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Essays in honor of Seppo Hassi

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Seppo Hassi

FOREWORD

With this Festschrift we celebrate the sixtieth birthday of our friend and colleague Professor Seppo Hassi of the University of Vaasa. It consists of papers written by colleagues outside Vaasa, who have been coauthors of Seppo, as well as by colleagues from Vaasa. Although many friends and colleagues have known and worked with Seppo for a long time, quite a few people answered in disbelief "What? Seppo 60?" when first approached about this project.

This collection of essays shows our appreciation of Seppo as a friend and as a colleague. From early on, his main activities have been in the branches of mathematics, known as operator theory and spectral theory, although his interests are much broader. Almost all of the included essays reflect these interests. Unfortunately, due to the consequences of the global pandemic some contributions could not be submitted in time to be part of our collection.

It is our pleasure to thank all the authors, both for contributing their work to this volume and for their readiness to respond to our questions and suggestions. Furthermore, we are grateful to Heinz Langer, Kenneth Nordström, Seppo Pynnönen, and Franek Szafraniec for answering our queries concerning several points about the past and present of the person to whom this volume is dedicated. Finally, our thanks go to the staff at the University of Vaasa, in particular to Riikka Kalmi, for their efficient production of this collection.

Groningen and Vaasa, April 2021

Henk de Snoo and Rudi Wietsma

SEPPO HASSI, 60 YEARS

Seppo Ortamo Hassi was born on July 2, 1961, in Hyvinkää in southern Finland. He received his secondary school education in Pori, located on the western coast of Finland, and in 1980 he went to the University of Helsinki to be a student in mathematics. There Seppo obtained his master's degree in 1984. He would stay at the university and eventually in 1985 became an assistant in the Department of Statistics (which was located at Yliopistonkatu, while the Department of Mathematics was located at Alexanterinkatu, a geographical gap).

The leading people in the statistics department were Hannu Niemi (a student of Louhivaara, whom we will meet below) and Seppo Mustonen. Mustonen somehow awakened Seppo's interest in singular values and canonical representations of operators. This eventually led to the dissertation *A singular value decomposition of matrices in a space with an indefinite scalar product*, with Ilppo Simo Louhivaara (1927 - 2008)[†] as adviser. This thesis in mathematics was approved by the University of Helsinki on January 31, 1991, at the time that Seppo served in the Finnish army (between August 1990 and April 1991). The opponent at the defence was Heinz Langer (originally from Dresden); Langer had first visited Louhivaara in Jyväskylä in 1969 and had been a frequent guest ever since.

Prior to finishing his dissertation, Seppo had been invited to participate in the *Schur Analysis* meeting (October 16 - October 20, 1989) at the Karl Marx Universität in Leipzig, Deutsche Demokratische Republik, organized by Bernd Kirstein and Bernd Fritzsche. This seminar brought together many people from East and West. It took place in the middle of the peaceful protests against the communist regime that had been going on in Leipzig for some time. Loudspeakers in empty streets would advise the public not to follow the protesting crowds: "They are misguided." On November 9, shortly after the conference, the Berlin wall came down. At the beginning of the conference it turned out that Heinz Langer had left the country and at its closing it was announced that the great mathematician Mark Grigorievich Kreĭn (1907-1989) had died. One of the people present from the East was Yury L'vovich Smul'yan (1927-1990), whose work played an important role in Seppo's dissertation and in his later articles.

With the dissertation completed, Seppo started some joint work with his colleague Kenneth Nordström, who was also an assistant in the Department of Statistics. Their interest focussed on antitonicity properties of operators and projections in indefinite inner product spaces. In the meantime Heinz Langer had obtained a professorship at the Technische Universität Wien in 1991. He invited Seppo to spend some weeks in Vienna in 1992 at the same time that Henk de Snoo from Groningen was also visiting. During that period Langer's Dutch and Finnish visitors started to work together, which led to many mutual

[†]Louhivaara had been one of the many students of Rolf Herman Nevanlinna (1895-1980). He was also interested in extension theory and indefinite metrics, like his contemporary fellow students Yrjö Kilpi (1924-2010) and Erkki Pesonen (1930-2006). Louhivaara had been a professor of mathematics at the universities in Helsinki and Jyväskylä, before moving to the Freie Universität Berlin.

visits to Holland and Finland over the years, up till the present day. It was during a number of subsequent conferences in or visits to Vienna, Pula, Timisoara, Warsaw, Krakow, and Budapest that it was possible to meet old and new acquaintances and lay foundations for future work. It is appropriate to mention in this context Michael Kaltenbäck, Harald Woracek, Henrik Winkler, Andreas Fleige, Franek Szafraniec, Zoltán Sebestyén (thanks to Jan Stochel), Jean-Philippe Labrousse, and last, but not least, Yury Arlinskiĭ, Vladimir Derkach, and Mark Malamud. A sabbatical visit to Groningen and Berlin in the academic year 2000-2001 made it possible to meet the group around Karl-Heinz Förster of the Technische Universität Berlin, which consisted of Peter Jonas and Peter's students Carsten Trunk and Jussi Behrndt. Peter Jonas was from East Berlin and had come to the Technische Universität via Ilppo Simo Louhivaara at the Freie Universität. Seppo's visit led to fruitful contacts; also the later December conferences in Berlin were very productive.

Seppo would remain at the Department of Statistics in Helsinki until 2001; in the meantime he had been formally named docent at the Department of Mathematics of the same university. In November-December 2000 there had been a longer visit to Manfred Möller at the University of Witwatersrand in South Africa and it was there that Seppo found out that the University of Vaasa was interested in his person. He obtained a professorship at that university in the spring of 2001. Seppo settled down in Vaasa during the summer and took up the usual teaching and administrative tasks. In the following years the number of coworkers increased with, for instance, Annemarie Luger, Adrian Sandovici, Sergey Belyi, Eduard Tsekanovskiĭ, and Sergii Kużel. As a consequence there has been a regular stream of visitors (all of whom think with a certain melancholy of the old wooden guestrooms of the University of Vaasa). In May 2003 Seppo was the organizer of an Operator Theory Symposium and, a little later, in 2005 he was one of the organizers of the Algorithmic Information Theory Conference, see Acta Wasaensia 124, 2005. Moreover, Seppo was one of the organizers of the conferences "Boundary relations" and "Operator realizations of analytic functions" at the Lorentz Center in Leiden in 2009 and 2013, respectively.

The main mathematical interest of Seppo circles around the topics of spectral theory, boundary value problems for differential equations, operator theory and its applications in analysis, mathematical physics, and system theory. This keeps him going with great dedication. In particular, right from the beginning Seppo looked into situations involving indefinite inner product spaces and this interest also led to several doctoral students, Rudi Wietsma, Dmytro Baidiuk, and Lassi Lilleberg, writing a dissertation on this topic under his direction. Being a rather prolific writer himself, he is furthermore an editor for a number of mathematical journals.

When Seppo first arrived in Vaasa he belonged to the Department of Mathematics and Statistics. As the century progresses, so does the university. Seppo now belongs to the School of Technology and Innovations, where he is the leader of the Mathematics and Statistics Research Group. He is also head of the Doctoral Programme in Technical Sciences. Moreover, there are duties beyond Vaasa. For many years Seppo has been involved with the nationwide entrance exam for Technical Sciences and Architecture studies of the member universities in Finland. And then there is the Academy of Finland: for some years now Seppo has been a member of its Research Council for Natural Sciences and Engineering, and a member of its steering group. All these things coming his way are done with his usual attention to detail.

Those who deal with Seppo, either as colleagues or as students, know that he provides a listening ear and is ready to help whenever needed. And those who are fortunate enough to work with him recognize his quiet determination. Uninterrupted, he can sit behind his desk for hours, like a sphinx – lost in thought (so we assume). But when he returns back to real life, you know that something is going to happen.

On behalf of all his many friends, whether in Vaasa or elsewhere in the world, we congratulate Seppo on reaching his sixtieth birthday and we wish him, together with his wife Merja and their son Leo, good health and happiness. May there be many more productive years to come!

CONTENTS

CONGRUENCE OF SELFADJOINT OPERATORS AND TRANSFORMATIONS OF OPERATOR-VALUED NEVANLINNA FUNCTIONS

Yury Arlinskiĭ

Dedicated to my colleague and friend Seppo Hassi on the occasion of his sixtieth birthday

1 Introduction

The Banach space of all continuous linear operators acting between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is denoted by $\mathbf{B}(\mathcal{H},\mathcal{K})$ and by $\mathbf{B}(\mathcal{H})$ if $\mathcal{K}=\mathcal{H}$. Likewise, the group of all invertible operators in $\mathbf{B}(\mathcal{H})$ is denoted by $\mathbf{GB}(\mathcal{H})$. Let $\mathfrak{N}$ be a Hilbert space. Recall that a $\mathbf{B}(\mathfrak{N})$-valued function M is called a *Nevanlinna function* (Behrndt, Hassi & de Snoo, 2020) (alternatively, an *R-function* (Allen & Narcowich, 1976; Derkach & Malamud, 2017; Kac & Kreĭn, 1968; Shmul'yan, 1971), a *Herglotz function* (Gesztesy & Tsekanovskiĭ, 2000), or a *Herglotz-Nevanlinna function* (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011; Arlinskiĭ & Klotz, 2010)) if it is holomorphic outside the real axis, symmetric $M(\lambda)^*=M(\bar\lambda)$, and satisfies the inequality $\operatorname{Im}\lambda\operatorname{Im}M(\lambda)\geq 0$ for all $\lambda\in\mathbb{C}\backslash\mathbb{R}$.

The class of Nevanlinna functions is often denoted by $\mathcal{R}[\mathfrak{N}]$. A function $M\in\mathcal{R}[\mathfrak{N}]$ admits the integral representation

$$M(\lambda)=A+B\lambda+\int_{\mathbb{R}}\left(\frac{1}{t-\lambda}-\frac{t}{t^2+1}\right)d\Sigma(t),\qquad \int_{\mathbb{R}}\frac{d\Sigma(t)}{t^2+1}\in\mathbf{B}(\mathfrak{N}),\ \lambda\in\mathbb{C}\backslash\mathbb{R},\tag{1.1}$$

where $A=A^*\in\mathbf{B}(\mathfrak{N})$, $0\leq B=B^*\in\mathbf{B}(\mathfrak{N})$, the $\mathbf{B}(\mathfrak{N})$-valued function $\Sigma(\cdot)$ is nondecreasing and $\Sigma(t)=\Sigma(t-0)$, see (Behrndt, Hassi & de Snoo, 2020; Derkach & Malamud, 2017; Kac & Kreĭn, 1968; Shmul'yan, 1971). The integral is uniformly convergent in the strong topology; cf. (Behrndt, Hassi & de Snoo, 2020; Brodskiĭ, 1969; Kac & Kreĭn, 1968).

It follows from (1.1) that

$$B=\operatorname*{s-lim}_{y\uparrow\infty}\frac{M(iy)}{y}\quad\text{and}\quad\operatorname{Im}M(iy)=B\,y+\int_{\mathbb{R}}\frac{y}{t^2+y^2}\,d\Sigma(t).$$

This implies that $\lim_{y\to\infty}y\operatorname{Im}M(iy)$ exists in the strong resolvent sense as a selfadjoint relation; see, e.g., (Behrndt, et al., 2010). This limit is a bounded selfadjoint operator if and only if $B=0$ and $\int_{\mathbb{R}}d\Sigma(t)\in\mathbf{B}(\mathfrak{N})$, in which case $\text{s-lim}_{y\to\infty}y\operatorname{Im}M(iy)=\int_{\mathbb{R}}d\Sigma(t)$. In this case one can rewrite the integral representation (1.1) in the form

$$M(\lambda)=E+\int_{\mathbb{R}}\frac{1}{t-\lambda}\,d\Sigma(t),\qquad\int_{\mathbb{R}}d\Sigma(t)\in\mathbf{B}(\mathfrak{N}),\tag{1.2}$$

where $E=\lim_{y\to\infty}M(iy)$ in $\mathbf{B}(\mathfrak{N})$.

The class of $\mathbf{B}(\mathfrak{N})$-valued Nevanlinna functions M with the integral representation (1.2) for which $E=0$ is denoted by $\mathcal{R}_0[\mathfrak{N}]$. In this paper we consider the following subclasses of the class $\mathcal{R}_0[\mathfrak{N}]$.

Definition 1.1. Let M belong to the class $\mathcal{R}_0[\mathfrak{N}]$. Then M is said to belong to $\mathcal{N}[\mathfrak{N}]$ if

$$\operatorname*{s-lim}_{y\to\infty} iyM(iy) = -I_{\mathfrak{N}}.$$

Moreover, M is said to belong to $\mathbf{N}^0_{\mathfrak{N}}$ if $M \in \mathcal{N}[\mathfrak{N}]$ and M is holomorphic at infinity.

If A is a selfadjoint operator in the Hilbert space $\mathfrak{H}$ and $\mathfrak{N}$ is a subspace (closed linear manifold) of $\mathfrak{H}$, then the compressed resolvent $M(\lambda)$, defined as

$$M(\lambda) = P_{\mathfrak{N}}(A-\lambda I)^{-1}\restriction\mathfrak{N}, \qquad \lambda\in\rho(A), \tag{1.3}$$

belongs to the class $\mathcal{N}[\mathfrak{N}]$. Moreover, M as in (1.3) belongs to the class $\mathbf{N}^0_{\mathfrak{N}} \subseteq \mathcal{N}[\mathfrak{N}]$ if and only if the selfadjoint operator A is bounded. Throughout this paper the representation of $M \in \mathcal{N}(\mathfrak{N})$ in the form (1.3) is called a *realization of the function* M. Note that the function M in (1.3) is often called the *compressed resolvent*, $\mathfrak{N}$*-resolvent*, *Weyl function*, or m*-function*; see (Berezansky, 1968; Gesztesy & Simon, 1997). Here, and throughout the paper, the notation $T\restriction\mathcal{N}$ denotes the restriction of a linear operator T to the set $\mathcal{N}\subset \operatorname{dom} T$ and $P_{\mathfrak{L}}$ denotes the orthogonal projection onto a subspace $\mathfrak{L}$ in the Hilbert space $\mathfrak{H}$.

Let $\mathfrak{H} = \mathfrak{N}\oplus\mathfrak{K}$ be a decomposition of a Hilbert space $\mathfrak{H}$, then a selfadjoint operator $A\in\mathfrak{H}$ is called *minimal with respect to* $\mathfrak{N}$, or $\mathfrak{N}$*-minimal*, if

$$\mathfrak{H} = \overline{\operatorname{span}}\left\{\mathfrak{N} + (A-\lambda I)^{-1}\mathfrak{N} :\ \lambda\in\mathbb{C}\setminus\mathbb{R}\right\}.$$

The next theorem follows from (Brodskiĭ, 1969: Theorem 4.8) and Naĭmark's dilation theorem (Brodskiĭ, 1969: Theorem 1, Appendix I); see (Arlinskiĭ, Hassi & de Snoo, 2006) and (Arlinskiĭ & Klotz, 2010) for the case $M\in\mathbf{N}^0_{\mathfrak{N}}$.

