j}_1\cdot,\cdot]_1\}$ to the Hilbert space $\{\mathfrak{K}_2, [\mathfrak{j}_2\cdot,\cdot]_2\}$, cf. (Calkin 1939a: Theorem 3.6). This observation will be used in Chapter 6 to obtain another useful graph decomposition of unitary relations.

**Remark 5.7.** Theorem 5.6 shows that unitary relations can be classified by the nature of the spectrum of an associated contraction $K$ at 1. In particular, the unitary operator $U$ is a standard unitary operator if and only if $K$ is a uniform contraction, see (Azizov & Iokhvidov 1989: Ch. 2, Theorem 5.10).

Theorem 5.6 can be interpreted as a realization result for maximal nonnegative and nonpositive subspaces (or, equivalently, for maximal dissipative or accumulative relations, see Proposition 2.20). Therefore observe first that if $\mathfrak{L}$ is a closed neutral subspace of the Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$ with fundamental symmetry $\mathfrak{j}$, then $\mathfrak{L}$ is a hyper-maximal neutral subspace of the Kreĭn space $\{\mathfrak{L} + \mathfrak{j}\mathfrak{L}, [\cdot,\cdot]\}$ and $\mathfrak{L} \times \mathfrak{L}$ is a unitary relation in $\{\mathfrak{L} + \mathfrak{j}\mathfrak{L}, [\cdot,\cdot]\}$, see e.g. Corollary 4.4.

**Theorem 5.8.** *Let $\mathfrak{M}^+$ and $\mathfrak{M}^-$ be a maximal nonnegative and nonpositive subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$, respectively, and let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then there exists a unitary relation $U$ in $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that*

$$\mathfrak{M}^+ = U(\operatorname{dom} U \cap \mathfrak{K}^+) \quad \textit{or} \quad \mathfrak{M}^- = U(\operatorname{dom} U \cap \mathfrak{K}^-),$$

*respectively. Moreover, If $U_1$ and $U_2$ are two unitary relations in $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that $U_1(\operatorname{dom} U_1 \cap \mathfrak{K}^+) = U_2(\operatorname{dom} U_2 \cap \mathfrak{K}^+)$ or $U_1(\operatorname{dom} U_1 \cap \mathfrak{K}^-) = U_2(\operatorname{dom} U_2 \cap \mathfrak{K}^-)$, then $\operatorname{clos}(U_2^{-1}U_1)$ is a unitary relation in $\{\mathfrak{K}, [\cdot,\cdot]\}$ with closed domain.*

*Proof.* Let $\mathfrak{j}$ be the fundamental symmetry associated with the canonical decomposition $\mathfrak{K}^+[+]\mathfrak{K}^-$ and let $\mathfrak{M}_0$ be defined as $\mathfrak{M}^+ \cap (\mathfrak{M}^+)^{[\perp]}$. Then $\mathfrak{M}_0$ is a closed neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and by means of this neutral subspace define $\mathfrak{K}_0 = \mathfrak{M}_0 + \mathrm{j}\mathfrak{M}_0$ and $\mathfrak{K}_r = \mathfrak{K} \cap \mathfrak{K}_0^{[\perp]}$. Then $\{\mathfrak{K}_0, [\cdot,\cdot]\}$ and $\{\mathfrak{K}_r, [\cdot,\cdot]\}$ are Kreĭn spaces and $\mathfrak{K}_0^+[+]\mathfrak{K}_0^- = (\mathfrak{K}^+ \cap \mathfrak{K}_0)[+](\mathfrak{K}^- \cap \mathfrak{K}_0)$ and $\mathfrak{K}_r^+[+]\mathfrak{K}_r^- = (\mathfrak{K}^+ \cap \mathfrak{K}_r)[+](\mathfrak{K}^- \cap \mathfrak{K}_r)$ are canonical decompositions of these spaces.

Now let $K_r$ be the angular operator of $\mathfrak{M}^+ \cap \mathfrak{K}_r$, i.e.,

$$\mathfrak{M}^+ \cap \mathfrak{K}_r = \{f_r^+ + K_r f_r^+ : f_r^+ \in \mathfrak{K}_r^+\}.$$

Since $\mathfrak{M}^+ \cap (\mathfrak{M}^+)^{[\perp]} \cap \mathfrak{K}_r = \{0\}$, it follows that $\ker(I - K_r^* K_r) = \{0\}$. Hence, $U_{K_r}$ is a unitary operator in $\{\mathfrak{K}_r, [\cdot,\cdot]\}$ such that $U_{K_r}(\mathrm{dom}\, U_{K_r} \cap \mathfrak{K}_r^+) = \mathfrak{M}^+ \cap \mathfrak{K}_r$, see Lemma 5.5. Since $U_0$ defined via $\mathrm{gr}\, U_0 = \mathfrak{M}_0 \times \mathfrak{M}_0$ is a unitary relation in $\{\mathfrak{K}_0, [\cdot,\cdot]\}$, Lemma 3.13 shows that $U$ defined via $\mathrm{gr}\, U = \mathrm{gr}\, U_0 + \mathrm{gr}\, U_{K_r}$ is a unitary relation in $\{\mathfrak{K}, [\cdot,\cdot]\}$, which satisfies $U(\mathrm{dom}\, U \cap \mathfrak{K}^+) = \mathfrak{M}^+$. Similar arguments can be used to show the existence of a unitary relation $U$ with $U(\mathrm{dom}\, U \cap \mathfrak{K}^-) = \mathfrak{M}^-$.

Next let $U_1$ and $U_2$ be unitary relations such that $U_1(\mathrm{dom}\, U_1 \cap \mathfrak{K}^+) = \mathfrak{M}^+ = U_2(\mathrm{dom}\, U_2 \cap \mathfrak{K}^+)$ and w.l.o.g. assume that $\ker U_1 = \{0\} = \ker U_2$, see the above arguments or Corollary 3.12. Then $U_1(\mathrm{dom}\, U_1 \cap \mathfrak{K}^-) = U_2(\mathrm{dom}\, U_2 \cap \mathfrak{K}^-)$, see Proposition 5.1. Hence $U_a := U_2^{-1} U_1$ maps $\mathrm{dom}\, U_1 \cap \mathfrak{K}^\pm$ onto $\mathrm{dom}\, U_2 \cap \mathfrak{K}^\pm$. Since $\mathrm{clos}\,(\mathrm{dom}\, U_i \cap \mathfrak{K}^\pm) = \mathfrak{K}^\pm$, for $i = 1, 2$, see (3.11), it follows that $\mathrm{clos}\,(U_a)$ is a standard unitary operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$. □

## 5.3 A quasi-block representation for maximal isometric operators

Next the quasi-block representation for unitary operators from Theorem 5.6 is generalized to a quasi-block representation for maximal isometric operators. That representation for maximal isometric operators implies that non-trivial properties of maximal isometric operators can be obtained from properties of unitary operators. Note also that a similar representation holds for an isometric operator $V$ whose domain contains a hyper-maximal semi-definite subspace, because in that case $\ker V + \mathrm{dom}\, V \cap \mathfrak{K}_1^+ + \mathrm{dom}\, V \cap \mathfrak{K}_1^-$ is dense in its domain.

**Theorem 5.9.** *Let $V$ be an isometric operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\ker V = (\mathrm{dom}\, V)^{[\perp]_1}$ and $\overline{\mathrm{ran}}\, V = \mathfrak{K}_2$. Moreover, let $\mathrm{j}_i$ be a fundamental symmetry of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be the associated canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$, for $i = 1, 2$. Then $V$ is a maximal isometric operator if and only if*

$$V = U_K V_t,$$

*where*

*(i)* $V_t$ *is a closed isometric operator from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *satisfying* $\ker V = \ker V_t$ *and* $(\ker V_t)^{[\perp]_1} = \operatorname{dom} V_t$*,* $\mathfrak{j}_2 V_t = V_t \mathfrak{j}_1$ *and*

$$P_2^+ \operatorname{ran} V_t = \mathfrak{K}_2^+ \quad \textit{or} \quad P_2^- \operatorname{ran} V_t = \mathfrak{K}_2^-;$$

*(ii)* $U_K$ *is the unitary operator in* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *associated with an everywhere defined contraction* $K$ *from (the Hilbert space)* $\{\mathfrak{K}_2^+, [\cdot,\cdot]_2\}$ *to (the Hilbert space)* $\{\mathfrak{K}_2^-, -[\cdot,\cdot]_2\}$ *with* $\ker(I - K^*K) = \{0\}$ *as in* (5.3).

*Proof.* W.l.o.g. assume that $\ker V = \{0\} = \ker V_t$, see Section 3.4.

First the sufficiency of the conditions is proven, where w.l.o.g. it is assumed that $P_2^- \operatorname{ran} V_t = \mathfrak{K}_2^-$. Note first that the assumption $\mathfrak{j}_2 V_t = V_t \mathfrak{j}_1$ together with the closedness of $V_t$ implies that $\operatorname{dom} V_t = \mathfrak{K}_1$. Hence, in particular, the assumptions imply that $V_t(\mathfrak{K}_1^+) = P_2^+ \operatorname{ran} V_t$ and $V_t(\mathfrak{K}_1^-) = \mathfrak{K}_2^-$. Furthermore, since $U_K$ is a unitary operator in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, Proposition 3.16 and Corollary 3.15 imply that

$$P_2^- U_K(\operatorname{dom} U_K \cap \mathfrak{K}_2^-) = \mathfrak{K}_2^- \quad \text{and} \quad P_2^+ U_K^{-1}(\operatorname{ran} U_K \cap \mathfrak{K}_2^+) = \mathfrak{K}_2^+,$$

Combining these equalities with $V_t(\mathfrak{K}_1^-) = \mathfrak{K}_2^-$ yields

$$\operatorname{dom} U_K \cap \mathfrak{K}_2^- \subseteq \operatorname{ran} V_t \quad \text{and} \quad P_2^+(U_K^{-1}(\operatorname{ran} U_K \cap \mathfrak{K}_2^+) \cap \operatorname{ran} V_t) = P_2^+ \operatorname{ran} V_t.$$

Consequently, since $V_t(\mathfrak{K}_1^+) = P_2^+ \operatorname{ran} V_t$ and $V_t(\mathfrak{K}_1^-) = \mathfrak{K}_2^-$, $V = U_K V_t$ satisfies

$$P_2^- V(\operatorname{dom} V \cap \mathfrak{K}_1^-) = \mathfrak{K}_2^- \quad \text{and} \quad P_1^+ V^{-1}(\operatorname{ran} V \cap \mathfrak{K}_2^+) = \mathfrak{K}_1^+.$$

This implies by Proposition 3.16 and Corollary 3.15 that $V$ is maximal isometric.

Next the necessity of the conditions is proven; w.l.o.g. that is only done for the case that $(P_1^- \times P_2^+)\operatorname{gr} V = \mathfrak{K}_1^- \times \mathfrak{K}_2^+$, see Corollary 3.15. If $(P_1^- \times P_2^+)\operatorname{gr} V = \mathfrak{K}_1^- \times \mathfrak{K}_2^+$, then $V(\operatorname{dom} V \cap \mathfrak{K}_1^+)$ is a maximal nonnegative subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ by Proposition 3.16. Let $K$ be its angular operator w.r.t. $\mathfrak{K}_2^+$, i.e., $K$ is the everywhere defined contraction from $\{\mathfrak{K}_2^+, [\cdot,\cdot]_2\}$ to $\{\mathfrak{K}_2^-, -[\cdot,\cdot]_2\}$ such that

$$V(\operatorname{dom} V \cap \mathfrak{K}_1^+) = \{f^+ + Kf^+ : f^+ \in \mathfrak{K}_2^+\}.$$

Moreover, $\ker(I - K^*K) = \{0\}$, because $V$ is a closed operator with $\operatorname{mul} V = \{0\}$, i.e. $V(\operatorname{dom} V \cap \mathfrak{K}_1^+)$ does not contain neutral vectors of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ (here $K^*$ is the adjoint of $K$ as an operator from $\{\mathfrak{K}_2^+, [\cdot,\cdot]_2\}$ to $\{\mathfrak{K}_2^-, -[\cdot,\cdot]_2\}$). Let $U_K$ be the unitary operator associated with $K$ as in (5.3). Then $U_K^{-1} V \upharpoonright_{\mathfrak{K}_1^+}$ is an isometric operator from $\{\mathfrak{K}_1^+, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2^+, [\cdot,\cdot]_2\}$ which maps the dense subspace $\operatorname{dom} V \cap \mathfrak{K}_1^+$ of $\mathfrak{K}_1^+$, see Lemma 3.19, onto the dense subspace $\operatorname{dom} U_K \cap \mathfrak{K}_2^+$ of $\mathfrak{K}_2^+$, see Corollary 3.20. I.e., $\dim \mathfrak{K}_1^+ = \dim \mathfrak{K}_2^+$ and $V_1^+ := \operatorname{clos}(U_K^{-1} V \upharpoonright_{\mathfrak{K}_1^+})$ is an everywhere defined unitary operator from the Hilbert space $\{\mathfrak{K}_1^+, [\cdot,\cdot]_1\}$ onto the Hilbert space $\{\mathfrak{K}_2^+, [\cdot,\cdot]_2\}$.

Similar arguments (applied to $V^{-1}$, $\operatorname{ran} V \cap \mathfrak{K}_2^-$ and $V^{-1}(\operatorname{ran} V \cap \mathfrak{K}_2^-)$) show that there exists an everywhere defined unitary operator $V_1^-$ from $\{\mathfrak{K}_1^-, -[\cdot,\cdot]_1\}$ onto $\{\mathfrak{K}_2^-, -[\cdot,\cdot]_2\}$. Then $V_1 := V_1^+ \times V_1^-$ is a standard unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ satisfying $V_1 \mathfrak{j}_1 = \mathfrak{j}_2 V_1$. Hence, by Lemma 3.10 $U := U_K V_1$ is a unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ which by definition of $V_1$ satisfies

$$Uf = Vf, \quad f \in \operatorname{dom} U \cap \mathfrak{K}_1^+ = \operatorname{dom} V \cap \mathfrak{K}_1^+. \tag{5.5}$$

Next let the definite inner products $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ on the closed subspaces $\mathfrak{K}_1^+ \times \mathfrak{K}_2^-$ and $\mathfrak{K}_1^- \times \mathfrak{K}_2^+$ be as in (4.11) and let $A_V$ and $A_U$ be the Potapov-Ginzburg transforms of $V$ and $U$:

$$\begin{aligned} \operatorname{gr} A_V &= \{\{P_1^+ f \times P_2^- f', P_1^- f \times P_2^+ f'\} : \{f, f'\} \in \operatorname{gr} V\}; \\ \operatorname{gr} A_U &= \{\{P_1^+ f \times P_2^- f', P_1^- f \times P_2^+ f'\} : \{f, f'\} \in \operatorname{gr} U\}, \end{aligned}$$

respectively, see Proposition 4.14. Then $A_V$ and $A_U$ are a maximal isometric and a unitary operator from the Hilbert space $\{\mathfrak{K}_1^+ \times \mathfrak{K}_2^-, (\cdot,\cdot)_1\}$ to the Hilbert space $\{\mathfrak{K}_1^- \times \mathfrak{K}_2^+, (\cdot,\cdot)_2\}$, respectively. Note that the assumption $(P_1^- \times P_2^+)\operatorname{gr} V = \mathfrak{K}_1^- \times \mathfrak{K}_2^+$ implies that $\operatorname{ran} A_V = \mathfrak{K}_1^- \times \mathfrak{K}_2^+$. Note also that $A_U^{-1}(\mathfrak{K}_1^-) \oplus A_U^{-1}(\mathfrak{K}_2^+) = \mathfrak{K}_1^+ \times \mathfrak{K}_2^-$, because $A_U$ is a (Hilbert space) unitary operator, and that $A_U^{-1}(\mathfrak{K}_2^+) = A_V^{-1}(\mathfrak{K}_2^+)$ by (5.5). Since $A_V$ is an isometric operator, the above observations yield

$$A_V^{-1}(\mathfrak{K}_1^-) \subseteq (A_V^{-1}(\mathfrak{K}_2^+))^{\perp_1} = (A_U^{-1}(\mathfrak{K}_2^+))^{\perp_1} = A_U^{-1}(\mathfrak{K}_1^-).$$

This shows that $V_2^- := A_U A_V^{-1} \upharpoonright_{\mathfrak{K}_1^-}$ is an everywhere defined isometric operator in $\{\mathfrak{K}_1^-, [\cdot,\cdot]_1\}$ and, hence, $V_2 := I_{\mathfrak{K}_1^+} \times V_2^-$ is an everywhere defined isometric operator in $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ which commutes with $\mathfrak{j}_1$. By definition of $V_2$

$$(UV_2)^{-1} f = V^{-1} f, \quad f \in \operatorname{ran} V \cap \mathfrak{K}_2^- = \operatorname{ran}(UV_2) \cap \mathfrak{K}_2^-. \tag{5.6}$$

Since $\operatorname{dom} V = \operatorname{dom} V \cap \mathfrak{K}_1^+ + V^{-1}(\operatorname{ran} V \cap \mathfrak{K}_2^-)$ by Proposition 3.16, (5.5) and (5.6) show that $UV_2 = U_K V_1 V_2$ and $V$ coincide on the domain of the maximal isometric operator $V$. Hence, the asserted decomposition holds with $V_t = V_1 V_2$. □

Although $U_K$ in Theorem 5.9 is a nonnegative selfadjoint operator in (the Hilbert space) $\{\mathfrak{K}_2, [\mathfrak{j}_2\cdot,\cdot]_2\}$ and $V_t$ an everywhere defined isometric operator from (the Hilbert space) $\{\mathfrak{K}_1, [\mathfrak{j}_1\cdot,\cdot]_1\}$ to (the Hilbert space) $\{\mathfrak{K}_2, [\mathfrak{j}_2\cdot,\cdot]_2\}$, the decomposition $V = U_K V_t$ is not, in general, a polar decomposition of $V$, because $\operatorname{dom} U_K \nsubseteq \overline{\operatorname{ran}}\, V_t$. However, if $\operatorname{ran} V_t = \mathfrak{K}_2$, then the decomposition in Theorem 5.9 is a polar decomposition. In fact, in that case $V$ is a unitary operator, see Section 5.2.

Recall that for a unitary relation $\ker U + \operatorname{dom} U \cap \mathfrak{K}_1^+ + \operatorname{dom} U \cap \mathfrak{K}_1^-$ is dense in the domain of $U$, see (3.10). In fact, Lemma 5.4 showed that the preceding subspace is a core for $U$. I.e., a unitary relation is completely determined by its behavior on the uniformly definite subspaces $\operatorname{dom} U \cap \mathfrak{K}_1^+$ and $\operatorname{dom} U \cap \mathfrak{K}_1^-$. For (maximal) isometric relations this does not in general hold as the following example shows.

**Example 5.10.** Let $U$ be a densely defined unbounded unitary operator in the separable (infinite-dimensional) Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then $\operatorname{dom} U \cap \mathfrak{K}^+$ and $\operatorname{dom} U \cap \mathfrak{K}^-$ are dense subspaces of $\mathfrak{K}^+$ and $\mathfrak{K}^-$, respectively, which are, moreover, operator ranges. Hence, there exists an infinite-dimensional closed subspace $\mathfrak{L}$ of $\mathfrak{K}^+$ such that $\mathfrak{L} \cap \operatorname{dom} U = \{0\}$, see Proposition 2.17 (vi). Now let $V_t$ be the everywhere defined isometric operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$ which is the identity mapping on $\mathfrak{K}^-$ and maps $\mathfrak{K}^+$ isometrically onto $\mathfrak{L} \subsetneq \mathfrak{K}^+$, then $V := UV_t$ is by (the first part of the proof of) Theorem 5.9 a maximal isometric operator and by construction $\operatorname{dom} V \cap \mathfrak{K}^+ = \{0\}$ (and $\overline{\operatorname{dom}}\, V = \mathfrak{K}$).

From the fact that $\overline{\operatorname{dom}}\, V = \mathfrak{K}$ and $\operatorname{dom} V \cap \mathfrak{K}^+ = \{0\}$ in Example 5.10, it follows that the domain of $V$ can not contain a hyper-maximal semi-definite subspace. Because if it would contain a hyper-maximal semi-definite subspace, then $\operatorname{dom} V \cap \mathfrak{K}^+$ and $\operatorname{dom} V \cap \mathfrak{K}^-$ should be dense in $\mathfrak{K}^+$ and $\mathfrak{K}^-$, respectively, see Corollary 2.14. Example 5.11 below shows that there exists a densely defined (non-maximal) closed isometric operator $V$ with dense range such that $\operatorname{dom} V \cap \mathfrak{K}^+ = \{0\} = \operatorname{dom} V \cap \mathfrak{K}^-$. In particular, the domain of the isometric operator in Example 5.11 also does not contain any hyper-maximal semi-definite subspace.

**Example 5.11.** Let $K$ be a compact nonnegative selfadjoint operator in the separable (infinite-dimensional) Hilbert space $\{\mathfrak{H}, (\cdot,\cdot)\}$ with $\operatorname{ran} K \neq \mathfrak{H} = \overline{\operatorname{ran}}\, K$. Then by (Brasche $\&$ Neidhardt 1993: Lemma 2), there exists a closed restriction $T$ of $K$ such that $\overline{\operatorname{ran}}\, T = \mathfrak{H}$, that $\operatorname{dom} T \cap \operatorname{ran} T = \{0\}$ and that $\dim(\operatorname{dom} T)^\perp = \infty$. Note that the operator range $\operatorname{dom} T + \operatorname{ran} T$, see Proposition 2.17 (i), is not equal to the whole space by Proposition 2.17 (iv).

Since $\operatorname{dom} T + \operatorname{ran} T$ is a nonclosed operator range, Proposition 2.17 (ii) and (v) implies that there exists an everywhere defined closed operator $B$ with $\ker B = \{0\}$ such that $\operatorname{ran} B \cap (\operatorname{dom} T + \operatorname{ran} T) = \{0\}$. Now $V$ defined as

$$V\{f, T^{-1}f + Bf'\} = \{f', Kf' - B^*f\}, \quad f \in \operatorname{ran} T,\ f' \in \operatorname{dom} K,$$

is a closed isometric operator in $\{\mathfrak{H}^2, <\cdot,\cdot>\}$, cf. (Derkach et al. 2006: Example 6.6) and (Derkach et al. 2012: Proposition 7.55). Clearly,

$$\begin{aligned} \operatorname{dom} V &= \operatorname{gr} T^{-1} + \{0\} \times B(\operatorname{dom} K); \\ \operatorname{ran} V &= \operatorname{gr} K + \{0\} \times B^*(\operatorname{ran} T). \end{aligned} \tag{5.7}$$

The above formulas imply that $\overline{\mathrm{dom}\,V} = \mathfrak{H}^2 = \overline{\mathrm{ran}\,V}$, because by assumption $\overline{\mathrm{dom}\,K} = \mathfrak{H} = \overline{\mathrm{ran}\,T}$, and $B$ and $B^*$ are closed everywhere defined operators with dense range. Recall that $\mathfrak{H}^+[+]\mathfrak{H}^-$, where $\mathfrak{H}^+ = \{\{f, if\} : f \in \mathfrak{H}\}$ and $\mathfrak{H}^- = \{\{f, -if\} : f \in \mathfrak{H}\}$, is a canonical decomposition of $\{\mathfrak{H}^2, < \cdot, \cdot >\}$, see Example 2.1. Hence, $\mathrm{dom}\,T \cap \mathrm{ran}\,T = \{0\}$, $\mathrm{ran}\,B \cap (\mathrm{dom}\,T + \mathrm{ran}\,T) = \{0\}$ and (5.7) yield $\mathrm{dom}\,V \cap \mathfrak{H}^+ = \{0\} = \mathrm{dom}\,V \cap \mathfrak{H}^-$.

Using compositions of unitary and maximal isometric operators as in Theorem 5.9, it can be shown that the domains of unitary relation do not differ essentially from the domains of isometric relations. I.e., to distinguish unitary relations from isometric relations their action also has to be considered.

**Example 5.12.** Let $U$ be an unbounded unitary operator between $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_2^+[+]\mathfrak{K}_2^-$ be a canonical decomposition of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Moreover, let $V$ be a closed everywhere defined isometric operator in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ which maps $\mathfrak{K}_2^+$ onto $\mathfrak{K}_2^+$ and $\mathfrak{K}_2^-$ onto $\mathfrak{D}_2^-$, where $\mathfrak{D}_2^- \subsetneq \mathfrak{K}_2^-$. Then $VU$ is an isometric operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\mathrm{dom}\,VU = \mathrm{dom}\,U$. Moreover, $VU$ is not unitary, because $P_2^- \mathrm{ran}\,VU = \mathfrak{D}_2^- \neq \mathfrak{K}_2^-$. In fact, arguments as in the proof of Theorem 5.9 show that $VU$ is a maximal isometric operator.

**Example 5.13.** Let $S$ be a densely defined closed symmetric operator in $\{\mathfrak{H}, (\cdot,\cdot)\}$ which does not have equal defect numbers. Then $\Upsilon_1(S)$ is a densely defined isometric operator in $\{\mathfrak{H}^2, < \cdot, \cdot >\}$ with dense range which can not be extended to a unitary operator. Next let $B$ be an everywhere defined closed operator such that $\mathrm{ran}\,B = \mathrm{dom}\,S$ and $\ker B = \{0\}$. Then $\Upsilon_2(B^{-1})$ is a unitary operator in $\{\mathfrak{H}^2, < \cdot, \cdot >\}$ with

$$\mathrm{dom}\,(\Upsilon_2(B^{-1})) = \mathrm{ran}\,B \oplus_2 \mathfrak{H} = \mathrm{dom}\,S \oplus_2 \mathfrak{H} = \mathrm{dom}\,(\Upsilon_1(S)).$$

## 5.4 Weyl identity and properties of unitary relations

The Weyl identity approach, which was shown to characterize unitary relations in Section 5.1, is now used to obtain two types of results on unitary relations: First it is shown that this approach indicates how unitary relations can be split and, secondly, it is shown how the approach can be used to determine the defect numbers of certain neutral subspaces after mapping them by a unitary relation. As a first step towards the splitting result, conditions (in terms of the Weyl identity) are presented for when a part of a unitary relation is itself a unitary relation between certain Kreĭn spaces.

**Proposition 5.14.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ with associated projections $P_i^+$ and $P_i^-$, for $i = 1, 2$. Moreover, let $\mathfrak{L}_d$ and $\mathfrak{L}_r$ be closed subspaces of* $\ker U$ *and* $\operatorname{mul} U$*, respectively, and assume that there exist subspaces $\mathfrak{M}^+ \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^+$ and $\mathfrak{M}^- \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-$ such that*

(i) $\operatorname{clos}(\mathfrak{M}^+) \cap \operatorname{dom} U = \mathfrak{M}^+$ *and* $\operatorname{clos}(\mathfrak{M}^-) \cap \operatorname{dom} U = \mathfrak{M}^-$*;*

(ii) $P_2^- U(\mathfrak{M}^+) \subseteq P_2^- U(\mathfrak{M}^-)$ *and* $P_2^+ U(\mathfrak{M}^-) \subseteq P_2^+ U(\mathfrak{M}^+)$.

*Then $\widetilde{U}$ defined via*

$$\operatorname{gr} \widetilde{U} = \{\{f+g, f'\} \in U : f \in \operatorname{clos}(\mathfrak{M}^+ + \mathfrak{M}^-),\ g \in \mathfrak{L}_d,\ f' \in \mathfrak{L}_r^{\perp_2}\}$$

*is a unitary relation from the Kreĭn space $\{\widetilde{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$ to the Kreĭn space $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$, where $\widetilde{\mathfrak{K}}_1 = \operatorname{clos}(\mathfrak{M}^+ + \mathfrak{M}^-) + (\mathfrak{L}_d + \mathfrak{j}_1\mathfrak{L}_d)$ and $\widetilde{\mathfrak{K}}_2 = (P_2^+ U(\mathfrak{M}^+) + P_2^- U(\mathfrak{M}^-)) \cap (\mathfrak{L}_r + \mathfrak{j}_2\mathfrak{L}_r)^{[\perp]_2}$.*

*Proof.* Note first that condition (i) implies that $P_2^+ U(\mathfrak{M}^+)$ and $P_2^- U(\mathfrak{M}^-)$ are closed, see Proposition 3.9. Hence, the assumptions (i) and (ii) together with the assumptions on $\mathfrak{L}_d$ and $\mathfrak{L}_r$ imply that $\{\widetilde{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$ and $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$ are Kreĭn spaces and that $\widetilde{U}$ is closed, because its graph is the intersection of the two closed subspaces $\operatorname{gr}(U)$ and $\widetilde{\mathfrak{K}}_1 \times \widetilde{\mathfrak{K}}_2$. Moreover, by construction $\ker \widetilde{U} = \mathfrak{L}_d = (\operatorname{dom} \widetilde{U})^{[\perp]_1} \cap \widetilde{\mathfrak{K}}_1$.

Next observe that the assumptions (i) and (ii) imply that $U(\mathfrak{M}^+)$ and $U(\mathfrak{M}^-)$ are a maximal nonnegative and maximal nonpositive subspace of the Kreĭn space $\{P_2^+ U(\mathfrak{M}^+) + P_2^- U(\mathfrak{M}^-), [\cdot,\cdot]_2\}$, respectively. Therefore, since $\mathfrak{L}_r \subseteq \operatorname{mul} U \subseteq U(\mathfrak{M}^+) \cap U(\mathfrak{M}^-)$, $\widetilde{U}(\mathfrak{M}^+) = U(\mathfrak{M}^+) \cap \widetilde{\mathfrak{K}}_2$ and $\widetilde{U}(\mathfrak{M}^-) = U(\mathfrak{M}^-) \cap \widetilde{\mathfrak{K}}_2$ are a maximal nonnegative and nonpositive subspace of the Kreĭn space $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$, respectively. Since $U(\mathfrak{M}^+) \subseteq (U(\mathfrak{M}^-))^{[\perp]_2}$, see Proposition 5.1, the maximality of $\widetilde{U}(\mathfrak{M}^+)$ and $\widetilde{U}(\mathfrak{M}^-)$ implies that $\widetilde{U}(\mathfrak{M}^+) = \widetilde{U}(\mathfrak{M}^-)^{[\perp]_2}$ in $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$. Hence, Theorem 5.3 yields that $\widetilde{U}$ is a unitary relation. □

The inverse to Proposition 5.14 also holds, i.e., if $\widetilde{U}$ is a unitary relation, then (i) and (ii) hold. Next Proposition 5.14 is used to obtain a result about the splitting of unitary relations which complements Lemma 3.13.

**Theorem 5.15.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, let $\{\widetilde{\mathfrak{K}}_i, [\cdot,\cdot]_i\}[+]\{\widehat{\mathfrak{K}}_i, [\cdot,\cdot]_i\}$ be an orthogonal decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ into two Kreĭn spaces, for $i = 1, 2$, and define $\widetilde{U}$ and $\widehat{U}$ via*

$$\operatorname{gr} \widetilde{U} = \operatorname{gr} U \cap (\widetilde{\mathfrak{K}}_1 \times \widetilde{\mathfrak{K}}_2) \quad \textit{and} \quad \operatorname{gr} \widehat{U} = \operatorname{gr} U \cap (\widehat{\mathfrak{K}}_1 \times \widehat{\mathfrak{K}}_2).$$

*Then $\widetilde{U}$ is unitary if and only if $\widehat{U}$ is unitary.*

*Proof.* W.l.o.g. assume that $\ker U = \{0\} = \mathrm{mul}\, U$ and let $\widetilde{\mathfrak{K}}_i^+[+]\widetilde{\mathfrak{K}}_i^-$ and $\widehat{\mathfrak{K}}_i^+[+]\widehat{\mathfrak{K}}_i^-$ be canonical decomposition of $\{\widetilde{\mathfrak{K}}_i, [\cdot,\cdot]_i\}$ and $\{\widetilde{\mathfrak{K}}_i, [\cdot,\cdot]_i\}$, respectively, for $i = 1,2$. Denote the associated canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ by $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ and let $P_i^+$ and $P_i^-$ denote the associated projections, for $i = 1,2$.

Clearly, to prove the equivalence it suffices to prove only one implication. Hence assume that $\widetilde{U}$ is unitary. Define $\widehat{U}_r^+$ via

$$\mathrm{gr}\, \widehat{U}_r^+ = \{\{f, f'\} \in \mathrm{gr}\, U : f \in \mathrm{dom}\, U \cap \mathfrak{K}_1^+ \text{ and } P_2^+ f' \in \widehat{\mathfrak{K}}_2^+\},$$

then $P_2^+ \mathrm{ran}\, \widehat{U}_r^+ = \widehat{\mathfrak{K}}_2^+$, see Proposition 5.1. If $\{f, f'\} \in \mathrm{gr}\, \widehat{U}_r^+$ and $\{g, g'\} \in \mathrm{gr}\, \widetilde{U}$ where $g \in \mathrm{dom}\, \widetilde{U} \cap \mathfrak{K}_1^-$, then $[f, g]_1 = 0$ and $[P_2^+ f', P_2^+ g']_2 = 0$. Therefore

$$0 = [f, g]_1 = [f', g']_2 = [P_2^- f', P_2^- g']_2.$$

Since $\widetilde{U}$ is unitary, $P_2^- \widetilde{U}(\mathrm{dom}\, \widetilde{U} \cap \mathfrak{K}_2^-) = \widetilde{\mathfrak{K}}_2^-$ and, hence, the previous equality implies that $P_2^- f' \in (\widetilde{\mathfrak{K}}_2^-)^{[\perp]_2} \cap \mathfrak{K}_2^- = \widehat{\mathfrak{K}}_2^-$. Consequently, $\mathrm{ran}\, \widehat{U}_r^+ \subseteq \widehat{\mathfrak{K}}_2$.

Now if $\{g, g'\} \in \widetilde{U}$, where $g \in \mathrm{dom}\, \widetilde{U} \cap \mathfrak{K}_1^+$, then, since $\mathrm{ran}\, \widehat{U}_r^+ \subseteq \widehat{\mathfrak{K}}_2 = \widetilde{\mathfrak{K}}_2^{[\perp]_2}$,

$$[f, g]_1 = [f', g']_2 = 0, \quad \{f, f'\} \in \mathrm{gr}\, \widehat{U}_r^+$$

This shows that $f \in (\mathrm{dom}\, \widetilde{U} \cap \mathfrak{K}_1^+)^{[\perp]_1} = (\widetilde{\mathfrak{K}}_1^+)^{[\perp]_1} = \widehat{\mathfrak{K}}_1^+$, see (3.11).

The above arguments show that $\widehat{U}_r^+ \subseteq \widehat{U}$ and, hence, $P_2^+ \widehat{U}(\mathrm{dom}\, \widehat{U} \cap \mathfrak{K}_1^+) = \widehat{\mathfrak{K}}_2^+$ and $P_2^- \widehat{U}(\mathrm{dom}\, \widehat{U} \cap \mathfrak{K}_1^+) \subseteq \widehat{\mathfrak{K}}_2^-$. By similar arguments $P_2^- \widehat{U}(\mathrm{dom}\, \widehat{U} \cap \mathfrak{K}_1^-) = \widehat{\mathfrak{K}}_2^-$ and $P_2^+ \widehat{U}(\mathrm{dom}\, \widehat{U} \cap \mathfrak{K}_1^-) \subseteq \widehat{\mathfrak{K}}_2^+$. Therefore $\widehat{U}$ is unitary by Proposition 5.14, because the condition (i) therein clearly holds. □

Next a very different application of the Weyl identity approach to unitary relations is presented. Namely this approach is now used to characterize the defect numbers of the pre-images of neutral subspaces under mapping by unitary relations; these results are an extension of Calkin's, cf. (Calkin 1939a: Theorem 4.8, Theorem 4.11 & Theorem 4.12). As a starting point, a simple observation on neutral subspaces contained in the domain of a unitary operator with a trivial kernel is stated.

**Lemma 5.16.** *Let $U$ be a unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\ker U = \{0\}$, let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$, for $i = 1,2$, and let $\mathfrak{L} \subseteq \mathrm{dom}\, U$ be a neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then there exists a subspace $\mathfrak{L}^+ \subseteq \mathrm{ran}\, U \cap \mathfrak{K}_2^+$, an injective mapping $L_+$ from $U^{-1}(\mathfrak{L}^+)$ to $\mathrm{dom}\, U \cap \mathfrak{K}_1^-$, and a subspace $\mathfrak{L}^- \subseteq \mathrm{ran}\, U \cap \mathfrak{K}_2^-$, an injective mapping $L_-$ from $U^{-1}(\mathfrak{L}^-)$ to $\mathrm{dom}\, U \cap \mathfrak{K}_1^+$ such that*

$$\{f_+ + L_+ f_+ : f_+ \in U^{-1}(\mathfrak{L}^+)\} = \mathfrak{L} = \{f_- + L_- f_- : f_- \in U^{-1}(\mathfrak{L}^-)\}. \quad (5.8)$$

*In particular,* $\mathfrak{L}$ *is closed if* $\operatorname{clos}(\mathfrak{L}^+) \cap \operatorname{ran} U = \mathfrak{L}^+$ *or if* $\operatorname{clos}(\mathfrak{L}^-) \cap \operatorname{ran} U = \mathfrak{L}^-$*,* $U(\mathfrak{L})$ *is closed if* $\operatorname{clos}(\operatorname{ran} L_+) \cap \operatorname{dom} U = \operatorname{ran} L_+$ *or if* $\operatorname{clos}(\operatorname{ran} L_-) \cap \operatorname{dom} U = \operatorname{ran} L_-$*. Moreover,*

$$n_+(\mathfrak{L}) = \dim(\mathfrak{K}_1^- \ominus_1 P_1^- U^{-1}(\mathfrak{L}^-)) \quad \textit{and} \quad n_-(\mathfrak{L}) = \dim(\mathfrak{K}_1^+ \ominus_1 P_1^+ U^{-1}(\mathfrak{L}^+)).$$

*Proof.* As a direct consequence of the decompositions of unitary relations in (3.8), the decompositions of $\mathfrak{L}$ in (5.8) hold. Recall that, since $\mathfrak{L}$ is neutral, $\mathfrak{L}$ is closed if and only if either $P_1^+\mathfrak{L}$ is closed or $P_1^-\mathfrak{L}$ is closed. This observation together with (5.8) implies that $\mathfrak{L}$ is closed if and only if either $P_1^+U^{-1}(\mathfrak{L}^+)$ is closed or $P_1^-U^{-1}(\mathfrak{L}^-)$ is closed. Therefore the stated conditions for the closedness of $\mathfrak{L}$ now follows from Proposition 3.9. Moreover, the stated conditions for the closedness of $U(\mathfrak{L})$ can be proven by similar arguments and the assertion about the defect numbers of $\mathfrak{L}$ follows straightforwardly from (5.8), because $L_+$ and $L_-$ map into $\mathfrak{K}_1^-$ and $\mathfrak{K}_1^+$, respectively. □

Combining the decomposition of unitary relations in (3.8) with the concept of angular operators, see Section 2.5, yields the following reformulation of Lemma 5.16 in terms of angular operators.

**Proposition 5.17.** *Let* $U$ *be a unitary relation from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$*, let* $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ *be a canonical decomposition of* $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$*, for* $i = 1, 2$*, and let* $K^+$ *and* $K^-$ *be the angular operators of* $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ *and* $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ *w.r.t.* $\mathfrak{K}_2^+$ *and* $\mathfrak{K}_2^-$*, respectively. Moreover, let* $\mathfrak{N} \subseteq \operatorname{ran} U$ *be a neutral subspace of* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *with angular operator* $K$ *w.r.t.* $\mathfrak{K}_2^+$*. Then the defect numbers of the neutral subspace* $U^{-1}(\mathfrak{N})$ *of* $\{\mathfrak{K}_1, [\cdot,\cdot]\}$ *are*

$$n_+(U^{-1}(\mathfrak{N})) = \dim\left(\mathfrak{K}_1^- \ominus_1 P_1^- U^{-1}(\operatorname{ran}(K^+ - K))\right);$$
$$n_-(U^{-1}(\mathfrak{N})) = \dim\left(\mathfrak{K}_1^+ \ominus_1 P_1^+ U^{-1}(\operatorname{ran}(K^- - K^{-1}))\right).$$

*Furthermore,* $U^{-1}(\mathfrak{N})$ *is closed if* $\overline{\operatorname{ran}}(K^+ - K) \cap \operatorname{ran} U = \operatorname{ran}(K^+ - K)$ *or if* $\overline{\operatorname{ran}}(K^- - K^{-1}) \cap \operatorname{ran} U = \operatorname{ran}(K^- - K^{-1})$*.*

*Proof.* W.l.o.g. assume that $\ker U = \{0\} = \operatorname{mul} U$, then to complete the proof it now suffices to note that if $\mathfrak{L} = U^{-1}(\mathfrak{N})$, then $\operatorname{ran}(K^+ - K) \subseteq \operatorname{ran} U \cap \mathfrak{K}_2^-$ and $\operatorname{ran}(K^- - K^{-1}) \subseteq \operatorname{ran} U \cap \mathfrak{K}_2^+$ correspond to the subspaces $\mathfrak{L}^-$ and $\mathfrak{L}^+$ from Lemma 5.16. □

The assumption $\mathfrak{N} \subseteq \operatorname{ran} U$ in Proposition 5.17 can be dropped if $\operatorname{ran}(K^+ - K)$ and $\operatorname{ran}(K^- - K^{-1})$ are replaced by $\operatorname{ran}(K^+ - K) \cap \operatorname{ran} U$ and $\operatorname{ran}(K^- - K^{-1}) \cap \operatorname{ran} U$, respectively. In particular, Proposition 5.17 yields the following conditions for the inverse image of a neutral subspace under a unitary relation to be maximal neutral.

**Corollary 5.18.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$, for $i = 1, 2$, and let $K^+$ and $K^-$ be the angular operators of $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ and $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ w.r.t. $\mathfrak{K}_2^+$ and $\mathfrak{K}_2^-$, respectively. Moreover, let $\mathfrak{N}$ be a neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with angular operator $K$ w.r.t. $\mathfrak{K}_2^+$. Then equivalent are:*

(i) $n_+(U^{-1}(\mathfrak{N} \cap \operatorname{ran} U)) = 0$ *and* $U^{-1}(\mathfrak{N} \cap \operatorname{ran} U)$ *is closed;*

(ii) $\operatorname{ran} U \cap \mathfrak{K}_2^- \subseteq \operatorname{ran}(K^+ - K)$*;*

(iii) $\operatorname{ran} U = (\mathfrak{N} \cap \operatorname{ran} U) + U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$.

*Similarly, equivalent are:*

(i) $n_-(U^{-1}(\mathfrak{N} \cap \operatorname{ran} U)) = 0$ *and* $U^{-1}(\mathfrak{N} \cap \operatorname{ran} U)$ *is closed;*

(ii) $\operatorname{ran} U \cap \mathfrak{K}_2^+ \subseteq \operatorname{ran}(K^- - K^{-1})$*;*

(iii) $\operatorname{ran} U = (\mathfrak{N} \cap \operatorname{ran} U) + U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$.

*Proof.* W.l.o.g. only the first set of equivalences will be proven. To prove the equivalence of (i) and (ii) recall first that $P_1^- U^{-1}(\mathfrak{M}^-) = \mathfrak{K}_1^-$ for $\mathfrak{M}^- \subseteq \operatorname{ran} U \cap \mathfrak{K}_2^-$ if and only if $\mathfrak{M}^- = \operatorname{ran} U \cap \mathfrak{K}_2^-$, cf. Proposition 5.1. In light of that observation, Proposition 5.17 together with the discussion following that statement show that (i) holds if and only if $\operatorname{ran} U \cap \mathfrak{K}_2^- \subseteq \operatorname{ran}(K^+ - K) \cap \operatorname{ran} U$; this latter condition is, clearly, equivalent to condition (ii). Finally, the equivalence of (ii) and (iii) follows directly from the fact that $\operatorname{ran} U = \operatorname{ran} U \cap \mathfrak{K}_2^- + U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ by (3.8). □

In fact, by means of direct arguments it can be shown that in the equivalences in Corollary 5.18 the assumption that $U$ is unitary is too strong. For instance if $V$ is an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and $\mathfrak{N}$ is a neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, then $n_+(V^{-1}(\mathfrak{N} \cap \operatorname{ran} V)) = 0$ and $V^{-1}(\mathfrak{N} \cap \operatorname{ran} V)$ is closed if and only if $P_1^- \operatorname{dom} V = \mathfrak{K}_1^-$ and $\operatorname{ran} V = (\mathfrak{N} \cap \operatorname{ran} V) + V(\operatorname{dom} V \cap \mathfrak{K}_1^+)$, and that $n_-(V^{-1}(\mathfrak{N} \cap \operatorname{ran} V)) = 0$ and $V^{-1}(\mathfrak{N} \cap \operatorname{ran} V)$ is closed if and only if $P_1^+ \operatorname{dom} V = \mathfrak{K}_1^+$ and $\operatorname{ran} V = (\mathfrak{N} \cap \operatorname{ran} V) + V(\operatorname{dom} V \cap \mathfrak{K}_1^-)$.

