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Long-range dependent completely correlated mixed fractional Brownian motion

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ABSTRACT

In this paper we introduce the long-range dependent completely correlated mixed fractional Brownian motion (ccmfBm). This is a process that is driven by a mixture of Brownian motion (Bm) and a long-range dependent completely correlated fractional Brownian motion (fBm, ccfBm) that is constructed from the Brownian motion via the Molchan–Golosov representation. Thus, there is a single Bm driving the mixed process. In the short time-scales the ccmfBm behaves like the Bm (it has Brownian Hölder index and quadratic variation). However, in the long time-scales it behaves like the fBm (it has long-range dependence governed by the fBms Hurst index). We provide a transfer principle for the ccmfBm and use it to construct the Cameron–Martin–Girsanov–Hitsuda theorem and prediction formulas. Finally, we illustrate the ccmfBm by simulations.

1. Introduction

Long range dependence have numerous applications in various models and have been a topic of active research, see e.g. monographs [9,30] and references therein.

The fractional Brownian motion (fBm) is maybe the simplest model for long-range dependence. Indeed, the fBm is Gaussian, has stationary increments, and continuous sample paths almost surely. For details on the fBm we refer to [10,23]. For the use of fBm in signal and image processing see e.g. [2,5,11,16,18,25–27,29,39].

The fBm is not a suitable model if one requires long-range dependence in long time scales and, at the same time, somehow different behaviour in the short time scales. For example in finance, empirical studies suggest long-range dependence while the standard Brownian motion (Bm) would be more suitable model in the short time scales. Indeed, the short time scale behaviour of the fBm makes it unsuitable for the use in finance since its paths are smooth enough to admit arbitrage. On the other hand, option prices indicate that the short time behaviour of the log-returns of financial time series are consistent with the Bm. We refer to [6,8] for details. The different long time and short time behaviour can be combined by a so-called mixed fractional Brownian motion (mfBm) that inherits short time behaviour from the Bm and long time behaviour from the fBm. Cheridito [13] introduced an independent mfBm as a sum of a Bm and an independent fBm that has received attention in the literature ever since.

One of the wanted features for a Gaussian model is the so-called transfer principle that allows to reconstruct underlying Brownian motion from the observations. Indeed, once the transfer principle is established, several applications including prediction formulas and likelihood ratios become rather straightforward. For this reason transfer principles are actively studied in the literature, see

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e.g. [12] for the transfer principle in the case of independent mfBm and [36] for the transfer principle for the general case of Gaussian processes. Let us note that the transfer principle is called canonical innovation representation in earlier works such as in Hida and Hitsuda [19].

In this article we introduce the long-range dependent completely correlated mixed fractional Brownian motion (ccmfBm). We study its basic properties and provide a transfer principle for it. In comparison to the independent mfBm, in our case the transfer principle is explicit allowing easily computable formulas and model fitting in different applications. The main contribution of this paper is the introduction of the new model, the ccmfBm, and its transfer principle. We show that ccmfBm share similar properties than the independent mixing mfBm, making it a possible model in finance. At the same time, an explicit transfer principle gives advantage of ccmfBm over mfBm. On the other hand, we show that ccmfBm do not have stationary increments which makes simulations and parameter estimation more complicated.

The rest of the paper is organized as follows: In Section 2 we recall what is the fractional Brownian motion (fBm) and recall its basic properties, and introduce the long-range dependent completely correlated mixed fractional Brownian motion (ccmfBm) and state its basic properties. In Section 3 we develop the transfer principle. In Section 4 we develop the Cameron–Martin–Girsanov–Hitsuda theorem for the ccmfBm by using the transfer principle. In Section 5 we state the prediction formula for the ccmfBm that follows directly from the transfer principle. In Section 6 we illustrate the ccmfBm by simulations. Finally, in Section 7 we summarize our findings.

2. Definitions and basic properties

Throughout, all our processes are defined on the same complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In Section 2.1 we first recall briefly what is the fractional Brownian motion (fBm) and recall its well-known basic properties. Then in Section 2.2 we introduce our new process, the long-range dependent completely correlated mixed fractional Brownian motion (ccmfBm), and derive its basic properties.

2.1. Fractional Brownian motion

The fractional Brownian motion (fBm) $B^H = (B^H_t)_{t\geq 0}$ with Hurst index $H \in (0,1)$ is the centred Gaussian process having the covariance function

$$R_H(t,s) = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right].$$

The fBm was introduced by Kolmogorov [21] for modelling turbulence under the name Wiener Spiral. The fBm was given its current name by Mandelbrot and Van Ness [22]. The fBm is the (up to a multiplicative constant) unique centred Gaussian process that has stationary increments and is self-similar with the Hurst index H. These facts follow immediately from the form of the covariance function R_H. Indeed, stationarity of the increments means that for all $t \geq s$,

$$R_H(t,s) = \frac{1}{2}\left[R_H(t,t) + R_H(s,s) - R_H(t-s,t-s)\right],$$

and the H-self-similarity means that for all $a > 0$,

$$R_H(at,as) = a^{2H} R_H(t,s).$$

In particular, if $H = \frac{1}{2}$, then the fBm is the standard Brownian motion (Bm), since

$$R_{\frac{1}{2}}(t,s) = \min(t,s).$$

The fBm admits a representation with respect to a Bm. Indeed for $H \in (\frac{1}{2}, 1)$, denote

$$c(H) = \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}}\left(H - \frac{1}{2}\right)$$

where Γ is the Gamma function. Set

$$K_H(t,s) = c(H)\frac{1}{s^{H-\frac{1}{2}}}\int_s^t \frac{u^{H-\frac{1}{2}}}{(u-s)^{\frac{3}{2}-H}}\,\mathrm{d}u. \tag{2.1}$$

Then $K_H(t,s)$ is a continuous-time analog of the Cholesky decomposition of $R_H(t,s)$ in the sense that

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,u)K_H(s,u)\,\mathrm{d}u,$$

where $t \wedge s = \min(t,s)$. Consequently the following representation due to Molchan and Golosov [24] holds:

$$B^H_t = \int_0^t K_H(t,s)\,\mathrm{d}W_s, \tag{2.2}$$

where W is a Bm. We note that the representation (2.2) is of Volterra type meaning that $K_H(t,s) = 0$ when $s > t$.