Theorem 1.2. *The following assertions are valid:*

(1) *If $M\in\mathcal{N}[\mathfrak{N}]$, then there exist a Hilbert space $\mathfrak{H}$ containing $\mathfrak{N}$ as a subspace and a selfadjoint operator A in $\mathfrak{H}$, such that A is $\mathfrak{N}$-minimal and $M(\lambda)$ is of the form* (1.3) *for λ in the domain of M. If $M\in\mathbf{N}^0_{\mathfrak{N}}$, then the selfadjoint operator A is bounded.*

(2) *If A_1 and A_2 are selfadjoint operators in the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, $\mathfrak{N}$ is a common subspace of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, A_1 and A_2 are $\mathfrak{N}$-minimal, and*

$$P_{\mathfrak{N}}(A_1-\lambda I_{\mathfrak{H}_1})^{-1}\restriction\mathfrak{N} = M(\lambda) = P_{\mathfrak{N}}(A_2-\lambda I_{\mathfrak{H}_2})^{-1}\restriction\mathfrak{N}, \qquad \lambda\in\mathbb{C}\backslash\mathbb{R},$$

then there exists a unitary operator U mapping $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ such that

$$U\restriction\mathfrak{N} = I_{\mathfrak{N}} \quad \textit{and} \quad UA_1 = A_2U.$$

The following linear transformations ω of the complex plane $\mathbb{C}$ will play an important role in this paper. Let

$$\omega(\lambda) = a\lambda + b, \qquad \lambda\in\mathbb{C}, \tag{1.4}$$

where $a\in\mathbb{R}_+$, $b\in\mathbb{R}$, then ω has the property

$$\omega(\mathbb{R}) = \mathbb{R}, \qquad \omega(\mathbb{C}_-) = \mathbb{C}_-, \qquad \omega(\mathbb{C}_+) = \mathbb{C}_+.$$

Note that if $\omega_1(\lambda) = a_1\lambda + b_1$ and $\omega_2(\lambda) = a_2\lambda + b_2$, then

$$\omega_1 \circ \omega_2(\lambda) := \omega_1(\omega_2(\lambda)) = a_1a_2\lambda + a_1b_2 + b_1. \tag{1.5}$$

Hence, these transformations form a group with respect to composition. The inverse transformation $\omega^{[-1]}$ corresponding to $\omega(\lambda) = a\lambda + b$ is

$$\omega^{[-1]}(\lambda) = a^{-1}\lambda - ba^{-1}. \tag{1.6}$$

This group is denoted by $\mathfrak{G}$. For $\omega(\lambda) = a\lambda + b \in \mathfrak{G}$ we define $a_\omega := a$. It follows from (1.5) that

$$a_{\omega_1\circ\omega_2} = a_{\omega_2\circ\omega_1} = a_{\omega_1}a_{\omega_2}.$$

Hence, the function $\mathfrak{G} \ni \omega \mapsto a_\omega \in \mathbb{R}_+$ is a character on the group $\mathfrak{G}$.

For a function $\omega(\lambda) = a\lambda + b \in \mathfrak{G}$ define the following transformations $\mathcal{G}_\omega$ on $\mathcal{N}(\mathfrak{N})$:

$$M(\lambda) \mapsto \mathcal{G}_\omega(M)(\lambda) := a_\omega M(\lambda)\left(I + (\lambda - \omega(\lambda))M(\lambda)\right)^{-1}. \tag{1.7}$$

The properties of this transformation are discussed in the theorem below. For this theorem also recall that two linear operators X and Y in $\mathcal{H}$ are said to be *congruent*, if there exists $U \in \mathbf{GB}(\mathcal{H})$ such that

$$Y = U^*XU;$$

see, e.g., (Patel, 1983). In the case of unbounded X and Y, the above equality means that

$$\operatorname{dom} Y = U^{-1}\operatorname{dom} X \quad \text{and} \quad YU^{-1}f = U^*Xf, \qquad \text{for all } f \in \operatorname{dom} X.$$

The main goal of this paper is to prove the following theorem.

Theorem 1.3. *For the transformations $\mathcal{G}_\omega$ defined in* (1.7)*, where ω is given by* (1.4)*, the following assertions are valid:*

(1) *For each $\omega \in \mathfrak{G}$ the transformation $\mathcal{G}_\omega$ is well-defined and maps $\mathcal{N}(\mathfrak{N})$ into $\mathcal{N}(\mathfrak{N})$, and $\mathbf{N}^0_{\mathfrak{N}}$ into $\mathbf{N}^0_{\mathfrak{N}}$.*

(2) *The set $\{\mathcal{G}_\omega : \omega \in \mathfrak{G}\}$ is a group with respect to composition:*

$$\begin{aligned} &\mathcal{G}_{\omega_2}(\mathcal{G}_{\omega_1}(M)) = \mathcal{G}_{\omega_1\circ\omega_2}(M), && \omega_1, \omega_2 \in \mathfrak{G}, \\ &\mathcal{G}_\omega^{-1}(M) = \mathcal{G}_{\omega^{[-1]}}(M), && \omega \in \mathfrak{G},\ M \in \mathcal{N}(\mathfrak{N}). \end{aligned}$$

In particular, for each $\omega \in \mathfrak{G}$ the transformation $\mathcal{G}_\omega$ maps $\mathcal{N}(\mathfrak{N})$ bijectively onto $\mathcal{N}(\mathfrak{N})$, and $\mathbf{N}^0_{\mathfrak{N}}$ bijectively onto $\mathbf{N}^0_{\mathfrak{N}}$.

(3) *If A is a $\mathfrak{N}$-minimal realization of $M \in \mathcal{N}(\mathfrak{N})$ and if $\omega(\lambda) = a\lambda + b \in \mathfrak{G}$, then any minimal realization of the function $\mathcal{G}_\omega(M)$ is congruent to $A - bP_{\mathfrak{N}}$. Moreover, if $\omega(\lambda) = \lambda + b$, $b \neq 0$, then any minimal realization of the function $\mathcal{G}_\omega(M)$ is unitarily equivalent to $A - bP_{\mathfrak{N}}$.*

Note that the transformations

$$\mathbf{N}^0_{\mathfrak{N}} \ni M(\lambda) \mapsto M_B(\lambda) := M(\lambda)\left(I_{\mathfrak{N}} + BM(\lambda)\right)^{-1} \in \mathbf{N}^0_{\mathfrak{N}}, \qquad B = B^* \in \mathbf{B}(\mathfrak{N}),$$

have been considered in Arlinskiĭ, Hassi & de Snoo (2006) and Arlinskiĭ & Klotz (2010). The transformations

$$\mathcal{R}[\mathfrak{N}] \ni m(\lambda) \mapsto \frac{m(\lambda)+t}{1-tm(\lambda)} \in \mathcal{R}[\mathfrak{N}], \qquad t \in \mathbb{R} \cup \{\infty\},$$

of scalar Nevanlinna functions and their connections with selfadjoint extensions of symmetric operators with deficiency indices $(1,1)$ have been studied in Behrndt, Hassi, de Snoo, Wietsma & Winkler (2013). Other transformations of Nevanlinna functions, or Nevanlinna families, and their fixed points have been examined in Arlinskiĭ (2017; 2020) and Arlinskiĭ & Hassi (2019).

This paper is organized as follows. In Section 2 we study properties of congruent operators; in particular, it is shown that congruence preserves the deficiency indices of densely defined closed symmetric operators. In Section 3 we define and examine special transformations of linear operators, which are used in Section 4 in the proof of Theorem 1.3.

2 Properties of congruent operators

Proposition 2.1. *The following assertions are valid:*

(1) *If the closed densely defined operators X and Y are congruent, then the adjoint operators X^* and Y^* are congruent.*

(2) *Congruence preserves the notions densely defined, closed, maximal dissipative, maximal accumulative, and selfadjoint.*

(3) *If the closed densely defined symmetric operators X and Y are congruent, then the deficiency indices of X and Y coincide.*

Proof. (1) If X and Y are densely defined and $Y = U^*XU$, then

$$Y^* = U^*X^*U, \tag{2.1}$$

as easily follows.

(2) Let $Y = U^*XU$, $U \in \mathbf{GB}(\mathcal{H})$. Then $(\mathrm{dom}\ Y)^\perp = U^*(\mathrm{dom}\ X)^\perp$ and it follows that X and Y are both densely defined or non-densely defined.

Next let X be a closed operator and suppose that $\{f_n\}$ and $\{U^*XUf_n\}$ are Cauchy sequences. Then, due to assumption $U \in \mathbf{GB}(\mathcal{H})$, it follows that $\{Uf_n\}$ and $\{XUf_n\}$ are Cauchy sequences. Since X is closed, we get that $g = \lim\limits_{n\to\infty} Uf_n \in \mathrm{dom}\ X$ and $Xg = \lim\limits_{n\to\infty} Uf_n$. Hence

$$U^{-1}g = \lim_{n\to\infty} f_n \in \mathrm{dom}\ Y \quad \text{and} \quad YU^{-1}g = U^*XU(U^{-1}g) = \lim_{n\to\infty} U^*XUf_n.$$

Thus, Y is closed.

The equality $(Yf,f) = (XUf,Uf)$, $f \in \mathrm{dom}\ Y$, yields that congruence preserves the notions Hermitian, dissipative, and accumulative. Thanks to (2.1) congruence preserves selfadjointness.

Finally, it is well-known that X is maximal dissipative if and only if X is dissipative and X^* is accumulative. Hence, we conclude that congruence also preserves the notions maximal dissipative and maximal accumulative.

(3) Suppose that X is a closed densely defined symmetric operator whose deficiency indices are $(n_+(X), n_-(X))$. Consider a maximal dissipative extension $\widetilde{X}$ of X. Then for any λ, $\operatorname{Im}\lambda < 0$, the following direct decomposition of $\operatorname{dom} X^*$ holds:

$$\operatorname{dom} X^* = \operatorname{dom} \widetilde{X} \dotplus \mathfrak{N}_\lambda(X). \tag{2.2}$$

Here $\mathfrak{N}_\lambda(X) = \ker(X^* - \lambda I)$ is the defect subspace of X corresponding to λ. In particular, if $h \in \operatorname{dom} X^*$, then there exists $\tilde{h} \in \operatorname{dom} \widetilde{X}$ and $\varphi_\lambda \in \mathfrak{N}_\lambda(X)$ such that

$$h = \widetilde{h} + \varphi_\lambda \quad \text{and} \quad X^* h = \widetilde{X}\widetilde{h} + \lambda\varphi_\lambda. \tag{2.3}$$

Next we will describe the defect subspace $\mathfrak{N}_\lambda(Y)$ for the symmetric operator Y congruent to X:

$$Y = U^* X U.$$

For this purpose, set $\widetilde{Y} = U^*\widetilde{X}U$. Then $\widetilde{Y}$ is a maximal dissipative extension of Y, see part (2) of this proposition. Hence, from the equality

$$\widetilde{Y} - \lambda I = U^*\widetilde{X}U - \lambda I = (U^*\widetilde{X} - \lambda U^{-1})U,$$

it follows that the operator $(U^*\widetilde{X} - \lambda U^{-1})^{-1}$ exists, is bounded, is defined on the whole $\mathfrak{H}$ for $\operatorname{Im}\lambda < 0$, and maps $\mathfrak{H}$ onto $\operatorname{dom} \widetilde{X}$.

Let $f_\lambda \in \mathfrak{N}_\lambda(Y)$. Then

$$0 = (Y^* - \lambda I)f_\lambda = (U^* X^* - \lambda U^{-1})U f_\lambda. \tag{2.4}$$

As $h_\lambda := U f_\lambda$ belongs to $\operatorname{dom} X^*$, (2.2)-(2.3) imply that the following decomposition holds

$$h_\lambda = h_{\widetilde{X}} + \varphi_\lambda \quad \text{and} \quad X^* h_\lambda = \widetilde{X} h_{\widetilde{X}} + \lambda\varphi_\lambda, \qquad h_{\widetilde{X}} \in \operatorname{dom} \widetilde{X},\ \varphi_\lambda \in \mathfrak{N}_\lambda(X).$$

Consequently,

$$\begin{aligned}
(Y^* - \lambda I)f_\lambda &= (U^* X^* U - \lambda I)f_\lambda \\
&= (U^* X^* - \lambda U^{-1})U f_\lambda \\
&= (U^* X^* - \lambda U^{-1})h_\lambda \\
&= U^* X^* h_\lambda - \lambda U^{-1} h_\lambda \\
&= U^*(\widetilde{X} h_{\widetilde{X}} + \lambda\varphi_\lambda) - \lambda U^{-1}(h_{\widetilde{X}} + \varphi_\lambda) \\
&= (U^*\widetilde{X} - \lambda U^{-1})h_{\widetilde{X}} + \lambda(U^* - U^{-1})\varphi_\lambda.
\end{aligned}$$

Combining the preceding result with (2.4) yields

$$h_{\widetilde{X}} = \lambda(U^*\widetilde{X} - \lambda U^{-1})^{-1}(U^{-1} - U^*)\varphi_\lambda.$$

Hence,

$$h_\lambda = \left(I + \lambda(U^*\widetilde{X} - \lambda U^{-1})^{-1}(U^{-1} - U^*)\right)\varphi_\lambda,$$
$$f_\lambda = U^{-1}\left(I + \lambda(U^*\widetilde{X} - \lambda U^{-1})^{-1}(U^{-1} - U^*)\right)\varphi_\lambda.$$

Thus, $\mathfrak{N}_\lambda(Y) \subseteq U^{-1}\left(I + \lambda(U^*\widetilde{X} - \lambda U^{-1})^{-1}(U^{-1} - U^*)\right)\mathfrak{N}_\lambda(X)$. One can verify that the converse inclusion is also true. Therefore

$$\mathfrak{N}_\lambda(Y) = U^{-1}\left(I + \lambda(U^*\widetilde{X} - \lambda U^{-1})^{-1}(U^{-1} - U^*)\right)\mathfrak{N}_\lambda(X).$$

This equality yields that $\dim \mathfrak{N}_\lambda(Y) = \dim \mathfrak{N}_\lambda(X)$, $\operatorname{Im}\lambda < 0$. Similarly, using the decomposition

$$\mathrm{dom}\ X^* = \mathrm{dom}\ \widetilde{X}^* \dotplus \mathfrak{N}_{\bar\lambda}, \qquad \operatorname{Im}\lambda < 0,$$

we obtain the equality $\dim \mathfrak{N}_\lambda(Y) = \dim \mathfrak{N}_\lambda(X)$. Consequently, the deficiency indices of Y are given by $(n_+(X), n_-(X))$. □

3 Special transformations of operators

Let $\mathfrak{H}$ be an infinite-dimensional separable complex Hilbert space, let $\mathfrak{N}$ be a subspace of $\mathfrak{H}$, and set $\mathfrak{N}^\perp := \mathfrak{H} \ominus \mathfrak{N}$. For $z \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ define the operator $\mathcal{U}_{z,\mathfrak{N}} \in \mathbf{B}(\mathfrak{H})$ as follows

$$\mathcal{U}_{z,\mathfrak{N}} := P_{\mathfrak{N}^\perp} + i\frac{\operatorname{Im} z}{\operatorname{Re} z}P_{\mathfrak{N}} = I - \frac{\bar z}{\operatorname{Re} z}P_{\mathfrak{N}}. \tag{3.1}$$

It is clear from the first equality in (3.1) that $\mathcal{U}_{z,\mathfrak{N}}\mathcal{U}_{i\bar z,\mathfrak{N}} = P_{\mathfrak{N}^\perp} - P_{\mathfrak{N}}$. Moreover, it follows from the second equality in (3.1) that $\operatorname{ran}\mathcal{U}_{z,\mathfrak{N}} = \mathfrak{H}$ and that $\mathcal{U}_{z,\mathfrak{N}} \in \mathbf{GB}(\mathfrak{H})$; in fact, one has

$$\mathcal{U}_{z,\mathfrak{N}}^{-1} = I - \frac{i\bar z}{\operatorname{Im} z}P_{\mathfrak{N}} = \mathcal{U}_{-iz,\mathfrak{N}}. \tag{3.2}$$

Hence, one also sees immediately that

$$\mathcal{U}_{z,\mathfrak{N}}^* = \mathcal{U}_{\bar z,\mathfrak{N}} \quad \text{and} \quad \mathcal{U}_{z,\mathfrak{N}}^{*-1} = \mathcal{U}_{i\bar z,\mathfrak{N}}. \tag{3.3}$$

Observe that

$$\mathcal{U}_{z,\mathfrak{N}}^*\mathcal{U}_{z,\mathfrak{N}} = \left(I - \frac{z}{\operatorname{Re} z}P_{\mathfrak{N}}\right)\left(I - \frac{\bar z}{\operatorname{Re} z}P_{\mathfrak{N}}\right) = I + \frac{|z|^2 - 2(\operatorname{Re} z)^2}{(\operatorname{Re} z)^2}P_{\mathfrak{N}}, \tag{3.4}$$

so that $\mathcal{U}_{z,\mathfrak{N}} \in \mathbf{GB}(\mathfrak{H})$ is unitary if and only if $(\operatorname{Re} z)^2 = (\operatorname{Im} z)^2$.