# 6 HYPER-MAXIMAL SEMI-DEFINITE SUBSPACES

As a preparation for Chapter 7, where block representation for certain classes of isometric operators are considered, here hyper-maximal semi-definite subspaces contained in the domains of isometric and unitary relations are investigated. More specifically, in the first section consequences of the existence of a hyper-maximal semi-definite subspace in the domain of an isometric relation are presented. Thereafter, in the second section, a graph decomposition of unitary relations is presented. That graph decomposition implies in particular that the domain or, equivalently, the range of a unitary relation always contains a hyper-maximal semi-definite subspace. In the third and final section of this chapter the graph decomposition approach to unitary relations from the first section is combined with the Weyl identity approach to unitary relations from Chapter 5 to obtain more insight into unitary relations.

## 6.1 Isometric relations and hyper-maximal semi-definite subspaces

Here some basic properties that an isometric relation possesses as a consequence of having a hyper-maximal semi-definite subspace in its domain are presented. Since hyper-maximal semi-definite subspaces are closed, a first consequence is that the kernels of those isometric relations are closed. Another connected consequence is contained in the following statement.

**Lemma 6.1.** *Let $V$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\operatorname{mul} V = (\operatorname{ran} V)^{[\perp]_2}$ and assume that there exists a hyper-maximal semi-definite subspace $\mathfrak{L}$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $\mathfrak{L} \subseteq \operatorname{dom} V$. Then $\ker V = (\operatorname{dom} V)^{[\perp]_1}$.*

*Proof.* Recall that $\mathfrak{L}^{[\perp]_1} \subseteq \mathfrak{L}$, because $\mathfrak{L}$ is hyper-maximal semi-definite, see e.g. Proposition 2.9. Hence, the assumption $\mathfrak{L} \subseteq \operatorname{dom} V$ implies that

$$(\operatorname{dom} V)^{[\perp]_1} \subseteq \mathfrak{L}^{[\perp]_1} \subseteq \mathfrak{L} \subseteq \operatorname{dom} V.$$

Consequently, Lemma 3.5 implies that $\ker V = (\operatorname{dom} V)^{[\perp]_1}$. □

Using the second von Neumann formula yields the following statement.

**Corollary 6.2.** *Let $V$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and assume that there exists a hyper-maximal semi-definite subspace $\mathfrak{L}$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $\mathfrak{L} \subseteq \operatorname{dom} V$. Then $\operatorname{clos}(\operatorname{dom} V \cap \mathfrak{K}_1^{\pm}) = \overline{\operatorname{dom}}\, V \cap \mathfrak{K}_1^{\pm}$.*

*Proof.* W.l.o.g. assume that $\mathfrak{L}$ is hyper-maximal neutral, then the statement follows directly from Corollary 2.14 (applied to $\mathfrak{L} = \operatorname{dom} V$ and $\mathfrak{M} = \mathfrak{L}$). □

In particular, if $V$ is an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $\ker V = (\operatorname{dom} V)^{[\perp]_1}$ and $\operatorname{dom} V$ contains a hyper-maximal semi-definite subspaces, then combining the first von Neumann formula (2.4) with Corollary 6.2 yields

$$\overline{\operatorname{dom}}\, V = \ker V \oplus_1 \operatorname{clos}(\operatorname{dom} V \cap \mathfrak{K}_1^{+}) \oplus_1 \operatorname{clos}(\operatorname{dom} V \cap \mathfrak{K}_1^{-}), \tag{6.1}$$

where $\mathfrak{K}_1^{+}[+]\mathfrak{K}_1^{-}$ is a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, cf. (3.10). The above formula together with Lemma 3.8 yields

$$\begin{aligned} \big(V(\operatorname{dom} V \cap \mathfrak{K}_1^{+})\big)^{[\perp]_2} \cap \operatorname{ran} V &= V(\operatorname{dom} V \cap \mathfrak{K}_1^{-}); \\ \big(V(\operatorname{dom} V \cap \mathfrak{K}_1^{-})\big)^{[\perp]_2} \cap \operatorname{ran} V &= V(\operatorname{dom} V \cap \mathfrak{K}_1^{+}). \end{aligned} \tag{6.2}$$

Observe that by Proposition 3.9 $V(\operatorname{dom} V \cap \mathfrak{K}_1^{+})$ and $V(\operatorname{dom} V \cap \mathfrak{K}_1^{-})$ are closed if $V$ is closed in addition to the previous conditions.

In fact, just as for unitary relations, the isometric relations under considerations are characterized by their behavior on the uniformly definite subspaces $\operatorname{dom} V \cap \mathfrak{K}_1^{+}$ and $\operatorname{dom} V \cap \mathfrak{K}_1^{-}$.

**Lemma 6.3.** *Let $V$ be a closed isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ satisfying $\ker V = (\operatorname{dom} V)^{[\perp]_1}$, assume that there exists a hyper-maximal semi-definite subspace $\mathfrak{L}$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $\mathfrak{L} \subseteq \operatorname{dom} U$ and let $\mathfrak{K}_1^{+}[+]\mathfrak{K}_1^{-}$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then $\ker V + \operatorname{dom} V \cap \mathfrak{K}_1^{+} + \operatorname{dom} V \cap \mathfrak{K}_1^{-}$ is a core for $V$.*

*Proof.* W.l.o.g. assume that $\ker V = \{0\} = \operatorname{mul} V$ and that $\mathfrak{L}$ is hyper-maximal neutral. Moreover, let $\mathfrak{j}_1$ be the fundamental symmetry corresponding to the canonical decomposition $\mathfrak{K}_1^{+}[+]\mathfrak{K}_1^{-}$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then $\operatorname{dom} V = \mathfrak{L} \oplus_1 \mathfrak{j}_1\mathfrak{L} \cap \operatorname{dom} V$. Moreover, since $\mathfrak{L}$ is closed and $V$ is a closed relation, it follows that $V$ restricted to $\mathfrak{L}$ is a bounded operator. Therefore the statement follows from the fact that $\operatorname{clos}(\mathfrak{j}_1\mathfrak{L} \cap \operatorname{dom} V) = \mathfrak{j}_1\mathfrak{L}$ and that $(\mathfrak{j}_1\mathfrak{L} \cap \operatorname{dom} V) + \mathfrak{j}_1(\mathfrak{j}_1\mathfrak{L} \cap \operatorname{dom} V) = \operatorname{dom} V \cap \mathfrak{K}_1^{+} + \operatorname{dom} V \cap \mathfrak{K}_1^{-}$. □

## 6.2 A graph decomposition of unitary relations

Here unitary relations are characterized by the fact that they have a special graph decomposition, see Lemma 6.4 and Theorem 6.8 below. This decomposition is the main result of this chapter and it will also play a major role in the next chapter. The decomposition result is based on the fact that unitary relations between Kreĭn spaces are connected to nonnegative selfadjoint operators in Hilbert spaces, see the discussion following Theorem 5.6. Note that Lemma 6.4 below is inspired by Calkin (1939a: Theorem 3.5); the difference is that here the graph of a unitary relation is decomposed whereas in (Calkin 1939a) only the domain of a unitary relation was decomposed.

**Lemma 6.4.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i,[\cdot,\cdot]_i\}$ associated to the fundamental symmetry $\mathrm{j}_i$ of $\{\mathfrak{K}_i,[\cdot,\cdot]_i\}$, for $i=1,2$. Define $U_c$ via*

$$\operatorname{gr} U_c=\{\{f,f'\}\in\operatorname{gr} U:[\mathrm{j}_1 f,g,]_1=[\mathrm{j}_2 f',g']_2,\quad \forall\{g,g'\}\in\operatorname{gr} U\}$$

*Moreover, with $\widetilde{\mathfrak{K}}_1:=\mathfrak{K}_1\cap(\ker U+\mathrm{j}_1\ker U+\operatorname{dom} U_c)^{[\perp]_1}$ and with $\widetilde{\mathfrak{K}}_2:=\mathfrak{K}_2\cap(\operatorname{mul} U+\mathrm{j}_2\operatorname{mul} U+\operatorname{ran} U_c)^{[\perp]_2}$, define $U_o$ via*

$$\operatorname{gr} U_o=\operatorname{gr} U\cap(\widetilde{\mathfrak{K}}_1\times\widetilde{\mathfrak{K}}_2).$$

*Then $U$ has the graph decomposition*

$$\operatorname{gr} U=(\ker U\times\operatorname{mul} U)\dotplus\operatorname{gr} U_c\dotplus\operatorname{gr} U_o,$$

*where*

(i) *$U_c$ is a standard unitary operator from the Kreĭn space $\{\operatorname{dom} U_c,[\cdot,\cdot]_1\}$ to the Kreĭn space $\{\operatorname{ran} U_c,[\cdot,\cdot]_2\}$. Moreover, $\operatorname{gr} U_c=\operatorname{gr} U_+ +\operatorname{gr} U_-$ where $U_+$ and $U_-$ are the Hilbert space unitary operators defined via*

$$\operatorname{gr} U_+=\operatorname{gr} U\cap(\mathfrak{K}_1^+\times\mathfrak{K}_2^+)\quad \textit{and}\quad \operatorname{gr} U_-=\operatorname{gr} U\cap(\mathfrak{K}_1^-\times\mathfrak{K}_2^-)$$

*from $\{\operatorname{dom} U_+,[\cdot,\cdot]_1\}$ onto $\{\operatorname{ran} U_+,[\cdot,\cdot]_2\}$ and from $\{\operatorname{dom} U_-,-[\cdot,\cdot]_1\}$ onto $\{\operatorname{ran} U_-,-[\cdot,\cdot]_2\}$, respectively.*

(ii) *$U_o$ is a unitary operator from the Kreĭn space $\{\widetilde{\mathfrak{K}}_1,[\cdot,\cdot]_1\}$ to the Kreĭn space $\{\widetilde{\mathfrak{K}}_2,[\cdot,\cdot]_2\}$ with dense domain and dense range. Moreover, there exist hypermaximal neutral subspaces $\mathfrak{L}_d\subseteq\operatorname{dom} U_o$ and $\mathfrak{L}_r\subseteq\operatorname{ran} U_o$ of $\{\widetilde{\mathfrak{K}}_1,[\cdot,\cdot]_1\}$ and $\{\widetilde{\mathfrak{K}}_2,[\cdot,\cdot]_2\}$, respectively, such that*

$$U_o(\mathfrak{L}_d)=\mathrm{j}_2\mathfrak{L}_r\cap\operatorname{ran} U_o\quad \textit{and}\quad U_o(\mathrm{j}_1\mathfrak{L}_d\cap\operatorname{dom} U_o)=\mathfrak{L}_r.$$

*In particular,*

$$\operatorname{dom} U_o = \mathfrak{L}_d \oplus_1 (\mathrm{j}_1\mathfrak{L}_d \cap \operatorname{dom} U_o) \quad \textit{and} \quad \operatorname{ran} U_o = \mathfrak{L}_r \oplus_2 (\mathrm{j}_2\mathfrak{L}_r \cap \operatorname{ran} U_o).$$

*Proof.* Note first that the stated graph decomposition of $U$ is a consequence of (i) and the fact that $\ker U \times \operatorname{mul} U$ is a unitary relation from the Kreĭn space $\{\ker U + \mathrm{j}_1 \ker U, [\cdot,\cdot]_1\}$ to the Kreĭn space $\{\operatorname{mul} U + \mathrm{j}_2 \operatorname{mul} U, [\cdot,\cdot]_2\}$, see e.g. Corollary 4.4.

(i), (ii): Define $\mathfrak{K}_{1,r} = \mathfrak{K}_1 \cap (\ker U + \mathrm{j}_1\ker U)^{[\perp]_1}$ and $\mathfrak{K}_{2,r} = \mathfrak{K}_2 \cap (\operatorname{mul} U + \mathrm{j}_2\operatorname{mul} U)^{[\perp]_2}$. Then Lemma 3.13 implies that $U_r$ defined via

$$\operatorname{gr} U_r = \operatorname{gr} U \cap (\mathfrak{K}_{1,r} \times \mathfrak{K}_{2,r})$$

is a unitary operator with a trivial kernel from the Kreĭn space $\{\mathfrak{K}_{1,r}, [\cdot,\cdot]_1\}$ to the Kreĭn space $\{\mathfrak{K}_{2,r}, [\cdot,\cdot]_2\}$. By Theorem 5.6 (applied to $U_r^{-1}$), see also the discussion following that statement, there exists a standard unitary operator $U_t$ from $\{\mathfrak{K}_{1,r}, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_{2,r}, [\cdot,\cdot]_2\}$, satisfying $U_t\mathrm{j}_1 = \mathrm{j}_2U_t$, such that $U_a := U_t^{-1}U_r$ is a unitary operator (without kernel) in $\{\mathfrak{K}_{1,r}, [\cdot,\cdot]_1\}$ which is additionally a nonnegative selfadjoint operator in (the Hilbert space) $\{\mathfrak{K}_{1,r}, [\mathrm{j}_1\cdot,\cdot]_1\}$.

Now let $\{E_t\}_{t\in\mathbb{R}}$ and $\{F_t\}_{t\in\mathbb{R}}$ be the spectral families of the nonnegative selfadjoint operators $U_a$ and $U_a^{-1}$ in (the Hilbert space) $\{\mathfrak{K}_{1,r}, [\mathrm{j}_1\cdot,\cdot]_1\}$, respectively, then $F_t = I - E_{(1/t)-}$ for $t > 0$. Moreover, $\mathfrak{L}_d := \operatorname{ran} E_{1-}$, $\mathfrak{M}_r := \operatorname{ran} F_{1-}$ and $\mathfrak{N}_d := \ker(U_a - I) = \operatorname{ran}(E_1 - E_{1-})$ are closed subspaces of $\{\mathfrak{K}_{1,r}, [\mathrm{j}_1\cdot,\cdot]_1\}$ such that

$$\operatorname{dom} U_a = \mathfrak{L}_d \oplus_1 \mathfrak{N}_d \oplus_1 U_a^{-1}(\mathfrak{M}_r) \quad \text{and} \quad \operatorname{ran} U_a = \mathfrak{M}_r \oplus_1 \mathfrak{N}_d \oplus_1 U_a(\mathfrak{L}_d). \tag{6.3}$$

Next note that $U_a^{-1} = \mathrm{j}_1U_a\mathrm{j}_1$, because $U_a$ is a selfadjoint operator in (the Hilbert space) $\{\mathfrak{K}_{1,r}, [\mathrm{j}_1\cdot,\cdot]_1\}$ and a unitary operator in $\{\mathfrak{K}_{1,r}, [\cdot,\cdot]_1\}$. The preceding equality together the before mentioned connection between the spectral measures of $U_a$ and $U_a^{-1}$, implies that

$$I - E_{(1/t)-} = \mathrm{j}_1E_t\mathrm{j}_1, \quad t > 0. \tag{6.4}$$

In particular, (6.4) yields $E_1 - E_{1-} = \mathrm{j}_1(E_1 - E_{1-})\mathrm{j}_1$. This implies that $\mathfrak{N}_d = \mathrm{j}_1\mathfrak{N}_d$ and, hence, $\{\mathfrak{N}_d, [\cdot,\cdot]_1\}$ is a Kreĭn space because $\mathfrak{N}_d$ is by definition closed. From (6.4) it also follows that $\mathrm{j}_1\operatorname{ran}(I - E_1) = \operatorname{ran}(E_{1-}\mathrm{j}_1) = \mathfrak{L}_d$. Since $\operatorname{ran}(I - E_1) \cap \operatorname{dom} U_a = U_a^{-1}(\mathfrak{M}_r)$, this implies that $U_a^{-1}(\mathfrak{M}_r) = \mathrm{j}_1\mathfrak{L}_d \cap \operatorname{dom} U_a$ and also that $\operatorname{clos}(\mathrm{j}_1\mathfrak{L}_d \cap \operatorname{dom} U_a) = \mathrm{j}_1\mathfrak{L}_d$. Consequently, (6.3) implies that

$$\mathfrak{L}_d^{[\perp]_1} = \mathrm{j}_1\mathfrak{L}_d^{\perp} = \mathrm{j}_1(\mathfrak{N}_d \oplus \operatorname{clos}(\mathrm{j}_1\mathfrak{L}_d \cap \operatorname{dom} U_a)) = \mathfrak{N}_d \oplus \mathfrak{L}_d.$$

The above formula implies that $\mathfrak{L}_d$ is a hyper-maximal neutral subspace of (the Kreĭn space) $\{\widetilde{\mathfrak{K}}_1, [\cdot,\cdot]_1\} := \{\overline{\operatorname{dom}} U_a \ominus_1 \mathfrak{N}_d, [\cdot,\cdot]_1\}$.

Similar arguments as above yield $U_a(\mathfrak{L}_d) = \mathrm{j}_1\mathfrak{M}_r \cap \operatorname{ran} U_a$ and that $\mathfrak{M}_r$ is a hyper-maximal neutral subspace in the Kreĭn space $\{\overline{\operatorname{ran}}\, U_a \ominus_1 \mathfrak{N}_d, [\cdot,\cdot]_1\}$. Hence, $\mathfrak{L}_r = U_t(\mathfrak{M}_r) = U_r(\mathrm{j}_1\mathfrak{L}_r \cap \operatorname{dom} U_r)$ is a hyper-maximal neutral subspace in $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\} := \{U_t(\overline{\operatorname{ran}}\, U_a \ominus_1 \mathfrak{N}_d), [\cdot,\cdot]_2\}$. Therefore, if $U_o$ and $U_c$ are defined via

$$\operatorname{gr} U_o = \operatorname{gr} U_r \cap (\widetilde{\mathfrak{K}}_1 \times \widetilde{\mathfrak{K}}_2) \quad \text{and} \quad \operatorname{gr} U_c = \operatorname{gr} U_r \cap (\mathfrak{N}_d \times U_t(\mathfrak{N}_d)),$$

then $\operatorname{gr} U_o + \operatorname{gr} U_c = \operatorname{gr} U_r$. Consequently, Lemma 3.13 shows that $U_o$ and $U_c$ are a unitary operator with a trivial kernel and a standard unitary operator, respectively. Moreover, the above arguments together with $\mathrm{j}_2 U_t = U_t \mathrm{j}_1$ show that (ii) holds with $\mathfrak{L}_d$ and $\mathfrak{L}_r$ as above. Finally, from the fact that $\mathfrak{N}_d = \mathrm{j}_1\mathfrak{N}_d$ and $\mathrm{j}_2 U_t = U_t \mathrm{j}_1$, it follows that the decomposition for $U_c$ as in (i) holds. □

Since the unitary relations $\ker U \times \operatorname{mul} U$ and $U_c$ are easily understood, Lemma 6.4 shows that, from a theoretical point of view, the most interesting unitary relations are those with dense domain and range in a Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$ with $k^+ = k^-$. In other words, to understand unitary relations it suffices for instance to consider only the unitary operators with a trivial kernel from Lemma 5.5. Lemma 6.4 also shows that if $U$ is a unitary relation such that $\ker U$ does not have equal defect numbers, then there exist uniformly definite subspaces $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $U(\mathfrak{D}_1) = \mathfrak{D}_2 + \operatorname{mul} U$ and $\widetilde{U}$ defined via $\operatorname{gr} \widetilde{U} = \operatorname{gr} U \cap (\mathfrak{D}_1^{[\perp]_1} \times \mathfrak{D}_2^{[\perp]_2})$ is a unitary relation from $\{\mathfrak{K}_1 \cap \mathfrak{D}_1^{[\perp]_1}, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2 \cap \mathfrak{D}_2^{[\perp]_2}, [\cdot,\cdot]_2\}$, see Corollary 3.14 whose kernel (and multi-valued part) has equal defect numbers.

From the graph decomposition of a unitary relation $U$ in Lemma 6.4 it follows, with the notation as in that statement, that

$$\begin{aligned} n_+(\ker U) &= \dim(\operatorname{dom} U_-) + \widetilde{k}_1^-, \quad & n_-(\ker U) &= \dim(\operatorname{dom} U_+) + \widetilde{k}_1^+; \\ n_+(\operatorname{mul} U) &= \dim(\operatorname{ran} U_-) + \widetilde{k}_2^-, \quad & n_-(\operatorname{mul} U) &= \dim(\operatorname{ran} U_+) + \widetilde{k}_2^+, \end{aligned} \tag{6.5}$$

where $\widetilde{k}_i^+$ and $\widetilde{k}_i^-$ are the dimensions of $\widetilde{\mathfrak{K}}_i^+$ and $\widetilde{\mathfrak{K}}_i^-$ for any canonical decomposition $\widetilde{\mathfrak{K}}_i^+[+]\widetilde{\mathfrak{K}}_i^-$ of $\{\widetilde{\mathfrak{K}}_i, [\cdot,\cdot]_i\}$, $i = 1,2$. Since $\dim(\operatorname{dom} U_\pm) = \dim(\operatorname{ran} U_\pm)$, cf. Proposition 4.5, and $\widetilde{k}_1^\pm = \widetilde{k}_2^\pm$ by Lemma 6.4 (ii), (6.5) shows that the defect numbers of the kernel and multi-valued part of a unitary operator $U$ are equal, cf. (Derkach et al. 2006: Lemma 2.14 (iii)).

**Corollary 6.5.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then*

$$n_+(\ker U) = n_+(\operatorname{mul} U) \quad \textit{and} \quad n_-(\ker U) = n_-(\operatorname{mul} U).$$

Next let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ with associated fundamental symmetry $\mathrm{j}_i$, for $i=1,2$. Then define $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2)$ and $d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)$ as

$$\begin{aligned} d_U^+(\mathfrak{j}_1,\mathfrak{j}_2)&=\dim\{f\in\mathfrak{K}_1^+ : \exists f'\in\mathfrak{K}_2^+ \text{ s.t. } \{f,f'\}\in\operatorname{gr}U\};\\ d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)&=\dim\{f\in\mathfrak{K}_1^- : \exists f'\in\mathfrak{K}_2^- \text{ s.t. } \{f,f'\}\in\operatorname{gr}U\}.\end{aligned} \tag{6.6}$$

I.e., with the notation as in Lemma 6.4, $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2)=\dim(\operatorname{dom}U_+)$ and $d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)=\dim(\operatorname{dom}U_-)$. Since $\widetilde{k}_1^-=\widetilde{k}_1^+$, (6.5) implies that if $n_-(\ker U)>n_+(\ker U)$ or $n_-(\ker U)<n_+(\ker U)$, then $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2)>d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)$ or $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2)<d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)$ for all $\mathfrak{j}_1$ and $\mathfrak{j}_2$, respectively. If $n_-(\ker U)=n_+(\ker U)$, then $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2)$ and $d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)$ can be ordered in an arbitrary manner, and differently for different fundamental symmetries $\mathfrak{j}_1$ and $\mathfrak{j}_2$ as Example 6.6 below shows.

**Example 6.6.** Let $U$ be a standard unitary operator from the separable (infinite-dimensional) Kreĭn space $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ to the separable (infinite-dimensional) Kreĭn space $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ such that $n_+(\ker U)=n_-(\ker U)$, i.e. $k_1^+=k_1^-=k_2^+=k_2^-$. If $\mathfrak{j}_1$ is a fundamental symmetry of $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$, then $\mathfrak{j}_2:=U\mathfrak{j}_1U^{-1}$ is a fundamental symmetry of $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$, see Lemma 4.12. With $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ the canonical decomposition of $\{\mathfrak{K}_i,[\cdot,\cdot]_i\}$ associated with $\mathfrak{j}_i$, for $i=1,2$, as a consequence of the above construction $U(\mathfrak{K}_1^\pm)=\mathfrak{K}_2^\pm$. Consequently, $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2)=k_1^+=k_1^-=d_U^-(\mathfrak{j}_2,\mathfrak{j}_2)$.

Next let $K$ be a uniform contraction from the Hilbert space $\{\mathfrak{K}_2^+,[\cdot,\cdot]_2\}$ to the Hilbert space $\{\mathfrak{K}_2^-,[\cdot,\cdot]_2\}$ with an $n$-dimensional kernel, $n\in\mathbb{N}$, such that $(\operatorname{ran}K)^\perp$ is infinite-dimensional. By means of $K$ define $\mathfrak{D}^+$ and $\mathfrak{D}^-$ as

$$\mathfrak{D}^+=\{f^++Kf^+ : f^+\in\mathfrak{K}_2^+\}\quad\text{and}\quad\mathfrak{D}^-=\{f^-+K^*f^- : f^-\in\mathfrak{K}_2^-\}.$$

Then $\mathfrak{D}^+$ and $\mathfrak{D}^-$ are a maximal uniformly positive and maximal uniformly negative subspace of $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ which are orthogonal. I.e., $\mathfrak{D}^+[+]\mathfrak{D}^-$ is a canonical decomposition of $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$. If $\mathfrak{j}_d$ is the corresponding fundamental symmetry, then by construction $d_U^+(\mathfrak{j}_1,\mathfrak{j}_d)=\dim(\ker K)=n$ and $d_U^-(\mathfrak{j}_1,\mathfrak{j}_d)=\dim(\ker K^*)=\dim(\operatorname{ran}K)^\perp=\infty\neq d_U^+(\mathfrak{j}_1,\mathfrak{j}_d)$.

If there exist $\mathfrak{j}_1$ and $\mathfrak{j}_2$ such that $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2)=d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)$, $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2)>d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)$ or $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2)<d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)$, then Lemma 6.4 implies that there exist hyper-maximal semi-definite subspaces in the domain and range of $U$ which are neutral, nonnegative or nonpositive, respectively; cf. (Calkin 1939a: Theorem 4.3 & Theorem 4.4). Importantly, those subspaces can be chosen to have more properties.

**Proposition 6.7.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ and let $\mathfrak{j}_i$ be a fundamental symmetry of $\{\mathfrak{K}_i,[\cdot,\cdot]_i\}$, for $i=1,2$. Then there exist hyper-maximal semi-definite subspaces $\mathfrak{L}\subseteq\operatorname{dom}U$ and $\mathfrak{M}\subseteq\operatorname{ran}U$ of $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$, respectively, such that*

$$\begin{aligned}\operatorname{dom}U&=\mathfrak{L}^{[\perp]_1}\oplus_1(\mathfrak{L}\cap\mathfrak{j}_1\mathfrak{L})\oplus_1(\mathfrak{j}_1\mathfrak{L}^{[\perp]_1}\cap\operatorname{dom}U);\\ \operatorname{ran}U&=\mathfrak{M}^{[\perp]_2}\oplus_2(\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M})\oplus_2(\mathfrak{j}_2\mathfrak{M}^{[\perp]_2}\cap\operatorname{ran}U),\end{aligned}$$

*where*

$$U(\mathfrak{L}^{[\perp]_1}) = \mathfrak{j}_2\mathfrak{M}^{[\perp]_2} \cap \operatorname{ran} U + \operatorname{mul} U;$$
$$U(\mathfrak{L} \cap \mathfrak{j}_1\mathfrak{L}) = \mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M} + \operatorname{mul} U;$$
$$U(\mathfrak{j}_1\mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} U) = \mathfrak{M}^{[\perp]_2} + \operatorname{mul} U.$$

*Here $\mathfrak{L}$ and $\mathfrak{M}$ can be taken to be hyper-maximal neutral, nonnegative or nonpositive subspaces of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ if $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2) = d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)$, $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2) > d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)$ or $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2) < d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)$, respectively.*

*Proof.* Using the notation of Lemma 6.4, recall first that the domain of the standard unitary operator $U_c$ is a Kreĭn space. Hence, there exists a hyper-maximal semi-definite subspace $\mathfrak{L}_c$ in $\{\operatorname{dom} U_c, [\cdot,\cdot]_1\}$, which can be taken to be neutral, nonnegative or nonpositive if $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2) = d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)$, $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2) > d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)$ or $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2) < d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)$, respectively, see the discussion following (6.6). Since $U_c$ is a standard unitary operator, $U_c(\mathfrak{L}_c)$ is a hyper-maximal semi-definite subspace in $\{\operatorname{ran} U_c, [\cdot,\cdot]_2\}$, see Proposition 4.5. Hence, using the fact that $U_c\mathfrak{j}_1 = \mathfrak{j}_2 U_c$,

$$\begin{aligned} \operatorname{dom} U_c &= \mathfrak{L}_c^{[\perp]_1} \oplus_1 (\mathfrak{L}_c \cap \mathfrak{j}_1\mathfrak{L}_c) \oplus_1 \mathfrak{j}_1\mathfrak{L}_c^{[\perp]_1}; \\ \operatorname{ran} U_c &= U_c(\mathfrak{j}_1\mathfrak{L}_c^{[\perp]_1}) \oplus_2 U_c(\mathfrak{L}_c \cap \mathfrak{j}_1\mathfrak{L}_c) \oplus_2 U_c(\mathfrak{L}_c^{[\perp]_1}), \end{aligned} \tag{6.7}$$

cf. Proposition 2.9 (iv). (Note that the orthogonal complement of $\mathfrak{L}_c = \mathfrak{L}_c^{[\perp]_1} \oplus_1 (\mathfrak{L}_c \cap \mathfrak{j}_1\mathfrak{L}_c)$ in the above equations is taken in $\{\operatorname{dom} U_c, [\cdot,\cdot]_1\}$.) With the above observation, the asserted decomposition of the domain and range of $U$ follows from (6.7) together with Lemma 6.4 (ii). Specifically, with $\mathfrak{L}_d$ and $\mathfrak{L}_r$ as in Lemma 6.4, $\mathfrak{L}$ and $\mathfrak{M}$ can be taken to be $\ker U + \mathfrak{L}_c + \mathfrak{L}_d$ and $U(\mathfrak{j}_1\mathfrak{L}_c) + \mathfrak{L}_r$, respectively. □

The hyper-maximal semi-definite subspace $\mathfrak{L}$ in Proposition 6.7 is shown to exist as an extension of the subspace $\mathfrak{L}_d$ as in Lemma 6.4. Not all hyper-maximal semi-definite subspaces contained in the domain of a unbounded unitary relation can be obtained in that manner. In view of Proposition 6.9 below, this follows for instance from the fact that every unitary operator has a hyper-maximal semi-definite subspace in its domain which it maps onto a hyper-maximal semi-definite subspace, see Corollary 7.25 below.

Combining Proposition 6.7 with Lemma 4.7 yields the following necessary and sufficient conditions for an isometric operator to be unitary are presented. In particular, they show that if an isometric relation has a graph decomposition as in Lemma 6.4, then it must be a unitary relation.

**Theorem 6.8.** *Let $U$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then $U$ is a unitary relation if and only if there exists a hyper-maximal semi-definite* *subspace* $\mathfrak{L} \subseteq \operatorname{dom} U$ *of* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *and a fundamental symmetry* $\mathrm{j}_1$ *of* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *such that* $U(\mathrm{j}_1\mathfrak{L} \cap \operatorname{dom} U)$ *is a hyper-maximal semi-definite subspace of* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$.

*Proof.* The existence of a subspace $\mathfrak{L}$ with the asserted conditions follows from Proposition 6.7. The sufficiency of the conditions in the case that $\mathfrak{L}$ is hyper-maximal neutral is the contents of Lemma 4.7 and the general case follows by arguments similar to those in Lemma 4.7. □

Finally, some special properties of the subspace $\mathfrak{L}_d$ in Lemma 6.4 are listed.

**Proposition 6.9.** *Let* $U$ *be a unitary relation from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *and let* $\mathfrak{L}_d$ *be the closed neutral subspace as in Lemma 6.4 for fixed fundamental symmetries* $\mathrm{j}_1$ *and* $\mathrm{j}_2$ *of* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *and* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$*, respectively. Then*

(i) $U$ *has closed domain if and only if* $U$ *maps some (any hence every) closed neutral subspace* $\mathfrak{L}$ *of* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *which extends* $\mathfrak{L}_d$ *onto a closed neutral subspace of* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$*;*

(ii) $\mathfrak{L} := \ker U + \mathfrak{L}_d$ *is such that* $\ker U \subseteq \mathfrak{L} \subseteq \mathfrak{L}^{[\perp]_1} \subseteq \operatorname{dom} U$*;*

(iii) *if* $\mathfrak{L}$ *is a neutral subspace of* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *such that* $\ker U \subseteq \mathfrak{L}$ *and* $\mathfrak{L}_d \subseteq \mathfrak{L}$ *or* $\mathrm{j}_1\mathfrak{L}_d \cap \operatorname{dom} U \subseteq \mathfrak{L}$*, then* $n_+(\mathfrak{L}) = n_+(U(\mathfrak{L}))$ *and* $n_-(\mathfrak{L}) = n_-(U(\mathfrak{L}))$.

*Proof.* In this proof the notation as in Lemma 6.4 is used.

(i): By Lemma 6.4 a closed neutral extension of $\mathfrak{L}_d$ can be written as $\ker U \oplus_1 \mathfrak{L}_d \oplus_1 \mathfrak{N}_1$, where $\mathfrak{N}_1 \subseteq \operatorname{dom} U_c$ is closed. It is mapped onto $\operatorname{mul} U \oplus_2 (\mathrm{j}_2\mathfrak{L}_r \cap \operatorname{dom} U) \oplus_2 \mathfrak{N}_2$, where $\mathfrak{N}_2 \subseteq \operatorname{ran} U_c$ is closed because $U_c$ is a standard unitary operator (in the appropriate space). Consequently, $U(\mathfrak{L})$ is closed if and only if $\mathrm{j}_2\mathfrak{L}_r \cap \operatorname{dom} U$ is closed, which by Lemma 6.4 is the case if and only if $\operatorname{ran} U_o$ is closed. Since $\operatorname{ran} U_o$ and $\operatorname{ran} U$ are simultaneously closed, this proves (i), see Proposition 4.2.

(ii): Since $\mathfrak{L}_d$ is hyper-maximal neutral in $\{\widetilde{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$, Lemma 6.4 implies that $(\mathfrak{L}_d)^{[\perp]_1} = \mathfrak{L}_d + \operatorname{dom} U_c + \ker U \subseteq \operatorname{dom} U$.

(iii): Only the case that $\mathfrak{L}_d \subseteq \mathfrak{L}$ is considered, the other case follows by similar arguments. If $\mathfrak{L}_d \subseteq \mathfrak{L}$, then note that the defect numbers of $\ker U + \mathfrak{L}_d$ and $U(\mathfrak{L}_d)$ coincide (since $\operatorname{clos}(\mathrm{j}_2\mathfrak{L}_r \cap \operatorname{ran} U)$ is hyper-maximal neutral in $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$). Now the desired conclusion is obtained by combining the preceding observation with with Proposition 4.5 and Lemma 3.13. □

## 6.3 Hyper-maximal semi-definite subspaces and the Weyl identity

The graph decomposition characterization of unitary operators, as expressed by Proposition 6.7, is combined with the Weyl identity approach to unitary relations from Chapter 5 in order to obtain conditions for the closure of an isometric relation to be unitary. Therefore recall that if $\mathfrak{L}$ is a hyper-maximal semi-definite subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$, then, for any fundamental symmetry $\mathrm{j}$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$, $\mathfrak{K}$ can be decomposed as $\mathfrak{K} = \mathfrak{L}^{[\perp]} \oplus (\mathfrak{L} \cap \mathrm{j}\mathfrak{L}) \oplus \mathrm{j}\mathfrak{L}^{[\perp]}$, see Proposition 2.9 (iv). In this connection $\mathcal{P}_{\mathfrak{L}^{[\perp]}}$ and $\mathcal{P}_{\mathrm{j}\mathfrak{L}^{[\perp]}}$ denote the orthogonal projections in $\mathfrak{K}$ w.r.t. $[\mathrm{j}\cdot,\cdot]$ onto $\mathfrak{L}^{[\perp]}$ and $\mathrm{j}\mathfrak{L}^{[\perp]}$, respectively.

As a starting point, some properties of the subspace $\mathfrak{M}$ from Proposition 6.7 are listed; the implications of these properties are investigated in this section.

**Lemma 6.10.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ associated to fundamental symmetry $\mathrm{j}_i$ of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$, for $i = 1,2$. Then there exists a subspace $\mathfrak{M}$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that*

(i) $\mathfrak{M} \subseteq \operatorname{ran} U$ *is a hyper-maximal semi-definite subspace of* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$*;*

(ii) $U^{-1}(\mathfrak{M} \cap \mathrm{j}_2\mathfrak{M}) \subseteq \ker U + \mathfrak{K}_1^+$ *or* $U^{-1}(\mathfrak{M} \cap \mathrm{j}_2\mathfrak{M}) \subseteq \ker U + \mathfrak{K}_1^-$*;*

(iii) $P_1^+ U^{-1}(\mathfrak{M}) = P_1^+ \ker U + \operatorname{dom} U \cap \mathfrak{K}_1^+$ *and* $P_1^- U^{-1}(\mathfrak{M}) = P_1^- \ker U + \operatorname{dom} U \cap \mathfrak{K}_1^-$*;*

(iv) $\mathfrak{N} := (U^{-1}(\mathrm{j}_2\mathfrak{M} \cap \operatorname{ran} U)) \cap (\ker U + \operatorname{dom} U \cap \mathfrak{K}_1^+ + \operatorname{dom} U \cap \mathfrak{K}_1^-)$ *is such that*

$$P_1^+\mathfrak{N} = P_1^+ \ker U + \operatorname{dom} U \cap \mathfrak{K}_1^+ \quad \textit{and} \quad P_1^-\mathfrak{N} = P_1^- \ker U + \operatorname{dom} U \cap \mathfrak{K}_1^-.$$

*Proof.* W.l.o.g. assume that $\mathfrak{M}$ as in Proposition 6.7 is hyper-maximal neutral. Then, clearly, $\mathfrak{M}$ satisfies (i) and (ii). Next note that $\mathfrak{L} := U^{-1}(\mathrm{j}_2\mathfrak{M} \cap \operatorname{ran} U)$ and the closure of $U^{-1}(\mathfrak{M}) = \ker U + \mathrm{j}_1\mathfrak{L} \cap \operatorname{dom} U$ are hyper-maximal neutral subspaces of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, see Proposition 6.7. The fact that $\mathfrak{L}$ is hyper-maximal neutral yields

$$\mathrm{j}_1(\mathrm{j}_1\mathfrak{L} \cap \operatorname{dom} U) + \mathrm{j}_1\mathfrak{L} \cap \operatorname{dom} U = \operatorname{dom} U \cap \mathfrak{K}_1^+ + \operatorname{dom} U \cap \mathfrak{K}_1^-.$$

Hence, the fact that (iii) and (iv) hold, follows from the preceding observations. □

Corollary 6.11 below shows that the properties (i) and (iii) of $\mathfrak{M}$ in Lemma 6.10 can be alternatively expressed by two equalities.

**Corollary 6.11.** *Let $V$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, and let $\mathfrak{M} \subseteq \operatorname{ran} V$ be a subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then*

$$\begin{aligned} P_1^+V^{-1}(\mathfrak{M}) &= P_1^+\ker V + \operatorname{dom} V \cap \mathfrak{K}_1^+; \\ P_1^-V^{-1}(\mathfrak{M}) &= P_1^-\ker V + \operatorname{dom} V \cap \mathfrak{K}_1^- \end{aligned} \tag{6.8}$$

*if and only if*

$$\begin{aligned} \mathfrak{M} + V(\operatorname{dom} V \cap \mathfrak{K}_1^+) &= V(\operatorname{dom} V \cap \mathfrak{K}_1^+) + V(\operatorname{dom} V \cap \mathfrak{K}_1^-); \\ \mathfrak{M} + V(\operatorname{dom} V \cap \mathfrak{K}_1^-) &= V(\operatorname{dom} V \cap \mathfrak{K}_1^+) + V(\operatorname{dom} V \cap \mathfrak{K}_1^-). \end{aligned} \tag{6.9}$$

*Proof.* If (6.8) holds, then for every $f^+ \in \operatorname{dom} V \cap \mathfrak{K}_1^+$ and $f^- \in \operatorname{dom} V \cap \mathfrak{K}_1^-$ there exists a $g^- \in \operatorname{dom} V \cap \mathfrak{K}_1^-$ such that $f^+ + g^- \in V^{-1}(\mathfrak{M})$. I.e. $f^+ + f^- = (f^+ + g^-) + (f^- - g^-)$, where $f^+ + g^- \in V^{-1}(\mathfrak{M})$ and $f^- - g^- \in \operatorname{dom} V \cap \mathfrak{K}_1^-$. This shows that $V(\operatorname{dom} V \cap \mathfrak{K}_1^+) + V(\operatorname{dom} V \cap \mathfrak{K}_1^-) \subseteq \mathfrak{M} + V(\operatorname{dom} V \cap \mathfrak{K}_1^-)$. On the other hand, if $f \in V^{-1}(\mathfrak{M})$, then by the assumptions there exists an $f_o \in \ker V$, an $f^+ \in \operatorname{dom} V \cap \mathfrak{K}_1^+$ and an $f^- \in \operatorname{dom} V \cap \mathfrak{K}_1^-$ such that $f = f_o + f^+ + f^-$. From this it follows that $\mathfrak{M} + V(\operatorname{dom} V \cap \mathfrak{K}_1^-) \subseteq V(\operatorname{dom} V \cap \mathfrak{K}_1^+) + V(\operatorname{dom} V \cap \mathfrak{K}_1^-)$. By similar arguments the second equality in (6.9) can be proven.

To prove the converse implication let $f^- \in \operatorname{dom} V \cap \mathfrak{K}_1^-$, then by the first equality in (6.9) there exists an $f^+ \in \operatorname{dom} V \cap \mathfrak{K}_1^+$ and an $f' \in \mathfrak{M}$ such that $V^{-1}f' + f^+ = f^- + \ker V$. Since $f^- \in \operatorname{dom} V \cap \mathfrak{K}_1^-$ was arbitrary, this implies that the first equality in (6.8) holds. Similar arguments show that the second equality in (6.8) holds. □

In particular, Corollary 6.11 shows that if $\mathfrak{M}$ is a hyper-maximal semi-definite subspace such that (6.9) holds, then $\operatorname{clos}(V^{-1}(\mathfrak{M}))$ is a hyper-maximal semi-definite subspace if and only if $\mathfrak{K}_1^{\pm} = \operatorname{clos}(P_1^{\pm}\ker V + \operatorname{dom} V \cap \mathfrak{K}_1^{\pm})$, cf. (3.11). In geometrical terminology the observation contained in Corollary 6.11 can be formulated as follows.

**Proposition 6.12.** *For every maximal nonnegative or nonpositive subspace $\mathfrak{M}$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$ there exists a hyper-maximal semi-definite subspace $\mathfrak{L}$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that*

$$\mathfrak{L} + \mathfrak{M} = \mathfrak{M} + \mathfrak{M}^{[\perp]} = \mathfrak{L} + \mathfrak{M}^{[\perp]}.$$

*Proof.* W.l.o.g. assume that $\mathfrak{M}$ is nonnegative and let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then by Theorem 5.8 there exists a unitary relation $U$ in $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that $U(\operatorname{dom} U \cap \mathfrak{K}^+) = \mathfrak{M}$ and $U(\operatorname{dom} U \cap \mathfrak{K}^-) = \mathfrak{M}^{[\perp]}$. Consequently, the statement follows from Lemma 6.10 and Corollary 6.11. □

Continuing the investigation of the properties of $\mathfrak{M}$ listed in Lemma 6.10, an alternative characterization of the properties (ii) and (iv) is given.