For $H \in (\frac{1}{2}, 1)$ the fractional Brownian motion is long-range dependent (see [9]) in the sense of the following definition: A stochastic process Y is long-range dependent if, for all $t_0 \geq 0$ and lags $\delta > 0$, the incremental covariance

$$\rho_Y(t_0, \delta; t) = \mathbb{E}\left[\left(Y_{t_0+\delta} - Y_{t_0}\right)\left(Y_{t+\delta} - Y_t\right)\right]$$

decay at most following a power law:

$$\rho_Y(t_0, \delta; t) \geq \frac{C_{t_0,\delta}}{t^\beta}$$

for some $\beta \in (0, 1)$. Indeed, for fBm with $H > 1/2$ we have

$$\begin{aligned}
&\rho_{B^H}(t_0, \delta; t) \\
&= \mathbb{E}\left[\left(B^H_{t_0+\delta} - B^H_{t_0}\right)\left(B^H_{t+\delta} - B^H_t\right)\right] \\
&= \frac{1}{\delta^{2H}}\mathbb{E}\left[B^H_1\left(B^H_{\frac{t-t_0}{\delta}+1} - B^H_{\frac{t-t_0}{\delta}}\right)\right] \\
&= \frac{1}{\delta^{2H}}\int_0^1\int_{\frac{t-t_0}{\delta}}^{\frac{t-t_0}{\delta}+1}\frac{\partial^2 R_H}{\partial u \partial v}(u, v)\,\mathrm{d}u\mathrm{d}v \\
&= \frac{H(2H-1)}{\delta^{2H}}\int_0^1\int_{\frac{t-t_0}{\delta}}^{\frac{t-t_0}{\delta}+1}(u-v)^{2H-2}\,\mathrm{d}u\mathrm{d}v \\
&\sim \frac{H(2H-1)\delta^2}{\left(t-t_0\right)^{2-2H}}.
\end{aligned}$$

Here and in what follows, we use the notation $f(t) \sim g(t)$ for asymptotic equivalence meaning that $\lim f(t)/g(t) = 1$.

For more details on the fBm, we refer to Biagini et al. [10] and Mishura [23].

2.2. Completely correlated mixed fractional Brownian motion

In this article, we consider a generalization of the fBm where the fBm is mixed with a Bm that generates the fBm. For other generalizations of the fBm, see e.g. [3,13,26,27].

Remark 2.1. We only consider the case $H \in (1/2, 1)$, even though some of the results would be true also for the short-range dependent case where $H \in (0, 1/2)$. See Section 7 for further discussion of the short-range completely correlated mixed fractional Brownian motions.

Definition 2.1. The completely correlated mixed fractional Brownian motion (ccmfBm) is

$$X = aW + bB^H,$$

where $a, b \in \mathbb{R}$ with $ab \neq 0$, W is a Bm, and B^H is a fBm constructed from W via (2.2).

Unlike the mixed fBm with independent summands (see e.g. Cheridito [13]), the ccmfBm does not have stationary increments. This can be seen directly from the form of the covariance function (2.3) below. However, the ccmfBm shares some path-properties and long-range dependence with the independent summands fBm.

In proving the properties of the ccmfBm the following Itô-isometry for the Bm is a central tool: for any L^2-function the Wiener integral satisfies the isometry

$$\mathbb{E}\left[\int_0^T f(t)\,\mathrm{d}W_t\right] = \int_0^T f(t)^2\,\mathrm{d}t.$$

We also recall that the Hölder index of a stochastic process is the supremum of exponents H such that the process is H-Hölder continuous. We recall that the Brownian motion has $1/2$ as its Hölder index, but is not $1/2$-Hölder continuous.

Proposition 2.1. *The ccmfBm is a centred Gaussian process with the covariance function*

$$\begin{aligned}
R(t, s) = {} & a^2(t \wedge s) \\
& + ab\int_0^{t\wedge s}\left[K_H(t, u) + K_H(s, u)\right]\,\mathrm{d}u \\
& + b^2 R_H(t, s).
\end{aligned} \tag{2.3}$$

Moreover, the ccmfBm

(i) has Hölder index $1/2$,
(ii) has quadratic variation $t \mapsto a^2 t$,
(iii) is long-range dependent having the same power law decay in its autocovariance as the fBm, if $ab > 0$.

Proof. It is clear that X is centred. The covariance (2.3) follows from the Itô-isometry

$$\begin{aligned}\mathbb{E}\left[W_t B_s^H\right] &= \mathbb{E}\left[\int_0^T \mathbf{1}_t(u)\,\mathrm{d}W_u \int_0^T K_H(s,u)\,\mathrm{d}W_u\right] \\ &= \int_0^{t\wedge s} K_H(s,u)\,\mathrm{d}u,\end{aligned}$$

where we have denoted $\mathbf{1}_t$ as the indicator of the set $[0,t]$.

(i) The Hölder index follows from Theorem 1 of [4] and (2.3). Indeed, let $s<t$. Then

$$\begin{aligned}\mathbb{E}\left[(X_t - X_s)^2\right] &= a^2|t-s| \\ &\quad + ab\mathbb{E}\left[\left(W_t - W_s\right)\left(B_t^H - B_s^H\right)\right] \\ &\quad + b^2|t-s|^{2H}\end{aligned}$$

and, by Itô-isometry and Volterra property,

$$\begin{aligned}&\mathbb{E}\left[\left(W_t - W_s\right)\left(B_t^H - B_s^H\right)\right] \\ &= \int_0^T \left(\mathbf{1}_t(u) - \mathbf{1}_s(u)\right)\left(K_H(t,u) - K_H(s,u)\right)\,\mathrm{d}u \\ &= \int_s^t K_H(t,u)\,\mathrm{d}u \\ &= K_H(t,u^*)|t-s|\end{aligned}$$

for some $u^* \in [s,t]$ as $K_H(t,s)$ is continuous in s. Since $K_H(t,s)\to 0$ as $s\to t$, we see that

$$\mathbb{E}\left[(X_t - X_s)^2\right] \sim a^2|t-s|,$$

which shows that the Hölder index is $1/2$.

(ii) The quadratic variation comes from the fact that B^H has zero quadratic variation for $H\in(1/2,1)$ applied to Example 1 of [7] (see also [15]).

(iii) Finally, let us show the long-range dependence. Now,

$$\begin{aligned}\rho_X(t_0,\delta;t) &= \mathbb{E}\left[\left(X_{t_0+\delta} - X_{t_0}\right)\left(X_{t+\delta} - X_t\right)\right] \\ &= a^2\mathbb{E}\left[\left(W_{t_0+\delta} - W_{t_0}\right)\left(W_{t+\delta} - W_t\right)\right] \\ &\quad + ab\mathbb{E}\left[\left(W_{t_0+\delta} - W_{t_0}\right)\left(B_{t+\delta}^H - B_t^H\right)\right] \\ &\quad + ab\mathbb{E}\left[\left(B_{t_0+\delta}^H - B_{t_0}^H\right)\left(W_{t+\delta} - W_t\right)\right] \\ &\quad + b^2\mathbb{E}\left[\left(B_{t_0+\delta}^H - B_{t_0}^H\right)\left(B_{t+\delta}^H - B_t^H\right)\right] \\ &\sim ab\mathbb{E}\left[\left(W_{t_0+\delta} - W_{t_0}\right)\left(B_{t+\delta}^H - B_t^H\right)\right] \\ &\quad + b^2\rho_{B^H}(t_0,\delta;t).\end{aligned}$$