For $z \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ we define the transformation $\mathcal{F}_{z,\mathfrak{N}}$ on the set of all linear operators A in $\mathfrak{H}$ as follows

$$\begin{cases} \mathrm{dom}\ \mathcal{F}_{z,\mathfrak{N}}(A) = \mathcal{U}_{z,\mathfrak{N}}\ \mathrm{dom}\ A, \\ \mathcal{F}_{z,\mathfrak{N}}(A)f = \left(A + \dfrac{iz}{\operatorname{Im} z}P_{\mathfrak{N}}(A - \bar z I)\right)\mathcal{U}_{z,\mathfrak{N}}^{-1}f, \quad f \in \mathrm{dom}\ \mathcal{F}_{z,\mathfrak{N}}(A). \end{cases} \tag{3.5}$$

Lemma 3.1. *Let A be an operator and let $z \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$. Then the operator $\mathcal{F}_{z,\mathfrak{N}}(A)$ satisfies*

$$\mathcal{F}_{z,\mathfrak{N}}(A) = \mathcal{U}_{z,\mathfrak{N}}^{*-1} \left(A - \frac{|z|^2}{\operatorname{Re} z} P_{\mathfrak{N}} \right) \mathcal{U}_{z,\mathfrak{N}}^{-1}, \tag{3.6}$$

i.e., $\mathcal{F}_{z,\mathfrak{N}}(A)$ is congruent to the operator $A - (|z|^2/\operatorname{Re} z)P_{\mathfrak{N}}$. Moreover, if $|\operatorname{Im} z| = |\operatorname{Re} z|$, then $\mathcal{F}_{z,\mathfrak{N}}(A)$ is unitarily equivalent to the operator $A - (|z|^2/\operatorname{Re} z)P_{\mathfrak{N}}$.

Proof. It follows from (3.1) that

$$\begin{aligned} A + \frac{iz}{\operatorname{Im} z} P_{\mathfrak{N}}(A - \bar{z}I) &= \left(I + i\frac{z}{\operatorname{Im} z} P_{\mathfrak{N}} \right) A + i\frac{|z|^2}{\operatorname{Im} z} P_{\mathfrak{N}} \\ &= \mathcal{U}_{i\bar{z},\mathfrak{N}} A + \mathcal{U}_{i\bar{z},\mathfrak{N}} \mathcal{U}_{z,\mathfrak{N}} i\frac{|z|^2}{\operatorname{Im} z} P_{\mathfrak{N}} \\ &= \mathcal{U}_{i\bar{z},\mathfrak{N}} \left[A + \mathcal{U}_{z,\mathfrak{N}} i\frac{|z|^2}{\operatorname{Im} z} P_{\mathfrak{N}} \right] = \mathcal{U}_{i\bar{z},\mathfrak{N}} \left(A - \frac{|z|^2}{\operatorname{Re} z} P_{\mathfrak{N}} \right). \end{aligned}$$

Consequently, the first statement about the congruence now follows from the definition of $\mathcal{F}_{z,\mathfrak{N}}(A)$ and (3.3). The last statement follows from the identity (3.4). □

It is clear from the definition in (3.5), that

$$(\operatorname{dom} \mathcal{F}_{z,\mathfrak{N}}(A))^{\perp} = \mathcal{U}_{z,\mathfrak{N}}^{*-1} (\operatorname{dom} A)^{\perp}.$$

Thus, the operator $\mathcal{F}_{z,\mathfrak{N}}(A)$ is densely defined if and only if the operator A is densely defined. Furthermore, the domain of $\operatorname{dom} \mathcal{F}_{z,\mathfrak{N}}(A)$ is closed if and only if $\operatorname{dom} A$ is closed.

The next corollary collects the basic properties of the transformation $\mathcal{F}_{z,\mathfrak{N}}$.

Corollary 3.2. *The transformation $\mathcal{F}_{z,\mathfrak{N}}$ in* (3.5) *possesses the following properties:*

(1) $\operatorname{dom} \mathcal{F}_{z,\mathfrak{N}}(A) \cap \operatorname{dom} \mathcal{F}_{z,\mathfrak{N}}(B) = \{0\}$ *if and only if* $\operatorname{dom} A \cap \operatorname{dom} B = \{0\}$.

(2) *The operator $\mathcal{F}_{z,\mathfrak{N}}(A)$ is bounded or closed if and only if A is bounded or closed, respectively.*

(3) *The operator $\mathcal{F}_{z,\mathfrak{N}}(A)$ is symmetric, dissipative, or accumulative if and only if A is symmetric, dissipative, or accumulative, respectively. Moreover, maximality with respect to these properties is preserved and selfadjointness is also preserved.*

(4) *The following relation holds*

$$(\mathcal{F}_{z,\mathfrak{N}}(A))^* = \mathcal{F}_{z,\mathfrak{N}}(A^*).$$

(5) *The following identities hold*

$$\operatorname{dom} \mathcal{F}_{z,\mathfrak{N}}(A) \cap \mathfrak{N}^{\perp} = \operatorname{dom} A \cap \mathfrak{N}^{\perp} \quad \textit{and} \quad P_{\mathfrak{N}^{\perp}} \mathcal{F}_{z,\mathfrak{N}}(A) {\upharpoonright} \mathfrak{N}^{\perp} = P_{\mathfrak{N}^{\perp}} A {\upharpoonright} \mathfrak{N}^{\perp}.$$

(6) *If A is a closed densely defined symmetric operator, then the deficiency indices of $\mathcal{F}_{z,\mathfrak{N}}(A)$ coincide with the deficiency indices of A.*

Proof. (1) Due to the identity

$$\operatorname{dom} \mathcal{F}_{z,\mathfrak{N}}(A) \cap \operatorname{dom} \mathcal{F}_{z,\mathfrak{N}}(B) = \mathcal{U}_{z,\mathfrak{N}} (\operatorname{dom} A \cap \operatorname{dom} B),$$

we obtain the equivalence

$$\operatorname{dom} \mathcal{F}_{z,\mathfrak{N}}(A) \cap \operatorname{dom} \mathcal{F}_{z,\mathfrak{N}}(B) = \{0\} \quad \Longleftrightarrow \quad \operatorname{dom} A \cap \operatorname{dom} B = \{0\}.$$

(2) – (5) These statements follow from Lemma 3.1, because $\mathcal{F}_{z,\mathfrak{N}}(A)$ is congruent to the operator $A(z,\mathfrak{N})$ given by (3.6), and $A(z,\mathfrak{N})$ is the additive perturbation of A by the bounded selfadjoint operator $(|z|^2/\operatorname{Re} z)P_{\mathfrak{N}}$.

(6) It is well known that the additive perturbation of a symmetric operator by a bounded selfadjoint operator preserves deficiency indices, see, e.g., (Akhiezer & Glazman, 1981). $\square$

Let $\mathfrak{N}$ be a subspace of the Hilbert space $\mathfrak{H}$. For a linear operator A in $\mathfrak{H}$ and $\lambda \in \rho(A)$ we define the transform $\mathcal{T}_{z,\mathfrak{N}}(A,\lambda)$ of A by

$$\mathcal{T}_{z,\mathfrak{N}}(A,\lambda) := P_{\mathfrak{N}^\perp} + i\frac{\operatorname{Re} z}{\operatorname{Im} z} P_{\mathfrak{N}}(A - \zeta_z(\lambda)I)(A-\lambda I)^{-1}, \tag{3.7}$$

where $\zeta_z(\lambda)$ is defined by

$$\zeta_z(\lambda) := \lambda \left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)^2 + \frac{|z|^2}{\operatorname{Re} z}. \tag{3.8}$$

From the definition in (3.7) it is clear that $\mathcal{T}_{z,\mathfrak{N}}(A,\lambda) \in \mathbf{B}(\mathfrak{H})$, since $\lambda \in \rho(A)$. Note that with respect to the orthogonal decomposition $\mathfrak{H} = \mathfrak{N}^\perp \oplus \mathfrak{N}$ one has

$$\mathcal{T}_{z,\mathfrak{N}}(A,\lambda) = \begin{pmatrix} I_{\mathfrak{N}^\perp} & 0 \\ \mathcal{A}_{z,\mathfrak{N}}(A,\lambda) & \mathcal{B}_{z,\mathfrak{N}}(A,\lambda) \end{pmatrix} : \begin{pmatrix} \mathfrak{N}^\perp \\ \mathfrak{N} \end{pmatrix} \to \begin{pmatrix} \mathfrak{N}^\perp \\ \mathfrak{N} \end{pmatrix}, \tag{3.9}$$

where $\mathcal{A}_{z,\mathfrak{N}}(A,\lambda)$ and $\mathcal{B}_{z,\mathfrak{N}}(A,\lambda)$ are defined by

$$\mathcal{A}_{z,\mathfrak{N}}(A,\lambda) := P_{\mathfrak{N}}\mathcal{T}_{z,\mathfrak{N}}(A,\lambda)\restriction \mathfrak{N}^\perp, \qquad \mathcal{B}_{z,\mathfrak{N}}(A,\lambda) := P_{\mathfrak{N}}\mathcal{T}_{z,\mathfrak{N}}(A,\lambda)\restriction \mathfrak{N},$$

so that $\mathcal{A}_{z,\mathfrak{N}}(A,\lambda) \in \mathbf{B}(\mathfrak{N}^\perp,\mathfrak{N})$ and $\mathcal{B}_{z,\mathfrak{N}}(A,\lambda) \in \mathbf{B}(\mathfrak{N})$. In particular, it is useful to observe that the compression $P_{\mathfrak{N}}\mathcal{T}_{z,\mathfrak{N}}(A,\lambda)\restriction \mathfrak{N}$ has the form

$$P_{\mathfrak{N}}\mathcal{T}_{z,\mathfrak{N}}(A,\lambda)\restriction \mathfrak{N} = i\,\frac{\operatorname{Re} z}{\operatorname{Im} z}\left(I + (\lambda - \zeta_z(\lambda))P_{\mathfrak{N}}(A-\lambda)^{-1}\right), \tag{3.10}$$

cf. (3.7). The properties of the transform $\mathcal{T}_{z,\mathfrak{N}}(A,\lambda)$ and its compression $\mathcal{B}_{z,\mathfrak{N}}(A,\lambda)$ to $\mathfrak{N}$ are stated in the following theorem.

Theorem 3.3. *Let A be a linear operator in the Hilbert space $\mathfrak{H}$, let $z \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, and let $\mathcal{F}_{z,\mathfrak{N}}(A)$ be defined as in* (3.5). *Let $\lambda \in \rho(A)$ and let the transformation $\mathcal{T}_{z,\mathfrak{N}}(A,\lambda)$ of A be defined as in* (3.7). *Then the following identity holds*

$$\mathcal{F}_{z,\mathfrak{N}}(A) - \lambda I = \mathcal{T}_{z,\mathfrak{N}}(A,\lambda)(A-\lambda I)U_{z,\mathfrak{N}}^{-1}, \qquad \lambda \in \rho(A). \tag{3.11}$$

Consequently, for $\lambda \in \rho(A)$ the following statements are equivalent:

(i) $\lambda \in \rho(\mathcal{F}_{z,\mathfrak{N}}(A))$;

(ii) *the operator* $\mathcal{T}_{z,\mathfrak{N}}(A,\lambda)$ *belongs to* $\mathbf{GB}(\mathfrak{H})$;

(iii) *the operator* $P_{\mathfrak{N}}\mathcal{T}_{z,\mathfrak{N}}(A,\lambda)\restriction\mathfrak{N}$ *belongs to* $\mathbf{GB}(\mathfrak{N})$.

Moreover, for $\lambda \in \rho(\mathcal{F}_{z,\mathfrak{N}}(A)) \cap \rho(A)$*, one has*

$$(\mathcal{F}_{z,\mathfrak{N}}(A) - \lambda I)^{-1} = \mathcal{U}_{z,\mathfrak{N}}(A-\lambda)^{-1}\mathcal{T}_{z,\mathfrak{N}}(A,\lambda)^{-1}, \tag{3.12}$$

while the compression of $(\mathcal{F}_{z,\mathfrak{N}}(A) - \lambda I)^{-1}$ *to* $\mathfrak{N}$ *is given by*

$$\begin{aligned} &P_{\mathfrak{N}}(\mathcal{F}_{z,\mathfrak{N}}(A) - \lambda I)^{-1}\restriction\mathfrak{N} \\ &\quad= \left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)^2 P_{\mathfrak{N}}(A-\lambda I)^{-1}\left(I_{\mathfrak{N}} + (\lambda - \zeta_z(\lambda))P_{\mathfrak{N}}(A-\lambda I)^{-1}\restriction\mathfrak{N}\right)^{-1}. \end{aligned} \tag{3.13}$$

In particular, if the operator A *is a maximal dissipative, maximal accumulative, or selfadjoint in the Hilbert space* $\mathfrak{H}$*, then* (3.12) *and* (3.13) *hold for each proper subspace* $\mathfrak{N}$*, for each* $z \in \mathbb{C}\setminus(\mathbb{R}\cup i\mathbb{R})$*, and for each* λ *in* $\mathbb{C}_-$*,* $\mathbb{C}_+$*, or* $\mathbb{C}_-\cup\mathbb{C}_+$*, respectively.*

Proof. Let $\widehat{A} := \mathcal{F}_{z,\mathfrak{N}}(A)$. It follows from $\operatorname{dom}\widehat{A} = \mathcal{U}_{z,\mathfrak{N}}\operatorname{dom} A$ that any $\widehat{f} \in \operatorname{dom}\widehat{A}$ is of the form $\widehat{f} = U_{z,\mathfrak{N}}f_A$ with a unique $f_A \in \operatorname{dom} A$ and conversely. From (3.5) one therefore sees that for all $\widehat{f} \in \operatorname{dom}\widehat{A}$

$$\begin{aligned} (\widehat{A} - \lambda I)\widehat{f} &= \left(A + \frac{iz}{\operatorname{Im} z}P_{\mathfrak{N}}(A - \bar{z}I)\right)f_A - \lambda\left(I - \frac{\bar{z}}{\operatorname{Re} z}P_{\mathfrak{N}}\right)f_A \\ &= (A-\lambda)f_A + \frac{iz}{\operatorname{Im} z}P_{\mathfrak{N}}(A-\bar{z}I)f_A + \lambda\frac{\bar{z}}{\operatorname{Re} z}P_{\mathfrak{N}}f_A \\ &= (A-\lambda I)f_A + \frac{iz}{\operatorname{Im} z}P_{\mathfrak{N}}(A-\lambda I)f_A + \left(\frac{iz}{\operatorname{Im} z}(\lambda-\bar{z}) + \lambda\frac{\bar{z}}{\operatorname{Re} z}\right)P_{\mathfrak{N}}f_A \\ &= \left(I + \frac{iz}{\operatorname{Im} z}P_{\mathfrak{N}} + \left(\frac{iz}{\operatorname{Im} z}(\lambda-\bar{z}) + \lambda\frac{\bar{z}}{\operatorname{Re} z}\right)P_{\mathfrak{N}}(A-\lambda I)^{-1}\right)(A-\lambda I)f_A. \end{aligned}$$

By writing $I = P_{\mathfrak{N}^\perp} + P_{\mathfrak{N}}$, we see that the first factor in the right-hand side of the last term is given by

$$\begin{aligned} P_{\mathfrak{N}^\perp} &+ \left(I + \frac{iz}{\operatorname{Im} z}\right)P_{\mathfrak{N}} + i\,\frac{\lambda\operatorname{Re} z^2 - |z|^2\operatorname{Re} z}{\operatorname{Re} z\operatorname{Im} z}P_{\mathfrak{N}}(A-\lambda)^{-1} \\ &= P_{\mathfrak{N}^\perp} + i\,\frac{\operatorname{Re} z}{\operatorname{Im} z}P_{\mathfrak{N}}\left[A - \lambda + \frac{\lambda\operatorname{Re} z^2 - |z|^2\operatorname{Re} z}{(\operatorname{Re} z)^2}\right](A-\lambda)^{-1} \\ &= P_{\mathfrak{N}^\perp} + i\,\frac{\operatorname{Re} z}{\operatorname{Im} z}P_{\mathfrak{N}}(A - \zeta_z(\lambda))(A-\lambda)^{-1} = \mathcal{T}_{z,\mathfrak{N}}(A,\lambda), \end{aligned}$$

where the following identities were used

$$I + \frac{iz}{\operatorname{Im} z} = i\,\frac{\operatorname{Re} z}{\operatorname{Im} z} \quad\text{and}\quad \zeta_z(\lambda) = \lambda - \frac{\lambda\operatorname{Re} z^2 - |z|^2\operatorname{Re} z}{(\operatorname{Re} z)^2}.$$

Therefore, (3.11) has been shown.