**Lemma 6.13.** *Let $V$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and let $\mathrm{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Moreover, let $\mathfrak{M}$ be a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $V^{-1}(\mathfrak{M} \cap \mathrm{j}_2\mathfrak{M}) \subseteq \ker V + \mathfrak{K}_1^+$ or $V^{-1}(\mathfrak{M} \cap \mathrm{j}_2\mathfrak{M}) \subseteq \ker V + \mathfrak{K}_1^-$ and let $\mathfrak{N} := V^{-1}(\mathrm{j}_2\mathfrak{M} \cap \operatorname{ran} V) \cap (\operatorname{dom} V \cap \mathfrak{K}_1^+ + \operatorname{dom} V \cap \mathfrak{K}_1^-)$. Then*

$$P_1^+\mathfrak{N} = \operatorname{dom} V \cap \mathfrak{K}_1^+ \iff \mathcal{P}_{\mathfrak{M}^{[\perp]_2}} V(\operatorname{dom} V \cap \mathfrak{K}_1^+) \subseteq \mathcal{P}_{\mathfrak{M}^{[\perp]_2}} V(\operatorname{dom} V \cap \mathfrak{K}_1^-);$$
$$P_1^-\mathfrak{N} = \operatorname{dom} V \cap \mathfrak{K}_1^- \iff \mathcal{P}_{\mathfrak{M}^{[\perp]_2}} V(\operatorname{dom} V \cap \mathfrak{K}_1^-) \subseteq \mathcal{P}_{\mathfrak{M}^{[\perp]_2}} V(\operatorname{dom} V \cap \mathfrak{K}_1^+).$$

*Proof.* As a consequence of the assumption that $V^{-1}(\mathfrak{M} \cap \mathrm{j}_2\mathfrak{M}) \subseteq \ker V + \mathfrak{K}_1^+$ or $V^{-1}(\mathfrak{M} \cap \mathrm{j}_2\mathfrak{M}) \subseteq \ker V + \mathfrak{K}_1^-$, assume w.l.o.g. that $\mathfrak{M}$ is a hyper-maximal neutral subspace. Since both equivalences are of a similar nature, only the first equivalence will be proven. Hence assume that $P_1^+\mathfrak{N} = \operatorname{dom} V \cap \mathfrak{K}_1^+$ and let $f_1^+ \in \operatorname{dom} V \cap \mathfrak{K}_1^+$. Then by the assumption there exists $\{f, f'\} \in \operatorname{gr} V$ such that $f \in \mathfrak{N}$, $P_1^+ f = f_1^+$ and $f' \in \mathrm{j}_2\mathfrak{M} \cap \operatorname{ran} V$. Then $\mathcal{P}_{\mathfrak{M}} f' = 0$ and, hence, one has shown that

$$\mathcal{P}_{\mathfrak{M}} V(\operatorname{dom} V \cap \mathfrak{K}_1^+) \subseteq \mathcal{P}_{\mathfrak{M}} V(\operatorname{dom} V \cap \mathfrak{K}_1^-).$$

Conversely, if the above inclusion holds, then for every $f^+ \in \operatorname{dom} V \cap \mathfrak{K}_1^+$, there exists an $f^- \in \operatorname{dom} V \cap \mathfrak{K}_1^-$ such that $V(f^+ + f^-) \in \mathrm{j}_2\mathfrak{M} \cap \operatorname{ran} V$. Hence, $f^+ + f^- \in \mathfrak{N}$ from which $P_1^+\mathfrak{N} = \operatorname{dom} V \cap \mathfrak{K}_1^+$ follows, because $f^+$ was taken arbitrarily. □

Lemma 6.13 implies that if

$$\mathcal{P}_{\mathfrak{M}^{[\perp]_2}} V(\operatorname{dom} V \cap \mathfrak{K}_1^+) = \mathcal{P}_{\mathfrak{M}^{[\perp]_2}} V(\operatorname{dom} V \cap \mathfrak{K}_1^-), \tag{6.10}$$

then $V^{-1}(\mathrm{j}_2\mathfrak{M} \cap \operatorname{ran} V)$ is an essentially hyper-maximal semi-definite if $P_1^\pm \ker V + \operatorname{clos}(\operatorname{dom} V \cap \mathfrak{K}_1^\pm) = \mathfrak{K}_1^\pm$, cf. (Derkach et al. 2006: Corollary 4.12). Proposition 6.15 below gives conditions for the hyper-maximal semi-definiteness of $\mathfrak{L} := U^{-1}(\mathrm{j}_2\mathfrak{M}^{[\perp]_2} \cap \operatorname{ran} U)$ for a unitary relation $U$ given that (6.10) holds, see (Derkach et al. 2006: Proposition 4.15 & Corollary 4.17). Therefore recall first the following result, see (Derkach et al. 2006: Lemma 4.10).

**Lemma 6.14.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and assume that there exists a hyper-maximal neutral subspace $\mathfrak{M}$ in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then, with $U_{\mathfrak{M}} := \mathcal{P}_{\mathfrak{M}}U$ considered as a mapping from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, the following statements hold:*

(i) *if $\operatorname{ran} U_{\mathfrak{M}}$ is closed, then $\ker U_{\mathfrak{M}}$ is closed;*

(ii) *if $U_{\mathfrak{M}}$ is closed, then $\operatorname{ran} U_{\mathfrak{M}}$ is closed if and only if $\ker U_{\mathfrak{M}}$ is closed.*

*Proof.* Observe that $\mathcal{P}_{\mathfrak{M}}^{[*]} = \mathcal{P}_{\mathrm{j}\mathfrak{M}}$ and hence by Lemma 2.15

$$\operatorname{ran} U_{\mathfrak{M}}^{[*]} = \operatorname{ran} (\mathcal{P}_{\mathfrak{M}}U)^{[*]} = \operatorname{ran} (U^{[*]}\mathcal{P}_{\mathfrak{M}}^{[*]}) = \operatorname{ran} (U^{-1}\mathcal{P}_{\mathrm{j}\mathfrak{M}}) = \ker U_{\mathfrak{M}}.$$

The above equality together with the fact that for a closed relation $H$ between Kreĭn spaces $\operatorname{ran} H$ is closed if and only if $\operatorname{ran} H^{[*]}$ is closed, see e.g. (Sorjonen 1978/1979), yields the statements. □

**Proposition 6.15.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ is a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and let $\mathrm{j}_2$ be a fixed fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Moreover assume that there exists a hyper-maximal neutral subspace $\mathfrak{M}$ in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then equivalent are*

(i) *$\mathcal{P}_{\mathfrak{M}}U(\operatorname{dom} U \cap \mathfrak{K}_1^+) = \mathcal{P}_{\mathfrak{M}}U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ and $\operatorname{ran} (\mathcal{P}_{\mathfrak{M}}U)$ is closed;*

(ii) *$\mathcal{P}_{\mathfrak{M}}U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ or $\mathcal{P}_{\mathfrak{M}}U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ is closed.*

*In particular, if either of the above conditions holds, then $U^{-1}(\mathrm{j}_2\mathfrak{M} \cap \operatorname{ran} U)$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$.*

*Proof.* (i) ⇒ (ii): By Lemma 6.14, the discussion preceding Lemma 6.14 and (3.11), the conditions in (i) imply that $\mathfrak{L} := U^{-1}(\mathrm{j}_2\mathfrak{M} \cap \operatorname{ran} U)$ is hyper-maximal neutral. Therefore

$$\mathfrak{L} + \operatorname{dom} U \cap \mathfrak{K}_1^+ = \operatorname{dom} U = \mathfrak{L} + \operatorname{dom} U \cap \mathfrak{K}_1^-.$$

From this it follows that

$$\mathcal{P}_{\mathfrak{M}}U(\operatorname{dom} U \cap \mathfrak{K}_1^+) = \operatorname{ran} (\mathcal{P}_{\mathfrak{M}}U) = \mathcal{P}_{\mathfrak{M}}U(\operatorname{dom} U \cap \mathfrak{K}_1^-).$$

This together with the assumption that $\operatorname{ran} (\mathcal{P}_{\mathfrak{M}}U)$ is closed implies that (ii) holds.

(ii) ⇒ (i): To prove this implication w.l.o.g. assume that $\operatorname{mul} U = \{0\}$. Since $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ and $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ are a maximal nonnegative and nonpositive subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]\}$, see Proposition 5.1, the quasi-angular operator $A$ and $A^*$ of $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ and $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ w.r.t. to $\mathfrak{M}$ are a maximal dissipative and a maximal accumulative relation, respectively, see Proposition 2.20. Moreover, $\overline{\operatorname{dom}} A = (\operatorname{mul} A^*)^{\perp_2} = \mathfrak{M}$ and $\overline{\operatorname{dom}} A^* = (\operatorname{mul} A)^{\perp_2} = \mathfrak{M}$, because $A$ and $A^*$ are operators as a consequence of the assumption that $\operatorname{mul} U = \{0\}$ and, hence, $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ and $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ do not contain any neutral vectors. Recall that $A$ and $A^*$ are defined as

$$\begin{aligned} \operatorname{gr} A &= \{\{\mathcal{P}_{\mathfrak{M}} f, i\mathcal{P}_{\mathfrak{M}} \mathfrak{j} f\} : f \in U(\operatorname{dom} U \cap \mathfrak{K}_1^+)\}; \\ \operatorname{gr} A^* &= \{\{\mathcal{P}_{\mathfrak{M}} f, i\mathcal{P}_{\mathfrak{M}} \mathfrak{j} f\} : f \in U(\operatorname{dom} U \cap \mathfrak{K}_1^-)\}. \end{aligned} \tag{6.11}$$

Hence, if the assumption in (ii) holds, then $\operatorname{dom} A$ and $\operatorname{dom} A^*$ closed. Since $\overline{\operatorname{dom}} A = \mathfrak{M} = \overline{\operatorname{dom}} A^*$, this implies that if (ii) holds, then $\operatorname{dom} A = \mathfrak{M} = \operatorname{dom} A^*$ which implies that (i) holds, see (6.11). □

The conclusion that $\mathfrak{L} := U^{-1}(\mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} U)$ is hyper-maximal neutral in Proposition 6.15 is stronger than the equality $\mathcal{P}_{\mathfrak{M}} U(\operatorname{dom} U \cap \mathfrak{K}_1^+) = \mathcal{P}_{\mathfrak{M}} U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$. To see this let $\mathfrak{K}_{1,a}^+[+]\mathfrak{K}_{1,a}^-$ and $\mathfrak{K}_{1,b}^+[+]\mathfrak{K}_{1,b}^-$ be two canonical decompositions of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, then the assumption that $\mathfrak{L}$ is hyper-maximal neutral implies that

$$\mathfrak{L} + \operatorname{dom} U \cap \mathfrak{K}_{1,a}^+ = \operatorname{dom} U = \mathfrak{L} + \operatorname{dom} U \cap \mathfrak{K}_{1,b}^-.$$

As a consequence of the definition of $\mathfrak{L}$, the above expression implies that

$$\mathcal{P}_{\mathfrak{M}} U(\operatorname{dom} U \cap \mathfrak{K}_{1,a}^+) = \mathcal{P}_{\mathfrak{M}} U(\operatorname{dom} U \cap \mathfrak{K}_{1,b}^-).$$

Next it is shown that if the hyper-maximal semi-definite subspace $\mathfrak{M}$ occurring in Lemma 6.13 is contained in $U(\mathfrak{K}_1^+ \cap \operatorname{dom} U) \cap U(\mathfrak{K}_1^- \cap \operatorname{dom} U)$, cf. Corollary 6.11, then $U^{-1}(\mathfrak{j}_2\mathfrak{M}^{[\perp]_2} \cap \operatorname{ran} U)$ is hyper-maximal semi-definite. In light of Lemma 4.7 this yields a necessary and sufficient condition for isometric relations to be unitary.

**Lemma 6.16.** *Let $V$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and let $\mathfrak{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Moreover, assume that $\mathfrak{M}$ is a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that*

(i) $\mathfrak{M} \subseteq V(\operatorname{dom} V \cap \mathfrak{K}_1^+) + V(\operatorname{dom} V \cap \mathfrak{K}_1^-)$;

(ii) $V^{-1}(\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}) \subseteq \ker V + \mathfrak{K}_1^+$ *or* $V^{-1}(\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}) \subseteq \ker V + \mathfrak{K}_1^-$;

(iii) $\mathcal{P}_{\mathfrak{M}^{[\perp]_2}} V(\operatorname{dom} V \cap \mathfrak{K}_1^+) = \mathcal{P}_{\mathfrak{M}^{[\perp]_2}} V(\operatorname{dom} V \cap \mathfrak{K}_1^-)$;

*Then* $V^{-1}(\mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} V)$ *is a hyper-maximal semi-definite subspace of* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *if and only if* $P_1^+\operatorname{dom} V = \mathfrak{K}_1^+$ *and* $P_1^-\operatorname{dom} V = \mathfrak{K}_1^-$.

*Proof.* The first assumption on $\mathfrak{M}$ implies that $P_1^{\pm}V^{-1}(\mathfrak{M}) \subseteq P_1^{\pm}\ker V + \operatorname{dom} V \cap \mathfrak{K}_1^{\pm}$, and from the second and third assumption on $\mathfrak{M}$ it follows by Lemma 6.13 that $P_1^{\pm}\ker V + \operatorname{dom} V \cap \mathfrak{K}_1^{\pm} \subseteq P_1^{\pm}V^{-1}(\mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} V)$. Consequently, $P_1^{\pm}V^{-1}(\mathfrak{M}) \subseteq P_1^{\pm}V^{-1}(\mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} V)$. Next note that the fact that $\mathfrak{M} \subseteq \operatorname{ran} V$ is hyper-maximal semi-definite implies that $\operatorname{ran} V = \mathfrak{M} + \mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} V$, i.e. $\operatorname{dom} V = V^{-1}(\mathfrak{M}) + V^{-1}(\mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} V)$. Combining the above observations yields

$$P_1^{\pm}\operatorname{dom} V = P_1^{\pm}V^{-1}(\mathfrak{M}) + P_1^{\pm}V^{-1}(\mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} V) = P_1^{\pm}V^{-1}(\mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} V).$$

From the above equality it follows that $V^{-1}(\mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} V)$ is a hyper-maximal semi-definite subspace if and only if $P_1^+\operatorname{dom} V = \mathfrak{K}_1^+$ and $P_1^-\operatorname{dom} V = \mathfrak{K}_1^-$. □

**Theorem 6.17.** *Let* $U$ *be an isometric relation from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, *let* $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ *be a canonical decomposition of* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *and let* $\mathfrak{j}_2$ *be a fundamental symmetry of* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. *Then* $U$ *is a unitary relation if and only if*

(i) *there exists a hyper-maximal semi-definite subspace* $\mathfrak{M}$ *of* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *such that*

  *(a)* $\mathfrak{M} \subseteq U(\operatorname{dom} U \cap \mathfrak{K}_1^+) + U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$;

  *(b)* $U^{-1}(\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}) \subseteq \ker U + \mathfrak{K}_1^+$ *or* $U^{-1}(\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}) \subseteq \ker U + \mathfrak{K}_1^-$;

  *(c)* $\mathcal{P}_{\mathfrak{M}^{[\perp]_2}}U(\operatorname{dom} U \cap \mathfrak{K}_1^+) = \mathcal{P}_{\mathfrak{M}^{[\perp]_2}}U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$;

(ii) $P_1^+\operatorname{dom} U = \mathfrak{K}_1^+$ *and* $P_1^-\operatorname{dom} U = \mathfrak{K}_1^-$.

*Proof.* The necessity of the conditions follows from Lemma 6.10, Lemma 6.13 and (3.7). The converse part follows directly from Theorem 6.8 after observing that Lemma 6.16 yields that the assumptions imply that $U^{-1}(\mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} U)$ is a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. □

In view of Proposition 6.18 below, the conditions in Theorem 6.17 (i) imply that $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ and $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ are maximal nonnegative and nonpositive, respectively. Hence, Theorem 5.3 shows that condition (ii) in Theorem 6.17 can be replaced by the conditions that $U$ is closed and that $\ker U = (\operatorname{dom} U)^{[\perp]_1}$.

Finally note that Lemma 6.10 combined with Lemma 6.13 implies that for a unitary relation $U$ there exists a hyper-maximal semi-definite subspace $\mathfrak{M}$ such that

$$\begin{aligned} \mathcal{P}_{\mathfrak{M}^{[\perp]_2}}U(\operatorname{dom} U \cap \mathfrak{K}_1^+) &= \mathcal{P}_{\mathfrak{M}^{[\perp]_2}}U(\operatorname{dom} U \cap \mathfrak{K}_1^-); \\ \mathcal{P}_{\mathfrak{j}_2\mathfrak{M}^{[\perp]_2}}U(\operatorname{dom} U \cap \mathfrak{K}_1^+) &= \mathcal{P}_{\mathfrak{j}_2\mathfrak{M}^{[\perp]_2}}U(\operatorname{dom} U \cap \mathfrak{K}_1^-). \end{aligned} \tag{6.12}$$

This observation yields half of the next geometrical statement, cf. Proposition 2.5.

**Proposition 6.18.** *Let* $\mathfrak{M}_+$ *and* $\mathfrak{M}_-$ *be a nonnegative and nonpositive subspace of* $\{\mathfrak{K}, [\cdot,\cdot]\}$*, respectively, such that* $\mathfrak{M}_+ \subseteq \mathfrak{M}_-^{[\perp]}$ *and* $\mathfrak{M}_- \subseteq \mathfrak{M}_+^{[\perp]}$ *and let* $\mathfrak{j}$ *be a fundamental symmetry of* $\{\mathfrak{K}, [\cdot,\cdot]\}$*. Then* $\mathfrak{M}_+$ *and* $\mathfrak{M}_-$ *are a maximal nonnegative and a maximal nonpositive subspace of* $\{\mathfrak{K}, [\cdot,\cdot]\}$*, respectively, if and only if there exists a hyper-maximal semi-definite subspace* $\mathfrak{L}$ *of* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *such that*

$$\mathcal{P}_{\mathfrak{L}}\mathfrak{M}_+ = \mathfrak{L} \quad \textit{and} \quad \mathcal{P}_{\mathfrak{L}}\mathfrak{M}_- = \mathfrak{L}^{[\perp]} \quad \textit{or} \quad \mathcal{P}_{\mathfrak{L}}\mathfrak{M}_+ = \mathfrak{L}^{[\perp]} \quad \textit{and} \quad \mathcal{P}_{\mathfrak{L}}\mathfrak{M}_- = \mathfrak{L},$$

*if* $\mathfrak{L}$ *is nonnegative or nonpositive, respectively.*

*Proof.* The necessity is clear by the discussion preceding the statement combined with Lemma 6.10 (ii) and Theorem 5.8. To prove the converse assume w.l.o.g. that $\mathfrak{L}$ is nonnegative. If $\mathfrak{M}_+$ is not maximal nonnegative, then $\mathfrak{M}_+$ can be nonnegatively extended by an element $h \in \mathfrak{K}$. In fact, as consequence of the assumption $\mathcal{P}_{\mathfrak{L}}\mathfrak{M}_+ = \mathfrak{L}$ one can assume that $h \in \mathfrak{L}^{\perp} = \mathfrak{j}\mathfrak{L}^{[\perp]}$. Consequently, there exists an $f \in \mathfrak{L}^{[\perp]}$ such that $h = \mathfrak{j}f$. On the other hand, by the assumption $\mathcal{P}_{\mathfrak{L}}\mathfrak{M}_+ = \mathfrak{L}$, there exists a $g \in \mathfrak{L}^{[\perp]}$ such that $f + \mathfrak{j}g \in \mathfrak{M}_+$. Hence, for any $c \in \mathbb{R}$

$$0 \leq [(f + \mathfrak{j}g) + c\mathfrak{j}f, (f + \mathfrak{j}g) + c\mathfrak{j}f] = 2c[\mathfrak{j}f, f] + [\mathfrak{j}g, f] + [f, \mathfrak{j}g].$$

Since $c$ is arbitrary, this implies that $f = 0$, i.e., $\mathfrak{M}_+$ is maximal nonnegative. The maximal nonpositivity of $\mathfrak{M}_-$ can be proven using similar arguments. □

If $\mathfrak{L}$ in Proposition 6.18 is neutral, then Proposition 6.18 can be interpreted as saying that a nonnegative (nonpositive) subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ is maximal nonnegative (nonpositive) if and only if it can be represented by an everywhere defined bounded operator in $\{\mathfrak{L}, [\mathfrak{j}\cdot,\cdot]\}$, cf. Section 2.5. Note also that there exists a subspace $\mathfrak{L}$ having the properties as in Proposition 6.18 which simultaneously has the properties of the subspace $\mathfrak{L}$ in Proposition 6.12.

# 7 BLOCK REPRESENTATIONS

In this chapter block representations will be given for certain classes of isometric operators; in particular, for unitary operators. Moreover, some consequences of those block representations are stated. As a preparation therefore the composition of archetypical unitary operators is studied in the first section. Note that those investigations yield simple examples of the peculiar mapping behavior of unitary relations. In the second section block representation for a special type of isometric operators, which are the abstract equivalent of the so-called quasi-boundary triplets, see Section A.2, are presented together with some consequences of their representation. In the third section it is shown that every unitary operator can be expressed as the composition of an archetypical unitary operator with a bounded unitary operator. This implies that the unboundedness of unitary operators can be understood by studying only unitary operators which have a diagonal block representation. As an application of these block representation approach to unitary operators, the main results from (Calkin 1939a) are proved in the fourth section with simple arguments. As another application of the obtained block representations for isometric and unitary operators, conditions for when their composition is (extendable to) a unitary operator are presented in the fifth and final section.

## 7.1 Compositions of archetypical unitary operators

Let $\mathfrak{j}$ be a fundamental symmetry of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and assume that there exists a hypermaximal neutral subspace $\mathfrak{M}$ in $\{\mathfrak{K}, [\cdot,\cdot]\}$. If $K_1$ and $K_2$ are selfadjoint relations in (the Hilbert space) $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$, then

$$\Upsilon_1(K_1)\Upsilon_1(K_2) = \Upsilon_1(K_1 + K_2),$$

see (Derkach et al. 2009: Example 2.11). This composition is (extendable to) a unitary relation if and only if $K_1 + K_2$ is (extendable to) a selfadjoint relation, see Proposition 4.8. Example 7.1 below provides an example of two selfadjoint operators $K_1$ and $K_2$ such that their sum cannot be extended to a selfadjoint relation, i.e., $\Upsilon_1(K_1 + K_2)$ can not be extended to a unitary relation.

**Example 7.1.** In the Hilbert space $L^2(\mathbb{R}_+)$ consider the differential expressions $\ell_1 f = -f'' - 2if' - f$ and $\ell_2 f = f'' + f$. Both expressions can be interpreted as canonical differential systems which are definite on $\mathbb{R}_+$, see e.g. (Behrndt et al.

2011b). With

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \Delta(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ H_1(t) = \begin{pmatrix} 2 & i \\ -i & 1 \end{pmatrix}, \ H_2(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

these systems are

$$\mathcal{J}F'(t) - H_i(t)F(t) = \lambda\Delta(t)F(t), \quad t \in \mathbb{R}_+ \text{ a.e.}, \quad \lambda \in \mathbb{C},$$

where $F = (f_1, f_2)^T$ and $i = 1, 2$. With $L^2_\Delta(\mathbb{R}_+)$ the Hilbert space (of equivalence classes) associated with $\Delta$, the minimal relations generated by the above canonical systems are symmetric operators in $L^2_\Delta(\mathbb{R}_+)$ with defect numbers $(1, 1)$, which follows e.g. from (Lesch & Malamud 2003: Proposition 5.25) together with the definiteness of the systems. In particular, for both systems $0$ is a regular endpoint and $\infty$ is an endpoint in the limit-point case. Therefore, properly understood, $K_1$ and $K_2$ defined via

$$\operatorname{gr} K_i = \{\{F, G\} \in L^2_\Delta(\mathbb{R}_+) \times L^2_\Delta(\mathbb{R}_+) : \ell_i f_1 = g_1, \ f_1(0) = 0\}, \quad i = 1, 2,$$

where $F = (f_1, f_2)^T$ and $G = (g_1, g_2)^T$, are selfadjoint operators in $L^2_\Delta(\mathbb{R}_+)$, see (Behrndt et al. 2011b: Section 4.1 and 5.3). Moreover, $\operatorname{dom} K_2 \subseteq \operatorname{dom} K_1$ and, hence, the sum of $K_1$ and $K_2$ is the symmetric operator $S$:

$$\operatorname{gr} S = \{\{F, G\} \in L^2_\Delta(\mathbb{R}_+) \times L^2_\Delta(\mathbb{R}_+) : F \in \operatorname{dom} K_2, \ \ell_S f_1 = g_1, \ f_1(0) = 0\},$$

where $\ell_S f = -2if'$, $F = (f_1, f_2)^T$ and $G = (g_1, g_2)^T$. Hence, the closure of $S$ is a well-known symmetric operator with defect numbers $n_+(S) = 0$ and $n_-(S) = 1$ corresponding to $\ell_S$.

The selfadjoint operators from Example 7.1 can also be used to show that there exist unitary operators which map hyper-maximal neutral subspaces onto (non-closed) neutral subspaces which can not be extended to hyper-maximal neutral subspaces.

**Example 7.2.** Let $K_1$ and $K_2$ be the selfadjoint operators in $L^2_\Delta(\mathbb{R}^+)$ as in Example 7.1 and let $\mathfrak{j}$ be the fundamental symmetry in $(L^2_\Delta(\mathbb{R}^+))^2$ as in Example 2.1, i.e., $\mathfrak{j}\{f, f'\} = i\{-f', f\}$. Then $\mathfrak{M} := L^2_\Delta(\mathbb{R}^+) \times 0$ is a hyper-maximal neutral subspace of the Kreĭn space $\{(L^2_\Delta(\mathbb{R}^+))^2, (\mathfrak{j}\cdot, \cdot)\}$, and $K_1$ and $K_2$ can be interpreted as selfadjoint operators (in the Hilbert space) $\{\mathfrak{M}, (\cdot, \cdot)\}$. Now $\Upsilon_1(K_1)$ is a unitary operator in $\{(L^2_\Delta(\mathbb{R}^+))^2, (\mathfrak{j}\cdot, \cdot)\}$ and $\mathfrak{L} := \operatorname{gr} K_2$ is a hyper-maximal neutral subspace of $\{(L^2_\Delta(\mathbb{R}^+))^2, (\mathfrak{j}\cdot, \cdot)\}$ such that $\mathfrak{L} \subseteq \operatorname{dom} \Upsilon_1(K_1) = \operatorname{dom} K_1 \oplus \mathfrak{j}\mathfrak{M}$, because $\operatorname{dom} K_2 \subseteq \operatorname{dom} K_1$. Moreover, $\Upsilon_1(K_1)\mathfrak{L} = \operatorname{gr}(K_1 + K_2)$ is a (non-closed) neutral subspace which can not be extended to a hyper-maximal neutral subspace, because $K_1 + K_2$ is a symmetric operator which can not be extended to a selfadjoint operator, see Example 7.1 and Proposition 2.20.

Example 7.1 can also be used to show that there exist isometric operators which cannot be extended to unitary relations such that the closure of their composition with a unitary relation is (extendable to) a unitary relation. Another example of this phenomenon is obtained by considering the composition of $\Upsilon_1(K)$ and $\Upsilon_1(S)$, where $K$ is a selfadjoint operator in the separable Hilbert space $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ and $S$ is a symmetric operator with unequal defect numbers in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ such that $\mathrm{dom}\, S \cap \mathrm{dom}\, K = \{0\}$, cf. Proposition 2.17 (v). Then, clearly, $\Upsilon_1(K)\Upsilon_1(S) = I_{\mathfrak{j}\mathfrak{M}}$ can be extended to a unitary operator.

Different from the composition of $\Upsilon_1(K_1)$ and $\Upsilon_1(K_2)$, the composition of the archetypical unitary relations $\Upsilon_2(B_1)$ and $\Upsilon_2(B_2)$, where $B_1$ and $B_2$ are closed operators (or relations), can always be extended to a unitary relation:

$$\Upsilon_2(B_1)\Upsilon_2(B_2) = \begin{pmatrix} B_1B_2 & 0 \\ 0 & \mathfrak{j}B_1^{-*}B_2^{-*}\mathfrak{j} \end{pmatrix} \subseteq \Upsilon_2(\mathrm{clos}\,(B_1B_2)).$$

Here it is used that $B_1^{-*}B_2^{-*} \subseteq (B_1B_2)^{-*}$, see Lemma 2.15.

Next compositions of the type $\Upsilon_1(S)\Upsilon_2(B)$ are considered. The following two statements give some conditions for when this composition is unitary.

**Proposition 7.3.** *Let* $\mathfrak{j}$ *be a fundamental symmetry of* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *and assume that there exists a hyper-maximal neutral subspace* $\mathfrak{M}$ *in* $\{\mathfrak{K}, [\cdot,\cdot]\}$*. Moreover, let* $B$ *be an operator in (the Hilbert space)* $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ *with* $\overline{\mathrm{dom}}\, B = \mathfrak{M} = \mathrm{ran}\,\mathrm{clos}\,(B)$ *and* $\ker\mathrm{clos}\,(B) = \{0\}$*, and let* $S$ *be a symmetric relation in* $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$*. Then* $\Upsilon_1(S)\Upsilon_2(\mathrm{clos}\, B)$ *is (extendable to) a unitary relation in* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *if and only if* $S$ *is (extendable to) a selfadjoint relation in* $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$*.*

*In particular,* $\Upsilon_1(S)\Upsilon_2(\mathrm{clos}\, B)$ *is a unitary operator if and only if* $S$ *is a selfadjoint operator in* $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ *with* $\mathrm{dom}\, S \cap \mathrm{mul}\,\mathrm{clos}\,(B) = \{0\}$*.*

*Proof.* Since the final equivalence evidently holds if the first equivalence holds, it suffices to prove only the first equivalence. Therefore note that if $T$ is a symmetric extension of $S$ (as in the statement), then $\Upsilon_1(T)\Upsilon_2(\mathrm{clos}\, B)$ is an isometric extension of $\Upsilon_1(S)\Upsilon_2(\mathrm{clos}\, B)$. Hence, to prove the first equivalence it suffices to show that $\Upsilon_1(S)\Upsilon_2(\mathrm{clos}\, B)$ is unitary if and only if $S$ is selfadjoint.

If $S$ is selfadjoint, then the fact that $\Upsilon_1(S)\Upsilon_2(\mathrm{clos}\, B)$ is unitary follows from Lemma 4.7 as in Proposition 4.8. To prove the converse assume that $S$ is a maximal symmetric relation which is not selfadjoint, and that $\Upsilon_1(S)\Upsilon_2(\mathrm{clos}\, B)$ is unitary. Then there exists $\{f, f'\} \in \mathrm{gr}\, S^*$ such that $\mathrm{Im}\,[f, f'] \neq 0$ and by the assumptions on $B$ there exists a $h \in \mathrm{dom}\,\mathrm{clos}\,(B)$ such that $\{h, f\} \in \mathrm{gr}\,\mathrm{clos}\,(B)$. Now a direct calculation shows that $[h, g] = [f + \mathfrak{j}if', g']$ for all $\{g, g'\} \in \mathrm{gr}\, U$, i.e,

$\{h, f + \mathfrak{j} i f'\} \in \operatorname{gr} U$ by Proposition 3.1. On the other hand, $[h, h] = 0$, because $h \in \mathfrak{M}$, and $[f + \mathfrak{j} i f', f + \mathfrak{j} i f'] = i([\mathfrak{j} f', f] - [f, \mathfrak{j} f']) \neq 0$, by the assumption on $\{f, f'\}$. This shows that $\{h, f + \mathfrak{j} i f'\}$ cannot be contained in the graph of an isometric relation. This contradiction completes the proof. □

**Corollary 7.4.** *Let* $\mathfrak{j}$ *be a fundamental symmetry of* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *and assume that there exists a hyper-maximal neutral subspace* $\mathfrak{M}$ *in* $\{\mathfrak{K}, [\cdot,\cdot]\}$*. Let* $B$ *be a closed operator in (the Hilbert space)* $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ *with* $\overline{\operatorname{dom}} B = \mathfrak{M} = \operatorname{ran} B$ *and* $\ker B = \{0\}$*, and let* $S$ *be a symmetric operator in* $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$*. Then* $\Upsilon_1(S)\Upsilon_2(B^{-1})$ *is (extendable to) a unitary operator in* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *if and only if* $B^{-*}SB^{-1}$ *is (extendable to) a selfadjoint operator in* $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$*.*

*Proof.* Note that

$$\Upsilon_1(S)\Upsilon_2(B^{-1}) = \begin{pmatrix} B^{-1} & 0 \\ \mathfrak{j} i S B^{-1} & \mathfrak{j} B^* \mathfrak{j} \end{pmatrix} = \begin{pmatrix} B^{-1} & 0 \\ 0 & \mathfrak{j} B^* \mathfrak{j} \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathfrak{j} i B^{-*} S B^{-1} & I \end{pmatrix}$$
$$= \Upsilon_2(B^{-1})\Upsilon_1(B^{-*}SB^{-1}) = \left(\Upsilon_1(-B^{-*}SB^{-1})\Upsilon_2(B)\right)^{-1}.$$

Here the second equality holds, because the assumptions on $B$ imply that $\operatorname{ran} B^* = \mathfrak{M}$. Since an isometric relation is unitary if and only if its inverse is unitary, the above equality together with Proposition 7.3 shows that the statement holds. □

Example 7.5 below shows that $S$ in Corollary 7.4 need not be a selfadjoint operator nor even a symmetric operator with equal defect numbers for $B^{-*}SB^{-1}$ to be selfadjoint and, hence, $\Upsilon_1(S)\Upsilon_2(B^{-1})$ to be unitary.

**Example 7.5.** Let $\mathfrak{j}$ be a fundamental symmetry of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and assume that there exists a hyper-maximal neutral subspace $\mathfrak{M}$ in $\{\mathfrak{K}, [\cdot,\cdot]\}$. Moreover, let $S$ be a closed symmetric operator in the Hilbert space $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ with $\overline{\operatorname{dom}} S = \mathfrak{M}$ and defect numbers $n_\pm(S) = n_\pm$, where $n_\pm \leq \aleph_0$, and let $B$ be a closed operator in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ with $\operatorname{dom} B = \operatorname{dom} S$, $\ker B = \{0\}$ and $\operatorname{ran} B = \mathfrak{M}$, see Proposition 2.17 (ii). Then $K := B^{-*}SB^{-1}$ is a symmetric operator with $\operatorname{dom} K = \mathfrak{M}$, i.e. $K$ is a selfadjoint operator in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$.

**Remark 7.6.** Note that if $S$ and $B$ are as in Example 7.5, then the unitary operator $\Upsilon_1(S)\Upsilon_2(B^{-1})$ maps the hyper-maximal neutral subspace $\mathfrak{M} \times \{0\}$ onto the closed neutral subspace $\{f + \mathfrak{j} i S f : f \in \operatorname{dom} S\}$ with defect numbers $n_+$ and $n_-$. Hence, unitary relations may map hyper-maximal neutral subspaces onto closed neutral subspaces with nonzero and arbitrary defect numbers ($\leq \aleph_0$), cf. Proposition 7.25 and Chapter 8.

Using archetypical unitary operator it is also easily shown that there exist unitary operators having the same domain which are really different, i.e. they do not differ by a composition with a standard unitary operator on the range side.

**Example 7.7.** Let $\mathrm{j}$ be a fundamental symmetry of $\{\mathfrak{K}, [\cdot,\cdot]\}$, assume that $\mathfrak{M}$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and, moreover, let $B$ be an arbitrary closed operator in the Hilbert space $\{\mathfrak{M}, [\mathrm{j}\cdot,\cdot]\}$ which satisfies $\overline{\mathrm{dom}}\, B = \mathfrak{M} = \mathrm{ran}\, B$ and $\ker B = \{0\}$. Then $U := \Upsilon_2(B)$ is a unitary operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$ with $\mathrm{dom}\, U = \mathrm{dom}\, B \oplus \mathrm{j}\mathfrak{M}$. Let $KX$ be the polar decomposition of $B$ in $\{\mathfrak{M}, [\mathrm{j}\cdot,\cdot]\}$, where $K$ is a nonnegative selfadjoint operator in $\{\mathfrak{M}, [\mathrm{j}\cdot,\cdot]\}$ (with $\mathrm{ran}\, K = \mathrm{ran}\, B = \mathfrak{M}$) and $X$ is a unitary operator in $\{\mathfrak{M}, [\mathrm{j}\cdot,\cdot]\}$. Then $U_a := \Upsilon_1(K)\Upsilon_2(X)$ is a unitary operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$ which has the same domain as $U$. Furthermore,

$$U_a^{-1}U = (\Upsilon_2(X))^{-1}(\Upsilon_1(K))^{-1}\Upsilon_2(K)\Upsilon_2(X) = \Upsilon_2(X^{-1})\Upsilon_1(-K)\Upsilon_2(K)\Upsilon_2(X),$$

is an unbounded unitary operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$, because $\Upsilon_2(X)$ and $\Upsilon_2(X^{-1})$ are standard unitary operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$ and $\Upsilon_1(-K)\Upsilon_2(K)$ is an (unbounded) unitary operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$ by Proposition 7.4.

## 7.2 Block representations for isometric operators

If for an isometric operator $V$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ there exists a hyper-maximal semi-definite subspace $\mathfrak{L} \subseteq \mathrm{dom}\, V$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, then by means of $\mathfrak{L}$ and a fundamental symmetry $\mathrm{j}_1$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, the domain of $V$ can be decomposed as

$$\mathrm{dom}\, V = \mathfrak{L}^{[\perp]_1} \oplus_1 (\mathfrak{L} \cap \mathrm{j}_1\mathfrak{L}) \oplus_1 \mathrm{j}_1\mathfrak{L}^{[\perp]_1} \cap \mathrm{dom}\, V, \tag{7.1}$$

see Proposition 2.9. If an isometric operator has a domain decomposition as in (7.1), then block representations (with respect to those coordinates) for it can be given. Since the main interest is in isometric operators which are closely connected to unitary operators, in addition to the assumption that the domain isometric operator can be decomposed as in (7.1), it will in this section also be assumed that $V(\mathfrak{L}^{[\perp]_1})$ is a neutral subspace with equal defect numbers. In other words, with $\mathrm{j}_2$ a fundamental symmetry for $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, isometric operators $V$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ are studied for which there exists a hyper-maximal semi-definite subspace $\mathfrak{M}$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $\mathfrak{M} \cap \mathrm{j}_2\mathfrak{M} \subseteq \mathrm{ran}\, V$ and $V^{-1}(\mathrm{j}_2\mathfrak{M} \cap \mathrm{ran}\, V)$ is a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. With respect to certain coordinates, the block representations of such isometric operators take a specific form: they can be written as the composition of two archetypical isometric operators and a bounded unitary operator. Note that the isometric operators studied here are the abstract analogue of so-called quasi-boundary triplets, see Definition A.11 below, and that for unitary operators the preceding conditions are always satisfied, see Proposition 6.7.

To obtain a block representation for isometric relations having the above mentioned property the following slightly technical lemma is used.

**Lemma 7.8.** *Let $V$ be an isometric operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\overline{\operatorname{ran}}\, V = \mathfrak{K}_2$ and assume that there exists a hyper-maximal neutral subspace $\mathfrak{L}$ in $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ with $\mathfrak{L} \subseteq \operatorname{dom} V$. Then there exists a bounded unitary operator $U_t$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ onto $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\operatorname{dom} V \subseteq \operatorname{dom} U_t$ such that $VU_t^{-1}$ is an isometric operator in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\overline{\operatorname{dom}}\,(VU_t^{-1}) = \mathfrak{K}_2 = \overline{\operatorname{ran}}\,(VU_t^{-1})$.*

*In particular, if $\mathfrak{M}$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and $\mathfrak{j}_1$ and $\mathfrak{j}_2$ are fundamental symmetries of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, respectively, then $U_t$ can be taken such that $U_t(\mathfrak{L}) = \mathfrak{M}$ and $U_t(\mathfrak{j}_1\mathfrak{L} \cap \operatorname{dom} U_t) = \mathfrak{j}_2\mathfrak{M}$.*

*Proof.* It is a direct consequence of the assumptions that $\ker V = (\operatorname{dom} V)^{[\perp]_1}$, see Lemma 6.1. Hence, $VU_1^{-1}$, where $U_1 := U_{\ker V}$ is as in Lemma 3.11, is an isometric operator from $\{\mathfrak{K}_3, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ which satisfies $\overline{\operatorname{dom}}\,(VU_1^{-1}) = \mathfrak{K}_3$ and $\overline{\operatorname{ran}}\,(VU_1^{-1}) = \mathfrak{K}_2$.

Since $\mathfrak{L}$ is a hyper-maximal neutral subspace and $U_1$ is a bounded unitary operator, $U_1(\mathfrak{L})$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_3, [\cdot,\cdot]_1\}$, see Proposition 4.5. In particular, $k_3^+ = k_3^-$, see e.g. (Azizov & Iokhvidov 1989: Ch. 1, Remark 4.16). Since $VU_1^{-1}$ maps $U_1(\mathfrak{L})$ injectively onto a neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, $k_3^\pm \leq k_2^\pm$. Moreover, the fact that $VU_1^{-1}$ is an injective operator together with $\overline{\operatorname{dom}}\,(VU_1^{-1}) = \mathfrak{K}_3$ and $\overline{\operatorname{ran}}\,(VU_1^{-1}) = \mathfrak{K}_2$ yields that $k_3^+ + k_3^- = k_2^+ + k_2^-$. The preceding arguments together show that $k_3^\pm = k_2^\pm$. Therefore there exists a standard unitary operator $U_2$ from $\{\mathfrak{K}_3, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and the first statement holds with $U_t := U_2U_1$.

Since $U_2U_1$ is a bounded unitary operator, $U_2U_1(\mathfrak{L})$ and $U_2U_1(\mathfrak{j}_1\mathfrak{L} \cap \operatorname{dom} U_1)$ are hyper-maximal neutral subspaces of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and there exists a fundamental symmetry $\mathfrak{j}_2'$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $U_2U_1(\mathfrak{j}_1\mathfrak{L} \cap \operatorname{dom} V) = \mathfrak{j}_2'U_2U_1(\mathfrak{L})$. Therefore, by Lemma 4.13, there exists a standard unitary operator $U_3$ in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $U_3(U_2U_1(\mathfrak{L})) = \mathfrak{M}$ and $U_3(U_2U_1(\mathfrak{j}_1\mathfrak{L} \cap \operatorname{dom} U_1)) = U_3(\mathfrak{j}_2'U_2U_1(\mathfrak{L})) = \mathfrak{j}_2\mathfrak{M}$. Hence, the final statement holds with $U_t := U_3U_2U_1$. □

Following is a representation for the isometric operators $V$ for which $V^{-1}(\mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} V)$ is a hyper-maximal neutral subspace. It is shown that such operators have, up to a bounded unitary transformation, a triangular representation which can be expressed in terms of archetypical isometric operators. Note that the isometric operators considered in Theorem 7.9 below are a coordinate free version of quasi-boundary triplets, see Definition A.11 below. To better see this connection, note that $V^{-1}(\mathfrak{j}_2\mathfrak{M}\cap \operatorname{ran} V)=\ker(\mathcal{P}_{\mathfrak{M}}V)$, where $\mathcal{P}_{\mathfrak{M}}$ is the orthogonal projection onto $\mathfrak{M}$ w.r.t. $[\mathfrak{j}_2\cdot,\cdot]_2$.

**Theorem 7.9.** *Let $V$ be an isometric operator from $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ with $\overline{\operatorname{ran}}\, V=\mathfrak{K}_2$, let $\mathfrak{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ and, moreover, assume that there exists a hyper-maximal neutral subspace $\mathfrak{M}$ in $\{\mathfrak{K}_2,[\cdot,\cdot]\}$ such that $\mathfrak{L}:=\ker(\mathcal{P}_{\mathfrak{M}}V)$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$. Then there exists an operator $B$ in the Hilbert space $\{\mathfrak{M},[\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\overline{\operatorname{dom}}\, B=\mathfrak{M}=\operatorname{ran}\operatorname{clos}(B)$ and $\ker\operatorname{clos}(B)=\{0\}$, a symmetric operator $S$ in $\{\mathfrak{M},[\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\operatorname{dom} S=\operatorname{ran} B$ and $\operatorname{dom} S^*\cap\operatorname{mul}\operatorname{clos}(B)=\{0\}$, and a bounded unitary operator $U_t$ from $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ onto $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ with $\operatorname{dom} V\subseteq\operatorname{dom} U_t$, mapping $\mathfrak{L}$ onto $\mathfrak{j}_2\mathfrak{M}$, such that*

$$VU_t^{-1}=\begin{pmatrix} B & 0\\ \mathfrak{j}_2 iSB & \mathfrak{j}_2B^{-*}\mathfrak{j}_2\end{pmatrix}=\Upsilon_1(S)\Upsilon_2(B). \tag{7.2}$$

*Furthermore,* $\operatorname{mul}\operatorname{clos}(B)=\{0\}$ *if and only if* $\operatorname{clos}(V(\mathfrak{L}))=\mathfrak{j}_2\mathfrak{M}$.

