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On Unitary Relations between Kreĭn Spaces

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Abstract

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Keywords

Unitary relation, Von Neumann formulas, Weyl function, Extension theory

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1. Introduction

As is well known, unitary operators between Hilbert spaces are bounded everywhere defined isometric mappings with bounded everywhere defined inverse. In Kreĭn spaces unitary operators were initially introduced as everywhere defined isometric operators with everywhere defined inverse; such operators are called standard unitary operators. Standard unitary operators are closely connected to unitary operators between Hilbert spaces and, consequently, they behave essentially in the same way as unitary operators between Hilbert spaces. R. Arens introduced in [1] an alternative, very general, definition of unitary relations (multivalued operators): a relation U between Kreĭn spaces is *unitary* if

$$U^{-1} = U^{[*]},$$

where the adjoint is taken with respect to the underlying inner products. This class of unitary relations contains the class of standard unitary operators. Each unitary relation is closed, however, they need not be bounded nor densely defined and they can be multivalued as the following example shows, cf. [7, Example 2.11].

Example 1.1. Let $\{\mathfrak{H}, (\cdot,\cdot)_{\mathfrak{H}}\}$ be a Hilbert space and define the indefinite inner product $[\cdot,\cdot]_{\mathfrak{H}}$ on $\mathfrak{H}^2$ by

$$[\{f_1,f_2\},\{g_1,g_2\}]_{\mathfrak{H}} = -i\left((f_2,g_1)_{\mathfrak{H}} - (f_1,g_2)_{\mathfrak{H}}\right), \quad f_1,f_2,g_1,g_2 \in \mathfrak{H}.$$

Moreover, let K be a selfadjoint operator in $\{\mathfrak{H}_0, (\cdot,\cdot)\}$, where $\mathfrak{H}_0$ is a closed subspace of $\mathfrak{H}$. Then U defined by

$$U\{f_1+f,f_2\} = \{f_1+Kf_2+f',f_2\}, \quad f_1 \in \mathfrak{H}_0,\ f_2 \in \operatorname{dom} K,\ f,f' \in \mathfrak{H}\ominus\mathfrak{H}_0\}$$

is a unitary relation from $\{\mathfrak{H}_0^2,[\cdot,\cdot]\}$ to $\{\mathfrak{H}^2,[\cdot,\cdot]\}$ with $\ker U = (\mathfrak{H}\ominus\mathfrak{H}_0)\times\{0\} = \operatorname{mul} U$. In particular, U has closed domain (and range) if and only if K is bounded, and U is a unitary operator with a trivial kernel if and only if $\mathfrak{H}_0 = \mathfrak{H}$.

The study of unitary relations in Kreĭn spaces is motivated by the extension theory of symmetric operators. These extensions have initially been studied by means of von Neumann's formula, see e.g. [4]. An alternative approach by means of so-called boundary triplets was introduced by V.M. Bruck and A.N. Kochubei (see [9] and the references therein). This approach was generalized by V.A. Derkach and M.M. Malamud who also associated so-called Weyl functions to boundary triplets, see e.g. [8]. Later those two authors together with S. Hassi and H.S.V. de Snoo developed this approach further by incorporating Kreĭn space methods into this approach, see [6, 7]. In fact, in [6] it was shown that the boundary triplet, and its generalizations, can be seen as unitary relations between Kreĭn spaces with a suitable, fixed, inner product.

In this paper unitary relations between Kreĭn spaces will be studied, foremost in geometrical terms, without auxiliary assumptions on the structure of the Kreĭn spaces. This simplifies proofs for various well-known statements, the obtained statements can also be easier generalized and superfluous conditions can be eliminated. Moreover, the developed theory can be used, without modifications, to study for instance extension theory of symmetric relations in Kreĭn space as developed by V.A. Derkach, see e.g. [5]. Similarly, D-boundary triplets, recently introduced by V. Mogilevskii in [11], are captured within the framework of this paper. It is interesting to note that J.W. Calkin had a similar, geometrical, approach in [4] where the structure of the inner product was only fixed on the side of the domain. As a consequence of the geometrical approach the Weyl function will not be introduced, although a similar (non-analytic) object will be introduced, which will be called the Weyl identity.

It is the purpose of this paper to investigate the general properties and structure of isometric and unitary relations in Kreĭn spaces. Therefore, the investigation of the difference between isometric and unitary relation is central. This difference will be expressed by giving criteria for isometric relations to be unitary. Those conditions are obtained by means of (abstract) von Neumann formulas, the Weyl identity, the Potapov-Ginzburg transformation and a graph decomposition of unitary relations. All these different investigations give also information about the structure of unitary relations.

Apart from the introduction this paper is organized into four sections. In the second section notation and general results about Kreĭn spaces and linear relations are recalled. In the third section basic properties of isometric and unitary relations are given. In the fourth and fifth section the major results are obtained, central in these two sections are the Weyl identity and a graph decomposition, respectively. In particular, in these last two sections several different necessary and sufficient conditions for isometric relations to be unitary are obtained.

2. Preliminary results

2.1. Some basic properties of Kreĭn spaces

A vector space $\mathfrak{K}$ with an indefinite inner product $[\cdot,\cdot]$ is called a *Kreĭn space* if there exist a decomposition of $\mathfrak{K}$ into the direct sum of two subspaces (linear subsets) $\mathfrak{K}^+$ and $\mathfrak{K}^-$ of $\mathfrak{K}$ such that $\{\mathfrak{K}^+,[\cdot,\cdot]\}$ and $\{\mathfrak{K}^-,-[\cdot,\cdot]\}$ are Hilbert spaces and $[f^+,f^-]=0$, $f^+\in\mathfrak{K}^+$ and $f^-\in\mathfrak{K}^-$: A decomposition $\mathfrak{K}^+[+]\mathfrak{K}^-$ of $\mathfrak{K}$ is called *a canonical decomposition* of $\{\mathfrak{K},[\cdot,\cdot]\}$. (Here the sum of two subspaces $\mathfrak{M}$ and $\mathfrak{N}$ is said to be *direct* if $\mathfrak{M}\cap\mathfrak{N}=\{0\}$, in which case the sum is denoted by $\mathfrak{M}\dot{+}\mathfrak{N}$.) The dimensions of $\mathfrak{K}^+$ and $\mathfrak{K}^-$ are independent of the canonical decomposition of $\{\mathfrak{K},[\cdot,\cdot]\}$ and are denoted by k^+ and k^-, respectively.

For a Kreĭn space $\{\mathfrak{K},[\cdot,\cdot]\}$ there exist a linear operator $\mathfrak{j}$ in $\mathfrak{K}$ such that $\{\mathfrak{K},[\mathfrak{j}\cdot,\cdot]\}$ is a Hilbert space and with respect to its inner product $\mathfrak{j}^*=\mathfrak{j}^{-1}=\mathfrak{j}$. Any mapping $\mathfrak{j}$ satisfying the preceding properties is called *a fundamental symmetry* for $\{\mathfrak{K},[\cdot,\cdot]\}$. Conversely, if $\{\mathfrak{H},(\cdot,\cdot)\}$ is a Hilbert space and $\mathfrak{j}$ is a fundamental symmetry in $\{\mathfrak{H},(\cdot,\cdot)\}$, then $\{\mathfrak{H},(\mathfrak{j}\cdot,\cdot)\}$ is a Kreĭn space. Each fundamental symmetry induces a canonical decomposition and, conversely, each canonical decomposition induces a fundamental symmetry. However, all the norms generated by fundamental symmetries are equivalent. Hence a subspace of the Kreĭn space $\{\mathfrak{K},[\cdot,\cdot]\}$ is called *closed* if it is closed with respect to the definite inner product $[\mathfrak{j}\cdot,\cdot]$ for one (and hence for every) fundamental symmetry of $\{\mathfrak{K},[\cdot,\cdot]\}$.

For a subspace $\mathfrak{L}$ of the Kreĭn space $\{\mathfrak{K},[\cdot,\cdot]\}$ the *orthogonal complement* of $\mathfrak{L}$, denoted by $\mathfrak{L}^{[\perp]}$, is the closed subspace defined by

$$\mathfrak{L}^{[\perp]}=\{f\in\mathfrak{K}:[f,g]=0,\quad\forall g\in\mathfrak{L}\}.$$

If $\mathfrak{j}$ is a (fixed) fundamental symmetry for $\{\mathfrak{K},[\cdot,\cdot]\}$, then the $\mathfrak{j}$-orthogonal complement of $\mathfrak{L}$, i.e. the orthogonal complement with respect to $[\mathfrak{j}\cdot,\cdot]$, is denoted by $\mathfrak{L}^{\perp}$. Clearly, $\mathfrak{L}^{[\perp]}=\mathfrak{j}\mathfrak{L}^{\perp}=(\mathfrak{j}\mathfrak{L})^{\perp}$.

For subspaces $\mathfrak{M}$ and $\mathfrak{N}$ of the Kreĭn space $\{\mathfrak{K},[\cdot,\cdot]\}$ with a fixed fundamental symmetry $\mathfrak{j}$ the notation $\mathfrak{M}[+]\mathfrak{N}$ and $\mathfrak{M}\oplus\mathfrak{N}$ is used to indicate that the sum of $\mathfrak{M}$ and $\mathfrak{N}$ is orthogonal or $\mathfrak{j}$-orthogonal, i.e. $\mathfrak{M}\subseteq\mathfrak{N}^{[\perp]}$ and $\mathfrak{N}\subseteq\mathfrak{M}^{[\perp]}$ or $\mathfrak{M}\subseteq\mathfrak{j}\mathfrak{N}^{[\perp]}$ and $\mathfrak{N}\subseteq\mathfrak{j}\mathfrak{M}^{[\perp]}$, respectively. Moreover,

$$\mathfrak{M}^{[\perp]}\cap\mathfrak{N}^{[\perp]}=(\mathfrak{M}+\mathfrak{N})^{[\perp]}\quad\text{and}\quad\mathfrak{M}^{[\perp]}+\mathfrak{N}^{[\perp]}\subseteq(\mathfrak{M}\cap\mathfrak{N})^{[\perp]}.\tag{2.1}$$

Furthermore, the following version of [10, Ch. IV: Theorem 4.8] holds.

Lemma 2.1. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed subspaces of the Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then $\mathfrak{M} + \mathfrak{N}$ is closed if and only if $\mathfrak{M}^{[\perp]} + \mathfrak{N}^{[\perp]}$ is closed.*

Moreover, if either of the above equivalent conditions holds, then

$$\mathfrak{M}^{[\perp]} + \mathfrak{N}^{[\perp]} = (\mathfrak{M} \cap \mathfrak{N})^{[\perp]}.$$

A projection P or $\mathcal{P}$ onto a closed subspace of the Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$ with fundamental symmetry $\mathfrak{j}$ is called *orthogonal* or $\mathfrak{j}$-*orthogonal* if

$$\mathfrak{K} = \ker P [+] \operatorname{ran} P \quad \text{or} \quad \mathfrak{K} = \ker \mathcal{P} \oplus \operatorname{ran} \mathcal{P},$$

respectively. Recall that $\{\ker P, [\cdot,\cdot]\}$ and $\{\operatorname{ran} P, [\cdot,\cdot]\}$ are Kreĭn spaces, see [2, Ch. I: Theorem 7.16]. In particular, for a canonical decomposition $\mathfrak{K}^+[+]\mathfrak{K}^-$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$, with associated fundamental symmetry $\mathfrak{j}$, the projections P^+ and P^- onto $\mathfrak{K}^+$ and $\mathfrak{K}^-$, respectively, are orthogonal and $\mathfrak{j}$-orthogonal projections.

Remark 2.2. In this paper the notation $\{\mathfrak{H}, (\cdot,\cdot)\}$ and $\{\mathfrak{K}, [\cdot,\cdot]\}$ is used to denote Hilbert and Kreĭn spaces, respectively. To distinguish different Hilbert and Kreĭn spaces subindexes are used: $\mathfrak{H}_1, \mathfrak{K}_1, \mathfrak{H}_2, \mathfrak{K}_2$, etc.. Closed subspaces of $\{\mathfrak{K}, [\cdot,\cdot]\}$ which are themselves Kreĭn (with the inner product $[\cdot,\cdot]$) are denoted by $\widetilde{\mathfrak{K}}$ or $\widehat{\mathfrak{K}}$. A canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ is denoted by $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$. Its associated fundamental symmetry is denoted by $\mathfrak{j}_i$ and the projections P_i^+ and P_i^- associated to a canonical decomposition $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ always mean the orthogonal projection onto $\mathfrak{K}_i^+$ and $\mathfrak{K}_i^+$, respectively.

A subspace $\mathfrak{L}$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$ is called *positive*, *negative*, *nonnegative*, *nonpositive* or *neutral* if $[f,f] > 0$, $[f,f] < 0$, $[f,f] \geq 0$, $[f,f] \leq 0$ or $[f,f] = 0$ for every $f \in \mathfrak{L}$, respectively. A positive or negative subspace $\mathfrak{L}$ is called *uniformly positive* or *negative* if there exists a constant $\alpha > 0$ such that $[\mathfrak{j}f,f] \leq \alpha[f,f]$ or $[\mathfrak{j}f,f] \leq -\alpha[f,f]$ for all $f \in \mathfrak{L}$ and one (and hence every) fundamental symmetry $\mathfrak{j}$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$, respectively. Furthermore, a subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ having a certain property is said to be *maximal* with respect to that property, if there exists no extensions of $\mathfrak{L}$ having the same property. A subspace is said to *essentially* having a certain property if its closure has the indicated property. Note that a subspace $\mathfrak{L}$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$ is neutral if and only if $\mathfrak{L} \subseteq \mathfrak{L}^{[\perp]}$.

Proposition 2.3. *(*[2, Ch. I: Corollary 5.8]*) Let $\mathfrak{L}$ be a neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then the space $\{\mathfrak{L}^{[\perp]}/\operatorname{clos}(\mathfrak{L}), [\cdot,\cdot]\}$ is a Kreĭn space*[1]*.*

Let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$ with associated projection P^+ and P^-, then for any closed subspace $\mathfrak{L}$

$$\mathfrak{L}^{[\perp]} \cap \mathfrak{K}^+ = \mathfrak{K}^+ \ominus P^+\mathfrak{L} \quad \text{and} \quad \mathfrak{L}^{[\perp]} \cap \mathfrak{K}^- = \mathfrak{K}^- \ominus P^-\mathfrak{L}. \tag{2.2}$$

Next some characterizations of maximal nonnegative and nonpositive subspaces are recalled, see [2, Ch. I: Section 8] and [3, Ch. V: Section 4].

[1] The indefinite inner product on the quotient space, induced by the indefinite inner product on the original space, is always indicated by the same symbol.

Proposition 2.4. *Let $\mathfrak{L}$ be a nonnegative (nonpositive) subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$ with associated projections P^+ and P^-. Then equivalent are*

(i) *$\mathfrak{L}$ is a maximal nonnegative (nonpositive) subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$;*
(ii) *$P^+\mathfrak{L} = \mathfrak{K}^+$ ($P^-\mathfrak{L} = \mathfrak{K}^-$);*
(iii) *$\mathfrak{L}$ is closed and $\mathfrak{L}^{[\perp]}$ is a nonpositive (nonnegative) subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$;*
(iv) *$\mathfrak{L}$ is closed and $\mathfrak{L}^{[\perp]}$ is a maximal nonpositive (nonnegative) subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$;*
(v) *$K_{\mathfrak{L}} := \{\{P^+f, P^-f\} : f \in \mathfrak{L}\}$ ($K_{\mathfrak{L}} := \{\{P^-f, P^+f\} : f \in \mathfrak{L}\}$) is an everywhere defined contraction from $\{\mathfrak{K}^+, [\cdot,\cdot]\}$ to $\{\mathfrak{K}^-, -[\cdot,\cdot]\}$ (from $\{\mathfrak{K}^-, -[\cdot,\cdot]\}$ to $\{\mathfrak{K}^+, [\cdot,\cdot]\}$).*

The operator $K_{\mathfrak{L}}$ in (v) is called the angular operator associated to $\mathfrak{L}$. If $K_{\mathfrak{L}}$ as above is the angular operator for a maximal nonnegative subspace $\mathfrak{L}$, then $(K_{\mathfrak{L}})^*$ is the angular operator for $\mathfrak{L}^{[\perp]}$, see [2, Ch. I: Theorem 8.11].

2.2. Hyper-maximal subspaces

Recall that a neutral subspace $\mathfrak{L}$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$ is called *hyper-maximal neutral* if it is simultaneously maximal nonnegative and maximal nonpositive. Equivalently, $\mathfrak{L}$ is hyper-maximal neutral if and only if $\mathfrak{L} = \mathfrak{L}^{[\perp]}$, see Proposition 2.4. I.e., if $\mathfrak{j}$ is a fundamental symmetry for $\{\mathfrak{K}, [\cdot,\cdot]\}$, then $\mathfrak{L}$ is hyper-maximal neutral if and only if $\mathfrak{K}$ has the $\mathfrak{j}$-orthogonal decomposition

$$\mathfrak{K} = \mathfrak{L} \oplus \mathfrak{j}\mathfrak{L}. \tag{2.3}$$

The following result gives additional characterizations of hyper-maximal neutrality by means of a canonical decomposition of the corresponding Kreĭn space, see [2, Ch. I: Theorem 4.13 & Theorem 8.10].

Proposition 2.5. *Let $\mathfrak{L}$ be a neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$ with associated projections P_1^+ and P_1^-. Then equivalent are*

(i) *$\mathfrak{L}$ is hyper-maximal neutral;*
(ii) *$P^+\mathfrak{L} = \mathfrak{K}^+$ and $P^-\mathfrak{L} = \mathfrak{K}^-$;*
(iii) *$U_{\mathfrak{L}} := \{\{P^+f, P^-f\} \in \mathfrak{K}^+ \times \mathfrak{K}^- : f \in \mathfrak{L}\}$ is a standard unitary operator from $\{\mathfrak{K}^+, [\cdot,\cdot]\}$ onto $\{\mathfrak{K}^-, -[\cdot,\cdot]\}$.*

As a consequence of Proposition 2.5, $k^+ = k^-$ if there exists a hyper-maximal neutral subspace in $\{\mathfrak{K}, [\cdot,\cdot]\}$. The converse also holds: If $k^+ = k^-$, then there exist hyper-maximal neutral subspaces in $\{\mathfrak{K}, [\cdot,\cdot]\}$. By definition hyper-maximal neutral subspaces are maximal neutral, the converse does not hold as the following example shows.

Example 2.6. Let $\{\mathfrak{H}, (\cdot,\cdot)\}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n\geq 0}$, $e_n \in \mathfrak{H}$. Define the indefinite inner product $[\cdot,\cdot]$ on $\mathfrak{H}^2$ by

$$[\{f_1, f_2\}, \{g_1, g_2\}] = (f_1, g_1) - (f_2, g_2), \quad f_1, f_2, g_1, g_2 \in \mathfrak{H}.$$

Then $\{\mathfrak{H}^2, [\cdot,\cdot]\}$ is a Kreĭn space. Define $\mathfrak{L}_1$ and $\mathfrak{L}_2$ by

$$\mathfrak{L}_1 = \overline{\operatorname{span}}\,\{\{e_n, e_{2n}\} : n \in \mathbb{N}\} \quad \text{and} \quad \mathfrak{L}_2 = \overline{\operatorname{span}}\,\{\{e_n, e_n\} : n \in \mathbb{N}\}.$$

Then $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are maximal neutral subspaces of $\{\mathfrak{H}^2, [\cdot,\cdot]\}$, but only $\mathfrak{L}_2$ is a hyper-maximal neutral subspace of $\{\mathfrak{H}^2, [\cdot,\cdot]\}$.

The above example can be modified to show that there also exist different types of maximal nonpositive and nonnegative subspaces of Kreĭn spaces. Hence the notion of hyper-maximality can meaningfully be extended to semi-definite subspaces.

Definition 2.7. Let $\mathfrak{L}$ be a nonnegative or nonpositive subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then $\mathfrak{L}$ is called *hyper-maximal nonnegative* or *hyper-maximal nonpositive* if $\mathfrak{L}$ is closed and $\mathfrak{L}^{[\perp]}$ is a neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$.

The following proposition gives alternative characterizations for a semi-definite subspace to be hyper-maximal semi-definite.

Proposition 2.8. *Let $\mathfrak{L}$ be a nonnegative (nonpositive) subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$ with associated fundamental symmetry $\mathfrak{j}$ and projections P^+ and P^-. Then equivalent are*

(i) *$\mathfrak{L}$ is hyper-maximal nonnegative (nonpositive);*
(ii) *$\mathfrak{L}$ is closed, $\mathfrak{L}^{[\perp]} \subseteq \mathfrak{L}$ and $P^-\mathfrak{L}^{[\perp]} = \mathfrak{K}^-$ ($P^+\mathfrak{L}^{[\perp]} = \mathfrak{K}^+$);*
(iii) *$\mathfrak{L}$ is closed, $\mathfrak{L} = \mathfrak{L}^{[\perp]} + \mathfrak{L} \cap \mathfrak{K}^+$ ($\mathfrak{L} = \mathfrak{L}^{[\perp]} + \mathfrak{L} \cap \mathfrak{K}^-$);*
(iv) *$\mathfrak{L}$ is closed and induces a $\mathfrak{j}$-orthogonal decomposition of $\mathfrak{K}$:*

$$\mathfrak{K} = \mathfrak{L}^{[\perp]} \oplus (\mathfrak{L} \cap \mathfrak{j}\mathfrak{L}) \oplus \mathfrak{j}\mathfrak{L}^{[\perp]}.$$

Proof. The statement will only be proven in the case that $\mathfrak{L}$ is a nonnegative subspace; the case $\mathfrak{L}$ is a nonpositive subspace can be proven by similar arguments.

(i) $\Rightarrow$ (ii): Since $\mathfrak{L}^{[\perp]}$ is neutral, $\mathfrak{L}^{[\perp]} \subseteq \mathfrak{L}^{[\perp][\perp]} = \operatorname{clos}\mathfrak{L} = \mathfrak{L}$. Next let $f^- \in \mathfrak{K}^- \ominus P^-\mathfrak{L}^{[\perp]} = \operatorname{clos}(\mathfrak{L}) \cap \mathfrak{K}^-$, see (2.2). Since $\mathfrak{L}$ is by assumption closed and nonnegative, it follows that $f^- = 0$, i.e., $\mathfrak{K}^- = P^-\mathfrak{L}^{[\perp]}$.

(ii) $\Rightarrow$ (iii): Clearly, it suffices to prove the inclusion $\mathfrak{L} \subseteq \mathfrak{L}^{[\perp]} + \mathfrak{L} \cap \mathfrak{K}^+$. Hence let $f \in \mathfrak{L}$ be decomposed as $f^+ + f^-$, where $f^\pm \in \mathfrak{K}^\pm$. Then the assumption $P^-\mathfrak{L}^{[\perp]} = \mathfrak{K}^-$ implies that there exists $g^+ \in \mathfrak{K}^+$ such that $g^+ + f^- \in \mathfrak{L}^{[\perp]}$ and, hence, $f - (g^+ + f^-) = f^+ - g^+ \in \mathfrak{L} \cap \mathfrak{K}^+$, because by assumption $\mathfrak{L}^{[\perp]} \subseteq \mathfrak{L}$.

(iii) $\Rightarrow$ (iv): Since $\mathfrak{L}$ is closed, $\mathfrak{L} \cap \mathfrak{K}^+ = \mathfrak{L} \cap \mathfrak{j}\mathfrak{L}$. Moreover, since $\mathfrak{L}$ is nonnegative, $\mathfrak{L}$ is in fact the $\mathfrak{j}$-orthogonal sum of $\mathfrak{L}^{[\perp]}$ and $\mathfrak{L} \cap \mathfrak{K}^+$. In other words, $\mathfrak{L}^{[\perp]}$ is a hyper-maximal neutral subspace of the Kreĭn space $\{\mathfrak{K} \ominus (\mathfrak{L} \cap \mathfrak{j}\mathfrak{L}), [\cdot,\cdot]\}$. Hence, (2.3) implies that (iv) holds.

(iv) $\Rightarrow$ (i): The decomposition in (iv) implies that $\mathfrak{L}^{[\perp]}$ is a hyper-maximal neutral neutral subspace of the Kreĭn space $\{\mathfrak{K} \ominus (\mathfrak{L} \cap \mathfrak{j}\mathfrak{L}), [\cdot,\cdot]\}$, see (2.3), and, hence, $\mathfrak{L}^{[\perp]}$ is a neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$. □

Corollary 2.9. *Let $\mathfrak{L}$ be a semi-definite subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then $\mathfrak{L}$ is hyper-maximal semi-definite if and only if $P^+\mathfrak{L} = \mathfrak{K}^+$ and $P^-\mathfrak{L} = \mathfrak{K}^-$ for every canonical decomposition $\mathfrak{K}^+[+]\mathfrak{K}^-$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$ with associated projections P^+ and P^-.*

Proof. Assume that $\mathfrak{L}$ is hyper-maximal semi-definite and let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$ with associated projections P^+ and P^-. Then Proposition 2.8 (iv) implies that $\mathfrak{L} \cap \mathrm{j}\mathfrak{L}$ is closed and that $\mathfrak{L}^{[\perp]}$ is a hyper-maximal neutral subspace of the Kreĭn space $\{\mathfrak{K} \ominus (\mathfrak{L} \cap \mathrm{j}\mathfrak{L}), [\cdot,\cdot]\}$. Hence, $P_1^{\pm}\mathfrak{L}^{[\perp]} = \mathfrak{K}^{\pm} \ominus (\mathfrak{L} \cap \mathrm{j}\mathfrak{L})$, see Proposition 2.5, and $\mathfrak{L}^{[\perp]} + \mathfrak{L} \cap \mathrm{j}\mathfrak{L} \subseteq \mathfrak{L}$, because $\mathfrak{L}$ is by assumption closed. These observations show that the stated characterization holds.

To prove the converse assume w.l.o.g. that $\mathfrak{L}$ is nonnegative. Then the assumption that $P^+\mathfrak{L} = \mathfrak{K}^+$ implies that $\mathfrak{L}$ is maximal nonnegative, and hence closed, and that $\mathfrak{L}^{[\perp]}$ is a maximal nonpositive subspace, see Proposition 2.4. Suppose that $f \in \mathfrak{L}^{[\perp]}$ is such that $[f,f] < 0$, then there exists a canonical decomposition $\widetilde{\mathfrak{K}}^+[+]\widetilde{\mathfrak{K}}^-$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that $f \in \widetilde{\mathfrak{K}}^-$, see [3, Ch. V: Theorem 5.6]. Then $f \in \widetilde{\mathfrak{K}}^- \ominus \widetilde{P}^-\mathfrak{L}$, see (2.2), which is in contradiction with the assumption $\widetilde{P}^-\mathfrak{L} = \widetilde{\mathfrak{K}}^-$. Consequently, $\mathfrak{L}^{[\perp]}$ is neutral and, hence, $\mathfrak{L}$ is hyper-maximal nonnegative. □

Corollary 2.9 shows that hyper-maximal nonnegative (nonpositive) subspaces are also maximal nonnegative (nonpositive), justifying the terminology. Moreover, it also shows that in a Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$ with $k^+ > k^-$ or $k^+ < k^-$ every hyper-maximal semi-definite subspace is nonnegative or nonpositive, respectively. If $k^+ = k^-$, then a hyper-maximal semi-definite subspace can be neutral, nonnegative or nonpositive, cf. Example 2.6. Note also that Proposition 2.8 (ii) shows that if $\mathfrak{L}$ is hyper-maximal semi-definite, then $\mathfrak{L}^{[\perp]}$ is maximal neutral. Clearly, the converse also holds: if $\mathfrak{L}$ is a maximal neutral subspace, then $\mathfrak{L}^{[\perp]}$ is a hyper-maximal semi-definite subspace.

