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GENERALIZED SCHUR FUNCTIONS AS MULTIVALENT FUNCTIONS

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Dedicated to Henk de Snoo on occasion of his 75th birthday.

Abstract. The multivalency approach to generalized Nevanlinna functions established in [13] is here extended to the related class of generalized Schur functions giving thereby rise to new characterizations for this class of functions as well as a straightforward function-theoretical proof of its factorization. In particular, this multivalency approach explains how the well-known factorizations of the two mentioned classes of functions differ from each other. Indeed, by this approach a new factorization of generalized Schur functions is obtained which is more directly connected to the factorization of generalized Nevanlinna functions. These results demonstrate that multivalency is a valuable concept for the complete understanding of the mentioned classes of functions.

1. Introduction

Generalized Schur functions were introduced as the characteristic functions of (maximal) isometric operators in Pontryagin spaces. A complex-valued function s meromorphic on (the unit disk) $\mathbb{D}$ is *a generalized Schur function with index* $\kappa \in \mathbb{N}_0$, $s \in \mathfrak{S}_\kappa$ for short, if its Schur kernel K_s, defined as

$$K_s(z,w) = \frac{1 - s(z)s(w)^*}{1 - z\overline{w}}, \tag{1.1}$$

has κ negative squares on $\mathbb{D} \cap \mathcal{D}(s)$, where $\mathcal{D}(s)$ is the domain of holomorphy of s; see [8] for details. Hereby $\mathfrak{S}_0$ coincides with the well-known class of ordinary Schur functions. In [8] it was shown that generalized Schur functions (even operator-valued ones) possess a factorization, the so-called Kreĭn-Langer factorization, by means of their poles and zeros in- and outside the unit circle. Following is the scalar version of that factorization; in it B_γ denotes a *Blaschke* factor:

$$B_\gamma(z) = \frac{z-\gamma}{1-\overline{\gamma}z}, \qquad z \in \mathbb{D}, \quad \gamma \in \mathbb{D}. \tag{1.2}$$

Theorem 1.1. *([8, Satz 3.2]) For every $s \in \mathfrak{S}_\kappa$ there exist $\gamma_1, \dots, \gamma_\kappa \in \mathbb{D}$ and an ordinary Schur function s_0 such that $s = \prod_{i=1}^{\kappa} (B_{\gamma_i})^{-1} s_0$.*

This factorization was obtained by relatively straightforward operator-theoretical arguments from the (essentially operator-theoretical) definition of generalized Schur

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functions in (1.1). Later it was attempted to use Theorem 1.1 to obtain the following analogous factorization result for the related class of generalized Nevanlinna functions with index κ ($\mathfrak{N}_\kappa$), see e.g. [9] for their definition.

Theorem 1.2. *For every $f \in \mathfrak{N}_\kappa$ there exists a rational function r of degree κ and an ordinary Nevanlinna function f_0 such that $f = r f_0 r^\#$. Here $r^\#(z) = \overline{r(\overline{z})}$.*

The two mentioned classes can be connected by the biholomorphic mapping ϕ,

$$\phi(z) = \frac{z-i}{z+i},$$

mapping $\mathbb{C}_+$ onto $\mathbb{D}$ and $\mathbb{C}_-$ onto $\mathbb{D}^c := \{z \in \mathbb{C} \cup \{\infty\} : |z| > 1\}$, in the following manner: $f \in \mathfrak{N}_\kappa$ if and only if there exists $s \in \mathfrak{S}_\kappa$ such that for $z \in \mathbb{C}_+$

$$f(z) = \phi^{-1} \circ s \circ \phi(z). \tag{1.3}$$

Despite this strong connection between the two classes of functions, it transpired to be difficult to obtain the preceding factorization result for generalized Nevanlinna functions (Theorem 1.2) straightforwardly from the factorization of generalized Schur functions (Theorem 1.1). Instead, the initial proofs for the factorization for generalized Nevanlinna functions in the papers [3, 4] depended heavily on the operator-theoretical results contained in [10]; cf. [14].

In [13] the observation was made that generalized Nevanlinna functions can also be understood as meromorphic functions having a certain multivalency property, see Theorem 4.2 below. In that manner a purely function-theoretical approach to generalized Nevanlinna functions was developed yielding new characterizations for them as well as facilitating the establishment of Theorem 1.2 by straightforward function-theoretical arguments. An additional advantage of this multivalency approach to generalized Nevanlinna functions is that multivalency behaves very regularly with respect to (1.3). Accordingly, a similar approach can also be applied to generalized Schur functions; this is done in the present paper. In fact, the multivalency approach is even more suitable for treating generalized Schur functions as their behavior at the boundary of their domain of holomorphy is of a simpler nature than that of generalized Nevanlinna functions, cf. Corollary 3.4 below.

By this approach several new characterizations of ordinary and generalized Schur functions in terms of multivalency are obtained. In particular, a purely function-theoretical proof for Theorem 1.1 is obtained based on the open mapping theorem (for holomorphic functions) and a weak form of the maximum modulus principle for bounded holomorphic functions. The multivalency-approach also shows that the presented factorizations, Theorem 1.1 and Theorem 1.2, are not equivalent (with respect to (1.3)). This observation is complemented by the establishment of an alternative factorization of generalized Schur functions, which explicitly reflects the connection between the two classes of functions. From that alternative factorization the presented factorization of generalized Nevanlinna functions is simply established.