*Proof.* Note first that if (7.2) holds, then $\mathfrak{j}_2V(\mathfrak{L})=\operatorname{dom} B^*$. This together with $\overline{\operatorname{dom}}\, B^*=(\operatorname{mul}\operatorname{clos}(B))^{\perp}$, see (2.6), shows that the final assertion holds. Next note that Lemma 7.8 implies the existence of a bounded unitary operator $U_t$ as in the statement. Then $W:=VU_t^{-1}$ is an isometric operator in $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ with $\overline{\operatorname{dom}}\, W=\mathfrak{K}_2=\overline{\operatorname{ran}}\, W$, $\mathfrak{j}_2\mathfrak{M}\subseteq\operatorname{dom} W$ and $W(\mathfrak{j}_2\mathfrak{M})=V(\mathfrak{L})\subseteq\mathfrak{j}_2\mathfrak{M}$.

*Step 1:* Since $\mathfrak{j}_2\mathfrak{M}\subseteq\operatorname{dom} W$ and $W(\mathfrak{j}_2\mathfrak{M})\subseteq\mathfrak{j}_2\mathfrak{M}$, $W$ has w.r.t. the decomposition $\mathfrak{M}\oplus\mathfrak{j}_2\mathfrak{M}$ of $\mathfrak{K}_2$ the following block representation:

$$W=\begin{pmatrix} B & 0\\ \mathfrak{j}_2 iC & \mathfrak{j}_2D\mathfrak{j}_2\end{pmatrix},$$

where $B$, $C$ and $D$ are operators in (the Hilbert space) $\{\mathfrak{M},[\mathfrak{j}_2\cdot,\cdot]_2\}$ which satisfy $\operatorname{dom} D=\mathfrak{M}$, $\ker D=\{0\}$ and $\operatorname{dom} B=\operatorname{dom} C$. Direct calculations shows that the fact that $W$ is isometric implies that $D\subseteq B^{-*}$ and that $C=SB$ for a symmetric operator $S$ with $\operatorname{dom} S=\operatorname{ran} B$, cf. Proposition 2.20.

*Step 2:* Next observe that $\overline{\operatorname{dom}}\, B=\mathfrak{M}$ and $\overline{\operatorname{ran}}\, B=\mathfrak{M}$, because $\overline{\operatorname{dom}}\, W=\mathfrak{K}_2=\overline{\operatorname{ran}}\, W$. Since $\operatorname{dom} D=\mathfrak{M}$, see Step 1, and $\operatorname{mul} B^{-*}=(\operatorname{ran} B)^{\perp}=\{0\}$, equality must hold in the inclusion $D\subseteq B^{-*}$ by (2.7): $D=B^{-*}$. Consequently, $\operatorname{ran} B^*=\operatorname{dom} D=\mathfrak{M}$ and combining this with $\overline{\operatorname{ran}}\, B=\mathfrak{M}$ yields $\operatorname{ran}\operatorname{clos}(B)=\mathfrak{M}$. Moreover, $\operatorname{ran} B^*=\mathfrak{M}$ also yields $\ker\operatorname{clos}(B)=\{0\}$, see (2.6).

*Step 3:* The arguments from step 1 and step 2 show that the asserted representation for $W = VU_t^{-1}$ holds. Therefore $\operatorname{ran} V = \{f + \mathfrak{j}_2 iSf : f \in \operatorname{dom} S\} + \mathfrak{j}_2 \operatorname{dom} B^*$. Since $\overline{\operatorname{ran}}\, V = \mathfrak{K}_2$, it now follows that

$$\begin{aligned}\{0\} = (\operatorname{ran} V)^{[\perp]_2} &= \{f + \mathfrak{j}_2 iSf : f \in \operatorname{dom} S\}^{[\perp]_2} \cap (\mathfrak{j}_2 \operatorname{dom} B^*)^{[\perp]_2} \\ &= \{f + \mathfrak{j}_2 iS^*f : f \in \operatorname{dom} S^*\} \cap (\operatorname{mul} \operatorname{clos}(B) \oplus \mathfrak{j}\mathfrak{M}),\end{aligned}$$

i.e. $\operatorname{dom} S^* \cap \operatorname{mul} \operatorname{clos}(B) = \{0\}$. This completes the proof. □

**Remark 7.10.** (i): Let $\mathfrak{j}_1$ be any fundamental symmetry of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then note that $U_t$ in Theorem 7.9 could have been chosen such that, in addition to the stated properties, $U_t(\mathfrak{j}_1\mathfrak{L} \cap \operatorname{dom} U_t) = \mathfrak{M}$, see Lemma 7.8. With that choice of $U_t$, (7.2) yields

$$V(\mathfrak{j}_1\mathfrak{L} \cap \operatorname{dom} V) = VU_t^{-1}(\mathfrak{M} \cap \operatorname{dom} VU_t^{-1}) = \{f + \mathfrak{j}_2 iSf : f \in \operatorname{dom} S\}.$$

In view of Proposition 7.3 and 2.20, this shows that the isometric operator in Theorem 7.9 is unitary if and only if $S$ is a selfadjoint operator or, equivalently, if and only if $V(\mathfrak{j}_1\mathfrak{L} \cap \operatorname{dom} V)$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$.

(ii): Using Corollary 3.14, Theorem 7.9 can be extended to the case that $\mathfrak{M}$ is a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M} \subseteq \operatorname{ran} V$ and $\mathfrak{L} := \ker(\mathcal{P}_{\mathfrak{M}^{[\perp]_2}} V) = V^{-1}(\mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} V)$ is a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Namely in that case there exist $S$ and $B$ as in Theorem 7.9 (with $\mathfrak{M}$ there replaced by $\mathfrak{M}^{[\perp]_2}$) and a bounded unitary operator $U_t$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\operatorname{dom} V \subseteq \operatorname{dom} U_t$, mapping $\mathfrak{L}$ onto $\mathfrak{j}_2\mathfrak{M}$, such that w.r.t. the decomposition $\mathfrak{M}^{[\perp]_2} \oplus \mathfrak{j}_2\mathfrak{M}^{[\perp]_2} \oplus (\mathfrak{j}_2\mathfrak{M} \cap \mathfrak{M})$ of $\mathfrak{K}$

$$VU_t^{-1} = \begin{pmatrix} B & 0 & 0 \\ \mathfrak{j}_2 iSB & \mathfrak{j}_2 B^{-*}\mathfrak{j}_2 & 0 \\ 0 & 0 & I_{\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}} \end{pmatrix} = \Upsilon_1(S)\Upsilon_2(B) \oplus I_{\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}}.$$

Next two consequences of Theorem 7.9 are given: The first shows that isometric operators as in Theorem 7.9 are closely connected to unitary relations and the second shows how the representation in Theorem 7.9 simplifies if it is assumed that $V$ maps $\mathfrak{L} := \ker(\mathcal{P}_{\mathfrak{M}^{[\perp]_2}} V)$ onto the hyper-maximal semi-definite subspace $\mathfrak{j}_2\mathfrak{M}$.

**Corollary 7.11.** *Let $V$ be an isometric operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\overline{\operatorname{ran}}\, V = \mathfrak{K}_2$, let $\mathfrak{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and assume that $\mathfrak{M}$ is a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]\}$ such that $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M} \subseteq \operatorname{ran} V$ and that $\mathfrak{L} := \ker(\mathcal{P}_{\mathfrak{M}^{[\perp]_2}} V)$ is a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then there exists a symmetric operator $T$ in the Hilbert space $\{\mathfrak{M}^{[\perp]_2}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\overline{\operatorname{dom}}\, T = \mathfrak{M}^{[\perp]_2}$ such that the closure of $(\Upsilon_1(T) \oplus I_{\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}})V$ is a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$.*

*Proof.* W.l.o.g. assume that $\mathfrak{L}$ and $\mathfrak{M}$ are hyper-maximal neutral subspaces, see Remark 7.10 (ii), then $VU_t^{-1} = \Upsilon_1(S)\Upsilon_2(B)$ by Theorem 7.9 (with $S$, $B$ and $U_t$ as in that statement). Next note that $(\Upsilon_1(S))^{-1} = \Upsilon_1(-S)$ and that $\overline{\operatorname{dom}} S = \mathfrak{M}$, because $\operatorname{dom} S = \operatorname{ran} B$ and $\operatorname{ran} \operatorname{clos}(B) = \mathfrak{M}$, see Theorem 7.9. Furthermore,

$$\begin{aligned}\operatorname{clos}(\Upsilon_1(-S)VU_t^{-1}) &= \operatorname{clos}(\Upsilon_1(-S)\Upsilon_1(S)\Upsilon_2(B)) = \operatorname{clos}(\Upsilon_2(B)) \\ &= \Upsilon_2(\operatorname{clos}(B)).\end{aligned}$$

Since $\Upsilon_2(\operatorname{clos}(B))$ is a unitary relation, see Proposition 4.9, and $U_t$ is a bounded unitary operator, the statement holds with $T = -S$ by Lemma 3.10. □

**Corollary 7.12.** *Let $V$ be an isometric operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\overline{\operatorname{ran}} V = \mathfrak{K}_2$ and let $\mathfrak{L} \subseteq \operatorname{dom} V$ be a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $\mathfrak{M} := \mathfrak{j}_2 V(\mathfrak{L})$ is a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then for every fundamental symmetry $\mathfrak{j}_2$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ there exists a symmetric operator $S$ in the Hilbert space $\{\mathfrak{M}^{[\perp]_2}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\overline{\operatorname{dom}} S = \mathfrak{M}^{[\perp]_2}$ and a bounded unitary operator $U_t$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ onto $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\operatorname{dom} V \subseteq \operatorname{dom} U_t$, mapping $\mathfrak{L}$ onto $\mathfrak{j}_2\mathfrak{M}$, such that $VU_t^{-1} = \Upsilon_1(S) \oplus I_{\mathfrak{M}\cap\mathfrak{j}\mathfrak{M}}$.*

*Proof.* As a consequence of Remark 7.10 (ii) assume w.l.o.g. that $\mathfrak{L}$ and $\mathfrak{M}$ are hyper-maximal neutral subspaces. Then the conditions of Theorem 7.9 are satisfied, i.e. $VU_t^{-1} = \Upsilon_1(S)\Upsilon_2(B)$. Moreover, the assumption that $V(\mathfrak{L})$ ($\subseteq \mathfrak{j}_2\mathfrak{M}$) is hyper-maximal neutral implies that $\operatorname{ran} B^{-*} = \mathfrak{j}_2 V(\mathfrak{L}) = \mathfrak{M}$. This, together with the other properties of $B$, see Theorem 7.9, implies that $\operatorname{clos}(B)$ is an operator with a trivial kernel satisfying $\operatorname{dom} \operatorname{clos}(B) = \mathfrak{M} = \operatorname{ran} \operatorname{clos}(B)$. Consequently, $\Upsilon_2(\operatorname{clos}(B)) = \operatorname{clos}(\Upsilon_2(B))$ is a standard unitary operator and, hence, $\Upsilon_2(\operatorname{clos}(B))U_t$ is a bounded unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ onto $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. This observation together with (7.2) shows that the statement holds with $S$ as in Theorem 7.9. □

If $V$ is as in Theorem 7.9 or, more generally, if $V = \Upsilon_1(S)\Upsilon_2(B)U_t$ for a symmetric operator $S$, an operator $B$ and a bounded unitary operator $U_t$, then $\ker(\mathcal{P}_{\mathfrak{j}\mathfrak{M}}V)$ is also a neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, $\ker V = \ker(\mathcal{P}_{\mathfrak{M}}V) \cap \ker(\mathcal{P}_{\mathfrak{j}\mathfrak{M}}V)$ and, moreover,

$$\ker(P_{\mathfrak{j}\mathfrak{M}}(VU_t^{-1})) = \{f + \mathfrak{j}iB^*(-S)Bf : f \in \operatorname{dom}(B^*SB)\} \tag{7.3}$$

Consequently, $\ker(P_{\mathfrak{j}\mathfrak{M}}(VU_t^{-1}))$, and hence also $\ker(P_{\mathfrak{j}\mathfrak{M}}V)$, is a hyper-maximal neutral subspace if and only if $B^*SB$ is a selfadjoint relation in $\{\mathfrak{M}, [\mathfrak{j}_2\cdot,\cdot]_2\}$, cf. Example 7.5. Moreover, $\ker(\mathcal{P}_{\mathfrak{j}\mathfrak{M}}V) = \ker V$ if and only if $\operatorname{dom}(B^*SB) = \{0\}$. Next an example of a unitary operator $U$ with $\ker(\mathcal{P}_{\mathfrak{j}\mathfrak{M}}U) = \ker U$ is presented, cf. (Derkach et al. 2006: Example 6.6).

**Example 7.13.** Let $\mathfrak{j}$ be a fundamental symmetry of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and assume that there exists a hyper-maximal neutral subspace $\mathfrak{M}$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ is a separable Hilbert space. Moreover, let $K$ be a selfadjoint operator in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ with $\operatorname{ran} K \neq \mathfrak{M}$ and $\overline{\operatorname{ran}}\, K = \mathfrak{M}$. Then there exists a closed operator $C$ in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ such that $\overline{\operatorname{ran}}\, C = \mathfrak{M}$, $\operatorname{ran} C \cap \operatorname{ran} K = \{0\}$, $\operatorname{dom} C = \mathfrak{M}$ and $\ker C = \{0\}$, see Proposition 2.17 (ii) and (v). Then $B = C^{-*}$ is a closed operator with $\overline{\operatorname{dom}}\, B = \mathfrak{M} = \operatorname{ran} B$, $\ker B = \{0\}$ and $\operatorname{dom} B^* \cap \operatorname{ran} K = \{0\}$. Now $\operatorname{dom}(B^*KB) = \{0\}$ and, hence, $U := \Upsilon_1(K)\Upsilon_2(B)$ is a unitary operator with $\ker(\mathcal{P}_{\mathfrak{j}\mathfrak{M}}U) = \ker U$, see Proposition 7.3 and (7.3).

Furthermore, if $V$ is as above, then

$$\ker(\mathcal{P}_{\mathfrak{M}}V) + \ker(\mathcal{P}_{\mathfrak{j}\mathfrak{M}}V) = \operatorname{dom} V \quad \text{if and only if} \quad \operatorname{ran}(SB) \subseteq \operatorname{dom} B^*. \tag{7.4}$$

Example 7.14 (i) below shows that for two hyper-maximal neutral subspaces $\mathfrak{L}_0$ and $\mathfrak{L}_1$ there always exists a unitary operator $U$ such that $\mathfrak{L}_0 = \ker(\mathcal{P}_{\mathfrak{M}}U)$, $\mathfrak{L}_1 = \ker(\mathcal{P}_{\mathfrak{j}\mathfrak{M}}U)$ and $\ker(\mathcal{P}_{\mathfrak{M}}U) + \ker(\mathcal{P}_{\mathfrak{j}\mathfrak{M}}U) = \operatorname{dom} U$. Also an isometric operator with the same properties is given which can not be extended to a unitary operator, see Example 7.14 (ii) below. Recall that if $\mathfrak{L}_0$ and $\mathfrak{L}_1$ are extension of a closed neutral subspace $\mathfrak{L}$ and their sum coincides with the orthogonal complement $\mathfrak{L}^{[\perp]}$, then $\mathfrak{L}_0$ and $\mathfrak{L}_1$ are traditionally called transversal extensions of $\mathfrak{L}$. For such cases it is well known that there exists a bounded unitary operator such that $\mathfrak{L}_0 = \ker(\mathcal{P}_{\mathfrak{M}}U)$ and $\mathfrak{L}_1 = \ker(\mathcal{P}_{\mathfrak{j}\mathfrak{M}}U)$, see (Derkach & Malamud 1995: Proposition 1.3).

**Example 7.14.** Let $\mathfrak{j}$ be a fundamental symmetry of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and assume that there exists an infinite-dimensional hyper-maximal neutral subspace $\mathfrak{M}$ in $\{\mathfrak{K}, [\cdot,\cdot]\}$.

(i) Let $\mathfrak{L}$ be an arbitrary hyper-maximal neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then, by Proposition 2.20, there exists a selfadjoint relation $K$ in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ such that $\mathfrak{L} = \{f + i\mathfrak{j}f' : \{f, f'\} \in \operatorname{gr} K\}$. Now a direct calculation shows that the unitary relation $U := \mathfrak{j}\Upsilon_1(K^{-1})\mathfrak{j}$ is such that $\mathfrak{M} = \ker(\mathcal{P}_{\mathfrak{j}\mathfrak{M}}U)$, $\mathfrak{L} = \ker(\mathcal{P}_{\mathfrak{M}}U)$ and $\ker(\mathcal{P}_{\mathfrak{M}}U) + \ker(\mathcal{P}_{\mathfrak{j}\mathfrak{M}}U) = \operatorname{dom} U$.

(ii) Let $S$ be a symmetric operator in the Hilbert space $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ with unequal defect numbers and $\overline{\operatorname{dom}}\, S = \mathfrak{M} = \overline{\operatorname{ran}}\, S$, and let $B$ be a closed operator with $\overline{\operatorname{dom}}\, B = \mathfrak{M} = \operatorname{ran} B$ and $\ker B = \{0\}$ such that $\operatorname{dom} B^* = \operatorname{ran} S$, see Proposition 2.17 (ii). Then $K := B^*SB$ is a selfadjoint operator in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$, because by the assumptions $\operatorname{ran} K = \mathfrak{M}$. Now $V := U_1(S)\Upsilon_2(B)$ is an isometric operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$ which cannot be extended to a unitary operator, see Proposition 7.3, while $\mathfrak{L}_0 := \ker(\mathcal{P}_{\mathfrak{M}}V) = \mathfrak{j}\mathfrak{M}$ and $\mathfrak{L}_1 := \ker(\mathcal{P}_{\mathfrak{j}\mathfrak{M}}V) = \{f + \mathfrak{j}iB^*(-S)Bf : f \in \operatorname{dom}(B^*SB)\} = \{f - \mathfrak{j}iKf : f \in \operatorname{dom} K\}$ are hyper-maximal neutral subspaces of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Finally, note that $\operatorname{dom} V = \mathfrak{L}_0 + \mathfrak{L}_1$ by (7.4), because $\operatorname{ran}(SB) = \operatorname{dom} B^*$ by construction.

## 7.3 Block representations for unitary operators

Continuing the investigations from the preceding section, here block representations for unitary operators are presented. For instance, it is shown that each unitary operator can be written (w.r.t. certain coordinates) as the composition of an archetypical unitary operator of the type $\Upsilon_2(B)$ and a bounded unitary operator. This shows that the unbounded part of a unitary operator can always be represented by a block diagonal unitary operator. To obtain the indicated represented the following lemma is needed; note that Theorem 7.23 below extends this lemma.

**Lemma 7.15.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$, let $\mathfrak{j}_1$ be a fundamental symmetry of $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ and let $\mathfrak{L}\subseteq \operatorname{dom} U$ be a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ such that $U(\mathfrak{L}^{[\perp]_1})$ is a neutral subspace of $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ with equal defect numbers in the Kreĭn space $\{U(\mathfrak{L}\cap \mathfrak{j}_1\mathfrak{L}))^{[\perp]_2},[\cdot,\cdot]_2\}$. Then $U(\mathfrak{j}_1\mathfrak{L}\cap \operatorname{dom} U)$ is a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$.*

*Proof.* Since $U$ is unitary, assume w.l.o.g. that $\operatorname{mul} U=\{0\}$ or, equivalently, that $\overline{\operatorname{ran}}\, U=\mathfrak{K}_2$. Then the statement follows directly from Remark 7.10 (i). □

**Theorem 7.16.** *Let $U$ be a unitary operator from $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ and let $\mathfrak{j}_i$ be a fundamental symmetry of $\{\mathfrak{K}_i,[\cdot,\cdot]_i\}$, for $i=1,2$. Then there exists a hyper-maximal semi-definite subspace $\mathfrak{M}\subseteq \operatorname{ran} U$ of $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$, a closed operator $B$ in the Hilbert space $\{\mathfrak{M}^{[\perp]_2},[\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\overline{\operatorname{dom}}\, B=\mathfrak{M}^{[\perp]_2}=\operatorname{ran} B$ and $\ker B=\{0\}$, and a bounded unitary operator $U_t$ from $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ onto $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ with $\operatorname{dom} U\subseteq \operatorname{dom} U_t$ such that*

$$UU_t^{-1}=\Upsilon_2(B)\oplus I_{\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}}.$$

*In particular, if $\mathfrak{L}$ is a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ such that $\mathfrak{L}\subseteq \operatorname{dom} U$ and that $U(\mathfrak{L}^{[\perp]_1})$ is a neutral subspace with equal defect numbers in the Kreĭn space $\{(U(\mathfrak{L}\cap \mathfrak{j}_1\mathfrak{L}))^{[\perp]_2},[\cdot,\cdot]_2\}$, then the subspace $\mathfrak{M}$ can be taken to be $U(\mathfrak{j}_1\mathfrak{L}\cap \operatorname{dom} U)$.*

*Proof.* The existence of a subspace $\mathfrak{L}$ as in the statement follows directly from Proposition 6.7 and by Lemma 7.15 $\mathfrak{M}:=U(\mathfrak{j}_1\mathfrak{L}\cap \operatorname{dom} U)$ is a hyper-maximal semi-definite subspace. To complete the proof it suffices to show that with this $\mathfrak{M}$ the indicated decomposition of $U$ holds. W.l.o.g. this will only be done in case that $\mathfrak{L}$, and hence also $\mathfrak{M}$, are hyper-maximal neutral subspaces, see Remark 7.10 (ii).

Now by Lemma 7.8 there exists a standard unitary operator $U_h$ from $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ with $\operatorname{dom} U\subseteq \operatorname{dom} U_h$ mapping $\mathfrak{L}$ onto $\mathfrak{j}_2\mathfrak{M}$ and $\mathfrak{j}_1\mathfrak{L}\cap\operatorname{dom} U_t$ onto $\mathfrak{M}$.

Therefore the unitary operator $UU_h^{-1}$ without kernel has w.r.t. the decomposition $\mathfrak{M} \oplus \mathfrak{j}_2\mathfrak{M}$ of $\mathfrak{K}_2$, the block representation

$$UU_h^{-1} = \begin{pmatrix} B & iC\mathfrak{j}_2 \\ 0 & \mathfrak{j}_2 D\mathfrak{j}_2 \end{pmatrix},$$

where $B$ is a closed operator satisfying $\overline{\operatorname{dom}} B = \operatorname{clos}(U_h(\mathfrak{j}_1\mathfrak{L} \cap \operatorname{dom} U)) = \mathfrak{M}$, $\operatorname{ran} B = U(\mathfrak{j}_1\mathfrak{L} \cap \operatorname{dom} U) = \mathfrak{M}$, $\ker B = \{0\} = \operatorname{mul} B$, and $C$ and $D$ are operators satisfying $\operatorname{dom} C = \operatorname{dom} D = U_h(\mathfrak{L} \cap \operatorname{dom} U) = U_h(\mathfrak{L}) = \mathfrak{j}_2\mathfrak{M}$. Now the arguments as in Theorem 7.9 show that $D = B^{-*}$ and that $C = SB^{-*}$ for a symmetric operator $S$ in $\{\mathfrak{M}, [\mathfrak{j}_2\cdot,\cdot]\}$. Hence, $UU_h^{-1}$ can be written as

$$UU_h^{-1} = \begin{pmatrix} B & iSB^{-*}\mathfrak{j}_2 \\ 0 & \mathfrak{j}_2 B^{-*}\mathfrak{j}_2 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & \mathfrak{j}_2 B^{-*}\mathfrak{j}_2 \end{pmatrix} \begin{pmatrix} I & iB^{-1}SB^{-*}\mathfrak{j}_2 \\ 0 & I \end{pmatrix}.$$

Here the second equality holds because $\operatorname{ran} B = \mathfrak{M}$. Next observe that $K := B^{-1}SB^{-*}$ is a symmetric operator, because $\operatorname{mul}(UU_h^{-1}) = \{0\}$, with $\operatorname{dom} K = \mathfrak{M}$, because $\operatorname{dom}(SB^{-*}) = \operatorname{dom} C = \mathfrak{M}$ and $\operatorname{ran} B = \mathfrak{M}$. This shows that $K$ is a everywhere defined selfadjoint operator and, hence, $\Upsilon_1(K)$ is a standard unitary operator. Therefore the statement holds with $U_t = \mathfrak{j}_2\Upsilon_1(K)\mathfrak{j}_2 U_h$. □

The diagonal block representation for $U$ in Theorem 7.16 holds only for special coordinates, it can be generalized to the case of arbitrary coordinates by means of standard unitary operators.

**Corollary 7.17.** *Let $U$ be a unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with strongly equal defect numbers (see Chapter 8 below) and let $\mathfrak{j}_2$ be a fundamental symmetry for $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then for every hyper-maximal neutral subspace $\mathfrak{N}$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ there exists a closed operator $B_{\mathfrak{N}}$ in $\{\mathfrak{N}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\overline{\operatorname{dom}} B_{\mathfrak{N}} = \mathfrak{N} = \operatorname{ran} B_{\mathfrak{N}}$ and $\ker B_{\mathfrak{N}} = \{0\}$ and a standard unitary operator $U_c$ in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that*

$$U_c U U_t^{-1} U_c^{-1} = \Upsilon_2(B_{\mathfrak{N}}).$$

*Proof.* Since $U$ has strongly equal defect numbers, there exists a hyper-maximal neutral subspace $\mathfrak{M}$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, a closed operator $B$ in $\{\mathfrak{M}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ satisfying $\overline{\operatorname{dom}} B = \mathfrak{M} = \operatorname{ran} B$ and $\ker B = \{0\}$, and a bounded unitary operator $U_t$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ onto $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\operatorname{dom} U \subseteq \operatorname{dom} U_t$ such that $UU_t^{-1} = \Upsilon_2(B)$, see Corollary 8.18 below. Therefore the statement follows by taking $U_c$ to be the standard unitary operator in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ which maps $\mathfrak{M}$ onto $\mathfrak{N}$ and $\mathfrak{j}_2\mathfrak{M}$ onto $\mathfrak{j}_2\mathfrak{N}$, see Lemma 4.13. □

Next an example is given of how a unitary operator with a block representation can be rewritten such that the unbounded part is in diagonal form. Note that the unitary operator under consideration is connected with so-called generalized boundary triplets, see (Derkach & Malamud 1995: Definition 6.1), see also Theorem 7.19 below.

**Example 7.18.** Let $\{\mathfrak{H}^2, < \cdot,\cdot >\}$ be the Kreĭn space with fundamental symmetry j associated to the Hilbert space $\{\mathfrak{H}, (\cdot,\cdot)\}$ as in Example 2.1. In $\{\mathfrak{H}^2, < \cdot,\cdot >\}$, consider the operator $U$ whose block decomposition w.r.t. $\mathfrak{H} \times \mathfrak{H}$ is given by

$$U = \begin{pmatrix} B & 0 \\ KB & B^{-*} \end{pmatrix},$$

where $B$ is an unbounded closed operator in $\{\mathfrak{H}, (\cdot,\cdot)\}$ with $\overline{\operatorname{dom}} B = \mathfrak{H} = \operatorname{ran} B$ and $\ker B = \{0\}$, and $K$ is a selfadjoint operator in $\{\mathfrak{H}, (\cdot,\cdot)\}$ with $\operatorname{dom} K = \mathfrak{H}$. Since $\Upsilon_1(K)$ is a standard unitary operator and $\Upsilon_2(B)$ is a unitary operator, it follows that $U = \Upsilon_1(K)\Upsilon_2(B)$ is a unitary operator.

To obtain a block representation of $U$ where the unbounded part is in diagonal form, note first that $\{0\} \times \mathfrak{H} \subseteq \operatorname{dom} U$ is a hyper-maximal neutral subspace of $\{\mathfrak{H}^2, < \cdot,\cdot >\}$ which is mapped onto the essentially hyper-maximal neutral subspace $\{0\} \times \operatorname{dom} B^*$. Therefore Theorem 7.16 implies that $U$ has a diagonal block representation with respect to hyper-maximal neutral subspace $\mathfrak{M} := \{\{Bf, KBf\} : f \in \operatorname{dom}(KB)\} = \operatorname{gr} K$ of $\{\mathfrak{H}^2, < \cdot,\cdot >\}$. Hence, to obtain a diagonal block representation with respect to the hyper-maximal neutral subspace $\mathfrak{H} \times \{0\}$ of $\{\mathfrak{H}^2, < \cdot,\cdot >\}$, a standard unitary operator in $\{\mathfrak{H}^2, < \cdot,\cdot >\}$ needs to be found which maps $\mathfrak{H} \times \{0\}$ onto $\mathfrak{M}$, see Corollary 7.17. A direct calculation shows that $U_{\mathfrak{M}}$ defined as :

$$U_{\mathfrak{M}} = \begin{pmatrix} C & -KC \\ KC & C \end{pmatrix}, \quad C = (I + KK)^{-1/2},$$

where the block representation is w.r.t. the decomposition $\mathfrak{H} \times \mathfrak{H}$ of $\mathfrak{H}^2$, is a standard unitary operator in $\{\mathfrak{H}^2, < \cdot,\cdot >\}$ with the desired properties. Now

$$\begin{aligned}
(U_{\mathfrak{M}})^{-1} U &= \begin{pmatrix} C & CK \\ -CK & C \end{pmatrix} \begin{pmatrix} B & 0 \\ KB & B^{-*} \end{pmatrix} \\
&= \begin{pmatrix} C(I + KK)B & CKB^{-*} \\ 0 & CB^{-*} \end{pmatrix} \\
&= \begin{pmatrix} C^{-1}B & C^{-1}BB^{-1}CCKB^{-*} \\ 0 & (C^{-1}B)^{-*} \end{pmatrix} \\
&= \begin{pmatrix} C^{-1}B & 0 \\ 0 & (C^{-1}B)^{-*} \end{pmatrix} \begin{pmatrix} I & B^{-1}CCKB^{-*} \\ 0 & I \end{pmatrix}.
\end{aligned}$$

Clearly, $B^{-1}CCKB^{-*} = B^{-1}CKCB^{-*}$ is an everywhere defined symmetric operator in the Hilbert space $\{\mathfrak{H}, (\cdot,\cdot)\}$, i.e. it is a bounded selfadjoint operator. Therefore the following decomposition of $U$ has been obtained:

$$U = \begin{pmatrix} C & -KC \\ KC & C \end{pmatrix} \begin{pmatrix} C^{-1}B & 0 \\ 0 & (C^{-1}B)^{-*} \end{pmatrix} \begin{pmatrix} I & B^{-1}CKCB^{-*} \\ 0 & I \end{pmatrix}.$$

The unboundedness of $U$ is now completely expressed by the unitary diagonal block operator, the other two operators in the righthand side of the above equality are standard unitary operators.

Next some further necessary and sufficient conditions for an isometric relation to be unitary are stated; note that the following result extends (Derkach et al. 2006: Lemma 5.5)[1]. Theorem 7.19 below shows that up to a standard unitary transformation each unitary boundary triplet whose domain contains a selfadjoint relation is a generalized boundary triplet, see also Section A.2.

**Theorem 7.19.** *Let $U$ be an isometric operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then $U$ is unitary if and only if there exists a hyper-maximal semi-definite subspace $\mathfrak{M}$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that*

(i) $\mathfrak{M} = \mathcal{P}_{\mathfrak{M}}\operatorname{ran} U$ *and* $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M} \subseteq \operatorname{ran} U$*;*

(ii) $\ker(\mathcal{P}_{\mathfrak{M}^{[\perp]_2}} U)$ *is a hyper-maximal semi-definite subspace of* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$.

*In particular, if (i)-(ii) hold, then there exists a closed operator $B$ in (the Hilbert space) $\{\mathfrak{M}^{[\perp]_2}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\overline{\operatorname{dom}} B = \mathfrak{M}^{[\perp]_2} = \operatorname{ran} B$ and $\ker B = \{0\}$, a selfadjoint operator $K$ in $\{\mathfrak{M}^{[\perp]_2}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\operatorname{dom} K = \mathfrak{M}^{[\perp]_2}$ and a bounded unitary operator $U_t$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_2\}$ onto $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\operatorname{dom} U \subseteq \operatorname{dom} U_t$, mapping $\ker(P_{\mathfrak{M}}U)$ onto $\mathfrak{j}_2\mathfrak{M}$, such that*

$$UU_t^{-1} = \Upsilon_1(K)\Upsilon_2(B) \oplus I_{\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}}. \tag{7.5}$$

*Proof.* If $U$ is unitary, then $\mathfrak{M}$ as in Theorem 6.8 satisfies (i)-(ii). In fact, in that case $\mathfrak{M} \subseteq \operatorname{ran} U$. To prove the sufficiency of the conditions, it suffices to prove that $U$ has the indicated block decomposition if the stated conditions hold, see Proposition 7.3. Since $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M} \subseteq \operatorname{ran} U$, this is w.l.o.g. only done in case that $\mathfrak{M}$, and hence also $\mathfrak{L} := \ker(\mathcal{P}_{\mathfrak{M}^{[\perp]_2}} U)$, is a hyper-maximal neutral subspace.

[1]Note that in (Derkach et al. 2006: Lemma 5.5) $A_0$ should be selfadjoint.

*Step 1:* Recall that by assumption $U(\mathfrak{L}) \subseteq \mathrm{j}_2\mathfrak{M}$. Next it is shown that the assumptions (i) and (ii) imply that $\mathrm{clos}\,(U(\mathfrak{L})) = \mathrm{j}_2\mathfrak{M}$. Therefore note first that the assumption that $\mathfrak{L}$ is hyper-maximal neutral implies that

$$\mathfrak{L} + \mathrm{dom}\,U \cap \mathfrak{K}_1^+ = \mathrm{dom}\,U = \mathfrak{L} + \mathrm{dom}\,U \cap \mathfrak{K}_1^-. \tag{7.6}$$

Now let $f_o \in \mathrm{j}_2\mathfrak{M}\ominus\mathrm{clos}\,(U(\mathfrak{L}))$, then by the assumption (i) together with (7.6) there exists an $f \in \mathrm{dom}\,U \cap \mathfrak{K}_1^+$ such that $\mathcal{P}_{\mathfrak{M}}Uf = \mathrm{j}_2 f_o$. Consequently, $[Uf, Ug]_2 = 0$ for every $g \in \mathfrak{L}$ and, hence, $[f,g]_1 = 0$ for every $g \in \mathfrak{L}$. Since $f \in \mathrm{dom}\,U \cap \mathfrak{K}_1^+$ and $\mathfrak{L}$ is hyper-maximal neutral, the preceding equality can only hold if $f = 0$. Consequently, $\mathrm{clos}\,(U(\mathfrak{L})) = \mathrm{j}_2\mathfrak{M}$. Now let $f' \in (\mathrm{ran}\,U)^{\perp_2}$, then $\mathrm{clos}\,(U(\mathfrak{L})) = \mathrm{j}_2\mathfrak{M}$ implies that $f' \in \mathfrak{M}$. Then (i) implies that $f' = 0$, i.e. $\overline{\mathrm{ran}}\,U = \mathfrak{K}_2$.

*Step 2:* Since it has been shown that $\overline{\mathrm{ran}}\,U = \mathfrak{K}_2$, Theorem 7.9 implies that there exists an operator $B$ in $\{\mathfrak{M}, [\mathrm{j}_2\cdot,\cdot]_2\}$ with $\overline{\mathrm{dom}}\,B = \mathfrak{M} = \mathrm{ran}\,\mathrm{clos}\,(B)$ and $\ker\mathrm{clos}\,(B) = \{0\}$, a symmetric operator $K$ in $\{\mathfrak{M}, [\mathrm{j}_2\cdot,\cdot]_2\}$ with $\mathrm{dom}\,K = \mathrm{ran}\,B$ and a standard unitary operator $U_3$ in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\mathrm{dom}\,U_i \subseteq \mathrm{dom}\,U_3$, mapping $\ker\,(P_{\mathfrak{M}}U_i)$ onto $\mathrm{j}_2\mathfrak{M}$, such that

$$U_a U_3^{-1} = \Upsilon_1(K)\Upsilon_2(B) = \begin{pmatrix} B & 0 \\ \mathrm{j}_2 iKB & \mathrm{j}_2 B^{-*}\mathrm{j}_2 \end{pmatrix}. \tag{7.7}$$

Now the assumption (i) implies that $\mathfrak{M} = \mathrm{ran}\,B$ and, hence, $\mathrm{dom}\,K = \mathfrak{M}$, i.e. $K$ is a bounded selfadjoint operator and $\mathrm{ran}\,B = \mathfrak{M}$ together with $\ker\,(\mathrm{clos}\,(B)) = \{0\}$ implies that $B$ is closed, see (2.8). This shows that (7.5) holds. □

The two conditions in Theorem 7.19 are independent of each other, i.e. there exist unitary operators for which either only (i) holds or only (ii) holds. First an example of a unitary operator is given which satisfies condition (i), but not condition (ii).

**Example 7.20.** Let $\{\mathfrak{H}^2, <\cdot,\cdot>\}$ be the Kreĭn space associated to the Hilbert space $\{\mathfrak{H}, (\cdot,\cdot)\}$ as in Example 2.1. Let $S$ be a closed symmetric operator in $\{\mathfrak{H}, (\cdot,\cdot)\}$ with defect numbers $n_+(S) = 1$ and $n_-(S) = 0$ such that $\overline{\mathrm{dom}}\,S = \mathfrak{H} = \overline{\mathrm{ran}}\,S$. Moreover, let $B$ be a closed operator in $\{\mathfrak{H}, (\cdot,\cdot)\}$ with $\mathrm{dom}\,B = \mathfrak{H}$, $\mathrm{ran}\,B = \mathrm{ran}\,S$ and $\ker B = \{0\}$, see Proposition 2.17 (ii), then $K := B^{-1}SB^{-*}$ is a selfadjoint operator in $\{\mathfrak{H}, (\cdot,\cdot)\}$ with $\mathrm{ran}\,K = \mathfrak{H}$. Now $U$ defined as

$$U = \begin{pmatrix} KB^{-*} & -B \\ B^{-*} & 0 \end{pmatrix},$$

where the block representation is w.r.t. the decomposition $\mathfrak{H} \times \mathfrak{H}$ of $\mathfrak{H}^2$, is a unitary operator in $\{\mathfrak{K}, <\cdot,\cdot>\}$, see e.g. Lemma 4.7. Clearly, $\mathcal{P}_{\mathfrak{H}\times\{0\}}U \supseteq \mathrm{ran}\,K = \mathfrak{H}$, while on the other hand

$$\begin{aligned}\ker(\mathcal{P}_{\mathfrak{H}\times\{0\}}U) &= \{\{f,f'\}\in \operatorname{dom} U : KB^{-*}f + Bf' = 0\}\\ &= \{\{f,f'\}\in \operatorname{dom} U : f' = -B^{-1}KB^{-*}f\} = \operatorname{gr}(-S).\end{aligned}$$

Since $S$ is by assumption not selfadjoint in $\{\mathfrak{H},(\cdot,\cdot)\}$, the above calculation shows that $\ker(\mathcal{P}_{\mathfrak{H}\times\{0\}}U)$ is not hyper-maximal neutral, see Proposition 2.20.

Instead of giving a concrete example of a unitary operator which satisfies condition (ii) in Theorem 7.19 but not condition (i), here a block representation characterization for unitary operators satisfying condition (ii) is given. In particular, this shows that such unitary operators are closely connected to those which do satisfy condition (i) and (ii) in Theorem 7.19. In that connection recall that the condition (ii) is a very strong one, i.e. isometric operators which satisfy it are quite close to being unitary, see Section 7.2.

**Corollary 7.21.** *Let $U$ be a unitary operator from $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$, let $\mathfrak{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ and let $\mathfrak{M}$ be a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_2,[\cdot,\cdot]\}$ such that $\mathfrak{M}\cap \mathfrak{j}_2\mathfrak{M}\subseteq \operatorname{ran} U$. Then equivalent are:*

(i) $\ker(\mathcal{P}_{\mathfrak{M}^{[\perp]_2}}U)$ *is a hyper-maximal semi-definite subspace of* $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$;

(ii) *there exists an operator $B$ in the Hilbert space $\{\mathfrak{M}^{[\perp]_2},[\mathfrak{j}_2\cdot,\cdot]_2\}$ satisfying $\overline{\operatorname{dom}}\, B = \mathfrak{M}^{[\perp]_2} = \operatorname{ran}\operatorname{clos}(B)$ and $\ker\operatorname{clos}(B)=\{0\}$, a selfadjoint operator $K$ in $\{\mathfrak{M}^{[\perp]_2},[\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\operatorname{dom} K = \operatorname{ran} B$ and a bounded unitary operator $U_t$ from $\{\mathfrak{K}_1,[\cdot,\cdot]_2\}$ onto $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ with $\operatorname{dom} U\subseteq \operatorname{dom} U_t$, mapping $\ker(P_{\mathfrak{M}}U)$ onto $\mathfrak{j}_2\mathfrak{M}$, such that*

$$UU_t^{-1} = \Upsilon_1(K)\Upsilon_2(B)\oplus I_{\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}}.$$

*Proof.* As a consequence of the assumption that $\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}\subseteq\operatorname{ran} U$ assume w.l.o.g. that $\mathfrak{M}$ is a hyper-maximal neutral subspace. If (i) holds, then (ii) follows from Theorem 7.9 and Remark 7.10 (i). Conversely, if (ii) holds, then $\ker(\mathcal{P}_{\mathfrak{M}^{[\perp]_2}}UU_t^{-1}) = \mathfrak{j}_2\operatorname{ran} B^* = \mathfrak{j}_2\mathfrak{M}$, i.e., $\ker(\mathcal{P}_{\mathfrak{M}^{[\perp]_2}}UU_t^{-1})$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$. Consequently, $\ker(\mathcal{P}_{\mathfrak{M}^{[\perp]_2}}U) = U_t^{-1}(\ker(\mathcal{P}_{\mathfrak{M}^{[\perp]_2}}UU_t^{-1}))$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$. □

Corollary 7.22 below contains conditions for the unitary operator in (7.5) to be a bounded unitary operator, which differ from the usual condition that the range of the unitary operator is onto, see also Section 4.1.

**Corollary 7.22.** *Let $U$ be a unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, let $\mathfrak{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{M}$ be a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that Theorem 7.19 (i) and (ii) hold. Then $U$ is a bounded unitary operator if and only if*

$$\mathfrak{j}_2\mathfrak{M} = \mathcal{P}_{\mathfrak{j}_2\mathfrak{M}}\mathrm{ran}\, U \quad \textit{and} \quad \ker(\mathcal{P}_{\mathfrak{M}}U) + \ker(\mathcal{P}_{\mathfrak{j}\mathfrak{M}}U) = \mathrm{dom}\, U.$$

*Proof.* By assumption $U$ has the representation in (7.5). In fact, since $K$ is a bounded selfadjoint operator, $\Upsilon_1(K)$ is a standard unitary operator therein. Moreover, since $B$ is closed and $\mathrm{ran}\, B = \mathfrak{M}^{[\perp]_2} = \overline{\mathrm{dom}}\, B$ in (7.5), $U$ is a bounded unitary operator, i.e. $\mathrm{ran}\, U = \mathfrak{K}_2$, if and only if $\mathrm{dom}\, B^* = \mathfrak{M}^{[\perp]_2}$. It is clear (see e.g. (7.7)) that $\mathrm{dom}\, B^* = \mathfrak{M}^{[\perp]_2}$ if and only if $\mathrm{ran}\,(KB) \subseteq \mathrm{dom}\, B^*$ and $\mathcal{P}_{\mathfrak{j}_2\mathfrak{M}}\mathrm{ran}\, U = \mathfrak{j}_2\mathfrak{M}$. In view of (7.4), this observation proves the equivalence. □

Finally, necessary and sufficient conditions for the isometric operators under consideration in Section 7.2 to be (extendable to) unitary relations are given. Note that Theorem 7.23 below is a (partial) inverse to Lemma 7.15.