2.3. Abstract von Neumann formulas

Let $\mathfrak{L}$ be a neutral subspace of the Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$ with a canonical decomposition $\mathfrak{K}^+[+]\mathfrak{K}^-$. Then the *first von Neumann formula holds:*

$$\mathfrak{L}^{[\perp]} = \operatorname{clos}(\mathfrak{L})[\oplus]\mathfrak{L}^{[\perp]} \cap \mathfrak{K}^+[\oplus]\mathfrak{L}^{[\perp]} \cap \mathfrak{K}^-, \tag{2.4}$$

see [2, Ch. 1 : 4.20] and (2.2). As a consequence of the first von Neumann formula and Proposition 2.3, the notion of defect numbers for neutral subspaces of Kreĭn spaces as introduced below is well-defined. This definition extends the usual definition of defect numbers for symmetric relations.

Definition 2.10. Let $\mathfrak{L}$ be a neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then the defect numbers $n_+(\mathfrak{L})$ and $n_-(\mathfrak{L})$ of $\mathfrak{L}$ are defined as

$$n_+(\mathfrak{L}) = \dim(\mathfrak{L}^{[\perp]} \cap \mathfrak{K}^-) \quad \text{and} \quad n_-(\mathfrak{L}) = \dim(\mathfrak{L}^{[\perp]} \cap \mathfrak{K}^+).$$

The following generalization of the second von Neumann formula will be of importance in the analysis of unitary relations, see Section 4.1.

Proposition 2.11. *Let $\mathfrak{L}$ and $\mathfrak{M}$ be subspaces of $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that $\mathfrak{M} \subseteq \mathfrak{L}$ and let P be an orthogonal projection in $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then*

$$P\mathfrak{L} \subseteq P\mathfrak{M} \quad \textit{if and only if} \quad \mathfrak{L} = \mathfrak{M} + \mathfrak{L} \cap \ker P. \tag{2.5}$$

Furthermore, if $\mathfrak{M}$ *is closed,* $P\mathfrak{L} \subseteq P\mathfrak{M}$ *and* $(I-P)\mathfrak{M}^{[\perp]}+(I-P)(\mathfrak{L}\cap\ker P)^{[\perp]}$ *is closed, then*

(i) $\mathfrak{L} \cap \ker P$ *is closed if and only if* $\mathfrak{L}$ *is closed;*
(ii) $\operatorname{clos}(\mathfrak{L} \cap \ker P) = (\operatorname{clos}\mathfrak{L}) \cap \ker P$.

Proof. The proof for the equivalence (2.5) is simple and left for the reader.

(i): If $\mathfrak{L}$ is closed, then clearly $\mathfrak{L}\cap\ker P$ is closed. To prove the converse note first that $\operatorname{ran} P \subseteq (\mathfrak{L} \cap \ker P)^{[\perp]}$. Therefore the assumption that $(I - P)\mathfrak{M}^{[\perp]} + (I - P)(\mathfrak{L} \cap \ker P)^{[\perp]}$ is closed implies that $\mathfrak{M}^{[\perp]} + (\mathfrak{L} \cap \ker P)^{[\perp]}$ is closed. This fact together with the assumption that $\mathfrak{L} \cap \ker P$ is closed implies that $\mathfrak{M} + \operatorname{clos}(\mathfrak{L} \cap \ker P)$ is closed, see Lemma 2.1. Consequently, $\mathfrak{M} + \mathfrak{L} \cap \ker P$ is closed and the closedness of $\mathfrak{L}$ now follows from (2.5).

(ii): The assumptions $P\mathfrak{L} \subseteq P\mathfrak{M}$ and $\mathfrak{M} \subseteq \mathfrak{L}$ together with (2.5) show that

$$\mathfrak{L} = \mathfrak{M} + (\mathfrak{L} \cap \ker P) \subseteq \mathfrak{M} + \operatorname{clos}(\mathfrak{L} \cap \ker P) \subseteq \operatorname{clos}\mathfrak{L}.$$

Since $\mathfrak{M} + \operatorname{clos}(\mathfrak{L} \cap \ker P)$ is closed (see the proof of (i)), taking closures in the above equation yields that $\operatorname{clos}(\mathfrak{L}) = \mathfrak{M} + \operatorname{clos}(\mathfrak{L} \cap \ker P)$ and therefore $P(\operatorname{clos}\mathfrak{L}) \subseteq P\mathfrak{M}$. Consequently, (2.5) implies that $\operatorname{clos}(\mathfrak{L}) = \mathfrak{M} + (\operatorname{clos}\mathfrak{L}) \cap \ker P$, i.e.,

$$\mathfrak{M} + \operatorname{clos}(\mathfrak{L} \cap \ker P) = \operatorname{clos}(\mathfrak{L}) = \mathfrak{M} + (\operatorname{clos}\mathfrak{L}) \cap \ker P.$$

This implies (ii). □

Let $\mathfrak{j}$ be a fundamental symmetry for $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then note that $(I - P)\mathfrak{M}^{[\perp]} + (I - P)(\mathfrak{L} \cap \ker P)^{[\perp]}$ is closed, if the following inclusion holds

$$\begin{aligned}(I - P)\mathfrak{M}^{[\perp]} &\supseteq ((I - P)(\mathfrak{L} \cap \ker P)^{[\perp]})^{\perp} \cap \ker P \\ &= \mathfrak{j}\operatorname{clos}(\mathfrak{L} \cap \ker P) \cap \ker P + (\mathfrak{j}\operatorname{ran} P) \cap \ker P.\end{aligned}$$

Corollary 2.12. *Let* $\mathfrak{L}$ *and* $\mathfrak{M}$ *be subspaces of* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *such that* $\mathfrak{M} \subseteq \mathfrak{L}$. *Moreover, let* $\mathfrak{K}^+[+]\mathfrak{K}^-$ *be a canonical decomposition of* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *with associated projections* P^+ *and* P^-. *Then*

$$\begin{cases} P^-\mathfrak{L} \subseteq P^-\mathfrak{M} & \text{if and only if} \quad \mathfrak{L} = \mathfrak{M} + \mathfrak{L} \cap \mathfrak{K}^+; \\ P^+\mathfrak{L} \subseteq P^+\mathfrak{M} & \text{if and only if} \quad \mathfrak{L} = \mathfrak{M} + \mathfrak{L} \cap \mathfrak{K}^-. \end{cases}$$

Furthermore, if $\mathfrak{M}$ *is closed,* $P^-\mathfrak{L} \subseteq P^-\mathfrak{M}$ *and* $\operatorname{clos}(\mathfrak{L} \cap \mathfrak{K}^+) \subseteq P^+\mathfrak{M}^{[\perp]}$, *then*

(i) $\mathfrak{L} \cap \mathfrak{K}^+$ *is closed if and only if* $\mathfrak{L}$ *is closed;*
(ii) $\operatorname{clos}(\mathfrak{L} \cap \mathfrak{K}^+) = (\operatorname{clos}\mathfrak{L}) \cap \mathfrak{K}^+$*;*

and if $\mathfrak{M}$ *is closed,* $P^+\mathfrak{L} \subseteq P^+\mathfrak{M}$ *and* $\operatorname{clos}(\mathfrak{L} \cap \mathfrak{K}^-) \subseteq P^-\mathfrak{M}^{[\perp]}$, *then*

(i') $\mathfrak{L} \cap \mathfrak{K}^-$ *is closed if and only if* $\mathfrak{L}$ *is closed;*
(ii') $\operatorname{clos}(\mathfrak{L} \cap \mathfrak{K}^-) = (\operatorname{clos}\mathfrak{L}) \cap \mathfrak{K}^-$.

Proof. The discussion preceding this corollary shows that the condition that $(I-P)\mathfrak{M}^{[\perp]}+(I-P)(\mathfrak{L}\cap\ker P)^{[\perp]}$ is closed (in Proposition 2.11) is satisfied for $P = P^-$ or $P = P^+$, if $\operatorname{clos}(\mathfrak{L} \cap \mathfrak{K}^+) \subseteq P^+\mathfrak{M}^{[\perp]}$ or $\operatorname{clos}(\mathfrak{L} \cap \mathfrak{K}^-) \subseteq P^-\mathfrak{M}^{[\perp]}$, respectively. Hence, this statement follows directly from Proposition 2.11 by taking P to be P^- and P^+. □

Note that if $\mathfrak{L}$ is an arbitrary subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$, then the conditions $P^-\mathfrak{L} \subseteq P^-\mathfrak{M}$ and $\operatorname{clos}(\mathfrak{L}\cap\mathfrak{K}^+) \subseteq P^+\mathfrak{M}^{[\perp]}$ are satisfied for any hyper-maximal nonpositive subspace $\mathfrak{M}$, and the conditions $P^+\mathfrak{L} \subseteq P^+\mathfrak{M}$ and $\operatorname{clos}(\mathfrak{L}\cap\mathfrak{K}^-) \subseteq P^-\mathfrak{M}^{[\perp]}$ are satisfied for any hyper-maximal nonnegative subspace $\mathfrak{M}$. Furthermore, Corollary 2.12 shows that if a subspace $\mathfrak{L}$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$ contains a maximal semi-definite subspace $\mathfrak{M}$ and its orthogonal complement, then $\mathfrak{L}^{[\perp]}$ is neutral. Because if e.g. $\mathfrak{M}$ is maximal nonnegative, then $P^-\mathfrak{M} = \mathfrak{K}^-$ and, hence, $\mathfrak{L} = \mathfrak{M} + \mathfrak{L}\cap\mathfrak{K}^+$ by Corollary 2.12. Consequently, using (2.1),

$$\mathfrak{L}^{[\perp]} = \left(\mathfrak{M} + \mathfrak{L}\cap\mathfrak{K}^+\right)^{[\perp]} = \mathfrak{M}^{[\perp]} \cap (\mathfrak{L}\cap\mathfrak{K}^+)^{[\perp]} \subseteq \mathfrak{M}^{[\perp]} \subseteq \mathfrak{L}.$$

2.4. Linear relations in Kreĭn spaces

Let $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ be Kreĭn spaces, then H is called a *(linear) relation* from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ if its graph, denoted by $\operatorname{gr}(H)$, is a subspace of $\mathfrak{K}_1 \times \mathfrak{K}_2$[2]. In particular, H is closed if and only if its graph is closed. The symbols $\operatorname{dom} H$, $\operatorname{ran} H$, $\ker H$, and $\operatorname{mul} H$ stand for the domain, range, kernel, and the multivalued part of H, respectively. In particular, $\operatorname{mul} H = \{0\}$ if and only if H is an operator. For a subspace $\mathfrak{L}$ of $\operatorname{dom} H$, $H(\mathfrak{L})$ denotes $\{f' \in \mathfrak{K}_2 : \exists f \in \mathfrak{L} \text{ s.t. } \{f, f'\} \in H\}$.

The inverse H^{-1} and the adjoint $H^{[*]}$ of H are defined as

$$H^{-1} = \{\{f, f'\} \in \mathfrak{K}_2 \times \mathfrak{K}_1 : \{f', f\} \in H\};$$

$$H^{[*]} = \{\{f, f'\} \in \mathfrak{K}_2 \times \mathfrak{K}_1 : [f', g]_1 = [f, g']_2, \ \forall \{g, g'\} \in H\}.$$

Using these definitions

$$(\operatorname{dom} H)^{[\perp]_1} = \operatorname{mul} H^{[*]} \quad \text{and} \quad (\operatorname{ran} H)^{[\perp]_2} = \ker H^{[*]}. \tag{2.6}$$

For linear relations G and H from $\mathfrak{K}_1$ to $\mathfrak{K}_2$ the notation $G + H$ stands for the usual operator-like sum of relations. The componentwise sum (linear span of the graphs) of G and H is the relation whose graph is given by $\operatorname{gr}(G) + \operatorname{gr}(H)$, where the sum is that of linear subspaces (of $\mathfrak{K}_1 \times \mathfrak{K}_2$. If the componentwise sum is direct, i.e. $\operatorname{gr}(G) \cap \operatorname{gr}(H) = \{0\}$, then the graph of the componentwise sum of G and H is denoted by $\operatorname{gr}(G) \dot{+} \operatorname{gr}(H)$. If G is a linear relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and H is a linear relation from $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ to $\{\mathfrak{K}_3, [\cdot,\cdot]_3\}$, then their product, denoted by HG, is defined by

$$HG = \{\{f, f'\} \in \mathfrak{K}_1 \times \mathfrak{K}_3 : \exists h \in \mathfrak{K}_2 \text{ s.t. } \{f, h\} \in G \quad \text{and} \quad \{h, f'\} \in H\}.$$

The following basic facts can be found in e.g. [1].

Lemma 2.13. *Let $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$, $i = 1, 2, 3$, be Kreĭn spaces and let $G : \mathfrak{K}_1 \to \mathfrak{K}_2$ and $H : \mathfrak{K}_2 \to \mathfrak{K}_3$ be linear relations. Then*

(i) $\left(H^{[*]}\right)^{-1} = \left(H^{-1}\right)^{[*]}$;
(ii) $(HG)^{-1} = G^{-1}H^{-1}$;
(iii) $G^{[*]}H^{[*]} \subseteq (HG)^{[*]}$;

---

[2]The interpretation of relations (operators) as subspaces in appropriate spaces, and vice versa, is used throughout the paper. These interpretations will not always be explicitly mentioned.

(iv) *if* G *is closed,* $\operatorname{ran} G$ *is closed and* $\operatorname{dom} H \subseteq \operatorname{ran} G$ *or* H *is closed,* $\operatorname{dom} H$ *is closed and* $\operatorname{ran} G \subseteq \operatorname{dom} H$*, then* $(HG)^{[*]} = G^{[*]}H^{[*]}$.

Let P_i be an orthogonal projection in $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$, $i = 1, 2$, then the projection $P_1 \times P_2$ in $\mathfrak{K}_1 \times \mathfrak{K}_2$ is defined as $(P_1 \times P_2)(f_1 \times f_2) = P_1 f_1 \times P_2 f_2$, $f_1 \in \mathfrak{K}_1$ and $f_2 \in \mathfrak{K}_2$. Note that for a relation H from $\mathfrak{K}_1$ to $\mathfrak{K}_2$ one has that

$$(P_1 \times P_2)\mathrm{gr}(H) = \{P_1 f \times P_2 f' : \{f, f'\} \in H\} \subseteq (P_1 \times P_2)(\operatorname{dom} H \times \operatorname{ran} H).$$

The following statement shows when the inverse inclusion holds.

Proposition 2.14. *Let* H *be a relation from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *and let* P_i *be an orthogonal projection in* $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$*, for* $i = 1, 2$*. Then equivalent are*

(i) $(P_1 \times P_2)(\operatorname{dom} H \times \operatorname{ran} H) \subseteq (P_1 \times P_2)\mathrm{gr}(H)$;
(ii) $P_1 H^{-1}(\operatorname{ran} H \cap \ker P_2) = P_1 \operatorname{dom} H$;
(iii) $P_2 H(\operatorname{dom} H \cap \ker P_1) = P_2 \operatorname{ran} H$;
(iv) $\operatorname{dom} H = H^{-1}(\operatorname{ran} H \cap \ker P_2) + (\operatorname{dom} H \cap \ker P_1)$;
(v) $\operatorname{ran} H = H(\operatorname{dom} H \cap \ker P_1) + (\operatorname{ran} H \cap \ker P_2)$.

Proof. (i) $\Rightarrow$ (ii): If (i) holds, then for every $f_1 \in P_1 \operatorname{dom} H$ there exists $\{f, f'\} \in H$ such that $P_1 f = f_1$ and $P_2 f' = 0$. Therefore $f' \in \operatorname{ran} H \cap \ker P_2$ and hence $P_1 \operatorname{dom} H \subseteq P_1 H^{-1}(\operatorname{ran} H \cap P_2)$. Since the inverse inclusion clearly holds, this shows that (ii) holds.

(ii) $\Leftrightarrow$ (iv): Let $f \in \operatorname{dom} H$, then by (ii) there exists $\{g, g'\} \in H$ such that $P_1 g = P_1 f$ and $g' \in \operatorname{ran} H \cap \ker P_2$. Hence $f = g + h$, where $h = f - g \in \operatorname{dom} H$ and $P_1 h = P_1(f - g) = 0$, i.e., $\operatorname{dom} H \subseteq H^{-1}(\operatorname{ran} H \cap \ker P_2) + (\operatorname{dom} H \cap \ker P_1)$. Since the inverse inclusion clearly holds, this proves the implication from (ii) to (iv). The reverse implication is direct.

(iii) $\Leftrightarrow$ (v): This proof is similar to the proof of the equivalence of (ii) and (iv).

(iv) $\Leftrightarrow$ (v): This follows by applying H and H^{-1}.

(ii) & (iii) $\Rightarrow$ (i): If $f_1 \in P_1 \operatorname{dom} H$ and $f_2 \in P_2 \operatorname{ran} H$, then by (ii) there exists $\{f, f'\} \in H$ such that $P_1 f = f_1$, $P_2 f' = 0$ and by (iii) there exists $\{g, g'\} \in H$ such that $P_1 g = 0$ and $P_2 g' = f_2$. Hence, $\{f + g, f' + g'\} \in H$, $P_1(f + g) = P_1 f = f_1$ and $P_2(f' + g') = P_2 g' = f_2$. □

3. Basic properties of isometric and unitary relations

Basic properties of isometric and unitary relations are given. In particular, it is shown how to reduce isometric (and unitary) relations by removing their easily understood parts. The properties of these separated-off parts, which are in fact unitary relations with closed domain and closed range, are investigated.

3.1. Isometric and unitary relations

A relation U from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ is called *isometric* or *unitary* if

$$U^{-1} \subseteq U^{[*]} \quad \text{or} \quad U^{-1} = U^{[*]}, \tag{3.1}$$

respectively, see [1]. Since $\left(U^{[*]}\right)^{-1} = (U^{-1})^{[*]}$, see Lemma 2.13, the above definition implies directly that a relation U is isometric (unitary) if and only if U^{-1} is isometric (unitary). In particular, the action of isometric (unitary) relations and their inverse are of the same type and, hence, the structure of their domain and range is of the same type.

For an isometric relation V from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ (2.6) becomes

$$\ker V \subseteq (\operatorname{dom} V)^{[\perp]_1} \quad \text{and} \quad \operatorname{mul} V \subseteq (\operatorname{ran} V)^{[\perp]_2}. \tag{3.2}$$

Hence, in particular, $\ker V$ and $\operatorname{mul} V$ are neutral subspaces of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, respectively. For a unitary relation U from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ the inequalities in (3.2) become equalities:

$$\ker U = (\operatorname{dom} U)^{[\perp]_1} \quad \text{and} \quad \operatorname{mul} U = (\operatorname{ran} U)^{[\perp]_2}. \tag{3.3}$$

Recall the following basic characterizations of isometric and unitary relations.

Proposition 3.1. *Let V be a relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then equivalent are*

(i) *V is isometric;*
(ii) *$[f,g]_1 = [f',g']_2$ for every $\{f,f'\},\{g,g'\} \in V$;*
(iii) *$[f,f]_1 = [f',f']_2$ for every $\{f,f'\} \in V$;*
(iv) *$\operatorname{clos} V$ is isometric.*

Proof. (i) $\Leftrightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii): These equivalences are clear by the definition (3.1) and by polarizing.

(i) $\Leftrightarrow$ (iv): If (i) holds, then, since $V^{[*]}$ is closed, $V^{-1} \subseteq V^{[*]}$ implies that $(\operatorname{clos} V)^{-1} = \operatorname{clos}\left(V^{-1}\right) \subseteq V^{[*]}$, i.e., $\operatorname{clos} V$ is isometric. The reverse implication follows from the fact that $V \subseteq \operatorname{clos} V$ and that $V^{[*]}$ is closed. □

Proposition 3.2. *Let U be a relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then U is unitary if and only if U is isometric and if $\{f,f'\} \in \mathfrak{K}_1 \times \mathfrak{K}_2$ is such that*

$$[f,g]_1 = [f',g']_2, \quad \forall \{g,g'\} \in U,$$

then $\{f,f'\} \in U$.

Proof. If $\{f,f'\} \in \mathfrak{K}_1 \times \mathfrak{K}_2$ satisfies the stated condition, then $\{f',f\} \in U^{[*]}$. Hence the equivalence is clear by the definition of unitary relations. □

For Kreĭn spaces $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, the notation $[\cdot,\cdot]_{1,-2}$ is used to denote the (indefinite) inner product on $\mathfrak{K}_1 \times \mathfrak{K}_2$ defined by

$$[f_1 \times f_2, g_1 \times g_2]_{1,-2} = [f_1,g_1]_1 - [f_2,g_2]_2, \quad f_1, g_1 \in \mathfrak{K}_1, f_2, g_2 \in \mathfrak{K}_2. \tag{3.4}$$

With this inner product, $\{\mathfrak{K}_1 \times \mathfrak{K}_2, [\cdot,\cdot]_{1,-2}\}$ is a Kreĭn space and for a relation H from $\mathfrak{K}_1$ to $\mathfrak{K}_2$ one has that $(\operatorname{gr}(H))^{[\perp]_{1,-2}} = \operatorname{gr}\left(H^{-[*]}\right)$. The preceding observation yields the following result which can be found in [12].

Proposition 3.3. *Let U be a relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then U is an isometric or a unitary relation if and only if $\operatorname{gr}(U)$ is a neutral or a hyper-maximal neutral subspace of $\{\mathfrak{K}_1 \times \mathfrak{K}_2, [\cdot,\cdot]_{1,-2}\}$, respectively.*

In light of Proposition 2.5 and the discussion following that statement, Proposition 3.3 implies that if U is a unitary relation from the Kreĭn space $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to the Kreĭn space $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, then $k_1^+ + k_2^- = k_1^- + k_2^+$.

The following statement, which generalizes Proposition 2.5, can be interpreted as an inverse to Proposition 3.3; it shows how hyper-maximal neutral subspaces can be interpreted (nonuniquely) as unitary relations.

Proposition 3.4. *Let $\mathfrak{L}$ be a subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and let P be an orthogonal projection in $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then $\mathfrak{L}$ is neutral or hyper-maximal neutral if and only if the relation $U_{\mathfrak{L}} := \{\{Pf, (I-P)f\} : f \in \mathfrak{L}\}$ is an isometric or a unitary relation from the Kreĭn space $\{\operatorname{ran} P, [\cdot,\cdot]\}$ to the Kreĭn space $\{\ker P, -[\cdot,\cdot]\}$, respectively.*

Proof. If $\mathfrak{L}$ is neutral, then for $f, g \in \mathfrak{L}$

$$0 = [f,g] = [(I-P)f + Pf, (I-P)g + Pg] = [Pf, Pg] + [(I-P)f, (I-P)g],$$

i.e. $U_{\mathfrak{L}}$ is an isometric relation. Next let $\{g, g'\} \in \operatorname{ran} P \times \ker P$ be such that $[Pf, g] = -[(I-P)f, g']$ for all $f \in \mathfrak{L}$. Then $[f, g+g'] = 0$ for all $f \in \mathfrak{L}$. If $\mathfrak{L}$ is hyper-maximal neutral, the preceding equation implies that $g + g' \in \mathfrak{L}$. Hence by Proposition 3.2 $U_{\mathfrak{L}}$ is an unitary relation.

The converse assertion is a direct consequence of Proposition 3.3. □

The previous two propositions together with Proposition 2.5 show that with each unitary relation between Kreĭn spaces one can associate a unitary relation between Hilbert spaces; that association is the Potapov-Ginzburg transformation, see Proposition 3.20.

The following proposition which can be found in [14, Proposition 2.3.1], gives some necessary and sufficient conditions for an isometric relation to be unitary. To prove the proposition, observe that if G and H are relations from $\mathfrak{K}_1$ to $\mathfrak{K}_2$, then

$$G = H \quad \text{if and only if} \quad G \subseteq H, \ \operatorname{dom} H \subseteq \operatorname{dom} G, \ \operatorname{mul} H \subseteq \operatorname{mul} G. \qquad (3.5)$$

Proposition 3.5. *Let U be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then U is a unitary if and only if $\operatorname{dom} U^{[*]} \subseteq \operatorname{ran} U$ and $(\operatorname{dom} U)^{[\perp]_1} \subseteq \ker U$ or, equivalently, $\operatorname{ran} U^{[*]} \subseteq \operatorname{dom} U$ and $(\operatorname{ran} U)^{[\perp]_2} \subseteq \operatorname{mul} U$.*

Proof. If U is unitary, then (3.1) and (3.3) imply that $\operatorname{dom} U^{[*]} = \operatorname{ran} U$ and $(\operatorname{dom} U)^{[\perp]_1} = \ker U$. Conversely, since U is isometric $U^{-1} \subseteq U^{[*]}$ and $\operatorname{mul} U \subseteq (\operatorname{dom} U)^{[\perp]_1}$, see (3.1) and (3.2). Therefore the assumptions imply that $\operatorname{dom} U^{[*]} = \operatorname{dom} U^{-1}$ and $\operatorname{mul} U^{-1} = \ker U = (\operatorname{dom} U)^{[\perp]_1} = \operatorname{mul} U^{[*]}$, see (2.6). Hence, the equality $U^{-1} = U^{[*]}$ holds due to (3.5), i.e., U is unitary.

The second equivalence is obtained from the first by passing to the inverses. □

3.2. Isometric relations and closures of subspaces

A standard unitary operator U from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ satisfies

$$U(\mathfrak{L}^{[\perp]_1}) = (U(\mathfrak{L}))^{[\perp]_2}, \quad \mathfrak{L} \subseteq \mathfrak{K}. \qquad (3.6)$$

Since a unitary relation between Kreĭn spaces need not be everywhere defined, (3.6) does not hold in general for unitary relations.

Lemma 3.6. *Let V be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{L} \subseteq \operatorname{dom} V$. Then*

$$V(\mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} V) = (V(\mathfrak{L}))^{[\perp]_2} \cap \operatorname{ran} V.$$

Proof. If $f' \in V\left(\mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} V\right)$, then there exists a $f \in \mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} V$ such that $\{f, f'\} \in V$. In particular, $[f,h]_1 = 0$ for all $h \in \mathfrak{L}$. Since V is isometric, this implies that $[f',h']_2 = 0$ for all $h' \in V(\mathfrak{L})$, i.e., $f' \in (V(\mathfrak{L}))^{[\perp]_2} \cap \operatorname{ran} V$. This shows that $V\left(\mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} V\right) \subseteq (V(\mathfrak{L}))^{[\perp]_2} \cap \operatorname{ran} V$. The inverse inclusion follows from the proven inclusion by applying it to V^{-1} and $V(\mathfrak{L})$. ☐

Lemma 3.6 can be used to show the under certain conditions an essentially hyper-maximal neutral subspace is mapped onto an essentially hyper-maximal neutral subspace.

Corollary 3.7. *Let V be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{L}$ be a neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $\operatorname{clos} \mathfrak{L} = \mathfrak{L}^{[\perp]_1}$ and $(\operatorname{clos} \mathfrak{L}) \cap \operatorname{dom} V = \mathfrak{L}$. Then equivalent are*

(i) $\operatorname{clos} V(\mathfrak{L}) = V(\mathfrak{L})^{[\perp]_2}$;

(ii) $\left(V(\mathfrak{L})^{[\perp]_2} \cap \operatorname{ran} V\right)^{[\perp]_2} = \operatorname{clos} V(\mathfrak{L})$.

If either of the above equivalent conditions holds, then $(\operatorname{clos} V(\mathfrak{L})) \cap \operatorname{ran} V = V(\mathfrak{L})$.

Proof. By the assumptions on $\mathfrak{L}$ and Lemma 3.6

$$V(\mathfrak{L}) = V((\operatorname{clos} \mathfrak{L}) \cap \operatorname{dom} V) = V(\mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} V) = V(\mathfrak{L})^{[\perp]_2} \cap \operatorname{ran} V. \quad (3.7)$$

If (i) holds, then (3.7) shows that (ii) and the final conclusion hold. Conversely, if (ii) holds, then by taking orthogonal complements in (3.7) (i) follows immediately. ☐

Since the above statement holds for all isometric relations, it shows that the (essential) hyper-maximal neutrality of a subspace $\mathfrak{L}$ is only weakly connected to the (essential) hyper-maximal neutrality of the subspace $V(\mathfrak{L})$.