(i) $\Leftrightarrow$ (ii) This equivalence follows from (3.11).

(ii) $\Leftrightarrow$ (iii) This equivalence follows from (3.9) and (3.11).

The resolvent formula (3.12) follows from (3.11). In order to see (3.13), first observe from (3.1) and (3.10) that

$$P_{\mathfrak{N}}\mathcal{U}_{z,\mathfrak{N}} = i\,\frac{\operatorname{Im} z}{\operatorname{Re} z}\,P_{\mathfrak{N}} \quad \text{and} \quad \mathcal{B}_{z,\mathfrak{N}}^{-1}(\lambda) = \frac{1}{i}\,\frac{\operatorname{Im} z}{\operatorname{Re} z}\left(I + (\lambda - \zeta_z(\lambda))P_{\mathfrak{N}}(A-\lambda)^{-1}\right)^{-1}.$$

Therefore, it is seen as a consequence of (3.9) and (3.12) that

$$\begin{aligned} P_{\mathfrak{N}}\left(\mathcal{F}_{z,\mathfrak{N}}(A) - \lambda I\right)^{-1}\restriction \mathfrak{N} &= P_{\mathfrak{N}}\mathcal{U}_{z,\mathfrak{N}}(A-\lambda)^{-1}\mathcal{T}_{z,\mathfrak{N}}^{-1}\restriction \mathfrak{N} = P_{\mathfrak{N}}\mathcal{U}_{z,\mathfrak{N}}(A-\lambda)^{-1}\mathcal{B}_{z,\mathfrak{N}}^{-1}(\lambda) \\ &= \left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)^2 P_{\mathfrak{N}}(A-\lambda)^{-1}\left(I + (\lambda - \zeta_z(\lambda))P_{\mathfrak{N}}(A-\lambda)^{-1}\right)^{-1}, \end{aligned}$$

which gives (3.13).

Next let A be a maximal dissipative operator. Then by Proposition 3.2 the operator $\widehat{A} := \mathcal{F}_{z,\mathfrak{N}}(A)$ is maximal dissipative too. Therefore the open lower half-plane $\mathbb{C}_-$ belongs to the resolvent set of A and $\widehat{A}$. As has been proven above, the operators $\mathcal{T}_{z,\mathfrak{N}}(A,\lambda)$ and $\mathcal{B}_{z,\mathfrak{N}}(A,\lambda)$ belong to $\mathbf{GB}(\mathfrak{H})$ and $\mathbf{GB}(\mathfrak{N})$, respectively for all $\lambda \in \mathbb{C}_-$. Hence the identities (3.12) and (3.13) are valid for all $\lambda \in \mathbb{C}_-$. The proofs of the statements for a maximal accumulative or selfadjoint operator A can be established in a similar way. □

Corollary 3.4. *Let A be a selfadjoint operator in the Hilbert space $\mathfrak{H}$ and let $\mathfrak{N}$ be a subspace of $\mathfrak{H}$. Then A is $\mathfrak{N}$-minimal if and only if $\mathcal{F}_{z,\mathfrak{N}}(A)$ is $\mathfrak{N}$-minimal.*

Proof. It follows from (3.9) and (3.11), and from the invertibility of $\mathcal{B}_{z,\mathfrak{N}}(A,\lambda)$ in $\mathfrak{H}$, that

$$(\mathcal{F}_{z,\mathfrak{N}}(A) - \lambda I)^{-1} = \mathcal{U}_{z,\mathfrak{N}}\,(A-\lambda)^{-1}\begin{pmatrix} I_{\mathfrak{N}^\perp} & 0 \\ -\mathcal{B}_{z,\mathfrak{N}}(A,\lambda)^{-1}\mathcal{A}_{z,\mathfrak{N}}(A,\lambda) & \mathcal{B}_{z,\mathfrak{N}}(A,\lambda)^{-1}\end{pmatrix}.$$

Thanks to the invertibility of $\mathcal{U}_{z,\mathfrak{N}}$ in $\mathfrak{H}$, the statement is clear from the above identity. □

Lemma 3.5. *Let A be an operator in the Hilbert space $\mathfrak{H}$ and let $z_1, z_2 \in \mathbb{C}\setminus(\mathbb{R}\cup i\mathbb{R})$. Then*

$$\begin{aligned} \mathcal{F}_{z_2,\mathfrak{N}}(\mathcal{F}_{z_1,\mathfrak{N}}(A)) &= \left(P_{\mathfrak{N}^\perp} - \frac{\operatorname{Re} z_1 \operatorname{Re} z_2}{\operatorname{Im} z_1 \operatorname{Im} z_2}P_{\mathfrak{N}}\right) \\ &\quad\times\left(A - \left(\frac{|z_1|^2}{\operatorname{Re} z_1} + \frac{|z_2|^2}{\operatorname{Re} z_2}\left(\frac{\operatorname{Im} z_1}{\operatorname{Re} z_1}\right)^2\right)P_{\mathfrak{N}}\right)\left(P_{\mathfrak{N}^\perp} - \frac{\operatorname{Re} z_1 \operatorname{Re} z_2}{\operatorname{Im} z_1 \operatorname{Im} z_2}P_{\mathfrak{N}}\right). \end{aligned}$$

Thus, the operator $\mathcal{F}_{z_2,\mathfrak{N}}(\mathcal{F}_{z_1,\mathfrak{N}}(A))$ is congruent to the operator

$$A - \left(\frac{|z_1|^2}{\operatorname{Re} z_1} + \frac{|z_2|^2}{\operatorname{Re} z_2}\left(\frac{\operatorname{Im} z_1}{\operatorname{Re} z_1}\right)^2\right)P_{\mathfrak{N}}.$$

Proof. Let A be a linear operator, then it follows from (3.6) that

$$\begin{aligned}
\mathcal{F}_{z_2,\mathfrak{N}}(\mathcal{F}_{z_1,\mathfrak{N}}(A)) = \mathcal{F}_{z_2,\mathfrak{N}}(\widehat{A}_1) &= \mathcal{U}^{*-1}_{z_2,\mathfrak{N}}\left(\widehat{A}_1 - \frac{|z_2|^2}{\operatorname{Re} z_2}P_{\mathfrak{N}}\right)\mathcal{U}^{-1}_{z_2,\mathfrak{N}} \\
&= \mathcal{U}^{*-1}_{z_2,\mathfrak{N}}\left(\mathcal{U}^{*-1}_{z_1,\mathfrak{N}}\left(A - \frac{|z_1|^2}{\operatorname{Re} z_1}P_{\mathfrak{N}}\right)\mathcal{U}^{-1}_{z_1,\mathfrak{N}} - \frac{|z_2|^2}{\operatorname{Re} z_2}P_{\mathfrak{N}}\right)\mathcal{U}^{-1}_{z_2,\mathfrak{N}} \\
&= \mathcal{U}^{*-1}_{z_2,\mathfrak{N}}\mathcal{U}^{*-1}_{z_1,\mathfrak{N}}\left(A - \frac{|z_1|^2}{\operatorname{Re} z_1}P_{\mathfrak{N}} - \frac{|z_2|^2}{\operatorname{Re} z_2}\mathcal{U}^{*}_{z_1,\mathfrak{N}}P_{\mathfrak{N}}\mathcal{U}_{z_1,\mathfrak{N}}\right)\mathcal{U}^{-1}_{z_1,\mathfrak{N}}\mathcal{U}^{-1}_{z_2,\mathfrak{N}},
\end{aligned}$$

which, thanks to (3.2) and (3.3), gives the required result. □

One can easily verify the identity

$$\mathcal{F}_{\mu(z),\mathfrak{N}} \circ \mathcal{F}_{z,\mathfrak{N}} = \mathcal{F}_{z,\mathfrak{N}} \circ \mathcal{F}_{\mu(z),\mathfrak{N}} = \mathrm{id},$$

where id is the identity transformation on the set of all linear operators in $\mathfrak{H}$ and the function $\mu(z)$ is defined as

$$\mu(z) := iz\frac{\operatorname{Re} z}{\operatorname{Im} z} = -\operatorname{Re} z + i\frac{(\operatorname{Re} z)^2}{\operatorname{Im} z} = z + i\frac{z^2}{\operatorname{Im} z}, \qquad z \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}).$$

Remark 3.6. Let S be a closed densely defined symmetric operator in the Hilbert space $\mathfrak{H}$, let $z \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, and let $\mathfrak{N}_z = \ker(S^* - zI) \neq \{0\}$ be the deficiency subspace of S corresponding to z. Define the associated operators $U_S(z)$ by

$$U_S(z) := \mathcal{U}_{z,\mathfrak{N}_z} = P_{\mathfrak{N}_z^{\perp}} + i\frac{\operatorname{Im} z}{\operatorname{Re} z}P_{\mathfrak{N}_z} = I - \frac{\bar{z}}{\operatorname{Re} z}P_{\mathfrak{N}_z}.$$

Then the symmetric operator

$$S(z) = \mathcal{U}_S(z)^{*-1}\left(S - \frac{|z|^2}{\operatorname{Re} z}P_{\mathfrak{N}_z}\right)\mathcal{U}_S(z)^{-1}$$

has been studied in Arlinskiĭ (2021) and it was established that $S(z)$ preserves various properties of S. When the deficiency indices of S are equal, then a bijection of the set of all selfadjoint extensions of S onto the set of all selfadjoint extensions of $S(z)$ was established.

4 Proof of Theorem 1.3

This section provides a proof of Theorem 1.3. It is based on the general constructions in Section 3, which are applied under the assumption that the underlying operator is selfadjoint.

(1) Let $M \in \mathcal{N}(\mathfrak{N})$ be arbitrary. Then by Theorem 1.2 there exists a selfadjoint operator A in the Hilbert space $\mathfrak{H}$, containing $\mathfrak{N}$ as a subspace, realizing M as follows

$$M(\lambda) = P_{\mathfrak{N}}(A - \lambda I)^{-1}\upharpoonright\mathfrak{N}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.1}$$

Let the transformation $\omega(\lambda) = a\lambda + b \in \mathfrak{G}$, where $a \in \mathbb{R}_+$ and $b \in \mathbb{R} \setminus \{0\}$, be arbitrary, cf. (1.4). Then define $z_\omega \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ as

$$z_\omega := \frac{b}{1+a} + i\frac{\sqrt{a}b}{1+a} \tag{4.2}$$

so that $\zeta_{z_\omega}(\lambda) = \omega(\lambda)$, see (3.8). Note that, conversely, a and b in (4.2) can be expressed in terms of z_ω as

$$a = \left(\frac{\operatorname{Im} z_\omega}{\operatorname{Re} z_\omega}\right)^2 \quad \text{and} \quad b = \frac{|z_\omega|^2}{\operatorname{Re} z_\omega}. \tag{4.3}$$

For the selfadjoint operator A in (4.1) let $\widehat{A} := \mathcal{F}_{z_\omega,\mathfrak{N}}(A)$, where z_ω is given by (4.2) and $\mathcal{F}_{z,\mathfrak{N}}$ is the transformation defined in (3.5). Then $\widehat{A}$ is selfadjoint by Corollary 3.2, and Theorem 3.3 implies that for $\lambda \in \mathbb{C}\backslash\mathbb{R}$

$$\begin{aligned} P_\mathfrak{N}(\widehat{A} - \lambda I)^{-1}{\restriction}\,\mathfrak{N} &= \left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)^2 P_\mathfrak{N}(A - \lambda I)^{-1}\left(I_\mathfrak{N} + (\lambda - \zeta_{z_\omega}(\lambda))P_\mathfrak{N}(A - \lambda I)^{-1}{\restriction}\,\mathfrak{N}\right)^{-1} \\ &= a_\omega M(\lambda)\left(I + (\lambda - \omega(\lambda))M(\lambda)\right)^{-1} \\ &= \mathcal{G}_\omega(M)(\lambda), \end{aligned} \tag{4.4}$$

see (4.1), (4.3), and (1.7). Now the representation (4.4) for $\mathcal{G}_\omega(M)$ shows that it belongs to the class $\mathcal{N}(\mathfrak{N})$, since it is the compression of the resolvent of the selfadjoint operator $\widehat{A}$. If the representing operator A is additionally assumed to be bounded, then Corollary 3.2 implies that also $\widehat{A}$ is bounded and, hence, $\mathcal{G}_\omega(M) \in \mathbf{N}^0_\mathfrak{N}$.

(2) Let $\omega_k(\lambda) = a_k\lambda + b_k \in \mathfrak{G}$ be arbitrary, for $k = 1, 2$. Then one observes

$$\omega_1 \circ \omega_2(\lambda) = \omega_1(\lambda) - a_{\omega_1}(\lambda - \omega_2(\lambda)), \tag{4.5}$$

cf. (1.5). For $M \in \mathcal{N}(\mathfrak{N})$ we have by (1) that $M_{\omega_1} := \mathcal{G}_{\omega_1}(M) \in \mathcal{N}(\mathfrak{N})$. Therefore, observe that

$$\begin{aligned} \mathcal{G}_{\omega_2}(\mathcal{G}_{\omega_1}(M))(\lambda) &= \mathcal{G}_{\omega_2}(M_{\omega_1})(\lambda) = a_{\omega_2}M_{\omega_1}(\lambda)\left(I + (\lambda - \omega_2(\lambda))M_{\omega_1}(\lambda)\right)^{-1} \\ &= a_{\omega_1}a_{\omega_2}M(\lambda)\left(I + (\lambda - \omega_1(\lambda))M(\lambda)\right)^{-1} \\ &\quad \times \left[I + (\lambda - \omega_2(\lambda))a_{\omega_1}M(\lambda)\left(I + (\lambda - \omega_1(\lambda))M(\lambda)\right)^{-1}\right]^{-1} \\ &= a_{\omega_1}a_{\omega_2}M(\lambda)\left(I + (\lambda - \omega_1(\lambda))M(\lambda)\right)^{-1}\left(I + (\lambda - \omega_1(\lambda))M(\lambda)\right) \\ &\quad \times \left[(I + (\lambda - \omega_1(\lambda))M(\lambda)) + a_{\omega_1}(\lambda - \omega_2(\lambda))M(\lambda)\right]^{-1} \\ &= a_{\omega_1}a_{\omega_2}M(\lambda)\big[I + \big(\lambda - \omega_1(\lambda) + a_{\omega_1}(\lambda - \omega_2(\lambda)\big)M(\lambda)\big]^{-1} \\ &= a_{\omega_1\circ\omega_2}M(\lambda)\big(I + \big(\lambda - \omega_1 \circ \omega_2(\lambda)\big)M(\lambda)\big)^{-1} = \mathcal{G}_{\omega_1\circ\omega_2}(M)(\lambda), \end{aligned}$$

where the penultimate identity follows thanks to (4.5). This shows that the first identity in Theorem 1.3 (2) holds. That identity implies the second identity in view of (1.5) and (1.6).