**Theorem 7.23.** *Let $U$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{j}_i$ be a fundamental symmetry of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$, for $i = 1, 2$. Moreover, assume that there exists a hyper-maximal semi-definite subspace $\mathfrak{M}$ of $\{\mathfrak{K}_2, [\cdot,\cdot]\}$ such that $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M} \subseteq \mathrm{ran}\, U$ and that $\mathfrak{L} := \ker(\mathcal{P}_{\mathfrak{M}^{[\perp]_2}}U)$ is a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then $U$ is (extendable to) a unitary relation if and only if $U(\mathfrak{j}_1\mathfrak{L} \cap \mathrm{dom}\, U)$ is (extendable to) a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$.*

*Proof.* Since $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M} \subseteq \mathrm{ran}\, U$, assume w.l.o.g. that $\mathfrak{M}$ and $\mathfrak{L}$ are hyper-maximal neutral subspaces. It can also be assumed that $U$ is closed, because if $U$ is not closed, then $\mathrm{clos}\,(U)$ clearly satisfies the same conditions. Moreover, $U$ can also w.l.o.g. be assumed to be an operator with a trivial kernel, see Lemma 3.11. Then arguments as in Step 1 of Theorem 7.9 show that w.r.t. to the decomposition $\mathfrak{L} \oplus_1 \mathfrak{j}_1\mathfrak{L}$ of $\mathfrak{K}_1$ and the decomposition $\mathfrak{M} \oplus_2 \mathfrak{j}_2\mathfrak{M}$ of $\mathfrak{K}_2$ $U$ has the following block representation:

$$U = \begin{pmatrix} 0 & C\mathfrak{j}_1 \\ \mathfrak{j}_2 B & \mathfrak{j}_2 iSC\mathfrak{j}_1 \end{pmatrix},$$

where $B$ and $C$ are operators from $\{\mathfrak{L}, [\mathfrak{j}_1\cdot,\cdot]_1\}$ to $\{\mathfrak{M}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\mathrm{dom}\, B = \mathfrak{L}$, $\ker B = \{0\} = \ker C$ and $C \subseteq B^{-*}$, and $S$ is a symmetric operator in $\{\mathfrak{M}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\mathrm{dom}\, S = \mathrm{ran}\, C$. Moreover, since $U$ is by assumption closed, $B$ needs to be closed. Now assume that $U(\mathfrak{j}_1\mathfrak{L} \cap \mathrm{dom}\, U)$ is (extendable to) a hyper-maximal neutral subspace, then the above representation shows that $S$ is (extendable to) a selfadjoint relation $K$ in $\{\mathfrak{M}, [\mathrm{j}_2\cdot,\cdot]_2\}$. Using $K$, $U_a$ defined via

$$\operatorname{gr} U_a = \{\{f + \mathrm{j}_1 g, B^{-*}g + \mathrm{j}_2(Bf + iKB^{-*}g)\} : f \in \mathfrak{M} \text{ and } g \in \operatorname{dom}(KB^{-*})\}$$

is a unitary extension of $U$, because $\mathfrak{L} \subseteq \operatorname{dom} U_a$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $U_a(\mathrm{j}_1\mathfrak{L} \cap \operatorname{dom} U_a) = \{f + \mathrm{j}_2 iKf : f \in \operatorname{dom} K\}$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, see Lemma 4.7. Note that here it was used that $\operatorname{ran} B^{-*} = \mathfrak{M}$. Hence, if $U(\mathrm{j}_1\mathfrak{L} \cap \operatorname{dom} U)$ is (extendable to) a hyper-maximal neutral subspace, then $U$ is (extendable to) a unitary relation. The converse implication is a direct consequence of Lemma 7.15. □

## 7.4 Block representations and Calkin

Here the block diagonal representation of unitary operators from Theorem 7.16 is used to furnish simple proofs for a number of statements from (Calkin 1939a). Starting with the following two statements which are the abstract analogues of (Calkin 1939a: Lemma 4.3 & Theorem 4.13) and of (Calkin 1939a: Lemma 4.4 & Theorem 4.15); they show how unitary relations can change the defect numbers of closed neutral subspaces.

**Proposition 7.24.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ which does not have a closed domain. Then there exists a maximal neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U$ in $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that*

(i) $\operatorname{clos}(U(\mathfrak{L}))$ *is a maximal neutral subspace of* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$*;*

(ii) *for every $0 \leq m \leq \aleph_0$ there exists a closed neutral subspace $\mathfrak{L}_m$ with $\ker U \subseteq \mathfrak{L}_m \subseteq \mathfrak{L}$ such that $\operatorname{clos}(U(\mathfrak{L}_m)) = \operatorname{clos}(U(\mathfrak{L}))$ and $n_\pm(\mathfrak{L}_m) = n_\pm(\mathfrak{L}) + m$.*

*Proof.* To prove the statement w.l.o.g. assume that $\ker U = \{0\} = \operatorname{mul} U$ and let $\mathrm{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then by Theorem 7.16, there exists a hyper-maximal semi-definite subspace $\mathfrak{M}$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, a closed operator $B$ in $\{\mathfrak{M}^{[\perp]_2}, [\mathrm{j}_2\cdot,\cdot]\}$ with $\overline{\operatorname{dom}} B = \mathfrak{M}^{[\perp]_2} = \operatorname{ran} B$ and $\ker B = \{0\}$, and a standard unitary operator $U_t$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ onto $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\operatorname{dom} U \subseteq \operatorname{dom} U_t$ such that $UU_t^{-1} = \Upsilon_2(B) \oplus I_{\mathfrak{M}\cap\mathrm{j}_2\mathfrak{M}}$. Since standard unitary operators do not change the defect numbers of neutral subspace, see Proposition 4.5, it suffice to proof the statement for the unitary operator $U_a := \Upsilon_2(B) \oplus I_{\mathfrak{M}\cap\mathrm{j}_2\mathfrak{M}}$ in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$.

From the properties of $B$, it follows that $\mathfrak{L} := \mathrm{j}_2\mathfrak{M} \subseteq \operatorname{dom} U_a$ is a maximal neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and that $\operatorname{clos}(U_a(\mathfrak{L})) = \operatorname{clos}(\mathrm{j}_2\operatorname{dom} B^*) = \mathrm{j}_2\mathfrak{M}$ is also a maximal neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Since $B$ is an unbounded operator (because $U$ by assumption does by not have closed domain), Corollary 2.18 implies that there exists for every $0 \leq m \leq \aleph_0$ an $m$-dimensional closed subspace $\mathfrak{N}_m$ of (the Hilbert space) $\{\mathfrak{M}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ such that $\mathfrak{M} = \operatorname{clos}(B^{-*}(\mathfrak{M} \ominus_2 \mathfrak{N}_m))$. This shows that, with $\mathfrak{L}$ as above, the statement holds for $\mathfrak{L}_m := \mathfrak{j}_2(\mathfrak{M} \ominus_2 \mathfrak{N}_m)$. □

**Proposition 7.25.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ which does not have a closed domain and let $\mathfrak{j}_1$ be a fundamental symmetry of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then for every $m \leq \aleph_0$ there exists a hyper-maximal semi-definite subspace $\mathfrak{L} \subseteq \operatorname{dom} U$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $U(\mathfrak{L}^{[\perp]_2})$ is a closed neutral subspace of the Kreĭn space $\{\mathfrak{K}_2 \cap (U(\mathfrak{L} \cap \mathfrak{j}_1\mathfrak{L}))^{[\perp]_2}, [\cdot,\cdot]_2\}$ with defect numbers $n_\pm(U(\mathfrak{L}^{[\perp]_2})) = m$.*

*Proof.* W.l.o.g. assume that $\operatorname{mul} U = \{0\}$. Then $U = (\Upsilon_2(B) \oplus I_{\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}})U_t$, where $\mathfrak{M}$, $B$ and $U_t$ are as in Theorem 7.16 and $\mathfrak{j}_2$ is a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. From the assumption that $U$ does not have closed domain it follows that $\Upsilon_2(B)$ is an unbounded unitary operator. Hence, by Proposition 4.10 there exists for every $m \leq \aleph_0$ a hyper-maximal neutral subspace $\mathfrak{M}_m \subseteq \operatorname{dom}(\Upsilon_2(B))$ such that $\Upsilon_2(B)(\mathfrak{M}_m)$ is a closed neutral subspace in $\{\mathfrak{K}_2 \cap (\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M})^{[\perp]_2}, [\cdot,\cdot]_2\}$ with defect numbers $n_\pm(\Upsilon_2(B)(\mathfrak{M}_m)) = m$. Consequently, the statement holds for $\mathfrak{L} := U_t^{-1}(\mathfrak{M}_m \oplus_2 (\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}))$, because $U_t$ is a bounded unitary operators, see Proposition 4.5. □

Proposition 7.25 implies in particular that the domain of a unitary relation always contains a hyper-maximal semi-definite subspace which is mapped onto a hyper-maximal semi-definite subspace. Combining this observation with Corollary 7.12 and Proposition 4.8 yields another representation for unitary relations.

**Corollary 7.26.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then there exists a hyper-maximal semi-definite subspace $\mathfrak{M}$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, a selfadjoint relation $K$ in the Hilbert space $\{\mathfrak{M}^{[\perp]_2}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ and a bounded unitary operator $U_t$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ onto $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\operatorname{dom} U \subseteq \operatorname{dom} U_t$ such that*

$$UU_t^{-1} = \Upsilon_1(K) \oplus I_{\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}}.$$

Now it is shown that for every unitary relation in a separable Kreĭn space which does not have closed domain there exists another unitary relation, also necessarily having a non-closed domain, having the same kernel such that the intersection of their domains is their kernel. This statement can be found from (Calkin 1939a: 416) where no proof for the assertion is given. In order to give a proof for that statement, the following lemma will be used.

**Lemma 7.27.** *Let* $\mathfrak{j}$ *be a fundamental symmetry of* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *and let* $\mathfrak{K}^+[+]\mathfrak{K}^-$ *be the associated canonical decomposition of* $\{\mathfrak{K}, [\cdot,\cdot]\}$. *Moreover, let* $\mathfrak{M}$ *be a hyper-maximal neutral subspace of* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *and let* $\mathfrak{D}$ *be a dense subspace of* $\mathfrak{K}^+$ *(*$\mathfrak{K}^-$*) which is an operator range. Then there exists a unitary operator* $U$ *in* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *satisfying*

$$\operatorname{dom} U = \mathfrak{j}\mathfrak{M} + \mathfrak{D} \quad \textit{and} \quad \operatorname{dom} U \cap \mathfrak{K}^+ = \mathfrak{D} \quad (\operatorname{dom} U \cap \mathfrak{K}^- = \mathfrak{D}).$$

*Proof.* Only the case $\mathfrak{D} \subseteq \mathfrak{K}^+$ is considered. First note that the assumption that $\mathfrak{D}$ is an operator range and that $\operatorname{clos}\mathfrak{D} = \mathfrak{K}^+$ implies that there exists a bounded operator $B$ in the Hilbert space $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ with $\operatorname{dom} B = \mathfrak{M} = \overline{\operatorname{ran}}\, B$ and $\ker B = \{0\}$ such that $\mathfrak{D} = \{Bf + \mathfrak{j}Bf : f \in \mathfrak{M}\}$, cf. Proposition 2.17 (ii). Then $U = \Upsilon_2(B^{-1})$ is a unitary operator with $\operatorname{dom} U = \operatorname{ran} B \oplus \mathfrak{j}\mathfrak{M}$, see Section 4.2. Moreover,

$$\operatorname{dom} U \cap \mathfrak{K}^+ = \operatorname{dom} U \cap \{f + \mathfrak{j}f : f \in \mathfrak{M}\} = \{f + \mathfrak{j}f : f \in \operatorname{ran} B\} = \mathfrak{D}.$$

Clearly, from the above calculation it also follows that $\operatorname{dom} U = \mathfrak{j}\mathfrak{M} + \mathfrak{D}$. □

**Theorem 7.28.** *Let* $U$ *be a unitary relation from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *which does not have a closed domain such that* $\{\overline{\operatorname{ran}}\, U/\operatorname{mul} U, [\cdot,\cdot]_2\}$ *is a separable Kreĭn space. Then there exists a unitary relation* $U_a$ *from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$*, which does not have a closed domain, with*

$$\ker U_a = \ker U \quad \textit{and} \quad \operatorname{dom} U \cap \operatorname{dom} U_a = \ker U.$$

*Proof.* In the proof let $\mathfrak{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, let $\mathfrak{K}_2^+[+]\mathfrak{K}_2^-$ be the corresponding canonical decomposition of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and w.l.o.g. assume that $U$ is an operator, see Corollary 3.12. Then by Theorem 7.16 there exists a hyper-maximal semi-definite subspace $\mathfrak{M}$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, an unbounded closed operator $B$ in the, by assumption, separable Hilbert space $\{\mathfrak{M}^{[\perp]_2}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\overline{\operatorname{dom}}\, B = \mathfrak{M}^{[\perp]_2} = \operatorname{ran} B$ and $\ker B = \{0\}$, and a bounded unitary operator $U_t$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\operatorname{dom} U \subseteq \operatorname{dom} U_t$ such that

$$UU_t^{-1} = \Upsilon_2(B) \oplus_2 I_{\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}}.$$

By Proposition 2.17 (ii) and (iv) there exists a bounded selfadjoint operator $K$ in $\{\mathfrak{M}^{[\perp]_2}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\overline{\operatorname{ran}}\, K = \mathfrak{M}^{[\perp]_2}$, $\operatorname{dom} B \cap \operatorname{ran} K = \{0\}$ and $\operatorname{dom} B + \operatorname{ran} K \neq \mathfrak{M}^{[\perp]_2}$. Then $\mathfrak{L} := \{f + \mathfrak{j}_2 iK^{-1}f : f \in \operatorname{ran} K\}$ is a hyper-maximal neutral subspace in $\{\mathfrak{K}_2 \ominus_2 (\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}), [\cdot,\cdot]_2\}$ with $\mathfrak{L} \cap \Upsilon_2(B) = \{0\}$.

Since $\operatorname{ran} K + \operatorname{dom} B$ is a nonclosed operator range, see Proposition 2.17 (i), using Proposition 2.17 (ii) and (v) once more yields the existence of a closed bounded (selfadjoint) operator $D$ in $\{\mathfrak{M}^{[\perp]_2}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\overline{\mathrm{ran}}\, D = \mathfrak{M}^{[\perp]_2}$ and $\ker D = \{0\}$, such that $\mathrm{ran}\, D \cap (\mathrm{ran}\, K + \mathrm{dom}\, B) = \{0\}$ and $\mathrm{ran}\, D + \mathrm{ran}\, K + \mathrm{dom}\, B \neq \mathfrak{M}^{[\perp]_2}$. Then the subspace $\mathfrak{D} := \{f + \mathfrak{j}_2 f : f \in \mathrm{ran}\, D\}$ is a uniformly definite subspace of $\{\mathfrak{K}_2 \ominus_2 (\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}), [\cdot,\cdot]_2\}$ such that $\mathrm{clos}\,(\mathfrak{D}) = \mathfrak{K}_2^+ \ominus_2 (\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M})$. Hence, by Lemma 7.27 there exists a unitary operator $U_2$ in $\{\mathfrak{K}_2 \ominus_2 (\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}), [\cdot,\cdot]_2\}$ with $\mathrm{dom}\, U_2 = \mathfrak{L} + \mathfrak{D}$. Moreover, since by construction $(\mathfrak{L}+\mathfrak{D}) \cap \mathrm{dom}\,(\Upsilon_2(B)) = \{0\}$, $\mathrm{dom}\, U_2 \cap \mathrm{dom}\,(\Upsilon_2(B)) = \{0\}$.

If $\mathfrak{M}$ is a (hyper-maximal) neutral subspace, then the statement holds with $U_a = U_2 U_t$. Next assume that $\mathfrak{M}$ is not neutral, but, w.l.o.g., assume that $\mathfrak{M}$ is nonnegative, i.e. $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M} \subseteq \mathfrak{K}_2^+$. Since $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ is a separable space, $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}$ has at most the dimension $\aleph_0$. Recall that

$$\begin{aligned}\mathrm{dom}\,(\Upsilon_2(B)) &= \mathrm{dom}\, B \oplus_2 \mathfrak{j}\mathfrak{M}^{[\perp]_2};\\ \mathrm{dom}\, U_2 &= \{Df + \mathfrak{j}_2 Df : f \in \mathfrak{M}^{[\perp]_2}\} + \{Kf + \mathfrak{j}_2 if : f \in \mathfrak{M}^{[\perp]_2}\}.\end{aligned}$$

Since also $\mathrm{dom}\, B + \mathrm{dom}\, D + \mathrm{ran}\, K$ is a nonclosed operator range, see Proposition 2.17 (i), there exists by Proposition 2.17 (vi) an infinite-dimensional closed subspace $\mathfrak{D}_e$ such that $\mathfrak{D}_e \cap (\mathrm{dom}\, B + \mathrm{dom}\, D + \mathrm{ran}\, K) = \{0\}$. Hence, $\mathfrak{D}_e^+ = \{f + \mathfrak{j}f : f \in \mathfrak{D}_e\}$ is an infinite-dimensional closed subspace of $\mathfrak{K}_2^+$ such that

$$\mathfrak{D}_e^+ \cap \mathrm{dom}\, U_2 = \{0\} \quad \text{and} \quad (\mathfrak{D}_e^+ + \mathrm{dom}\, U_2) \cap \Upsilon_2(B) = \{0\}.$$

Now let $U_i$ be the standard unitary operator in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ which is the identity mapping on $\mathfrak{K}_2 \ominus_2 (\mathfrak{D}_e^+ \oplus_2 \mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M})$, maps $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}$ onto $\mathfrak{D}_e^+$ and $\mathfrak{D}_e^+$ onto $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}$. Then $U_m := (U_2 \oplus I_{\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}})U_i$ is a unitary operator in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $\mathrm{dom}\, U_m \cap \mathrm{dom}\,(UU_t^{-1}) = \{0\}$. Consequently, $U_a := U_m U_t$ satisfies the conditions. □

Combining Theorem 7.28 with Proposition 6.7 yields the following statement, see (Calkin 1939a: Theorem 4.6).

**Corollary 7.29.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ which does not have a closed domain such that $\{\overline{\mathrm{ran}}\, U/\mathrm{mul}\, U, [\cdot,\cdot]_2\}$ is a separable Kreĭn space. Then there exists a hyper-maximal semi-definite subspace $\mathfrak{L}$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $\mathfrak{L} \cap \mathrm{dom}\, U = \ker U$.*

## 7.5 Compositions of unitary operators

As a further application of the block representations for isometric operators presented in Section 7.2 and 7.3, here conditions for when the composition of a unitary operator with an isometric operator is (extendable to) a unitary operator are given. Two distinct cases are considered: The composition of unitary operators with isometric operators with a trivial kernel and, secondly, the composition of unitary operators with bounded unitary operators with a non-trivial kernel.

**Proposition 7.30.** *Let $U$ be a unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, let $\mathfrak{j}_2$ a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, let $\mathfrak{M}$ be a hyper-maximal neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $\ker(\mathcal{P}_{\mathfrak{M}}U)$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and let $V$ be a closed isometric operator in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\ker V = \{0\}$. Moreover, let $B$, $K$ and $U_t$ be as in Corollary 7.21 (ii) such that*

$$UU_t^{-1} = \Upsilon_1(K)\Upsilon_2(B). \tag{7.8}$$

*Then $VU$ can be extended to a unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\ker(\mathcal{P}_{\mathfrak{M}}VU) = \ker(\mathcal{P}_{\mathfrak{M}}U)$ if and only if there exists a closed relation $D$ in the Hilbert space $\{\mathfrak{M}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ such that $D^{-*}B^{-*}$ is a closed operator satisfying $\operatorname{dom}(D^{-*}B^{-*}) = \mathfrak{M}$ and $\ker(D^{-*}B^{-*}) = \{0\}$ , and a symmetric operator $S$ in $\{\mathfrak{M}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ which has a selfadjoint extension $K_S$ satisfying $\operatorname{dom} K_S \cap \ker(D^{-*}B^{-*})^* = \{0\}$, such that $V$ is an extension of*

$$\Upsilon_1(S)\Upsilon_2(D)\Upsilon_1(-K).$$

*In particular, $\operatorname{clos}(VU)$ is a unitary operator if and only if $V$ is an isometric extension of $\Upsilon_1(S)\Upsilon_2(D)\Upsilon_1(-K)$ as above and, additionally, $\operatorname{clos}(S)$ is selfadjoint and $\operatorname{clos}(DI_{\operatorname{dom} K}B)) = (D^{-*}B^{-*})^{-*}$.*

*Proof.* If $VU$ can be extended to a unitary operator and $\ker(\mathcal{P}_{\mathfrak{M}}VU) = \ker(\mathcal{P}_{\mathfrak{M}}U)$, then $VUU_t^{-1}$, where $U_t$ is as in (7.8), is an isometric operator in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $\ker(\mathcal{P}_{\mathfrak{M}}VUU_t^{-1}) = \mathfrak{j}_2\mathfrak{M}$. Hence, as in step 1 of the proof of Theorem 7.9, there exist operators $B_1$ and $C$ in $\{\mathfrak{M}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ with $B_1 \subseteq C^{-*}$, $\operatorname{dom} C = \mathfrak{M}$, $\ker C = \{0\} = \operatorname{mul} C$ and a symmetric operator $T$ in $\{\mathfrak{M}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\operatorname{dom} T = \operatorname{ran} B_1$ such that

$$VUU_t^{-1} = \begin{pmatrix} B_1 & 0 \\ \mathfrak{j}_2 iTB_1 & \mathfrak{j}_2C\mathfrak{j}_2 \end{pmatrix} = \Upsilon_1(T)\begin{pmatrix} B_1 & 0 \\ 0 & \mathfrak{j}_2C\mathfrak{j}_2 \end{pmatrix}. \tag{7.9}$$

Since $VU$, and hence also $VUU_t^{-1}$, is extendable to a unitary operator, it follows that $\operatorname{mul}\operatorname{clos} C = \{0\}$. This observation together with $\operatorname{dom} C = \mathfrak{M}$ yields that $C$ is a closed operator. Moreover, since $VU$ is extendable to a unitary operator, $T$ is extendable to a selfadjoint operator $K_S$ such that $\operatorname{dom} K_S \cap (\operatorname{ran} C)^\perp = \{0\}$, see Remark 7.9 (i) and step 3 of the proof of Theorem 7.9.

Combining (7.8) and (7.9) yields

$$V \upharpoonright_{\operatorname{ran} U} = \Upsilon_1(T) \begin{pmatrix} B_1 B^{-1} & 0 \\ 0 & \mathfrak{j}_2 C B^* \mathfrak{j}_2 \end{pmatrix} \Upsilon_1(-K). \tag{7.10}$$

Since $V$ is by assumption closed, the closure of the righthand side of (7.10) is contained in $V$. Hence, the assumption that $V$ is an operator with a trivial kernel implies that the operator $E := CB^*$ satisfies $\ker \operatorname{clos}(E) = \{0\} = \operatorname{mul} \operatorname{clos}(E)$. Hence, $D := E^{-*}$ is a relation which satisfies the stated conditions, because

$$D^{-*}B^{-*} = \operatorname{clos}(E)B^{-*} = C + \{0\} \times \operatorname{mul} \operatorname{clos}(E) = C.$$

Hence, by taking $S$ to be the restriction of $T$ to $\operatorname{ran}(B_1 B^{-1})$ the necessity of the conditions is clear.

Conversely, let $D$ and $S$ be as in the statement, then with $\Delta := \operatorname{dom} K \oplus \mathfrak{j}_2 \mathfrak{M}$

$$\Upsilon_1(S)\Upsilon_2(D)\Upsilon_1(-K)U = \Upsilon_1(S)\Upsilon_2(D) I_\Delta \Upsilon_2(B) U_t.$$

Now observe that

$$\Upsilon_2(D) I_\Delta \Upsilon_2(B) = \begin{pmatrix} D I_{\operatorname{dom} K} B & 0 \\ 0 & \mathfrak{j}_2 D^{-*} B^{-*} \mathfrak{j}_2 \end{pmatrix} \subseteq \Upsilon_2((D^{-*}B^{-*})^{-*}).$$

By the assumptions $E := (D^{-*}B^{-*})^{-*}$ is a (closed) relation in $\{\mathfrak{M}, [\mathfrak{j}_2 \cdot, \cdot]_2\}$ satisfying $\overline{\operatorname{dom}} E = \mathfrak{M} = \operatorname{ran} E$ and $\ker E = \{0\}$. Hence, if $K_S$ is a selfadjoint extension of $S$ such that $\operatorname{dom} K_S \cap \operatorname{mul} E = \{0\}$, then the above calculations show that $\Upsilon_1(S)\Upsilon_2(D)\Upsilon_1(-K)UU_t^{-1}$ can be extended to the unitary operator $\Upsilon_1(K_S)\Upsilon_2(E)$, see Proposition 7.3, i.e., $VU$ can be extended to the unitary operator $\Upsilon_1(K_S)\Upsilon_2(E)U_t$, see Lemma 3.10.

The final equivalence is clear by the above observations. □

Note that the isometric operator $\Upsilon_1(S)\Upsilon_2(D)\Upsilon_1(-K)$ in Proposition 7.30 need not be extendable to a unitary operator. Consider for instance the case that $D = I$, and that $S$ and $-K$ are the selfadjoint operators $K_1$ and $K_2$ from Example 7.1. However, in the case that $U$ and $VU$ in Proposition 7.30 are the abstract equivalent of generalized boundary triplets, then $V$ must be a unitary operator.

**Corollary 7.31.** *Let $U$ be a unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, let $\mathfrak{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{M}$ be a hyper-maximal neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $\ker(\mathcal{P}_\mathfrak{M} U)$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and that $\mathcal{P}_\mathfrak{M} \operatorname{ran} U = \mathfrak{M}$. Moreover, let $V$ be a closed isometric operator in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\ker V = \{0\}$ such that $\ker(\mathcal{P}_\mathfrak{M} VU) = \ker(\mathcal{P}_\mathfrak{M} U)$ and $\mathcal{P}_\mathfrak{M} \operatorname{ran}(VU) = \mathfrak{M}$. Then $VU$ is a unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and $V$ is a unitary relation in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$.*

*Proof.* The assumptions on $VU$ imply by Theorem 7.19 that $VU$ is a unitary operator. Moreover, Theorem 7.19 implies that $K$ and $\operatorname{clos}(S)$ in Proposition 7.30 are bounded selfadjoint operators in the Hilbert space $\{\mathfrak{M}, [\mathfrak{j}_2\cdot,\cdot]\}$ and, therefore, $\Upsilon_1(S)$ and $\Upsilon_1(-K)$ are standard unitary operators in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. From this it follows that $\operatorname{clos}(\Upsilon_1(S)\Upsilon_2(D)\Upsilon_1(-K)) = \Upsilon_1(S)\Upsilon_2(\operatorname{clos}(D))\Upsilon_1(-K)$ is a unitary relation in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Since $\Upsilon_1(S)\Upsilon_2(D)\Upsilon_1(-K) \subseteq V$ and $V$ is by assumption closed, this implies that $V$ itself is a unitary operator in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. □

In Proposition 7.30 the composition of a unitary operator with a closed isometric operator with a trivial kernel was considered. Next the composition of a unitary operator with a bounded unitary operator with a non-trivial kernel is considered.

**Proposition 7.32.** *Let $U$ be a unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Let $\mathfrak{M}$ be a hyper-maximal neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $\mathfrak{L} := \ker(\mathcal{P}_{\mathfrak{M}}U)$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and let $U_b$ be a bounded unitary operator from $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ onto $\{\mathfrak{K}_3, [\cdot,\cdot]_3\}$ such that $\mathfrak{j}_2\mathfrak{M} \subseteq \operatorname{dom} U_b$ or, equivalently, $\ker U_b \subseteq \mathfrak{j}_2\mathfrak{M}$. Then $U_bU$ is an isometric operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_3, [\cdot,\cdot]_3\}$ which can be extended to a unitary relation. In particular, $U_bU$ is a unitary operator if and only if there exists a fundamental symmetry $\mathfrak{j}_1$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $U(\mathfrak{j}_1\mathfrak{L}\cap\operatorname{dom} U)\cap\operatorname{dom} U_b+\ker U_b$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$.*

*Proof.* Note first that if $\mathfrak{j}_2\mathfrak{M} \subseteq \operatorname{dom} U_b$, then $\ker U_b = (\operatorname{dom} U_b)^{[\perp]_2} \subseteq (\mathfrak{j}_2\mathfrak{M})^{[\perp]_2} = \mathfrak{j}_2\mathfrak{M}$ and, conversely, if $\ker U_b \subseteq \mathfrak{j}_2\mathfrak{M}$, then $\mathfrak{j}_2\mathfrak{M} = (\mathfrak{j}_2\mathfrak{M})^{[\perp]_2} \subseteq (\ker U_b)^{[\perp]_2} = \overline{\operatorname{dom}} U_b = \operatorname{dom} U_b$, where in the last step the boundedness of $U_b$ is used.

Since $U(\mathfrak{L}) \subseteq \mathfrak{j}_2\mathfrak{M}$ ($\subseteq \operatorname{dom} U_b$) is a neutral subspace with equal defect numbers and $U_b$ is a bounded unitary operator, $U_b(U(\mathfrak{L}))$ is a neutral subspace with equal defect numbers. Hence, by Theorem 7.23, $U_bU$ is (extendable to) a unitary relation if and only if $U_bU((\mathfrak{j}_1\mathfrak{L}\cap\operatorname{dom} U))$ is (extendable to) a hyper-maximal neutral subspace of $\{\mathfrak{K}_3, [\cdot,\cdot]_3\}$. Since $U_b$ is a bounded unitary operator, this last condition is equivalent to $U(\mathfrak{j}_1\mathfrak{L}\cap\operatorname{dom} U)\cap\operatorname{dom} U_b$ ($+\ker U_b$) being (extendable to) a hyper-maximal neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. But that follows immediately from the fact that $U(\mathfrak{j}_1\mathfrak{L}\cap\operatorname{dom} U)\cap\operatorname{dom} U_b$ is a restriction of $U(\mathfrak{j}_1\mathfrak{L}\cap\operatorname{dom} U)$ which is a hyper-maximal neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ by Lemma 7.15, because $U$ is unitary and $\mathfrak{L} := \ker(\mathcal{P}_{\mathfrak{M}}U)$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. □

Not every composition of a unitary operator with a unitary operator with closed domain can be extended to a unitary operator as the following example shows.

**Example 7.33.** By Example 7.2 there exists a unitary operator $U$ in $\{\mathfrak{K}, [\cdot,\cdot]\}$ which maps a neutral subspace $\mathfrak{L}$ with unequal defect numbers onto a hyper-maximal neutral subspace. Now let $U_b$ be the unitary operator from $\{\mathfrak{K}, [\cdot,\cdot]\}$ to $\{0\}$ whose graph is $U(\mathfrak{L}) \times \{0\}$. Then $U_bU$ is an isometric operator from $\{\mathfrak{K}, [\cdot,\cdot]\}$ to $\{0\}$ whose graph is given by $\mathfrak{L} \times \{0\}$. Clearly, $U_bU$ cannot be extended to a unitary operator, because $\mathfrak{L}$ can not be extended to a hyper-maximal neutral subspace.

Finally, Proposition 7.32 is applied to the abstract equivalent of generalized boundary triplets. The following result will be used in Section A.4 to obtain results on the boundary relations for intermediate extensions.

**Corollary 7.34.** *Let $U$ be a unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, let $\mathrm{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{M}$ be a hyper-maximal neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $\mathcal{P}_{\mathfrak{M}}\mathrm{ran}\, U = \mathfrak{M}$ and that $\ker(\mathcal{P}_{\mathfrak{M}}U)$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Moreover, let $U_b$ be a bounded unitary operator from $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ onto $\{\mathfrak{K}_3, [\cdot,\cdot]_3\}$ such that $\mathrm{j}_2\mathfrak{M} \subseteq \mathrm{dom}\, U_b$ or, equivalently, $\ker U_b \subseteq \mathrm{j}_2\mathfrak{M}$. Then $U_bU$ is a unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_3, [\cdot,\cdot]_3\}$ and $\mathfrak{N} := U_b(\mathfrak{M} \cap \mathrm{dom}\, U_b)$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_3, [\cdot,\cdot]_3\}$ such that,*

$$\mathcal{P}_{\mathfrak{N}}(\mathrm{ran}\,(U_bU)) = \mathfrak{N} \quad \textit{and} \quad \ker(\mathcal{P}_{\mathfrak{N}}(U_bU)) = \ker(\mathcal{P}_{\mathfrak{M}}U),$$

*where $\mathcal{P}_{\mathfrak{N}}$ the orthogonal projection onto $\mathfrak{N}$ w.r.t. $[\mathrm{j}_3\cdot,\cdot]_3$, $\mathrm{j}_3 := U_b\mathrm{j}_2U_b^{-1}$.*

*Proof.* Theorem 7.19 shows that to prove the statement it suffices to shows that the last two equalities hold. Note therefore first that since by assumption $\ker U_b \subseteq \mathrm{j}_2\mathfrak{M}$, $\mathfrak{M}_r := \mathfrak{M} \cap \mathrm{dom}\, U_b$ is a closed subspace such that

$$\mathrm{dom}\, U_b = \mathfrak{M}_r \oplus_2 \mathrm{j}_2\mathfrak{M} = \mathfrak{M}_r \oplus_2 \mathrm{j}_2\mathfrak{M}_r \oplus_2 \ker U_b.$$

Since $\mathfrak{K}_2 \ominus_2 \mathrm{dom}\, U_b = \mathrm{j}_2\ker U_b$, the above formula shows that $\mathfrak{M}_r + \ker U_b \subseteq \mathrm{dom}\, U_b$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and, hence, $\mathfrak{N} := U_b(\mathfrak{M}_r)$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_3, [\cdot,\cdot]_3\}$.

Next note that the assumption $\mathcal{P}_{\mathfrak{M}}\mathrm{ran}\, U = \mathfrak{M}$ together with $\mathrm{j}_2\mathfrak{M} \subseteq \mathrm{dom}\, U_b$ implies that $\mathcal{P}_{\mathfrak{M}_r}(\mathrm{ran}\, U \cap \mathrm{dom}\, U_b) = \mathfrak{M}_r$. Since $\mathrm{j}_3\mathfrak{N} = U_b(\mathrm{j}_2\mathfrak{M})$ by definition of $\mathrm{j}_3$, the preceding observations imply that $\mathcal{P}_{\mathfrak{N}}(\mathrm{ran}\,(U_bU)) = \mathfrak{N}$. Moreover, $\mathrm{j}_3\mathfrak{N} = U_b(\mathrm{j}_2\mathfrak{M})$ together with the assumption $\mathrm{j}_2\mathfrak{M} \subseteq \mathrm{dom}\, U_b$ yields

$$\ker(\mathcal{P}_{\mathfrak{M}}U) = U^{-1}(\mathrm{j}_2\mathfrak{M} \cap \mathrm{ran}\, U) = (U_bU)^{-1}(\mathrm{j}_3\mathfrak{N} \cap \mathrm{ran}\,(U_bU)) = \ker(\mathcal{P}_{\mathfrak{N}}(U_bU)).$$

This completes the proof. □

# 8 A CLASSIFICATION OF UNITARY RELATIONS

Extending upon the work of Calkin (1939a: Ch. 3, Section 4), here a classification of unitary relations into three types is presented and characterized. This classification is introduced and analyzed in order to describe what kind of closed neutral subspaces the domain of a unitary relation can contain. In particular, this approach is used to characterize when the domain of a unitary relation contains a hyper-maximal neutral subspace. More specifically, in the first section a classification of unitary relations is introduced and investigated, and the concept of strongly equal defect numbers is introduced. In the second and third section unitary relations of type I (type Ia and type Ib) and II, respectively, are studied. In particular, these classes of unitary relations are characterized by the closed neutral subspaces contained in their domain and by their diagonal block representation.

## 8.1 Basic properties of the classification

The discussion in Section 6.2 shows that even if the kernel of a unitary relation has equal defect numbers, then it is not a priori clear whether there exist hyper-maximal neutral extensions of the kernel which are contained in the domain of the unitary relation. As is shown in this chapter, that need not be the case, see e.g. Example 8.11 below. Therefore it makes sense to introduce the following definition.

**Definition 8.1.** Let $U$ be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then $\ker U$ is said to have *strongly equal defect numbers* if there exists a hyper-maximal neutral subspace $\mathfrak{L}$ in $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $\mathfrak{L} \subseteq \operatorname{dom} U$.

Clearly, if the kernel of an isometric relation has strongly equal defect numbers, then it also has equal defect numbers. To describe whether the kernel of a unitary relation $U$ has strongly equal defect numbers, in (Calkin 1939a) it was shown that the dimensions of closed subspaces contained in $\operatorname{dom} U \cap \mathfrak{K}_1^+$ and $\operatorname{dom} U \cap \mathfrak{K}_1^-$ need to be considered. Therefore, following Calkin, unitary relations are subdivided into different types according to whether $\operatorname{dom} U \cap \mathfrak{K}_1^+$ and $\operatorname{dom} U \cap \mathfrak{K}_1^-$ contain finite-dimensional of infinite-dimensional closed subspaces, cf. (Calkin 1939a: Definition 3.5). Note that here the definition is stated only for unitary relations, but that most statements that follow only make use of the structure of the domain of the unitary relation and therefore also hold for certain isometric relations.

**Definition 8.2.** Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let

$\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then $U$ is said to be of *type II* if $\operatorname{dom} U \cap \mathfrak{K}_1^+$ and $\operatorname{dom} U \cap \mathfrak{K}_1^-$ both contain infinite-dimensional closed subspaces and of *type I* otherwise. A unitary relation $U$ of type I is said to be of *type Ia* if $\operatorname{dom} U \cap \mathfrak{K}_1^+$ and $\operatorname{dom} U \cap \mathfrak{K}_1^-$ contain both only finite-dimensional closed subspaces and of *type Ib* otherwise.

The well-definedness of Definition 8.2, i.e. the independence of the type of a unitary relation from the canonical decomposition, is not a priori clear. To prove this Proposition 8.3 below suffices; it characterizes the introduced types of unitary relations by means of closed neutral subspaces contained in their domain. Note first however that if $U$ is a unitary relation with closed domain, then $U$ is of type Ia if and only if both defect numbers are finite, of type Ib if and only if precisely one of the defect numbers is finite and of type II if and only if both defect numbers are infinite. Moreover, in that case $\ker U$ has strongly equal defect numbers if and only if $\ker U$ has equal defect numbers.

Note also that if Definition 8.2 is well defined, then Proposition 3.9 implies directly that a unitary relation $U$ is of type Ia, Ib or II if and only if $U^{-1}$ is of type Ia, Ib or II, respectively. The same proposition also shows that if $U_t$ is a bounded unitary operator from $\{\mathfrak{K}_3, [\cdot,\cdot]_3\}$ to $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $\operatorname{dom} U \subseteq \operatorname{ran} U_t$, then $U$ is of type Ia, Ib or II if and only if $UU_t$ is of type Ia, Ib or II, respectively.

**Proposition 8.3.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then*

(i) *$U$ is of type II if and only if there exists a closed neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ with $n_+(\mathfrak{L}) = \infty$ and $n_-(\mathfrak{L}) = \infty$ such that $\mathfrak{L}^{[\perp]_1} \subseteq \operatorname{dom} U$;*

(ii) *$U$ is of type Ib if and only if $U$ is not of type II and there exists a closed neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ with $n_+(\mathfrak{L}) = \infty$ and $n_-(\mathfrak{L}) < \infty$ or $n_+(\mathfrak{L}) < \infty$ and $n_-(\mathfrak{L}) = \infty$ such that $\mathfrak{L}^{[\perp]_1} \subseteq \operatorname{dom} U$;*

(iii) *$U$ is of type Ia if and only if every neutral subspace $\mathfrak{L}$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ with $\mathfrak{L}^{[\perp]_1} \subseteq \operatorname{dom} U$ has finite defect numbers.*

*Proof.* Since the characterizations can be proven by similar arguments, only the equivalence in (i) is proven. First the sufficiency of the condition in (i) is proven, which at the same time proves the well-definedness of Definition 8.2. Hence, let $\mathfrak{L} \subseteq \operatorname{dom} U$ be a closed neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ with $n_+(\mathfrak{L}) = \infty = n_-(\mathfrak{L})$ such that $\mathfrak{L}^{[\perp]_1} \subseteq \operatorname{dom} U$ and let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then $\mathfrak{L}^{[\perp]_1} \cap \mathfrak{K}_1^+$ and $\mathfrak{L}^{[\perp]_1} \cap \mathfrak{K}_1^-$ are infinite-dimensional closed subspaces contained in $\operatorname{dom} U \cap \mathfrak{K}_1^+$ and $\operatorname{dom} U \cap \mathfrak{K}_1^-$, respectively, i.e., $U$ is of type II.

Conversely, assume that $U$ is of type II and let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, with associated fundamental symmetry $\mathrm{j}_1$, such that $\operatorname{dom} U \cap \mathfrak{K}_1^+$ and $\operatorname{dom} U \cap \mathfrak{K}_1^-$ both contain infinite-dimensional closed subspaces. Moreover, let $\mathfrak{M} \subseteq \operatorname{dom} U$ be a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, see Proposition 6.7, and w.l.o.g. assume that $\mathfrak{M}$ is a hyper-maximal nonnegative subspace, i.e. $\mathfrak{M} \cap \mathrm{j}_1\mathfrak{M} \subseteq \mathfrak{K}_1^+$. Then $\mathfrak{M}^{[\perp]_1} \subseteq \mathfrak{M} \subseteq \operatorname{dom} U$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_1 \ominus_1 (\mathfrak{M} \cap \mathrm{j}_1\mathfrak{M}), [\cdot,\cdot]_1\}$. Let $K$ be the angular operator of $\mathfrak{M}^{[\perp]_1}$ w.r.t. $\mathfrak{K}_1^-$:

$$\mathfrak{M}^{[\perp]_1} = \{f^- + Kf^- : f^- \in P_1^-\mathfrak{M}^{[\perp]_1} = \mathfrak{K}_1^-\}.$$

Now let $\mathfrak{D}_1^- \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-$ be an infinite-dimensional closed subspace, which exists by the assumption that $U$ is of type II. Then, since $K$ is a unitary operator from the Hilbert space $\{\mathfrak{K}_1^-, -[\cdot,\cdot]_1\}$ to the Hilbert space $\{\mathfrak{K}_1^+ \ominus_1 (\mathfrak{M} \cap \mathrm{j}_1\mathfrak{M}), [\cdot,\cdot]_1\}$, $K(\mathfrak{D}_1^-)$ is an infinite-dimensional closed subspace of $\operatorname{dom} U \cap (\mathfrak{K}_1^+ \ominus_1 (\mathfrak{M} \cap \mathrm{j}_1\mathfrak{M}))$. Consequently, $\mathfrak{L} := \{f^- + Kf^- : f^- \in \mathfrak{K}_1^- \ominus_1 \mathfrak{D}_1^-\}$ is a closed neutral subspace which satisfies the requirements, because by construction

$$\mathfrak{L}^{[\perp]_1} = \mathfrak{L} + (\mathfrak{M} \cap \mathrm{j}_1\mathfrak{M}) + \mathfrak{D}_1^- + K(\mathfrak{D}_1^-) \subseteq \operatorname{dom} U.$$

This completes the proof. □

If $\operatorname{dom} U \cap \mathfrak{K}_1^+$ and $\operatorname{dom} U \cap \mathfrak{K}_1^-$ contain both one vector, then the proof of Proposition 8.3 shows that there exists a closed neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ with $n_+(\mathfrak{L}) \geq 1$ and $n_-(\mathfrak{L}) \geq 1$ such that $\mathfrak{L}^{[\perp]_1} \subseteq \operatorname{dom} U$. This shows that the number of maximal neutral subspaces contained in the domain of a unitary relation whose kernel has nonzero defect numbers is uncountable, see (Calkin 1939a: Theorem 4.3 & 4.4).