Finally, the contents of this paper are outlined. In Section 2 the present multivalency approach is compared to the function-theoretical approach of [2]; herefore a weak version of the maximum modulus principle and the Inner-Outer factorization of H_p-functions are recalled. After establishing basic properties of generalized Schur functions in the first half of Section 3, valency, boundary behavior and factorization characterizations of generalized Schur functions are established in this section's second half. In Section 4 the obtained results are compared with the valency characterizations of generalized Nevanlinna functions in order to explain the difference between their factorizations. That discussion is complemented by first establishing an alternative factorization for generalized Schur functions and, thereafter, using that new factorization to obtain the factorization result of generalized Nevanlinna functions.

2. Weak maximum modulus principle and the Inner-Outer factorization

Here first a weak version of the maximum modulus principle and the Inner-Outer factorization are recalled. To formulate the former result recall that a function f belongs to the class H_p, for $0 < p < \infty$, if it is holomorphic on $\mathbb{D}$ and satisfies

$$\sup_{0 \le r < 1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \mathrm{d}\theta < \infty.$$

Moreover, H_∞ denotes the class of bounded holomorphic functions in $\mathbb{D}$:

$$\sup_{z \in \mathbb{D}} |f(z)| < \infty.$$

For H_p-functions the following weak form of the maximum modulus principle holds; hereby the notation $\widehat{\rightarrow}$ denotes a non-tangential limit from inside the unit disk.

Proposition 2.1. *Let $f \in H_p$, $0 < p \le \infty$, and let $M > 0$. Then f is bounded in absolute value on $\mathbb{D}$ by M if and only if $\lim_{z \widehat{\rightarrow} x} |f(z)| \le M$ for almost all $x \in \mathbb{T}$.*

Proposition 2.1 can in the H_∞-case be found in [12, Theorem 11.32]; that is the only case of Proposition 2.1 needed in this paper. From that special case the general case follows by making use of [12, Theorem 17.18]. In fact, Privalov's uniqueness theorem implies that the preceding statement holds for all analytic functions.

Next the Inner-Outer factorization of H_p-functions is stated. Recall therefore that the class of the functions of bounded type on $\mathbb{D}$, $\mathcal{B}$, consists of functions meromorphic on $\mathbb{D}$ representable as the quotient of H_∞-functions. In particular, R. and F. Nevanlinna showed that the class of functions of bounded type which are holomorphic on $\mathbb{D}$ coincides with the class of functions of bounded characteristic. Consequently, $\mathcal{B}$ contains all H_p-classes.

Theorem 2.2. *(Inner-Outer factorization) For every $f \in \mathcal{B}$ there exist mutually prime Blaschke products B_0 and B_∞, $c \in \mathbb{R}$ and positive measures $\mathrm{d}\sigma_+$ and $\mathrm{d}\sigma_-$ on $(-\pi, \pi]$ with disjoint supports singular w.r.t. the Lebesgue measure such that*

$$f(z) = \frac{B_0(z)}{B_\infty(z)} e^{ic} e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}[\mathrm{d}\sigma_+(\theta) - \mathrm{d}\sigma_-(\theta)]} e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|f(e^{i\theta})|\mathrm{d}\theta};$$

here $\log|f(e^{i\theta})| := \lim_{z \widehat{\rightarrow} e^{i\theta}} \log|f(z)|$ exists for almost all $\theta \in (-\pi, \pi]$. If $f \in H_p$, where $0 < p \le \infty$, then the representation holds with $B_\infty \equiv 1$ and $\mathrm{d}\sigma_+ \equiv 0$.

Here Blaschke products are (possibly infinite) products of Blaschke factors as in (1.2).

Despite not being a central concern of this paper, it is an interesting observation that the weak maximum moduls principle and the Inner-Outer factorization are equivalent, see Appendix A. This equivalence is of interest, because it shows how the current multivalency approach (which is connected to the behavior of the function at the boundary of the domain of holomorphy) is connected to the Inner-Outer factorization approach of Delsarte, Genin and Kamp in [2]. Although initially overlooked, their interesting paper contains a function-theoretical proof for the factorization of a class of functions which can, with some effort, be shown to be equal to the class of generalized Caratheodory functions. As the class of generalized Caratheodory functions behaves essentially in the same way as generalized Nevanlinna functions, their result would also imply the factorization result Theorem 1.2.

More specifically, in [2] the classes of pseudo-Schur and pseudo-Caratheodory functions were introduced. The former class consist of all functions s of bounded type ($\mathcal{B}$) satisfying $\lim_{z \widehat{\to} x} |s(z)| \leq 1$, for almost all $x \in \mathbb{T}$. The Inner-Outer factorization implies that a pseudo-Schur function s can be factorized as $s_u s_b$, where

$$\begin{aligned} s_b(z) &= B_0(z) e^{ic} e^{-\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \mathrm{d}\sigma_-(\theta)} e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|s(e^{i\theta})| \mathrm{d}\theta}; \\ s_u(z) &= (B_\infty(z))^{-1} e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \mathrm{d}\sigma_+(\theta)}. \end{aligned}$$

The assumed boundary behavior of pseudo-Schur functions implies that s_b is an ordinary Schur function. Consequently, in light of Theorem 1.1, a pseudo-Schur function s is a generalized Schur function if and only if s_u is a rational function. To describe this subclass of pseudo-Schur functions Delsarte, Genin and Kamp introduce the concept of an index. That concept is introduced both for pseudo-Schur functions as well as for pseudo-Caratheodory functions.