If U is a standard unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, then (3.6) implies that $U(\operatorname{clos} \mathfrak{L}) = \operatorname{clos} U(\mathfrak{L})$ for any subspace $\mathfrak{L}$ of $\mathfrak{K}_1$. Clearly, this equality does not hold for general unitary relations and a similar result only holds for certain subspaces. For instance if V is an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and $\ker V \subseteq \mathfrak{L} \subseteq \operatorname{dom} V$ is such that

$$\operatorname{clos} \mathfrak{L} = (\mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} V)^{[\perp]_1} \quad \text{and} \quad \operatorname{clos} V(\mathfrak{L}) = (V(\mathfrak{L})^{[\perp]_2} \cap \operatorname{ran} V)^{[\perp]_2}.$$

Then applying Lemma 3.6 twice yields that

$$V(\operatorname{clos} \mathfrak{L} \cap \operatorname{dom} V) = (\operatorname{clos} V(\mathfrak{L})) \cap \operatorname{ran} V.$$

The above example indicates that the behavior of isometric relations with respect to the closure of subspaces is in general not easy to describe. However, for uniformly definite subspaces this behavior is specific.

Proposition 3.8. *Let V be a closed and isometric operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{D} \subseteq \operatorname{dom} V$ be a uniformly definite subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then*

(i) *if $\mathfrak{D} = \operatorname{clos}(\mathfrak{D}) \cap \operatorname{dom} V$, then $V(\mathfrak{D})$ is closed;*
(ii) *$\mathfrak{D}$ is closed if and only if $V(\mathfrak{D}) + [\operatorname{mul} V]$ is a closed uniformly definite subspace of $\{(\operatorname{mul} V)^{[\perp]_2}/\operatorname{mul} V, [\cdot,\cdot]_2\}$.*

Proof. To prove the statements w.l.o.g. assume $\mathfrak{D}$ to be uniformly positive and let $\mathfrak{j}_1$ and $\mathfrak{j}_2$ be fundamental symmetries of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, respectively.

(i) : Let $f' \in \operatorname{clos} V(\mathfrak{D})$, then there exists a sequence $\{\{f_n, f_n'\}\}_{n\geq 0}$, where $f_n \in \mathfrak{D}$ and $\{f_n, f_n'\} \in V$, such that $f' = \lim_{n\to\infty} f_n'$. By the isometry of V

$$[\mathfrak{j}_2(f_m' - f_n'), f_m' - f_n']_2 \geq [f_m' - f_n', f_m' - f_n']_2 = [f_m - f_n, f_m - f_n]_1.$$

Since $\mathfrak{D}$ is uniformly positive, there exists a constant $\alpha > 0$ such that $\alpha[\mathfrak{j}_1 g, g]_1 \leq [g,g]_1$ for all $g \in \mathfrak{D}$. Combining this fact with the above inequality yields

$$[\mathfrak{j}_2(f_m' - f_n'), f_m' - f_n']_2 \geq \alpha[\mathfrak{j}_1(f_m - f_n), f_m - f_n]_1.$$

Since $\{f_n'\}_{n\geq 0}$ converges by assumption in the Hilbert space $\{\mathfrak{K}_2, [\mathfrak{j}_2\cdot,\cdot]_2\}$, the preceding inequality shows that $\{f_n\}_{n\geq 0}$ is a Cauchy-sequence in the Hilbert space $\{\mathfrak{K}_1, [\mathfrak{j}_1\cdot,\cdot]_1\}$ and, hence, converges to a $f \in \operatorname{clos}(\mathfrak{D})$. Consequently, $\{\{f_n, f_n'\}\}_{n\geq 0}$ converges (in the graph norm) to $\{f, f'\} \in \mathfrak{K}_1 \times \mathfrak{K}_2$ and, hence, $\{f, f'\} \in V$ by the closedness of V. Therefore $f \in \operatorname{clos}(\mathfrak{D}) \cap \operatorname{dom} V = \mathfrak{D}$ and, hence, $f' \in V(\mathfrak{D})$.

(ii) : For simplicity assume that $\operatorname{mul} V = \{0\}$. Let $\mathfrak{D} \subseteq \operatorname{dom} V$ be closed, then $V\restriction_{\mathfrak{D}}$ is an everywhere defined closed (isometric) operator from the Hilbert space $\{\mathfrak{D}, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. I.e., there exists a $M > 0$ such that

$$[\mathfrak{j}_2 Vf, Vf]_2 \leq M[f,f]_1, \quad \forall f \in \mathfrak{D}.$$

Moreover, since $\mathfrak{D}$ is uniformly positive, there exists a constant $\alpha > 0$ such that $[\mathfrak{j}_1 f, f]_1 \leq \alpha[f,f]_1$ for all $f \in \mathfrak{D}$. Consequently,

$$[\mathfrak{j}_2 Vf, Vf]_2 \leq M[\mathfrak{j}_1 f, f]_1 \leq \alpha M[f,f]_1 = \alpha M[Vf, Vf]_2, \quad f \in \mathfrak{D}.$$

I.e., $V(\mathfrak{D})$ is a uniformly definite subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. The converse implication is obtained by applying (i) to V^{-1} and $V(\mathfrak{D})$. □

3.3. Reduction of unitary relations

Here unitary relations are reduced in two different ways: by means of neutral subspaces contained in their domain (or range) and by splitting them. These reductions allow us to remove from unitary relations that part of their behavior which is well understood. To start with, observe the following basic composition results for isometric relations, see [6, Section 2.2].

Lemma 3.9. *Let S be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let T an isometric relation from $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ to $\{\mathfrak{K}_3, [\cdot,\cdot]\}$. Then*

(i) *TS is isometric;*
(ii) *if S and T are unitary,* $\operatorname{ran} S \subseteq \operatorname{dom} T$ *and* $\operatorname{dom} T$ *is closed or* $\operatorname{dom} T \subseteq \operatorname{ran} S$ *and* $\operatorname{ran} S$ *is closed, then TS is unitary.*

Proof. Combine Lemma 2.13 with (3.1). □

The following lemma associates to each neutral subspace a unitary relation which can be used to reduce unitary relations, see Corollary 3.11 below.

Lemma 3.10. *Let $\mathfrak{L}$ be a closed neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then $U_{\mathfrak{L}}$ defined by*

$$U_{\mathfrak{L}} : \mathfrak{K} \to \mathfrak{L}^{[\perp]}/\mathfrak{L}, \quad f \in \operatorname{dom} U_{\mathfrak{L}} = \mathfrak{L}^{[\perp]} \mapsto f + [\mathfrak{L}]$$

is a unitary operator from $\{\mathfrak{K}, [\cdot,\cdot]\}$ onto the Kreĭn space $\{\mathfrak{L}^{[\perp]}/\mathfrak{L}, [\cdot,\cdot]\}$.

Proof. Note first that by Proposition 2.3 $\{\mathfrak{L}^{[\perp]}/\mathfrak{L}, [\cdot,\cdot]\}$ is a Kreĭn space and that the isometry of $U_{\mathfrak{L}}$ is a direct consequence of the neutrality of $\mathfrak{L}$. Next let $h \in \mathfrak{K}$ and $k \in \mathfrak{L}^{[\perp]}/\mathfrak{L}$ be such that $[f,h] = [U_{\mathfrak{L}} f, k]$ for all $f \in \mathfrak{L}^{[\perp]} = \operatorname{dom} U_{\mathfrak{L}}$. Since $U_{\mathfrak{L}}$ maps onto $\mathfrak{L}^{[\perp]}/\mathfrak{L}$ by its definition, there exists $g \in \operatorname{dom} U_{\mathfrak{L}}$ such that $U_{\mathfrak{L}} g = k$ and, hence, $[f, h-g] = 0$ for all $f \in \mathfrak{L}^{[\perp]}$. This shows that $h - g \in \operatorname{clos}(\mathfrak{L}) = \mathfrak{L} \subseteq \ker U_{\mathfrak{L}}$. Consequently, $\{h,k\} = \{g, U_{\mathfrak{L}} g\} + \{h-g, 0\} \in U_{\mathfrak{L}}$ and, hence, Proposition 3.2 implies that $U_{\mathfrak{L}}$ is unitary. □

Since $\ker V$ and $\operatorname{mul} V$ are neutral subspaces for an isometric relation V, see (3.2), composing isometric relations with the unitary relations provided by Corollary 3.10 yields isometric mappings without kernel and multivalued part. In other words the interesting behavior of isometric relations takes place on the quotient spaces $\overline{\operatorname{dom}}\, V/\ker V$ and $\overline{\operatorname{ran}}\, V/\operatorname{mul} V$. In particular, the following corollary can be used to simplify proofs for statements concerning unitary relations.

Corollary 3.11. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then*

$$U_r = \{\{f, f' + [\operatorname{mul} U]\} \in \mathfrak{K}_1 \times \overline{\operatorname{ran}}\, U/\operatorname{mul} U : \{f,f'\} \in U\}$$

is a unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to the Kreĭn space $\{\overline{\operatorname{ran}}\, U/\operatorname{mul} U, [\cdot,\cdot]_2\}$ with dense range and

$$U_d = \{\{f + [\ker U], f'\} \in \overline{\operatorname{dom}}\, U/\ker U \times \mathfrak{K}_2 : \{f,f'\} \in U\}$$

is a unitary relation from the Kreĭn space $\{\overline{\operatorname{dom}}\, U/\ker U, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with dense domain. In particular, $(U_r)_d = (U_d)_r$ is a unitary operator from $\{\overline{\operatorname{dom}}\, U/\ker U, [\cdot,\cdot]_1\}$ to $\{\overline{\operatorname{ran}}\, U/\operatorname{mul} U, [\cdot,\cdot]_2\}$ with dense domain and dense range.

Proof. Since $\ker U$ and $\operatorname{mul} U$ are closed neutral subspaces, $U_{\ker U}$ and $U_{\operatorname{mul} U}$ are unitary operators by Lemma 3.10 with closed domain and closed range. Consequently, Lemma 3.9 implies that $U_r := U_{\operatorname{mul} U} U$ and $U_d := U (U_{\ker U})^{-1}$ are unitary relations which clearly have the stated properties. □

Next observe the following simple but useful result.

Lemma 3.12. *Let U be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\{\widetilde{\mathfrak{K}}_i, [\cdot,\cdot]_i\}[+]\{\widehat{\mathfrak{K}}_i, [\cdot,\cdot]_i\}$ be an orthogonal decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ into two Kreĭn spaces, for $i = 1, 2$, such that $\mathrm{gr}(U) = \mathrm{gr}(\widetilde{U}) + \mathrm{gr}(\widehat{U})$, where $\mathrm{gr}(\widetilde{U}) := \mathrm{gr}(U) \cap \widetilde{\mathfrak{K}}_1 \times \widetilde{\mathfrak{K}}_2$ and $\mathrm{gr}(\widehat{U}) := \mathrm{gr}(U) \cap \widehat{\mathfrak{K}}_1 \times \widehat{\mathfrak{K}}_2$. Then U is unitary if and only if $\widetilde{U}$ and $\widehat{U}$ are unitary.*

Proof. This follows from the definition of unitary relations ($U^{[*]} = U^{-1}$) and the orthogonal decomposition of U. □

Corollary 3.13. *Let U be a closed and isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Moreover, let $\mathfrak{D}_1 \subseteq \mathrm{dom}\, U$ be a closed uniformly definite subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and let $\mathfrak{D}_2$ be the closed uniformly definite subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $U(\mathfrak{D}_1) = \mathfrak{D}_2 + \mathrm{mul}\, U$. Then U is a unitary relation if and only if*

$$\widetilde{U} = \{\{f, f'\} \in U : f \in \mathfrak{D}_1^{[\perp]_1} \text{ and } f' \in \mathfrak{D}_2^{[\perp]_2}\}$$

is a unitary relation from $\{\mathfrak{K}_1 \cap \mathfrak{D}_1^{[\perp]_1}, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2 \cap \mathfrak{D}_2^{[\perp]_2}, [\cdot,\cdot]_2\}$.

Proof. Note first that the existence of $\mathfrak{D}_2$ as stated follows from Proposition 3.8 and that $\widehat{U} := \{\{f, f'\} \in U : f \in \mathfrak{D}_1$ and $f' \in \mathfrak{D}_2\}$ is an everywhere defined isometric operator from the Hilbert space $\{\mathfrak{D}_1, [\cdot,\cdot]_1\}$ onto the Hilbert space $\{\mathfrak{D}_2, [\cdot,\cdot]_2\}$ and, hence, unitary. Since, clearly, $\mathrm{gr}(U) = \mathrm{gr}(\widetilde{U}) + \mathrm{gr}(\widehat{U})$, the statement follows now directly from Lemma 3.12. □

3.4. Unitary operators with closed domain and range

The following proposition shows that unitary relations with closed domain and range have almost the same properties as standard unitary operators (everywhere defined unitary operators with everywhere define inverse.)

Proposition 3.14. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with closed domain and range. Let $\mathfrak{L}$ be a subspace of $\mathfrak{K}_1$ with $\ker U \subseteq \mathfrak{L} \subseteq \mathrm{dom}\, U$, then*

$$U(\mathfrak{L}^{[\perp]_1}) = (U(\mathfrak{L}))^{[\perp]_2}.$$

In particular, $\mathfrak{L}$ is a (essentially, closed) (hyper-maximal, maximal) nonnegative, nonpositive or neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ if and only if $U(\mathfrak{L})$ is a (essentially, closed) (hyper-maximal, maximal) nonnegative, nonpositive or neutral of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, respectively.

Proof. Let $\ker U \subseteq \mathfrak{L} \subseteq \mathrm{dom}\, U$, then $(\mathrm{dom}\, U)^{[\perp]_1} \subseteq \mathfrak{L}^{[\perp]_1} \subseteq (\ker U)^{[\perp]_1}$. Hence, using (3.3) and the closedness of the domain of U, it follows that $\ker U \subseteq \mathfrak{L}^{[\perp]_1} \subseteq \mathrm{dom}\, U$. Similar arguments show that $\mathrm{mul}\, U \subseteq (U(\mathfrak{L}))^{[\perp]_2} \subseteq \mathrm{ran}\, U$. Consequently, the equality $U(\mathfrak{L}^{[\perp]_1}) = (U(\mathfrak{L}))^{[\perp]_2}$ follows directly from Lemma 3.6, cf. (3.6). □

The condition that the domain and range of U are closed in Proposition 3.14 is too strong; they are always closed simultaneously as the following result from [12] shows, cf. [13]. Here an elementary proof is given.

Proposition 3.15. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then* $\operatorname{dom} U$ *is closed if and only if* $\operatorname{ran} U$ *is closed.*

Proof. Since U is unitary if and only if U^{-1} is unitary, it suffices to prove that if $\operatorname{dom} U$ is closed, then $\operatorname{ran} U$ is closed.

Assume that $\operatorname{dom} U$ is closed, then $U_c := U_{\operatorname{mul} U} U (U_{\ker U})^{-1}$ is an everywhere defined unitary operator from the Kreĭn space $\{\operatorname{dom} U/\ker U, [\cdot,\cdot]_1\}$ to the Kreĭn space $\{\overline{\operatorname{ran}}\, U/\operatorname{mul} U, [\cdot,\cdot]_2\}$ with dense range, see Lemma 3.10. Let $\mathfrak{D}_1^+[+]\mathfrak{D}_1^-$ be a canonical decomposition of $\{\overline{\operatorname{dom}}\, U/\ker U, [\cdot,\cdot]_1\}$, then $U_c(\mathfrak{D}_1^+)$ and $U_c(\mathfrak{D}_1^-)$ are a closed uniformly positive and negative subspace of $\{\overline{\operatorname{ran}}\, U/\operatorname{mul} U, [\cdot,\cdot]_2\}$ which are orthogonal to each other, see Proposition 3.8 and Lemma 3.6. Therefore, $\operatorname{ran} U_c = U_c(\mathfrak{D}_1^+) + U_c(\mathfrak{D}_1^-)$ is closed, see [3, Ch. V: Theorem 5.3]. Since $(U_{\operatorname{mul} U})^{-1}$ is a unitary operator, Proposition 3.14 implies that $(U_{\operatorname{mul} U})^{-1} \operatorname{ran} U_c = \operatorname{ran} U (U_{\ker U})^{-1} = \operatorname{ran} U$ is closed. □

Corollary 3.16. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with closed domain. Then for any neutral subspace $\mathfrak{L}$, $\ker U \subseteq \mathfrak{L} \subseteq \operatorname{dom} U$, of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$*

$$n_+(\mathfrak{L}) = n_+(U(\mathfrak{L})) \quad \textit{and} \quad n_-(\mathfrak{L}) = n_-(U(\mathfrak{L})).$$

Proof. Since $\operatorname{dom} U$ is closed, $(U_d)_r$ is a standard unitary operator from $\{\operatorname{dom} U/\ker U, [\cdot,\cdot]_1\}$ to $\{\operatorname{ran} U/\operatorname{mul} U, [\cdot,\cdot]_2\}$, see Proposition 3.15 and Corollary 3.11. Hence, $n_\pm(0_d) = n_\pm(0_r)$, where 0_d and 0_r are the trivial subspaces in $\{\operatorname{dom} U/\ker U, [\cdot,\cdot]_1\}$ and $\{\operatorname{ran} U/\operatorname{mul} U, [\cdot,\cdot]_2\}$, respectively. Furthermore, the first von Neumann formula ((2.4)) implies that $n_\pm(0_d) = n_\pm(\ker U)$ and $n_\pm(0_r) = n_\pm(\operatorname{ran} U)$. Together these observations yield the statement. □

Let U be a standard unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then the proof of Proposition 3.15 shows that $U(\mathfrak{K}_1^+)[+]U(\mathfrak{K}_1^-)$ is a canonical decomposition of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Consequently, standard unitary operators in Kreĭn spaces are the orthogonal sum of two Hilbert space unitary operators. This shows that standard unitary operators give a one-to-one correspondence between fundamental symmetries.

Proposition 3.17. *Let U be a standard unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ onto $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then the mapping $\mathfrak{j}_1 \mapsto U\mathfrak{j}_1U^{-1}$ is a bijective mapping from the set of all fundamental symmetries of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ onto the set of all fundamental symmetries of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$.*

Proof. Let $\mathfrak{j}_1$ be a fundamental symmetry of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, then

$$U\mathfrak{j}_1U^{-1}U\mathfrak{j}_1U^{-1} = I \quad \text{and} \quad (U\mathfrak{j}_1U^{-1})^{[*]} = U^{-[*]}\mathfrak{j}_1U^{[*]} = U\mathfrak{j}_1U^{-1}.$$

This shows that $U\mathfrak{j}_1U^{-1}$ is a fundamental symmetry for $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. The bijectivity of the mapping is clear, because for any fundamental symmetry $\mathfrak{j}_2$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ one has that $\mathfrak{j}_2 = UU^{-1}\mathfrak{j}_2UU^{-1}$ and similar arguments as above show that $U^{-1}\mathfrak{j}_2U$ is a fundamental symmetry for $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. □

Analogues of Proposition 3.17 hold for unitary relations with closed domain and range. For instance, if $\operatorname{ran} U = \mathfrak{K}_2$, then the indicated mapping is surjective.

The following proposition gives a characterization for unitary relations with closed domain, cf. [3, Ch. VI: Theorem 3.5].

Proposition 3.18. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then U has closed domain if and only if U maps every uniformly positive (negative) subspace $\mathfrak{D} \subset \operatorname{dom} U$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ onto the sum of* $\operatorname{mul} U$ *and a uniformly positive (negative) subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$.*

Proof. $(U_r)_d$ is a unitary operator from the Kreĭn space $\{\overline{\operatorname{dom}} U/\ker U, [\cdot,\cdot]_1\}$ to the Kreĭn space $\{\overline{\operatorname{ran}} U/\operatorname{mul} U, [\cdot,\cdot]_2\}$ with dense domain and dense range, see Corollary 3.11. Clearly, $(U_r)_d$ has closed domain if and only if U has closed domain. Hence, it suffices to prove the statement for a densely defined unitary operator.

If U has closed domain and $\mathfrak{D} \subseteq \operatorname{dom} U$ is a uniformly positive (negative) subspace, then $\operatorname{clos}(\mathfrak{D}) \subseteq \operatorname{dom} U$ is a uniformly positive (negative) subspace which is mapped by U onto a uniformly positive (negative) subspace, see Proposition 3.8. Hence $\mathfrak{D}$ is also mapped onto a uniformly positive subspace. To prove the converse implication let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$, then $\mathfrak{K}_1 = \overline{\operatorname{dom}} U = \operatorname{clos}(\operatorname{dom} U \cap \mathfrak{K}_1^+) + \operatorname{clos}(\operatorname{dom} U \cap \mathfrak{K}_1^-)$, see (4.1) below. By the assumption together with Proposition 3.8 $\operatorname{dom} U \cap \mathfrak{K}_1^+$ and $\operatorname{dom} U \cap \mathfrak{K}_1^-$ are closed. Hence, $\mathfrak{K}_1 = \operatorname{dom} U \cap \mathfrak{K}_1^+ + \operatorname{dom} U \cap \mathfrak{K}_1^- \subseteq \operatorname{dom} U$ shows that $\operatorname{dom} U$ is closed. □

Corollary 3.19. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then*

(i) *if $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ is a Hilbert space, then* $\operatorname{dom} U = \mathfrak{K}_1$*;*
(ii) *if $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ are Hilbert spaces, then U is a standard unitary operator.*

Proof. Clearly, (ii) follows from (i). If the assumption in (i) holds, then by Proposition 3.18 (applied to U^{-1}) U has closed range and, hence, closed domain, see Proposition 3.15. Since $\ker U = (\operatorname{dom} U)^{[\perp]_1}$ is a neutral subspace, see (3.3), the assumption also implies that $\ker U = \{0\}$ and, hence, $\operatorname{dom} U = \overline{\operatorname{dom}} U = \mathfrak{K}_1$. □

Potapov-Ginzburg transformations, see [2, Ch. I: Section 1], can be interpreted as a standard unitary operator. Therefore introduce the Hilbert spaces $\{\mathfrak{H}_1, (\cdot,\cdot)_1\} := \{\mathfrak{K}_1^+ \times \mathfrak{K}_2^-, (\cdot,\cdot)_1\}$ and $\{\mathfrak{H}_2, (\cdot,\cdot)_2\} := \{\mathfrak{K}_2^+ \times \mathfrak{K}_1^-, (\cdot,\cdot)_2\}$, where

$$\begin{aligned} (f \times f', g \times g')_1 &= [f,g]_1 - [f',g']_2, \quad f,g \in \mathfrak{K}_1^+, f',g' \in \mathfrak{K}_2^-; \\ (f \times f', g \times g')_2 &= [f,g]_2 - [f',g']_1, \quad f,g \in \mathfrak{K}_2^+, f',g' \in \mathfrak{K}_1^-. \end{aligned} \tag{3.8}$$

Here $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ is a canonical decomposition of the Kreĭn space $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$, for $i = 1, 2$.

Proposition 3.20. *Let* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *and* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *be Kreĭn spaces with associated Hilbert spaces* $\{\mathfrak{H}_1, (\cdot,\cdot)_1\}$ *and* $\{\mathfrak{H}_2, (\cdot,\cdot)_2\}$ *as defined above for fundamental symmetries* $\mathfrak{j}_1$ *and* $\mathfrak{j}_2$*. Then the Potapov-Ginzburg transformation* $\mathfrak{P}_{\mathfrak{j}_1,\mathfrak{j}_2}$ *defined by*

$$\mathfrak{P}_{\mathfrak{j}_1,\mathfrak{j}_2}\{f,g\} = \{P_1^+ f \times P_2^- g, P_2^+ g \times P_1^- f\}$$

is a standard unitary operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\} \times \{\mathfrak{K}_2, -[\cdot,\cdot]_2\}$ *to* $\{\mathfrak{H}_1, (\cdot,\cdot)_1\} \times \{\mathfrak{H}_2, -(\cdot,\cdot)_2\}$ *and for any relation* H *from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$

$$\mathfrak{P}_{\mathfrak{j}_1,\mathfrak{j}_2}(H^{-[*]}) = (\mathfrak{P}_{\mathfrak{j}_1,\mathfrak{j}_2}(H))^{-*}.$$

In particular, $\mathfrak{P}_{\mathfrak{j}_1,\mathfrak{j}_2}$ *maps the graph of (closed) isometric and unitary relations from the Kreĭn space* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to the Kreĭn space* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *to the graph of (closed) isometric and unitary relations from the Hilbert space* $\{\mathfrak{H}_1, (\cdot,\cdot)_1\}$ *to the Hilbert space* $\{\mathfrak{H}_2, (\cdot,\cdot)_2\}$*, respectively.*

Proof. Let $f, g \in \mathfrak{K}_1$ and $f', g' \in \mathfrak{K}_2$, then with the introduced inner products

$$\begin{aligned}
&[f,g]_1 - [f',g']_2 \\
&\quad = [P_1^+ f, P_1^+ g]_1 + [P_1^- f, P_1^- g]_1 - [P_2^+ f', P_2^+ g']_2 - [P_2^- f', P_2^- g']_2 \\
&\quad = (P_1^+ f \times P_2^- f', P_1^+ g \times P_2^- g')_1 - (P_2^+ f' \times P_1^- f, P_2^+ g' \times P_1^- g)_2.
\end{aligned}$$

Hence the Potapov-Ginzburg transformation $\mathfrak{P}_{\mathfrak{j}_1,\mathfrak{j}_2}$ is an everywhere defined isometric operator from the Kreĭn space $\{\mathfrak{K}_1, [\cdot,\cdot]_1\} \times \{\mathfrak{K}_2, -[\cdot,\cdot]_2\}$ onto the Kreĭn space $\{\mathfrak{H}_1, (\cdot,\cdot)_1\} \times \{\mathfrak{H}_2, -(\cdot,\cdot)_2\}$, i.e., it is a standard unitary operator, see e.g. Proposition 3.5. Finally, the equality $\mathfrak{P}_{\mathfrak{j}_1,\mathfrak{j}_2}(H^{-[*]}) = (\mathfrak{P}_{\mathfrak{j}_1,\mathfrak{j}_2}(H))^{-*}$ follows from Lemma 3.6 combined with an interpretation of the orthogonal complement, cf. the arguments preceding Proposition 3.3. □

3.5. Kernels and multivalued parts of isometric relations

Recall that for an isometric operator from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ one has that $\ker V \subseteq (\operatorname{dom} V)^{[\perp]_1}$ and $\operatorname{mul} V \subseteq (\operatorname{ran} V)^{[\perp]_2}$. The following statement contains a useful consequence of the equality $\ker V = (\operatorname{dom} V)^{[\perp]_1}$.

Corollary 3.21. *Let* V *be an isometric relation from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *which satisfies* $\ker V = (\operatorname{dom} V)^{[\perp]_1}$ *and let* $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ *be a canonical decomposition of* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *with associated orthogonal projections* P_1^+ *and* P_1^-*. Then* $P_1^+\overline{\operatorname{dom}}\, V = \mathfrak{K}_1^+$ *and* $P_1^-\overline{\operatorname{dom}}\, V = \mathfrak{K}_1^-$.

Proof. The assumption $\ker V = (\operatorname{dom} V)^{[\perp]_1}$ together with equation (2.2) implies that $\overline{\operatorname{dom}}\, V \cap \mathfrak{K}_1^\pm = \mathfrak{K}_1^\pm \ominus \ker V$. Since $\ker V \subseteq \operatorname{dom} V$, the conclusion follows from the preceding equalities. □

The following statement gives conditions under which the inequalities in (3.2) become equalities given that equality holds in either of the two inclusion.