Let $\widehat{M} \in \mathcal{N}(\mathfrak{N})$ be arbitrary. Then by the above composition result $M := \mathcal{G}_{\omega^{[-1]}}(\widehat{M}) \in \mathcal{N}(\mathfrak{N})$ and $\mathcal{G}_\omega(M) = \widehat{M}$, showing that $\mathcal{G}$ is surjective. Likewise, if $M_1, M_2 \in \mathcal{N}(\mathfrak{N})$ satisfy the equality $\mathcal{G}_\omega(M_1) = \mathcal{G}_\omega(M_2)$, then composing the preceding equality with $\mathcal{G}_{\omega^{[-1]}}$ yields that $M_1 = M_2$. Thus $\mathcal{G}_\omega$ is bijective on the set $\mathcal{N}(\mathfrak{N})$. The bijectivity of $\mathcal{G}_\omega$ restricted to the set $\mathbf{N}^0_\mathfrak{N}$ can be established in exactly the same manner.

(3) Let A be a $\mathfrak{N}$-minimal realization of $M \in \mathcal{N}(\mathfrak{N})$ and let $\omega(\lambda) = a\lambda + b \in \mathfrak{G}$ be arbitrary. Moreover, define z_w as in (4.2) and let $\widehat{A} := \mathcal{F}_{z_\omega,\mathfrak{N}}(A)$. Then the identity (4.4) holds. Lemma 3.1 now yields that the operator $\widehat{A} = \mathcal{F}_{z_\omega,\mathfrak{N}}(A)$ is congruent to the operator

$$A(z,\mathfrak{N}) = A - \frac{|z|^2}{\operatorname{Re} z} P_{\mathfrak{N}} = A - bP_{\mathfrak{N}},$$

see also (4.2) and (4.3), via

$$\mathcal{U}_{z,\mathfrak{N}}^{-1} = P_{\mathfrak{N}^\perp} - i\frac{\operatorname{Re} z}{\operatorname{Im} z} P_{\mathfrak{N}} = P_{\mathfrak{N}^\perp} - i\frac{1}{\sqrt{a}} P_{\mathfrak{N}}.$$

Moreover, the operator A is $\mathfrak{N}$-minimal if and only if the operator $\widehat{A} := \mathcal{F}_{z_\omega,\mathfrak{N}}(A)$ is $\mathfrak{N}$-minimal, see Corollary 3.4. Finally, recall that by Theorem 1.2 (2) any $\mathfrak{N}$-minimal realization of $\mathcal{G}_\omega(M)$ is unitary equivalent to $\widehat{A} = \mathcal{F}_{z_\omega,\mathfrak{N}}(A)$ and, hence, is congruent to $A - bP_{\mathfrak{N}}$. This establishes the first part of this assertion.

Next assume that $\omega(\lambda) = \lambda + b$. If z_ω is such that $\zeta_{z_\omega}(\lambda) = \omega(\lambda)$, then equation (4.3) implies that $(\operatorname{Re} z)^2 = (\operatorname{Im} z)^2$. Therefore (3.3) yields that the operator $U_{z,\mathfrak{N}}^{*-1}$ is equal to $P_{\mathfrak{N}^\perp} + iP_{\mathfrak{N}}$ and, hence, is unitary. Consequently, Lemma 3.1 shows that $\widehat{A} := \mathcal{F}_{z_\omega,\mathfrak{N}}(A)$ is unitary equivalent to $A - bP_{\mathfrak{N}}$. This establishes the second part of the assertion by Theorem 1.2 (2).

Remark 4.1. The composition formula $\mathcal{G}_{\omega_2} \circ \mathcal{G}_{\omega_1} = \mathcal{G}_{\omega_1 \circ \omega_2}$ in Theorem 1.3 has a counterpart for the operator representations. With the transformations

$$\omega_1(\lambda) = a_1\lambda + b_1 \quad \text{and} \quad \omega_2(\lambda) = a_2\lambda + b_2,$$

define the corresponding parameters

$$z_1 = \frac{b_1}{1+a_1} + i\frac{\sqrt{a_1}b_1}{1+a_1} \quad \text{and} \quad z_2 = \frac{b_2}{1+a_2} + i\frac{\sqrt{a_2}b_2}{1+a_2},$$

cf. (4.2). Then the composition of $\mathcal{F}_{z_2,\mathfrak{N}}(\mathcal{F}_{z_1,\mathfrak{N}}(A))$ in Lemma 3.5 is given in terms of ω_1 and ω_2 by

$$\left(P_{\mathfrak{N}^\perp} - \frac{1}{\sqrt{a_1a_2}} P_{\mathfrak{N}}\right)\left(A - (a_1b_2 + b_1)P_{\mathfrak{N}}\right)\left(P_{\mathfrak{N}^\perp} - \frac{1}{\sqrt{a_1a_2}} P_{\mathfrak{N}}\right).$$

Note that $a_1b_2 + b_1$ is the constant term of the composition $\omega_1 \circ \omega_2$, see (1.5).

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A CLASS OF SINGULAR PERTURBATIONS OF THE DIRAC OPERATOR: BOUNDARY TRIPLETS AND WEYL FUNCTIONS

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Dedicated to our friend and colleague Seppo Hassi on the occasion of his 60th birthday!

1 Introduction

Singular perturbations of self-adjoint operators play an important role in the description of idealized quantum systems, where a localized short-range potential is often replaced by a more singular model potential. More precisely, assume that A_0 is a self-adjoint differential operator in an L^2-Hilbert space which is viewed as the Hamiltonian of an unperturbed quantum system and suppose that V is some potential such that the formal sum $A_V = A_0 + V$ describes the quantum system under investigation. Standard operator theory techniques ensure that for potentials V belonging to certain function spaces the perturbed operator A_V is again self-adjoint; we refer the reader to the monographs of Reed & Simon (1972; 1975; 1979; 1978) or Kato (1995). However, a detailed spectral analysis of A_V is typically very difficult, and for this reason the potential V is often replaced by an idealized perturbation term of δ-type, which is then regarded as an approximation of the real model, see (Behrndt et al., 2017; Exner, 2008). On the one hand, this procedure may simplify the spectral analysis considerably, see (Albeverio et al., 2005; Behrndt, Langer & Lotoreichik, 2013; Brasche et al., 1994; Holzmann & Unger, 2020), but, on the other hand, it may lead to new technical difficulties in the mathematically rigorous definition of the Hamiltonian itself.

In the case that A_0 is the Laplacian in an L^2-space and the δ-potential is supported on hypersurfaces in $\mathbb{R}^d$ (e.g., curves in $\mathbb{R}^2$, or surfaces in $\mathbb{R}^3$) the standard quadratic form approach is useful. Roughly speaking, the perturbed operator $A_\tau = A_0 + \tau\delta_\Sigma$ is in this situation viewed as the self-adjoint operator corresponding to the form

$$\mathfrak{a}[f, g] = (\nabla f, \nabla g)_{L^2} + \int_\Sigma \tau\, f|_\Sigma\, \overline{g|_\Sigma}\; dx, \tag{1.1}$$

where $(\nabla f, \nabla g)_{L^2}$ is the quadratic form defined on the Sobolev space H^1 associated with the Laplacian, and the singular perturbation is encoded in the additive form perturbation with Σ denoting the support of the δ-distribution, τ is some real (position dependent) coefficient, and $f|_\Sigma$ and $g|_\Sigma$ denote the traces of the Sobolev space functions f and g, respectively, defined in an appropriate way. Of course, one has to impose certain assumptions on the support Σ of the δ-potential and the coefficient τ to ensure that $\mathfrak{a}$ in (1.1) is a densely defined closed semibounded form (which then gives rise to a self-adjoint operator A_τ); we refer to (Brasche et al., 1994; Exner, 2008; Exner & Kovarik, 2015; Herczyński, 1989; Stollmann & Voigt, 1996) for a detailed treatment and further references. A different approach to the operator A_τ is via extension theory techniques in general, and boundary triplet methods in particular, see the recent monograph (Behrndt, Hassi & de Snoo, 2020) and (Derkach, Hassi & Malamud, 2020; Derkach et al., 2000; 2006; 2009; 2012; Derkach, Hassi & de Snoo, 2001; 2003) by Seppo Hassi and his coauthors for an extensive treatment of boundary triplets and further developments. For the case of point interactions it is well known what type of transmission or jump conditions the functions in the domain of A_τ satisfy; cf. (Albeverio et al., 2005) for a comprehensive treatment of point interactions. In the case that the δ-distribution is supported on a hypersurface we refer to (Behrndt, Langer & Lotoreichik, 2013), where quasi boundary triplets were used for the first time to define A_τ as a self-adjoint restriction of a Laplacian that is decoupled along the support Σ. As in the case of point interactions, also in the multi-dimensional setting one ends up with transmission and jump conditions for the functions in the domain of A_τ along the support Σ of the δ-distribution, see also (Behrndt et al., 2020; 2018; Mantile, Posilicano & Sini, 2016). In conclusion, for the case that A_0 is the Laplacian (or some more general semibounded Schrödinger operator) nowadays one may efficiently apply form techniques or boundary triplet methods to define and study the perturbed operator A_τ; depending on the particular problem under consideration one method may prove more useful than the other.

Now assume that the unperturbed operator A_0 is the Dirac operator instead of the Laplacian or the Schrödinger operator. While the Dirac operator describes a similar physical system as the Laplace operator including relativistic effects (see Section 3 for more details), the mathematical situation is entirely different: The free Dirac operator A_0 is not semibounded from below and, hence, standard quadratic form methods are not applicable. Therefore, it is most natural to try to apply boundary triplet techniques, since these methods do not require any type of semiboundedness of the operators under consideration. In fact, Dirac operators with singular interactions supported on points and spheres were already treated with direct methods in (Albeverio et al., 2005; Dittrich, Exner & Šeba, 1989; Gesztesy & Šeba, 1987), but for more general supports of the singular potential only recently a series of papers was published (Arrizabalaga, Mas & Vega, 2014; 2015; 2016), which in turn led to our publications (Behrndt et al., 2018; Behrndt & Holzmann, 2020; Behrndt, Holzmann & Mas, 2020; Behrndt et al., 2020) employing the quasi boundary triplet technique. We also emphasize the recent papers (Behrndt et al., 2019; 2020; Holzmann, Ourmières-Bonafos & Pankrashkin, 2018; Mas & Pizzichillo, 2018; Ourmières-Bonafos & Vega, 2018; Pankrashkin & Richard, 2014) where closely related techniques were used to study Dirac operators with δ-shell interactions.

The main objective of this note is to provide boundary triplets for Dirac operators with Lorentz scalar interactions supported on a point in the one-dimensional case, and supported on curves and surfaces in the two- and three-dimensional situation. This operator is formally given by

$$A_\tau = A_0 + \tau\alpha_0\delta_\Sigma,$$

where α_0 is a Dirac matrix defined in Section 3, and $\tau\alpha_0\delta_\Sigma$ describes the Lorentz scalar δ-shell interaction supported on Σ. The one-dimensional setting with a single point interaction is particularly easy to treat and we discuss in Section 4 a possible choice of an ordinary boundary triplet, which was also used in Pankrashkin & Richard (2014). We compute the corresponding γ-field and Weyl function, and give an expression for the resolvent of the singularly perturbed one-dimensional Dirac operator. In the multi-dimensional setting one observes typical analytic difficulties with trace maps and integration by parts formulas on maximal operator domains, similar to the case of the Laplacian or more general elliptic operators; cf. (Behrndt & Langer, 2007; 2012). It is convenient to extend the notion of ordinary boundary triplet in such a way that these analytic difficulties can be circumvented. As in the case of symmetric second order elliptic operators, the concepts of quasi boundary triplets and generalized boundary triplets are useful and fit in this setting very well. In the present manuscript we allow some flexibility in the domain of the boundary maps and obtain a family of quasi boundary triplets that reduce to a generalized boundary triplet in the limit case, where the parameter describing regularity of the operator domain is minimal; cf. Theorem 5.3. As in the one-dimensional situation, we provide the corresponding γ-fields and Weyl functions, we discuss the self-adjointness of the operator A_τ, and list some of its spectral properties. An interesting issue in the multi-dimensional setting is the regularity of the support Σ of the Lorentz scalar δ-perturbation: From C^2-curves and hypersurfaces treated earlier in Arrizabalaga, Mas & Vega (2014; 2015); Behrndt et al. (2018; 2019); Behrndt & Holzmann (2020); Ourmières-Bonafos & Vega (2018) and piecewise C^2-curves studied in Pizzichillo & Van Den Bosch (2019), we make a substantial step towards more rough supports and discuss in Theorem 5.4 the case that Σ is the boundary of a bounded Lipschitz domain.

The paper is organized as follows. In Section 2 we briefly recall some basic definitions and abstract facts about ordinary, generalized, and quasi boundary triplets. Section 3 is devoted to regular Dirac operators: We collect some required notations, state the well-known properties of the unperturbed Dirac operator A_0, and shortly describe the physical interpretation of the objects of interest. In Section 4 we study the one-dimensional case, provide an ordinary boundary triplet suitable to treat singular perturbations of the free Dirac operator in $\mathbb{R}$, and investigate Dirac operators with Lorentz scalar δ-point interactions. Finally, Section 5 is devoted to the multi-dimensional case. We construct a family of quasi boundary triplets that are suitable to prove the self-adjointness of Dirac operators with Lorentz scalar δ-shell interactions supported on arbitrary closed compact Lipschitz smooth hypersurfaces in $\mathbb{R}^2$ and $\mathbb{R}^3$.

2 Ordinary, generalized, and quasi boundary triplets

In this section we briefly recall basic definitions of ordinary and generalized boundary triplets, quasi boundary triplets, and some related techniques in extension and spectral theory of symmetric and self-adjoint operators in Hilbert spaces. The concepts will be presented such that they can be applied directly to Dirac operators with singular interactions in the next sections. We refer the reader to (Behrndt, Hassi & de Snoo, 2020; Behrndt & Langer, 2007; 2012; Brüning, Geyler & Pankrashkin, 2008; Derkach & Malamud, 1991; 1995; Gorbachuk & Gorbachuk, 1991) for more details on boundary triplet techniques. Throughout this section $\mathcal{H}$ denotes a complex Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{H}}$ and S is a densely defined closed symmetric operator with adjoint S^*.

Definition 2.1. Let T be a linear operator in $\mathcal{H}$ such that $\overline{T} = S^*$. A triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ consisting of a Hilbert space $\mathcal{G}$ and linear mappings $\Gamma_0, \Gamma_1 : \mathrm{dom}\ T \to \mathcal{G}$ is called a *quasi boundary triplet* for S^* if it has the following properties:

(i) For all $f, g \in \mathrm{dom}\ T$ the abstract Green's identity

$$(Tf, g)_{\mathcal{H}} - (f, Tg)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}}$$

is true.

(ii) The range of $\Gamma = (\Gamma_0, \Gamma_1)^\top$ is dense in $\mathcal{G} \times \mathcal{G}$.

(iii) The restriction $A_0 := T \upharpoonright \ker \Gamma_0$ is a self-adjoint operator in $\mathcal{H}$.

If (i) and (iii) hold, and the mapping $\Gamma_0 : \mathrm{dom}\ T \to \mathcal{G}$ is surjective, then $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called a *generalized boundary triplet*; if (i) and (iii) hold, and the mapping

$$\Gamma = (\Gamma_0, \Gamma_1)^\top : \mathrm{dom}\ T \to \mathcal{G} \times \mathcal{G}$$

is surjective, then $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called an *ordinary boundary triplet*.