The long-range dependence follows by noting that

$$\begin{aligned}&\mathbb{E}\left[\left(W_{t_0+\delta} - W_{t_0}\right)\left(B_{t+\delta}^H - B_t^H\right)\right] \\ &= \int_{t_0}^{t_0+\delta} \left[K_H(t+\delta,u) - K_H(t,u)\right]\,\mathrm{d}u \\ &= \int_{t_0}^{t_0+\delta}\int_t^{t+\delta} \frac{\partial K_H}{\partial v}(v,u)\,\mathrm{d}v\,\mathrm{d}u \\ &\sim \delta^2 \frac{\partial K_H}{\partial t}(t,t_0) \\ &= \delta^2 c(H)\left(\frac{t}{t_0}\right)^{H-\frac{1}{2}} \frac{1}{(t-t_0)^{\frac{3}{2}-H}},\end{aligned}$$

which is the same power decay law, namely t^{2H-2}, as the fBm part has: $\rho_{B^H}(t_0,\delta;t)$. Since $ab>0$ the last two terms

$$ab\mathbb{E}\left[\left(W_{t_0+\delta} - W_{t_0}\right)\left(B_{t+\delta}^H - B_t^H\right)\right]$$

and

$$b^2\rho_{B^H}(t_0,\delta;t)$$

do not cancel out. □

Remark 2.2. From Proposition 2.1 part (ii) we see that the mixing magnitude a has a clear meaning in mathematical finance context. Indeed, if the log-returns of an underlying asset follow a ccmfBm, then the asset's derivative prices are completely determined by the parameter a and they are independent of the parameters b and H. Consequently, the parameter a is given by the volatility implied by the derivative prices. The parameter H is connected to the long-range dependence. In particular, the power law decay of the incremental autocovariance of the log-returns is given by $2-2H$. Finally, the parameter b can be taken as a tuning parameter to fix the autocovariance to the given implied volatility a and the power-decay $2-2H$.

Unlike the case of independent summands mfBm with $a=b$ and $H \in (3/4,1)$ (see Cheridito [13]), the ccmfBm is not equivalent to a Bm in any range of $H \in (1/2,1)$. Indeed, by Theorem 1.1 of Chong et al. [14] we have the following:

Proposition 2.2. *Let $b \neq 0$. Then the law of the ccmfBm on $[0,T]$ is singular to the law of any multiple of Bm on $[0,T]$.*

Remark 2.3. By Proposition 2.2 the presence or absence of the fBm part in the ccmfBm mixture can be determined almost surely if one has continuous observations.

3. Transfer principle

A non-anticipative transformation $F(X)$ of a signal X is such that it does not look into future: $F_t(X) = F(X_s; s \leq t)$. For a general anticipative transfer principle we refer to [36].

The transfer principle of this section states that from the ccmfBm we can construct a Bm in a non-anticipative way (the inverse transfer principle) and then represent the ccmfBm in a non-anticipative way by using the constructed Bm (the direct transfer principle). We consider processes on a compact time interval $[0,T]$ in this section.

Let $L^2 = L^2([0,T])$ and $\|\cdot\|_2 = \|\cdot\|_{L^2}$.

For a kernel $K : [0,T]^2 \to \mathbb{R}$ its associated operator is

$$\mathrm{K}f(t) = \int_0^T f(s)K(t,s)\,\mathrm{d}s.$$

The adjoint associated operator K^* of a kernel K is defined by linearly extending the relation

$$\mathrm{K}^*\mathbf{1}_t(s) = K(t,s), \tag{3.1}$$

where $\mathbf{1}_t = \mathbf{1}_{[0,t)}$ is the indicator function.

Remark 3.1. The name adjoint comes from the following identity. Let ϕ be simple and let $h \in L^2$. Then, by Lemma 1 of Alòs et al. [1] we have

$$\int_0^T \mathrm{K}^*\phi(t)h(t)\,\mathrm{d}t = \int_0^T \phi(t)\mathrm{K}h(\mathrm{d}t).$$

Since the kernel $K_H(t,s)$ is differentiable in t, Volterra, and $K_H(t,t-)=0$, its adjoint associated operator can be written as

$$\mathrm{K}_H^* f(t) = \int_t^T f(u)\frac{\partial K_H}{\partial u}(u,t)\,\mathrm{d}u. \tag{3.2}$$

Indeed, to verify formula (3.2) it is enough to check that it satisfies the relation (3.1).

Let Λ be the closure of the indicator functions $\mathbf{1}_t$, $t \in [0,T]$, under the inner product generated by the relation

$$\langle \mathbf{1}_t, \mathbf{1}_s \rangle_\Lambda = R(t,s).$$

Let $\mathcal{H}_1$ be the linear space of X, i.e., the closure of the random variables X_t, $t \in [0,T]$, in $L^2(\Omega)$.

For $f \in \Lambda$ the abstract Wiener integral

$$\int_0^T f(t)\,\mathrm{d}X_t$$

is the image of the isometry $\mathbf{1}_t \mapsto X_t$ from Λ to $\mathcal{H}_1$.

Denote $L(t,s) = a\mathbf{1}_t(s) + bK_H(t,s)$ and let L and L^* be the associated and adjoint associated operators of L.

Before characterizing the operator L^* we recall that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t f(u)\,\mathrm{d}B_u^H \int_0^s g(v)\,\mathrm{d}B_v^H\right] \\
&= \int_0^{t\wedge s}\left(\int_\tau^t f(u)\frac{\partial K_H}{\partial u}(u,\tau)\mathrm{d}u\right)\left(\int_\tau^s f(v)\frac{\partial K_H}{\partial v}(v,\tau)\mathrm{d}v\right)\mathrm{d}\tau.
\end{aligned}$$

Lemma 3.1. *L^* is a bounded operator on L^2 and it can be represented as*

$$\begin{aligned}
\mathrm{L}^* f(t) &= af(t) + b\int_t^T f(u)\frac{\partial K_H}{\partial u}(u,t)\,\mathrm{d}u \\
&= af(t) + \frac{bc(H)}{t^{H-\frac{1}{2}}}\int_t^T f(u)\frac{u^{H-\frac{1}{2}}}{(u-t)^{\frac{3}{2}-H}}\,\mathrm{d}u.
\end{aligned} \tag{3.3}$$

Proof. Let us first note that if $I(t,s) = \mathbf{1}_t(s)$, then I is the integral operator and I* is the identity operator. Let us then note that $\mathrm{L}^* = a\mathrm{I}^* + b\mathrm{K}_H^*$, where I* is the identity operator and K_H^* is the adjoint associated operator defined in (3.2), where $K_H(t,s)$ is the Molchan–Golosov kernel K_H defined in (2.1). Since

$$\|\mathrm{L}^* f\|_2 \le |a|\|f\|_2 + |b|\|\mathrm{K}_H^* f\|_2,$$

L* is bounded in L^2, if K_H^* is bounded in L^2. Also, we note that (3.3) is true for step functions. So, if L* is bounded in L^2, the formula (3.3) extends to all functions in L^2 Finally we note that K_H^* is bounded on L^2, because (for step functions f)

$$\begin{aligned}
\|\mathrm{K}_H^* f\|_2^2 &= \int_0^T \left[\mathrm{K}_H^* f(t)\right]^2 \mathrm{d}t \\
&= \int_0^T \int_0^T f(t) f(s) \frac{\partial^2 R_H}{\partial s \partial t}(t,s)\,\mathrm{d}s\mathrm{d}t \\
&= H(2H-1) \int_0^T \int_0^T \frac{f(t)f(s)}{|t-s|^{2-2H}}\,\mathrm{d}s\mathrm{d}t \\
&\le H(2H-1) \int_0^T \int_0^T \frac{f(t)^2}{|t-s|^{2-2H}}\,\mathrm{d}s\mathrm{d}t \\
&\le H(2H-1) \frac{T^{2H-1}}{H-\frac{1}{2}} \|f\|_2^2,
\end{aligned}$$