Lemma 3.22. *Let* V *be a closed and isometric relation from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$*. Then*

$$\left\{ \begin{array}{l} \ker V = (\operatorname{dom} V)^{[\perp]_1} \\ (\operatorname{ran} V)^{[\perp]_2} \subseteq \operatorname{ran} V \end{array} \right. \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \operatorname{mul} V = (\operatorname{ran} V)^{[\perp]_2} \\ (\operatorname{dom} V)^{[\perp]_1} \subseteq \operatorname{dom} V \end{array} \right.$$

Proof. Because V^{-1} is an isometric relations if V is an isometric relation, it suffices to prove only one implication. Therefore, assume that $\ker V = (\operatorname{dom} V)^{[\perp]_1}$ and $(\operatorname{ran} V)^{[\perp]_2} \subseteq \operatorname{ran} V$. Then, clearly, $(\operatorname{dom} V)^{[\perp]_1} \subseteq \operatorname{dom} V$. Furthermore, the assumption $\ker V = (\operatorname{dom} V)^{[\perp]_1}$ and Lemma 3.6 yield that

$$\begin{aligned}\operatorname{mul} V = V(\ker V) = V((\operatorname{dom} V)^{[\perp]_1}) &= V(\operatorname{dom} V)^{[\perp]_2} \cap \operatorname{ran} V \\ &= (\operatorname{ran} V)^{[\perp]_2} \cap \operatorname{ran} V.\end{aligned}$$

Hence, the assumption $(\operatorname{ran} V)^{[\perp]_2} \subseteq \operatorname{ran} V$ yields $\operatorname{mul} V = (\operatorname{ran} V)^{[\perp]_2}$. □

Corollary 3.23. *Let V be a closed and isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and assume that there exists a hyper-maximal semi-definite subspace $\mathfrak{L}$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $\mathfrak{L} \subseteq \operatorname{dom} V$. Then $\operatorname{mul} U = (\operatorname{ran} U)^{[\perp]_2}$ implies that $\ker U = (\operatorname{dom} U)^{[\perp]_1}$.*

Proof. Recall that $\mathfrak{L}^{[\perp]_1} \subseteq \mathfrak{L}$ for hyper-maximal semi-definite subspaces, see Proposition 2.8. Hence, the assumption $\mathfrak{L} \subseteq \operatorname{dom} V$ implies that

$$(\operatorname{dom} V)^{[\perp]_1} \subseteq \mathfrak{L}^{[\perp]_1} \subseteq \mathfrak{L} \subseteq \operatorname{dom} V.$$

Consequently, Lemma 3.22 implies that $\ker V = (\operatorname{dom} V)^{[\perp]_1}$. □

A further condition for the equality $\ker U = (\operatorname{dom} U)^{[\perp]_1}$ to hold is furnished by the following lemma which will be used later.

Lemma 3.24. *Let V be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{j}_1$ be a fundamental symmetry of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Moreover, assume that there exists a hyper-maximal semi-definite subspace $\mathfrak{L}$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $\ker V \subseteq \mathfrak{L} \subseteq \operatorname{dom} V$. Then $\ker V = (\operatorname{dom} V)^{[\perp]_1}$ if and only if $\mathfrak{j}_1\mathfrak{L} \cap \operatorname{dom} V + \ker V$ is an essentially hyper-maximal semi-definite subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$.*

Proof. Since $\mathfrak{L}$, being hyper-maximal semi-definite, is closed, the inclusion $\ker V \subseteq \mathfrak{L}$ implies that $\ker V$ is closed. Using this observation and the hyper-maximality of $\mathfrak{L}$, it follows that $\mathfrak{K}_1$ has the following $\mathfrak{j}_1$-orthogonal decomposition:

$$\mathfrak{K}_1 = \mathfrak{L}^{[\perp]_1} \oplus_1 (\mathfrak{L} \cap \mathfrak{j}_1\mathfrak{L}) \oplus_1 \mathfrak{j}_1(\mathfrak{L}^{[\perp]_1} \ominus_1 \ker V) \oplus_1 \mathfrak{j}_1\ker V, \tag{3.9}$$

cf. Proposition 2.8 (iv). Hence $\operatorname{dom} V$ $(\subseteq (\ker V)^{[\perp]_1})$ and $(\ker V)^{[\perp]_1}$ have the decompositions:

$$\begin{aligned}\operatorname{dom} V &= \ker V \oplus_1 (\mathfrak{L}^{[\perp]_1} \ominus_1 \ker V) \oplus_1 (\mathfrak{L} \cap \mathfrak{j}_1\mathfrak{L}) \oplus_1 (\mathfrak{j}_1\mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} V), \\ (\ker V)^{[\perp]_1} &= \ker V \oplus_1 (\mathfrak{L}^{[\perp]_1} \ominus_1 \ker V) \oplus_1 (\mathfrak{L} \cap \mathfrak{j}_1\mathfrak{L}) \oplus_1 \mathfrak{j}_1(\mathfrak{L}^{[\perp]_1} \ominus_1 \ker V).\end{aligned}$$

Since $\ker V$ is closed, $\ker V = (\operatorname{dom} V)^{[\perp]_1}$ if and only if $(\ker V)^{[\perp]_1} = \overline{\operatorname{dom}} V$. Hence, the above two formula lines show that $\ker V = (\operatorname{dom} V)^{[\perp]_1}$ if and only if $\operatorname{clos}(\mathfrak{j}_1\mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} V) = \mathfrak{j}_1\mathfrak{L}^{[\perp]_1} \ominus_1 \mathfrak{j}_1\ker V$. Since, $\mathfrak{j}_1\mathfrak{L} \ominus_1 \mathfrak{j}_1\ker V = (\mathfrak{L} \cap \mathfrak{j}_1\mathfrak{L}) \oplus (\mathfrak{j}_1\mathfrak{L}^{[\perp]_1} \ominus_1 \mathfrak{j}_1\ker V)$, it follows from (3.9) together with Proposition 2.8 that the statement holds. □

If equalities hold in (3.2) for an isometric relation and additionally its domain or range is closed, then the isometry must be unitary. Note that without the condition that the domain or range is closed this is not in general true, see e.g. Example 5.17.

Lemma 3.25. *Let U be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ which has closed domain or closed range and satisfies*

$$\ker U = (\operatorname{dom} U)^{[\perp]_1} \quad \textit{and} \quad \operatorname{mul} U = (\operatorname{ran} U)^{[\perp]_2}.$$

Then U is a unitary relation with closed domain and range.

Proof. Assume that U has closed range, then the assumptions and (2.6) yield

$$\operatorname{dom} U^{[*]} \subseteq \overline{\operatorname{dom}}\, U^{[*]} = (\operatorname{mul} \operatorname{clos}(U))^{[\perp]_2} \subseteq (\operatorname{mul} U)^{[\perp]_2} = \operatorname{ran} U.$$

Therefore U is unitary by Proposition 3.5 and U has closed domain by Proposition 3.15. The case that U has closed domain follows by passing to the inverse. □

Finally, some additions to Proposition 3.5 are given, cf. [6, Section 2.3].

Corollary 3.26. *Let U be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then*

(i) *U is a unitary operator (not necessarily densely defined) if and only if $\operatorname{ran} U^{[*]} \subseteq \operatorname{dom} U$ and $\overline{\operatorname{ran}}\, U = \mathfrak{K}_2$;*
(ii) *U is a unitary operator with closed domain (not necessarily densely defined) if and only if $(\operatorname{dom} U)^{[\perp]_1} \subseteq \operatorname{dom} U$ and $\operatorname{ran} U = \mathfrak{K}_2$.*

Proof. (i): Since $\overline{\operatorname{ran}}\, U \subseteq (\operatorname{mul} U)^{[\perp]_2}$ for any isometric relation U, see (3.2), (i) can be obtained from Proposition 3.5.

(ii): If U is a unitary operator, then by (3.3) $(\operatorname{dom} U)^{[\perp]_1} = \ker U \subseteq \operatorname{dom} U$ and by (i) $\overline{\operatorname{ran}}\, U = \mathfrak{K}_2$. Hence, necessity of the conditions follows from the fact that the domain and range of a unitary relation are simultaneously closed, see Proposition 3.15.

Conversely, if $(\operatorname{dom} U)^{[\perp]_1} \subseteq \operatorname{dom} U$ and $\operatorname{ran} U = \mathfrak{K}_2$, then $\operatorname{mul} U = \{0\}$ by (3.2) and, hence, $\ker \operatorname{clos}(U) = (\operatorname{dom} U)^{[\perp]_1}$ by Lemma 3.22 (applied to $\operatorname{clos} U$). Moreover, the assumption $(\operatorname{dom} U)^{[\perp]_1} \subseteq \operatorname{dom} U$ also yields that

$$\ker \operatorname{clos}(U) = (\operatorname{dom} \operatorname{clos}(U))^{[\perp]_1} = (\operatorname{dom} U)^{[\perp]_1} \subseteq \operatorname{dom} U.$$

Consequently, $\ker U$ is closed and, hence, U is unitary by Lemma 3.25. □

In view of Proposition 3.5 and Lemma 3.22, the following statement is also true (cf. [6, Corollary 2.6]):

$$\text{(i)} \quad \Leftrightarrow \quad \operatorname{dom} U^{[*]} \subseteq \operatorname{ran} U, \quad (\operatorname{dom} U)^{[\perp]_1} \subseteq \operatorname{dom} U, \quad \overline{\operatorname{ran}}\, U = \mathfrak{K}_2.$$

4. Unitary relations and the Weyl identity

In order to further investigate unitary relations it is shown that they satisfy a geometrical identity, the *Weyl identity*, which is shown to characterize unitary relations. This identity is then used to obtain a quasi-block decomposition of unitary relations and to split unitary relations.

4.1. Unitary relations and the von Neumann formula

As a consequence of interpreting unitary relations as hyper-maximal neutral subspaces, the second von Neumann formula gives rise to the following statement.

Proposition 4.1. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ with associated orthogonal projections P_i^+ and P_i^-, $i = 1, 2$. Then*

$$(P_1^+ \times P_2^-)\mathrm{gr}(U) = \mathfrak{K}_1^+ \times \mathfrak{K}_2^- \quad \textit{and} \quad (P_1^- \times P_2^+)\mathrm{gr}(U) = \mathfrak{K}_1^- \times \mathfrak{K}_2^+.$$

In particular,

$$\mathrm{clos}\,(\mathrm{dom}\,U \cap \mathfrak{K}_1^\pm) = \overline{\mathrm{dom}}\,U \cap \mathfrak{K}_1^\pm \quad \textit{and} \quad \mathrm{clos}\,(\mathrm{ran}\,U \cap \mathfrak{K}_2^\pm) = \overline{\mathrm{ran}}\,U \cap \mathfrak{K}_2^\pm.$$

Proof. Note first that $(\mathfrak{K}_1^+ \times \mathfrak{K}_2^-)[+](\mathfrak{K}_1^- \times \mathfrak{K}_2^+)$ is a canonical decomposition of $\{\mathfrak{K}_1 \times \mathfrak{K}_2, [\cdot,\cdot]_{1,-2}\}$, see (3.4), with associated orthogonal projections $P_1^+ \times P_2^-$ and $P_1^- \times P_2^+$, and recall that $\mathrm{gr}(U)$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_1 \times \mathfrak{K}_2, [\cdot,\cdot]_{1,-2}\}$, see Proposition 3.3. These observations together with Proposition 2.5 show that the first assertion holds.

Using the proven equalities and the hyper-maximal neutrality of $\mathrm{gr}(U)$, Corollary 2.12 (applied to $\mathfrak{L} = \mathrm{dom}\,U \times \mathrm{ran}\,U$ and $\mathfrak{M} = \mathrm{gr}(U)$ with $P^+ = P_1^+ \times P_2^-$ and $P^- = P_1^- \times P_2^+$) yields that

$$\mathrm{clos}\,((\mathrm{dom}\,U \times \mathrm{ran}\,U) \cap (\mathfrak{K}_1^+ \times \mathfrak{K}_2^-)) = \mathrm{clos}\,(\mathrm{dom}\,U \times \mathrm{ran}\,U) \cap (\mathfrak{K}_1^+ \times \mathfrak{K}_2^-);$$
$$\mathrm{clos}\,((\mathrm{dom}\,U \times \mathrm{ran}\,U) \cap (\mathfrak{K}_1^- \times \mathfrak{K}_2^+)) = \mathrm{clos}\,(\mathrm{dom}\,U \times \mathrm{ran}\,U) \cap (\mathfrak{K}_1^- \times \mathfrak{K}_2^+).$$

This shows that the final assertion holds. □

With the notation as in Proposition 4.1, Proposition 4.1 combined with Proposition 2.14 shows $\mathrm{dom}\,U$ has the following direct sum decompositions:

$$\mathrm{dom}\,U \cap \mathfrak{K}_1^+ + U^{-1}(\mathrm{ran}\,U \cap \mathfrak{K}_2^-) = \mathrm{dom}\,U = \mathrm{dom}\,U \cap \mathfrak{K}_1^- + U^{-1}(\mathrm{ran}\,U \cap \mathfrak{K}_2^+),$$

see also [4, Theorem 3.9]. Furthermore, Proposition 4.1 combined with the first von Neumann formula (2.4) yields

$$\overline{\mathrm{dom}}\,U = \ker U + \mathrm{clos}\,(\mathrm{dom}\,U \cap \mathfrak{K}_1^+) + \mathrm{clos}\,(\mathrm{dom}\,U \cap \mathfrak{K}_1^-), \tag{4.1}$$

see also [6, Lemma 2.14 (ii)]. Combining (4.1) with the last equalities in Proposition 4.1 yields the following equalities:

$$\mathfrak{K}_1^\pm = P_1^\pm \ker U + \mathrm{clos}\,(\mathrm{dom}\,U \cap \mathfrak{K}_1^\pm). \tag{4.2}$$

On the other hand, if an isometric relation V satisfies (4.1) and $\ker V = (\operatorname{dom} V)^{[\perp]_1}$, then, clearly,

$$\begin{aligned} \operatorname{clos}(\operatorname{dom} V \cap \mathfrak{K}_1^{\pm}) &= \overline{\operatorname{dom}}\, V \cap \mathfrak{K}_1^{\pm}; \\ \operatorname{clos}(\operatorname{ran} V \cap \mathfrak{K}_2^{\pm}) &= \overline{\operatorname{ran}}\, V \cap \mathfrak{K}_2^{\pm}. \end{aligned} \tag{4.3}$$

The following statement gives an alternative condition for (4.3) to hold.

Corollary 4.2. *Let V be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and assume that there exists a hyper-maximal semi-definite subspace $\mathfrak{L}$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $\mathfrak{L} \subseteq \operatorname{dom} V$. Then* $\operatorname{clos}(\operatorname{dom} V \cap \mathfrak{K}_1^{\pm}) = \overline{\operatorname{dom}}\, V \cap \mathfrak{K}_1^{\pm}$.

Proof. This follows directly from Corollary 2.12 (applied to $\mathfrak{L} = \operatorname{dom} V$ and $\mathfrak{M} = \mathfrak{L}$, and $\mathfrak{L} = \operatorname{dom} V$ and $\mathfrak{M} = \mathfrak{L}^{[\perp]_1}$). □

The following theorem shows that the first two equalities in Proposition 4.1 almost characterize unitary relations among the set of all isometric relations.

Theorem 4.3. *Let U be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then U is unitary if and only if there exist orthogonal projections P_1 and P_2 in $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, respectively, such that*

(i) $(\operatorname{dom} U)^{[\perp]_1} \subseteq \ker U$ *and* $(\operatorname{ran} U)^{[\perp]_2} \subseteq \operatorname{mul} U$*;*
(ii) $(P_1 \times P_2)(\operatorname{dom} U \times \operatorname{ran} U) \subseteq (P_1 \times P_2)\operatorname{gr}(U)$*;*
(iii) $((I-P_1) \times (I-P_2))(\operatorname{dom} U \times \operatorname{ran} U) \subseteq ((I-P_1) \times (I-P_2))\operatorname{gr}(U)$*;*
(iv) $P_1 \operatorname{dom} U = P_1 \overline{\operatorname{dom}}\, U$ *and* $P_2 \operatorname{ran} U = P_2 \overline{\operatorname{ran}}\, U$*.*

Proof. Let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ with associated projections P_i^+ and P_i^-, $i = 1, 2$. Then necessity of the condition (i) is clear by (3.3) and the conditions (ii) and (iii) are satisfied for $P_1 = P_1^+$ and $P_2 = P_2^+$ by the first two equalities in Proposition 4.1. Those same equalities in Proposition 4.1 show that $P_1^+ \operatorname{dom} U = \mathfrak{K}_1^+$ and $P_2^- \operatorname{ran} U = \mathfrak{K}_2^-$, which shows that (iv) holds.

Conversely, observe first that by Proposition 2.14 condition (iii) is equivalent to

$$\operatorname{dom} U = U^{-1}(\operatorname{ran} U \cap \operatorname{ran} P_2) + (\operatorname{dom} U \cap \operatorname{ran} P_1). \tag{4.4}$$

Now assume that $f \in \mathfrak{K}_1$ and $f' \in \mathfrak{K}_2$ satisfy $[f, g]_1 = [f', g']_2$, for all $\{g, g'\} \in U$. Then $f \in (\ker U)^{[\perp]_1} \subseteq \overline{\operatorname{dom}}\, U$ and $f' \in (\operatorname{mul} U)^{[\perp]_2} \subseteq \overline{\operatorname{ran}}\, U$ by (i). By (ii) and (iv) there exists a $\{h, h'\} \in U$ such that $P_1 h = P_1 f$ and $P_2 h' = P_2 f'$. Consequently,

$$[(I-P_1)(f-h), g]_1 = [f-h, g]_1 = [f'-h', g']_2 = [(I-P_2)(f'-h'), g']_2, \tag{4.5}$$

for all $\{g, g'\} \in U$. If $g \in \operatorname{dom}(U \cap \operatorname{ran} P_1)$, then the inner-product on the lefthand side of (4.5) is zero. If $g' \in \operatorname{ran}(U \cap \operatorname{ran} P_2)$, then the inner product on the righthand side of (4.5) is zero. Hence, (4.4), (4.5) and (i) imply that $f-h \in (\operatorname{dom} U)^{[\perp]_1} \subseteq \ker U$ and $f'-h' \in (\operatorname{ran} U)^{[\perp]_2} \subseteq \operatorname{mul} U$. Consequently, $\{f, f'\} = \{h, h'\} + \{f-h, 0\} + \{0, f'-h'\} \in U$ and, hence, U is unitary, see Proposition 3.2. □

4.2. The Weyl identity

Here it is shown that a unitary relation satisfies an identity which will be called the *Weyl identity.* The reason for this name is that it is the abstract equivalent of the so-called Weyl function (or family) which plays an important role in the study of spectral properties of extensions of symmetric operators, see e.g. [6, Definition 3.3].

Proposition 4.4. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then the Weyl identity holds:*

$$U(\operatorname{dom} U \cap \mathfrak{K}_1^+) = \big(U(\operatorname{dom} U \cap \mathfrak{K}_1^-)\big)^{[\perp]_2}.$$

In particular, $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ and $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ are a maximal nonnegative and maximal nonpositive subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, respectively.

Proof. Let $\mathfrak{K}_2^+[+]\mathfrak{K}_2^-$ be a canonical decomposition of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let P_i^+ and P_i^- be the projections associated to $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$, $i = 1, 2$. Then by Proposition 4.1

$$(P_1^+ \times P_2^-)(\operatorname{dom} U \times \operatorname{ran} U) \subseteq (P_1^+ \times P_2^-)\operatorname{gr}(U) = \mathfrak{K}_1^+ \times \mathfrak{K}_2^-.$$

Hence Proposition 2.14 (with $P_1 = P_1^+$ and $P_2 = P_2^-$) implies that

$$P_2^- U(\operatorname{dom} U \cap \ker P_1^+) = P_2^- \operatorname{ran} U = \mathfrak{K}_2^-.$$

Hence, $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ is a maximal nonpositive subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Similar arguments show that $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ is a maximal nonnegative subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Moreover, evidently $\operatorname{dom} U \cap \mathfrak{K}_1^+ \subseteq (\operatorname{dom} U \cap \mathfrak{K}_1^-)^{[\perp]_1}$. Therefore, applying U and using Lemma 3.6 yields

$$U(\operatorname{dom} U \cap \mathfrak{K}_1^+) \subseteq (U(\operatorname{dom} U \cap \mathfrak{K}_1^-))^{[\perp]_2} \cap \operatorname{ran} U.$$

Since $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ and $(U(\operatorname{dom} U \cap \mathfrak{K}_1^-))^{[\perp]_2}$ are both maximal nonnegative, see Proposition 2.4, the Weyl identity follows from the previous inclusion. □

Note also that the equality $(\operatorname{dom} U \cap \mathfrak{K}_1^+) \cap (\operatorname{dom} U \cap \mathfrak{K}_1^-) = \{0\}$ yields

$$U(\operatorname{dom} U \cap \mathfrak{K}_1^+) \cap U(\operatorname{dom} U \cap \mathfrak{K}_1^-) = \operatorname{mul} U. \tag{4.6}$$

Remark 4.5. For an isometric relation V (4.6) remains true. In general Proposition 4.4 does not hold, but instead the following inclusions hold:

$$\begin{aligned} V(\operatorname{dom} V \cap \mathfrak{K}_1^+) &\subseteq \big(V(\operatorname{dom} V \cap \mathfrak{K}_1^-)\big)^{[\perp]_2} \cap \operatorname{ran} V; \\ V(\operatorname{dom} V \cap \mathfrak{K}_1^-) &\subseteq \big(V(\operatorname{dom} V \cap \mathfrak{K}_1^+)\big)^{[\perp]_2} \cap \operatorname{ran} V. \end{aligned} \tag{4.7}$$

Since $\operatorname{clos}(\operatorname{dom} V \cap \mathfrak{K}_1^\pm) \cap \operatorname{dom} V = \operatorname{dom} V \cap \mathfrak{K}_1^\pm$, Proposition Proposition 3.8 implies that $V(\operatorname{dom} V \cap \mathfrak{K}_1^+)$ and $V(\operatorname{dom} V \cap \mathfrak{K}_1^+)$ are closed if V is a closed isometric relation. Furthermore, equalities hold in (4.7) when $\operatorname{clos}(\operatorname{dom} V \cap \mathfrak{K}_1^\pm) = \operatorname{clos}(\operatorname{dom} V) \cap \mathfrak{K}_1^\pm$. These last equalities hold for instance if there exists a hyper-maximal semi-definite subspace $\mathfrak{L} \subseteq \operatorname{dom} V$, see Corollary 4.2.

Combining Proposition 4.4 with Proposition 4.1 yields the following result for the defect numbers of $\ker U$ and $\operatorname{mul} U$, see Definition 2.10; it is an analog of [6, Lemma 2.14 (iii)].

Corollary 4.6. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then*

$$n_+(\ker U) = n_+(\operatorname{mul} U) \quad \text{and} \quad n_-(\ker U) = n_-(\operatorname{mul} U).$$

Proof. Let $U_{\ker U}$ and $U_{\operatorname{mul} U}$ be the unitary operators (with closed domain and range) associated to $\ker U$ and $\operatorname{mul} U$ as in Corollary 3.10. Then Corollary 3.16 implies that

$$n_\pm(\ker U) = n_\pm(U_{\ker U}(\ker U)) = n_\pm(0_d);$$
$$n_\pm(\operatorname{mul} U) = n_\pm(U_{\operatorname{mul} U}(\operatorname{mul} U)) = n_\pm(0_r).$$

Here 0_d and 0_r denote the trivial subspaces in $\{\overline{\operatorname{dom}} U/\ker U, [\cdot,\cdot]_1\}$ and $\{\overline{\operatorname{ran}} U/\operatorname{mul} U, [\cdot,\cdot]_2\}$, respectively. Now note that $(U_d)_r = U_{\operatorname{mul} U} U (U_{\ker U})^{-1}$ is a unitary operator with dense domain and dense range, see Corollary 3.11, which maps 0_d onto 0_r and that $n_\pm(0_d) = k_d^\mp$ and $n_\pm(0_r) = k_r^\mp$. Here for a canonical decomposition $\mathfrak{K}_d^+[+]\mathfrak{K}_d^-$ of $\{\overline{\operatorname{dom}} U/\ker U, [\cdot,\cdot]_1\}$ and a canonical decomposition $\mathfrak{K}_r^+[+]\mathfrak{K}_r^-$ of $\{\overline{\operatorname{ran}} U/\operatorname{mul} U, [\cdot,\cdot]_2\}$, $k_d^\pm = \dim \mathfrak{K}_d^\pm$ and $k_r^\pm = \dim \mathfrak{K}_r^\pm$. This discussion shows that the statement is proven if it is shown that for a unitary operator with dense domain from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ one has $k_1^\pm = k_2^\pm$.

Under that assumption, let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, then $\dim(\operatorname{dom} U \cap \mathfrak{K}_1^+) \leq k_1^+$. Moreover, Proposition 4.4 implies that $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ is a maximal nonnegative subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and, hence, $\dim U(\operatorname{dom} U \cap \mathfrak{K}_1^+) = k_2^+$. Since U is a injective mapping this yields $k_2^+ \leq k_1^+$ and a similar argument yields the inequality $k_2^- \leq k_1^-$. Since the inequalities $k_1^+ \leq k_2^+$ and $k_1^- \leq k_2^-$ follow by the same arguments applied to U^{-1}, this completes the proof. □

Now some further characterization for the closedness of the domain of a unitary relation can be given; they are closely related to results on Weyl families of boundary relations studied in [6]. Moreover, note that the equivalence of (i), (ii) and (iii) goes back to [4, Theorem 3.10] and that the characterization (vii) gives an inverse to a statement in [6, Lemma 4.4].

Proposition 4.7. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then equivalent are*

(i) $\operatorname{dom} U$ *is closed;*
(ii) $\operatorname{dom} U \cap \mathfrak{K}_1^+$ *is closed;*
(iii) $\operatorname{dom} U \cap \mathfrak{K}_1^-$ *is closed;*
(iv) $U(\operatorname{dom} U \cap \mathfrak{K}_1^+) + [\operatorname{mul} U]$ *is uniformly positive in* $\{\overline{\operatorname{ran}} U/\operatorname{mul} U, [\cdot,\cdot]_2\}$*;*
(v) $U(\operatorname{dom} U \cap \mathfrak{K}_1^-) + [\operatorname{mul} U]$ *is uniformly negative in* $\{\overline{\operatorname{ran}} U/\operatorname{mul} U, [\cdot,\cdot]_2\}$*;*
(vi) $\operatorname{dom} U = \ker U + \operatorname{dom} U \cap \mathfrak{K}_1^+ + \operatorname{dom} U \cap \mathfrak{K}_1^-$*;*
(vii) $\operatorname{ran} U = U(\operatorname{dom} U \cap \mathfrak{K}_1^+) + U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$*;*

Proof. (i)-(v): The implication from (i) to (ii) and (iii) is clear, the equivalences of (ii) and (iv), and (iii) and (v) follows from Proposition 3.8. The equivalence of (iv) and (v) follows from Proposition 4.4, and (4.1) shows that (ii) and (iii) imply (i).

(i)-(v) $\Leftrightarrow$ (vi) : By (4.1) the conditions (i)-(iii) imply (vi). Conversely, if (vi) holds, then, using Proposition 4.1, it follows that

$$\mathfrak{K}_1^+ = P_1^+ \mathrm{dom}\, U = P_1^+ \ker U + \mathrm{dom}\, U \cap \mathfrak{K}_1^+,$$

where P_1^+ is the orthogonal projection onto $\mathfrak{K}_1^+$. Comparing this with (4.2) shows that (ii) holds.

(vi) $\Leftrightarrow$ (vii) : This is obvious. □

Using the Potapov-Ginzburg transformation the following necessary and sufficiency conditions for isometric relations to be unitary are obtained. Those conditions are subsequently used to prove that the Weyl identity characterizes unitary relations almost completely.

Lemma 4.8. *Let U be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ with associated projection P_i^+ and P_i^-, $i = 1, 2$. Then U is unitary if and only if*

(i) *U is closed, and* $\ker U = (\mathrm{dom}\, U)^{[\perp]_1}$ *and* $\mathrm{mul}\, U = (\mathrm{ran}\, U)^{[\perp]_1}$*;*
(ii) *there exists a subspace $\mathfrak{M}^+ \subseteq \mathrm{dom}\, U \cap \mathfrak{K}_1^+$ with* $\mathrm{clos}\,(P_2^+ U(\mathfrak{M}^+)) = \mathfrak{K}_2^+$*;*
(iii) *there exists a subspace $\mathfrak{M}^- \subseteq \mathrm{dom}\, U \cap \mathfrak{K}_1^-$ with* $\mathrm{clos}\,(P_2^- U(\mathfrak{M}^-)) = \mathfrak{K}_2^-$*.*

Proof. Necessity of (i) is clear by (3.1) and (3.3). Since $P_2^\pm U(\mathrm{dom}\, U \cap \mathfrak{K}_\pm^+) = \mathfrak{K}_2^\pm$ by Proposition 4.4, (ii) and (iii) hold for $\mathfrak{M}^\pm = \mathrm{dom}\, U \cap \mathfrak{K}_1^\pm$.