If the defect numbers of the kernel of a unitary relation $U$ are different, then the dimension of maximal closed subspaces contained in $\operatorname{dom} U \cap \mathfrak{K}_1^+$ and $\operatorname{dom} U \cap \mathfrak{K}_1^-$ are different.

**Lemma 8.4.** *Let $U$ be unitary relation between $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ with associated fundamental symmetry $\mathrm{j}_1$. If $n_+(\ker U) > n_-(\ker U)$ or $n_+(\ker U) < n_-(\ker U)$, then there exists a closed subspace $\mathfrak{D}_1^- \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-$ or closed subspace $\mathfrak{D}_1^+ \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^+$ such that $\dim(\mathfrak{D}_1^-) = n_+(\ker U)$ or $\dim(\mathfrak{D}_1^+) = n_-(\ker U)$, respectively.*

*Proof.* W.l.o.g. assume that $\ker U = \{0\}$, and that $\operatorname{dom} U$ is not closed, because otherwise there is nothing to prove by the definition of defect numbers. Recall that, since $U$ is a unitary relation, there exists a hyper-maximal semi-definite subspace

$\mathfrak{M} = \mathfrak{M}^{[\perp]_1} \oplus_1 (\mathfrak{M} \cap \mathfrak{j}_1 \mathfrak{M})$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $\mathfrak{M} \subseteq \operatorname{dom} U$, see Proposition 6.7. If $\mathfrak{M}$ is nonnegative, then $\mathfrak{M} \cap \mathfrak{j}_1 \mathfrak{M} \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^+$,

$$\dim(\mathfrak{K}_1^+) = \dim(\mathfrak{M}^{[\perp]_1}) + \dim(\mathfrak{M} \cap \mathfrak{j}_1 \mathfrak{M}) \quad \text{and} \quad \dim(\mathfrak{K}_1^-) = \dim(\mathfrak{M}^{[\perp]_1}).$$

Similarly, if $\mathfrak{M}$ is nonpositive, then $\mathfrak{M} \cap \mathfrak{j}_1 \mathfrak{M} \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-$,

$$\dim(\mathfrak{K}_1^+) = \dim(\mathfrak{M}^{[\perp]_1}) \quad \text{and} \quad \dim(\mathfrak{K}_1^-) = \dim(\mathfrak{M}^{[\perp]_1}) + \dim(\mathfrak{M} \cap \mathfrak{j}_1 \mathfrak{M}).$$

These observations imply that if $n_+(\ker U) \neq n_-(\ker U)$, then $\dim(\mathfrak{M} \cap \mathfrak{j}_1 \mathfrak{M}) > \dim(\mathfrak{M}^{[\perp]_1})$ (note that here it is used that $\dim(\mathfrak{K}_1^+) = \infty = \dim(\mathfrak{K}_1^-)$, because $\operatorname{dom} U$ is not closed). Since $\mathfrak{M} \cap \mathfrak{j}_1 \mathfrak{M}$ is a closed subspace, the statement follows from the above discussion. □

The difference between unitary relations of type I and II can be characterized by looking at the closed neutral subspaces contained in their domain. In particular, Lemma 8.5 and Theorem 8.6 below give such type of sufficient conditions for a unitary operator to be of type II; note that these results are an extension of (Calkin 1939a: Lemma 4.1 & part of Theorem 4.4).

**Lemma 8.5.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_1^+ [+] \mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Moreover, let $\mathfrak{L} \subseteq \operatorname{dom} U$ be a closed neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then*

(i) *if $\operatorname{dom} U \cap \mathfrak{K}_1^-$ contains an infinite-dimensional closed subspace and $n_+(\mathfrak{L}) < \infty$, then $\operatorname{dom} U \cap \mathfrak{K}_1^+$ contains an infinite-dimensional closed subspace;*

(ii) *if $\operatorname{dom} U \cap \mathfrak{K}_1^+$ contains an infinite-dimensional closed subspace and $n_-(\mathfrak{L}) < \infty$, then $\operatorname{dom} U \cap \mathfrak{K}_1^-$ contains an infinite-dimensional closed subspace.*

*Proof.* Clearly, it suffices to prove only one of the two assertions. Hence, assume that $\mathfrak{L} \subseteq \operatorname{dom} U$ is a closed neutral subspace with $n_+(\mathfrak{L}) < \infty$ and that $\mathfrak{D}_1^- \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-$ is an infinite-dimensional closed subspace. Together the two assumptions imply that $\mathfrak{D}_1^- \cap P_1^- \mathfrak{L}$ is an infinite-dimensional closed subspace contained in $\operatorname{dom} U \cap \mathfrak{K}_1^-$. Now let $K$ be the angular operator for $\mathfrak{L}$ w.r.t. $\mathfrak{K}_1^-$:

$$\mathfrak{L} = \{f^- + K f^- : f^- \in P_1^- \mathfrak{L}\}.$$

Since, $K$ is a closed isometric operator from the Hilbert space $\{\mathfrak{K}_1^-, -[\cdot,\cdot]_1\}$ to the Hilbert space $\{\mathfrak{K}_1^+, [\cdot,\cdot]_1\}$, $K$ maps $\mathfrak{D}_1^- \cap P_1^- \mathfrak{L} \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-$ onto an infinite-dimensional closed subspace of $\operatorname{dom} U \cap \mathfrak{K}_1^+$ (because $\mathfrak{L} \subseteq \operatorname{dom} U$). □

**Theorem 8.6.** *Let* $U$ *be a unitary relation between* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *and* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$*, let* $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ *be a canonical decomposition of* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *with associated projections* $P_1^+$ *and* $P_1^-$*, and assume that there exist closed neutral subspaces* $\mathfrak{L}_1 \subseteq \operatorname{dom} U$ *and* $\mathfrak{L}_2 \subseteq \operatorname{dom} U$ *of* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *satisfying either of the following conditions:*

*(a)* $n_+(\mathfrak{L}_1) < \infty$*,* $\dim\left(P_1^+\mathfrak{L}_1 \ominus_1 \mathcal{I}^+\right) < \infty$*, where* $\mathcal{I}^+ = P_1^+\mathfrak{L}_1 \cap P_1^+\mathfrak{L}_2$*, and*

$$n_+(\mathfrak{L}_1) + \dim\left(P_1^+\mathfrak{L}_1 \ominus_1 \mathcal{I}^+\right) < n_+(\mathfrak{L}_2) + \dim\left(P_1^+\mathfrak{L}_2 \ominus_1 \mathcal{I}^+\right);$$

*(b)* $n_-(\mathfrak{L}_1) < \infty$*,* $\dim\left(P_1^-\mathfrak{L}_1 \ominus_1 \mathcal{I}^-\right) < \infty$*, where* $\mathcal{I}^- = P_1^-\mathfrak{L}_1 \cap P_1^-\mathfrak{L}_2$*, and*

$$n_-(\mathfrak{L}_1) + \dim\left(P_1^-\mathfrak{L}_1 \ominus_1 \mathcal{I}^-\right) < n_-(\mathfrak{L}_2) + \dim\left(P_1^-\mathfrak{L}_2 \ominus_1 \mathcal{I}^-\right).$$

*Then* $\operatorname{dom} U \cap \mathfrak{K}_1^+$ *and* $\operatorname{dom} U \cap \mathfrak{K}_1^-$ *contain infinite-dimensional closed subspaces.*

*Proof.* To prove the statement it suffices to consider only the case that $\mathfrak{L}_1$ and $\mathfrak{L}_2$ satisfy (a). Hence, let the assumptions in (a) hold and denote the (closed) angular operators of $\mathfrak{L}_1$ and $\mathfrak{L}_2$ w.r.t. $\mathfrak{K}_1^+$ by $K_1$ and $K_2$:

$$\mathfrak{L}_1 = \{f^+ + K_1 f^+ : f^+ \in P^+\mathfrak{L}_1\} \quad \text{and} \quad \mathfrak{L}_2 = \{f^+ + K_2 f^+ : f^+ \in P^+\mathfrak{L}_2\}.$$

Then $X := K_2 K_1^{-1}$ is a closed isometric operator in the Hilbert space $\{\mathfrak{K}_1^-, -[\cdot,\cdot]_1\}$ which, because of the assumptions in (a), satisfies

$$\dim(\operatorname{dom} X)^{\perp_1} < \dim(\operatorname{ran} X)^{\perp_1} \quad \text{and} \quad \dim(\operatorname{dom} X)^{\perp_1} < \infty.$$

This implies that there exists a finite-dimensional (closed) isometric extension $Y$ of $X$ such that $\operatorname{dom} Y = \mathfrak{K}_1^-$ and $\operatorname{ran} Y \neq \mathfrak{K}_1^-$. If $\operatorname{ran}(I - Y)$ does not contain an infinite-dimensional closed subspace, then $I - Y$ is a compact operator, see e.g. (Calkin 1939a: Lemma 3.1). Therefore $\operatorname{ran} Y \neq \mathfrak{K}_1^-$ implies by the Fredholm alternative that $\ker Y \neq \{0\}$. Since $Y$ is an isometric operator in a Hilbert space, this is impossible. Consequently, $\operatorname{ran}(I - Y)$ and, hence, also $\operatorname{ran}(I - X)$ contain an infinite-dimensional closed subspace.

Next note that the assumptions $\mathfrak{L}_1 \subseteq \operatorname{dom} U$ and $\mathfrak{L}_2 \subseteq \operatorname{dom} U$ together imply that $\operatorname{ran}(K_1 - K_2) \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-$ and therefore

$$\operatorname{ran}(I - X) = \operatorname{ran}(I - K_2K_1^{-1}) = \operatorname{ran}((K_1 - K_2)K_1^{-1}) \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-.$$

Consequently, the above arguments show that $\operatorname{dom} U \cap \mathfrak{K}_1^-$ contains an infinite-dimensional closed subspaces. In view of Lemma 8.5, this completes the proof. □

## 8.2 Unitary relations of type I

Now unitary relation of type Ia and Ib are considered. In particular, two characterizations for them are given: First by means of the defect numbers of neutral subspaces contained in their domain and, secondly, by specifying their block decomposition. In order to prove the first mentioned characterization, it is shown that, as a consequence of Theorem 8.6, closed neutral subspaces contained in the domain (or range) of a unitary relation of type I have specific defect numbers.

**Proposition 8.7.** *Let* $U$ *be a unitary relation from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *of type I and let* $\mathfrak{L}_1 \subseteq \operatorname{dom} U$ *and* $\mathfrak{L}_2 \subseteq \operatorname{dom} U$ *be closed neutral subspaces of* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *satisfying either of the following conditions:*

*(a)* $n_+(\mathfrak{L}_1) = n_+(\mathfrak{L}_2) < \infty$ *and* $n_-(\mathfrak{L}_2) < \infty$*;*

*(b)* $n_-(\mathfrak{L}_1) = n_-(\mathfrak{L}_2) < \infty$ *and* $n_+(\mathfrak{L}_2) < \infty$*.*

*Then* $n_+(\mathfrak{L}_1) = n_+(\mathfrak{L}_2)$ *and* $n_-(\mathfrak{L}_1) = n_-(\mathfrak{L}_2)$*.*

*Proof.* W.l.o.g. only case (a) is considered. Let $\mathcal{I}^+ := P_1^+\mathfrak{L}_1 \cap P_1^+\mathfrak{L}_2$, then the assumption $n_-(\mathfrak{L}_2) < \infty$ implies that $P_1^+\mathfrak{L}_1 \ominus_1 \mathcal{I}^+$ is a finite-dimensional subspace. Since $U$ is of type I, Theorem 8.6 implies that $n_+(\mathfrak{L}_2) + \dim(P_1^+\mathfrak{L}_2 \ominus_1 \mathcal{I}^+) \leq n_+(\mathfrak{L}_1) + \dim(P_1^+\mathfrak{L}_1 \ominus_1 \mathcal{I}^+) < \infty$. In particular, $\dim(P_1^+\mathfrak{L}_2 \ominus_1 \mathcal{I}^+) < \infty$. Using Theorem 8.6 once more (with $\mathfrak{L}_1 = \mathfrak{L}_2$ and $\mathfrak{L}_2 = \mathfrak{L}_1$) yields that $n_+(\mathfrak{L}_1) + \dim(P_1^+\mathfrak{L}_1 \ominus_1 \mathcal{I}^+) \leq n_+(\mathfrak{L}_2) + \dim(P_1^+\mathfrak{L}_2 \ominus_1 \mathcal{I}^+)$, i.e.

$$n_+(\mathfrak{L}_1) + \dim(P_1^+\mathfrak{L}_1 \ominus_1 \mathcal{I}^+) = n_+(\mathfrak{L}_2) + \dim(P_1^+\mathfrak{L}_2 \ominus_1 \mathcal{I}^+).$$

The above equality together with the assumption that $n_+(\mathfrak{L}_1) = n_+(\mathfrak{L}_2) < \infty$ yields that $\dim(P_1^+\mathfrak{L}_1 \ominus_1 \mathcal{I}^+) = \dim(P_1^+\mathfrak{L}_2 \ominus_1 \mathcal{I}^+)$. Clearly,

$$P_1^+\mathfrak{L}_1 = (P_1^+\mathfrak{L}_1 \ominus_1 \mathcal{I}^+) \oplus_1 \mathcal{I}^+ \quad \text{and} \quad P_1^+\mathfrak{L}_2 = (P_1^+\mathfrak{L}_2 \ominus_1 \mathcal{I}^+) \oplus_1 \mathcal{I}^+. \tag{8.1}$$

Since $n_-(\mathfrak{L}_2) < \infty$ and $\dim(P_1^+\mathfrak{L}_2 \ominus_1 \mathcal{I}^+) = \dim(P_1^+\mathfrak{L}_1 \ominus_1 \mathcal{I}^+) < \infty$, (8.1) implies that $\dim(\mathfrak{K}_1^+ \ominus_1 \mathcal{I}^+) < \infty$. This observation together with (8.1) and the proven fact that $\dim(P_1^+\mathfrak{L}_2 \ominus_1 \mathcal{I}^+) = \dim(P_1^+\mathfrak{L}_1 \ominus_1 \mathcal{I}^+) < \infty$ yields $n_-(\mathfrak{L}_1) = n_-(\mathfrak{L}_2)$. □

In particular, Proposition 8.7 implies that in a separable Hilbert space each maximal neutral subspace contained in the domain of a unitary relation of type I has the same defect numbers, see (Calkin 1939a: Theorem 4.4). Next further properties of the closed neutral subspaces contained in the domain of unitary relations of type I are stated.

**Proposition 8.8.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and, moreover, let $N_U := \min\{\dim(\operatorname{dom} U \cap \mathfrak{K}_1^+), \dim(\operatorname{dom} U \cap \mathfrak{K}_1^-)\}$. If $U$ is of type Ia, then there exists a $d \in \mathbb{N}$ such that either of the following two alternatives holds:*

- *(a1) for every $n \in \mathbb{N}$, $n \leq N_U$, there exists a closed neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U$ with $n_+(\mathfrak{L}) = n$ and $n_-(\mathfrak{L}) = n + d$ and, conversely, if $\mathfrak{L} \subseteq \operatorname{dom} U$ is a closed neutral subspace and $n_+(\mathfrak{L}) < \infty$ or $n_-(\mathfrak{L}) < \infty$, then $n_-(\mathfrak{L}) = n_+(\mathfrak{L}) + d$;*
- *(a2) for every $n \in \mathbb{N}$, $n \leq N_U$, there exists a closed neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U$ with $n_+(\mathfrak{L}) = n + d$ and $n_-(\mathfrak{L}) = n$ and, conversely, if $\mathfrak{L} \subseteq \operatorname{dom} U$ is a closed neutral subspace and $n_+(\mathfrak{L}) < \infty$ or $n_-(\mathfrak{L}) < \infty$, then $n_+(\mathfrak{L}) = n_-(\mathfrak{L}) + d$.*

*If $U$ is of type Ib, then either of the following two alternatives holds:*

- *(b1) for every $n \in \mathbb{N}$, $n \leq N_U$, there exists a closed neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U$ with $n_+(\mathfrak{L}) = n$ and $n_-(\mathfrak{L}) = \infty$ and, conversely, if $\mathfrak{L} \subseteq \operatorname{dom} U$ is a closed neutral subspace, then $n_-(\mathfrak{L}) = \infty$;*
- *(b2) for every $n \in \mathbb{N}$, $n \leq N_U$, there exists a closed neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U$ with $n_+(\mathfrak{L}) = \infty$ and $n_-(\mathfrak{L}) = n$ and, conversely, if $\mathfrak{L} \subseteq \operatorname{dom} U$ is a closed neutral subspace, then $n_+(\mathfrak{L}) = \infty$.*

*Proof.* Let $\mathfrak{M} \subseteq \operatorname{dom} U$ be a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, see Proposition 6.7, and w.l.o.g. assume that $\mathfrak{M}$ is a nonnegative subspace, i.e. $\mathfrak{M} = \mathfrak{M}^{[\perp]_1} + \mathfrak{M} \cap \mathfrak{K}_1^+$. Next let $K$ be the angular operator of $\mathfrak{M}^{[\perp]_1}$ w.r.t. $\mathfrak{K}_1^-$:

$$\mathfrak{M}^{[\perp]_1} = \{f^- + Kf^- : f^- \in P_1^-\mathfrak{M} = \mathfrak{K}_1^-\}.$$

Since $K$ maps closed subspaces of $\operatorname{dom} U \cap \mathfrak{K}_1^-$ onto closed subspaces of $\operatorname{dom} U \cap \mathfrak{K}_1^+$, because $\mathfrak{M}^{[\perp]_1} \subseteq \mathfrak{M} \subseteq \operatorname{dom} U$, the closed subspace $\mathfrak{M} \cap \mathfrak{K}_1^+$ of $\operatorname{dom} U \cap \mathfrak{K}_1^+$ is finite-dimensional if $U$ is of type Ia and infinite-dimensional if $U$ is of type Ib, see Definition 8.2. Denote the dimension of $\mathfrak{M} \cap \mathfrak{K}_1^+$ by $d$.

Clearly, there exists an $n$-dimensional (closed) subspace $\mathfrak{D}_n^-$ of $\operatorname{dom} U \cap \mathfrak{K}_1^-$, where $n$ is as in the statement. Now $\mathfrak{L}$ defined as

$$\mathfrak{L} = \{f^- + Kf^- : f \in \mathfrak{K}_1^- \ominus_1 \mathfrak{D}_n^-\}$$

can be easily seen to satisfy the first condition in (a1) and (b1), because $n_+(\mathfrak{L}) = \dim \mathfrak{D}_n^-$ and $n_-(\mathfrak{L}) = \dim(K(\mathfrak{D}_n^-)) + \dim(\mathfrak{M} \cap \mathfrak{K}_1^+) = \dim \mathfrak{D}_n^- + d$. The second assertion in (a1) follows from the first assertion together with Proposition 8.7. For the second assertion in (b1) note that by the first assertion therein $\operatorname{dom} U \cap \mathfrak{K}_1^-$ can only contain finite-dimensional closed subspaces, see Lemma 8.5. Since $U$ is of type Ib, that implies that $\operatorname{dom} U \cap \mathfrak{K}_1^+$ must contain infinite-dimensional closed subspace. Hence, Lemma 8.6 implies that $n_-(\mathfrak{L}) = \infty$ for any closed neutral subspace which is contained in the domain of $U$; otherwise $U$ would be of type II.

Similar arguments show that (a2) and (b2) hold if $\mathfrak{M}$ is a nonpositive subspace. □

In fact, from Lemma 8.15 below it follows that Proposition 8.8 yields a characterization of unitary relations of type Ia, but not of type Ib. Note also that Proposition 8.8 implies that if $U$ is a unitary relation of type I, then $U$ is of type Ia if and only if there exists a closed neutral subspace in the domain of $U$ with finite defect numbers. As a further consequence of Proposition 8.8, a characterization of unitary relations of type I with strongly equal defect numbers is obtained.

**Corollary 8.9.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ of type I. Then equivalent are:*

(i) *$U$ has strongly equal defect numbers;*

(ii) *there exists a closed neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ with finite and equal defect numbers;*

(iii) *if $\mathfrak{L} \subseteq \operatorname{dom} U$ is a closed neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ with $n_+(\mathfrak{L}) < \infty$ or $n_-(\mathfrak{L}) < \infty$, then $n_+(\mathfrak{L}) = n_-(\mathfrak{L})$.*

*Proof.* The equivalences are all a direct consequence of Proposition 8.8, because all the conditions imply that $U$ is of type Ia and that $d$ in Proposition 8.8 is zero. □

Using Proposition 8.8, a block decomposition characterization of unitary operators of type I can be obtained. That characterization shows that such unitary operators are closely connected to compact operators, cf. (Calkin 1939a: Theorem 3.13).

**Theorem 8.10.** *Let $U$ be an isometric operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then $U$ is a unitary operator of type Ia or Ib if and only if there exists a hyper-maximal semi-definite subspace $\mathfrak{M}$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\dim(\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}) < \infty$, if $U$ is of type Ia, or $\dim(\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}) = \infty$, if $U$ is of type Ib, a closed operator $B$ in (the Hilbert space) $\{\mathfrak{M}^{[\perp]_2}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\overline{\operatorname{dom} B} = \mathfrak{M}^{[\perp]_2} = \operatorname{ran} B$ and $\ker B = \{0\}$ such that $B^{-1}$ is a compact operator,* *and a bounded unitary operator from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *with* $\operatorname{dom} U \subseteq \operatorname{dom} U_t$ *such that*

$$UU_t^{-1} = \Upsilon_2(B) \oplus I_{\mathfrak{M}\cap \mathfrak{j}_2\mathfrak{M}}.$$

*In particular,* $U$ *has strongly equal defect numbers if and only if* $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M} = \{0\}$.

*Proof.* Clearly, $U$ has the stated representation for a hyper-maximal semi-definite subspace $\mathfrak{M}$ if and only if $U$ is unitary, see Theorem 7.16. Therefore the first statement is proven by observing that if $\mathfrak{M}$ is hyper-maximal nonnegative, then

$$\begin{aligned}
\operatorname{dom}(UU_t^{-1}) \cap \mathfrak{K}_2^+ &= \{f + \mathfrak{j}_2 f + g : f \in \operatorname{dom} B \text{ and } g \in \mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}\};\\
\operatorname{dom}(UU_t^{-1}) \cap \mathfrak{K}_2^- &= \{f - \mathfrak{j}_2 f : f \in \operatorname{dom} B\},
\end{aligned}$$

and if $\mathfrak{M}$ is hyper-maximal nonpositive, then

$$\begin{aligned}
\operatorname{dom}(UU_t^{-1}) \cap \mathfrak{K}_2^+ &= \{f + \mathfrak{j}_2 f : f \in \operatorname{dom} B\};\\
\operatorname{dom}(UU_t^{-1}) \cap \mathfrak{K}_2^- &= \{f - \mathfrak{j}_2 f + g : f \in \operatorname{dom} B \text{ and } g \in \mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}\}.
\end{aligned}$$

Here $\mathfrak{K}_2^+[+]\mathfrak{K}_2^-$ is the canonical decomposition of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ corresponding to $\mathfrak{j}_2$. The above equalities show that $U$ is of type I if and only if $\operatorname{dom} B$ contains only finite dimensional closed subspaces, i.e. if and only if $B^{-1}$ is a compact operator, see e.g. (Calkin 1939a: Lemma 3.1). Moreover, the same equalities show that $U$ is of type Ia or Ib if and only if $\dim(\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}) < \infty$ or $\dim(\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}) = \infty$, respectively. This proves the first part of the statement.

The necessity of the condition in the last equivalence in the statement is clear by definition, because if $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M} = \{0\}$, then $U_t^{-1}(\mathfrak{j}_2\mathfrak{M})$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ which is contained in the domain of $U$. Conversely, assume that $U$ has strongly equal defect numbers and $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M} \neq \{0\}$. Then by the proven decomposition $U_t^{-1}(\mathfrak{M}^{[\perp]_2}) \subseteq \operatorname{dom} U$ is a maximal neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ which is not hyper-maximal neutral. Hence, Corollary 8.9 implies that $U$ does not have strongly equal defect numbers, which is in contradiction with the assumption. □

Note that if $U$ is a unitary operator of type I and $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are two subspaces such that the decomposition in Theorem 8.10 holds with respect to them, then Proposition 8.7 implies that $\dim(\mathfrak{M}_1 \cap \mathfrak{j}_2\mathfrak{M}_1) = \dim(\mathfrak{M}_2 \cap \mathfrak{j}_2\mathfrak{M}_2)$. Furthermore, if $U$ is an unbounded unitary operator of type Ia, then Theorem 8.10 shows that $n_+(\ker U) = n_-(\ker U)$. If $U$ is an unbounded operator of type Ib, then $\ker U$ need not have equal defect numbers as the following example illustrates.

**Example 8.11.** Let $B$ be an everywhere defined compact operator in the separable (infinite-dimensional) Hilbert space $\{\mathfrak{H}, (\cdot,\cdot)\}$ with $\ker B = \{0\}$ and $\overline{\operatorname{ran}}\, B = \mathfrak{H}$.

Moreover, let $U_2$ be a unitary operator in the Hilbert space $\{\mathcal{H}, (\cdot,\cdot)\}$ and define $[\cdot,\cdot]$ on $\mathfrak{K} := \mathfrak{H}^2 \times \mathcal{H}$ by

$$[\{f, f', f''\}, \{g, g', g''\}] = i\,[(f, g') - (f', g)] + (f'', g''),$$

where $f, f', g, g' \in \mathfrak{H}$ and $f'', g'' \in \mathcal{H}$. Then $\{\mathfrak{K}, [\cdot,\cdot]\}$ is a Kreĭn space, cf. Example 2.1. W.r.t. the decomposition $\mathfrak{H} \times \mathfrak{H} \times \mathcal{H}$ of $\mathfrak{K}$ define $U$ as

$$U = \begin{pmatrix} B^{-1} & 0 & 0 \\ 0 & B^* & 0 \\ 0 & 0 & U_2 \end{pmatrix}.$$

Then $U$ is a unitary operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$ with $\ker U = \{0\}$ and

$$n_+(\ker U) = \dim(\mathfrak{H}) \quad \text{and} \quad n_-(\ker U) = \dim(\mathfrak{H}) + \dim(\mathcal{H}). \tag{8.2}$$

Now Theorem 8.10 implies that $U$ is a unitary operator of type Ib if and only if $\mathcal{H}$ is infinite-dimensional. Combining this with (8.2) shows that $U$ is a unitary operator of type Ib with $n_+(\ker U) = n_-(\ker U)$ if $\infty = \dim(\mathcal{H}) \leq \dim(\mathfrak{H})$ and that $U$ is a unitary operator of type Ib with $n_+(\ker U) < n_-(\ker U)$ if $\dim(\mathcal{H}) > \dim(\mathfrak{H})$.

**Remark 8.12.** Example 8.11 shows, in light of Lemma 3.11, that for every neutral subspace $\mathfrak{L}$ in a Kreĭn space $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ with defect numbers $n_+(\mathfrak{L}) = \aleph_0 = n_-(\mathfrak{L})$ there exists an (unbounded) unitary operator $U$ (of type I) from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $\mathfrak{L} = \ker U$ and that there does not exist a hyper-maximal neutral extension of $\mathfrak{L}$ which is contained in the domain of $U$, cf. Corollary 7.29.

## 8.3 Unitary relations of type II

In this section two classes of unitary relations of type II are studied: Those with strongly equal defect numbers and those without strongly equal defect numbers. As in the preceding section, two characterization of these classes of unitary relations are given: First by means of the defect numbers of closed neutral subspaces contained in their domain and, secondly, by specifying their block representation.

Proposition 8.13 below gives a characterization of unitary relations of type II with strongly equal defect numbers among other things in terms of the closed neutral subspaces contained in their domain.

**Proposition 8.13.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ of type II, let $\mathfrak{j}_1$ be a fundamental symmetry of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be the associated canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then equivalent are:*

(i) *$U$ has strongly equal defect numbers;*

(ii) *for every $n_\pm \in \mathbb{N}$ there exists a closed neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U$ such that $n_+(\mathfrak{L}) = n_+$ and $n_-(\mathfrak{L}) = n_-$;*

(iii) *there exists a closed neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ with finite defect numbers;*

(iv) *there exist closed neutral subspaces $\mathfrak{L}_1 \subseteq \operatorname{dom} U$ and $\mathfrak{L}_2 \subseteq \operatorname{dom} U$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ with $n_+(\mathfrak{L}_1) < \infty$ and $n_-(\mathfrak{L}_2) < \infty$;*

(v) *for every closed subspace $\mathfrak{D}_1^+ \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^+$ and every closed subspace $\mathfrak{D}_1^- \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-$ there exists a closed subspace $\mathfrak{D}_{1,a}^+ \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^+$ and a closed subspace $\mathfrak{D}_{1,a}^- \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-$ such that $\dim(\mathfrak{D}_1^+) = \dim(\mathfrak{D}_{1,a}^-)$ and $\dim(\mathfrak{D}_1^-) = \dim(\mathfrak{D}_{1,a}^+)$;*

(vi) *for every hyper-maximal semi-definite subspace $\mathfrak{M} \subseteq \operatorname{dom} U$ there exists a closed subspace $\mathfrak{D}_1^+ \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^+$ and a closed subspace $\mathfrak{D}_1^- \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-$ such that $\dim(\mathfrak{M} \cap \mathfrak{j}_1\mathfrak{M}) \leq \dim \mathfrak{D}_1^+$ and $\dim(\mathfrak{M} \cap \mathfrak{j}_1\mathfrak{M}) \leq \dim \mathfrak{D}_1^-$.*

*Proof.* (i) ⇒ (ii): If $U$ has strongly equal defect numbers, then there exists a hyper-maximal neutral subspace $\mathfrak{M} \subseteq \operatorname{dom} U$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Let $K$ be the angular operator of $\mathfrak{M}$ w.r.t. $\mathfrak{K}_1^+$: $\mathfrak{M} = \{f^+ + Kf^+ : f^+ \in \mathfrak{K}_1^+\}$. Then $K$ is a unitary operator from (the Hilbert space) $\{\mathfrak{K}_1^+, [\cdot,\cdot]_1\}$ onto (the Hilbert space) $\{\mathfrak{K}_1^-, -[\cdot,\cdot]_1\}$. Since $U$ is of type II, there exists an infinite-dimensional closed subspace $\mathfrak{D}_1^+ \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^+$ and, hence, $\mathfrak{D}_1^- := K(\mathfrak{D}_1^+) \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-$ is an infinite-dimensional closed subspace of the same dimension as $\mathfrak{D}_1^+$. For every $n_\pm \in \mathbb{N}$ there exists a closed isometric operator $V$ from (the Hilbert space) $\{\mathfrak{D}_1^+, [\cdot,\cdot]_1\}$ to (the Hilbert space) $\{\mathfrak{D}_1^-, -[\cdot,\cdot]_1\}$ such that $\dim(\operatorname{dom} V)^\perp = n_-$ and $\dim(\operatorname{ran} V)^\perp = n_+$. Hence, $\mathfrak{L}$ defined as

$$\mathfrak{L} = \{f^+ + Kf^+ : f^+ \in \mathfrak{K}_1^+ \ominus_1 \mathfrak{D}_1^+\} + \{f^+ + Vf^+ : f^+ \in \operatorname{dom} V\}$$

is a closed neutral subspace contained in $\operatorname{dom} U$ with $n_\pm(\mathfrak{L}) = n_\pm$.

(ii) ⇒ (iii) ⇒ (iv): These implications evidently hold.

(iv) ⇒ (v): Let $\mathfrak{D}_1^+$ be a closed subspace of $\operatorname{dom} U \cap \mathfrak{K}_1^+$ and w.l.o.g. assume that $\mathfrak{D}_1^+$ is infinite-dimensional, because otherwise by the definition of type II there is nothing to prove. Moreover, let $\mathfrak{L}_2$ be a closed neutral subspace as in (iv) and let $K$ be its angular operator w.r.t. $\mathfrak{K}_1^+$: $\mathfrak{L}_2 = \{f^+ + Kf^+ : f^+ : P_1^+\mathfrak{L}_2\}$. Since $n_-(\mathfrak{L}_2) = n_- < \infty$, $\mathfrak{D}_1^+ \cap \operatorname{dom} K$ is a subspace with the same dimension as $\mathfrak{D}_1^+$ which is mapped onto a subspace $\mathfrak{D}_{1,a}^-$ of $\mathfrak{K}_1^-$ of the same dimension, because $K$ is a closed isometric operator between Hilbert spaces. Moreover, $\mathfrak{D}_{1,a}^- \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-$, because $\mathfrak{L}_2 \subseteq \operatorname{dom} U$. Since a similar reasoning can be used for subspaces $\mathfrak{D}_1^-$ as in (v), this shows that (v) holds.

(v) $\Rightarrow$ (vi): This is evident from the fact that either $\mathfrak{M} \cap \mathrm{j}_1\mathfrak{M} \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^+$ or $\mathfrak{M} \cap \mathrm{j}_1\mathfrak{M} \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-$.

(vi) $\Rightarrow$ (i): Let $\mathfrak{M} \subseteq \operatorname{dom} U$ be hyper-maximal semi-definite, see Proposition 6.7. If $\mathfrak{M}$ is hyper-maximal neutral, then there is nothing to prove. Hence, w.l.o.g., assume that $\mathfrak{M}$ is hyper-maximal nonnegative, then $\mathfrak{D}_1^+ := \mathfrak{M} \cap \mathrm{j}_1\mathfrak{M} \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^+$ is a closed positive definite subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\mathfrak{M}^{[\perp]_1}$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_1 \ominus_1 \mathfrak{D}_1^+, [\cdot,\cdot]_1\}$. Let $K$ be the angular operator of $\mathfrak{M}^{[\perp]_1}$ w.r.t. $\mathfrak{K}_1^+ \ominus \mathfrak{D}_1^+$:

$$\mathfrak{M}^{[\perp]_1} = \{f^+ + Kf^+ : f^+ \in P_1^+\mathfrak{M}^{[\perp]_1}\}.$$

Then $K$ is a Hilbert space unitary operator from $\{\mathfrak{K}_1^+ \ominus_1 \mathfrak{D}_1^+, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_1^-, -[\cdot,\cdot]_1\}$. Now by assumption there exists a closed subspace $\mathfrak{D}_{1,a}^- \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-$ such that $\dim(\mathfrak{D}_{1,a}^-) = \dim(\mathfrak{D}_1^+)$, if $\dim(\mathfrak{D}_1^+) = \infty$, or $\mathfrak{D}_{1,a}^-$ is infinite-dimensional, if $\mathfrak{D}_1^+$ is finite-dimensional (note that here the fact that $U$ is of type II is used). Since $K$ is a Hilbert space unitary operator, $\mathfrak{D}_{1,a}^+ := K^{-1}(\mathfrak{D}_{1,a}^-) \subseteq \operatorname{dom} U \cap (\mathfrak{K}_1^+ \ominus \mathfrak{D}_1^+)$ is an infinite-dimensional closed subspace with the same dimension as $\mathfrak{D}_{1,a}^-$. Hence, by construction, $\mathfrak{D}_1^+ + \mathfrak{D}_{1,a}^+ \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^+$ is a closed subspace of the same dimension as $\mathfrak{D}_{1,a}^- \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^-$. Now let $U_r$ be any Hilbert space unitary operator from $\{\mathfrak{D}_1^+ + \mathfrak{D}_{1,a}^+, [\cdot,\cdot]_1\}$ onto $\{\mathfrak{D}_{1,a}^-, -[\cdot,\cdot]_1\}$. Then $\mathfrak{L}$ defined as

$$\mathfrak{L} = \{f^+ + Kf : f^+ \in \mathfrak{K}_1^+ \ominus_1 (\mathfrak{D}_1^+ + \mathfrak{D}_{1,a}^+)\} + \{f^+ + U_r f^+ : f^+ \in \mathfrak{D}_1^+ + \mathfrak{D}_{1,a}^+\}$$

is by construction a hyper-maximal neutral subspace such that $\mathfrak{L} \subseteq \operatorname{dom} U$. □

The sixth characterization in Proposition 8.13 implies that in the separable case the concepts of strongly equal defect numbers and equal defect numbers coincide for unitary relations of type II. Recall that for unitary relations of type I this is not true, see e.g. Example 8.11.

**Corollary 8.14.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ of type II and assume that $n_+(\ker U) = \aleph_0 = n_-(\ker U)$. Then $U$ has strongly equal defect numbers.*

*Proof.* Recall that $\operatorname{dom} U \cap \mathfrak{K}_1^+$ and $\operatorname{dom} U \cap \mathfrak{K}_1^-$ contain infinite-dimensional closed subspaces, because $U$ is of type II. Hence, the statement is a direct consequence of the characterization (vi) in Proposition 8.13 together with the assumption that $n_+(\ker U) = n_-(\ker U) = \aleph_0$. □

The following statement shows that unitary relations of type II which do not have strongly equal defect numbers have the same kind of closed neutral subspaces in their domain as those of type Ib.

**Lemma 8.15.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ of type II which does not have strongly equal defect numbers. Then either of the following two alternatives holds:*

*(a) every hyper-maximal semi-definite subspace contained in the domain of $U$ is nonnegative, for every $n_+ \in \mathbb{N}$ there exists a closed neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U$ such that $n_+(\mathfrak{L}) = n_+$ and $n_-(\mathfrak{L}) = \infty$, and if $\mathfrak{L} \subseteq \operatorname{dom} U$ is a closed neutral subspace, then $n_-(\mathfrak{L}) = \infty$;*

*(b) every hyper-maximal semi-definite subspace contained in the domain of $U$ is nonpositive, for every $n_- \in \mathbb{N}$ there exists a closed neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U$ such that $n_-(\mathfrak{L}) = n_-$ and $n_+(\mathfrak{L}) = \infty$, and if $\mathfrak{L} \subseteq \operatorname{dom} U$ is a closed neutral subspace, then $n_+(\mathfrak{L}) = \infty$.*

*Proof.* Let $\mathfrak{j}_1$ be a fundamental symmetry of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be the associated canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and recall that the domain of $U$ contains a hyper-maximal semi-definite subspace $\mathfrak{M}$, see Proposition 6.7. Assume that $\mathfrak{M}$ is hyper-maximal nonnegative, i.e. $\mathfrak{M} \cap \mathfrak{j}_1\mathfrak{M} \subseteq \operatorname{dom} U \cap \mathfrak{K}_1^+$. Then $\mathfrak{M}^{[\perp]_1}$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_1 \ominus_1 (\mathfrak{M} \cap \mathfrak{j}_1\mathfrak{M}), [\cdot,\cdot]_1\}$ and

$$\operatorname{gr} U_a = \operatorname{gr} U \cap ((\mathfrak{M} \cap \mathfrak{j}_1\mathfrak{M})^{[\perp]_1} \times (U(\mathfrak{M} \cap \mathfrak{j}_1\mathfrak{M}))^{[\perp]_2})$$

is a unitary relation from the Kreĭn space $\{\mathfrak{K}_1 \cap (\mathfrak{M} \cap \mathfrak{j}_1\mathfrak{M})^{[\perp]_1}, [\cdot,\cdot]_1\}$ to the Kreĭn space $\{\mathfrak{K}_2 \cap (U(\mathfrak{M} \cap \mathfrak{j}_1\mathfrak{M}))^{[\perp]_2}, [\cdot,\cdot]_2\}$, see Corollary 3.14, with strongly equal defect numbers. Hence, Proposition 8.13 implies that for every $n_+, n_- \in \mathbb{N}$ there exists a closed neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U_a$ of $\{\mathfrak{K}_1 \ominus_1 (\mathfrak{M} \cap \mathfrak{j}_1\mathfrak{M}), [\cdot,\cdot]_1\}$ such that $n_+(\mathfrak{L}) = n_+$ and $n_-(\mathfrak{L}) = n_-$. Now $\mathfrak{L}$ considered as a subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ has the defect numbers $n_+(\mathfrak{L}) = n_+$ and $n_-(\mathfrak{L}) = n_- + \dim(\mathfrak{M} \cap \mathfrak{j}_1\mathfrak{M}) = \infty$. Note that here $\dim(\mathfrak{M} \cap \mathfrak{j}_1\mathfrak{M}) = \infty$, because $U$ does not have strongly equal defect numbers, cf. Proposition 8.13. Finally, since $n_+(\mathfrak{M}^{[\perp]}) = 0$, Proposition 8.13 (iv) implies that $n_-(\mathfrak{L}) = \infty$ for every closed neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U$. This, in particular, implies that every hyper-maximal semi-definite subspace contained in the domain of $U$ is nonnegative.

The above arguments show that alternative (a) hold if $\mathfrak{M}$ is hyper-maximal nonnegative. Similar arguments show that alternative (b) holds if $\mathfrak{M}$ is hyper-maximal nonpositive. □

The following two statements contain a characterization of unitary operators of type II in terms of their associated diagonal block representation.

**Theorem 8.16.** *Let $U$ be an isometric operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathrm{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then $U$ is a unitary operator of type II with strongly equal defect numbers if and only if there exists a hyper-maximal neutral subspace $\mathfrak{M}$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, a closed operator $B$ in the Hilbert space $\{\mathfrak{M}, [\mathrm{j}_2\cdot,\cdot]_2\}$ with $\overline{\mathrm{dom}}\, B = \mathfrak{M} = \mathrm{ran}\, B$ and $\ker B = \{0\}$ such that $B^{-1}$ is a noncompact operator, and a bounded unitary operator $U_t$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ onto $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\mathrm{dom}\, U \subseteq \mathrm{dom}\, U_t$ such that*

$$UU_t^{-1} = \Upsilon_2(B).$$

*Proof.* If $U$ has the indicated representation, then $U$ is clearly a unitary operator with strongly equal defect numbers, because the hyper-maximal neutral subspace $U_t^{-1}(\mathrm{j}_2\mathfrak{M})$ is contained in the domain of $U$. Furthermore, $\Upsilon_2(B)$, and hence also $U$, is not of type I, because $B^{-1}$ is not compact, see Theorem 8.10.

Conversely, assume that $U$ is a unitary operator and w.l.o.g. assume that $\ker U = \{0\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be the canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ associated with the fundamental symmetry $\mathrm{j}_i$ of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$, for $i = 1, 2$. Moreover, recall that by Proposition 7.25 there exists a hyper-maximal semi-definite subspace $\mathfrak{L} \subseteq \mathrm{dom}\, U$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $U(\mathfrak{L})$ is a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. W.l.o.g. assume that $\mathfrak{L} = \mathfrak{L}^{[\perp]_1} + (\mathfrak{L} \cap \mathrm{j}_1\mathfrak{L})$ is hyper-maximal nonnegative, i.e. $\mathfrak{L} \cap \mathrm{j}_1\mathfrak{L} \subseteq \mathrm{dom}\, U \cap \mathfrak{K}_1^+$. Now let $K$ be the angular operator of $\mathfrak{L}^{[\perp]_1} \subseteq \mathfrak{L}$ w.r.t. to $\mathfrak{K}_1^+$:

$$\mathfrak{L}^{[\perp]_1} = \{f^+ + Kf^+ : f^+ \in P_1^+\mathfrak{L}^{[\perp]_1} = \mathfrak{K}_1^+ \ominus_1 (\mathfrak{L} \cap \mathrm{j}_1\mathfrak{L})\}.$$

Since $\mathfrak{L} \cap \mathrm{j}_1\mathfrak{L} \subseteq \mathrm{dom}\, U \cap \mathfrak{K}_1^+$, $U$ is of type II and has strongly equal defect numbers, there exists a closed subspace $\mathfrak{D}_1^- \subseteq \mathrm{dom}\, U \cap \mathfrak{K}_1^-$ such that $\dim(\mathfrak{D}_1^-) = \infty$, if $\dim(\mathfrak{L} \cap \mathrm{j}_1\mathfrak{L}) < \infty$, or $\dim(\mathfrak{D}_1^-) = \dim(\mathfrak{L} \cap \mathrm{j}_1\mathfrak{L})$, if $\dim(\mathfrak{L} \cap \mathrm{j}_1\mathfrak{L}) = \infty$, see Proposition 8.13. Since the angular operator $K$ is a closed isometric operator between Hilbert spaces, $K^{-1}(\mathfrak{D}_1^-) \subseteq \mathrm{dom}\, U \cap \mathfrak{K}_1^+$ is a closed subspace of the same dimension as $\mathfrak{D}_1^-$. Hence, $\mathfrak{D}_1^+ := K^{-1}(\mathfrak{D}_1^-) + \mathfrak{L} \cap \mathrm{j}_1\mathfrak{L} \subseteq \mathrm{dom}\, U \cap \mathfrak{K}_1^+$ is a closed subspace of the same dimension as $\mathfrak{D}_1^-$.