where we have used the elementary estimate

$$2|f(t)f(s)| \le f(t)^2 + f(s)^2$$

and symmetry. □

Lemma 3.2. *For each $t \in [0,T]$, the integral equation*

$$\mathbf{1}_t(s) = aL^{-1}(t,s) + b\int_s^t L^{-1}(t,u)\frac{\partial K_H}{\partial u}(u,s)\,\mathrm{d}u \tag{3.4}$$

admits the unique L^2-solution given by

$$L^{-1}(t,s) = \frac{1}{a}\mathbf{1}_t(s) + \frac{1}{a}\sum_{k=1}^{\infty}(-1)^k\left(\frac{b}{a}\right)^k \gamma_k(t,s), \tag{3.5}$$

where

$$\begin{aligned}
\gamma_k(t,s) = &\frac{c(H)^k \Gamma(H-\frac{1}{2})^k}{\Gamma\left(k\left(H-\frac{1}{2}\right)\right)} \\
&\times \frac{1}{s^{H-\frac{1}{2}}}\int_s^t u^{H-\frac{1}{2}}(u-s)^{k(H-\frac{1}{2})-1}\,\mathrm{d}u.
\end{aligned}$$

Proof. Denote

$$G(s,u) = -\frac{bc(H)}{a}\frac{u^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}(u-s)^{\frac{3}{2}-H}}.$$

Then (3.4) is the anti-Volterra equation of the second kind

$$\frac{1}{a}\mathbf{1}_t(s) = L^{-1}(t,s) - \int_s^t L^{-1}(t,u)G(s,u)\,\mathrm{d}u.$$

Since $\mathrm{L}^* = a\mathrm{I}^* - a\mathrm{G}$, Lemma 3.1 implies that the solution of Eq. (3.4) is given by the L^2-convergent Liouville–Neumann series

$$L^{-1}(t,s) = \sum_{k=1}^{\infty} \mathrm{G}^k\left[\frac{1}{a}\mathbf{1}_t\right](s), \tag{3.6}$$

where G^0 is the identity operator and $\mathrm{G}^{k+1} = \mathrm{G}\mathrm{G}^k$. Indeed, the series converge by [31, Theorem 2.7.1] which can be seen by writing the anti-Volterra operator G as Volterra operator via time reversal Formula (3.5) follows from formula (3.6) by induction by using the formula

$$\int_s^u (v-s)^{k\alpha-1}(u-v)^{\alpha-1}\,\mathrm{d}v = \frac{\Gamma(k\alpha)\Gamma(\alpha)}{\Gamma((k+1)\alpha)}(u-s)^{(k+1)\alpha-1},$$

where $\alpha = H - \frac{1}{2}$. □

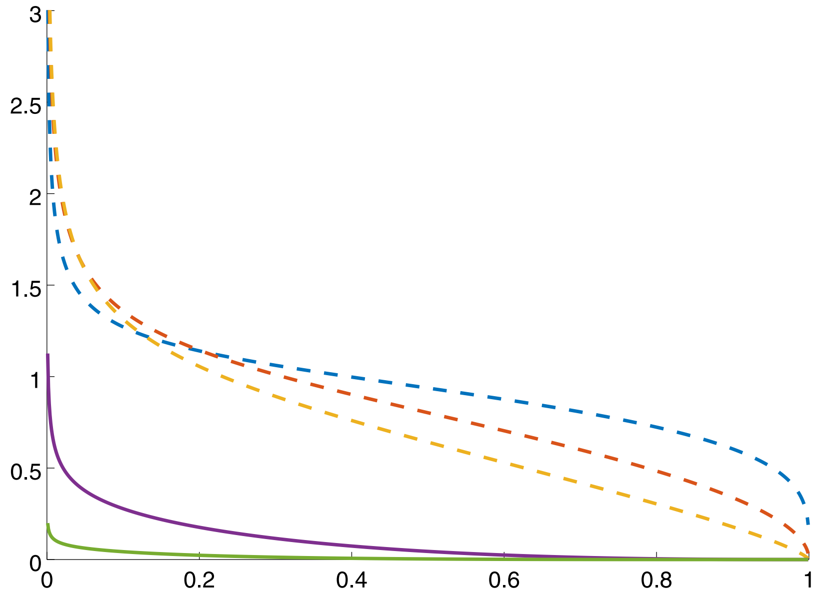

Fig. 1. The summands $\gamma_k(t,s)$ for $k=1,2,3$ (dashed lines) and $k=10,15$ (solid lines) with $t=1$, $a=b=1$ and $H=0.75$.

Remark 3.2. The series (3.5) converges fast. Indeed, by using the Stirling's approximation

$$\Gamma\left(k\left(H-\frac{1}{2}\right)\right) \sim \sqrt{2\pi}\left(k\left(H-\frac{1}{2}\right)\right)^{k\left(H-\frac{1}{2}\right)-\frac{1}{2}} \times e^{-k\left(H-\frac{1}{2}\right)}$$

and the estimate

$$\frac{1}{s^{H-\frac{1}{2}}}\int_s^t u^{H-\frac{1}{2}}(u-s)^{k\left(H-\frac{1}{2}\right)-1}\,\mathrm{d}u \le \left(\frac{t}{s}\right)^{H-\frac{1}{2}} \frac{1}{k\left(H-\frac{1}{2}\right)}(t-s)^{k\left(H-\frac{1}{2}\right)}$$

we obtain

$$\gamma_k(t,s) \le \frac{C^k}{\left(k\left(H-\frac{1}{2}\right)\right)^{k\left(H-\frac{1}{2}\right)+\frac{1}{2}}}\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{k\left(H-\frac{1}{2}\right)},$$

which also shows that (3.5) converges uniformly for all $s \in [\varepsilon, T]$.

We have illustrated the summands γ_k and the kernels L and L^{-1} in Figs. 1–3.

From Lemma 3.2 we obtain directly the following basic form of the transfer principle that is the main result of this paper.

Theorem 3.1. *Let $L^{-1}(t,s)$ be given by (3.5). The ccmfBm X is an invertible Gaussian Volterra process in the sense that the process W defined as the abstract Wiener integral*

$$W_t = \int_0^t L^{-1}(t,s)\,\mathrm{d}X_s$$

is a Bm, and the ccmfBm can be reconstructed from it by the Wiener integral

$$X_t = \int_0^t L(t,s)\,\mathrm{d}W_s,$$

where

$$L(t,s) = a\mathbf{1}_t(s) + bK_H(t,s)$$

and $K_H(t,s)$ is the Molchan–Golosov kernel given by (2.1).