Conversely, if (i)-(iii) hold, then by (i) U is a closed isometric relation. Now let U_{PG} be the Potapov-Ginzburg transformation of U, i.e.,

$$\mathrm{gr}(U_{PG}) = \{\{P_1^+ f \times P_2^- f', P_2^+ f' \times P_1^- f\} : \{f, f'\} \in U\},$$

see Proposition 3.20. Then U_{PG} is a closed isometric operator from the Hilbert space $\{\mathfrak{K}_1^+ \times \mathfrak{K}_2^-, (\cdot,\cdot)_1\}$ to the Hilbert space $\{\mathfrak{K}_2^+ \times \mathfrak{K}_1^-, (\cdot,\cdot)_2\}$, see (3.8). Now observe that the assumption (ii) implies that $\mathfrak{K}_2^+ \times \{0\} \subseteq \overline{\mathrm{ran}\, U_{PG}}$. Moreover, the assumption $\ker U = (\mathrm{dom}\, U)^{[\perp]_1}$ implies that $P_1^- \overline{\mathrm{dom}\, U} = \mathfrak{K}_1^-$, see Corollary 3.21, and, hence, there exists a subspace $\mathfrak{N}_1^- \subseteq \mathfrak{K}_1^-$ satisfying $\mathrm{clos}\, \mathfrak{N}_1^- = \mathfrak{K}_1^-$, i.e., $P_1^- \mathrm{ran}\, U_{PG} = \mathfrak{N}_1^-$. Combining the preceding observations shows that $\mathfrak{K}_2^+ \times \mathfrak{N}_1^- \subseteq \overline{\mathrm{ran}\, U_{PG}}$ and, hence, $\overline{\mathrm{ran}}\, U_{PG} = \mathfrak{K}_2^+ \times \mathfrak{K}_1^-$. Similar arguments show that $\overline{\mathrm{dom}}\, U_{PG} = \mathfrak{K}_1^+ \times \mathfrak{K}_2^-$. Consequently, $\mathrm{clos}\,(U_{PG}) = U_{PG}$ is a unitary operator and therefore, using the inverse Potapov-Ginzburg transformation, U is unitary. □

Theorem 4.9. *Let U be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then U is unitary if and only if*

(i) *U is closed;*
(ii) $\ker U = (\mathrm{dom}\, U)^{[\perp]_1}$*;*
(iii) $U(\mathrm{dom}\, U \cap \mathfrak{K}_1^+) = \big(U(\mathrm{dom}\, U \cap \mathfrak{K}_1^-)\big)^{[\perp]_2}$*.*

Proof. Necessity of the conditions (i)-(iii) follows from (3.1), (3.3) and Proposition 4.4. Conversely, if (iii) holds, then

$$(\operatorname{ran} U)^{[\perp]_2} \subseteq \big(U(\operatorname{dom} U \cap \mathfrak{K}_1^-)\big)^{[\perp]_2} = U(\operatorname{dom} U \cap \mathfrak{K}_1^+) \subseteq \operatorname{ran} U.$$

By Lemma 3.22 this result combined with the assumptions (i) and (ii) show that $\operatorname{mul} U = (\operatorname{ran} U)^{[\perp]_2}$. As a consequence of Proposition 3.8 $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ and $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ are closed and, hence, assumption (iii) combined with Proposition 2.4 implies that $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ and $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ are a maximal nonnegative and nonpositive subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, respectively. Hence the sufficiency of the conditions (i)-(iii) follows now from Lemma 4.8. □

Geometrically Theorem 4.9 says that closed isometric operators with dense domain are unitary precisely when they map essentially uniformly definite subspaces onto maximal definite subspaces. It can be seen as an abstract extension of [6, Proposition 3.6].

4.3. A quasi-block representation for unitary relations

Here a quasi-block representation for unitary relations is given which is based on the Weyl identity. That representation is an extension of the representation for standard unitary operators in [2, Ch. II: Theorem 5.10]. To obtain that representation two lemmas will be used.

Lemma 4.10. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then $\mathfrak{L} = \ker U + \operatorname{dom} U \cap \mathfrak{K}_1^+ + \operatorname{dom} U \cap \mathfrak{K}_1^-$ is a core for U, i.e., $\operatorname{clos}(U \restriction_{\mathfrak{L}}) = U$.*

Proof. Note that $U_r := U \restriction_{\mathfrak{L}}$ is an isometric relation which satisfies

$$U_r(\operatorname{dom} U_r \cap \mathfrak{K}_1^+) = \big(U_r(\operatorname{dom} U_r \cap \mathfrak{K}_1^-)\big)^{[\perp]_2}. \tag{4.8}$$

Because, by definition, $\operatorname{dom} U_r \cap \mathfrak{K}_1^\pm = \operatorname{dom} U \cap \mathfrak{K}_1^\pm$ and U satisfies (4.8) by Proposition 4.4. Since $\overline{\operatorname{dom}}\, U_r = \overline{\operatorname{dom}}\, U$, see (4.1), it follows from (3.3) that

$$\ker U_r = \ker U = (\operatorname{dom} U)^{[\perp]_1} = (\operatorname{dom} U_r)^{[\perp]_1}.$$

Consequently, $\operatorname{clos} U_r$ is a closed isometric relation satisfying the conditions of Theorem 4.9, i.e., U_r is unitary. Since $U_r \subseteq U$, this completes the proof. □

For the following statement recall that for an everywhere defined contraction K from the Hilbert space $\{\mathfrak{H}_1, (\cdot,\cdot)_1\}$ to the Hilbert space $\{\mathfrak{H}_2, (\cdot,\cdot)_2\}$

$$\ker(I - K^*K) = \{0\} \quad \text{if and only if} \quad \ker(I - KK^*) = \{0\}. \tag{4.9}$$

Lemma 4.11. *Let $\{\mathfrak{K}, [\cdot,\cdot]\}$ be a Kreĭn space with fundamental symmetry $\mathfrak{j}$ and let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be associated canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Moreover, let K be an everywhere defined contractive operator K from $\{\mathfrak{K}^+, [\cdot,\cdot]\}$ to $\{\mathfrak{K}^-, -[\cdot,\cdot]\}$ with $\ker(I - K^*K) = \{0\}$. Then U_K defined as*

$$U_K = \operatorname{clos}\left(\begin{pmatrix} I & K^* \\ K & I \end{pmatrix}\begin{pmatrix} (I-K^*K)^{-\frac{1}{2}} & 0 \\ 0 & (I-KK^*)^{-\frac{1}{2}} \end{pmatrix}\right)$$

is a unitary relation in $\{\mathfrak{K}, [\cdot,\cdot]\}$ *with* $\ker U_K = \{0\} = \operatorname{mul} U_K$ *and*

$$U_K(\operatorname{dom} U_K \cap \mathfrak{K}^+) = \{f^+ + Kf^+ : f^+ \in \mathfrak{K}^+\};$$
$$U_K(\operatorname{dom} U_K \cap \mathfrak{K}^-) = \{f^- + K^*f^- : f^- \in \mathfrak{K}^-\}.$$

Furthermore, $U_K = \mathfrak{j}U_K^{[*]}\mathfrak{j}$.

Proof. In this proof the following notation will be used

$$D_K = (I - K^*K)^{-\frac{1}{2}} \quad \text{and} \quad D_{K^*} = (I - KK^*)^{-\frac{1}{2}}. \tag{4.10}$$

Step 1: With respect to the decomposition $\mathfrak{K}^+ \times \mathfrak{K}^-$ of $\mathfrak{K}$, define S and T as

$$S = \begin{pmatrix} I & K^* \\ K & I \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} D_K & 0 \\ 0 & D_{K^*} \end{pmatrix}.$$

Then S is an everywhere defined operator and, hence, by Lemma 2.13

$$(ST)^{[*]} = T^{[*]}S^{[*]} = \begin{pmatrix} D_K & 0 \\ 0 & D_{K^*} \end{pmatrix} \begin{pmatrix} I & -K^* \\ -K & I \end{pmatrix}. \tag{4.11}$$

Consequently, $V := ST$ satisfies

$$\begin{aligned} V^{[*]}V &= \begin{pmatrix} D_K & 0 \\ 0 & D_{K^*} \end{pmatrix} \begin{pmatrix} I & -K^* \\ -K & I \end{pmatrix} \begin{pmatrix} I & K^* \\ K & I \end{pmatrix} \begin{pmatrix} D_K & 0 \\ 0 & D_{K^*} \end{pmatrix} \\ &= \begin{pmatrix} D_K & 0 \\ 0 & D_{K^*} \end{pmatrix} \begin{pmatrix} I - K^*K & 0 \\ 0 & I - KK^* \end{pmatrix} \begin{pmatrix} D_K & 0 \\ 0 & D_{K^*} \end{pmatrix} = I_{\operatorname{dom} V}. \end{aligned}$$

This shows that V is an isometric operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$. Moreover, the condition $\ker(I - K^*K) = \{0\}$ implies that V has dense domain, see (4.9) and (2.6). Consequently, (3.2) implies that $(\operatorname{dom} V)^{[\perp]_1} = \{0\} = \ker V$.

Moreover, evidently, $\operatorname{dom} V \cap \mathfrak{K}^+ = D_K$ and $\operatorname{dom} V \cap \mathfrak{K}^- = D_{K^*}$, and $\operatorname{ran}(I - K^*K)^{-\frac{1}{2}} = \mathfrak{K}^+$ and $\operatorname{ran}(I - KK^*)^{-\frac{1}{2}} = \mathfrak{K}_2^-$. Hence

$$V(\operatorname{dom} V \cap \mathfrak{K}^+) = \{f^+ + Kf^+ : f^+ \in \mathfrak{K}^+\};$$
$$V(\operatorname{dom} V \cap \mathfrak{K}^-) = \{f^- + K^*f^- : f^- \in \mathfrak{K}^-\}.$$

These equations show that $V(\operatorname{dom} V \cap \mathfrak{K}_2^+)$ and $V(\operatorname{dom} V \cap \mathfrak{K}_2^-)$ are a maximal nonnegative and a maximal nonpositive subspace, respectively, and that

$$V(\operatorname{dom} V \cap \mathfrak{K}_2^+) = (V(\operatorname{dom} V \cap \mathfrak{K}_2^-))^{[\perp]_2},$$

see Proposition 2.4 and the discussion following that statement. Consequently, $U_K = \operatorname{clos}(V)$ is unitary by Theorem 4.9 and, since $V(\operatorname{dom} V \cap \mathfrak{K}_2^+) \cap V(\operatorname{dom} V \cap \mathfrak{K}_2^-) = \{0\}$, it follows from (4.6) that $\operatorname{mul} U_K = \{0\}$.

Step 2: Next observe that $(I - KK^*)K = K(I - K^*K)$ and that $(I - K^*K)K^* = K^*(I - KK^*)$. Hence,

$$(I - KK^*)^{\frac{1}{2}}K = K(I - K^*K)^{\frac{1}{2}} \quad \text{and} \quad (I - K^*K)^{\frac{1}{2}}K^* = K^*(I - KK^*)^{\frac{1}{2}},$$

and, consequently,

$$KD_K = D_{K^*}KI_{\operatorname{dom} D_K} \quad \text{and} \quad K^*D_{K^*} = D_KK^*I_{\operatorname{dom} D_{K^*}}.$$

Now define $\mathfrak{L} = \operatorname{dom} D_K \times \operatorname{dom} D_{K^*}$, then using the above equalities

$$V = \begin{pmatrix} D_K & K^* D_{K^*} \\ K D_K & D_{K^*} \end{pmatrix} = \begin{pmatrix} D_K & 0 \\ 0 & D_{K^*} \end{pmatrix} \begin{pmatrix} I & K^* \\ K & I \end{pmatrix} \restriction_{\mathfrak{L}} = \mathfrak{j} U_K^{[*]} \mathfrak{j} \restriction_{\mathfrak{L}} .$$

Since $\mathfrak{j} U_K^{[*]} \mathfrak{j}$ is a unitary operator, see Lemma 3.9, and, hence, closed, the above equation shows that $U_K = \operatorname{clos}(V) \subseteq \mathfrak{j} U_K^{[*]} \mathfrak{j}$. In fact, since U_K is also a unitary operator, the preceding inclusion yields that $U_K = \mathfrak{j} U_K^{[*]} \mathfrak{j}$. □

Theorem 4.12. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\ker U = \{0\} = \operatorname{mul} U$ and let $\mathfrak{K}_2^+[+]\mathfrak{K}_2^-$ be a canonical decompositions of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Then there exists a standard unitary operator U_t from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and an everywhere defined contractive operator K from $\{\mathfrak{K}_2^+, [\cdot,\cdot]_2\}$ to $\{\mathfrak{K}_2^-, -[\cdot,\cdot]_2\}$ with $\ker(I - K^*K) = \{0\}$ such that with respect to the decomposition $\mathfrak{K}_2^+ \times \mathfrak{K}_2^-$ of $\mathfrak{K}_2$*

$$UU_t^{-1} = \operatorname{clos}\left(\begin{pmatrix} I & K^* \\ K & I \end{pmatrix} \begin{pmatrix} (I - K^*K)^{-\frac{1}{2}} & 0 \\ 0 & (I - KK^*)^{-\frac{1}{2}} \end{pmatrix} \right).$$

Proof. Let U be a unitary operator with $\ker U = \{0\}$ and let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$. Then by Proposition 4.4 and 2.4 there exists a (unique) contractive operator K from $\{\mathfrak{K}_2^+, [\cdot,\cdot]_2\}$ to $\{\mathfrak{K}_2^-, -[\cdot,\cdot]_2\}$ such that

$$U(\operatorname{dom} U \cap \mathfrak{K}_1^+) = \{f_2^+ + K f_2^+ : f_2^+ \in \mathfrak{K}_2^+\};$$
$$U(\operatorname{dom} U \cap \mathfrak{K}_1^-) = \{f_2^- + K^* f_2^- : f_2^- \in \mathfrak{K}_2^-\}.$$

Here $\ker(I - K^*K) = \{0\}$, because by (4.6) $\{0\} = U(\operatorname{dom} U \cap \mathfrak{K}_1^+) \cap U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$. With this K, let U_K be the unitary operator in $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with $\operatorname{mul} U_K = \{0\}$ given by Lemma 4.11 and define $\mathfrak{L}$ to be $\operatorname{dom} U \cap \mathfrak{K}_1^+ + \operatorname{dom} U \cap \mathfrak{K}_1^-$. Then ran $(U \restriction_{\mathfrak{L}}) \subseteq \operatorname{ran} U_K$ and, hence, $U_K^{-1} U \restriction_{\mathfrak{L}}$ is an isometric operator, see Lemma 3.9, which satisfies

$$U_K^{-1}(U \restriction_{\mathfrak{L}})(\operatorname{dom} U \cap \mathfrak{K}_1^+) = \operatorname{dom}(I - K^*K)^{-\frac{1}{2}} \times \{0\};$$
$$U_K^{-1}(U \restriction_{\mathfrak{L}})(\operatorname{dom} U \cap \mathfrak{K}_1^-) = \{0\} \times \operatorname{dom}(I - KK^*)^{-\frac{1}{2}}.$$

Now observe that $\operatorname{dom} U \cap \mathfrak{K}_1^+$ and $\operatorname{dom}(I - K^*K)^{-\frac{1}{2}}$ are dense in the Hilbert spaces $\{\mathfrak{K}_1^+, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2^+, [\cdot,\cdot]_2\}$, respectively, and $\operatorname{dom} U \cap \mathfrak{K}_1^-$ and $\operatorname{dom}(I - KK^*)^{-\frac{1}{2}}$ are dense in the Hilbert spaces $\{\mathfrak{K}_1^-, -[\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2^-, -[\cdot,\cdot]_2\}$, respectively, see Proposition 4.1. Hence there exists standard unitary operators U_t^+ and U_t^- from $\{\mathfrak{K}_1^+, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2^+, [\cdot,\cdot]_2\}$ and from $\{\mathfrak{K}_1^-, -[\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2^-, -[\cdot,\cdot]_2\}$, respectively, such that with respect to the decompositions $\mathfrak{K}_1^+ \times \mathfrak{K}_1^-$ and $\mathfrak{K}_2^+ \times \mathfrak{K}_2^-$ of $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively, $U_t := \operatorname{clos}(U_K^{-1} U \restriction_{\mathfrak{L}}) = U_t^+ \times U_t^-$, i.e., $U \subseteq U_K U_t$. Since U_t is a standard unitary operator, $U_K U_t$ is unitary by Lemma 3.9 and, hence, the inclusion $U \subseteq U_K U_t$ yields that $U = U_K U_t$. □

In particular, Theorem 4.12 combined with Lemma 4.11 shows that if U is a unitary operator in $\{\mathfrak{K}, [\cdot,\cdot]\}$ without kernel and $\mathfrak{j}$ is a fundamental symmetry of $\{\mathfrak{K}, [\cdot,\cdot]\}$, then there exists a standard unitary operator U_t in $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that $(UU_t^{-1}) = (UU_t^{-1})^{-[*]} = (UU_t^{-1})^*$ (where the $*$-adjoint is with respect to the inner product $\{\mathfrak{K}, [\mathfrak{j}\cdot,\cdot]\}$). I.e., up to standard unitary transformation unitary operators without kernel are "unbounded fundamental symmetries".

Remark 4.13. Theorem 4.12, see also the discussion below, shows that unitary relations can be classified by the nature of the spectrum of an associated contraction K at 1.

The condition $\ker(I - K^*K) = \{0\}$ in Theorem 4.12 can be dropped by allowing U to have a kernel and a multivalued part. In that case the representation for unitary operators in Theorem 4.12 remains valid for arbitrary everywhere defined contractions K if the operator

$$\begin{pmatrix} (I - K^*K)^{-\frac{1}{2}} & 0 \\ 0 & (I - KK^*)^{-\frac{1}{2}} \end{pmatrix}$$

occurring in that representation is interpreted as having the following graph

$$\operatorname{gr}\left(\begin{pmatrix} (I - K^*K)^{(-\frac{1}{2})} & 0 \\ 0 & (I - KK^*)^{(-\frac{1}{2})} \end{pmatrix}\right) + \left\{\begin{pmatrix} f \\ U_K f \end{pmatrix} : f \in \mathcal{K}\right\} \times \begin{pmatrix} \mathcal{K} \\ \widetilde{\mathcal{K}} \end{pmatrix},$$

where $\mathcal{K} = \ker(I - K^*K)$ and $\widetilde{\mathcal{K}} = \ker(I - KK^*)$, $(I - K^*K)^{(-\frac{1}{2})}$ and $(I - KK^*)^{(-\frac{1}{2})}$ denote the Moore-Penrose inverses of the respective operators, and U_K is a unitary operator from $\{\mathfrak{K}_2^+, [\cdot,\cdot]_2\}$ to $\{\mathfrak{K}_2^-, -[\cdot,\cdot]_2\}$ such that $U_K(\mathcal{K}) = \widetilde{\mathcal{K}}$. In particular,

$$\begin{aligned} \operatorname{mul} U &= \{f + Kf : f \in \mathcal{K}\} = \{f + K^*f : f \in \widetilde{\mathcal{K}}\}; \\ \ker U &= U_t^{-1}(\{f + U_K f : f \in \mathcal{K}\}). \end{aligned}$$

The above discussion shows how to obtain representations for a unitary relation U from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ when $k_1^{\pm} = k_2^{\pm}$ and $\dim P_1^{\pm} \ker U = \dim P_2^{\pm} \operatorname{mul} U$. Representation of unitary relations for which these equalities do not hold can be obtained by composing them (on the domain side) with unitary operators with closed domain as furnished by Lemma 3.10.

Theorem 4.12 together with the above discussion can also be interpreted as a realization result for maximal nonnegative and nonpositive subspaces (or, equivalently, for maximal dissipative or accumulative relations). Therefore observe that if $\mathfrak{L}$ is a closed neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ with fundamental symmetry $\mathfrak{j}$, then $\mathfrak{L}$ is a hyper-maximal neutral subspace of the Kreĭn space $\{\mathfrak{L} + \mathfrak{j}\mathfrak{L}, [\cdot,\cdot]\}$ and $\mathfrak{L} \times \mathfrak{L}$ is a unitary relation in $\{\mathfrak{L} + \mathfrak{j}\mathfrak{L}, [\cdot,\cdot]\}$, see e.g. Lemma 3.25.

Theorem 4.14. *Let $\mathfrak{M}^+$ or $\mathfrak{M}^-$ be a maximal nonnegative or nonpositive subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$, respectively, and let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then there exists a unitary relation U in $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that*

$$\mathfrak{M}^+ = U(\operatorname{dom} U \cap \mathfrak{K}^+) \quad \text{or} \quad \mathfrak{M}^- = U(\operatorname{dom} U \cap \mathfrak{K}^-),$$

respectively. If U_1 *and* U_2 *are two unitary relations satisfying the above conditions, then* $\operatorname{clos}(U_2^{-1}U_1)$ *is a unitary relation in* $\{\mathfrak{K}, [\cdot,\cdot]\}$ *with closed domain.*

Proof. Let $\mathfrak{j}$ be the fundamental symmetry associated with the canonical decomposition $\mathfrak{K}^+[+]\mathfrak{K}^-$ and let $\mathfrak{M}_0$ be defined as $\mathfrak{M}^+\cap(\mathfrak{M}^+)^{[\perp]}$. Clearly, $\mathfrak{M}_0$ is a closed neutral subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ and by means of this neutral subspace define $\mathfrak{K}_0 = \mathfrak{M}_0 + \mathfrak{j}\mathfrak{M}_0$ and $\mathfrak{K}_r = \mathfrak{K} \cap \mathfrak{K}_0^{[\perp]}$. Then $\{\mathfrak{K}_0, [\cdot,\cdot]\}$ and $\{\mathfrak{K}_r, [\cdot,\cdot]\}$ are Kreĭn spaces and $\mathfrak{K}_0^+[+]\mathfrak{K}_0^- = (\mathfrak{K}^+\cap\mathfrak{K}_0)[+](\mathfrak{K}^-\cap\mathfrak{K}_0)$ and $\mathfrak{K}_r^+[+]\mathfrak{K}_r^- = (\mathfrak{K}^+\cap\mathfrak{K}_r)[+](\mathfrak{K}^-\cap\mathfrak{K}_r)$ are canonical decompositions for these spaces.

Now let K_r be the angular operator of $\mathfrak{M}^+\cap\mathfrak{K}_r$, i.e.,

$$\mathfrak{M}^+\cap\mathfrak{K}_r = \{f_r^+ + K_r f_r^+ : f_r^+ \in \mathfrak{K}_r^+\}.$$

Since $\mathfrak{M}^+\cap(\mathfrak{M}^+)^{[\perp]}\cap\mathfrak{K}_r = \{0\}$, it follows that $\ker(I - K_r^*K_r) = \{0\}$. Hence, U_{K_r} is a unitary operator in $\{\mathfrak{K}_r, [\cdot,\cdot]\}$ such that $U_{K_r}(\operatorname{dom} U_{K_r}\cap\mathfrak{K}_r^+) = \mathfrak{M}^+\cap\mathfrak{K}_r$, see Lemma 4.11. Since U_0 defined via $\operatorname{gr}(U_0) = \mathfrak{M}_0\times\mathfrak{M}_0$ is a unitary relation in $\{\mathfrak{K}_0, [\cdot,\cdot]\}$, Lemma 3.12 shows that U defined via $\operatorname{gr}(U) = \operatorname{gr}(U_0) + \operatorname{gr}(U_{K_r})$ is a unitary relation in $\{\mathfrak{K}, [\cdot,\cdot]\}$, which clearly satisfies $U(\operatorname{dom} U\cap\mathfrak{K}^+) = \mathfrak{M}^+$. Similar arguments can be used to show the existence of a unitary relation U such that $U(\operatorname{dom} U\cap\mathfrak{K}^-) = \mathfrak{M}^-$.

Finally, let U_1 and U_2 be unitary relations such that $U_1(\operatorname{dom} U_1\cap\mathfrak{K}^+) = \mathfrak{M}^+ = U_2(\operatorname{dom} U_2\cap\mathfrak{K}^+)$, then $U_1(\operatorname{dom} U_1\cap\mathfrak{K}^-) = U_2(\operatorname{dom} U_2\cap\mathfrak{K}^-)$, see Proposition 4.4. Hence, $\operatorname{clos}(U_2^{-1}U_1)$ is an isometric relation which maps $\operatorname{clos}(\operatorname{dom} U_1\cap\mathfrak{K}^\pm)$ onto $\operatorname{clos}(\operatorname{dom} U_2\cap\mathfrak{K}^\pm) + \ker U_2$. In particular, since $(\operatorname{clos}(\operatorname{dom} U_2\cap\mathfrak{K}^+) + \ker U_2)^{[\perp]_2} = \operatorname{clos}(\operatorname{dom} U_2\cap\mathfrak{K}^-)$ by (4.1), $\operatorname{clos}(U_2^{-1}U_1)$ satisfies the Weyl identity. Furthermore, since $\operatorname{clos}(\ker U_1 + \operatorname{dom} U_1\cap\mathfrak{K}^+ + \operatorname{dom} U_1\cap\mathfrak{K}^-) = \overline{\operatorname{dom}}\, U_1$ by (4.1), it follows that

$$\ker U_2^{-1}U_1 = \ker U_1 = (\operatorname{dom} U_1)^{[\perp]_1} = (\operatorname{dom} U_2^{-1}U_1)^{[\perp]_1}.$$

Consequently, Theorem 4.9 implies that $\operatorname{clos}(U_2^{-1}U_1)$ is a unitary operator which has a closed domain by (the proof of) Proposition 3.18. □

4.4. Intermediate extensions of unitary relations

A unitary relation $\widetilde{U}$ is called *an intermediate extension* of the unitary relation U if $\ker U \subseteq \ker\widetilde{U}$ and the action of $\widetilde{U}$ coincides with the action of U on $\operatorname{dom}\widetilde{U}\ominus\ker\widetilde{U}$. Here it will be shown, using the Weyl identity, that intermediate extensions are closely related to the splitting of unitary relations into two unitary relations.

Therefore first observe the following simple result.

Proposition 4.15. *Let* U *be a unitary relation from* $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ *to* $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ *and let* $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ *be a canonical decomposition of* $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ *with associated projections* P_i^+ *and* P_i^-, $i = 1,2$. *Moreover, let* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$ *be closed subspaces of* $\ker U$ *and* $\operatorname{mul} U$, *respectively, and assume that there exist subspaces* $\mathfrak{M}^+\subseteq \operatorname{dom} U\cap\mathfrak{K}_1^+$ *and* $\mathfrak{M}^-\subseteq\operatorname{dom} U\cap\mathfrak{K}_1^-$ *such that*

(i) $P_2^+U(\mathfrak{M}^+)$ *and* $P_2^-U(\mathfrak{M}^-)$ *are closed;*
(ii) $P_2^-U(\mathfrak{M}^+)\subseteq P_2^-U(\mathfrak{M}^-)$ *and* $P_2^+U(\mathfrak{M}^-)\subseteq P_2^+U(\mathfrak{M}^+)$.

Then $\widetilde{U}$ defined by

$$\widetilde{U} = \{\{f+g, f'\} \in U : f \in \operatorname{clos}(\mathfrak{M}^+ + \mathfrak{M}^-),\ f' \in \mathfrak{L}_2^{\perp_2},\ g \in \mathfrak{L}_1\}$$

is a unitary relation from the Kreĭn space $\{\widetilde{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$ to the Kreĭn space $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$, where $\widetilde{\mathfrak{K}}_1 = \operatorname{clos}(\mathfrak{M}^+ + \mathfrak{M}^-) + (\mathfrak{L}_1 + \mathfrak{j}_1\mathfrak{L}_1)$ and $\widetilde{\mathfrak{K}}_2 = (P_2^+ U(\mathfrak{M}^+) + P_2^- U(\mathfrak{M}^-)) \cap (\mathfrak{L}_2 + \mathfrak{j}_2\mathfrak{L}_2)^{[\perp]_2}$.