Note that the above (non-standard) definition of generalized and ordinary boundary triplets is equivalent to the usual one given in, e.g., (Behrndt, Hassi & de Snoo, 2020; Brüning, Geyler & Pankrashkin, 2008; Derkach & Malamud, 1991; 1995; Gorbachuk & Gorbachuk, 1991), see (Behrndt & Langer, 2007: Corollary 3.2 & Corollary 3.7). In particular, if $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triplet, then $T = S^*$. Note that a quasi boundary triplet, generalized boundary triplet, or ordinary boundary triplet for S^* exists if and only if the defect numbers $\dim \ker(S^* \pm i)$ coincide, i.e., if and only if S admits self-adjoint extensions in $\mathcal{H}$. Moreover, the operator T in Definition 2.1 is in general not unique.

Next, we recall the definition of the γ-field and the Weyl function associated with the quasi boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$. These mappings will allow us to describe spectral properties of self-adjoint extensions of S. With $A_0 = T \upharpoonright \ker \Gamma_0$ the direct sum decomposition

$$\operatorname{dom} T = \operatorname{dom} A_0 \dotplus \ker(T - \lambda) = \ker \Gamma_0 \dotplus \ker(T - \lambda), \qquad \lambda \in \rho(A_0), \tag{2.1}$$

holds. The definition of the γ-field and Weyl function for quasi boundary triplets is in accordance with the definition for ordinary and generalized boundary triplets in Derkach & Malamud (1991; 1995).

Definition 2.2. Assume that T is a linear operator in $\mathcal{H}$ satisfying $\overline{T} = S^*$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triplet for S^*. Then the corresponding γ-field γ and Weyl function M are defined by

$$\rho(A_0) \ni \lambda \mapsto \gamma(\lambda) := \big(\Gamma_0 \upharpoonright \ker(T - \lambda)\big)^{-1}$$

and

$$\rho(A_0) \ni \lambda \mapsto M(\lambda) := \Gamma_1 \big(\Gamma_0 \upharpoonright \ker(T - \lambda)\big)^{-1},$$

respectively.

From (2.1) we see that the γ-field is well defined and that $\operatorname{ran} \gamma(\lambda) = \ker(T - \lambda)$ holds for all $\lambda \in \rho(A_0)$. Moreover, $\operatorname{dom} \gamma(\lambda) = \operatorname{ran} \Gamma_0$ is dense in $\mathcal{G}$ by Definition 2.1. With the help of the abstract Green's identity in Definition 2.1 (i) one verifies that

$$\gamma(\lambda)^* = \Gamma_1 (A_0 - \overline{\lambda})^{-1}, \qquad \lambda \in \rho(A_0). \tag{2.2}$$

Thus $\gamma(\lambda)^*$ is a bounded and everywhere defined operator from $\mathcal{H}$ to $\mathcal{G}$. Therefore, $\gamma(\lambda)$ is a, in general not everywhere defined, bounded operator; cf. (Behrndt & Langer, 2007: Proposition 2.6) or (Behrndt & Langer, 2012: Proposition 6.13). If $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a generalized or ordinary boundary triplet, then $\gamma(\lambda)$ is automatically bounded and everywhere defined.

Next, we state some useful properties of the Weyl function M corresponding to the quasi boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$; see, e.g., (Behrndt & Langer, 2007: Proposition 2.6) for proofs of these statements. For any $\lambda \in \rho(A_0)$ the operator $M(\lambda)$ is densely defined in $\mathcal{G}$ with $\operatorname{dom} M(\lambda) = \operatorname{ran} \Gamma_0$ and $\operatorname{ran} M(\lambda) \subset \operatorname{ran} \Gamma_1$. Moreover, for all $\lambda, \mu \in \rho(A_0)$ and $\varphi \in \operatorname{ran} \Gamma_0$ one has

$$M(\lambda)\varphi - M(\mu)^*\varphi = (\lambda - \overline{\mu})\gamma(\mu)^*\gamma(\lambda)\varphi. \tag{2.3}$$

Therefore, we see that $M(\lambda) \subset M(\overline{\lambda})^*$ for any $\lambda \in \rho(A_0)$ and hence $M(\lambda)$ is a closable, but, in general, unbounded linear operator in $\mathcal{G}$. If $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a generalized or ordinary boundary triplet, then $M(\lambda)$ is bounded and everywhere defined.

In the main part of this paper we are going to use ordinary boundary triplets, generalized boundary triplets, quasi boundary triplets, and their Weyl functions to define and study self-adjoint extensions of the underlying symmetry S. Let again T be a linear operator in $\mathcal{H}$ such that $\overline{T} = S^*$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triplet for S^*, and let ϑ be a linear operator in $\mathcal{G}$. Then we define the extension A_ϑ of S by

$$A_\vartheta := T \upharpoonright \ker(\Gamma_1 - \vartheta\Gamma_0), \tag{2.4}$$

i.e., $f \in \mathrm{dom}\, T$ belongs to $\mathrm{dom}\, A_\vartheta$ if and only if f satisfies $\Gamma_1 f = \vartheta\Gamma_0 f$. If ϑ is a symmetric operator in $\mathcal{G}$, then Green's identity implies

$$(A_\vartheta f, g)_\mathcal{H} - (f, A_\vartheta g)_\mathcal{H} = (\vartheta\Gamma_0 f, \Gamma_0 g)_\mathcal{G} - (\Gamma_0 f, \vartheta\Gamma_0 g)_\mathcal{G} = 0 \tag{2.5}$$

for all $f, g \in \mathrm{dom}\, A_\vartheta$ and, hence, the extension A_ϑ is symmetric in $\mathcal{H}$.

Of course, one is mostly interested in the self-adjointness of A_ϑ. If $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triplet, then the situation is simple: Here one has a one-to-one correspondence between self-adjoint realizations A_ϑ as in (2.4) and self-adjoint operators and relations ϑ in $\mathcal{G}$. In particular, if ϑ is a self-adjoint operator in $\mathcal{G}$, then A_ϑ is self-adjoint in $\mathcal{H}$, see, e.g., (Behrndt, Hassi & de Snoo, 2020: Theorem 2.1.3) for more details.

If $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a generalized or a quasi boundary triplet, then the self-adjointness of ϑ does, in general, not imply the self-adjointness of A_ϑ, or vice versa. However, the following theorem, where we also state an abstract version of the Birman-Schwinger principle and a Kreĭn type resolvent formula for canonical extensions A_ϑ, will allow us to give conditions for the self-adjointness of A_ϑ; for the proof we refer to (Behrndt & Langer, 2007: Theorem 2.8) or (Behrndt & Langer, 2012: Theorem 6.16).

Theorem 2.3. *Let T be a linear operator in $\mathcal{H}$ satisfying $\overline{T} = S^*$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triplet for S^* with $A_0 = T \upharpoonright \ker\Gamma_0$, and denote the associated γ-field and Weyl function by γ and M, respectively. Let A_ϑ be the extension of S associated with an operator ϑ in $\mathcal{G}$ as in* (2.4)*. Then the following statements hold for all $\lambda \in \rho(A_0)$:*

(i) $\lambda \in \sigma_\mathrm{p}(A_\vartheta)$ *if and only if* $0 \in \sigma_\mathrm{p}(\vartheta - M(\lambda))$*. Moreover,*

$$\ker(A_\vartheta - \lambda) = \big\{\gamma(\lambda)\varphi : \varphi \in \ker(\vartheta - M(\lambda))\big\}.$$

(ii) *If $\lambda \notin \sigma_\mathrm{p}(A_\vartheta)$, then $g \in \mathrm{ran}\,(A_\vartheta - \lambda)$ if and only if $\gamma(\overline{\lambda})^* g \in \mathrm{ran}\,(\vartheta - M(\lambda))$.*

(iii) *If $\lambda \notin \sigma_\mathrm{p}(A_\vartheta)$, then*

$$(A_\vartheta - \lambda)^{-1} g = (A_0 - \lambda)^{-1} g + \gamma(\lambda)\big(\vartheta - M(\lambda)\big)^{-1}\gamma(\overline{\lambda})^* g$$

holds for all $g \in \mathrm{ran}\,(A_\vartheta - \lambda)$.

Assertion (ii) of the previous theorem shows how the self-adjointness of an extension A_ϑ can be proven if $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a generalized or a quasi boundary triplet. If ϑ is symmetric in $\mathcal{G}$, then A_ϑ is symmetric in $\mathcal{H}$ by (2.5), and hence A_ϑ is self-adjoint if, in addition, $\mathrm{ran}\,(A_\vartheta \mp i) = \mathcal{H}$. According to Theorem 2.3 (ii) the latter is the case if $\mathrm{ran}\ \gamma(\mp i)^* \subset \mathrm{ran}\,(\vartheta - M(\pm i))$.

3 Some facts about Dirac operators

In this section, a brief introduction to Dirac operators will be presented. These operators correspond to the right-hand side of the Dirac equation. The free Dirac equation was derived by P. Dirac when linearising the relativistic energy-momentum relationship

$$E^2 = \sum_{j=1}^{d} p_j^2 + m^2, \tag{3.1}$$

where E denotes the energy and $p = (p_1, \ldots, p_d)$ denotes the momentum. Here, and in the subsequent sections, d is the space dimension and $m > 0$ is the mass of the particle. Furthermore, the speed of light c and Planck's constant $\hbar$ are set to 1 for simplicity. This can always be realized by a suitable choice of units. The usual linearization approach, as it is carried out for instance in Thaller (1992), corresponds to

$$\left(E - \sum_{j=1}^{d} \alpha_j p_j - m\alpha_0\right)\left(E + \sum_{j=1}^{d} \alpha_j p_j + m\alpha_0\right) = 0 \tag{3.2}$$

with matrices $\alpha_j \in \mathbb{C}^{N\times N}$, where $N = 2^{[(d+1)/2]}$ and $[\cdot]$ is the Gauss bracket. For the cases relevant to us we have $N = 2$ for $d \in \{1,2\}$ and $N = 4$ for $d = 3$. A comparison of (3.2) with the energy-momentum relationship (3.1) shows that the matrices α_j must be chosen such that they satisfy the anti-commutation relations

$$\alpha_k\alpha_j + \alpha_j\alpha_k = 2\delta_{kj} I_N \quad \text{for all} \quad k, j \in \{0, 1, \ldots, d\}, \tag{3.3}$$

where I_n denotes the $n \times n$-identity matrix. For $d \in \{1,2\}$ the matrices α_j can be chosen as the Pauli spin matrices

$$\alpha_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \alpha_0 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and for $d = 3$ as the so-called Dirac matrices

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad \text{and} \quad \alpha_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

If one now applies the usual substitution rules $i\frac{\partial}{\partial t}$ and $-i\frac{\partial}{\partial x_j}$ for E and p_j in one of the factors in (3.2), one obtains the free Dirac equation

$$i\frac{\partial}{\partial t}\Psi = \left(-i\sum_{j=1}^{d} \alpha_j \frac{\partial}{\partial x_j} + m\alpha_0\right)\Psi,$$

which describes a particle with spin $1/2$, such as an electron, that moves in $\mathbb{R}^d$. Here, and in the following, we use for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ the formal notations

$$\alpha \cdot x := \sum_{j=1}^{d} \alpha_j x_j \quad \text{and} \quad \alpha \cdot \nabla := \sum_{j=1}^{d} \alpha_j \frac{\partial}{\partial x_j}.$$

As in the case of the Schrödinger equation, one now defines the free Dirac operator as the right-hand side of the free Dirac equation by

$$A_0 f := (-i(\alpha\cdot\nabla) + m\alpha_0)\, f, \qquad \mathrm{dom}\; A_0 = H^1(\mathbb{R}^d;\mathbb{C}^N). \tag{3.4}$$

With the help of the Fourier transform it is not difficult to verify that A_0 is self-adjoint in $L^2(\mathbb{R}^d;\mathbb{C}^N)$ with purely essential spectrum

$$\sigma(A_0) = (-\infty,-m]\cup[m,\infty); \tag{3.5}$$

cf. (Thaller, 1992) or (Weidmann, 2003). From a physical point of view there are possible energy states of the system that are negative and these energies are not bounded from below. This led to the discovery of anti-particles, as, e.g., in the case of the electron, the positron.

To derive an explicit representation of the resolvent $(A_0-\lambda)^{-1}$ for $\lambda\in\rho(A_0)$, one uses that (3.3) implies the relation

$$(A_0-\lambda)(A_0+\lambda) = \left(-\Delta + m^2 - \lambda^2\right) I_N,$$

where $-\Delta$ is the free Laplace operator defined on $\mathrm{dom}\,(-\Delta) = H^2(\mathbb{R}^d)$. This implies

$$(A_0-\lambda)^{-1} = \big(-i(\alpha\cdot\nabla) + m\alpha_0 + \lambda I_N\big)(-\Delta+m^2-\lambda^2)^{-1} I_N. \tag{3.6}$$

Using the well-known form of the resolvent of $-\Delta$, one finds that $(A_0-\lambda)^{-1}$ is an integral operator in $L^2(\mathbb{R}^d;\mathbb{C}^N)$. In order to describe its integral kernel $G_{\lambda,d}(x-y)$, we write K_j for the modified Bessel functions of the second kind and

$$k(\lambda) = \sqrt{\lambda^2-m^2} \quad\text{and}\quad \zeta(\lambda) = \frac{\lambda+m}{k(\lambda)} = \frac{\lambda+m}{\sqrt{\lambda^2-m^2}}; \tag{3.7}$$

here $\sqrt{z}$ is chosen for $z\in\mathbb{C}\setminus[0,\infty)$ such that $\mathrm{Im}\,\sqrt{z}>0$. For $d\in\{1,2,3\}$ the integral kernel $G_{\lambda,d}$ is explicitly given by

$$\begin{aligned}
G_{\lambda,1}(x) &= \frac{i}{2} e^{ik(\lambda)|x|} \begin{pmatrix} \zeta(\lambda) & \mathrm{sgn}\,(x) \\ \mathrm{sgn}\,(x) & \zeta(\lambda)^{-1} \end{pmatrix}, \\
G_{\lambda,2}(x) &= \frac{k(\lambda)}{2\pi} K_1\big(-ik(\lambda)|x|\big)\frac{\sigma\cdot x}{|x|} + \frac{1}{2\pi} K_0\big(-ik(\lambda)|x|\big)\big(\lambda I_2 + m\sigma_3\big), \\
G_{\lambda,3}(x) &= \left(\lambda I_4 + m\alpha_0 + (1-ik(\lambda)|x|)\,\frac{i(\alpha\cdot x)}{|x|^2}\right)\frac{1}{4\pi|x|} e^{ik(\lambda)|x|};
\end{aligned} \tag{3.8}$$

cf. (Albeverio et al., 2005; Behrndt et al., 2020; Thaller, 1992; Weidmann, 2003).

Next, we consider external potential fields in which the particle moves. Since we are studying relativistic effects, these potentials must be invariant under Lorentz transformations. For a given scalar potential Φ_s the quantity $V = \Phi_s\alpha_0$ is Lorentz invariant as shown in Thaller (1992). This motivates the following formal ansatz for the Dirac operator corresponding to a relativistic quantum particle with spin $1/2$ moving in an external field consisting of a scalar potential Φ_s:

$$A = A_0 + \Phi_s\alpha_0.$$

Of particular interest are strongly localized fields, i.e., fields that only have an effect in a small neighborhood of a set $\Sigma\subset\mathbb{R}^d$ with measure 0. An example of a field of this kind is the quark confinement inside a nucleon in the form of the MIT bag model. To describe these strongly localized fields it is often a useful simplification to replace them by δ-potentials which are supported on Σ. In the following we consider a Lorentz scalar potential which is strongly localized in a neighborhood of the hypersurface $\Sigma \subset \mathbb{R}^d$ and approximate it by a δ-potential supported on Σ. Applying the formal ansatz above for the Dirac operator yields the formal expression

$$A_\tau = A_0 + \tau\alpha_0\delta_\Sigma \tag{3.9}$$

with interaction strength $\tau \in \mathbb{R}$. In the following sections, this operator will be defined in a mathematically rigorous way and its properties will be studied. Recall from (3.5) that the free Dirac operator A_0 is not bounded from below and hence the usual form approach to construct self-adjoint realizations with singular perturbations is not applicable.