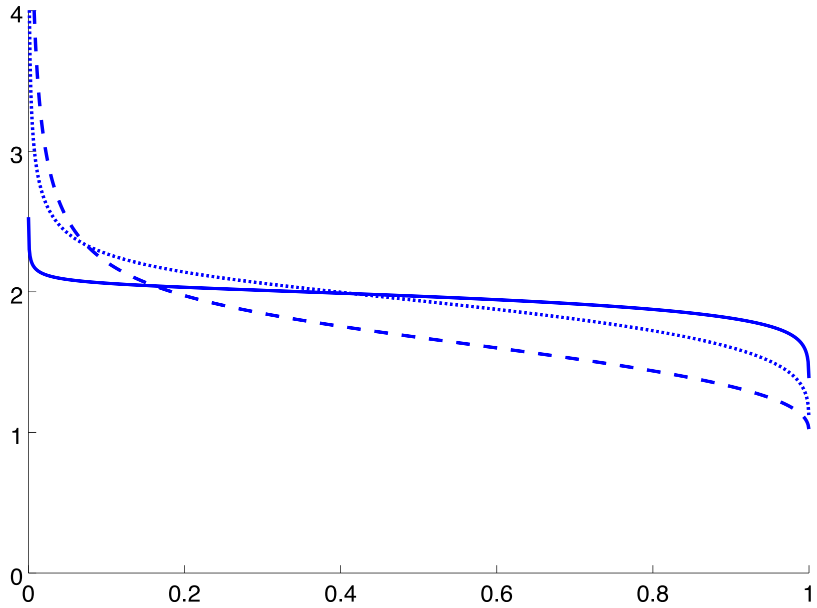

Fig. 2. The kernels $L(t,s)$ for $t=1$, $a=b=1$ and $H=0.6$ (solid line), $H=0.75$ (dotted line) and $H=0.9$ (dashed line).

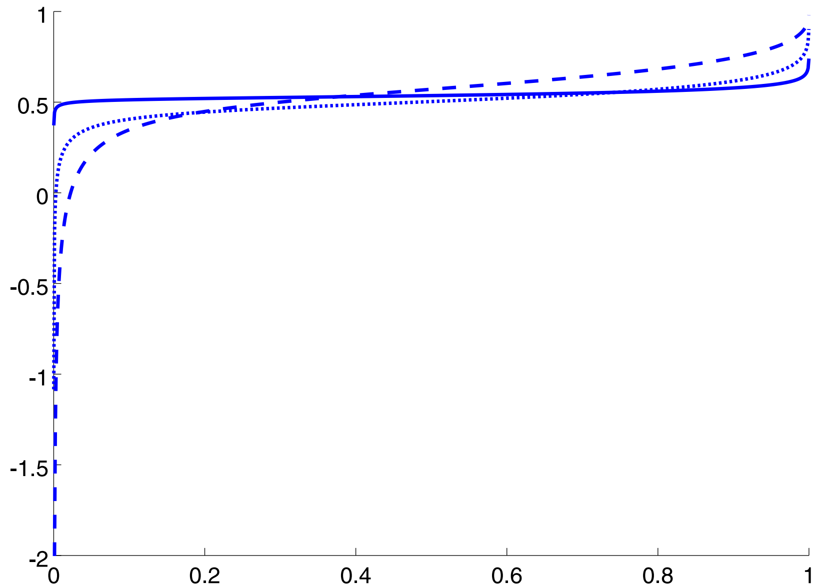

Fig. 3. The kernels $L^{-1}(t,s)$ for $t=1$, $a=b=1$ and $H=0.6$ (solid line), $H=0.75$ (dotted line) and $H=0.9$ (dashed line).

Proof. The representation

$$X_t = \int_0^t L(t,s)\,\mathrm{d}W_s$$

holds by definition.

For the inverse representation, let us first note that

$$\int_0^t g(s)\,\mathrm{d}X_s = \int_0^T \mathrm{L}^* g(s)\,\mathrm{d}W_s \tag{3.7}$$

holds for $g = \mathbf{1}_t$ by definition, since $\mathrm{L}^*\mathbf{1}_t(s) = L(t,s)$. Consequently, since L^* is linear, (3.7) holds for all simple $g \in L^2$. Finally, since L^* is bounded by Lemma 3.1 and the Wiener integral is an isometry, (3.7) holds for all $g \in L^2$. Let us then note that by Lemma 3.2.

we have

$$(\mathrm{L}^*)L^{-1}(t,\cdot)(s) = \mathbf{1}_t(s).$$

The representation

$$W_t = \int_0^t L^{-1}(t,s)\,\mathrm{d}X_s$$

follows now from (3.7) by setting $g = L^{-1}(t,\cdot)$. □

Remark 3.3. The key point for having the inverse representation

$$W_t = \int_0^t L^{-1}(t,s)\,\mathrm{d}X_s \tag{3.8}$$

is Lemma 3.2 which tells us that we have an inverse operator $(\mathrm{L}^*)^{-1}$ such that

$$\mathrm{L}^*(\mathrm{L}^*)^{-1}\mathbf{1}_t = \mathbf{1}_t. \tag{3.9}$$

In general, suppose that we have

$$X_t = \int_0^t L(t,s)\,\mathrm{d}W_s$$

for some square-integrable kernel $L(t,s)$. Then, if and only if, $\mathcal{F}_t^W = \mathcal{F}_t^X$, we still would have (3.8) and an inverse operator $(\mathrm{L}^*)^{-1}$ such that (3.9) holds, but the kernel $L^{-1}(t,s) = (\mathrm{L}^*)^{-1}\mathbf{1}_t(s)$ can be a distribution and not a genuine function since the space Λ may not be a space of functions. Indeed, for fBm with $H > 1/2$ Λ is not a function space, see [28]. Nevertheless, in that case there still exists a genuine function kernel $K_H^{-1}(t,s)$ for fBm with $H > 1/2$.

By Lemmas 3.1 and 3.2 we have the following result.

Lemma 3.3. *We have $\Lambda = L^2$.*

Proof. By Lemma 3.1, we have $L^2 \subset \Lambda$. For the other direction, a similar argument as in Lemma 3.2 and Theorem 3.1 implies that it suffices to prove that, for any $f \in L^2$, the equation

$$f(t) = v(t) - \int_s^t v(u)G(s,u)\,\mathrm{d}u,$$

where

$$G(s,u) = -\frac{bc(H)}{a}\frac{u^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}(u-s)^{\frac{3}{2}-H}},$$

admits a solution $v \in L^2$. Now the solution is given by

$$v(s) = \sum_{k=1}^{\infty} \mathrm{G}^k[f](s),$$

where the series converges again by [31, Theorem 2.7.1]. This completes the proof. □

Utilizing Lemma 3.3, Theorem 3.1 extends to a transfer principle for deterministic integrands on L^2 with respect to a Bm and ccmfBm. Actually, we could extend the transfer principle to stochastic integrands by using Malliavin calculus and Skorokhod integration as explained in [35,36]. However, we omit that extension here.