In other words, the results in [2] imply that the class of generalized Schur functions (and, similarly, the classes of generalized Caratheodory or Nevanlinna functions) can be described via their (non-tangential) boundary behavior together with an additional index condition. Our multivalency approach can be understood as consisting of a combination of those two conditions into one valency condition on the function under consideration.

3. Generalized Schur functions

For a complex-valued function f, any subset Δ of $\mathbb{C}$ and any subset A of $\mathbb{C} \cup \{\infty\}$, the notation $V(f, \Delta, A) = \kappa$ (or $V(f, \Delta, A) \leq \kappa$) is used to denote that for every fixed $w \in A$ the multiplicities of the zeros of $f(z) - w$ on $\Delta \cap \mathcal{D}(f)$ add up exactly (or add up at most) to $\kappa \in \mathbb{N}_0$. Hereby $V(f, \Delta, \infty) = \kappa$ and $V(f, \Delta, \infty) \leq \kappa$ mean that $V(1/f, \Delta, 0) = \kappa$ and $V(1/f, \Delta, 0) \leq \kappa$, respectively. In this paper, Δ will almost without exception be $\mathbb{D}$ and A will generally be (a subset of) $\mathbb{D}^c$:

$$\mathbb{D}^c := \{z \in \mathbb{C} \cup \{\infty\} : |z| > 1\}.$$

Before proving characterizations of generalized Schur functions, two properties of generalized Schur functions following from their definition as in (1.1) are established.

Lemma 3.1. *Let $s \in \mathfrak{S}_\kappa$. Then the following statements hold:*

(i) $V(s, \mathbb{D}, \mathbb{D}^c) \leq \kappa$;
(ii) $\limsup_{\mathbb{D} \ni z \to x} |s(x)| \leq 1$ *for all but at most κ points x of $\mathbb{T}$.*

Note that generalized Schur functions can easily be proven to be of bounded type as a consequence of property (i) in Lemma 3.1.

Proof. (i): Assume that $V(s,\Delta,w_0)=\kappa_0\in\mathbb{N}$ for any $w_0\in\mathbb{D}^c\setminus\{\infty\}$ and any open subset Δ of $\mathbb{D}$, then, by definition, there exist distinct zeros $z_1,\ldots,z_n\in\Delta$ of $s-w_0$ having multiplicities $\pi_1,\ldots,\pi_n$ such that $\sum_{i=1}^n\pi_i=\kappa_0$. Now let $r>0$ be such that for $1\le i\le n$ there exist open sets U_i such that

(a) $U_i:=\{z\in\mathbb{C}:|z-z_i|<r\}$ is contained in $\Delta\cap\mathcal{D}(s)$;
(b) s' is not zero on $U_i\setminus\{z_i\}$;
(c) $U_i\cap U_j=\emptyset$, if $i\neq j$.

Then by the open mapping theorem there exists a non-empty open subset A of $\bigcap_{i=1}^n s(U_i)$ such that s takes each value of A precisely $\kappa_0=\sum_{i=1}^n\pi_i$-times (counting multiplicities) on an open subset U of $\bigcup_{i=1}^n U_i$ (i.e., $V(s,U,A)=\kappa_0$), see e.g. [13, Corollary 2.2]. Note that A contains w_0, as $s(z_i)=w_0$ for $i=1,\ldots,n$, and that $U\subseteq\Delta$ as $U_i\subseteq\Delta$ for $i=1,\ldots,n$.

Property (b) guarantees that the multiplicities of the zeros of $s-w_1$ are one for any $w_1\in A\setminus\{w_0\}$. As $V(s,U,w_1)=\kappa_0$ by the above construction of the sets U and A, there exists κ_0 distinct elements $z_1,\ldots,z_{\kappa_0}$ in U such that $s(z_i)=w_1$. The Schur kernel K_s of s, see (1.1), has at these points the following expression:

$$(K_s(z_i,z_j))_{i,j=1,\ldots,\kappa_0}=(1-w_1\overline{w_1})\left(\frac{1}{1-z_i\overline{z_j}}\right)_{i,j=1,\ldots,\kappa_0}. \tag{3.1}$$

That is, the Schur kernel evaluated at $z_1,\ldots,z_{\kappa_0}$ coincides with a negative constant, $1-|w_1|^2$, times the Szegö kernel evaluated at those points. Since the latter kernel is seen to be positive definite by a series development of $(1-\zeta\overline{\xi})^{-1}$, (3.1) implies that $\kappa_0\le\kappa$, because $s\in\mathfrak{S}_\kappa$, cf. (1.1). Since $w_0\in\mathbb{D}^c\setminus\{\infty\}$ and $\Delta\subseteq\mathbb{D}$ were arbitrary, we have proven that $V(s,\mathbb{D},w_0)\le\kappa$ for every $w_0\in\mathbb{D}^c\setminus\{\infty\}$.

Finally, if $V(s,\mathbb{D},\infty)>\kappa$, then $V(s^{-1},\mathbb{D},0)>\kappa$ and, hence, by the open mapping theorem there exists a neighborhood A of 0 such that $V(s^{-1},\mathbb{D},A)>\kappa$; in contradiction to the proven fact that $V(s,\mathbb{D},w_0)\le\kappa$ for every $w_0\in\mathbb{D}^c\setminus\{\infty\}$.