Consequently, $\mathfrak{L}_r = \mathfrak{L} \cap (\mathfrak{D}_1^+ + \mathfrak{D}_1^-)^{[\perp]_1}$ is a closed neutral subspace with defect numbers $n_+(\mathfrak{L}_r) = \dim(\mathfrak{D}_1^+) = \dim(\mathfrak{D}_1^-) = n_-(\mathfrak{L}_r)$ and

$$\mathfrak{L}_r^{[\perp]_1} = \mathfrak{L}_r \oplus_1 \mathfrak{D}_1^+ \oplus_1 \mathfrak{D}_1^- = \mathfrak{L} + \mathfrak{D}_1^- \subseteq \mathrm{dom}\, U.$$

Recall that by assumption $U(\mathfrak{L})$ is a hyper-maximal nonnegative subspace and that $U(\mathfrak{D}_1^-)$ is a closed uniformly definite subspace by Proposition 3.9. Hence,

$U(\mathfrak{L}_r^{[\perp]_1}) = U(\mathfrak{L}) + U(\mathfrak{D}_1^-)$ is closed, because, clearly, $(U(\mathfrak{L}))^{[\perp]_2} + (U(\mathfrak{D}_1^-))^{[\perp]_2} = \mathfrak{K}_2$, see Lemma 2.2. Moreover, since $U(\mathfrak{L})$ is by assumption a hyper-maximal semi-definite subspace contained in the range of $U$, one has

$$\begin{aligned} U(\mathfrak{L}_r^{[\perp]_1})^{[\perp]_2} &= U(\mathfrak{L} + \mathfrak{D}_1^-)^{[\perp]_2} \\ &= U(\mathfrak{L})^{[\perp]_2} \cap U(\mathfrak{D}_1^-)^{[\perp]_2} \\ &= U(\mathfrak{L}^{[\perp]_1}) \cap U(\mathfrak{D}_1^-)^{[\perp]_2} \\ &= U(\mathfrak{L}^{[\perp]_1}) \cap U(\mathfrak{D}_1^-)^{[\perp]_2} \cap \operatorname{ran} U \\ &= U(\mathfrak{L}^{[\perp]_1}) \cap U((\mathfrak{D}_1^-)^{[\perp]_1} \cap \operatorname{dom} U) \\ &= U(\mathfrak{L}^{[\perp]_1} \cap (\mathfrak{D}_1^-)^{[\perp]_1}) = U(\mathfrak{L}_r). \end{aligned}$$

Since $U(\mathfrak{L}_r^{[\perp]_1}) = U(\mathfrak{L}) + U(\mathfrak{D}_1^+)$ has been shown to be closed, the above calculation implies that $U(\mathfrak{L}_r)$ is closed and that $U(\mathfrak{L}_r^{[\perp]_1}) = U(\mathfrak{L}_r)^{[\perp]_2}$, i.e. $U(\mathfrak{L}_r)^{[\perp]_2} = U(\mathfrak{L}_r) + U(\mathfrak{D}_1^+) + U(\mathfrak{D}_1^-)$. From these observations it follows that any hyper-maximal neutral extension of $\mathfrak{L}_r$, which exists because $\dim(\mathfrak{D}_1^+) = \dim(\mathfrak{D}_1^-)$, is mapped onto a hyper-maximal neutral extension of the closed neutral subspace $U(\mathfrak{L}_r)$. Consequently, the stated representation holds by Theorem 7.16. □

Using the above characterization for unitary operators of type II with strongly equal defect numbers, one can easily obtain a characterization for unitary operators of type II without strongly equal defect numbers.

**Corollary 8.17.** *Let $U$ be an isometric operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then $U$ is a unitary operator of type II without strongly equal defect numbers if and only if there exists a hyper-maximal semi-definite subspace $\mathfrak{M}$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that either $\dim(\mathfrak{D}_2^+) < \dim(\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M})$ for every closed subspace $\mathfrak{D}_2^+$ of $\operatorname{ran} U \cap \mathfrak{K}_2^+$, or $\dim(\mathfrak{D}_2^-) < \dim(\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M})$, for every closed subspace $\mathfrak{D}_2^-$ of $\operatorname{ran} U \cap \mathfrak{K}_2^-$, a closed operator $B$ in the Hilbert space $\{\mathfrak{M}^{[\perp]_2}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\overline{\operatorname{dom}} B = \mathfrak{M}^{[\perp]_2} = \operatorname{ran} B$ and $\ker B = \{0\}$ such that $B^{-1}$ is a noncompact operator, and a bounded unitary operator $U_t$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ onto $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\operatorname{dom} U \subseteq \operatorname{dom} U_t$ such that*

$$UU_t^{-1} = \Upsilon_2(B) \oplus I_{\mathfrak{M} \cap \mathfrak{j}_2 \mathfrak{M}}.$$

*Proof.* W.l.o.g. assume that $\ker U = \{0\}$, then by Proposition 6.7 there exists a hyper-maximal semi-definite subspace $\mathfrak{M} \subseteq \operatorname{ran} U$ in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Now $\mathfrak{D}_2 := \mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}$ is a closed uniformly definite subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and, hence, $\mathfrak{D}_1 := U^{-1}(\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M})$ is a closed uniformly definite subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, see Proposition 3.9. Therefore, $\widetilde{U}$ and $\widehat{U}$ defined via

$$\operatorname{gr} \widetilde{U} = \operatorname{gr} U \cap (\mathfrak{D}_1 \times \mathfrak{D}_2) \quad \text{and} \quad \operatorname{gr} \widehat{U} = \operatorname{gr} U \cap (\mathfrak{K}_1 \cap \mathfrak{D}_1^{[\perp]_1} \times \mathfrak{K}_2 \cap \mathfrak{D}_2^{[\perp]_2})$$

are a standard unitary operator from $\{\mathfrak{D}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{D}_2, [\cdot,\cdot]_2\}$ and a unitary operator from $\{\mathfrak{K}_1 \cap \mathfrak{D}_1^{[\perp]_1}, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2 \cap \mathfrak{D}_2^{[\perp]_2}, [\cdot,\cdot]_2\}$ with strongly equal defect numbers, respectively, see Lemma 3.13 and Corollary 3.14. Therefore the representation in the statement can be obtained via Theorem 8.16. Finally, the conditions on the dimension of $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}$ are a direct consequence of the fact that $U$ does not have strongly equal defect numbers, cf. Proposition 8.13.

Conversely, if $U$ has the indicated representation, then by Theorem 8.10 $U$ is of type II and the assumptions on the dimension of $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}$ imply that $U^{-1}$, and hence also $U$, does not have strongly equal defect numbers, see Proposition 8.13. □

Combining some of the above results it is possible to characterize when a unitary relation has strongly equal defect numbers in the general case.

**Corollary 8.18.** *Let $U$ be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then equivalent are:*

(i) *$U$ has strongly equal defect numbers;*

(ii) *there exists a closed neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U$ in $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ with finite and equal defect numbers;*

(iii) *there exists a hyper-maximal neutral subspace $\mathfrak{M}$ in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, a closed operator $B$ in the Hilbert space $\{\mathfrak{M}, [\mathfrak{j}_2\cdot,\cdot]_2\}$ with $\overline{\operatorname{dom} B} = \mathfrak{M} = \operatorname{ran} B$ and $\ker B = \{0\}$, and a bounded unitary operator $U_t$ from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ onto $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\operatorname{dom} U \subseteq \operatorname{dom} U_t$ such that*

$$UU_t^{-1} = \Upsilon_2(B).$$

*Proof.* (i) ⇔ (ii): If $U$ has strongly equal defect numbers, then by Definition 8.1, there exists a hyper-maximal neutral subspace $\mathfrak{L} \subseteq \operatorname{dom} U$, i.e., (ii) holds. Conversely, if (ii) holds, and $U$ is of type I or II, then $U$ has strongly equal defect numbers by Corollary 8.9 or Proposition 8.13, respectively.

(i) ⇔ (iii): If (i) holds, then there exists a representation as in (iii) by Theorem 8.10, if $U$ is of type I, or by Theorem 8.16, if $U$ is of type II. Conversely, if (iii) holds, then $U_t^{-1}(\mathfrak{j}_2\mathfrak{M})$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ which is contained in the domain of $U$. □

# 9 SUMMARY

In order to obtain more insight into the properties and structure of unitary relations, broadly speaking two approaches, and their interaction, to unitary (and isometric) relations were presented in this dissertation. In the first approach the behavior of unitary (and isometric) relations with respect to uniformly definite subspaces was considered and in the second approach the behavior of unitary (and isometric) relations with respect to hyper-maximal semi-definite subspaces was considered. These approaches were used to understand the difference between isometric and unitary relations and, secondly, to investigate their essential mapping properties.

## Weyl identity approach

In the first approach, presented mainly in Chapter 5, it was shown that unitary relations are characterized by their behavior on uniformly definite subspaces and that this characterization can be expressed by means of the Weyl identity. If $U$ is unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, then this identity is given by

$$U(\operatorname{dom} U \cap \mathfrak{K}_1^+) = U(\operatorname{dom} U \cap \mathfrak{K}_1^-)^{[\perp]_2}. \tag{9.1}$$

As a consequence of this identity, a known quasi-block representation for unitary operators can be obtained. That representation was shown to be extendable to the case of maximal isometric operators: $V$ is a maximal isometric operator if and only if there exists a unitary operator $U_K$ with $\ker U_K = \{0\}$ and a maximal isometric operator $V_t$ with closed domain and $\ker V_t = \ker V$ such that

$$V = U_K V_t. \tag{9.2}$$

Note that if (9.2) holds, then $V$ is a unitary operator if and only if $V_t$ is a unitary operator. Since isometric operators with a closed domain have a relatively simple geometrical behavior, (9.2) implies that certain properties of unitary operators (or relations) also hold for maximal isometric relations.

The representation in (9.2) shows that in general (maximal) isometric operators (or relations) can not be characterized by the Weyl identity (9.1), because that representation implies that there exist maximal isometric operators (or relations) $V$ such that $\operatorname{dom} V \cap \mathfrak{K}_1^+ = \{0\}$ or $\operatorname{dom} V \cap \mathfrak{K}_1^- = \{0\}$. In fact, it can be shown that there exist (non-maximal) closed isometric operators $V$ with dense domain and range such that $\operatorname{dom} V \cap \mathfrak{K}_1^+ = \{0\} = \operatorname{dom} V \cap \mathfrak{K}_1^-$. I.e., isometric relations can in general not be completely understood by considering only their behavior with respect to uniformly definite subspaces. An exception to that case is provided by isometric operators whose domain is dense and contains a hyper-maximal semi-definite subspace.

This Weyl identity approach to unitary relations was also used to give, based on the work of J.W. Calkin, an expression for the defect numbers of the pre-image of a neutral subspace under a unitary relation. Therein it was essential to compare the angular operators of the subspace with the angular operators of $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ and $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$. In particular, in that way conditions for the pre-image of a neutral subspace under a unitary relation to be (hyper-)maximal neutral were obtained

## Block representation approach

Secondly, the behavior of unitary (and isometric) relations with respect to hyper-maximal semi-definite subspaces was investigated. In Chapter 6 a graph decomposition characterization of unitary relations was presented, extending a domain decomposition result of J.W. Calkin (1939a). That graph decomposition implied, in particular, that the domain and range of a unitary relation always contain a hyper-maximal semi-definite subspace. Note that by means of a simple example it was shown that there exist densely defined (maximal) isometric relations whose domains do not contain a hyper-maximal semi-definite subspace, see Example 5.10 and the discussion following it.

In the same chapter also some implications of the existence of a hyper-maximal semi-definite subspace in the domain of an isometric relation were presented, but, more importantly, the already mentioned graph decomposition was combined with the Weyl identity approach to obtain necessary and sufficient conditions for an isometric relation to be unitary and to give characterizations for the pre-image of a neutral subspace to be (essentially) hyper-maximal neutral.

Using the above mentioned graph decomposition of unitary relations, in Chapter 7 the main contribution of this dissertation to the understanding of (unbounded) unitary relations was presented. Namely, there it was shown that unitary operators can be represented by means of operator block matrices. More specifically, it was shown that unitary operators can be written as the composition of bounded unitary operators, whose mapping behavior is easily understood, and two types of unitary operators which have a simple block structure and reflect the possibly unbounded behavior of unitary operators. Those latter unitary operators are the so-called archetypical unitary operators, see Chapter 4.2. For a Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$ with fundamental symmetry $\mathfrak{j}$ which contains a hyper-maximal neutral subspace $\mathfrak{M}$, these archetypical unitary operators have w.r.t. the decomposition $\mathfrak{M} \oplus \mathfrak{j}\mathfrak{M}$ of $\mathfrak{K}$ the block representation

$$\begin{pmatrix} I & 0 \\ i\mathfrak{j}K & I \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} B & 0 \\ 0 & \mathfrak{j}B^{-*}\mathfrak{j} \end{pmatrix}, \tag{9.3}$$

where $K$ is a selfadjoint operator in $\{\mathfrak{M}, [\mathfrak{j}\cdot,\cdot]\}$ and $B$ is a closed operator in $\mathfrak{M}$ with $\overline{\mathrm{dom}}\, B = \mathfrak{M} = \mathrm{ran}\, B$ and $\ker B = \{0\}$.

Using the above mentioned representations for unitary operators simple proofs were obtained for the main results from (Calkin 1939a), see Section 7.4. Moreover, in Chapter 8 it was shown that the classification of unitary operators occurring in (Calkin 1939a) can be characterized by the type of the operator $B$ appearing in the archetypical unitary operator, see (9.3), which characterizes the unitary operator. Note that in Chapter 8 also new characterizations and properties of the classification of unitary operators from (Calkin 1939a) were presented and that most of the statements proven in that chapter can directly be generalized to the case of isometric relations which have a hyper-maximal semi-definite subspace in their domain.

Another manner in which archetypical unitary operators, and their compositions, were used, was to give elementary examples of the behavior of unitary relations. For instance, it was shown that a unitary relation may map a hyper-maximal neutral subspace onto a neutral subspace with essentially arbitrary defect numbers and that the domains of unitary relations can not be distinguished from the domains isometric relations. I.e., isometric and unitary relations can only be distinguished by their graphs (action). The block representations were also used to give different necessary and sufficient conditions for an isometric relation to be (extendable to) a unitary relation and, moreover, it was shown that they can be used to investigate when the composition of a unitary and an isometric relation is (extendable to) a unitary relation.

The above indicated block representation approach was not limited to the investigation of unitary operators. Namely, it was also shown that isometric operators whose domain contains a hyper-maximal semi-definite subspace which is mapped by the isometry to a subspace which is extendable to hyper-maximal semi-definite subspace, can be represented as the composition of bounded unitary operators and isometric operators having a block representation as in (9.3). That showed that such isometric operators, which are the abstract equivalent of the class of quasi-boundary triplets, see (Behrndt & Langer 2007), are closely related to unitary relations.

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# A BOUNDARY TRIPLETS

Here it is shortly illustrated how the results obtained for isometric and unitary relations can be applied to the different types of boundary triplets appearing in the literature, see e.g. (Behrndt & Kreusler 2007; Behrndt & Langer 2007; Derkach 1995; Derkach & Hassi 2003; Derkach et al. 2006; 2009; Derkach & Malamud 1991; 1995; Mogilevskii 2006; 2011). Therefore in the first section some basic results on symmetric relations in Kreĭn spaces are recalled. In the second section the various notions of boundary triplets occurring in the literature are recalled and it is shown how they are connected to each other by archetypical isometric operators. In the third section this connection between the various types of boundary triplets is lifted to their Weyl functions. Finally, in the fourth section an application of the composition results obtained in Section 7.5 is presented. Namely, there it is shown that the results on boundary triplets for intermediate extensions of symmetric relations in a Hilbert space presented in (Derkach et al. 2009: Section 4) remain without change valid in the Kreĭn space setting.

## A.1 Preliminaries for boundary triplets

The definition of symmetric and selfadjoint relations in Kreĭn spaces are recalled and those relations are via their graph connected to neutral and hyper-maximal neutral subspaces of a Kreĭn space. Moreover, hyper-maximal nonnegative and nonpositive subspaces are shown to be interpretable as a special type of maximal dissipative or accumulative relations, respectively, and, finally, some statements on defect subspaces of relations are presented.

**Symmetric relations in Kreĭn spaces:** A relation $S$ in $\{\mathfrak{K}, [\cdot,\cdot]\}$ is called *symmetric* or *selfadjoint* if

$$S \subseteq S^{[*]} \quad \text{or} \quad S = S^{[*]},$$

respectively. A symmetric relation is called *maximal symmetric* if it has no symmetric extensions. For a symmetric relation $S$ in $\{\mathfrak{K}, [\cdot,\cdot]\}$, the notation $\widehat{\mathfrak{N}}_\lambda(S^{[*]})$ is used to denote its defect spaces:

$$\widehat{\mathfrak{N}}_\lambda(S^{[*]}) = \{\{f_\lambda, \lambda f_\lambda\} : f_\lambda \in \mathfrak{N}_\lambda(S^{[*]}) := \ker(S^{[*]} - \lambda)\}, \quad \lambda \in \mathbb{C}. \qquad \text{(A.1)}$$

Note that (2.6) implies that for a symmetric relation $S$

$$\ker S \subseteq \ker S^{[*]} = (\operatorname{ran} S)^{[\perp]} \quad \text{and} \quad \operatorname{mul} S \subseteq \operatorname{mul} S^{[*]} = (\operatorname{dom} S)^{[\perp]}. \qquad \text{(A.2)}$$

In particular, the adjoint of a densely defined symmetric operator is an operator.

For a Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$ define the operator $\mathfrak{j}_{\mathfrak{K}}$ on $\mathfrak{K}^2$ as

$$\mathfrak{j}_{\mathfrak{K}}\{f, f'\} = i\{-f', f\}. \tag{A.3}$$

Clearly, if $\mathfrak{j}$ is a fundamental symmetry of $\{\mathfrak{K}, [\cdot,\cdot]\}$, then $\mathfrak{j}_{\mathfrak{K}}(\mathfrak{j} \times \mathfrak{j}) = (\mathfrak{j} \times \mathfrak{j})\mathfrak{j}_{\mathfrak{K}}$. As a consequence of this observation it follows that $\{\mathfrak{K}^2, [\mathfrak{j}_{\mathfrak{K}}\cdot,\cdot]\}$ is a Kreĭn space. This introduced Kreĭn space can be used to connect symmetric and selfadjoint relations to neutral and hyper-maximal neutral subspaces, respectively.

**Proposition A.1.** *Let* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *be a Kreĭn space, let* $\mathfrak{j}_{\mathfrak{K}}$ *be as in* (A.3) *and let* $\langle\!\langle\perp\rangle\!\rangle$ *denote the orthogonal complement of a subspace of* $\mathfrak{K}^2$ *w.r.t.* $[\mathfrak{j}_{\mathfrak{K}}\cdot,\cdot]$. *Then for any relation* $H$ *in* $\{\mathfrak{K}, [\cdot,\cdot]\}$

$$(\mathrm{gr}\, H)^{\langle\!\langle\perp\rangle\!\rangle} = \mathrm{gr}\, H^{[*]}.$$

*In particular,* $S$ *is a (closed, maximal) symmetric or selfadjoint relation in* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *if and only if* $\mathrm{gr}\, S$ *is a (closed, maximal) neutral or hyper-maximal neutral subspace of* $\{\mathfrak{K}^2, [\mathfrak{j}_{\mathfrak{K}}\cdot,\cdot]\}$*, respectively.*

*Proof.* Since the final statements follow essentially from $(\mathrm{gr}\, H)^{\langle\!\langle\perp\rangle\!\rangle} = \mathrm{gr}\, H^{[*]}$, only that equality will be proven. By definition $\{f, f'\} \in (\mathrm{gr}\, H)^{\langle\!\langle\perp\rangle\!\rangle}$ if and only if

$$0 = [\mathfrak{j}_{\mathfrak{K}}\{f, f'\}, \{g, g'\}] = [i\{-f', f\}, \{g, g'\}] = i\left([f, g'] - [f', g]\right),$$

for all $\{g, g'\} \in \mathrm{gr}\, H$. By definition this implies that $\{f, f'\} \in (\mathrm{gr}\, H)^{\langle\!\langle\perp\rangle\!\rangle}$ if and only if $\{f, f'\} \in \mathrm{gr}\, H^{[*]}$. □

Next recall that there exists a direct connection between symmetric relations in Hilbert spaces and symmetric relations in Kreĭn spaces by means of a fundamental symmetry of the Kreĭn space, see (Behrndt et al. 2011a).

**Proposition A.2.** *Let* $\mathfrak{j}$ *be a fundamental symmetry of* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *and let* $\{\mathfrak{H}, (\cdot,\cdot)\}$ *be the Hilbert space* $\{\mathfrak{K}, [\mathfrak{j}\cdot,\cdot]\}$. *Then* $U_{\mathfrak{j}}$ *defined as*

$$U_{\mathfrak{j}}\{f, f'\} = \{f, \mathfrak{j}f'\}, \quad f, f' \in \mathfrak{K}$$

*is a standard unitary operator from the Kreĭn space* $\{\mathfrak{K}^2, [\mathfrak{j}_{\mathfrak{K}}\cdot,\cdot]\}$ *to the Kreĭn space* $\{\mathfrak{H}^2, (\mathfrak{j}_{\mathfrak{H}}\cdot,\cdot)\}$. *Moreover, if* $K$ *is a relation in* $\mathfrak{K}$ *and* $H$ *is the relation in* $\mathfrak{H}$ *such that* $\mathrm{gr}\, H = U_{\mathfrak{j}}(\mathrm{gr}\, K)$*, then*

$$U_{\mathfrak{j}}(\mathrm{gr}\, K^{[*]}) = \mathrm{gr}\, H^*.$$

*In particular,* $U_{\mathfrak{j}}$ *establishes a bijective correspondence between the (closed, maximal) symmetric and selfadjoint relations in* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *and the (closed, maximal) symmetric and selfadjoint relations in* $\{\mathfrak{H}, (\cdot,\cdot)\}$*, respectively.*

Using Proposition A.2 the defect numbers of a symmetric relation $S$ in the Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$ are defined to be the defect numbers of the symmetric relation $S_h$, defined via $\mathrm{gr}\, S_h = U_{\mathfrak{j}}(\mathrm{gr}\, S)$, in the Hilbert space $\{\mathfrak{K}, [\mathfrak{j}\cdot,\cdot]\}$, see (A.6) below.

**Symmetric relations in Hilbert spaces:** Next shortly the main difference between symmetric relations in Kreĭn spaces and Hilbert spaces is recalled. Namely, in the Hilbert space case the defect spaces $\widehat{\mathfrak{N}}_\lambda(S^*)$ of the symmetric relation $S$ are, outside the real line, uniformly definite whilst in the Kreĭn space case they are in general not. In particular, for a Hilbert space $\{\mathfrak{H}, (\cdot,\cdot)\}$, define $\mathfrak{H}_\lambda^+ = \{\{f, \lambda f\} : f \in \mathfrak{H}\}$ and $\mathfrak{H}_\lambda^- = \{\{f, \overline{\lambda} f\} : f \in \mathfrak{H}\}$, for $\lambda \in \mathbb{C}_+$. Then a direct calculation shows that $\mathfrak{H}_\lambda^+ + \mathfrak{H}_\lambda^-$ is a canonical decomposition of $\{\mathfrak{H}^2, (\mathfrak{j}_{\mathfrak{H}}\cdot,\cdot)\}$. Evidently, for a symmetric relation $S$ in $\{\mathfrak{H}, (\cdot,\cdot)\}$

$$\widehat{\mathfrak{N}}_\lambda(S^*) = \mathrm{gr}\, S^* \cap \mathfrak{H}_\lambda^+ \quad \text{and} \quad \widehat{\mathfrak{N}}_{\overline{\lambda}}(S^*) = \mathrm{gr}\, S^* \cap \mathfrak{H}_\lambda^-, \quad \lambda \in \mathbb{C}_+. \tag{A.4}$$

Observe also that $\mathfrak{H}_i^+ + \mathfrak{H}_i^-$ is the canonical decomposition of $\{\mathfrak{H}^2, (\mathfrak{j}_{\mathfrak{H}}\cdot,\cdot)\}$ corresponding to $\mathfrak{j}_{\mathfrak{H}}$ as in (A.3). The above observations together with Proposition A.1 and (2.4) shows that for a symmetric relation $S$ in $\{\mathfrak{H}, (\cdot,\cdot)\}$ the first von Neumann formula holds:

$$\mathrm{gr}\, S^* = \mathrm{gr}\, S \dot{+} \widehat{\mathfrak{N}}_\lambda(S^*) \dot{+} \widehat{\mathfrak{N}}_{\overline{\lambda}}(S^*), \quad \lambda \in \mathbb{C}_+. \tag{A.5}$$

The defect numbers $n_+(S)$ and $n_-(S)$ for $S$ are in this case defined as usual:

$$n_+(S) = \dim \widehat{\mathfrak{N}}_{\overline{\lambda}}(S^*) \quad \text{and} \quad n_-(S) = \dim \widehat{\mathfrak{N}}_\lambda(S^*), \quad \lambda \in \mathbb{C}_+. \tag{A.6}$$

**Defect subspaces and dissipative relations:** Recall that Proposition 2.20 implies that a relation $A$ is a (closed, maximal) dissipative or accumulative relation if and only if $\mathrm{gr}\, A$ is a (closed, maximal) nonnegative or nonpositive subspace of $\{\mathfrak{K}^2, [\mathfrak{j}_{\mathfrak{K}}\cdot,\cdot]\}$, respectively. As a generalization of these concepts, a dissipative or accumulative relation $A$ is called *hyper-maximal dissipative* or *hyper-maximal accumulative* if $\mathrm{gr}\, A$ is a hyper-maximal nonnegative or nonpositive subspace, respectively. Proposition A.3 below contains a property of hyper-maximal dissipative and accumulative relations in the Hilbert space case.

**Proposition A.3.** *Let $A$ be a hyper-maximal dissipative (accumulative) relation in the Hilbert space $\{\mathfrak{H}, (\cdot,\cdot)\}$. Then*

(i) $\mathbb{C}_- \subseteq \rho(A)$ *(*$\mathbb{C}_+ \subseteq \rho(A)$*);*

(ii) $\mathrm{ran}\,(A - \lambda) = \mathfrak{H}$ *for* $\lambda \in \mathbb{C}_+$ *(*$\lambda \in \mathbb{C}_-$*).*

*Proof.* For $\lambda \in \mathbb{C}_+$ define $P_\lambda$ on $\mathfrak{H}^2$ as $P_\lambda\{f, f'\} = \frac{1}{\lambda-\overline{\lambda}}\{\lambda f - f', \overline{\lambda}(\lambda f - f')\}$. Then a direct calculation shows that

$$\ker P_\lambda = \{\{f, \lambda f\} : f \in \mathfrak{H}\} \quad \text{and} \quad \operatorname{ran} P_\lambda = \{\{f, \overline{\lambda} f\} : f \in \mathfrak{H}\}.$$

Moreover, with $\mathfrak{H}_\lambda^+ := \ker P_\lambda$ and $\mathfrak{H}_\lambda^- := \operatorname{ran} P_\lambda$, $\mathfrak{H}_\lambda^+ + \mathfrak{H}_\lambda^-$ is a canonical decomposition of $\{\mathfrak{H}^2, (\mathfrak{j}_{\mathfrak{H}}\cdot, \cdot)\}$ with associated projections $I - P_\lambda = P_{\overline{\lambda}}$ and $P_\lambda$.

Now assume w.l.o.g. that $A$ is a hyper-maximal dissipative relation, i.e. $\operatorname{gr} A$ is a hyper-maximal nonnegative subspace of $\{\mathfrak{H}^2, (\mathfrak{j}_{\mathfrak{H}}\cdot, \cdot)\}$. Then $P^+(\operatorname{gr} A) = \mathfrak{K}^+$, $P^-(\operatorname{gr} A) = \mathfrak{K}^-$ and $P^-((\operatorname{gr} A)^{<\perp>}) = P^-(\operatorname{gr} A^*) = \mathfrak{K}^-$ for any canonical decomposition $\mathfrak{K}^+ + \mathfrak{K}^-$ of $\{\mathfrak{H}^2, (\mathfrak{j}_{\mathfrak{H}}\cdot, \cdot)\}$ with associated projections $P^+$ and $P^-$, see Section 2.2. Hence, by taking $\mathfrak{K}^+$ and $\mathfrak{K}^-$ as $\mathfrak{H}_\lambda^+$ and $\mathfrak{H}_\lambda^-$ as above, the aforementioned conditions become:

$$\begin{aligned}
\{\{f, \lambda f\} : f \in \mathfrak{H}\} &= \operatorname{ran} P_{\overline{\lambda}} = P_{\overline{\lambda}} \operatorname{gr} A = \{\{f, \lambda f\} : f \in \operatorname{ran}(A - \overline{\lambda})\};\\
\{\{f, \overline{\lambda} f\} : f \in \mathfrak{H}\} &= \operatorname{ran} P_\lambda = P_\lambda \operatorname{gr} A = \{\{f, \overline{\lambda} f\} : f \in \operatorname{ran}(A - \lambda)\};\\
\{\{f, \overline{\lambda} f\} : f \in \mathfrak{H}\} &= \operatorname{ran} P_\lambda = P_\lambda \operatorname{gr} A^* = \{\{f, \overline{\lambda} f\} : f \in \operatorname{ran}(A^* - \lambda)\}.
\end{aligned}$$

In other words, $\operatorname{ran}(A - \overline{\lambda})$, $\operatorname{ran}(A - \lambda)$ and $\operatorname{ran}(A^* - \lambda) = (\ker(A - \overline{\lambda}))^\perp$ are all equal to $\mathfrak{H}$. This shows that the statement holds. □

In particular, by means of the canonical decomposition in the above proof it follows that if $A$ is a hyper-maximal dissipative or accumulative relation, then

$$\operatorname{gr} A = \operatorname{gr} A^* + \widehat{\mathfrak{N}}_\lambda(A), \quad \lambda \in \mathbb{C}_+, \quad \text{or} \quad \operatorname{gr} A = \operatorname{gr} A^* + \widehat{\mathfrak{N}}_\lambda(A), \quad \lambda \in \mathbb{C}_-,$$

respectively, see Proposition 2.9 (iii).

Next a special case of Proposition 2.13 is presented, which in particular shows that if a relation extends a hyper-maximal dissipative or accumulative relation in a Hilbert space, then the graph of the extension can be decomposed with respect to the dissipative or accumulative relation. Note further that the first assertion in Corollary A.4 also follows easily from direct arguments; see e.g. (Hassi et al. 2007: Lemma 1.4).

**Corollary A.4.** *Let $H$ and $A$ be relations in $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that $A \subseteq H$. Then*

$$\operatorname{ran}(H - \lambda) = \operatorname{ran}(A - \lambda) \quad \textit{if and only if} \quad \operatorname{gr} H = \operatorname{gr} A + \widehat{\mathfrak{N}}_\lambda(H), \quad \lambda \in \mathbb{C}.$$

*Furthermore, if $\rho(A) \cap (\mathbb{C} \setminus \mathbb{R}) \neq \emptyset$, then for $\lambda \in \rho(A) \cap (\mathbb{C} \setminus \mathbb{R})$ the following statements hold:*

(i) *$H$ is closed if and only if $\mathfrak{N}_\lambda(H)$ is closed;*

(ii) $\mathfrak{N}_\lambda(H)$ *is dense in* $\mathfrak{N}_\lambda(\operatorname{clos} H)$.

*Proof.* For $\lambda \in \rho(A) \cap (\mathbb{C} \setminus \mathbb{R})$ define the projection $P_\lambda$ on $\mathfrak{K}^2$ as $P_\lambda\{f, f'\} = \frac{1}{\lambda - \overline{\lambda}}\{\lambda f - f', \overline{\lambda}(\lambda f - f')\}$. Then a direct calculation shows that $\ker P_\lambda = \{\{f, \lambda f\} : f \in \mathfrak{K}\}$ and $\operatorname{ran} P_\lambda = \{\{f, \overline{\lambda} f\} : f \in \mathfrak{K}\}$ and that $\mathfrak{K}^2 = \ker P_\lambda \ll + \gg \operatorname{ran} P_\lambda$, i.e. $P_\lambda$ is an orthogonal projection in the Kreĭn space $\{\mathfrak{K}^2, [\mathfrak{j}_{\mathfrak{K}}\cdot, \cdot]\}$. Moreover,

$$P_\lambda \operatorname{gr} A = \{\{f, \overline{\lambda} f\} : f \in \operatorname{ran}(A - \lambda)\};$$
$$(I - P_\lambda)(\operatorname{gr} A)^{\ll\perp\gg} = P_{\overline{\lambda}}(\operatorname{gr} A^{[*]}) = \{\{f, \lambda f\} : f \in \operatorname{ran}(A^{[*]} - \overline{\lambda})\},$$

see Proposition A.1. Since by assumption $\lambda \in \rho(A)$, the above equalities imply that $P_\lambda(\operatorname{gr} A) = \operatorname{ran} P_\lambda$ and $(I - P_\lambda)(\operatorname{gr} A)^{\ll\perp\gg} = \ker P_\lambda$. Hence, the statement is now a direct consequence of Corollary 2.14. □

Next it is shown how the defect subspaces of $H$ are holomorphically connected, cf. (Derkach et al. 2006: Proposition 4.1). In particular, this observation explains why the Weyl function of a (quasi-)boundary triplet is a holomorphic function, see Section A.3,

**Lemma A.5.** *Let $H$ and $A$ be relations in $\{\mathfrak{K}, [\cdot, \cdot]\}$ and let $\lambda, \mu \in \mathbb{C}$. Then*

$$\widehat{\mathfrak{N}}_\lambda(H) \subseteq \operatorname{gr} A + \widehat{\mathfrak{N}}_\mu(H) \quad \Longleftrightarrow \quad \mathfrak{N}_\lambda(H) \subseteq (I + (\lambda - \mu)(A - \lambda)^{-1})\mathfrak{N}_\mu(H).$$

*In particular,* $\operatorname{gr} A + \widehat{\mathfrak{N}}_\lambda(H) = \operatorname{gr} A + \widehat{\mathfrak{N}}_\mu(H)$ *if and only if*

$$\mathfrak{N}_\lambda(H) = (I + (\lambda - \mu)(A - \lambda)^{-1})\mathfrak{N}_\mu(H).$$

*Proof.* Assume that $\widehat{\mathfrak{N}}_\lambda(H) \subseteq \operatorname{gr} A + \widehat{\mathfrak{N}}_\mu(H)$, then for every $f_\lambda \in \mathfrak{N}_\lambda(H)$ there exists an $f_\mu \in \mathfrak{N}_\mu(H)$ such that

$$\{f_\lambda - f_\mu, \lambda f_\lambda - \mu f_\mu\} = \{f_\lambda, \lambda f_\lambda\} - \{f_\mu, \mu f_\mu\} \in \operatorname{gr} A.$$

I.e,, $\{f_\lambda - f_\mu, (\lambda - \mu) f_\mu\} \in \operatorname{gr}(A - \lambda)$ or, equivalently, $\{(\lambda - \mu) f_\mu, f_\lambda - f_\mu\} \in \operatorname{gr}((A - \lambda)^{-1})$. From this the inclusion $\mathfrak{N}_\lambda(H) \subseteq (I + (\lambda - \mu)(A - \lambda)^{-1})\mathfrak{N}_\mu(H)$ follows (note that $(A - \lambda)^{-1}$ need not be an operator). The converse implication in the first equivalence can be proven by reversing the above arguments and the second equivalence follows from the first equivalence by symmetry. □

In particular, if $U$ is an isometric relation from $\{\mathfrak{H}^2, (\mathfrak{j}_{\mathfrak{H}}\cdot, \cdot)\}$ to $\{\mathcal{H}^2, (\mathfrak{j}_{\mathcal{H}}\cdot, \cdot)\}$, $S$ and $T$ are the relations such that $\operatorname{gr} S = \ker U$ and $\operatorname{gr} T = \operatorname{dom} U$, and there exists a hyper-maximal dissipative relation $A$ such that $S \subseteq A \subseteq T$, then $A^* \subseteq A$ and by Proposition A.3 combined with Corollary A.4

$$\operatorname{gr} T = \operatorname{gr} A \dot{+} \widehat{\mathfrak{N}}_\lambda(T), \quad \lambda \in \mathbb{C}_-, \quad \text{and} \quad \operatorname{gr} T = \operatorname{gr} A^* \dot{+} \widehat{\mathfrak{N}}_\lambda(T), \quad \lambda \in \mathbb{C}_+.$$

Therefore for such isometric operators

$$\begin{aligned} \mathfrak{N}_\lambda(T) &= (I + (\lambda - \mu)(A^* - \lambda)^{-1})\mathfrak{N}_\mu(T), \quad \lambda, \mu \in \mathbb{C}_+; \\ \mathfrak{N}_\lambda(T) &= (I + (\lambda - \mu)(A - \lambda)^{-1})\mathfrak{N}_\mu(T), \quad \lambda, \mu \in \mathbb{C}_-. \end{aligned} \tag{A.7}$$

If $A$ is a hyper-maximal accumulative relation, then a similar result holds, and if $A$ is hyper-maximal dissipative and accumulative at the same time, i.e. if $A$ is selfadjoint, then

$$\mathfrak{N}_\lambda(T) = (I + (\lambda - \mu)(A - \lambda)^{-1})\mathfrak{N}_\mu(T), \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}. \tag{A.8}$$

## A.2 Basic properties of boundary triplets

Here the various notions of boundary triplets occurring in the literature are recalled and it is shown how they can be interpreted as unitary or isometric operators.

**Ordinary boundary triplets:** First the definition of an ordinary (or standard) boundary triplet is presented, see (Gorbachuk & Gorbachuk 1991: Ch 3: Section 1.4) and (Derkach 1995: Definition 2.1).

**Definition A.6.** Let $S$ be a closed symmetric operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$ with $\overline{\operatorname{dom} S} = \mathfrak{K}$. Then the triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\{\mathcal{H}, (\cdot,\cdot)\}$ is a Hilbert space and $\Gamma_i : \mathfrak{K} \to \mathcal{H}$ is a linear operator for $i = 0, 1$, is called an *ordinary boundary triplet* for $S^{[*]}$ if

(i) the Lagrange identity (or Greens identity) holds: For every $f, g \in \operatorname{dom} S^{[*]}$

$$[S^{[*]}f, g] - [f, S^{[*]}g] = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g);$$

(ii) the mapping $\Gamma : \{f, S^{[*]}f\} \to \{\Gamma_0 f, \Gamma_1 f\}$ from $\operatorname{gr} S^{[*]}$ to $\mathcal{H}^2$ is surjective.

Since $S$ is a symmetric operator, $\operatorname{gr} S^{[*]} = (\operatorname{gr} S)^{\ll\perp\gg}$, see Proposition A.1. Therefore Definition A.6 implies that

$$\ker \Gamma = \operatorname{gr} S = (\operatorname{gr} S^{[*]})^{\ll\perp\gg} = (\operatorname{dom} \Gamma)^{\ll\perp\gg}. \tag{A.9}$$

Using the operators $\mathrm{j}_{\mathfrak{K}}$ and $\mathrm{j}_{\mathcal{H}}$ for $\mathfrak{K}^2$ and $\mathcal{H}^2$ as defined in (A.3) condition (i) is saying that $\Gamma$ defined as $\Gamma : \{f, S^* f\} \subseteq \mathfrak{K}^2 \to \{\Gamma_0 f, \Gamma_1 f\}$ is an isometric operator from the Kreĭn space $\{\mathfrak{K}^2, [\mathrm{j}_{\mathfrak{K}}\cdot,\cdot]\}$ to the Kreĭn space $\{\mathcal{H}^2, (\mathrm{j}_{\mathcal{H}}\cdot,\cdot)\}$. Combining that observation with (A.9) and the assumption that $\Gamma$ is surjective yields that $\Gamma$ is a bounded unitary operator from $\{\mathfrak{K}^2, [\mathrm{j}_{\mathfrak{K}}\cdot,\cdot]\}$ onto $\{\mathcal{H}^2, (\mathrm{j}_{\mathcal{H}}\cdot,\cdot)\}$, see Corollary 4.4. In fact, since $\operatorname{gr} S^{[*]}$ is a closed subspace of $\{\mathfrak{K}^2, [\mathrm{j}_{\mathfrak{K}}\cdot,\cdot]\}$, that statement shows that condition (ii) can be weakened to $\overline{\operatorname{ran}}\,\Gamma = \mathcal{H}^2$. Also the condition that $\overline{\operatorname{dom}}\,S = \mathfrak{K}$, i.e. that $S^{[*]}$ is an operator, can be dropped without difficulties. By means of these observations the following more general definition of an ordinary boundary triplet is obtained, cf. (Derkach & Malamud 1995: Definition 1.6).

**Definition A.7.** Let $S$ be a closed symmetric relation in $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then the triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\{\mathcal{H}, (\cdot,\cdot)\}$ is a Hilbert space and $\Gamma_i : \mathfrak{K}^2 \to \mathcal{H}$ is a linear operator for $i = 0, 1$, is called an *ordinary boundary triplet* for $S^{[*]}$ if

(i) the Lagrange identity (or Greens identity) holds: For every $\{f, f'\}, \{g, g'\} \in \operatorname{gr} S^{[*]}$

$$[f', g] - [f, g'] = (\Gamma_1\{f, f'\}, \Gamma_0\{g, g'\}) - (\Gamma_0\{f, f'\}, \Gamma_1\{g, g'\});$$

(ii) the mapping $\Gamma : \{f, f'\} \to \{\Gamma_0\{f, f'\}, \Gamma_1\{f, f'\}\}$ from $\operatorname{gr} S^{[*]}$ to $\mathcal{H}^2$ is surjective.

Note that if $\mathfrak{H}^+ \dot{+} \mathfrak{H}^-$ is a canonical decomposition of $\{\mathcal{H}^2, (\mathfrak{j}_{\mathcal{H}}\cdot, \cdot)\}$, then $\dim \mathfrak{H}^+ = \dim \mathfrak{H}^-$. Hence, Corollary 6.5 implies that there exist ordinary boundary triplets only for symmetric relations with equal defect numbers.

**Generalized boundary triplets:** Next a generalization of the ordinary boundary triplet is presented, the so-called generalized boundary triplet, see (Derkach & Malamud 1995: Definition 6.1); note that here the Kreĭn space analogue of that definition is stated.

**Definition A.8.** Let $S$ be a closed symmetric relation in $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then the triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\{\mathcal{H}, (\cdot,\cdot)\}$ is a Hilbert space and $\Gamma_i : \mathfrak{K}^2 \to \mathcal{H}$ is a linear operator for $i = 0, 1$, is called a *generalized boundary triplet* for $S^{[*]}$ if

(i) $\overline{\operatorname{dom}}\,\Gamma = \operatorname{gr} S^{[*]}$ and the Lagrange identity (or Greens identity) holds: For every $\{f, f'\}, \{g, g'\} \in \operatorname{dom} \Gamma$

$$[f', g] - [f, g'] = (\Gamma_1\{f, f'\}, \Gamma_0\{g, g'\}) - (\Gamma_0\{f, f'\}, \Gamma_1\{g, g'\});$$

(ii) $\operatorname{ran} \Gamma_0 = \mathcal{H}$ and $\ker \Gamma_0$ is the graph of a selfadjoint relation in $\{\mathfrak{K}, [\cdot,\cdot]\}$.