4 One-dimensional Dirac operators with Lorentz scalar δ-point interactions

In this section, one-dimensional Dirac operators with Lorentz scalar δ-interactions supported on $\Sigma = \{0\}$ will be investigated. The following results are well known, see for instance (Pankrashkin & Richard, 2014), but are presented here for the sake of completeness. In particular, the methods used and the results obtained in the discussion will serve as a motivation for the analysis of two- and three-dimensional Dirac operators in the following section.

As already mentioned in the previous section, it is well known that the free Dirac operator

$$A_0 f = -i\sigma_1 \frac{\mathrm{d}}{\mathrm{d}x} f + m\sigma_3 f, \qquad \mathrm{dom}\ A_0 = H^1(\mathbb{R};\mathbb{C}^2),$$

is self-adjoint in the Hilbert space $L^2(\mathbb{R};\mathbb{C}^2)$. In accordance with (3.9), Lorentz scalar δ-interactions will now be considered, which are represented by the formal expression

$$A_\tau = A_0 + \tau\sigma_3\delta_\Sigma. \tag{4.1}$$

Here $\tau \in \mathbb{R}$ corresponds to the constant interaction strength. Following the usual construction of self-adjoint realizations of the expression above as in Albeverio et al. (2005), one first defines the symmetric operator

$$\begin{aligned} Sf &:= -i\sigma_1 \frac{\mathrm{d}}{\mathrm{d}x} f + m\sigma_3 f, \\ \mathrm{dom}\ S &:= H_0^1\left((0,\infty);\mathbb{C}^2\right) \oplus H_0^1((-\infty,0);\mathbb{C}^2). \end{aligned}$$

It can be shown that the adjoint operator S^* acts in the same way as S, but has the larger domain

$$\mathrm{dom}\ S^* = H^1((0,\infty);\mathbb{C}^2) \oplus H^1((-\infty,0);\mathbb{C}^2).$$

In the next step, self-adjoint extensions of S are defined by restricting S^* to a suitable domain of definition. This domain is characterized by imposing certain coupling conditions on $\Sigma = \{0\}$, which are found by a formal integration of the expression (4.1). In the present case the coupling conditions for a spinor $f = (f_1, f_2)$ have the form

$$\begin{aligned} i\left(f_2(0+) - f_2(0-)\right) &= \frac{\tau}{2}\left(f_1(0+) + f_1(0-)\right), \\ i\left(f_1(0+) - f_1(0-)\right) &= -\frac{\tau}{2}\left(f_2(0+) + f_2(0-)\right). \end{aligned} \tag{4.2}$$

Next, we define the two linear mappings $\Gamma_0, \Gamma_1 : \mathrm{dom}\ S^* \to \mathbb{C}^2$ by the assignments

$$\Gamma_0 f := -i \begin{pmatrix} f_2(0+) - f_2(0-) \\ f_1(0+) - f_1(0-) \end{pmatrix} \quad \text{and} \quad \Gamma_1 f := \frac{1}{2} \begin{pmatrix} f_1(0+) + f_1(0-) \\ f_2(0+) + f_2(0-) \end{pmatrix}. \tag{4.3}$$

Using these boundary maps one obtains the equivalent representation

$$\Gamma_0 f + \tau\sigma_3\Gamma_1 f = 0, \qquad f \in \mathrm{dom}\ S^*,$$

of the coupling conditions in (4.2).

Proposition 4.1. *The triplet $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triplet for S^*.*

Proof. Integration by parts and a straightforward computation shows that the abstract Green's identity in Definition 2.1 is valid. If one defines the function

$$f(x) = \frac{i}{2}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} \mathrm{sgn}\,(x) e^{-|x|} + \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} e^{-|x|}, \qquad x \in \mathbb{R},$$

for a given vector $(c_1, c_2, c_3, c_4) \in \mathbb{C}^4$, then $f \in \mathrm{dom}\ S^*$ and the surjectivity of the mapping $(\Gamma_0, \Gamma_1)^\top : \mathrm{dom}\ S^* \to \mathbb{C}^4$ follows. This shows (ii) in Definition 2.1. Finally, to show that Definition 2.1 (iii) holds, notice that the restriction $A_0 = S^* \upharpoonright \ker \Gamma_0$ corresponds to the free Dirac operator. Hence, it follows that the triplet is an ordinary boundary triplet. □

Using the ordinary boundary triplet from Proposition 4.1, one can now define the operator

$$A_\tau = S^* \upharpoonright \ker(\Gamma_0 + \tau\sigma_3\Gamma_1),$$

which is interpreted as the realization of the formal expression (4.1) on the basis of the coupling conditions (4.2). Due to $\tau \in \mathbb{R}$ it follows immediately that A_τ is a self-adjoint operator in $L^2(\mathbb{R}; \mathbb{C}^2)$; see the discussion before Theorem 2.3 with $\vartheta = -\tau^{-1}\sigma_3$, which is self-adjoint.

Next we derive an explicit resolvent formula for A_τ and characterize its spectrum. For this purpose, the first step is to determine the γ-field and the Weyl function of the ordinary boundary triplet from Proposition 4.1. To simplify the presentation, we first define the two functions

$$f_1(x) := \frac{i}{2}\begin{pmatrix} \zeta(\lambda) \\ \mathrm{sgn}\,(x) \end{pmatrix} e^{ik(\lambda)|x|} \quad \text{and} \quad f_2(x) := \frac{i}{2}\begin{pmatrix} \mathrm{sgn}\,(x) \\ \zeta(\lambda)^{-1} \end{pmatrix} e^{ik(\lambda)|x|}$$

with $k(\lambda)$ and $\zeta(\lambda)$ defined as in (3.7). Note that these functions form a basis of $\ker(S^* - \lambda)$ for all $\lambda \in \rho(A_0)$ and are mapped to the basis vectors $(1, 0)$ and $(0, 1)$ of $\mathbb{C}^2$ by Γ_0. A simple computation now shows that the γ-field is given by

$$\left[\gamma(\lambda)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}\right](x) = \xi_1 f_1(x) + \xi_2 f_2(x) = \frac{i}{2} e^{ik(\lambda)|x|} \begin{pmatrix} \zeta(\lambda) & \mathrm{sgn}\,(x) \\ \mathrm{sgn}\,(x) & \zeta(\lambda)^{-1} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

for $(\xi_1, \xi_2) \in \mathbb{C}^2$ and $x \in \mathbb{R}$, while the Weyl function corresponds to the matrix

$$M(\lambda) = \frac{i}{2} \begin{pmatrix} \zeta(\lambda) & 0 \\ 0 & \zeta(\lambda)^{-1} \end{pmatrix}.$$

Note that the x-dependent part in the representation of the γ-field corresponds to the Green's function of the free Dirac operator. This will remain valid also in the multi-dimensional considerations in the next section.

Using the above representations of the γ-field and the Weyl function the next result follows from Theorem 2.3.

Proposition 4.2. *For all $\lambda \in \rho(A_\tau) \cap \rho(A_0)$ and $f \in L^2(\mathbb{R}; \mathbb{C}^2)$ the resolvent formula*

$$\begin{aligned}
(A_\tau - \lambda)^{-1} f(x) &= (A_0 - \lambda)^{-1} f(x) \\
&+ \frac{\tau}{2\,(2 + i\tau\zeta(\lambda))} \left(\begin{pmatrix} \zeta(\lambda) \\ -\mathrm{sgn}\,(\cdot) \end{pmatrix} e^{ik(\lambda)|\cdot|}, \overline{f} \right)_{L^2(\mathbb{R};\mathbb{C}^2)} \begin{pmatrix} \zeta(\lambda) \\ \mathrm{sgn}\,(x) \end{pmatrix} e^{ik(\lambda)|x|} \\
&- \frac{\tau\zeta(\lambda)}{2\,(2\zeta(\lambda) - i\tau)} \left(\begin{pmatrix} -\mathrm{sgn}\,(\cdot) \\ \zeta(\lambda)^{-1} \end{pmatrix} e^{ik(\lambda)|\cdot|}, \overline{f} \right)_{L^2(\mathbb{R};\mathbb{C}^2)} \begin{pmatrix} \mathrm{sgn}\,(x) \\ \zeta(\lambda)^{-1} \end{pmatrix} e^{ik(\lambda)|x|}
\end{aligned}$$

is valid for all $x \in \mathbb{R}$. Furthermore, the spectrum of A_τ is given by

$$\begin{aligned}
\sigma_{\mathrm{ess}}(A_\tau) &= (-\infty, -m] \cup [m, \infty), \\
\sigma_{\mathrm{disc}}(A_\tau) &= \begin{cases} \emptyset, & \text{if } \tau \geq 0, \\ \left\{ \pm m \frac{4-\tau^2}{4+\tau^2} \right\}, & \text{if } \tau < 0. \end{cases}
\end{aligned}$$

Proof. From Theorem 2.3 (iii) the representation

$$(A_\tau - \lambda)^{-1} f = (A_0 - \lambda)^{-1} f - \gamma(\lambda)\tau\sigma_3(I + \tau M(\lambda)\sigma_3)^{-1}\gamma(\overline{\lambda})^* f$$

follows for all $\lambda \in \rho(A_\tau) \cap \rho(A_0)$. After a simple calculation, using the above expressions for the γ-field and the Weyl function, one obtains the claimed resolvent representation for all $f \in L^2(\mathbb{R}; \mathbb{C}^2)$. The statement about the essential spectrum follows from the fact that both A_τ and A_0 are self-adjoint extensions of the operator S, which has the finite defect indices $(2, 2)$. It remains to show the claim about the discrete spectrum. Notice first that

$$\sigma_{\mathrm{disc}}(A_\tau) \subseteq (-m, m) \subseteq \rho(A_0).$$

Thus, it follows from Theorem 2.3 (i) that $\lambda \in \sigma_{\mathrm{disc}}(A_\tau)$ if and only if $0 \in \sigma(I + \tau M(\lambda)\sigma_3)$. The eigenvalues of this matrix can be determined in an elementary way and one obtains the defining equations

$$2 + i\tau\zeta(\lambda) = 0 \quad \text{or} \quad 2\zeta(\lambda) - i\tau = 0.$$

If the first equality holds, then there exist an eigenvalue if and only if $\tau < 0$ (due to the choice of the complex square root in (3.7)). This eigenvalue is then given by

$$\lambda_1 = m\frac{4 - \tau^2}{4 + \tau^2}.$$

If the second equation holds, then a similar reasoning yields the eigenvalue $\lambda_2 = -\lambda_1$. $\square$

5 Boundary triplets for two- and three-dimensional Dirac operators with singular interactions

In this section we use boundary mappings similar to those in Section 4 to construct boundary triplets for Dirac operators with δ-shell interactions in $\mathbb{R}^2$ and $\mathbb{R}^3$. However, by translating the boundary mappings in (4.3) directly to the higher-dimensional setting one obtains a generalized or quasi boundary triplet instead of an ordinary boundary triplet. Before we can introduce the boundary triplets, some preliminaries related to function spaces and trace theorems are needed. For smooth surfaces similar boundary triplets and Sobolev spaces were used in (Behrndt et al., 2018; 2019; Behrndt & Holzmann, 2020; Holzmann, Ourmières-Bonafos & Pankrashkin, 2018) and (Behrndt et al., 2020; Benguria et al., 2017; Ourmières-Bonafos & Vega, 2018), respectively; it is one of the main goals of this note to extend these constructions to closed Lipschitz smooth hypersurfaces. As an application we prove that Dirac operators with Lorentz scalar δ-shell interactions supported on general compact Lipschitz hypersurfaces are self-adjoint.

5.1 Sobolev spaces for Dirac operators and related trace theorems

As in Section 3 the space dimension is denoted by $d \in \{2, 3\}$ while $N := 2^{[(d+1)/2]}$, where $[\cdot]$ is the Gauss bracket. Consequently, we have $N = 2$ for $d = 2$ and $N = 4$ for $d = 3$. Let $\alpha_0, \dots, \alpha_d$ be the $d + 1$ anti-commuting $\mathbb{C}^{N\times N}$-valued Dirac matrices defined in Section 3.

Throughout this subsection let $\Omega \subset \mathbb{R}^d$ be a bounded or unbounded Lipschitz domain with compact boundary and denote by ν the unit normal vector field at $\partial\Omega$. For $s \in [0, 1]$ we define the space

$$H^s_\alpha(\Omega; \mathbb{C}^N) := \big\{ f \in H^s(\Omega; \mathbb{C}^N) : (\alpha \cdot \nabla) f \in L^2(\Omega; \mathbb{C}^N) \big\},$$

where the derivatives are understood in the distributional sense and $H^s(\Omega; \mathbb{C}^N)$ is the standard L^2-based Sobolev space of order s of $\mathbb{C}^N$-valued functions, and we endow it with the norm

$$\|f\|^2_{H^s_\alpha(\Omega;\mathbb{C}^N)} := \|f\|^2_{H^s(\Omega;\mathbb{C}^N)} + \|(\alpha \cdot \nabla) f\|^2_{L^2(\Omega;\mathbb{C}^N)}.$$

One can show with standard techniques that $H^s_\alpha(\Omega; \mathbb{C}^N)$ is a Hilbert space and that $C^\infty_0(\overline{\Omega}; \mathbb{C}^N)$ is dense in $H^s_\alpha(\Omega; \mathbb{C}^N)$; cf. (Benguria et al., 2017: Lemma 2.1), (Behrndt & Holzmann, 2020: Lemma 3.2), or (Ourmières-Bonafos & Vega, 2018: Proposition 2.12) for similar arguments. Moreover, with the help of the Fourier transform one sees that $H^s_\alpha(\mathbb{R}^d; \mathbb{C}^N) = H^1(\mathbb{R}^d; \mathbb{C}^N)$ for any $s \in [0, 1]$. In the following lemma we state a trace theorem for $H^s_\alpha(\Omega; \mathbb{C}^N)$ when $s \geq \frac{1}{2}$.