Theorem 3.2. *Let $f \in L^2$. Let X and W be connected by Theorem 3.1. Then*

$$\begin{aligned}\int_0^T f(t)\,\mathrm{d}X_t &= \int_0^T \mathrm{L}^* f(t)\,\mathrm{d}W_t,\\ \int_0^T f(t)\,\mathrm{d}W_t &= \int_0^T (\mathrm{L}^*)^{-1} f(t)\,\mathrm{d}X_t,\end{aligned}$$

where

$$\begin{aligned}\mathrm{L}^* f(t) &= af(t) + b\int_t^T f(s)\frac{\partial K_H}{\partial s}(s,t)\,\mathrm{d}s,\\ (\mathrm{L}^*)^{-1} f(t) &= f(t)L^{-1}(T,t)\\ &\quad + \int_t^T [f(s)-f(t)]\,L^{-1}(\mathrm{d}s,t).\end{aligned}$$

Proof. Let us first note that the formula

$$\int_0^T f(t)\,\mathrm{d}X_t = \int_0^T \mathrm{L}^* f(t)\,\mathrm{d}W_t$$

was shown in the proof of Theorem 3.1.

Let us then consider the formula

$$\int_0^T f(t)\,\mathrm{d}W_t = \int_0^T (\mathrm{L}^*)^{-1} f(t)\,\mathrm{d}X_t. \tag{3.10}$$

First we note that by using (3.5) we can verify that the kernel $L^{-1}(t,s)$ is of bounded variation in t. Hence the operator $(\mathrm{L}^*)^{-1}$ is well-defined and we have $(\mathrm{L}^*)^{-1}\mathbf{1}_t(s) = L^{-1}(t,s)$. Moreover, if f is a simple function, (3.10) follows from the formula

$$\begin{aligned}\int_0^T \mathbf{1}_t(s)\,\mathrm{d}W_s &= W_t \\ &= \int_0^t L^{-1}(t,s)\,\mathrm{d}X_s \\ &= \int_0^t (\mathrm{L}^*)^{-1}\mathbf{1}_t(s)\,\mathrm{d}X_s\end{aligned}$$

of Theorem 3.1 by linearity. Finally, formula (3.10) for $g \in L^2$ follows from the facts that $L^2 = \Lambda$ by Lemma 3.3, $(\mathrm{L}^*)^{-1}$ is a bounded operator, and the abstract Wiener integral is an isometry (see [36] for details). □

Remark 3.4. Unlike the independent mfBm introduced by Cheridito [13], the ccmfBm does not have stationary increments. Hence, it could be argued that the independent mfBm is more natural. However, the inverse transfer principle for the ccmfBm has a rather explicit and convenient form. For the independent mfBm the inverse transfer principle is implicit and involves solving integral equations numerically, see Cai et al. [12]. Therefore, the ccmfBm is more convenient for applications. We also note that the Hölder continuity properties, quadratic variation, and the long-range dependence of the independent mfBm and ccmfBm are the same.

4. Cameron–Martin–Girsanov–Hitsuda theorem

In this section we show how the transfer principle of Theorem 3.2 can be used to characterize Gaussian processes that are equivalent in law to the ccmfBm and to provide the corresponding Cameron–Martin–Girsanov–Hitsuda theorem in the same way as in [33,34]. In this section we consider processes on a compact time interval $[0,T]$.

By the Hitsuda representation theorem [20], a Gaussian process $\tilde{W}$ is equivalent to a Bm W if and only if it can be represented

$$\tilde{W}_t = W_t - \int_0^t \int_0^s \ell(s,u)\,\mathrm{d}W_u\,\mathrm{d}s - \int_0^t g(s)\,\mathrm{d}s \tag{4.1}$$

for some $\ell \in L^2([0,T]^2)$ and $g \in L^2([0,T])$. Here W is a Bm that is constructed from $\tilde{W}$ by

$$\begin{aligned}W_t &= \tilde{W}_t - \int_0^t \int_0^s \tilde{\ell}(s,u)\,\mathrm{d}\tilde{W}_u\,\mathrm{d}s \\ &\quad - \int_0^t \left[g(s) - \int_0^u \tilde{\ell}(s,u)\,\mathrm{d}u \right] \mathrm{d}s,\end{aligned} \tag{4.2}$$

where $\tilde{\ell}$, the resolvent of ℓ, is given by the Neumann series (see [31])

$$\tilde{\ell}(t,s) = \sum_{k=1}^{\infty} \ell^k(t,s), \tag{4.3}$$

where

$$\begin{aligned}\ell^1(t,s) &= \ell(t,s), \\ \ell^{k+1}(t,s) &= \int_s^t \ell(t,u)\ell^k(u,s)\,\mathrm{d}u.\end{aligned}$$

Let φ_t denote the likelihood ratio of $\tilde{W}$ over W given observation $\mathcal{F}_t$ on the interval $[0,t]$. Then by [20]

$$\begin{aligned}\varphi_t &= \frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\Big|\mathcal{F}_t \\ &= \exp\left\{ \int_0^t \left[\int_0^s \ell(s,u)\mathrm{d}W_u + g(s) \right] \mathrm{d}W_s \right. \\ &\quad \left. - \frac{1}{2}\int_0^t \left[\int_0^s \ell(s,u)\mathrm{d}W_u - g(s) \right]^2 \mathrm{d}s \right\} \\ &= \exp\left\{ \int_0^t \left[\int_0^s \tilde{\ell}(s,u)\mathrm{d}\tilde{W}_u + g(s) \right.\right.\end{aligned} \tag{4.4}$$

$$- \int_0^s \tilde{\ell}(s,u)g(u)\,\mathrm{d}u\Big]\mathrm{d}\tilde{W}_s$$
$$-\frac{1}{2}\int_0^t \Big[\int_0^s \tilde{\ell}(s,u)\mathrm{d}\tilde{W}_u + g(s)$$
$$- \int_0^s \tilde{\ell}(s,u)g(u)\,\mathrm{d}u\Big]^2\mathrm{d}s\Big\}. \tag{4.5}$$

Let then $\tilde{X}$ be a Gaussian process. Set

$$\tilde{W}_t = \int_0^t L^{-1}(t,s)\,\mathrm{d}\tilde{X}_s. \tag{4.6}$$

By the transfer principle Theorem 3.2 and Hitsuda representation theorem, we then have the following Cameron–Martin–Girsanov–Hitsuda theorem.

Proposition 4.1. *A Gaussian process $\tilde{X}$ is equivalent in law to a ccmfBm if and only if the process $\tilde{W}$ given by (4.6) is of form (4.1) and in this case the process*

$$X_t = \int_0^t L(t,s)\,\mathrm{d}W_s \tag{4.7}$$

is a ccmfBm, where W is a Bm constructed from $\tilde{X}$ via (4.2) and (4.6) The likelihood ratio φ_t between $\tilde{X}$ and X given the observations on $[0,t]$ is given by (4.4) and (4.5).

Proof. Since the transformations

$$\tilde{W}_t = \int_0^t L^{-1}(t,s)\,\mathrm{d}\tilde{X}_s,$$
$$W_t = \int_0^t L^{-1}(t,s)\,\mathrm{d}X_s$$

are one-to-one, linear, and non-anticipative, the claim follows from the classical Cameron–Martin–Girsanov–Hitsuda theorem for the Bm. □

The Hitsuda representation can also be given without constructing the intermediate Bms as follows:

Proposition 4.2. *A Gaussian process $\tilde{X}$ is equivalent in law to the ccmfBm X if and only if it admits the representation*

$$\tilde{X}_t = X_t - \int_0^t f(t,s)\,\mathrm{d}X_s - \mathrm{L}g(t),$$

where $g \in L^2([0,T])$ and the Voterra kernel f is defined by the equation

$$\mathrm{L}^* f(t,\cdot)(s) = \mathrm{L}\ell(\cdot,s)(t) \tag{4.8}$$

for some Volterra kernel $\ell \in L^2([0,T]^2)$.