(ii): Let $x_1,\ldots,x_n$ be distinct points of $\mathbb{T}$ for which there exist sequences $\{z_{i,k}\}_{k\in\mathbb{N}}$ in $\mathbb{D}\cap\mathcal{D}(s)$ such that $\lim_{k\to\infty}z_{i,k}=x_i$ and $\lim_{k\to\infty}|s(z_{i,k})|>1$, for $i=1,\ldots,n$. Then consider

$$(K_s(z_{i,k},z_{j,k}))_{i,j=1,\ldots,n}=\left(\frac{1-s(z_{i,k})\overline{s(z_{j,k})}}{1-z_{i,k}\overline{z_{j,k}}}\right)_{i,j=1,\ldots,n}.$$

As k tends to infinity the diagonal elements tend to minus infinity while the off-diagonal elements are bounded. Thus (ii) holds, because by assumption the index of s is κ, cf. (1.1). □

As the second and final preparation, the following characterizations of ordinary Schur functions are needed.

Theorem 3.2. *Let s be meromorphic on $\mathbb{D}$. Then equivalent are:*

(i) $s\in\mathfrak{S}_0$;
(ii) $V(s,\mathbb{D},\mathbb{D}^c)=0$;

(iii) $s \in H_p$, $0 < p \leq \infty$, *and* $\lim_{z \rightrightarrows x} |s(z)| \leq 1$ *for almost all* $x \in \mathbb{T}$.

Proof. The equivalence between (i) and (ii) (the latter condition usually being formulated as $|s| \leq 1$ on $\mathbb{D}$) is well known, see e.g. [1, Chapter 4: Theorem 2.1], and the equivalence between (ii) and (iii) follows directly from Proposition 2.1. □

Note that if the condition $s \in H_p$ in Theorem 3.2 (iii) is replaced by the stronger condition $s \in H_\infty$, which is the version of Theorem 3.2 needed in Theorem 3.3 below, then Proposition 2.1 needs only be used in case that $s \in H_\infty$. That case can be established by a simple argument.

The obtained results are now combined to furnish characterizations of generalized Schur functions. Recall that in the following theorem the notation B_γ denotes the (scalar) Blaschke function in (1.2).

Theorem 3.3. *Let* s *be meromorphic on* $\mathbb{D}$. *Then equivalent are:*

(i) $s \in \mathfrak{S}_\kappa$;
(ii) $V(s, \mathbb{D}, \mathbb{D}^c) = \kappa$;
(iii) (a) *there exists* $w \in \mathbb{D}^c$ *such that* $V(s, \mathbb{D}, w) = \kappa$;
(b) $\limsup_{\mathbb{D} \ni z \to x} |s(z)| \leq 1$ *for all* $x \in \mathbb{T}$;
(iv) *there exists* $s_0 \in \mathfrak{S}_0$ *and* $\gamma_1, \ldots, \gamma_\kappa \in \mathbb{D}$, *with* $s_0(\gamma_i) \neq 0$, *such that*

$$s = \left(\prod_{i=1}^{\kappa} B_{\gamma_i}^{-1} \right) s_0;$$

(v) (a) *there exists an open set* A *of* $\mathbb{D}^c$ *such that* $V(s, \mathbb{D}, A) = \kappa$;
(b) $\lim_{z \rightrightarrows x} |s(z)| \leq 1$ *for almost all* $x \in \mathbb{T}$.

Proof. (ii) ⇔ (iii): Necessity of the conditions is a consequence of the open mapping theorem, while the sufficiency can be established by the argument used in [13, Corollary 2.4].

(ii), (iii) ⇒ (v): This is obvious.

(v) ⇒ (iv): Assume first that (v) holds with $\infty \in A$, then (v)(a) implies that $\{\gamma_1, \ldots, \gamma_\kappa\} = s^{-1}(\infty) \cap \mathbb{D}$ is such that $s_0 := (\prod_{i=1}^{\kappa} B_{\gamma_i}) s$ is holomorphic on $\mathbb{D}$, is bounded on $\mathbb{D}$, satisfies $s_0(\gamma_i) \neq 0$ and has the property (v)(b). Consequently, Theorem 3.2 implies that $s_0 \in \mathfrak{S}_0$ and, hence, (iv) holds.

Next assume that (v) holds with $\infty \notin A$. Then consider $s_\alpha := \phi_{-1/\overline{\alpha}}(s)$ for any $\alpha \in A$; here the notation ϕ_α denotes the following automorphism of the unit circle:

$$\phi_\alpha(z) = \frac{z + \alpha}{1 + \overline{\alpha} z}, \qquad z, \alpha \in \mathbb{D}.$$

The function s_α has by construction property (v)(a) for an open subset A_α containing ∞ and, evidently, also has property (v)(b). Hence, $V(s_\alpha, \mathbb{D}, \mathbb{D}^c) = \kappa$ by the proven implications. Since $s = \phi_{1/\overline{\alpha}}(s_\alpha)$, one now easily sees that $V(s, \mathbb{D}, \mathbb{D}^c) = \kappa$ and, hence, (iv) holds by the arguments in the first paragraph.

(iv) ⇒ (iii): If (iv) holds, then it is evident that (iii) holds with $w = \infty$.