Proof. Note first that by definition and the assumptions (i) and (ii) $\{\widetilde{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$ and $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$ are Kreĭn spaces and that $\widetilde{U}$ is closed being the intersection of U with $\widetilde{\mathfrak{K}}_1 \times \widetilde{\mathfrak{K}}_2$. Moreover, since $\mathfrak{M}^+ + \mathfrak{M}^- \in (\ker U)^{[\perp]_1}$, it follows that $\ker \widetilde{U} = (\operatorname{dom} \widetilde{U})^{[\perp]_1} \cap \widetilde{\mathfrak{K}}_1$. Furthermore, the assumptions (i) and (ii) imply that $U(\mathfrak{M}^+)$ and $U(\mathfrak{M}^-)$ are a maximal nonnegative and maximal nonpositive subspace of the Kreĭn space $P_2^+ U(\mathfrak{M}^+) + P_2^- U(\mathfrak{M}^-))$. Since $\mathfrak{L}_2 \subseteq \operatorname{mul} U \subseteq U(\mathfrak{M}^+) \cap U(\mathfrak{M}^-)$, it follows that $\widetilde{U}(\mathfrak{M}^+) = U(\mathfrak{M}^+) \cap \widetilde{\mathfrak{K}}_2$ and $\widetilde{U}(\mathfrak{M}^-) = U(\mathfrak{M}^-) \cap \widetilde{\mathfrak{K}}_2$ are a maximal nonnegative and maximal nonpositive subspace of $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$. Since $U(\mathfrak{M}^+) \subseteq (U(\mathfrak{M}^-))^{[\perp]_2}$, see Proposition 4.4, the maximality of $\widetilde{U}(\mathfrak{M}^+)$ and $\widetilde{U}(\mathfrak{M}^-)$ now implies that $\widetilde{U}(\mathfrak{M}^+) = \widetilde{U}(\mathfrak{M}^-)^{[\perp]_2} \cap \widetilde{\mathfrak{K}}_2$. Hence, Theorem 4.9 yields that $\widetilde{U}$ is a unitary relation. □

The splitting result is now given; it partially strengthens Lemma 3.12.

Theorem 4.16. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, let $\{\widetilde{\mathfrak{K}}_i, [\cdot,\cdot]_i\}[+]\{\widehat{\mathfrak{K}}_i, [\cdot,\cdot]_i\}$ be an orthogonal decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ into two Kreĭn spaces, for $i = 1, 2$, and define $\widetilde{U}$ and $\widehat{U}$ via*

$$\operatorname{gr}(\widetilde{U}) = \operatorname{gr}(U) \cap (\widetilde{\mathfrak{K}}_1 \times \widetilde{\mathfrak{K}}_2) \quad \text{and} \quad \operatorname{gr}(\widehat{U}) = \operatorname{gr}(U) \cap (\widehat{\mathfrak{K}}_1 \times \widehat{\mathfrak{K}}_2).$$

Then $\widetilde{U}$ is unitary if and only if $\widehat{U}$ is unitary.

Proof. W.l.o.g. assume that $\ker U = \{0\} = \operatorname{mul} U$. Furthermore, let $\widetilde{\mathfrak{K}}_i^+[+]\widetilde{\mathfrak{K}}_i^-$ and $\widehat{\mathfrak{K}}_i^+[+]\widehat{\mathfrak{K}}_i^-$ be canonical decomposition of $\{\widetilde{\mathfrak{K}}_i, [\cdot,\cdot]_i\}$ and $\{\widetilde{\mathfrak{K}}_i, [\cdot,\cdot]_i\}$, respectively, for $i = 1, 2$. Denote the associated canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ by $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ and let P_i^+ and P_i^- denote the associated projections, $i = 1, 2$.

Clearly, to prove the equivalence it suffices to prove only one implication. Hence assume that $\widetilde{U}$ is unitary. Define $\widehat{U}_r^+$ as

$$\widehat{U}_r^+ = \{\{f, f'\} \in U : f \in \operatorname{dom} U \cap \mathfrak{K}_1^+ \text{ and } P_2^+ f' \in \widehat{\mathfrak{K}}_2^+\},$$

then $P_2^+ \operatorname{ran} \widehat{U}_r^+ = \widehat{\mathfrak{K}}_2^+$, see Proposition 4.4. If $\{f, f'\} \in \widehat{U}_r^+$ and $\{g, g'\} \in \widetilde{U}$ where $g \in \operatorname{dom} \widetilde{U} \cap \mathfrak{K}_1^-$, then $[f, g]_1 = 0$ and $[P_2^+ f', P_2^+ g']_2 = 0$. Therefore

$$0 = [f, g]_1 = [f', g']_2 = [P_2^- f', P_2^- g']_2.$$

Since $\widetilde{U}$ is unitary, $P_2^- U(\operatorname{dom} \widetilde{U} \cap \mathfrak{K}_2^-) = \widetilde{\mathfrak{K}}_2^-$ and, hence, the previous equality implies that $P_2^- f' \in (\widetilde{\mathfrak{K}}_2^-)^{[\perp]_2} \cap \mathfrak{K}_2^- = \widehat{\mathfrak{K}}_2^-$, i.e., $\operatorname{ran} \widehat{U}_r^+ \subseteq \widehat{\mathfrak{K}}_2$. Now if $\{g, g'\} \in \widetilde{U}$, where $g \in \operatorname{dom} \widetilde{U} \cap \mathfrak{K}_1^+$, then, since $\operatorname{ran} \widehat{U}_r^+ \subseteq \widehat{\mathfrak{K}}_2 = \widetilde{\mathfrak{K}}_2^{[\perp]_2}$,

$$[f, g]_1 = [f', g']_2 = 0.$$

This shows that $f \in (\operatorname{dom} \widetilde{U} \cap \mathfrak{K}_1^+)^{[\perp]_1} = (\widetilde{\mathfrak{K}}_1^+)^{[\perp]_1} = \widehat{\mathfrak{K}}_1^+$, see (4.2).

The above arguments show that $\widehat{U}_r^+ \subseteq \widehat{U}$ and, hence, $P_2^+ U(\operatorname{dom} \widehat{U} \cap \mathfrak{K}_1^+) = \widehat{\mathfrak{K}}_2^+$ and $P_2^- U(\operatorname{dom} \widehat{U} \cap \mathfrak{K}_1^+) \subseteq \widehat{\mathfrak{K}}_2^-$. By similar arguments $P_2^- U(\operatorname{dom} \widehat{U} \cap \mathfrak{K}_1^-) = \widehat{\mathfrak{K}}_2^-$ and $P_2^+ U(\operatorname{dom} \widehat{U} \cap \mathfrak{K}_1^-) \subseteq \widehat{\mathfrak{K}}_2^+$. Therefore $\widehat{U}$ is unitary by Proposition 4.15. □

Next it is indicated how the splitting of unitary relations into two unitary relations can be used to obtain intermediate extension. Therefore let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Moreover, $\widetilde{U}$ and $\widehat{U}$ be as in Theorem 4.16 and assume that they are unitary. Then by Proposition 5.4 below there exists hyper-maximal semi-definite subspaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$ of $\{\widehat{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$ and $\{\widehat{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$ such that $\ker \widehat{U} \subseteq \mathfrak{L}_1 \subseteq \operatorname{dom} \widehat{U}$ and $\operatorname{mul} \widehat{U} \subseteq \mathfrak{L}_2 \subseteq \operatorname{ran} \widehat{U}$, respectively, and $\widehat{U}(\mathfrak{L}_1 \cap \mathfrak{j}_1 \mathfrak{L}_1) = (\mathfrak{L}_2 \cap \mathfrak{j}_2 \mathfrak{L}_2) + \operatorname{mul} \widehat{U}$. Hence

$$\{\{f+g+h, f'+g'+k\} : \{f,f'\} \in \widetilde{U},\ \{g,g'\} \in \widehat{U},\\ g \in \mathfrak{L}_1 \cap \mathfrak{j}_1 \mathfrak{L}_1,\ h \in \mathfrak{L}_1^{[\perp]_1},\ k \in \mathfrak{L}_2^{[\perp]_2}\}$$

is an intermediate extension of U. This shows how a splitting of a unitary relations gives rises to intermediate extensions; the converse is obvious.

5. Structure of unitary relations

In this section the structure of unitary relations is investigated by means of a graph decomposition. This decomposition is thereafter used to obtain different necessary and sufficiency conditions for isometric relations to be unitary. In particular, a block decomposition for unitary operators without kernel is obtained.

5.1. A graph decomposition of unitary relations

The graph decomposition of unitary relations in Theorem 5.1 below is the main result here; it is inspired by [4, Lemma 4.4]. The difference is that here the graph of a unitary relation is decomposed whereas in [4] only the domain of a unitary relation was decomposed.

Theorem 5.1. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ associated to the fundamental symmetry $\mathfrak{j}_i$, for $i = 1,2$. Define U_+ and U_- as*

$$\operatorname{gr}(U_+) = \operatorname{gr}(U) \cap (\mathfrak{K}_1^+ \times \mathfrak{K}_2^+) \quad \text{and} \quad \operatorname{gr}(U_-) = \operatorname{gr}(U) \cap (\mathfrak{K}_1^- \times \mathfrak{K}_2^-).$$

Moreover, with $\widetilde{\mathfrak{K}}_1 := \mathfrak{K}_1 \cap (\ker U + \mathfrak{j}_1 \ker U + \operatorname{dom} U_+ + \operatorname{dom} U_-)^{[\perp]_1}$ and $\widetilde{\mathfrak{K}}_2 := \mathfrak{K}_2 \cap (\operatorname{mul} U + \mathfrak{j}_2 \operatorname{mul} U + \operatorname{ran} U_+ + \operatorname{ran} U_-)^{[\perp]_2}$, define U_o as

$$\operatorname{gr}(U_o) = \operatorname{gr}(U) \cap (\widetilde{\mathfrak{K}}_1 \times \widetilde{\mathfrak{K}}_2).$$

Then U has the graph decomposition

$$\operatorname{gr}(U) = (\ker U \times \operatorname{mul} U) \dot{+} \operatorname{gr}(U_+) \dot{+} \operatorname{gr}(U_-) \dot{+} \operatorname{gr}(U_o),$$

where

(i) U_+ *and* U_- *are (Hilbert space) unitary operators from* $\{\mathrm{dom}\, U_+, [\cdot,\cdot]_1\}$ *and* $\{\mathrm{dom}\, U_-, -[\cdot,\cdot]_1\}$ *to* $\{\mathrm{ran}\, U_+, [\cdot,\cdot]_2\}$ *and* $\{\mathrm{ran}\, U_-, -[\cdot,\cdot]_2\}$, *respectively. In particular,* $\mathrm{gr}(U_+)\dot{+}\mathrm{gr}(U_-)$ *is the graph of the standard unitary operator* U_c *defined by*

$$U_c = \{\{f,f'\} \in U : [\mathfrak{j}_1 f, g,]_1 = [\mathfrak{j}_2 f', g']_2, \quad \forall \{g,g'\} \in U\}$$

from the Kreĭn space $\{\mathrm{dom}\, U_c, [\cdot,\cdot]_1\}$ *to the Kreĭn space* $\{\mathrm{ran}\, U_c, [\cdot,\cdot]_2\}$;

(ii) U_o *is a unitary operator from the Kreĭn space* $\{\widetilde{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$ *to the Kreĭn space* $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$ *with dense domain and dense range. Moreover, there exist hyper-maximal neutral subspaces* $\mathfrak{L}_d$ *and* $\mathfrak{L}_r$ *of* $\{\widetilde{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$ *and* $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$, *respectively, such that*

$$\mathrm{dom}\, U_o = \mathfrak{L}_d \oplus_1 (\mathfrak{j}_1\mathfrak{L}_d \cap \mathrm{dom}\, U_o) \quad \textit{and} \quad \mathrm{ran}\, U_o = \mathfrak{L}_r \oplus_2 (\mathfrak{j}_2\mathfrak{L}_r \cap \mathrm{ran}\, U_o)$$

$$U_o(\mathfrak{L}_d) = \mathfrak{j}_2\mathfrak{L}_r \cap \mathrm{ran}\, U_o \quad \textit{and} \quad U_o(\mathfrak{j}_1\mathfrak{L}_d \cap \mathrm{dom}\, U_o) = \mathfrak{L}_r.$$

In particular, $\widetilde{k}_1^+ = \widetilde{k}_1^- = \widetilde{k}_2^+ = \widetilde{k}_2^-$.

Proof. Note first that the stated graph decomposition of U is a consequence of (i). Secondly, note that $\ker U \times \mathrm{mul}\, U$ is the graph of a unitary relation from $\{\ker U + \mathfrak{j}_1\ker U, [\cdot,\cdot]_1\}$ to $\{\mathrm{mul}\, U + \mathfrak{j}_2\mathrm{mul}\, U, [\cdot,\cdot]_2\}$, see the discussion preceding Theorem 4.14.

(i): The first part of the statement is immediate by Proposition 3.8. Furthermore, by Proposition 3.2 it follows that $\{f,f'\} \in U_c$ if and only if $\{\mathfrak{j}_1 f, \mathfrak{j}_2 f'\} \in U$. Consequently, $\{f,f'\} \in U_c$ if and only if $\{\mathfrak{j}_1 f, \mathfrak{j}_2 f'\} \in U_c$. By means of this observation the second part of (i) is clear. (Note that $\mathrm{gr}(U_c) = \mathrm{gr}(U) \cap \mathrm{gr}(U^*)$.)

(ii): Since $\ker U \times \mathrm{mul}\, U$, U_+ and U_- are unitary relations, Lemma 3.12 implies that U_o is a unitary relation. In fact, it is evident that U_o and also U_o^{-1} are unitary operators. The final dimension result follows directly from the existence of the indicated $\mathfrak{L}_d$ and $\mathfrak{L}_r$, cf. Lemma 3.24. The existence of these sets and their indicated properties will be proven next.

Step 1: Let $A := \mathfrak{j}_1 U_o^{-1}\mathfrak{j}_2 U_o$, then $A = U_o^* U_o$ where U_o^* is the adjoint of U_o as an operator from $\{\mathfrak{K}_1, [\mathfrak{j}_1\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\mathfrak{j}_2\cdot,\cdot]_2\}$. Since U_o is a closed operator, A is nonnegative sefadjoint operator in $\{\mathfrak{K}_1, [\mathfrak{j}_1\cdot,\cdot]_1\}$. Moreover, a direct calculation shows that $A^{-1} = \mathfrak{j}_1 A \mathfrak{j}_1$.

Step 2: Define the operators V_k and V_g as

$$V_k = \{\{f, U_o f\} \in U_o : [\mathfrak{j}_1 f, g]_1 \leq [\mathfrak{j}_2 U_o f, U_o g]_2, \quad \forall g \in \mathrm{dom}\, U_o\};$$
$$V_g = \{\{f, U_o f\} \in U_o : [\mathfrak{j}_1 f, g]_1 \geq [\mathfrak{j}_2 U_o f, U_o g]_2, \quad \forall g \in \mathrm{dom}\, U_o\}.$$

Then $V_k \cap V_g = \{0\}$, see the definition of U_c in (i), and being restrictions of the unitary operator U_o, V_k and V_g are isometric operators. Moreover, by their definition and the closedness of U_o, it follows that V_k and V_g are closed operators.

Next let $\mathfrak{L} := \mathrm{dom}\,(U_o^{-1}\mathfrak{j}_2 U_o)$, then

$$U_o(\mathfrak{L}) = \mathrm{ran}\, U_o \cap \mathrm{dom}\,(U_o^{-1}\mathfrak{j}_2) = \mathrm{ran}\, U_o \cap \mathfrak{j}_2\mathrm{ran}\, U_o \supseteq \mathrm{ran}\, U_o \cap \widetilde{\mathfrak{K}}_2^+ + \mathrm{ran}\, U_o \cap \widetilde{\mathfrak{K}}_2^-.$$

Hence, Lemma 4.10 implies that $U_o(\mathfrak{L})$ is a core for U_o^{-1}, i.e. $\mathfrak{L}$ is a core for U_o. Now, with $A = \mathfrak{j}_1 U_o^{-1} \mathfrak{j}_2 U_o$ as in Step 1, one has that

$$\operatorname{dom} V_k \cap \mathfrak{L} = \{f \in \operatorname{dom} A : [\mathfrak{j}_1 f, g]_1 \leq [\mathfrak{j}_1 A f, g]_1, \ \forall g \in \operatorname{dom} U_o\};$$
$$\operatorname{dom} V_g \cap \mathfrak{L} = \{f \in \operatorname{dom} A : [\mathfrak{j}_1 f, g]_1 \geq [\mathfrak{j}_1 A f, g]_1, \ \forall g \in \operatorname{dom} U_o\}.$$

Step 3: Now using the spectral family of A (see Step 1) yields that

$$\mathfrak{L} = \operatorname{dom} V_k \cap \mathfrak{L} \oplus_1 \operatorname{dom} V_g \cap \mathfrak{L}.$$

Consequently, by definition of V_k and V_g

$$U_o(\mathfrak{L}) = \operatorname{ran} V_k \cap U_o(\mathfrak{L}) \oplus_2 \operatorname{ran} V_g \cap U_o(\mathfrak{L}).$$

Hence, using the fact that $\mathfrak{L}$ is a core for U_o and that V_k and V_g are closed isometric operators, one has that

$$\operatorname{gr}(U_o) = \operatorname{clos} \operatorname{gr}(U_o \upharpoonright_{\mathfrak{L}}) = \operatorname{clos} \operatorname{gr}(V_k \upharpoonright_{\mathfrak{L}}) + \operatorname{clos} \operatorname{gr}(V_g \upharpoonright_{\mathfrak{L}}) = \operatorname{gr}(V_k) + \operatorname{gr}(V_g)$$

and, consequently,

$$\operatorname{dom} V_k \oplus_1 \operatorname{dom} V_g = \operatorname{dom} U_o \quad \text{and} \quad \operatorname{ran} V_k \oplus_2 \operatorname{ran} V_g = \operatorname{ran} U_o. \tag{5.1}$$

Step 4: Next observe that the equality $A^{-1} = \mathfrak{j}_1 A \mathfrak{j}_1$ (see Step 1) implies that

$$\operatorname{clos}(\operatorname{dom} V_k \cap \mathfrak{L}) = \mathfrak{j}_1 \operatorname{clos}(\operatorname{dom} V_g \cap \mathfrak{L}). \tag{5.2}$$

This equation together with (5.1) shows that $\operatorname{dom} V_k$ and $\operatorname{dom} V_g$ are neutral subspaces of $\{\widetilde{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$ and, hence, the definition of V_k and V_g imply that $\operatorname{ran} V_k$ and $\operatorname{ran} V_g$ are neutral subspaces of $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$. Since V_g is by definition a closed bounded operator from $\{\widetilde{\mathfrak{K}}_1, [\mathfrak{j}_1\cdot,\cdot]_1\}$ to $\{\operatorname{clos}(\operatorname{ran} V_g), [\mathfrak{j}_2\cdot,\cdot]_2\}$, $\mathfrak{L}_d := \operatorname{dom} V_g$ is a closed neutral subspace of $\{\widetilde{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$. Hence, (5.1), (5.2) and the fact that $\ker U_o = \{0\}$ imply that

$$\widetilde{\mathfrak{K}}_1 = \overline{\operatorname{dom}}\, U_o = \operatorname{dom} V_k \oplus_1 \operatorname{clos}(\operatorname{dom} V_g) = \mathfrak{L}_d \oplus \mathfrak{j}_1 \mathfrak{L}_d.$$

This shows that $\mathfrak{L}_d$ is a hyper-maximal neutral subspace of $\{\widetilde{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$. A similar argument shows that $\mathfrak{L}_r := \operatorname{ran} V_k$ is a hyper-maximal neutral subspace of $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$. Finally, $\mathfrak{L}_d$ and $\mathfrak{L}_r$ satisfy the further stated properties as a consequence of (5.1). □

Remark 5.2. The above proof shows that there is a close connection between unitary relations without kernel and multivalued part, and nonnegative self-adjoint operators in Hilbert spaces, cf. [4, Theorem 3.6] and Section 4.3.

Since the unitary relations $\ker U \times \operatorname{mul} U$, U_+ and U_- are easily understood, the above theorem shows that (from a theoretical point of view) the most interesting unitary relations are those with dense domain and range in a Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$ with $k^+ = k^-$. Furthermore, Theorem 5.1 shows that if U is a unitary relation such that $\ker U$ does not have equal defect numbers, then there exists uniformly definite subspaces $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $U(\mathfrak{D}_1) = \mathfrak{D}_2 + \operatorname{mul} U$ and $\widetilde{U}$ defined via $\operatorname{gr}(\widetilde{U}) = \operatorname{gr}(U) \cap (\mathfrak{D}_1^{[\perp]_1} \times \mathfrak{D}_2^{[\perp]_2})$ is a unitary relation from $\{\mathfrak{K}_1 \cap \mathfrak{D}_1^{[\perp]_1}, [\cdot,\cdot]_1\}$

to $\{\mathfrak{K}_2 \cap \mathfrak{D}_2^{[\perp]_2}, [\cdot,\cdot]_2\}$ whose kernel (and multivalued part) has equal defect numbers.

The subspace $\mathfrak{L}_d$ in Theorem 5.1 has the following special properties.

Proposition 5.3. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{L}_d$ be the closed neutral subspace as in Theorem* 5.1 *for fixed fundamental symmetries $\mathfrak{j}_1$ and $\mathfrak{j}_2$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, respectively. Then*

- (i) *U has closed domain if and only if U maps some (any hence every) closed neutral subspace $\mathfrak{L}$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ which extends $\mathfrak{L}_d$ onto a closed neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$;*
- (ii) *$\mathfrak{L} := \ker U + \mathfrak{L}_d$ is such that $\ker U \subseteq \mathfrak{L} \subseteq \mathfrak{L}^{[\perp]_1} \subseteq \operatorname{dom} U$;*
- (iii) *if $\mathfrak{L}$ is a neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $\ker U \subseteq \mathfrak{L}$ and $\mathfrak{L}_d \subseteq \mathfrak{L}$ or $\mathfrak{j}_1\mathfrak{L}_d \cap \operatorname{dom} U \subseteq \mathfrak{L}$, then*

$$n_+(\mathfrak{L}) = n_+(U(\mathfrak{L})) \quad \text{and} \quad n_-(\mathfrak{L}) = n_-(U(\mathfrak{L})).$$

Proof. In this proof the notation as in Theorem 5.1 is used.

(i): By Theorem 5.1 a closed neutral extension of $\mathfrak{L}_d$ can be written as $\ker U \oplus_1 \mathfrak{L}_d \oplus_1 \mathfrak{N}_1$, where $\mathfrak{N}_1$ is a closed neutral subspace of $\operatorname{dom} U_+ + \operatorname{dom} U_-$. This extension is mapped onto $\operatorname{mul} U \oplus_2 (\mathfrak{j}_2\mathfrak{L}_r \cap \operatorname{dom} U) \oplus \mathfrak{N}_2$, where $\mathfrak{N}_2$ is a closed neutral subspace of $\operatorname{ran} U_+ + \operatorname{ran} U_-$ (because $U_+ + U_-$ is a standard unitary operator). Consequently, this subspace is closed if and only if $\mathfrak{j}_2\mathfrak{L}_r \cap \operatorname{dom} U$ is closed, which by Theorem 5.1 is the case if and only if $\operatorname{ran} U_o$. Since $\operatorname{ran} U_o$ and $\operatorname{ran} U$ are simultaneously closed, this proves the statement, see Proposition 3.15.

(ii): Since $\mathfrak{L}_d$ is hyper-maximal neutral in $\{\widetilde{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$, it follows from Theorem 5.1 that $(\mathfrak{L}_d)^{[\perp]_1} = \mathfrak{L}_d + \operatorname{dom} U_+ + \operatorname{dom} U_- + \ker U \subseteq \operatorname{dom} U$.

(iii): This follows from the fact that the defect numbers of $\ker U + \mathfrak{L}_d$ and $U(\mathfrak{L}_d)$ coincide (since $\operatorname{clos}(\mathfrak{j}_2\mathfrak{L}_r \cap \operatorname{ran} U)$ is hyper-maximal neutral in $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$), combined with Corollary 3.16 and Lemma 3.12. □

Next let $\mathfrak{L}_d$, U_+ and U_- be as in Theorem 5.1, and let $\mathfrak{D}^+ := \operatorname{dom} U_+$ and $\mathfrak{D}^- := \operatorname{dom} U_-$. Then, see Section 4.4, U_i defined as

$$U_i = \{\{f + h, f' + k\} : h \in \ker U + \mathfrak{L}_d,\ k \in \operatorname{clos}(U(\mathfrak{L}_d)), \\ \{f, f'\} \in U,\ f \in \mathfrak{D}^+ + \mathfrak{D}^-\}$$

is a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. In particular, if $\mathfrak{D}^+$ and $\mathfrak{D}^-$ have the same dimension, then there exists a hyper-maximal neutral extension of $\ker U$ which is contained in $\operatorname{dom} U$. If their dimensions are greater than the cardinality of the continuum, then, clearly, there exists maximal neutral subspaces contained in the domain of U_i (and hence in the domain of U) with arbitrary defect numbers smaller than or equal to $\max(\mathfrak{D}^+, \mathfrak{D}^-)$, cf. [4, Theorem 4.3].

5.2. A block representation for unitary relations

Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ with associated fundamental symmetry $\mathfrak{j}_i$, $i = 1, 2$. Then define $d_U^+(\mathfrak{j}_1, \mathfrak{j}_2)$ and $d_U^-(\mathfrak{j}_1, \mathfrak{j}_2)$ as

$$\begin{aligned} d_U^+(\mathfrak{j}_1, \mathfrak{j}_2) &= \dim\{f \in \mathfrak{K}_1^+ : \exists f' \in \mathfrak{K}_2^+ \text{ s.t. } \{f, f'\} \in U\}; \\ d_U^-(\mathfrak{j}_1, \mathfrak{j}_2) &= \dim\{f \in \mathfrak{K}_1^- : \exists f' \in \mathfrak{K}_2^- \text{ s.t. } \{f, f'\} \in U\}. \end{aligned} \tag{5.3}$$

Now Theorem 5.1 shows that if $n_-(\ker U) > n_+(\ker U)$ or $n_-(\ker U) < n_+(\ker U)$, see Definition 2.10, then $d_U^+(\mathfrak{j}_1, \mathfrak{j}_2) > d_U^-(\mathfrak{j}_1, \mathfrak{j}_2)$ or $d_U^+(\mathfrak{j}_1, \mathfrak{j}_2) < d_U^-(\mathfrak{j}_1, \mathfrak{j}_2)$ for all $\mathfrak{j}_1$ and $\mathfrak{j}_2$, respectively. Furthermore, if there exist $\mathfrak{j}_1$ and $\mathfrak{j}_2$ such that $d_U^+(\mathfrak{j}_1, \mathfrak{j}_2) = d_U^-(\mathfrak{j}_1, \mathfrak{j}_2)$, $d_U^+(\mathfrak{j}_1, \mathfrak{j}_2) > d_U^-(\mathfrak{j}_1, \mathfrak{j}_2)$ or $d_U^+(\mathfrak{j}_1, \mathfrak{j}_2) < d_U^-(\mathfrak{j}_1, \mathfrak{j}_2)$, then Theorem 5.1 shows that there exists a hyper-maximal semi-definite subspaces $\mathfrak{L}$ and $\mathfrak{M}$ satisfying $\ker U \subseteq \mathfrak{L} \subseteq \operatorname{dom} U$ and $\operatorname{mul} U \subseteq \mathfrak{M} \subseteq \operatorname{ran} U$ which are neutral, nonnegative or nonpositive, respectively; cf. [4, Theorem 4.4].