Proof. By Hitsuda representation [19], a process $\tilde{W}$ is equivalent to W if and only if there exists a Volterra kernel $\ell \in L^2([0,T]^2)$ and a function $g \in L^2([0,T])$ such that

$$\tilde{W}_t = W_t - \int_0^t \left(\int_0^s \ell(s,u)\,\mathrm{d}W_u - g(s)\right)\mathrm{d}s.$$

Operating both sides with kernel $L(t,s)$, changing the order of integration, and using the transfer principle of Theorem 3.2 gives the result. □

Remark 4.1. If X is the Bm, then $\mathrm{L} = \mathrm{I}$ is simply the integral operator $\mathrm{I}f(t) = \int_0^t f(s)\mathrm{d}s$ and its adjoint $\mathrm{L}^* = \mathrm{I}^*$ is simply the identity operator. Thus for the Bm Eq. (4.8) takes the familiar form of the Hitsuda's representation theorem

$$f(t,s) = \int_s^t \ell(u,s)\,\mathrm{d}u,$$

where $\ell \in L^2([0,T]^2)$.

5. Prediction

Let $\mathcal{F}_u^X = \sigma\{X_u; u \le t\}$ denote the σ-algebra of observing the ccmfBm over the interval $[0,t]$. Naturally, we are interested in predicting the future, i.e., we are interested in the conditional future probability law of the process X, given the information $\mathcal{F}_u^X$. The transfer principle of Theorem 3.2 provides us these prediction formulas for the ccmfBm in the same way as in [37,38]:

Theorem 5.1. *The conditional process* $t \mapsto X_t(u) = X_t|\mathcal{F}_u^X$, $t \geq u$, *is Gaussian with stochastic mean*

$$\hat{m}_t(u) = \mathbb{E}\left[X_t \middle| \mathcal{F}_u^X\right] = X_u + \int_0^u \Psi(t,s|u)\,\mathrm{d}X_s,$$

where

$$\Psi(t,s|u) = (\mathrm{L}^*)^{-1}\left[L(t,\cdot) - L(u,\cdot)\right](s);$$

and with deterministic covariance

$$\hat{R}(t,s|u) = \mathbb{C}\mathrm{ov}\left[X_t, X_s \middle| \mathcal{F}_u^X\right] = R(s,t) - \int_0^u L(t,v)L(s,v)\,\mathrm{d}v.$$

6. Simulation

We illustrate the paths of ccmfBm by simulations. The paths are simulated on $N = 500$ equidistant time points $t_k = k/N$ on the interval $[0,1]$ by using the Cholesky decomposition $\mathbf{R}_H = \mathbf{L}_H \mathbf{L}_H^\top$ of the covariance matrix $\mathbf{R}_H(k,j) = R_H(t_k, t_j)$ of the fBm:

$$X_{t_k} = \frac{a}{\sqrt{N}} \sum_{j=1}^{k} \xi_j + b \sum_{j=1}^{k} \mathbf{L}_H(k,j)\xi_j, \tag{6.1}$$

where ξ_j's are i.i.d. standard random variables.

Remark 6.1. Eq. (6.1) provides exact simulation. The cost of having exact simulation is of course in calculating the Cholesky decomposition. One can also use the Molchan–Golosov representation of the fBm for simulation. One way of doing this is to use the approximation

$$X_t \approx \sum_{j=1}^{[Nt]} \left[a + bN \int_{\frac{j-1}{N}}^{\frac{j}{N}} K_H\left(\frac{[Nt]}{N}, s\right) \mathrm{d}s \right] \frac{1}{\sqrt{N}} \xi_j \tag{6.2}$$

and approximate the integral above taking into account that it has singularity at $s = 0$. If the integral is approximated in an efficient way, then this approach can be fast as it avoids calculating the Cholesky decomposition and instead uses the Molchan–Golosov kernel K_H as a proxy for the Cholesky square root. See [32] for more information on the convergence of the approximation (6.2). We note however, that approximating singular integrals in (6.2) accurately may also be computationally heavy. At the same time, efficient simulation methods based on the FFT are not at our disposal, since ccmfBm is not a stationary-increment process.

Remark 6.2. The integral representation

$$X_t = \int_0^t L(t,s)\,\mathrm{d}W_s$$

provides us with a series expansion. Indeed, let $(\tilde{e}_k)_{k=1}^\infty$ be your favourite orthonormal basis on $L^2([0,T])$ (note that the basis functions $\tilde{e}_k$ depend on T) and let $(\xi_k)_{k=1}^\infty$ be a sequence of i.i.d. standard normal random variables. Let

$$e_k(t) = \int_0^t L(t,s)\tilde{e}_k(s)\,\mathrm{d}s. \tag{6.3}$$

Then it follows that

$$X_t = \sum_{k=1}^{\infty} e_k(t)\xi_k, \tag{6.4}$$

where the series (6.4) converges both in L^2 and pointwise See [17] for details and for an explicit series expansion for the fBm that can be extended to the ccmfBm in a straightforward manner Cutting the series (6.4) and approximating the integral (6.3) provides us yet another way to approximately simulate the ccmfBm process. Let us note, however, that the use of (6.4) requires both cutting the series and approximating the integral in (6.3). Hence, simulating with (6.2) seems a better way to go.

In Figs. 4–6 we have plotted simulated paths of the completely correlated mixed fractional Brownian motion (ccmfBm) together with its components, the Brownian motion (Bm) and the completely correlated fractional Brownian motion (ccfBm) that is constructed from the Bm. The plots were created by using the exact simulation formula (6.1). Figs. 4–6 illustrate that the Hölder continuity of the ccmfBm is the same as the Hölder continuity of the driving Bm. Consequently, in order to distinguish between Bm and cccfBm one needs to analyse the long range behaviour.

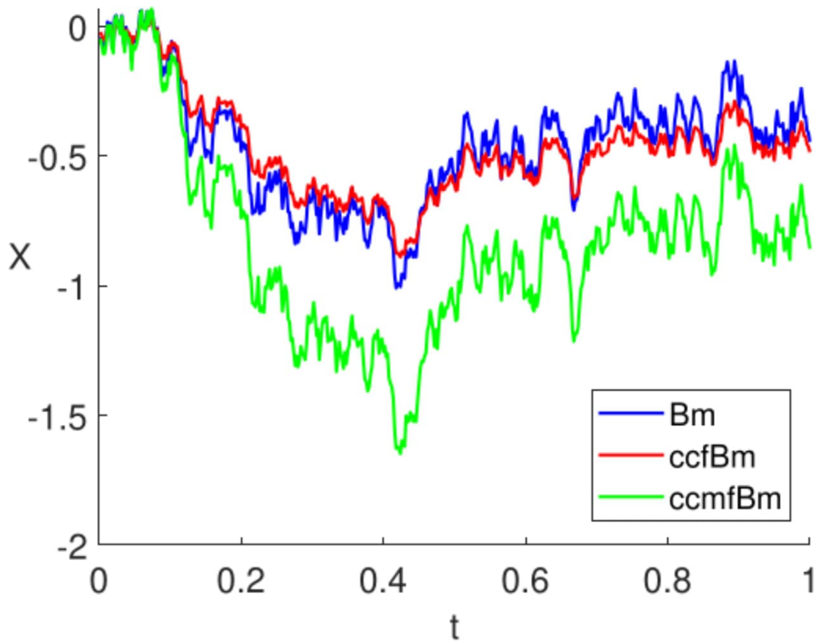

Fig. 4. ccmfBm with $a = 0.4$, $b = 1.4$, $H = 0.6$.