(i) $\Rightarrow$ (iv): Since $V(s,\mathbb{D},\mathbb{D}^c)\le\kappa$ by Lemma 3.1 (i), κ_0 defined as

$$\kappa_0 := \sup_{w\in\mathbb{D}^c} V(s,\mathbb{D},w) \tag{3.2}$$

is well-defined, $\kappa_0\le\kappa$ and there exists $w_0\in\mathbb{D}^c$ such that $V(s,\mathbb{D},w_0)=\kappa_0$. Thus there exists an open neighborhood A of w_0 in $\mathbb{D}^c$ such that $V(s,\mathbb{D},A)=\kappa_0$ by the open mapping theorem, cf. [13, Corollary 2.2]. Since Lemma 3.1 (ii) implies that s has the property (v)(b), the established equivalence of (iv) and (v) shows that there exist $\gamma_1,\ldots,\gamma_{\kappa_0}\in\mathbb{D}$ such that $s_0(\gamma_i)\neq 0$ and $s=(\prod_{i=1}^{\kappa_0}B_{\gamma_i}^{-1})s_0$.

Next recall that if $s_1\in\mathfrak{S}_{\kappa_1}$ and $s_2\in\mathfrak{S}_{\kappa_2}$, then the straightforwardly established, and well-known, Schur-kernel identity

$$K_{s_1s_2}(z,w)=K_{s_1}(z,w)+s_1(z)K_{s_2}(z,w)s_1(w)^*, \tag{3.3}$$

implies that $s_1s_2\in\mathfrak{S}_{\kappa_{\mathrm{p}}}$ where $\kappa_{\mathrm{p}}\le\kappa_1+\kappa_2$. Since it is easily established that $B_\gamma^{-1}\in\mathfrak{S}_1$ for every $\gamma\in\mathbb{D}$, repeatedly using the preceding observation yields that the index of $s=(\prod_{i=1}^{\kappa_0}B_{\gamma_i}^{-1})s_0$ (assumed to be κ) is smaller than κ_0. Since $\kappa_0\le\kappa$ by definition of κ_0, see (3.2), the implication (i) $\Rightarrow$ (iv) has now been established.

(iv) $\Rightarrow$ (i): If s is as in (iv), then the argument surrounding (3.3) yields that $s\in\mathfrak{S}_{\kappa_0}$, where $\kappa_0\le\kappa$. Thus $V(s,\mathbb{D},\mathbb{D}^c)=\kappa_0$ by the proven implication (i) $\Rightarrow$ (iv) together with the established equivalence (iv) $\Leftrightarrow$ (ii). On the other hand, the proven equivalence (iv) $\Leftrightarrow$ (ii) yields that $V(s,\mathbb{D},\mathbb{D}^c)=\kappa$. Accordingly, $\kappa_0=\kappa$ and, hence, the implication is established. $\square$

As a consequence of the preceding theorem, the class of all generalized Schur functions is characterizable in terms of their boundary behavior, cf. Proposition 2.1. Different from the characterization of generalized Nevanlinna functions in terms of their boundary behavior, see [6] and [11, p. 5], the following characterization of generalized Schur functions reflects that the behavior at the boundary is not contributing to the negative index of the function.

Corollary 3.4. *Let s be a function meromorphic on $\mathbb{D}$. Then $s\in\mathfrak{S}_\kappa$ for some $\kappa\in\mathbb{N}_0$ if and only if* $\limsup_{\mathbb{D}\ni z\to x}|s(z)|\le 1$ *for every $x\in\mathbb{T}$.*

Proof. Necessity of the condition is obvious by Theorem 3.3. Next suppose that s has the stated boundary behavior, but that $s\notin\mathfrak{S}_\kappa$ for any $\kappa\in\mathbb{N}_0$ and let $w_0\in\mathbb{D}^c$ be arbitrary. Then the equivalence of (i) and (iii) in Theorem 3.3 implies that there exists an infinite sequence $\{z_i\}_{i\in\mathbb{N}}$, $z_i\in\mathbb{D}$, of points such that $s(z_i)=w_0$. Clearly, the sequence has an accumulation point in $\mathbb{D}\cup\mathbb{T}$. On the one hand, it cannot have an accumulation point in $\mathbb{T}$, because then the limit of $|s(z)|$ along that subsequence would be $|w_0|>1$; in contradiction with the assumed boundary behavior of s. On the other hand, it also can not have any accumulation point in $\mathbb{T}$, because then it would be identically equal to w_0 and, hence, generate the same contradiction as in the preceding case. Thus $V(s,\mathbb{D},w_0)<\infty$ and, hence, $s\in\mathfrak{S}_\kappa$ for some $\kappa\in\mathbb{N}_0$ by Theorem 3.3. $\square$

4. Connection between the factorizations of generalized Schur functions and generalized Nevanlinna fuctions

By means of the established multivalency characterization of generalized Schur functions it is now possible to explain how the factorizations of generalized Schur and generalized Nevanlinna functions differs from each other. Therefore recall that generalized Schur and Nevanlinna functions can be characterized in terms of multivalency as follows.