Again, the first condition in Definition A.8 implies that $\Gamma$ defined as $\Gamma : \{f, f'\} \in \operatorname{dom} \Gamma \subseteq \mathfrak{K}^2 \to \{\Gamma_0\{f, f'\}, \Gamma_1\{f, f'\}\}$ is an isometric operator from the Kreĭn space $\{\mathfrak{K}^2, [\mathfrak{j}_{\mathfrak{K}}\cdot, \cdot]\}$ to the Kreĭn space $\{\mathcal{H}^2, (\mathfrak{j}_{\mathcal{H}}\cdot, \cdot)\}$ and the second condition implies that $\Gamma$ is a unitary operator from $\{\mathfrak{K}^2, [\mathfrak{j}_{\mathfrak{K}}\cdot, \cdot]\}$ to $\{\mathcal{H}^2, (\mathfrak{j}_{\mathcal{H}}\cdot, \cdot)\}$, see Theorem 7.19. Note that if the triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a generalized boundary triplet for

$S^{[*]}$, then by (the second part of) Theorem 7.19 there exists an ordinary boundary triplet $\{\mathcal{H}, \Gamma_0^o, \Gamma_1^o\}$ for $S^{[*]}$, a bounded selfadjoint operator $K$ in $\{\mathcal{H}, (\cdot,\cdot)\}$ and a closed operator $B$ in $\{\mathcal{H}, (\cdot,\cdot)\}$ with $\overline{\operatorname{dom}} B = \mathcal{H} = \operatorname{ran} B$ and $\ker B = \{0\}$ such that

$$\begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} = \begin{pmatrix} B & 0 \\ KB & B^{-*} \end{pmatrix} \begin{pmatrix} \Gamma_0^o \\ \Gamma_1^o \end{pmatrix}. \tag{A.10}$$

Conversely, if a triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ has the above representation, then direct arguments show that it is a generalized boundary triplet.

**Remark A.9.** Since a generalized boundary triplet can be interpreted as a unitary operator, a generalized boundary triplet is said to be of type Ia, type Ib or type II if its interpretation as a unitary operator is of type Ia, type Ib or type II, respectively. In fact every generalized boundary triplet can only be of type Ia or type II (with strongly equal defect numbers), because by definition there exists a hypermaximal neutral subspace in the domain of every generalized boundary triplet, see Corollary 8.18. Since composition with bounded unitary operators does not change the type of a unitary relation, (A.10) shows that a generalized boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with the representation (A.10) is of type I if and only if $B^{-1}$ is a compact operator, cf. Theorem 8.10.

**Unitary boundary triplets:** As a further generalization of generalized boundary triplets, the notion of a unitary boundary triplet for the adjoint of a symmetric relation $S$ was introduced, see (Derkach et al. 2006: Definition 3.1) and (Behrndt et al. 2011a: Definition 3.1).

**Definition A.10.** Let $S$ be a closed symmetric linear relation in $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then the triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\{\mathcal{H}, (\cdot,\cdot)\}$ is a Hilbert space and $\Gamma_i : \mathfrak{K}^2 \to \mathcal{H}$ is a linear operator for $i = 0, 1$, is called a *unitary boundary triplet* for $S^{[*]}$ if

(i) $\overline{\operatorname{dom}} \Gamma = \operatorname{gr} S^{[*]}$ and the Lagrange identity (or Greens identity) holds: For every $\{f, f'\}, \{g, g'\} \in \operatorname{dom} \Gamma$

$$[f', g] - [f, g'] = (\Gamma_1\{f, f'\}, \Gamma_0\{g, g'\}) - (\Gamma_0\{f, f'\}, \Gamma_1\{g, g'\});$$

(ii) if $g, g' \in \mathfrak{H}$ and $k, k' \in \mathcal{H}$ are such that

$$[f', g] - [f, g'] = (\Gamma_1\{f, f'\}, k) - (\Gamma_0\{f, f'\}, k'), \quad \forall \{f, f'\} \in \operatorname{dom} \Gamma,$$

then $\{g, g'\} \in \operatorname{dom} \Gamma$ and $\{k, k'\} = \Gamma\{g, g'\} = \{\Gamma_0\{g, g'\}, \Gamma_1\{g, g'\}\}$.

The first condition in Definition A.8 implies that $\Gamma$ defined as $\Gamma : \{f, f'\} \in \operatorname{gr} T \subseteq \mathfrak{K}^2 \to \{\Gamma_0\{f, f'\}, \Gamma_1\{f, f'\}\}$ is an isometric operator from the Kreĭn space $\{\mathfrak{K}^2, [\mathfrak{j}_{\mathfrak{K}}\cdot, \cdot]\}$ to the Kreĭn space $\{\mathcal{H}^2, (\mathfrak{j}_{\mathcal{H}}\cdot, \cdot)\}$ and the second condition implies that $\Gamma$ is a unitary operator from $\{\mathfrak{K}^2, [\mathfrak{j}_{\mathfrak{K}}\cdot, \cdot]\}$ to $\{\mathcal{H}^2, (\mathfrak{j}_{\mathcal{H}}\cdot, \cdot)\}$, see Proposition 3.1. Consequently, the condition that $\overline{\operatorname{dom}}\,\Gamma = \operatorname{gr} S^{[*]}$ implies that $\ker \Gamma = \operatorname{gr} S$, see Proposition A.1 and (3.4). As in the preceding cases, unitary boundary triplets only exist for symmetric relation with equal defect numbers. For symmetric relations with unequal defect numbers boundary relations or D-boundary triplets are needed, see (Derkach et al. 2006: Proposition 3.7) or (Mogilevskii 2006), respectively.

Corollary 7.17 implies that $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a unitary boundary triplet for $S^{[*]}$, where the symmetric relation $S$ has strongly equal defect numbers, if and only if there exists an ordinary boundary triplet $\{\mathcal{H}, \Gamma_0^o, \Gamma_1^o\}$, a closed operator $B$ in $\{\mathcal{H}, (\cdot, \cdot)\}$ with $\overline{\operatorname{dom}}\, B = \mathcal{H} = \operatorname{ran} B$ and $\ker B = \{0\}$, and a standard unitary operator $U_a$ in $\{\mathcal{H}^2, (\mathfrak{j}_{\mathcal{H}}\cdot, \cdot)\}$ such that

$$\begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} = U_a \begin{pmatrix} B & 0 \\ 0 & B^{-*} \end{pmatrix} \begin{pmatrix} \Gamma_0^o \\ \Gamma_1^o \end{pmatrix}. \tag{A.11}$$

Note also that Corollary 7.21 shows that $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a unitary boundary triplet for $S^{[*]}$ such that $\ker \Gamma_0$ is the graph of a selfadjoint relation in $\{\mathfrak{K}, [\cdot, \cdot]\}$ if and only if there exists an ordinary boundary triplet $\{\mathcal{H}, \Gamma_0^o, \Gamma_1^o\}$, an operator $B$ in $\{\mathcal{H}, (\cdot, \cdot)\}$ with $\overline{\operatorname{dom}}\, B = \mathcal{H} = \operatorname{ran} \operatorname{clos}(B)$ and $\ker \operatorname{clos}(B) = \{0\}$, and a selfadjoint operator $K$ in $\{\mathfrak{H}, (\cdot, \cdot)\}$ with $\operatorname{dom} K = \operatorname{ran} B$ such that

$$\begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} = \begin{pmatrix} B & 0 \\ KB & B^{-*} \end{pmatrix} \begin{pmatrix} \Gamma_0^o \\ \Gamma_1^o \end{pmatrix}. \tag{A.12}$$

**Quasi-boundary triplet:** In (Behrndt & Langer 2007: Definition 2.1) the concept of an ordinary boundary triplet for the adjoint of a symmetric relation in a Hilbert space was generalized to the concept of a quasi-boundary triplet; below the natural generalization to the Kreĭn space case is presented.

**Definition A.11.** Let $S$ be a closed symmetric relation in $\{\mathfrak{K}, [\cdot, \cdot]\}$. Then the triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\{\mathcal{H}, (\cdot, \cdot)\}$ is a Hilbert space and $\Gamma_i : \mathfrak{K}^2 \to \mathcal{H}$ is a linear operator for $i = 0, 1$, is called a *quasi-boundary triplet* for $S^{[*]}$ if

(i) $\overline{\operatorname{dom}}\,\Gamma = \operatorname{gr} S^{[*]}$ and the Lagrange identity (or Greens identity) holds: For every $\{f, f'\}, \{g, g'\} \in \operatorname{dom} \Gamma$

$$[f', g] - [f, g'] = (\Gamma_1\{f, f'\}, \Gamma_0\{g, g'\}) - (\Gamma_0\{f, f'\}, \Gamma_1\{g, g'\});$$

(ii) $\ker \Gamma_0$ is the graph of a selfadjoint relation in $\{\mathfrak{K}, [\cdot,\cdot]\}$;

(iii) $\overline{\operatorname{ran}}\,\Gamma = \mathcal{H}^2$, where $\Gamma : \{f, f'\} \in \operatorname{dom} U \to \{\Gamma_0\{f, f'\}, \Gamma_1\{f, f'\}\}$.

Condition (i) in Definition A.11 implies that $\Gamma$ is an isometric operator from the Kreĭn space $\{\mathfrak{K}^2, [\mathrm{j}_{\mathfrak{K}}\cdot,\cdot]\}$ to the Kreĭn space $\{\mathcal{H}^2, (\mathrm{j}_{\mathcal{H}}\cdot,\cdot)\}$. Conditions (ii) and (iii) do not guaranty that $\Gamma$ is a unitary operator as the following example shows.

**Example A.12.** Let $\{\mathcal{H}, (\cdot,\cdot)\}$ be a Hilbert space and let $T$ be a symmetric operator in $\{\mathcal{H}, (\cdot,\cdot)\}$ with $\overline{\operatorname{dom}}\,T = \mathcal{H}$ which is not a selfadjoint operator. Then define the linear operator $\Gamma$ in $\{\mathcal{H}^2, (\mathrm{j}_{\mathcal{H}}\cdot,\cdot)\}$ as

$$\Gamma = \begin{pmatrix} I & 0 \\ T & I \end{pmatrix},$$

where the block representation of $\Gamma$ is w.r.t. the decomposition $\mathcal{H} \times \mathcal{H}$ of $\mathcal{H}^2$. Then a direct calculation shows that $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\Gamma_0 = \mathcal{P}_{\mathcal{H}\times\{0\}}\Gamma$ and $\Gamma_1 = \mathcal{P}_{\{0\}\times\mathcal{H}}\Gamma$, is a quasi-boundary triplet for $S^*$, where $\operatorname{gr} S^* = \mathcal{H} \times \mathcal{H}$. Moreover, $\Gamma$ is (extendable to) a unitary operator in $\{\mathcal{H}^2, (\mathrm{j}_{\mathcal{H}}\cdot,\cdot)\}$ if and only if $T$ is (extendable to) a selfadjoint operator, see Proposition 4.8.

Like Definition A.10, Definition A.11 can be extended by allowing $\Gamma$ to be a relation. In that case condition (iii) should be replaced by the condition that $\operatorname{mul}\Gamma = (\operatorname{ran}\Gamma)^{<\perp>}$, where $<\perp>$ is the orthogonal complement in $\mathcal{H}^2$ w.r.t. $(\mathrm{j}_{\mathcal{H}}\cdot,\cdot)$. The conditions (ii) and (iii) in Definition A.11 imply that $\ker\Gamma = (\operatorname{dom}\Gamma)^{\ll\perp\gg}$, where $\ll\perp\gg$ is the orthogonal complement w.r.t. $[\mathrm{j}_{\mathfrak{K}}\cdot,\cdot]$, see Lemma 6.1 and Proposition A.1. Therefore, as for boundary triplets, $\ker\Gamma = \operatorname{gr} S$. Note further that if $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a quasi-boundary triplet for $S^{[*]}$, then also $\{\mathcal{H}, \operatorname{clos}(\Gamma_0), \operatorname{clos}(\Gamma_1)\}$ is a quasi-boundary triplet for $S^{[*]}$.

Theorem 7.9 implies that $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a quasi-boundary triplet for $S^{[*]}$ if and only if there exists an ordinary boundary triplet $\{\mathcal{H}, \Gamma_0^o, \Gamma_1^o\}$ for $S^{[*]}$, an operator $B$ in $\{\mathcal{H}, (\cdot,\cdot)\}$ with $\overline{\operatorname{dom}}\,B = \mathcal{H} = \operatorname{ran}\operatorname{clos}(B)$ and $\ker\operatorname{clos}(B) = \{0\}$, and a symmetric operator $T$ in $\{\mathcal{H}, (\cdot,\cdot)\}$ with $\operatorname{dom} T = \operatorname{ran} B$ and $\operatorname{dom} T^* \cap \operatorname{mul}\operatorname{clos}(B) = \{0\}$ such that

$$\begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} = \begin{pmatrix} B & 0 \\ TB & B^{-*} \end{pmatrix} \begin{pmatrix} \Gamma_0^o \\ \Gamma_1^o \end{pmatrix}. \tag{A.13}$$

In particular, $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is extendable to a unitary boundary triplet if and only if $T$ is extendable to a selfadjoint operator, see Proposition 7.3 and Remark 7.10 (i). I.e., the following necessary and sufficient conditions for a quasi-boundary triplet to be (extendable to) a unitary boundary triplet hold.

**Proposition A.13.** *Let $S$ be a closed and symmetric relation in $\{\mathfrak{K}, [\cdot,\cdot]\}$ and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a quasi-boundary triplet for $S^{[*]}$. Then $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is (extendable to) a unitary boundary triplet for $S^{[*]}$ if and only if $\Gamma(\mathfrak{j}\ker\Gamma_0 \cap \operatorname{dom}\Gamma)$ is (extendable to) a hyper-maximal neutral subspace of $\{\mathcal{H}^2, (\mathfrak{j}_{\mathcal{H}}\cdot,\cdot)\}$ for some (and hence for every) fundamental symmetry $\mathfrak{j}$ of $\{\mathfrak{K}^2, [\mathfrak{j}_{\mathfrak{K}}\cdot,\cdot]\}$.*

The characterization of quasi-boundary triplets in (A.13) shows that they are very closely connected to generalized boundary triplets, the following statement makes that connection precise. Therefore note that for a symmetric relation $S$ in (the Hilbert space) $\{\mathcal{H}, (\cdot,\cdot)\}$ and a relation $B$ in $\{\mathcal{H}, (\cdot,\cdot)\}$, the archetypical isometric relations $\Upsilon_1(S)$ and $\Upsilon_2(B)$ in $\{\mathcal{H}^2, (\mathfrak{j}_{\mathcal{H}}\cdot,\cdot)\}$ take the form

$$\Upsilon_1(S)\{f,g\} = \{f, Sf+g\}, \quad f \in \operatorname{dom} S,\ g \in \mathfrak{M};$$
$$\Upsilon_2(B)\{f,g\} = \{Bf, B^{-*}g\}, \quad f \in \operatorname{dom} B,\ g \in \operatorname{ran} B^*.$$

cf. Section 4.2. In particular, if $\Upsilon_1(S)$ and $\Upsilon_2(B)$ are operators, then w.r.t. the decomposition $\mathcal{H}\times\mathcal{H}$ of $\mathcal{H}^2$, they have the following block representation:

$$\Upsilon_1(S) = \begin{pmatrix} I & 0 \\ S & I \end{pmatrix} \quad \text{and} \quad \Upsilon_2(B) = \begin{pmatrix} B & 0 \\ 0 & B^{-*} \end{pmatrix},$$

**Proposition A.14.** *Let $\{\mathcal{H}, \Gamma_0^q, \Gamma_1^q\}$ be a quasi-boundary triplet for the adjoint of the closed symmetric relation $S$ in $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then there exists a boundary relation*[1] *$\{\mathcal{H}, \Gamma\}$ for $S^{[*]}$ with $\mathcal{H}\times\{0\} \subseteq \operatorname{ran}\Gamma$ and $\ker\Gamma_0 = (\ker\Gamma_0)^{\ll\perp\gg}$, and a symmetric operator $T$ in $\{\mathcal{H}, (\cdot,\cdot)\}$ with $\overline{\operatorname{dom}}\,T = \mathcal{H}$ and $\operatorname{dom} T^* \cap \operatorname{mul}\Gamma_0 = \{0\}$ such that $\Gamma_q = \Upsilon_1(T)\Gamma$. Conversely, if $T$ and $\Gamma$ are as above, then $\{\mathcal{H}, \Gamma_0^q, \Gamma_1^q\}$, where $\Gamma_0^q = \mathcal{P}_{\mathcal{H}\times\{0\}}\Upsilon_1(T)\Gamma$ and $\Gamma_1^q = \mathcal{P}_{\{0\}\times\mathcal{H}}\Upsilon_1(T)\Gamma$, is a quasi-boundary triplet for $S^{[*]}$.*

*Proof.* For the direct part recall that by Theorem 7.9 there exists an operator $B$ in $\{\mathcal{H}, (\cdot,\cdot)\}$ with $\overline{\operatorname{dom}}\,B = \mathcal{H} = \operatorname{ran}\operatorname{clos}(B)$ and $\ker\operatorname{clos}(B) = \{0\}$, a symmetric operator $T$ in $\{\mathcal{H}, (\cdot,\cdot)\}$ with $\operatorname{dom} T = \operatorname{ran} B$ and $\operatorname{dom} T^* \cap \operatorname{mul}\operatorname{clos}(B) = \{0\}$, and a bounded unitary operator $\Gamma$ from $\{\mathfrak{K}^2, [\mathfrak{j}_{\mathfrak{K}}\cdot,\cdot]\}$ onto $\{\mathcal{H}^2, (\mathfrak{j}_{\mathcal{H}}\cdot,\cdot)\}$ with $\operatorname{dom}\Gamma^q \subseteq \operatorname{dom}\Gamma$ such that

$$\Gamma^q = \begin{pmatrix} \Gamma_0^q \\ \Gamma_1^q \end{pmatrix} = \Upsilon_1(T)\Upsilon_2(B)\Gamma.$$

Consequently, $\Gamma := \Upsilon_2(\operatorname{clos}(B))\Gamma$ satisfies the stated conditions.

[1] A boundary relation is a unitary boundary triplet which is allowed to be multi-valued, see (Derkach et al. 2006: Definition 3.1).

To prove the converse note that by the assumptions on $\Gamma$, $\operatorname{mul}\Gamma \subseteq \mathcal{H} \times \{0\}$. Consequently, arguments as in Theorem 7.19 show that there exists a closed relation $B$ in $\{\mathcal{H}, (\cdot,\cdot)\}$ with $\overline{\operatorname{dom}} B = \mathcal{H} = \operatorname{ran} B$ and $\ker B = \{0\}$, a bounded selfadjoint operator $K$ in $\{\mathcal{H}, (\cdot,\cdot)\}$ and a standard boundary triplet $\{\mathcal{H}, \Gamma_0^o, \Gamma_1^o\}$ such that

$$\Gamma = \Upsilon_1(K)\Upsilon_2(B)\begin{pmatrix}\Gamma_0^o \\ \Gamma_1^o\end{pmatrix}.$$

Consequently,

$$\begin{pmatrix}\Gamma_0^q \\ \Gamma_1^q\end{pmatrix} = \Upsilon_1(T+K)\Upsilon_2(B)\begin{pmatrix}\Gamma_0^o \\ \Gamma_1^o\end{pmatrix},$$

where the righthand side is an operator as a consequence of the assumption that $\operatorname{dom} T \cap \operatorname{mul}\Gamma_0 = \{0\}$. Clearly, $\ker\Gamma_0^q = \Gamma^{-1}(\{0\}\times\mathcal{H})$ is the graph of a selfadjoint relation in $\{\mathfrak{K}, [\cdot,\cdot]\}$ and $\overline{\operatorname{ran}}\,\Gamma^q = \mathcal{H}^2$ as a consequence of the assumption that $\operatorname{dom} T^* \cap \operatorname{mul}\Gamma_0 = \{0\}$, see Step 3 of the proof of Theorem 7.9. □

Note that if the symmetric operator $T$ in Proposition A.14 has equal defect numbers, then the quasi-boundary triplet can be extended to a unitary boundary triplet, see e.g. Proposition 7.3

## A.3 Weyl functions of boundary triplets

Here the Weyl function of boundary triplets for the adjoint of a symmetric relation in a Hilbert space are shortly described and, in particular, it is shown how each Weyl function is the transformation of a bounded and boundedly invertible Nevanlinna function. Therefore recall first that by means of eigenspaces, see (A.1), a Weyl family can be associated with boundary triplets, see (Derkach et al. 2006; Behrndt & Langer 2007; Behrndt et al. 2011a).

**Definition A.15.** Let $S$ be a closed symmetric relation in the Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$, let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a unitary boundary triplet or a quasi-boundary triplet for $S^{[*]}$ and let $T$ be the relation in $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that $\operatorname{gr} T = \operatorname{dom} U$. Then the *Weyl family* associated with $\Gamma$ is the operator-valued function $M(\lambda)$ defined for $\lambda \in \mathbb{C}$ via

$$\operatorname{gr}(M(\lambda)) = \Gamma(\widehat{\mathfrak{N}}_\lambda(T)) = \{\{\Gamma_0\{f_\lambda, \lambda f_\lambda\}, \Gamma_1\{f_\lambda, \lambda f_\lambda\}\} : \{f_\lambda, \lambda f_\lambda\} \in \widehat{\mathfrak{N}}_\lambda(T)\},$$

or, equivalently,

$$M(\lambda) = \Gamma_1(\Gamma_0 \upharpoonright_{\widehat{\mathfrak{N}}_\lambda(T)})^{-1}, \quad \lambda \in \mathbb{C}.$$

Recall also the definition of the so-called Nevanlinna family, see (Derkach et al. 2006: Section 2.6).

**Definition A.16.** A family of linear relations $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, in $\{\mathcal{H}, (\cdot,\cdot)\}$ is called a *Nevanlinna family* if it has the following properties:

(i) for every $\lambda \in \mathbb{C}_+$ ($\mathbb{C}_-$) the relation $M(\lambda)$ is maximal dissipative (resp. accumulative);

(ii) $M(\lambda)^* = M(\overline{\lambda})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(iii) for some, and hence for all, $\mu \in \mathbb{C}_-$ ($\mathbb{C}_-$) the operator family $(M(\lambda) - \mu)^{-1}$ is an everywhere defined operator function which depends holomorphically on $\lambda$ for $\lambda \in \mathbb{C}_+$ ($\mathbb{C}_-$).

With this definition it can easily be seen that every Weyl family of a boundary relation for the adjoint of a symmetric relation in a Hilbert space is a Nevanlinna function, i.e., a Nevanlinna family whose values are operators. Namely the conditions (i) and (ii) are satisfied as a consequence of Proposition 5.1, Proposition A.1 and (A.4). In light of the fact that $(M(\lambda) - \mu)^{-1}$ is everywhere defined for $\lambda \in \mathbb{C}_+$ and $\mu \in \mathbb{C}_-$, because $M(\lambda)$ is maximal dissipative for $\lambda \in \mathbb{C}_+$, the third condition holds as a consequence of the definition of $M(\lambda)$, see Definition A.15, and (A.7). Conversely, every Nevanlinna family can be realized (nonuniquely) as the Weyl family of a boundary relation, see (Derkach et al. 2006: Theorem 3.9). Note also that as a consequence of the fact that the Weyl function associated to a boundary triplet satisfies $M(\lambda)^* = M(\overline{\lambda})$, the identity in Proposition 5.1 is called the Weyl identity.

**Weyl functions of ordinary boundary triplets:** Let the triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for the adjoint of a closed symmetric relation $S$ in $\{\mathfrak{H}, (\cdot,\cdot)\}$. Then $\mathrm{ran}\,\Gamma = \mathrm{ran}\,(\Gamma_0 \times \Gamma_1) = \mathcal{H}^2$ implies that the hyper-maximal neutral subspaces $\mathcal{H} \times \{0\}$ and $\{0\} \times \mathcal{H}$ of $\{\mathcal{H}^2, (\mathfrak{j}_{\mathcal{H}}\cdot,\cdot)\}$ are contained in the range of $\Gamma$. Hence, $A_0$ and $A_1$ defined via

$$\mathrm{gr}\,A_0 = \Gamma^{-1}(\{0\} \times \mathcal{H}) \quad \text{and} \quad \mathrm{gr}\,A_1 = \Gamma^{-1}(\mathcal{H} \times \{0\})$$

are selfadjoint relations in $\{\mathfrak{H}, (\cdot,\cdot)\}$, see Proposition A.1 and Proposition 4.5. Consequently, for $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$\mathrm{dom}\,\Gamma = \mathrm{gr}\,A_0 \widehat{+} \widehat{\mathfrak{N}}_\lambda(S^*) \quad \text{and} \quad \mathrm{dom}\,\Gamma = \mathrm{gr}\,A_1 \widehat{+} \widehat{\mathfrak{N}}_\lambda(S^*), \tag{A.14}$$

see Corollary A.4. From (A.14) it follows that $\mathcal{P}_{\mathcal{H}\times\{0\}}\Gamma(\widehat{\mathfrak{N}}_\lambda(S^*)) = \mathcal{P}_{\mathcal{H}\times\{0\}}\mathrm{ran}\,\Gamma$ and $\mathcal{P}_{\{0\}\times\mathcal{H}}\Gamma(\widehat{\mathfrak{N}}_\lambda(S^*)) = \mathcal{P}_{\{0\}\times\mathcal{H}}\mathrm{ran}\,\Gamma$, i.e. (A.14) implies that

$$\mathrm{dom}\,M(\lambda) = \mathrm{dom}\,M(\mu) \quad \text{and} \quad \mathrm{ran}\,M(\lambda) = \mathrm{ran}\,M(\mu) \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}.$$

In fact, since $\operatorname{ran}\Gamma = \mathcal{H}^2$, the above observation yields $\operatorname{dom} M(\lambda) = \mathcal{H} = \operatorname{ran} M(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, i.e. the Weyl function associated to an ordinary boundary triplet is a bounded and boundedly invertible Nevanlinna function.

**Weyl functions of generalized boundary triplets:** Using the above arguments, one can show that the Weyl function of a generalized boundary triplet is an everywhere defined Nevanlinna function. This can also be seen from the connection of generalized boundary triplets to ordinary boundary triplets presented above. Namely, (A.10) implies that $M(\cdot)$ is the Weyl function of a generalized boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ if and only if there exists a bounded and boundedly invertible Nevanlinna function $M_o(\cdot)$, a closed operator $B$ in $\{\mathcal{H}, (\cdot,\cdot)\}$ with $\overline{\operatorname{dom}} B = \mathcal{H} = \operatorname{ran} B$ and $\ker B = \{0\}$, and a bounded selfadjoint operator $K$ in $\{\mathcal{H}, (\cdot,\cdot)\}$ such that

$$M(\lambda) = K + B^{-*} M_o(\lambda) B^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

As a consequence of Remark A.9, $\operatorname{Im} M(\lambda)$ is a compact operator if $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a generalized boundary triplet of type I; the converse also holds. Therefore recall that $\gamma_\lambda$, the mapping from $\mathcal{H}$ onto $\mathfrak{N}_\lambda(T)$, where $\operatorname{gr} T = \operatorname{dom}\Gamma$, such that $\Gamma_0\{\gamma_\lambda h, \lambda g_\lambda h\} = h$, satisfies

$$(\lambda - \overline{\lambda})\gamma_{\overline{\lambda}}^* \gamma_\lambda = M(\lambda) - M(\lambda)^*,$$

see (Derkach & Malamud 1995: (6.7)). Since by assumption $\operatorname{Im} M(\lambda) = (M(\lambda) - M(\lambda)^*)/(2i)$ is an everywhere defined compact operators, the above equality shows that $\gamma_{\overline{\lambda}}^* \gamma_\lambda$ is a compact operator. From this it follows immediately that $\gamma_\lambda$ is a compact operator and, hence, $\mathfrak{N}_\lambda(T)$ as the range of a compact operator contains only finite-dimensional closed subspaces for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. I.e. $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a generalized boundary triplet of type I, see Remark A.9 and (A.4). Note that this situation occurs for instance in the case of partial differential equations, see (Behrndt & Langer 2007) and the references therein.

**Weyl functions of unitary boundary triplets:** From (A.11) it follows that $M(\lambda)$ is the Weyl function of a unitary boundary triplet for the adjoint of a closed symmetric relation with strongly equal defect numbers if and only if there exists a closed operator $B$ in $\{\mathcal{H}, (\cdot,\cdot)\}$ with $\overline{\operatorname{dom}} B = \mathcal{H} = \operatorname{ran} B$ and $\ker B = \{0\} = \operatorname{mul} B$, everywhere defined operators $A_{ij}$, $1 \le i, j \le 2$ such that

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

is a unitary operator in $\{\mathcal{H}^2, (\cdot,\cdot)\}$, and a bounded and boundedly invertible Nevanlinna function $M_o$ such that

$$M(\lambda) = (A_{21} + A_{22} B^{-*} M_o(\lambda) B^{-1})(A_{11} + A_{12} B^{-*} M_o(\lambda) B^{-1})^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

In the special case that $A_0$ defined via $\operatorname{gr} A_0 = \Gamma^{-1}(\{0\} \times \mathcal{H})$ is selfadjoint, $M(\lambda)$ is as a consequence of (A.12) the Weyl function of a unitary boundary triplet if and only if there exists a bounded and boundedly invertible Nevanlinna function $M_o(\lambda)$, a closed relation $B$ in $\{\mathcal{H}, (\cdot,\cdot)\}$ with $\overline{\operatorname{dom}} B = \mathcal{H} = \operatorname{ran} B$ and $\ker B = \{0\} = \operatorname{mul} B$, and a selfadjoint operator $K$ in $\{\mathcal{H}, (\cdot,\cdot)\}$ with $\operatorname{dom} K \cap \operatorname{mul} B = \{0\}$ such that

$$M(\lambda) = K + B^{-*} M_o(\lambda) B^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Note that in the preceding case the domain of $\operatorname{clos}(\operatorname{Im} M(\lambda))$ is independent of $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and equal to $\mathcal{H}$. The converse also holds, if for a Weyl function $M(\lambda)$ $\operatorname{dom}(\operatorname{clos}(\operatorname{Im} M(\lambda))) = \mathcal{H}$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $M(\lambda)$ is the Weyl function of a boundary relation for which $A_0$ is selfadjoint, see (Derkach et al. 2012).

**Weyl functions of quasi-boundary triplets:** Quasi-boundary triplets can also be characterized by their associated Weyl functions, cf. (Behrndt & Langer 2007: Proposition 2.6) and (Alpay & Behrndt 2009: Proposition 2.6).

**Proposition A.17.** *Let $\{\mathcal{H}, (\cdot,\cdot)\}$ be a Hilbert space and let $M(\cdot)$ be a $\mathcal{H}$-valued operator function. Then $M(\cdot)$ is the Weyl family of a quasi-boundary triplet (for the adjoint of a certain closed symmetric relation) if and only if there exists a symmetric operator $T$ in $\{\mathcal{H}, (\cdot,\cdot)\}$ such that $\operatorname{dom} M(\lambda) \subseteq \operatorname{dom} T$ and that $M_{\Gamma'}(\cdot) := \operatorname{clos}(M(\cdot) + T)$ is a Nevanlinna family which satisfies $\operatorname{dom} M_{\Gamma'}(\lambda) = \mathcal{H}$ and $\ker M_{\Gamma'}(\lambda) \cap \operatorname{dom} T^* = \{0\}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

*Proof.* If $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a quasi-boundary triplet for the adjoint of a symmetric relation $S$ in a Hilbert space $\{\mathfrak{H}, (\cdot,\cdot)\}$, then by Proposition A.14 there exists a symmetric operator $T$ in $\{\mathcal{H}, (\cdot,\cdot)\}$ with $\overline{\operatorname{dom}} T = \mathcal{H}$ and a boundary relation $\{\mathcal{H}, \Gamma'\}$ for $S^*$ with $(\ker \Gamma'_0)^* = \ker \Gamma'_0$, $\operatorname{ran} \Gamma'_0 = \mathcal{H}$ and $\operatorname{dom} T^* \cap \operatorname{mul} \Gamma_0 = \{0\}$ such that $\Gamma = \Upsilon_1(T)\Gamma'$. The Weyl family $M_{\Gamma'}(\cdot)$ associated to $\Gamma'$ is a Nevanlinna family of bounded operators, i.e. $\operatorname{dom} M_{\Gamma'}(\lambda) = \mathcal{H}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, see (Derkach et al. 2009: Proposition 3.15). Note also that the condition $\operatorname{dom} T^* \cap \operatorname{mul} \Gamma_0 = \{0\}$ implies that $\ker M_{\Gamma'}(\lambda) \cap \operatorname{dom} T^* = \{0\}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Finally, a direct calculation shows that the Weyl family $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, associated to $\Gamma = \Gamma_0 \times \Gamma_1$ is

$$M(\lambda) = T + M_{\Gamma'}(\lambda), \quad \operatorname{dom} M(\lambda) = \operatorname{dom} T.$$

Since $\overline{\operatorname{dom}} T = \mathcal{H}$ and $\operatorname{dom} M_{\Gamma'}(\cdot) = \mathcal{H}$, the above equality implies that $M_{\Gamma'}(\cdot) = \operatorname{clos}(M(\cdot) - T)$.

Conversely, if $M_{\Gamma'} := \operatorname{clos}(M(\cdot) + T)$ is a Nevanlinna family which satisfies $\operatorname{dom} M_{\Gamma'}(\lambda) = \mathcal{H}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then, see (Derkach et al. 2009: Proposition 3.15), there exists a closed symmetric relation $S$ in a Hilbert space $\{\mathfrak{H}, (\cdot,\cdot)\}$ and a boundary relation $\{\mathcal{H}, \Gamma'\}$ for $S^*$ satisfying $(\ker \Gamma_0')^* = \ker \Gamma_0'$ and $\operatorname{ran} \Gamma_0' = \mathcal{H}$ such that its associated Weyl family is $M_{\Gamma'}$. Then, since $\overline{\operatorname{dom}} T = \mathcal{H}$ and $\ker M_{\Gamma'}(\lambda) \cap \operatorname{dom} T^* = \{0\}$, $\{\mathcal{H}, \mathcal{P}_{\mathcal{H}\times\{0\}}\Upsilon_1(-T)\Gamma', \mathcal{P}_{\{0\}\times\mathcal{H}}\Upsilon_1(-T)\Gamma'\}$ is a quasi-boundary triplet for $S^*$ by Proposition A.14 and a calculation shows that its Weyl family is $M_{\Gamma'}(\cdot) - T = M(\cdot)$. $\square$

Note that if $T$ has equal defect numbers in the above statement, then the quasi-boundary triplet for $M(\cdot)$ can be extended to a boundary relation.

## A.4 Boundary triplets for intermediate extensions

The results in (Derkach et al. 2009: Section 4) for boundary relations in the Hilbert space setting are here shown to remain valid in the Kreĭn space setting. Therefore first observe the following simple statement about the renormalization of the Weyl function of a unitary boundary triplet, see (Derkach et al. 2009: Proposition 3.11).

**Lemma A.18.** *Let $S$ be a closed symmetric relation in $\{\mathfrak{K}, [\cdot,\cdot]\}$ and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a unitary boundary triplet for $S^{[*]}$ with associated Weyl function $M_\Gamma(\cdot)$. Moreover, let $B$ be a closed operator in $\{\mathcal{H}, (\cdot,\cdot)\}$ with $\operatorname{dom} B = \mathcal{H} = \operatorname{ran} B$ and $\ker B = \{0\}$, and let $K$ be a bounded selfadjoint operator in $\{\mathcal{H}, (\cdot,\cdot)\}$. Then, with*

$$\begin{pmatrix} \Gamma_0' \\ \Gamma_1' \end{pmatrix} := \Upsilon_1(K)\Upsilon_2(B) \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix},$$

*also $\{\mathcal{H}, \Gamma_0', \Gamma_1'\}$ is a unitary boundary triplet for $S^{[*]}$. Its Weyl function $M_{\Gamma'}(\cdot)$ is*

$$M_{\Gamma'}(\lambda) = K + B^{-*}M(\lambda)B^{-1}, \quad \operatorname{dom} M_{\Gamma'} = \operatorname{dom}(M(\lambda)B^{-1}), \quad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

*Proof.* Since $\Upsilon_1(K)$ and $\Upsilon_2(B)$ are standard unitary operators in $\{\mathcal{H}^2, (\mathfrak{j}_{\mathcal{H}}\cdot,\cdot)\}$, $\Upsilon_1(K)\Upsilon_2(B)(\Gamma_0 \times \Gamma_1)$ is a unitary operator from $\{\mathfrak{K}^2, (\mathfrak{j}_{\mathfrak{K}}\cdot,\cdot)\}$ to $\{\mathcal{H}^2, (\mathfrak{j}_{\mathcal{H}}\cdot,\cdot)\}$, see Lemma 3.10. Consequently, $\{\mathcal{H}, \Gamma_0', \Gamma_1'\}$ is a unitary boundary triplet for $S^{[*]}$. The expression for $M_{\Gamma'}$ follows from a direct calculation after the observation that $\operatorname{dom}\Gamma = \operatorname{dom}\Gamma'$ and, hence, $\widehat{\mathfrak{N}}_\lambda(T) = \widehat{\mathfrak{N}}_\lambda(T')$, where $T$ and $T'$ are the relations in $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that $\operatorname{gr} T = \operatorname{dom}\Gamma$ and $\operatorname{gr} T' = \operatorname{dom}\Gamma'$. $\square$

To obtain results on generalized boundary triplets for intermediate extensions, the above lemma is combined with Proposition A.19 below. Note that the following statement is a generalization of a similar statement for generalized boundary triplets from the Hilbert space setting to the Kreĭn space setting.

**Proposition A.19.** *Let $S$ be a closed and symmetric relation in $\{\mathfrak{K}, [\cdot,\cdot]\}$ and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for $S^{[*]}$ with associated Weyl function $M_\Gamma(\cdot)$. Moreover, let $\mathcal{H}'$ be a closed subspace of $\mathcal{H}$ and define the operators $\Gamma_0'$ and $\Gamma_1'$ from $\mathfrak{K}^2$ to $\mathcal{H}'$ as*

$$\Gamma_0'\{f,f'\} = \Gamma_0\{f,f'\} \quad \textit{and} \quad \Gamma_1'\{f,f'\} = \mathcal{P}_{\mathcal{H}'}\Gamma_1\{f,f'\}$$

*for all $\{f,f'\} \in \operatorname{dom}\Gamma$ such that $\Gamma_0\{f,f'\} \in \mathcal{H}'$. Then $\{\mathcal{H}', \Gamma_0', \Gamma_1'\}$ is a generalized boundary triplet for $S_r^{[*]} \subseteq S^{[*]}$, where $\operatorname{gr} S_r = \ker \Gamma'$. Its associated Weyl function $M_{\Gamma'}(\cdot)$ is*

$$M_{\Gamma'}(\lambda) = P_{\mathcal{H}'}M_\Gamma(\lambda), \quad \operatorname{dom} M_{\Gamma'} = \operatorname{dom} M_\Gamma(\lambda) \cap \mathcal{H}' = \mathcal{H}', \quad \lambda \in \mathbb{C}.$$

*Proof.* The first part is a direct consequence of Corollary 7.34 with $U_b$ defined as $U_b\{f,f'\} = \{f, \mathcal{P}_{\mathcal{H}'}f'\}$, $f \in \mathcal{H}'$ and $f' \in \mathcal{H}$. The formula for the Weyl function is a direct consequence of the definition of $\Gamma'$ together with the observation that $\operatorname{dom}\Gamma' \subseteq \operatorname{dom}\Gamma$ and, hence, $\widehat{\mathfrak{N}}_\lambda(T') \subseteq \widehat{\mathfrak{N}}_\lambda(T)$, where $T$ and $T'$ are the relations in $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that $\operatorname{gr} T = \operatorname{dom}\Gamma$ and $\operatorname{gr} T' = \operatorname{dom}\Gamma'$. □

In the Hilbert space setting the above result corresponds to (Derkach et al. 2009: Proposition 4.1). The other statements from (Derkach et al. 2009: Section 4) can be obtained by combining Proposition A.19 with Lemma A.18; following is an example, cf. (Derkach et al. 2009: Corollary 4.5).

**Corollary A.20.** *Let $S_i$ be a closed and symmetric relation in $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ and let $\{\mathcal{H}, \Gamma_0^i, \Gamma_1^i\}$ be a generalized boundary triplet for $S_i^{[*]}$ with associated Weyl function $M_i$, for $i = 1, 2$. With $\mathfrak{K} := \mathfrak{K}_1 \oplus \mathfrak{K}_2$, define the operators $\Gamma_0$ and $\Gamma_1$ from $\mathfrak{K}^2$ to $\mathcal{H}$ as*

$$\Gamma_0\{f_1 \oplus f_2, f_1' \oplus f_2'\} = \Gamma_0^1\{f_1, f_1'\}$$

*and*

$$\Gamma_1\{f_1 \oplus f_2, f_1' \oplus f_2'\} = \Gamma_1^1\{f_1, f_1'\} + \Gamma_1^2\{f_2, f_2'\},$$

*where*

$$\operatorname{dom}\Gamma = \{\{f_1 \oplus f_2, f_1' \oplus f_2'\} \in \mathfrak{K}^2 : \{f_1, f_1'\} \in \operatorname{dom}\Gamma^1,\ \{f_2, f_2'\} \in \operatorname{dom}\Gamma^2 \\ \textit{and } \Gamma_0^1\{f_1, f_1'\} = \Gamma_0^2\{f_2, f_2'\}\}.$$

*Then $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a generalized boundary triplet for $S^{[*]} \subseteq S_1^{[*]} \oplus S_2^{[*]}$, where $\operatorname{gr} S = \ker\Gamma$, and its associated Weyl function is $M_1 + M_2$.*

*Proof.* Define $\Gamma_0'$ and $\Gamma_1'$ as

$$\Gamma_0'\{f_1 \oplus f_2, f_1' \oplus f_2'\} = \begin{pmatrix} \Gamma_0^1\{f_1, f_1'\} \\ \Gamma_0^2\{f_2, f_2'\} \end{pmatrix}, \quad \Gamma_1'\{f_1 \oplus f_2, f_1' \oplus f_2'\} = \begin{pmatrix} \Gamma_1^1\{f_1, f_1'\} \\ \Gamma_1^2\{f_2, f_2'\} \end{pmatrix},$$

where $\{f_1, f_1'\} \in \operatorname{dom}\Gamma^1$ and $\{f_2, f_2'\} \in \operatorname{dom}\Gamma^2$. Then $\{\mathcal{H}^2, \Gamma_0', \Gamma_1'\}$ is a generalized boundary triplet for $S_1^{[*]} \oplus S_2^{[*]}$ with associated Weyl function $M_1(\cdot) \oplus M_2(\cdot)$.

Next define the operator $B$ on $\mathcal{H}^2$ by $B\{f, f'\} = \{f', f - f'\}$, $f, f' \in \mathcal{H}$. Then with $\Gamma_0^B$ and $\Gamma_1^B$ defined via $\Gamma_0^B \times \Gamma_1^B = \Upsilon_2(B)(\Gamma_0' \times \Gamma_1')$, $\{\mathcal{H}^2, \Gamma_0^B, \Gamma_1^B\}$ is a generalized boundary triplet for $S_1^{[*]} \oplus S_2^{[*]}$. Here

$$\Gamma_0^B\{f_1 \oplus f_2, f_1' \oplus f_2'\} = \begin{pmatrix} \Gamma_0^2\{f_2, f_2'\} \\ \Gamma_0^1\{f_2, f_2'\} - \Gamma_0^2\{f_1, f_1'\} \end{pmatrix}$$

and

$$\Gamma_0^B\{f_1 \oplus f_2, f_1' \oplus f_2'\} = \begin{pmatrix} \Gamma_1^1\{f_2, f_2'\} + \Gamma_1^2\{f_1, f_1'\} \\ \Gamma_1^1\{f_2, f_2'\} \end{pmatrix},$$

for $\{f_1, f_1'\} \in \operatorname{dom}\Gamma^1$ and $\{f_2, f_2'\} \in \operatorname{dom}\Gamma^2$. Its associated Weyl function is

$$M_B(\lambda) = B^{-*} \begin{pmatrix} M_1(\lambda) & 0 \\ 0 & M_2(\lambda) \end{pmatrix} B^{-1} = \begin{pmatrix} M_1(\lambda) + M_2(\lambda) & M_1(\lambda) \\ M_1(\lambda) & M_1(\lambda) \end{pmatrix},$$

see Lemma A.18. After these observations the statement follows from Proposition A.19. □