If $n_+(\ker U) = n_-(\ker U)$, then it is not clear whether there exist hyper-maximal neutral subspaces contained in the domain and range of U. Though there always exist hyper-maximal semi-definite subspaces contained in its domain and range. In fact, as the next statement shows, these subspaces can be chosen to have more properties, which in what follows will be of great importance.

Proposition 5.4. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{j}_i$ be a fundamental symmetry for $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$, for $i = 1, 2$. Then there exists hyper-maximal semi-definite subspaces $\mathfrak{L}$ and $\mathfrak{M}$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, respectively, such that*

$$\operatorname{dom} U = \mathfrak{L}^{[\perp]_1} \oplus_1 (\mathfrak{L} \cap \mathfrak{j}_1\mathfrak{L}) \oplus_1 (\mathfrak{j}_1\mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} U);$$

$$\operatorname{ran} U = \mathfrak{M}^{[\perp]_2} \oplus_2 (\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}) \oplus_2 (\mathfrak{j}_2\mathfrak{M}^{[\perp]_2} \cap \operatorname{ran} U),$$

where

$$\begin{aligned} U(\mathfrak{L}^{[\perp]_1}) &= \mathfrak{j}_2\mathfrak{M}^{[\perp]_2} \cap \operatorname{ran} U + \operatorname{mul} U; \\ U(\mathfrak{L} \cap \mathfrak{j}_1\mathfrak{L}) &= \mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M} + \operatorname{mul} U; \\ U(\mathfrak{j}_1\mathfrak{L}^{[\perp]_1} \cap \operatorname{dom} U) &= \mathfrak{M}^{[\perp]_2} + \operatorname{mul} U. \end{aligned}$$

Here $\mathfrak{L}$ and $\mathfrak{M}$ are hyper-maximal neutral, nonnegative or nonpositive subspaces of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ if $d_U^+(\mathfrak{j}_1, \mathfrak{j}_2) = d_U^-(\mathfrak{j}_1, \mathfrak{j}_2)$, $d_U^+(\mathfrak{j}_1, \mathfrak{j}_2) > d_U^-(\mathfrak{j}_1, \mathfrak{j}_2)$ or $d_U^+(\mathfrak{j}_1, \mathfrak{j}_2) < d_U^-(\mathfrak{j}_1, \mathfrak{j}_2)$, respectively.

Proof. Using the notation of Theorem 5.1, it clearly suffices to prove the statement only for U_o. In that case the statement holds by taking $\mathfrak{L}$ to be $\mathfrak{L}_d$ in which case $\mathfrak{M} = \mathfrak{L}_r$. The decompositions are a direct consequence of Proposition 2.8 (iv). □

Remark 5.5. In particular, if $d_U^+(\mathfrak{j}_1, \mathfrak{j}_2) \neq d_U^-(\mathfrak{j}_1, \mathfrak{j}_2)$, then note that the unitary relation $\widetilde{U}$ defined via $\operatorname{gr}(\widetilde{U}) = \operatorname{gr}(U) \cap ((\mathfrak{K}_1 \cap (\mathfrak{L} \cap \mathfrak{j}_1\mathfrak{L})^{[\perp]_1}) \times (\mathfrak{K}_2 \cap (\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M})^{[\perp]_2}))$, see Corollary 3.13, satisfies $d_{\widetilde{U}}^+(\mathfrak{j}_1, \mathfrak{j}_2) = d_{\widetilde{U}}^-(\mathfrak{j}_1, \mathfrak{j}_2)$.

The hyper-maximal semi-definite subspace $\mathfrak{L}$ in Proposition 5.4 is shown to exists as an extension of the subspace $\mathfrak{L}_d$ as in Theorem 5.1. Not all hyper-maximal semi-definite subspaces contained in the domain of a unitary relation can be obtained in this manner. Namely, consider Example 1.1 with K an unbounded operator and $\mathfrak{H}_0 = \mathfrak{H}$, then $\mathfrak{H} \times \{0\}$ is a hyper-maximal neutral subspace which is mapped by U onto the hyper-maximal neutral subspace $\mathfrak{H} \times \{0\}$. Since the unitary relation does not have a closed domain, it follows from Proposition 5.3 that $\mathfrak{H} \times \{0\}$ is not an extension of any $\mathfrak{L}_d$ as in Theorem 5.1.

By means of Proposition 5.4 the following characterizations for isometric relations to be unitary is obtained. Note that it in particular shows that if an isometric relation has a decomposition as in Theorem 5.1, then it is a unitary relation.

Theorem 5.6. *Let U be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{j}_i$ be a fundamental symmetry of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$, for $i = 1, 2$. Then U is unitary if and only if there exists a decomposition $\widetilde{\mathfrak{K}}_i \oplus_i \mathfrak{D}_i^+ \oplus_i \mathfrak{D}_i^-$ of $\mathfrak{K}_i$, for $i = 1, 2$, such that*

- (i) *$\mathfrak{D}_i^+$ is a closed uniformly positive subspaces of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ for $i = 1, 2$, $\mathfrak{D}_1^+ \subseteq \operatorname{dom} U$, $\mathfrak{D}_2^+ \subseteq \operatorname{ran} U$ and $U(\mathfrak{D}_1^+) = \mathfrak{D}_2^+ + \operatorname{mul} U$;*
- (ii) *$\mathfrak{D}_i^-$ is a closed uniformly negative subspaces of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ for $i = 1, 2$, $\mathfrak{D}_1^- \subseteq \operatorname{dom} U$, $\mathfrak{D}_2^- \subseteq \operatorname{ran} U$ and $U(\mathfrak{D}_1^-) = \mathfrak{D}_2^- + \operatorname{mul} U$;*
- (iii) *there exists hyper-maximal neutral subspaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$ of $\{\widetilde{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$ and $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$, respectively, such that*
  - (a) *$\mathfrak{L}_1 = U^{-1}(\mathfrak{j}_2\mathfrak{L}_2 \cap \operatorname{ran} U)$ and $\mathfrak{L}_2 = U(\mathfrak{j}_1\mathfrak{L}_1 \cap \operatorname{dom} U)$;*
  - (b) *$\ker U + \operatorname{clos}(\mathfrak{j}_1\mathfrak{L}_1 \cap \operatorname{dom} U)$ and $\operatorname{mul} U + \operatorname{clos}(\mathfrak{j}_2\mathfrak{L}_2 \cap \operatorname{ran} U)$ are hyper-maximal neutral subspaces of $\{\widetilde{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$ and $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$, respectively.*

Proof. The necessity of the conditions is clear by Theorem 5.1. In particular, one can take $\mathfrak{L}_1 = \mathfrak{L}_d + \ker U$, $\mathfrak{L}_2 = \mathfrak{L}_r + \operatorname{mul} U$, $\mathfrak{D}_1^+ = \operatorname{dom} U_+$ and $\mathfrak{D}_1^- = \operatorname{dom} U_-$.

To prove the converse observe first that $\{\{f, f'\} \in U : f \in \mathfrak{D}_1^+ + \mathfrak{D}_1^-,\ f' \in \mathfrak{D}_2^+ + \mathfrak{D}_2^-\}$ is a standard unitary operator from the Kreĭn space $\{\mathfrak{D}_1^+ + \mathfrak{D}_1^-, [\cdot,\cdot]_1\}$ to the Kreĭn space $\{\mathfrak{D}_2^+ + \mathfrak{D}_2^-, [\cdot,\cdot]_2\}$. Hence, Lemma 3.12 (see also Corollary 3.13) shows that to prove the statement, it suffices to show that $\widetilde{U}$ defined via $\operatorname{gr}(\widetilde{U}) = \operatorname{gr}(U) \cap (\widetilde{\mathfrak{K}}_1 \times \widetilde{\mathfrak{K}}_2)$ is a unitary relation.

Therefore note first that by (iii)(a)

$$\operatorname{dom} \widetilde{U} = \mathfrak{L}_1 + \mathfrak{j}_1\mathfrak{L}_1 \cap \operatorname{dom} \widetilde{U} = \widetilde{U}^{-1}(\mathfrak{j}_2\mathfrak{L}_2 \cap \operatorname{ran} \widetilde{U}) + \mathfrak{j}_1\mathfrak{L}_1 \cap \operatorname{dom} \widetilde{U}. \tag{5.4}$$

Now let $f \in \mathfrak{K}_1$ and $f' \in \mathfrak{K}_2$ be such that $[f, g]_1 = [f', g']_2$ for all $\{g, g'\} \in \widetilde{U}$. Then, with $\mathcal{P}_{\mathfrak{L}_i}$ the $\mathfrak{j}_i$-orthogonal projection onto $\mathfrak{L}_i$ in $\mathfrak{K}_i$, there exist by (iii)(a) $\{h, h'\}, \{k, k'\} \in \widetilde{U}$ such that $\mathcal{P}_{\mathfrak{L}_1} h = \mathcal{P}_{\mathfrak{L}_1} f$, $\mathcal{P}_{\mathfrak{L}_2} h' = 0$, $\mathcal{P}_{\mathfrak{L}_1} k = 0$ and $\mathcal{P}_{\mathfrak{L}_2} k' = \mathcal{P}_{\mathfrak{L}_2} f'$. Hence,

$$\{f, f'\} = \{h, h'\} + \{k, k'\} + \{f - h - k, f' - h' - k'\},$$

where $\mathcal{P}_{\mathfrak{L}_1}(f-h-k)=0$ and $\mathcal{P}_{\mathfrak{L}_2}(f'-h'-k')=0$. Clearly,

$$[f-h-k,g]_1=[f'-h'-k',g']_2, \quad \forall\{g,g'\}\in\widetilde{U}.$$

Since the lefthand side is zero for all $g\in \mathfrak{j}_1\mathfrak{L}_1\cap\operatorname{dom}\widetilde{U}$ and the righthand side is zero for all $g'\in\mathfrak{j}_2\mathfrak{L}_2\cap\operatorname{ran}\widetilde{U}$, (5.4) implies that $f-h-k\in(\operatorname{dom}\widetilde{U})^{[\perp]_1}$ and, consequently, $f'-h'-k'\in(\operatorname{ran}\widetilde{U})^{[\perp]_2}$. Since $(\operatorname{dom}\widetilde{U})^{[\perp]_1}=\ker\widetilde{U}$ and $(\operatorname{ran}\widetilde{U})^{[\perp]_2}=\operatorname{mul}\widetilde{U}$ by Lemma 3.24 (using the assumption (iii)(b)), it follows that $\{f,f'\}\in\widetilde{U}$. Consequently, Proposition 3.2 implies that $\widetilde{U}$ is unitary. □

Proposition 5.4 gives also naturally rise to a block representation for unitary relations which is different from the quasi-block representation in Theorem 4.12; there the "coordinates" are uniformly definite and here they are neutral.

Theorem 5.7. *Let U be a unitary relation from $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ with* $\ker U=\{0\}=\operatorname{mul}U$. *Then there exists a hyper-maximal semi-definite subspace $\mathfrak{M}$ of $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$, a fundamental symmetry $\mathfrak{j}_2$ of $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$, a densely defined closed operator B in $\{\mathfrak{M}^{[\perp]_2},[\mathfrak{j}_2\cdot,\cdot]_2\}$ with* $\operatorname{ran}B=\mathfrak{M}^{[\perp]_2}$, *a standard unitary operator U_h in $\{\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M},[\cdot,\cdot]_2\}$ and a standard unitary operator U_t from $\{\mathfrak{K}_1,[\cdot,\cdot]_2\}$ onto $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ such that with respect to the decomposition $\mathfrak{M}^{[\perp]_2}\oplus_2\mathfrak{j}_2\mathfrak{M}^{[\perp]_2}\oplus_2(\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M})$ of $\mathfrak{K}_2$, see Proposition* 2.8 *(iv), one has*

$$UU_t^{-1}=\begin{pmatrix} B & 0 & 0\\ 0 & \mathfrak{j}_2B^{-*}\mathfrak{j}_2 & 0\\ 0 & 0 & U_h\end{pmatrix}.$$

Conversely, for B, $\mathfrak{j}_2$ and U_h as above, the righthand side of the above equation represents a unitary relation in $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ such that $\ker U=\{0\}=\operatorname{mul}U$.

Proof. The converse part can be proven by a straightforward calculation and Theorem 5.6. Therefore only the existence of the block representation will be shown. Hence, let $\mathfrak{j}_1$ and $\mathfrak{j}_2$ be fundamental symmetries for $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ and $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$, and let $\mathfrak{L}$ and $\mathfrak{M}$ be hyper-maximal semi-definite subspaces associated to the unitary relations (for the indicated fundamental symmetries) as in Proposition 5.4. In particular, $\mathfrak{L}\subseteq\operatorname{dom}U$, $\mathfrak{M}\subseteq\operatorname{ran}U$ and $U(\mathfrak{L}\cap\mathfrak{j}_1\mathfrak{L})=\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}$.

Let U_0 and U_1 be Hilbert space unitary operators from $\{\mathfrak{L}^{[\perp]_1},[\mathfrak{j}_1\cdot,\cdot]_1\}$ onto $\{\mathfrak{j}_2\mathfrak{M}^{[\perp]_2},[\mathfrak{j}_2\cdot,\cdot]_2\}$ and from $\{\mathfrak{L}\cap\mathfrak{j}_1\mathfrak{L},[\mathfrak{j}_1\cdot,\cdot]_1\}$ onto $\{\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M},[\mathfrak{j}_2\cdot,\cdot]_1\}$, respectively. Then U_t defined as

$$U_t(f_0+f_1+f_2)=U_0(f_0)+U_1(f_1)+\mathfrak{j}_2U_0(\mathfrak{j}_1f_2)$$

for $f_0\in\mathfrak{L}^{[\perp]_1}$, $f_1\in\mathfrak{L}\cap\mathfrak{j}_1\mathfrak{L}$ and $f_2\in\mathfrak{j}_1\mathfrak{L}^{[\perp]_1}$ is a standard unitary operator from $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$. Note that, since UU_t^{-1} is isometric and maps $\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}$ onto $\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}$, the restriction of UU_t^{-1} to $\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}$, denote it by U_h, is an standard unitary operator in $\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}$. Furthermore, recall that by Proposition 5.4 $U(\mathfrak{L}^{[\perp]_1})=\mathfrak{j}_2\mathfrak{M}^{[\perp]_2}\cap\operatorname{ran}U$ and $U(\mathfrak{j}_1\mathfrak{L}^{[\perp]_1}\cap\operatorname{dom}U)=\mathfrak{M}^{[\perp]_2}$.

This yields the existence of operators A and C such that with respect to the decomposition $\mathfrak{M}^{[\perp]_2} \oplus_2 \mathfrak{j}_2\mathfrak{M}^{[\perp]_2} \oplus_2 (\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M})$ of $\mathfrak{K}_2$

$$UU_t^{-1} = \begin{pmatrix} A & 0 & 0 \\ 0 & \mathfrak{j}_2 C \mathfrak{j}_2 & 0 \\ 0 & 0 & U_h \end{pmatrix},$$

where $\operatorname{ran} A = \mathfrak{M}^{[\perp]_2} = \operatorname{dom} C$. Moreover, A and C are closed, because UU_t^{-1} is unitary (see Lemma 3.9) and, hence, closed. Moreover, since UU_t^{-1} is isometric, it follows that

$$[f, \mathfrak{j}_2 g]_2 = [Af, \mathfrak{j}_2 Cg]_2, \quad \forall f \in \operatorname{dom} A, \forall g \in \mathfrak{M}^{[\perp]_2}.$$

This equation shows that $A^{-1} \subseteq C^*$, where the adjoint of C is taken in $\{\mathfrak{M}^{[\perp]_2}, [\mathfrak{j}_2\cdot, \cdot]_2\}$. Since $\operatorname{ran} A = \mathfrak{M}^{[\perp]_2} = \operatorname{dom} C^*$, $\operatorname{mul} A^{-1} = \ker A = \{0\}$ (because $\ker U = \{0\}$) and $\ker C^{-*} = (\operatorname{dom} C)^{\perp_2} = \{0\}$, (3.5) implies that $A^{-1} = C^*$. □

Note that $\mathfrak{M}$ in Theorem 5.7 can be chosen to be neutral, nonnegative or nonpositive, if and only if there exists a fundamental symmetry $\mathfrak{j}_1$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\mathfrak{j}_2$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $d_U^+(\mathfrak{j}_1, \mathfrak{j}_2) = d_U^-(\mathfrak{j}_1, \mathfrak{j}_2)$, $d^+(\mathfrak{j}_1, \mathfrak{j}_2) > d^-(\mathfrak{j}_1, \mathfrak{j}_2)$ or $d^+(\mathfrak{j}_1, \mathfrak{j}_2) < d^-(\mathfrak{j}_1, \mathfrak{j}_2)$, respectively.

The assumption $\ker U = \{0\} = \operatorname{mul} U$ in Theorem 5.7 can be dropped by allowing B and its adjoint B^* to have nontrivial kernels. Note also that the above theorem gives a one-to-one correspondence between unitary relations and bounded everywhere defined operators (with kernels). That connection can also be understood from Theorem 4.14 and Remark 5.12 below.

5.3. Essentially unitary relations

Here Proposition 5.4 is further analyzed in order to obtain conditions for the closure of an isometric relation to be unitary. Therefore recall that for a hyper-maximal semi-definite subspace $\mathfrak{L}$ in the Kreĭn space $\{\mathfrak{K}, [\cdot,\cdot]\}$ with associated fundamental symmetry $\mathfrak{j}$, the space $\mathfrak{K}$ can be decomposed as

$$\mathfrak{K} = \mathfrak{L}^{[\perp]} \oplus (\mathfrak{L} \cap \mathfrak{j}\mathfrak{L}) \oplus \mathfrak{j}\mathfrak{L}^{[\perp]}, \tag{5.5}$$

see Proposition 2.8 (iv). In this connection $\mathcal{P}_{\mathfrak{L}^{[\perp]}}$ and $\mathcal{P}_{\mathfrak{j}\mathfrak{L}^{[\perp]}}$ denote the $\mathfrak{j}$-orthogonal projections in $\mathfrak{K}$ onto $\mathfrak{L}^{[\perp]}$ and $\mathfrak{j}\mathfrak{L}^{[\perp]}$, respectively.

Now observe the following consequence of Proposition 5.4.

Proposition 5.8. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ with associated fundamental symmetry $\mathfrak{j}_i$ and associated projections P_i^+ and P_i^-, for $i = 1, 2$. Then there exists a subspace $\mathfrak{M}$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that*

(i) *$\mathfrak{M} \subseteq \operatorname{ran} U$ is a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$;*
(ii) *$U^{-1}(\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}) \subseteq \ker U + \mathfrak{K}_1^+$ or $U^{-1}(\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}) \subseteq \ker U + \mathfrak{K}_1^-$;*
(iii) *$\mathfrak{N} := U^{-1}(\mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} U) \cap (\ker U + \operatorname{dom} U \cap \mathfrak{K}_1^+ + \operatorname{dom} U \cap \mathfrak{K}_1^-)$ is such that*

$$P_1^+ U^{-1}(\mathfrak{M}) = P_1^+ \ker U + \operatorname{dom} U \cap \mathfrak{K}_1^+ = P_1^+ \mathfrak{N};$$
$$P_1^- U^{-1}(\mathfrak{M}) = P_1^- \ker U + \operatorname{dom} U \cap \mathfrak{K}_1^- = P_1^- \mathfrak{N}.$$

Proof. Clearly, $\mathfrak{M}$ as in Proposition 5.4 satisfies condition (i) and (ii), and, moreover, $U^{-1}(\mathfrak{j}_2\mathfrak{M}\cap\operatorname{ran} U)$ coincides with $\mathfrak{L}$ in Proposition 5.4. If $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2) = d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)$, then $\mathfrak{M}$ and $\mathfrak{N}$ are neutral, $\ker U \subseteq \mathfrak{N}, U^{-1}(\mathfrak{M})$ and

$$\mathfrak{N} + U^{-1}(\mathfrak{M}) = \ker U + \operatorname{dom} U \cap \mathfrak{K}_1^+ + \operatorname{dom} U \cap \mathfrak{K}_1^-,$$

because $U^{-1}(\mathfrak{j}_2\mathfrak{M}\cap\operatorname{ran} U) + U^{-1}(\mathfrak{M}) = \operatorname{dom} U$. The above equality together with the neutrality of $\mathfrak{M}$ and $\mathfrak{N}$ shows that (iii) holds. If $d_U^+(\mathfrak{j}_1,\mathfrak{j}_2) \neq d_U^-(\mathfrak{j}_1,\mathfrak{j}_2)$, then the situation can be reduced to the above case, because in that case $\mathfrak{N}$ and $U^{-1}(\mathfrak{M})$ both contain the same (closed) uniformly definite subspace $\mathfrak{L}\cap\mathfrak{j}_1\mathfrak{L}$, see Remark 5.5. □

In Proposition 5.8 $\mathfrak{M}$, $U^{-1}(\mathfrak{j}_2\mathfrak{M}\cap\operatorname{ran} U)$, $\widetilde{\mathfrak{L}}$ and $U^{-1}(\mathfrak{M})$ are all either neutral, nonnegative or nonpositive; they reflect the defect numbers of $\ker U$. Furthermore, as a consequence of (i) and (iii) $\mathfrak{M}$ satisfies

$$\begin{aligned} \mathfrak{M} + U(\operatorname{dom} U\cap\mathfrak{K}_1^+) &= U(\operatorname{dom} U\cap\mathfrak{K}_1^+) + U(\operatorname{dom} U\cap\mathfrak{K}_1^-);\\ \mathfrak{M} + U(\operatorname{dom} U\cap\mathfrak{K}_1^-) &= U(\operatorname{dom} U\cap\mathfrak{K}_1^+) + U(\operatorname{dom} U\cap\mathfrak{K}_1^-). \end{aligned} \tag{5.6}$$

This observation can be generalized to the following geometrical result.

Proposition 5.9. *For every maximal nonnegative or nonpositive subspace* $\mathfrak{M}$ *of* $\{\mathfrak{K},[\cdot,\cdot]\}$ *there exists a hyper-maximal semi-definite subspace* $\mathfrak{L}$ *of* $\{\mathfrak{K},[\cdot,\cdot]\}$ *such that*

$$\mathfrak{L} + \mathfrak{M} = \mathfrak{M} + \mathfrak{M}^{[\perp]} = \mathfrak{L} + \mathfrak{M}^{[\perp]}.$$

Proof. W.l.o.g. assume that $\mathfrak{M}$ is nonnegative and let $\mathfrak{K}^+[+]\mathfrak{K}^-$ be a canonical decomposition of $\{\mathfrak{K},[\cdot,\cdot]\}$. Then by Theorem 4.14 there exists a unitary relation U in $\{\mathfrak{K},[\cdot,\cdot]\}$ such that $U(\operatorname{dom} U\cap\mathfrak{K}^+) = \mathfrak{M}$ and $U(\operatorname{dom} U\cap\mathfrak{K}^-) = \mathfrak{M}^{[\perp]}$. Consequently, the statement follows from the discussion preceding this statement. □

The following statement shows that Proposition 5.8 (ii) and (iii) implies that $U^{-1}(\mathfrak{j}_2\mathfrak{M}\cap\operatorname{ran} U)$ is hyper-maximal semi-definite.

Proposition 5.10. *Let U be unitary relation from $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ and let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be a canonical decomposition of $\{\mathfrak{K}_i,[\cdot,\cdot]_i\}$ with associated fundamental symmetry $\mathfrak{j}_i$ and associated projections P_i^+ and P_i^-, for $i=1,2$. Moreover, let $\mathfrak{M}$ be a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ such that*

(i) $\mathfrak{M} \subseteq U(\operatorname{dom} U\cap\mathfrak{K}_1^+) + U(\operatorname{dom} U\cap\mathfrak{K}_1^-)$*;*
(ii) $U^{-1}(\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}) \subseteq \ker U + \mathfrak{K}_1^+$ *or* $U^{-1}(\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}) \subseteq \ker U + \mathfrak{K}_1^-$.

Then $\mathfrak{L} := U^{-1}(\mathfrak{j}_2\mathfrak{M}\cap\operatorname{ran} U)$ is a hyper-maximal semi-definite subspace of $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ if and only if

$$P_1^+\ker U + \operatorname{dom} U\cap\mathfrak{K}_1^+ \subseteq P_1^+\mathfrak{L} \quad \textit{and} \quad P_1^-\ker U + \operatorname{dom} U\cap\mathfrak{K}_1^- \subseteq P_1^-\mathfrak{L}. \tag{5.7}$$

Proof. Necessity of the condition (5.7) is obvious by Corollary 2.9 and (4.2) so only sufficiency of the condition needs to be proven. Therefore first observe that $\mathfrak{M} = \mathfrak{M}^{[\perp]_2} + (\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M})$, where $\mathfrak{M}^{[\perp]_2}$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_2\cap(\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M})^{[\perp]_2},[\cdot,\cdot]_2\}$. By assumption (ii) $\mathfrak{M}\cap\mathfrak{j}_2\mathfrak{M}$ is mapped onto the sum of $\ker U$ and a closed uniformly definite subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ contained in either $\mathfrak{K}_1^+$ or $\mathfrak{K}_1^-$. This shows that w.l.o.g. one can assume that $\mathfrak{M}$ is a hyper-maximal neutral subspace and, hence, one needs only to prove that $\mathfrak{L}$ is a hyper-maximal neutral subspace if and only if (5.7) holds.

If $\mathfrak{M} \subseteq \operatorname{ran} U$ is hyper-maximal neutral, then $\operatorname{ran} U = \mathfrak{M} + \mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} U$ and, hence, $\operatorname{dom} U = \mathfrak{L} + U^{-1}(\mathfrak{M})$. Furthermore, by the assumption (i) and (5.7)

$$P_1^{\pm} U^{-1}(\mathfrak{M}) \subseteq P_1^{\pm} \ker U + \operatorname{dom} U \cap \mathfrak{K}_1^{\pm} \subseteq P_1^{\pm}\mathfrak{L}.$$

The preceding inclusions together with Proposition 4.1 yield that

$$\mathfrak{K}_1^{\pm} = P_1^{\pm} \operatorname{dom} U = P_1^{\pm}(\mathfrak{L} + U^{-1}(\mathfrak{M})) \subseteq P_1^{\pm}\mathfrak{L}.$$

This completes the proof, see Proposition 2.5. □

Continuing the investigation of the statements occurring in Proposition 5.8, observe next that (ii) and (iii) imply that

$$\begin{aligned} \mathcal{P}_{\mathfrak{M}^{[\perp]_2}} U(\operatorname{dom} U \cap \mathfrak{K}_1^+) &= \mathcal{P}_{\mathfrak{M}^{[\perp]_2}} U(\operatorname{dom} U \cap \mathfrak{K}_1^-); \\ \mathcal{P}_{\mathfrak{j}_2\mathfrak{M}^{[\perp]_2}} U(\operatorname{dom} U \cap \mathfrak{K}_1^+) &= \mathcal{P}_{\mathfrak{j}_2\mathfrak{M}^{[\perp]_2}} U(\operatorname{dom} U \cap \mathfrak{K}_1^-), \end{aligned} \tag{5.8}$$

where $\mathcal{P}_{\mathfrak{M}^{[\perp]_2}}$ and $\mathcal{P}_{\mathfrak{j}_2\mathfrak{M}^{[\perp]_2}}$ are the $\mathfrak{j}_2$-orthogonal projections onto $\mathfrak{M}^{[\perp]_2}$ and $\mathfrak{j}\mathfrak{M}^{[\perp]_2}$ in $\mathfrak{K}_2$ respectively. To see that equalities in (5.8) hold, let $f_1^+ \in \operatorname{dom} U \cap \mathfrak{K}_1^+$, then by Proposition 5.8 (iii) there exists $\{f, f'\} \in U$ such that $f' \in \mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} U$, $f \in \operatorname{dom} U \cap \mathfrak{K}_1^+ + \operatorname{dom} U \cap \mathfrak{K}_1^-$ and $P_1^+ f = f_1^+$. Since, clearly, $\mathcal{P}_{\mathfrak{M}^{[\perp]_2}} f' = 0$, this shows that

$$\mathcal{P}_{\mathfrak{M}^{[\perp]_2}} U(\operatorname{dom} U \cap \mathfrak{K}_1^+) \subseteq \mathcal{P}_{\mathfrak{M}^{[\perp]_2}} U(\operatorname{dom} U \cap \mathfrak{K}_1^-).$$

The other inclusions in (5.8) can be seen to hold using similar arguments.