Theorem 4.1. *Let s be meromorphic on $\mathbb{D}$ and let $\kappa \in \mathbb{N}_0$. Then $s \in \mathfrak{S}_\kappa$ if and only if $V(s, \mathbb{D}, \mathbb{D}^c) = \kappa$.*

Theorem 4.2. *(*[13, Theorem 1.1]*) Let f be a symmetric function meromorphic on $\mathbb{C} \setminus \mathbb{R}$ and let $\kappa \in \mathbb{N}_0$. Then $f \in \mathfrak{N}_\kappa$ if and only if $V(f, \mathbb{C}_+, \mathbb{C}_-) = \kappa$.*

Different from their respective factorizations, the equivalence of the preceding theorems is a direct consequence of the elementary observation expressed by (1.3).

The factorizations of generalized Schur functions and generalized Nevanlinna functions have a similar structure: A generalized Schur function is factorized as the product of a symmetric rational function with an ordinary Schur function and a generalized Nevanlinna functions as the product of a symmetric rational function with an ordinary Nevanlinna function, see Theorems 1.1 and 1.2, respectively. However, Theorems 4.1 and 4.2 show that the valency of generalized Schur functions and generalized Nevanlinna is very different for the value ∞: In the case of a generalized Schur function s there exists an open neighborhood A of ∞ such that $V(s, \mathbb{D}, A)$ is constant, while for a generalized Nevanlinna function f there does not in general exists an open neighborhood A of ∞ such that $V(f, \mathbb{C}_+, A)$ is constant. In other words, the poles of generalized Schur functions are isolated while those of generalized Nevanlinna functions need not be. As the factorizations are made with respect to the value ∞, their structure most inevitably differ from each other.

The preceding difference explains why the factorization of generalized Schur functions is easily established (see e.g. the proof of the implication (v) $\Rightarrow$ (iv) in Theorem 3.3), while the factorization of generalized Nevanlinna function is more difficult to establish. This difference is also reflected by the invariant subspace property for contractive operators in Pontryagin spaces, by means of which generalized Schur functions are realized, being easily established, while the same invariant subspace property for selfadjoint relations in Pontryagin spaces, by means of which generalized Nevanlinna functions are realized, being more involved to established; cf. [14].

The preceding discussion shows that the factorization of generalized Nevanlinna functions, i.e. Theorem 1.2, cannot be expected to be straightforwardly derivable from the seemingly similar factorization of generalized Schur functions contained in Theorem 1.1. Indeed, a factorization for generalized Schur functions equivalent (from the perspective of (1.3)) to the factorization for generalized Nevanlinna functions contained in Theorem 1.2 would be a factorization not with respect to the value $\infty \in \mathbb{D}^c$, because for that value there always exists an open neighborhood A such that $V(s, \mathbb{D}, A)$ is constant for every generalized Schur funtion s. Instead, it would be a factorization with respect to a value from $\mathbb{T}$; for such a value the preceding valency statement does not in general hold. Such a factorization is contained in the following statement.

Proposition 4.3. *Let $s \in \mathfrak{S}_\kappa$ and let ϕ be as in* (1.3)*. Then there exists a rational function r of degree κ with poles in $\mathbb{D}$ and zeros in $\mathbb{D} \cup \mathbb{T}$, and $f_0 \in \mathfrak{N}_0$ such that*

$$s(z) = 1 + ir(z)\overline{r(1/\overline{z})}f_0(\phi^{-1}(z)), \qquad z \in \mathbb{D}.$$

Proof. If $s \in \mathfrak{S}_0$, then $\mathrm{Re}\,(s-1) < 0$ on $\mathbb{D}$. Consequently, $\mathrm{Im}\,(-i(s-1)) > 0$ on $\mathbb{D}$ and, hence, there exists $f_0 \in \mathfrak{N}_0$ such that $-i(s-1) = f_0 \circ \phi^{-1}$. This shows that the statement holds in this case. Next assume that $s \in \mathfrak{S}_\kappa$, with $\kappa \neq 0$, and define

$$s_c(z) := s(z) - c, \qquad z \in \mathbb{D} \text{ and } c \in (1,2]. \tag{4.1}$$

By construction of s_c there exists open neighborhoods A_∞ and A_0 of ∞ and 0 in $\mathbb{C} \cup \{\infty\}$, respectively, such that $V(s_c, \mathbb{D}, A_\infty) = \kappa$ and $V(s_c, \mathbb{D}, A_0) = \kappa$, see Theorem 4.1. The former equality implies that there exists a function $q(z) = \prod_{k=1}^{\kappa}(z - p_k)$, where $p_k \in \mathbb{D}$, such that $qs_c \in H_\infty$, cf. Theorem 1.2. And the latter equality means that s_c has zeros of total multiplicity κ in $\mathbb{D}$. Let $p_c(q) = \prod_{k=1}^{\kappa}(z - n_k^c)$, where $n_1^c, \ldots, n_\kappa^c$ denote the zeros of s_c in $\mathbb{D}$. Then one can factorize s_c as:

$$s_c(z) = ir_c(z)\overline{r_c(1/\overline{z})}f_c(z), \tag{4.2}$$

where

$$r_c(z) := \frac{p_c(z)}{q(z)} \qquad \text{and} \qquad f_c(z) := \frac{1}{i}\frac{\overline{q(1/\overline{z})}q(z)s_c(z)}{\overline{p_c(1/\overline{z})}p_c(z)}.$$