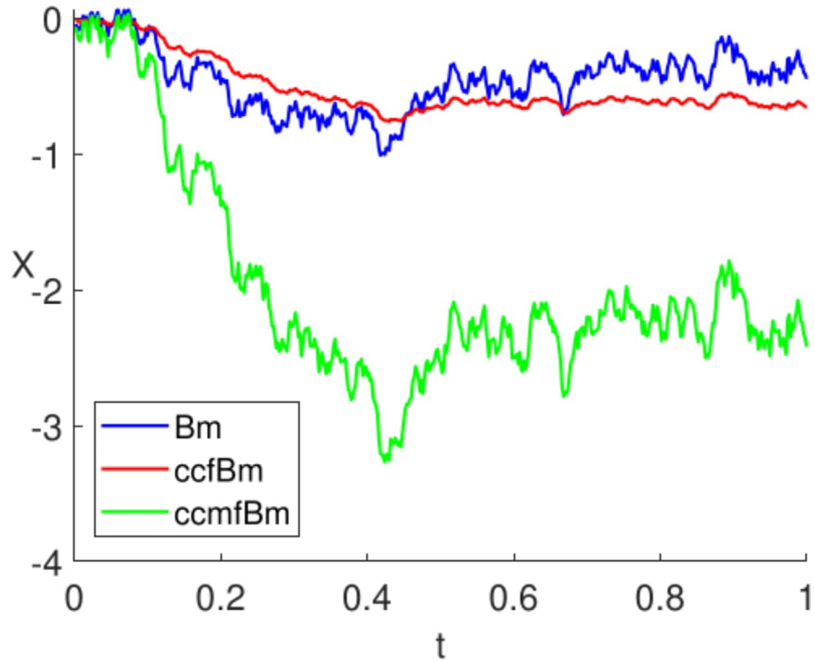

Fig. 5. ccmfBm with $a = 1$, $b = 3$, $H = 0.75$.

7. Conclusion

We have considered the long-range dependent completely correlated mixed fractional Brownian motion (ccmfBm) that is constructed by using a single Brownian motion (Bm) and a completely correlated fractional Brownian motion (ccfBm). We have shown that the short-time path behaviour (Hölder continuity and quadratic variation) of the ccmfBm are the same as those of the Bm, but unlike the Bm, the ccmfBm has long-range dependence that is characterized by the fBm part. We have constructed explicitly the transfer principle for the ccmfBm, i.e., we have constructed explicitly a kernel such that given the ccmfBm, the driving Bm can be

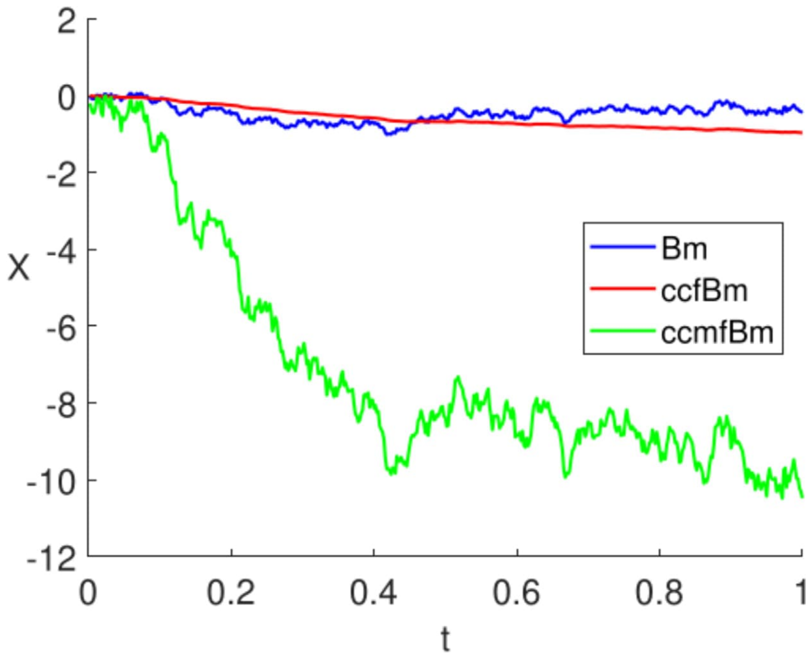

Fig. 6. ccmfBm with $a = 4,\ b = 9,\ H = 0.9$.

recovered from it. We also noted that the transfer principle for ccmfBm has a more convenient form than the corresponding principle of independent mixture mfBm. Indeed, our inverse kernel have a series expansion that converges fast, while for the independent mfBm it is only known that such an inverse kernel exists and it is a solution to a certain integral equation. This fact gives an advantage of ccmfBm over the independent mixture mfBm in modelling, and provides an alternative way to capture correct short- and long scale behaviours in mixed models in finance. As straightforward application of our transfer principle, we have considered the Cameron–Martin–Girsanov–Hitsuda theorem and prediction. Finally, we have illustrated the paths of ccmfBm by simulations.

Our results provides several natural directions of further studies. First of all, while in the present paper we have proposed the new model ccmfBm and studied its basic properties, it is an interesting topic how the paths of ccmfBm can be simulated efficiently. Indeed, non-stationary increments caused by the same driving Brownian motion in both components imply that methods based on Fourier transforms cannot be applied in a straightforward manner. Secondly, in the present paper we have only considered the long-range dependent case $H > \frac{1}{2}$. In the short-range dependent case when $H \in (0, 1/2)$, the Molchan–Golosov kernel takes a different form and the adjoint associated operator of the ccmfBm kernel is no longer a bounded operator on L^2. Consequently, Lemma 3.1 is no longer true. In particular, the integrand space Λ will be a strict subset of L^2 and the transfer principle of Theorem 3.2 is no longer true without radical modifications. Also, the short time-scale behaviour of the short-range dependent ccmfBm will be governed by the fBm part, not the Brownian part: H will be the Hölder index of the ccmfBm and the quadratic variation will be infinite. This causes additional technical difficulties. Other extensions of the ccmfBm could include mixing multiple completely correlated fBm's with different Hurst indices. In this case it is expected that the smallest Hurst index would dominate the small time scale behaviour and the largest Hurst index would dominate the large time scale behaviour. However, it is not clear if one can construct a transfer principle for this multi-mixed process.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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