The equalities in (5.6) and (5.8) together with Proposition 5.8 (ii) imply that

$$\begin{aligned} \mathcal{P}_{\mathfrak{M}} U(\operatorname{dom} U \cap \mathfrak{K}_1^+) &= \mathfrak{M}^{[\perp]_2} + P_2^+(\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}); \\ \mathcal{P}_{\mathfrak{M}} U(\operatorname{dom} U \cap \mathfrak{K}_1^-) &= \mathfrak{M}^{[\perp]_2} + P_2^-(\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M}). \end{aligned} \tag{5.9}$$

The preceding observations yield half of the following geometrical statement, cf. Proposition 2.4.

Proposition 5.11. *Let $\mathfrak{M}_+$ and $\mathfrak{M}_-$ be a nonnegative and nonpositive subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$, respectively, such that $\mathfrak{M}_+ \subseteq \mathfrak{M}_-^{[\perp]}$ and $\mathfrak{M}_- \subseteq \mathfrak{M}_+^{[\perp]}$ and let $\mathfrak{j}$ be a fundamental symmetry of $\{\mathfrak{K}, [\cdot,\cdot]\}$. Then $\mathfrak{M}_+$ and $\mathfrak{M}_-$ are a maximal nonnegative and a maximal nonpositive subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$, respectively, if and only if there exists a hyper-maximal semi-definite subspace $\mathfrak{L}$ of $\{\mathfrak{K}, [\cdot,\cdot]\}$ such that*

$$\mathcal{P}_{\mathfrak{L}}\mathfrak{M}_+ = \mathfrak{L} \quad \textit{and} \quad \mathcal{P}_{\mathfrak{L}}\mathfrak{M}_- = \mathfrak{L}^{[\perp]} \quad \textit{or} \quad \mathcal{P}_{\mathfrak{L}}\mathfrak{M}_+ = \mathfrak{L}^{[\perp]} \quad \textit{and} \quad \mathcal{P}_{\mathfrak{L}}\mathfrak{M}_- = \mathfrak{L},$$

if $\mathfrak{L}$ is nonnegative or nonpositive, respectively.

Proof. The necessity is clear by the discussion preceding the statement combined with Theorem 4.14. To prove the converse assume w.l.o.g. that $\mathfrak{L}$ is nonnegative. If $\mathfrak{M}_+$ is not maximal nonnegative, then $\mathfrak{M}_+$ can be nonnegatively extended by an element $h \in \mathfrak{K}$. In fact, by the assumption $\mathcal{P}_{\mathfrak{L}}\mathfrak{M}_+ = \mathfrak{L}$ and (5.5), one can assume that $h \in \mathfrak{L}^{\perp} = \mathfrak{j}\mathfrak{L}^{[\perp]}$. In particular, there exists $f \in \mathfrak{L}^{[\perp]}$ such that $h = \mathfrak{j}f$. On the other hand, by the assumption $\mathcal{P}_{\mathfrak{L}}\mathfrak{M}_+ = \mathfrak{L}$ there also exists $g \in \mathfrak{L}^{[\perp]}$ such that $f + \mathfrak{j}g \in \mathfrak{M}_+$. Hence, for all $a \in \mathbb{R}$

$$0 \leq [(f + \mathfrak{j}g) + a\mathfrak{j}f, (f + \mathfrak{j}g) + a\mathfrak{j}f] = 2a[\mathfrak{j}f, f] + [\mathfrak{j}g, f] + [f, \mathfrak{j}g].$$

Since a is arbitrary, this implies that $f = 0$, i.e., $\mathfrak{M}_+$ is maximal nonnegative. The maximal nonpositivity of $\mathfrak{M}_-$ can be proven using similar argument. $\square$

Note that there exists a subspace $\mathfrak{L}$ having the properties as in Proposition 5.9 which simultaneously has the properties of the subspace $\mathfrak{L}$ in Proposition 5.11.

Remark 5.12. If $\mathfrak{L}$ in Proposition 5.11 is neutral, then the statement says that a nonnegative (nonpositive) subspace of $\{\mathfrak{K}, [\cdot,\cdot]\}$ is maximal nonnegative (nonpositive) if it is the graph of an everywhere defined bounded operator with respect to the neutral coordinates $\mathfrak{L}$ and $\mathfrak{j}\mathfrak{L}$.

The following statement shows that the equalities in (5.8) imply that the closure of a certain associated subspace is hyper-maximal semi-definite. For simplicity, the subspace $\mathfrak{M}$ (in (5.8)) will here be assumed to be hyper-maximal neutral. Note that following statement generalizes a part of [6, Corollary 4.12].

Proposition 5.13. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and let $\mathfrak{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$. Moreover, let $\mathfrak{M}$ be a hyper-maximal neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that*

$$\mathcal{P}_{\mathfrak{M}}U(\operatorname{dom} U \cap \mathfrak{K}_1^+) = \mathcal{P}_{\mathfrak{M}}U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$$

Then the closure of $\mathfrak{N} := U^{-1}(\mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} U)$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ and $\mathfrak{N}$ satisfies

$$P_1^+\ker U + \operatorname{dom} U \cap \mathfrak{K}_1^+ \subseteq P_1^+\mathfrak{N} \quad \text{and} \quad P_1^-\ker U + \operatorname{dom} U \cap \mathfrak{K}_1^- \subseteq P_1^-\mathfrak{N}.$$

Proof. For every $f^+ \in \operatorname{dom} U \cap \mathfrak{K}_1^+$ and for every $f^- \in \operatorname{dom} U \cap \mathfrak{K}_1^-$, there exists by the assumption $g^- \in \operatorname{dom} U \cap \mathfrak{K}_1^-$ and $g^+ \in \operatorname{dom} U \cap \mathfrak{K}_1^+$ such that $U(f^+ + g^-), U(f^- + g^+) \in \mathfrak{j}_2\mathfrak{M}$. These arguments show that the asserted inclusions hold. Consequently, (4.2) implies that

$$\mathfrak{K}_1^{\pm} = P_1^+\ker U + \operatorname{clos}(\operatorname{dom} U \cap \mathfrak{K}_1^{\pm}) = \operatorname{clos}(P_1^{\pm}\mathfrak{N}).$$

This shows that $\operatorname{clos}(\mathfrak{N})$ is hyper-maximal neutral, see Proposition 2.5. $\square$

Next necessary and sufficiency conditions are given for the closure of isometric relations to be unitary; they are based on Theorem 4.9 and Proposition 5.11.

Theorem 5.14. *Let U be an isometric relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$, let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ with associated projection P_1^+ and P_1^-, and let $\mathfrak{j}_2$ be a fundamental symmetry of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$.*

Then U is unitary if and only if

(i) *U is closed;*
(ii) $\ker U = (\operatorname{dom} U)^{[\perp]_1}$*;*
(iii) *there exists a hyper-maximal semi-definite subspace $\mathfrak{M}$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that*
   (a) $\mathfrak{M} \subseteq U(\operatorname{dom} U \cap \mathfrak{K}_1^+) + U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$*;*
   (b) $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M} \subseteq U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ *or* $\mathfrak{M} \cap \mathfrak{j}_2\mathfrak{M} \subseteq U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$*;*
   (c) $\mathfrak{N} := U^{-1}(\mathfrak{j}_2\mathfrak{M} \cap \operatorname{ran} U) \cap (\operatorname{dom} U \cap \mathfrak{K}_1^+ + \operatorname{dom} U \cap \mathfrak{K}_1^-)$ *satisfies* $P_1^+\mathfrak{N} = \operatorname{dom} U \cap \mathfrak{K}_1^+$ *and* $P_1^-\mathfrak{N} = \operatorname{dom} U \cap \mathfrak{K}_1^-$.

Proof. The necessity of the conditions (i) and (ii) is clear and the necessity of the condition (iii) follows from Proposition 5.8, see also (5.6). To prove the converse note first that the assumptions imply that (5.9) holds. Therefore Proposition 5.11 (see also the discussion preceding it) implies that $U(\operatorname{dom} U \cap \mathfrak{K}_1^+)$ and $U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$ are maximal nonnegative and nonpositive subspace and, hence, $U(\operatorname{dom} U \cap \mathfrak{K}_1^+) = U(\operatorname{dom} U \cap \mathfrak{K}_1^-)$, see (the proof of) Proposition 4.4. Consequently, Theorem 4.9 implies that U is unitary. □

Remark 5.15. The previous theorem was proven by showing that (ii) and (iii) imply that $\mathcal{P}_{\mathfrak{M}}U(\operatorname{dom} \mathfrak{K}_1^+) = \mathfrak{M}$ and $\mathcal{P}_{\mathfrak{M}}U(\operatorname{dom} \mathfrak{K}_1^-) = \mathfrak{M}^{[\perp]_2}$ if $\mathfrak{M}$ is nonnegative and $\mathcal{P}_{\mathfrak{M}}U(\operatorname{dom} \mathfrak{K}_1^+) = \mathfrak{M}^{[\perp]_2}$ and $\mathcal{P}_{\mathfrak{M}^{[\perp]_2}}U(\operatorname{dom} \mathfrak{K}_1^-) = \mathfrak{M}$ if $\mathfrak{M}$ is nonpositive. Hence Theorem 5.14 is partially a reformulation of [7, Proposition 3.15].

Since the subspace $\mathfrak{M}$ in Theorem 5.14 can be taken to be such that it satisfies the conclusions of Proposition 5.8, condition (iii) in Theorem 5.14 can be replaced by the condition (iii') below which describes the behavior of unitary relations in a different manner. Here (iii') is formulated only for the case that $\mathfrak{M}$ in Theorem 5.14 is hyper-maximal neutral.

(iii') there exist mappings V and W from $\operatorname{dom} U \cap \mathfrak{K}_1^+$ onto $\operatorname{dom} U \cap \mathfrak{K}_1^-$ such that $\mathfrak{L}$ and $\mathfrak{N}$ defined as

$$\mathfrak{L} = \{f + Vf : f \in \operatorname{dom} U \cap \mathfrak{K}_1^+\} \quad \text{and} \quad \mathfrak{N} = \{f + Wf : f \in \operatorname{dom} U \cap \mathfrak{K}_1^+\}$$

are neutral subspaces of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$, $U(\mathfrak{L})$ is a hyper-maximal neutral subspace of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ and $U(\mathfrak{N})$ is $\mathfrak{j}_2$-orthogonal to $U(\mathfrak{L})$.

In particular, W can be taken to be $-V$.

5.4. A further representation for unitary relations

As a consequence of the existence of a hyper-maximal semi-definite subspace in the domain of a unitary relation, the domain has the following "quasi-von Neumann" decomposition.

Proposition 5.16. *Let U be a unitary relation from $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ and let $\mathfrak{K}_1^+[+]\mathfrak{K}_1^-$ be a canonical decomposition of $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ with associated fundamental symmetry $\mathfrak{j}_1$. Then there exists a neutral subspace $\mathfrak{N}$ of $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ such that*

$$\overline{\operatorname{dom}} U = \operatorname{dom} U \dot{+} \mathfrak{j}_1\mathfrak{N}$$

and

$$\operatorname{dom} U = \ker U[\oplus_1]\left((\operatorname{dom} U \cap \mathfrak{K}_1^+[\oplus_1]\operatorname{dom} U \cap \mathfrak{K}_1^-)\dot{+}\mathfrak{N}\right).$$

Proof. Let $\mathfrak{j}_2$ be a fixed fundamental decomposition of $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ and denote by $\perp_{1,2}$ the $(\mathfrak{j}_1\times\mathfrak{j}_2)$-orthogonal complement in $\mathfrak{K}_1\times\mathfrak{K}_2$. Then define $\mathfrak{M}$ by

$$\mathfrak{M} = \mathfrak{j}_1 P_{\mathfrak{K}_1}\left((\overline{\operatorname{dom}} U \times \overline{\operatorname{ran}} U) \cap (\operatorname{gr}(U))^{\perp_{1,2}}\right),$$

where $P_{\mathfrak{K}_1}$ is the projection onto $\mathfrak{K}_1\times 0$ in the indicated Cartesian product space. Since $\overline{\operatorname{dom}} U \times \overline{\operatorname{ran}} U$ and $\operatorname{gr}(U)$ are closed in the product space it follows that $\overline{\operatorname{dom}} U = \operatorname{dom} U\dot{+}\mathfrak{j}_1\mathfrak{M}$. Next let $\mathfrak{L}$ be as in Proposition 5.4, i.e.

$$\operatorname{dom} U = \ker U \oplus_1 (\mathfrak{L}^{[\perp]_1} \ominus_1 \ker U) \oplus_1 (\mathfrak{L}\cap\mathfrak{j}_1\mathfrak{L}) \oplus_1 (\mathfrak{j}_1\mathfrak{L}^{[\perp]_1}\cap \operatorname{dom} U).$$

Then $\mathfrak{N} := \mathfrak{M}\cap(\mathfrak{L}^{[\perp]_1}\ominus_1 \ker U)$ is neutral, $\mathfrak{j}_1\mathfrak{N}\cap\operatorname{dom} U = \{0\}$ and $(\mathfrak{j}_1\mathfrak{L}^{[\perp]_1}\cap \operatorname{dom} U)\dot{+}\mathfrak{j}_1\mathfrak{N} = \mathfrak{j}_1(\mathfrak{L}^{[\perp]}\ominus_1\ker U)$. From these properties the statement follows. □

Note that the domain decomposition presented in Proposition 5.16 also holds for isometric relations V from $\{\mathfrak{K}_1,[\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2,[\cdot,\cdot]_2\}$ which satisfy

(i) V is closed;
(ii) $\ker V = (\operatorname{dom} V)^{[\perp]_1}$;
(iii) there exists a subspace $\mathfrak{L}$, $\ker U \subseteq \mathfrak{L} \subseteq \operatorname{dom} U$ such that $\mathfrak{L}$ is a hyper-maximal neutral, nonnegative or nonpositive subspace if $k_1^+ = k_1^-$, $k_1^+ > k_1^-$ or $k_1^+ < k_1^-$, respectively.

A subspace $\mathfrak{N}$ satisfying the conclusions of Proposition 5.16 is minimal, i.e., there does not exist a subspace of $\mathfrak{N}$ satisfying the same conclusions. Moreover, the above proof shows that there exists a subspace $\mathfrak{N}$ satisfying the conclusions of Proposition 5.16 which possesses a hyper-maximal semi-definite extension contained in the domain of U. The decomposition in Proposition 5.16 also provides an alternative proof for some of the equivalences in Proposition 4.7, because Proposition 5.16 shows that $\operatorname{dom} U$ has the "von Neumann" decomposition

$$\operatorname{dom} U = \ker U[\oplus_1](\operatorname{dom} U\cap\mathfrak{K}_1^+)[\oplus_1](\operatorname{dom} U\cap\mathfrak{K}_1^-),$$

i.e. $\mathfrak{N} = \{0\}$, if and only if $\operatorname{dom} U$ is closed.

The above properties (i)-(iii) are not sufficient for an isometric relation to be unitary as the following example shows. In particular, there exist isometric relations whose domains and ranges do not differ from the domains and range of unitary relations; they can only be distinguished by considering their graphs.

Example 5.17. Let $\{\mathfrak{H}, (\cdot,\cdot)\}$ be a Hilbert space and define the indefinite inner product $[\cdot,\cdot]$ on $\mathfrak{H}^2$ by

$$[\{f_1, f_2\}, \{g_1, g_2\}] = -i\left((f_2, g_1) - (f_1, g_2)\right), \quad f_1, f_2, g_1, g_2 \in \mathfrak{H}.$$

Moreover, let S be a closed symmetric operator in $\{\mathfrak{H}, (\cdot,\cdot)\}$ with dense domain. Then

$$V(\{f_1, f_2\}) = \{f_1 + Sf_2, f_2\}, \quad f_1 \in \mathfrak{H}, f_2 \in \operatorname{dom} S\}.$$

is an isometric operator from $\{\mathfrak{H}^2, [\cdot,\cdot]\}$ to $\{\mathfrak{H}^2, [\cdot,\cdot]\}$. Clearly $\operatorname{dom} V = \mathfrak{H} \times \operatorname{dom} S = \operatorname{ran} V$, so that the domain and range are dense in $\{\mathfrak{H}^2, [\cdot,\cdot]\}$. A direct calculation shows that V is unitary if and only if S is selfadjoint. In particular, if S does not have equal defect numbers, then V can not be extended to a unitary relation. However, independent of the nature of S, V maps the hyper-maximal neutral subspace $\mathfrak{H} \times \{0\}$ onto the hyper-maximal neutral subspace $\mathfrak{H} \times \{0\}$.

Hence the domain decomposition given in Proposition 5.16 is not specific enough to characterize unitary relations. However, combining Proposition 5.16 with Proposition 5.4, a characterizing graph decomposition can be obtained.

Theorem 5.18. *Let U be a unitary relation from $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ to $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ with associated fundamental symmetries $\mathfrak{j}_1$ and $\mathfrak{j}_2$, respectively. Then there exist a closed uniformly positive and negative subspace $\mathfrak{D}_i^+$ and $\mathfrak{D}_i^-$ of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ and neutral subspaces $\mathfrak{N}_i$ and $\mathfrak{O}_i$ of $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$, for $i = 1, 2$ such that*

$$\operatorname{dom} U = \ker U \oplus_1 \mathfrak{D}_1^+ \oplus_1 \mathfrak{D}_1^- \oplus_1 \left(\mathfrak{O}_1 \dot{+} \mathfrak{j}_1 \mathfrak{O}_1 \dot{+} \mathfrak{N}_1\right);$$
$$\operatorname{ran} U = \operatorname{mul} U \oplus_2 \mathfrak{D}_2^+ \oplus_2 \mathfrak{D}_2^- \oplus_2 \left(\mathfrak{O}_2 \dot{+} \mathfrak{j}_2 \mathfrak{O}_2 \dot{+} \mathfrak{N}_2\right),$$

where

(i) $U(\mathfrak{D}_1^+ + \mathfrak{D}_1^-) = \mathfrak{D}_2^+ + \mathfrak{D}_2^- + \operatorname{mul} U$;
(ii) $\operatorname{clos}(\mathfrak{N}_1) = \mathfrak{N}_1 \dot{+} \mathfrak{O}_1 = \operatorname{clos}(\mathfrak{O}_1)$ *and* $\mathfrak{N}_2 \dot{+} \mathfrak{O}_2 = \operatorname{clos}(\mathfrak{O}_2)$;
(iii) $U(\mathfrak{N}_1 \dot{+} \mathfrak{O}_1) = \mathfrak{j}_2 \mathfrak{O}_2 + \operatorname{mul} U$ *and* $U(\mathfrak{j}_1 \mathfrak{O}_1) = \mathfrak{N}_2 \dot{+} \mathfrak{O}_2 + \operatorname{mul} U$.

Conversely, if U is an isometric relation which has a decomposition as above and additionally $\ker U = (\operatorname{dom} U)^{[\perp]_1}$ and $\operatorname{mul} U = (\operatorname{ran} U)^{[\perp]_2}$, then U is unitary.

Proof. Since the converse statement is clear by Theorem 5.6, only the existence of the decomposition needs to be proven. For convenience of the reader the proof is split into three steps; in the first two steps it will be shown that $\operatorname{dom} U$ and $\operatorname{ran} U$ have the indicated decompositions such that (ii) is satisfied and in the third step it will be shown how the decompositions for the domain and range can be chosen such that they satisfy (i) and (iii).

Step 1: Let $\mathfrak{K}_i^+[+]\mathfrak{K}_i^-$ be the canonical decomposition for $\{\mathfrak{K}_i, [\cdot,\cdot]_i\}$ associated with the fundamental symmetry $\mathfrak{j}_i$ and let P_i^+ and P_i^- be the associated projections, $i = 1, 2$. Then by Proposition 5.16 there exists a neutral subspace $\mathfrak{N}_1$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that

$$\operatorname{dom} U = (\operatorname{dom} U \cap \mathfrak{K}_1^+ \oplus_1 \operatorname{dom} U \cap \mathfrak{K}_1^-) \dot{+} \mathfrak{N}_1 \tag{5.10}$$

and, moreover, $\mathfrak{N}_1$ has a hyper-maximal semi-definite extension $\mathfrak{L}$, $\mathfrak{L} \subseteq \operatorname{dom} U$, see the discussion following Proposition 5.16. The decomposition (5.10) and the existence of the hyper-maximal semi-definite extension imply that $\mathfrak{D}_1^+ := (\operatorname{dom} U \cap \mathfrak{K}_1^+) \cap (\mathfrak{N}_1)^{[\perp]_1}$ and $\mathfrak{D}_1^- := (\operatorname{dom} U \cap \mathfrak{K}_1^-) \cap (\mathfrak{N}_1)^{[\perp]_1}$ are closed uniformly definite subspaces. Hence (5.10) can be rewritten as

$$\operatorname{dom} U = \mathfrak{D}_1^+ \oplus_1 \mathfrak{D}_1^- \oplus_1 \left(\operatorname{dom} U \cap (\mathfrak{K}_1^+ \ominus \mathfrak{D}_1^+) \oplus_1 \operatorname{dom} U \cap (\mathfrak{K}_1^- \ominus \mathfrak{D}_1^-) \dot{+} \mathfrak{N}_1\right).$$

Step 2: Define $\widetilde{\mathfrak{K}}_1 = \ker U + \mathfrak{j}_1 \ker U + \mathfrak{D}_1^+ + \mathfrak{D}_1^-$, $\widehat{\mathfrak{K}}_1 = \mathfrak{K}_1 \cap \widetilde{\mathfrak{K}}_1^{[\perp]_1}$, $\widetilde{\mathfrak{K}}_2 = U(\mathfrak{D}_1^+ + \mathfrak{D}_1^-) + \mathfrak{j}_2 \operatorname{mul} U$ and $\widehat{\mathfrak{K}}_2 = \mathfrak{K}_2 \cap \widetilde{\mathfrak{K}}_2^{[\perp]_2}$. Then $\{\widetilde{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$, $\{\widehat{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$, $\{\widetilde{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$ and $\{\widehat{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$ are Kreĭn spaces. Now, define $\widetilde{U}$ and $\widehat{U}$ via

$$\operatorname{gr}(\widetilde{U}) = \operatorname{gr}(U) \cap (\widetilde{\mathfrak{K}}_1 \times \widetilde{\mathfrak{K}}_2) \quad \text{and} \quad \operatorname{gr}(\widehat{U}) = \operatorname{gr}(U) \cap (\widehat{\mathfrak{K}}_1 \times \widehat{\mathfrak{K}}_2).$$

Then since U and $\widetilde{U}$ are unitary, and $\operatorname{gr}(U) = \operatorname{gr}(\widetilde{U}) + \operatorname{gr}(\widehat{U})$, Lemma 3.12 implies that $\widehat{U}$ is a unitary operator without kernel. The domain of $\widehat{U}$ has the decomposition

$$\operatorname{dom} \widehat{U} = \operatorname{dom} \widehat{U} \cap \mathfrak{K}_1^+ [\oplus_1] \operatorname{dom} \widehat{U} \cap \mathfrak{K}_1^- \dot{+} \mathfrak{N}_1.$$

In particular, $\operatorname{clos}(\mathfrak{N}_1)$ is hyper-maximal neutral in $\{\widehat{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$ and, hence, $\mathfrak{O}_1 := \mathfrak{j}_1(\mathfrak{j}_1 \operatorname{clos}(\mathfrak{N}_1) \cap \operatorname{dom} U)$ is such that $\operatorname{clos}(\mathfrak{N}_1) = \mathfrak{N}_1 \dot{+} \mathfrak{O}_1$ and $\mathfrak{O}_1 + \mathfrak{j}_1 \mathfrak{O}_1 = \operatorname{dom} \widehat{U} \cap \mathfrak{K}_1^+ [\oplus_1] \operatorname{dom} \widehat{U} \cap \mathfrak{K}_1^-$. Therefore

$$\operatorname{dom} \widehat{U} = \mathfrak{O}_1 \dot{+} \mathfrak{j}_1 \mathfrak{O}_1 \dot{+} \mathfrak{N}_1.$$

Moreover, from $\operatorname{clos}(\operatorname{dom} \widehat{U} \cap \mathfrak{K}_1^\pm) = \overline{\operatorname{dom}} \widehat{U} \cap \mathfrak{K}_1^\pm$, see Proposition 4.1, it follows that $\operatorname{clos}(\mathfrak{O}_1)$ is also hyper-maximal neutral subspace of $\{\widehat{\mathfrak{K}}_1, [\cdot,\cdot]_1\}$ and coincides with $\operatorname{clos}(\mathfrak{N}_1)$, because $\operatorname{clos} \mathfrak{N}_1 = \mathfrak{O}_1 + \mathfrak{N}_1$. Consequently, $\operatorname{dom} U$ and, hence, also $\operatorname{ran} U$ have the indicated decompositions such that (ii) holds.

Step 3: By Proposition 5.4 (and Remark 5.5) there exists a hyper-maximal neutral subspace $\widehat{\mathfrak{L}} \subseteq \operatorname{dom} \widehat{U}$ of $\{\mathfrak{K}_1, [\cdot,\cdot]_1\}$ such that $\widehat{\mathfrak{M}} = \widehat{U}(\mathfrak{j}_1 \widehat{\mathfrak{L}} \cap \operatorname{dom} \widehat{U})$ is a hyper-maximal neutral subspace of $\{\widehat{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$ and $\widehat{U}(\widehat{\mathfrak{L}}) = \mathfrak{j}_2 \widehat{\mathfrak{M}} \cap \operatorname{ran} \widehat{U}$. Note that by Proposition 5.4 there exists also hyper-maximal semi-definite subspace $\widetilde{\mathfrak{L}}$ and $\widetilde{\mathfrak{M}}$ for $\widetilde{U}$ having the same properties. Consequently, $\mathfrak{L} := \widehat{\mathfrak{L}} + \widetilde{\mathfrak{L}}$ and $\mathfrak{M} := \widehat{\mathfrak{M}} + \widetilde{\mathfrak{M}}$ satisfy the criteria in Proposition 5.4 for U.

By means of this observation it is evident, using steps 1 and 2, the following domain and range decomposition of $\widehat{U}$ can be obtained:

$$\operatorname{dom} \widehat{U} = \mathfrak{N}_1 \dot{+} \mathfrak{O}_1 \dot{+} \mathfrak{j}_1 \mathfrak{O}_1 \quad \text{and} \quad \operatorname{ran} \widehat{U} = \mathfrak{D}_2^+ \oplus_2 \mathfrak{D}_2^- \oplus_2 \left(\mathfrak{N}_2 \dot{+} \mathfrak{O}_2 \dot{+} \mathfrak{j}_2 \mathfrak{O}_2\right),$$

where (because $\mathfrak{M}$ is hyper-maximal neutral in $\{\widehat{\mathfrak{K}}_2, [\cdot,\cdot]_2\}$) there exists a neutral subspace $\widetilde{\mathfrak{O}}_2$ of $\{\mathfrak{K}_2, [\cdot,\cdot]_2\}$ such that $\mathfrak{D}_2^+ \oplus_2 \mathfrak{D}_2^- = \widetilde{\mathfrak{O}}_2 + \mathfrak{j}_2 \widetilde{\mathfrak{O}}_2$ and

$$\begin{aligned} \mathfrak{L} &= \mathfrak{N}_1 \dot{+} \mathfrak{O}_1 && \text{and} \quad \mathfrak{j}_1 \mathfrak{L} \cap \operatorname{dom} \widehat{U} = \mathfrak{j}_1 \mathfrak{O}_1; \\ \mathfrak{M} &= \widetilde{\mathfrak{O}}_2 \oplus_2 (\mathfrak{N}_2 \dot{+} \mathfrak{O}_2) && \text{and} \quad \mathfrak{j}_2 \mathfrak{M} \cap \operatorname{ran} \widehat{U} = \mathfrak{j}_2 \widetilde{\mathfrak{O}}_2 \oplus_2 \mathfrak{j}_2 \mathfrak{O}_2. \end{aligned}$$

This completes the proof. ☐

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