Since the zeros of p_c and q are contained in $\mathbb{D}$, the function $\overline{q(1/\overline{z})}/\overline{p_c(1/\overline{z})}$ is bounded on $\mathbb{D}$. As $qs_c \in H_\infty$ by construction and p_c contains the zeros of s_c, the function $f_c(z)$ is bounded on $\mathbb{D}$ for every $c \in (1,2]$. Since we have that $\lim_{z \hat{\to} x} r_c(z)\overline{r_c(1/\overline{z})} > 0$ for all $x \in \mathbb{T}$ and $\lim_{z \hat{\to} x} \mathrm{Re}\,(s_c) < 0$ (cf. Theorem 3.3 (v)(b) and (4.1)), (4.2) shows that $\lim_{z \hat{\to} x} \mathrm{Im}\, f_c(z) > 0$ for almost all $x \in \mathbb{T}$. Consequently, $f_c \circ \phi \in \mathfrak{N}_0$ by [14, Proposition 3.2] for every $c \in (1,2]$.

Finally, let $\{c_l\}$, $c_l \in (1,2]$ be a sequence converging to 1 such that $n_k^{c_l}$ converges to some element $n_k \in \mathbb{D} \cup \mathbb{T}$ for $k = 1, \ldots, \kappa$. Moreover, let $z_0 \in \mathbb{D}$ be such that $s(z_0) = 10$, then $s_c(z)$ and $(r_c(z)\overline{r_c(1/\overline{z})})^{-1}$ are evidently bounded at z_0 for all $c \in (1,2]$. Hence, $f_c(z)$ is uniformly bounded at z_0 for all $c \in (1,2]$. Accordingly, there exists by [5, Lemma 3 on p. 32] a subsequence $\{c_{l_k}\}_{k \in \mathbb{N}}$ of $\{c_l\}_{l \in \mathbb{N}}$ such that $f_{c_{l_k}} \circ \phi$ converges to some ordinary Nevanlinna function f_0 as k tends to ∞. Therefore the statement holds by taking the limit in (4.2) along the sequence $\{c_{l_k}\}$. □

Proof of Theorem 1.2 via the factorization of generalized Schur functions: Let $f \in \mathfrak{N}_\kappa$ be arbitrary with $\kappa \neq 0$, because otherwise there is nothing to prove. In that case there exists $s \in \mathfrak{S}_\kappa$, $\kappa \neq 0$, such that

$$f(z) = i\frac{1 + s\,(\phi(z))}{1 - s\,(\phi(z))}, \qquad \phi(z) = \frac{z - i}{z + i}. \tag{4.3}$$

see (1.3). By Proposition 4.3 (applied to s and $-s$) there exists rational functions r_+ and r_- of degree κ and ordinary Nevanlinna functions f_+ and f_- such that

$$s(z) = 1 + ir_+(z)\overline{r_+(1/\overline{z})}f_+(\phi^{-1}(z)) \qquad \text{and} \qquad s(z) = -1 - ir_-(z)\overline{r_-(1/\overline{z})}f_-(\phi^{-1}(z)).$$

Plugging the preceding expressions into (4.3) yields

$$f(z) \quad = i\frac{-ir_-(\phi(z))\overline{r_-(1/\overline{\phi(z)})}f_-(z)}{-ir_+(\phi(z))\overline{r_+(1/\overline{\phi(z)})}f_+(z)} = \frac{r_-(\phi(z))\overline{r_-(1/\overline{\phi(z)})}}{r_+(\phi(z))\overline{r_+(1/\overline{\phi(z)})}}\frac{if_-(z)}{f_+(z)} = \frac{r_-(\phi(z))r_-^\#(\phi(z))}{r_+(\phi(z))r_+^\#(\phi(z))}\frac{if_-(z)}{f_+(z)}.$$

As r_+ and r_- have the same poles (see the proof of Proposition 4.3),

$$r := \frac{r_- \circ \phi}{r_+ \circ \phi}$$

is a rational function of degree at most κ. Under the assumption that if_-/f_+ is an ordinary Nevanlinna function, [3, (3.14)] implies that the degree of r should also be at least κ (since $f \in \mathfrak{N}_\kappa$). Consequently, Theorem 1.2 is established when it has been shown that $if_-/f_+ \in \mathfrak{N}_0$. As $if_-/f_+ = f/rr^{\#}$ and $f \in \mathfrak{N}_\kappa$, [3, (3.14)] implies that if_-/f_+ is a generalized Nevanlinna function. Therefore s defined as

$$s := \frac{\frac{if_-(\phi^{-1}(z))}{f_+(\phi^{-1}(z))} - i}{\frac{if_-(\phi^{-1}(z))}{f_+(\phi^{-1}(z))} + i} = \frac{if_-(\phi^{-1}(z)) - if_+(\phi^{-1}(z))}{if_-(\phi^{-1}(z)) + if_+(\phi^{-1}(z))} = \frac{f_-(\phi^{-1}(z)) - f_+(\phi^{-1}(z))}{f_-(\phi^{-1}(z)) + f_+(\phi^{-1}(z))},$$

is a generalized Schur function, see (1.3). As f_+ and f_- are ordinary Nevanlinna functions, the equation $f_-(z) = -f_+(z)$ has a solution in $\mathbb{C}_+$ if and only if f_- and f_+ are constant on $\mathbb{C}_+$: $f_+ = -f_- = c \in \mathbb{R}$. Accordingly, either s has no poles in $\mathbb{D}$ or $f_+ = -f_- = c \in \mathbb{R}$ on $\mathbb{C}_+$. In the latter case, $-if_-/f_+ = i$ on $\mathbb{C}_+$ and, hence, $-f_-/f_+ \in \mathfrak{N}_0$. In the former case, $s \in \mathfrak{S}_0$ and, hence, $if_-/f_+ = \phi^{-1} \circ s \circ \phi \in \mathfrak{N}_0$. $\square$

Appendix A

Here the weak form of the maximum modulus principle, Proposition 2.1, and the Inner-Outer factorization, Theorem 2.2, are shown to imply each other. To do so the class of (ordinary) Caratheodory functions on $\mathbb{D}$, $\mathfrak{C}$ is used. This class, which consists of functions holomorphic on $\mathbb{D}$ whose real part is nonnegative, is closely related to the classes of ordinary Schur and Nevanlinna functions. The following properties of this class are needed, see e.g. [5].

Theorem A.1. *A function f belongs to the class of Caratheodory functions on $\mathbb{D}$, $f \in \mathfrak{C}$, if and only if there exists a real constant c and a finite positive Borel measure* $\mathrm{d}\sigma$ *defined on $(-\pi,\pi]$ such that*

$$f(z) = ic + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\mathrm{d}\sigma(\theta).$$

Moreover, if $f \in \mathfrak{C}$ has the above representation, then

$$\lim_{z \widehat{\to} e^{i\theta}} \operatorname{Re} f(z) = \sigma'(\theta).$$

In particular, $\lim_{z \widehat{\to} e^{i\theta}} \operatorname{Re} f(z)$ *exists for almost all* $\theta \in (-\pi,\pi]$.

Proof of the Inner-Outer factorization by means of Proposition 2.1. **Case $f \in H_\infty$:** By [12, Theorem 17.9] there exists a Blaschke product B_0 containing all zeros of f (in $\mathbb{D}$) such that $f/B_0 \in H_\infty$ and, evidently, there exists $M > 1$ such that $|f/(B_0M)| \leq 1$ on $\mathbb{D}$. Since $f/(B_0M)$ is by construction bounded in absolute value by one and zero free, $-\log(f/(B_0M)) \in \mathfrak{C}$. Thus, in light of the Lebesgue decomposition theorem, there exists by Theorem A.1 a finite positive singular measure $\mathrm{d}\sigma_-$ on $(-\pi,\pi]$ and a real constant d such that

$$-\log\left(\frac{f(z)}{MB_0(z)}\right) = id + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\left[\mathrm{d}\sigma_-(\theta) + \log\left|\frac{f(e^{i\theta})}{B_0(e^{i\theta})M}\right|\mathrm{d}\theta\right].$$

The preceding equality implies that the statement holds in this case (with $c = -d$), because a direct calculation yields that $\log(M) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log(M)\mathrm{d}\theta$ and, evidently, $|B_0(x)| = 1$ for almost all $x \in \mathbb{T}$.

Case $f \in \mathcal{B}$: In light of the mentioned characterization of $\mathcal{B}$-functions as the quotient of H_∞-functions, this case follows immediately from the established one.

Case $f \in H_p$: As a consequence of [12, Theorem 17.9] one may without loss of generality assume that f is zero-free (in $\mathbb{D}$). Then the assumption $f \in H_p$ implies via the inequality between the arithmetic and geometric means that

$$\sup_{0 \leq r < 1} \int_{-\pi}^{\pi} \max\{\log |f(re^{i\theta})|, 0\} \mathrm{d}\theta < \infty,$$

see e.g. [7, p. 76]. Hence, if $f \in H_p$, then h_f, defined as

$$h_f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \max\{\log |f(e^{i\theta})|, 0\} \mathrm{d}\theta,$$

is a well-defined Caratheodory function, see Theorem A.1. Thus $f_+ := e^{-h_f}$ is bounded in norm by 1 on $\mathbb{D}$. Consequently, $ff_+ \in H_p$ and, by construction,

$$\lim_{z \widehat{\rightarrow} x} |f(z) f_+(z)| \leq 1, \qquad \text{for almost all } x \in \mathbb{T}.$$

Therefore Proposition 2.1 yields that ff_+ is uniformly bounded in norm by 1 and, hence, the statement holds by the established representation of H_∞-functions. □

Proof of Proposition 2.1 via the Theorem 2.2: Necessity of the condition in Proposition 2.1 is obvious and, without loss of generality, sufficiency is only proven in case $M = 1$. In that case Theorem 2.2 guarantees the existence of a Blaschke product B_0, a $c \in \mathbb{R}$ and a singular positive measure $\mathrm{d}\sigma$ such that

$$f(z) = B_0(z) e^{ic} e^{-h(z)}, \quad h(z) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} [\mathrm{d}\sigma(\theta) - \log |f(e^{i\theta})| \mathrm{d}\theta]. \tag{A.1}$$

It is well known that the Blaschke product B_0 is bounded in norm by 1 on $\mathbb{D}$. Moreover, the assumption on the boundary values implies that $\log |f(x)|$ is a nonpositive function a.e. on $\mathbb{T}$ and, hence, $h \in \mathfrak{C}$ by Theorem A.1. Hence, $|e^{-h(z)}| \leq 1$ on $\mathbb{D}$. Thus the factorization of f in (A.1) yields that $|f|$ is bounded by 1 on $\mathbb{D}$. □

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