Lemma 5.1. *For $s \in [\frac{1}{2}, 1]$ the map $C^\infty_0(\overline{\Omega}; \mathbb{C}^N) \ni f \mapsto f|_{\partial\Omega}$ extends to a unique continuous operator $\gamma_D : H^s_\alpha(\Omega; \mathbb{C}^N) \to H^{s-1/2}(\partial\Omega; \mathbb{C}^N)$.*

Proof. For $s \in (\frac{1}{2}, 1]$ the claim follows from the classical trace theorem (McLean, 2000: Theorem 3.38), as $H^s_\alpha(\Omega; \mathbb{C}^N)$ is continuously embedded in $H^s(\Omega; \mathbb{C}^N)$. For $s = \frac{1}{2}$ we consider for $s_1, s_2 \in \mathbb{R}$ the Hilbert space

$$H^{s_1,s_2}_\Delta(\Omega; \mathbb{C}^N) := \big\{ f \in H^{s_1}(\Omega; \mathbb{C}^N) : -\Delta f \in H^{s_2}(\Omega; \mathbb{C}^N) \big\} \tag{5.1}$$

endowed with the norm

$$\|f\|^2_{H^{s_1,s_2}_\Delta(\Omega;\mathbb{C}^N)} := \|f\|^2_{H^{s_1}(\Omega;\mathbb{C}^N)} + \|\Delta f\|^2_{H^{s_2}(\Omega;\mathbb{C}^N)}.$$

It follows from Gesztesy & Mitrea (2011: Lemma 3.1) that there exists a continuous trace map from $H^{1/2,-1}_\Delta(\Omega)$ to $L^2(\partial\Omega)$. Since (3.3) implies $(\alpha\cdot\nabla)^2 = -\Delta$ in the distributional sense, $H^{1/2}_\alpha(\Omega;\mathbb{C}^N)$ is continuously embedded in $H^{1/2,-1}_\Delta(\Omega;\mathbb{C}^N)$. This yields the claim also for $s = \frac{1}{2}$. □

Using Lemma 5.1 as well as the fact that $C_0^\infty(\overline{\Omega};\mathbb{C}^N)$ is dense in $H^s_\alpha(\Omega;\mathbb{C}^N)$, one can show for all $f, g \in H^s_\alpha(\Omega;\mathbb{C}^N)$, $s \in [\frac{1}{2}, 1]$, the following integration by parts formula

$$\int_\Omega i(\alpha\cdot\nabla)f\cdot\overline{g}\,dx = \int_{\partial\Omega} i(\alpha\cdot\nu)f\cdot\overline{g}\,d\sigma + \int_\Omega f\cdot\overline{i(\alpha\cdot\nabla)g}\,dx. \tag{5.2}$$

In the construction of boundary triplets for Dirac operators with singular interactions some families of integral operators related to the fundamental solution $G_{\lambda,d}$ given in (3.8) are required. Assume that $\Sigma \subset \mathbb{R}^d$ is a closed bounded Lipschitz hypersurface and that Ω_+ is the bounded Lipschitz domain with $\partial\Omega_+ = \Sigma$, let ν be the unit normal vector field at Σ pointing outwards of Ω_+, and let $\Omega_- := \mathbb{R}^d \setminus \overline{\Omega_+}$. We introduce the potential operator $\Phi_\lambda : L^2(\Sigma;\mathbb{C}^N) \to L^2(\mathbb{R}^d;\mathbb{C}^N)$ for $\lambda \notin (-\infty,-m]\cup[m,\infty)$ by

$$\Phi_\lambda\varphi(x) := \int_\Sigma G_{\lambda,d}(x-y)\varphi(y)\,\mathrm{d}\sigma(y), \qquad \varphi\in L^2(\Sigma;\mathbb{C}^N),\ x\in\mathbb{R}^d\setminus\Sigma, \tag{5.3}$$

and the strongly singular boundary integral operator $\mathcal{C}_\lambda : L^2(\Sigma;\mathbb{C}^N) \to L^2(\Sigma;\mathbb{C}^N)$ by the following limit

$$\mathcal{C}_\lambda\varphi(x) := \lim_{\varepsilon\searrow 0}\int_{\Sigma\setminus B(x,\varepsilon)} G_{\lambda,d}(x-y)\varphi(y)\,\mathrm{d}\sigma(y), \qquad \varphi\in L^2(\Sigma;\mathbb{C}^N),\ x\in\Sigma, \tag{5.4}$$

where $B(x,\varepsilon)$ is the ball of radius ε centered at x. Both operators Φ_λ and $\mathcal{C}_\lambda$ are well defined and bounded, see (Arrizabalaga, Mas & Vega, 2014: Lemma 3.3) and the references there. Moreover, for $\lambda \in (-m,m)$ the operator $\mathcal{C}_\lambda$ is self-adjoint in $L^2(\Sigma;\mathbb{C}^N)$. In the next lemma we improve the mapping properties for Φ_λ.

Lemma 5.2. *For any $\lambda\in\rho(A_0)$ the operator Φ_λ gives rise to a bounded map*

$$\Phi_\lambda : L^2(\Sigma;\mathbb{C}^N) \to H^{1/2}_\alpha(\mathbb{R}^d\setminus\Sigma;\mathbb{C}^N).$$

Proof. Let $\mathrm{SL}(\mu) = (-\Delta-\mu)^{-1}\gamma_D'$ be the single layer potential for $-\Delta-\mu$, where γ_D' is the dual of the Dirichlet trace operator. Using that (3.3) implies $(\alpha\cdot\nabla)^2 = \Delta$ in the distributional sense one gets

$$\Phi_\lambda = \big(-i\alpha\cdot\nabla + m\alpha_0 + \lambda I_N\big)\mathrm{SL}(\lambda^2-m^2)I_N,$$

see also (3.6). Since

$$\mathrm{SL}(\lambda^2-m^2) : L^2(\Sigma) \to H^{3/2,0}_\Delta(\mathbb{R}^d\setminus\Sigma)$$

is bounded, where $H^{3/2,0}_\Delta(\mathbb{R}^d\setminus\Sigma)$ is defined by (5.1) (this follows, e.g., from (Gesztesy & Mitrea, 2009: Equation (2.127))), the claimed result follows. □

Finally, we note that for $\varphi \in L^2(\Sigma;\mathbb{C}^N)$ the trace of $\Phi_\lambda\varphi$, which is well defined by Lemmas 5.1 and Lemma 5.2, is given by

$$\gamma_D^{\pm}\Phi_\lambda\varphi = \mp\frac{i}{2}(\alpha\cdot\nu)\varphi + \mathcal{C}_\lambda\varphi, \tag{5.5}$$

where $\gamma_D^{\pm}$ denotes the trace operator for $\Omega_\pm$; this can be shown in the same way as in (Arrizabalaga, Mas & Vega, 2014: Lemma 3.3) or (Behrndt et al., 2020: Proposition 3.4).

5.2 Quasi boundary triplets and generalized boundary triplets for Dirac operators with singular interactions

In this subsection we follow ideas from Section 4 and introduce a family of quasi boundary triplets for Dirac operators; similar constructions can also be found in Behrndt et al. (2018) and Behrndt & Holzmann (2020). Let $\Omega_+ \subset \mathbb{R}^d$ be a bounded Lipschitz domain and set $\Omega_- := \mathbb{R}^d \setminus \overline{\Omega_+}$, $\Sigma := \partial\Omega_+ = \partial\Omega_-$. We denote by ν the unit normal vector field at Σ that is pointing outwards of Ω_+. In the following we will often denote the restriction of a function f defined on $\mathbb{R}^d$ onto $\Omega_\pm$ by $f_\pm$.

We introduce for $s \in [0,1]$ the operators $T^{(s)}$ in $L^2(\mathbb{R}^d;\mathbb{C}^N)$ by

$$\begin{aligned} T^{(s)}f &:= (-i(\alpha\cdot\nabla) + m\alpha_0)f_+ \oplus (-i(\alpha\cdot\nabla) + m\alpha_0)f_-, \\ \operatorname{dom} T^{(s)} &:= H^s_\alpha(\Omega_+;\mathbb{C}^N) \oplus H^s_\alpha(\Omega_-;\mathbb{C}^N), \end{aligned}$$

and $S := T^{(s)} \upharpoonright H^1_0(\mathbb{R}^d \setminus \Sigma;\mathbb{C}^N)$, which is given more explicitly by

$$Sf = (-i(\alpha\cdot\nabla) + m\alpha_0)f, \qquad \operatorname{dom} S = H^1_0(\mathbb{R}^d \setminus \Sigma;\mathbb{C}^N).$$

The operator S is densely defined, closed, and symmetric. Using standard arguments and distributional derivatives one verifies that

$$S^* = T^{(0)} \quad \text{and} \quad (T^{(0)})^* = S.$$

Next, we introduce for $s \in [\frac{1}{2}, 1]$ the mappings $\Gamma_0^{(s)}, \Gamma_1^{(s)} : \operatorname{dom} T^{(s)} \to L^2(\Sigma;\mathbb{C}^N)$ by

$$\Gamma_0^{(s)}f := i(\alpha\cdot\nu)(f_+|_\Sigma - f_-|_\Sigma) \quad \text{and} \quad \Gamma_1^{(s)}f := \frac{1}{2}(f_+|_\Sigma + f_-|_\Sigma), \tag{5.6}$$

and note that $\Gamma_0^{(s)}$ and $\Gamma_1^{(s)}$ are well defined due to Lemma 5.1. In order to characterize the range of $\Gamma_0^{(s)}$, we introduce the space

$$H^s_\alpha(\Sigma;\mathbb{C}^N) := \big\{\varphi \in L^2(\Sigma;\mathbb{C}^N) : (\alpha\cdot\nu)\varphi \in H^s(\Sigma;\mathbb{C}^N)\big\},$$

where $H^s(\Sigma;\mathbb{C}^N)$ denotes the standard Sobolev space on Σ of $\mathbb{C}^N$-valued functions. If Σ is $C^{1,s+\varepsilon}$-smooth for some $\varepsilon > 0$, then $H^s_\alpha(\Sigma;\mathbb{C}^N) = H^s(\Sigma;\mathbb{C}^N)$, cf. (Behrndt, Holzmann & Mas, 2020: Lemma A.2).

In the following theorem we show that the mappings $\Gamma_0^{(s)}$ and $\Gamma_1^{(s)}$ in (5.6) give rise to a quasi boundary triplet for S^* and we compute the associated γ-field and Weyl function. Recall that A_0 is the free Dirac operator defined in (3.4), and that Φ_λ and $\mathcal{C}_\lambda$ are the mappings introduced in (5.3) and (5.4), respectively.

Theorem 5.3. *Let $s \in [\frac{1}{2}, 1]$. Then the following statements hold:*

(i) *The triplet $\{L^2(\Sigma;\mathbb{C}^N), \Gamma_0^{(s)}, \Gamma_1^{(s)}\}$ is a quasi boundary triplet for $S^* = \overline{T^{(s)}}$ such that $T^{(s)} \upharpoonright \ker \Gamma_0^{(s)} = A_0$, and one has*

$$\operatorname{ran} \Gamma_0^{(s)} = H_\alpha^{s-1/2}(\Sigma;\mathbb{C}^N). \tag{5.7}$$

In particular, $\{L^2(\Sigma;\mathbb{C}^N), \Gamma_0^{(1/2)}, \Gamma_1^{(1/2)}\}$ is a generalized boundary triplet.

(ii) *For $\lambda \in \rho(A_0) = \mathbb{C} \setminus ((-\infty, -m] \cup [m, \infty))$ the values $\gamma^{(s)}(\lambda)$ of the γ-field are given by*

$$\gamma^{(s)}(\lambda) = \Phi_\lambda \upharpoonright H_\alpha^{s-1/2}(\Sigma;\mathbb{C}^N).$$

Each $\gamma^{(s)}(\lambda)$ is a densely defined bounded operator from $L^2(\Sigma;\mathbb{C}^N)$ to $L^2(\mathbb{R}^d;\mathbb{C}^N)$ and an everywhere defined bounded operator from $H_\alpha^{s-1/2}(\Sigma;\mathbb{C}^N)$ to $H_\alpha^s(\mathbb{R}^d \setminus \Sigma;\mathbb{C}^N)$. Moreover,

$$\gamma^{(s)}(\lambda)^* : L^2(\mathbb{R}^d;\mathbb{C}^N) \to L^2(\Sigma;\mathbb{C}^N)$$

is compact.

(iii) *For $\lambda \in \rho(A_0) = \mathbb{C} \setminus ((-\infty, -m] \cup [m, \infty))$ the values $M^{(s)}(\lambda)$ of the Weyl function are given by*

$$M^{(s)}(\lambda) = \mathcal{C}_\lambda \upharpoonright H_\alpha^{s-1/2}(\Sigma;\mathbb{C}^N).$$

Each $M^{(s)}(\lambda)$ is a densely defined bounded operator in $L^2(\Sigma;\mathbb{C}^N)$ and a bounded everywhere defined operator from $H_\alpha^{s-1/2}(\Sigma;\mathbb{C}^N)$ to $H^{s-1/2}(\Sigma;\mathbb{C}^N)$.

Proof. Let $s \in [\frac{1}{2}, 1]$ be fixed. First, we show that $\{L^2(\Sigma;\mathbb{C}^N), \Gamma_0^{(s)}, \Gamma_1^{(s)}\}$ is a quasi boundary triplet. For this we note that $\overline{T^{(s)}} = T^{(0)} = S^*$, as $C_0^\infty(\overline{\Omega_\pm};\mathbb{C}^N)$ is dense in $H_\alpha^0(\Omega_\pm;\mathbb{C}^N)$, while the norm in $H_\alpha^0(\mathbb{R}^3 \setminus \Sigma;\mathbb{C}^N)$ and the graph norm induced by $T^{(0)}$ are equivalent. Next, we verify that Green's identity in Definition 2.1 (i) is fulfilled. For this let

$$f = f_+ \oplus f_-,\ g = g_+ \oplus g_- \in \operatorname{dom} T^{(s)} = H_\alpha^s(\Omega_+;\mathbb{C}^N) \oplus H_\alpha^s(\Omega_-;\mathbb{C}^N).$$

Then integration by parts (5.2) applied in $\Omega_\pm$ yields

$$\begin{aligned}\big((-i(\alpha\cdot\nabla) + m\alpha_0)f_\pm, g_\pm\big)_{L^2(\Omega_\pm;\mathbb{C}^N)} - \big(f_\pm, (-i(\alpha\cdot\nabla) + m\alpha_0)g_\pm\big)_{L^2(\Omega_\pm;\mathbb{C}^N)}&\\ = \pm\big(-i(\alpha\cdot\nu)f_\pm|_\Sigma, g_\pm|_\Sigma\big)_{L^2(\Sigma;\mathbb{C}^N)},&\end{aligned}$$

where it is used that $-\nu$ is the normal vector field pointing outwards of Ω_-. By adding these two formulas for Ω_+ and Ω_- one arrives at Green's identity.

Next, we show that $T^{(s)} \upharpoonright \ker \Gamma_0^{(s)} = A_0$. As the free Dirac operator A_0 is self-adjoint, this shows that $T^{(s)} \upharpoonright \ker \Gamma_0^{(s)}$ is self-adjoint. The inclusion $A_0 \subset T^{(s)} \upharpoonright \ker \Gamma_0^{(s)}$ is clear. To verify the converse inclusion, let $f \in \ker \Gamma_0^{(s)}$. Then Green's identity yields for any $\varphi \in C_0^\infty(\mathbb{R}^d;\mathbb{C}^N)$

$$\big(f, -i(\alpha\cdot\nabla)\varphi\big)_{L^2(\mathbb{R}^d;\mathbb{C}^N)} = \big((T^{(s)} - m\alpha_0)f, \varphi\big)_{L^2(\mathbb{R}^d;\mathbb{C}^N)}. \tag{5.8}$$

Hence, $(\alpha\cdot\nabla)f \in L^2(\mathbb{R}^d;\mathbb{C}^N)$, which shows $f \in H_\alpha^s(\mathbb{R}^d;\mathbb{C}^N) = H^1(\mathbb{R}^d;\mathbb{C}^N) = \operatorname{dom} A_0$. Therefore, we conclude that $T^{(s)} \upharpoonright \ker \Gamma_0^{(s)} = A_0$ holds.

It remains to prove that $\operatorname{ran}(\Gamma_0^{(s)}, \Gamma_1^{(s)})$ is dense in $L^2(\Sigma;\mathbb{C}^N) \times L^2(\Sigma;\mathbb{C}^N)$. For this, we prove

$$\operatorname{ran}(\Gamma_0^{(s)} \upharpoonright \ker \Gamma_1^{(s)}) = H_\alpha^{1/2}(\Sigma;\mathbb{C}^N) \tag{5.9}$$

and

$$\operatorname{ran}(\Gamma_1^{(s)} \upharpoonright \ker \Gamma_0^{(s)}) = H^{1/2}(\Sigma;\mathbb{C}^N). \tag{5.10}$$

To establish the inclusion "$\subset$" in (5.9) we note first that any function $f \in \ker \Gamma_1^{(s)}$ satisfies the equality $f_+|_\Sigma = -f_-|_\Sigma$. One can show as in (5.8) that $f_+ \oplus (-f_-) \in H_\alpha^s(\mathbb{R}^d;\mathbb{C}^N) = H^1(\mathbb{R}^d;\mathbb{C}^N)$ and thus, $f \in H^1(\Omega_+;\mathbb{C}^N) \oplus H^1(\Omega_-;\mathbb{C}^N)$. Therefore, the definition of $\Gamma_0^{(s)}$ yields the claimed inclusion. For the converse inclusion "$\supset$" let $\varphi \in H_\alpha^{1/2}(\Sigma;\mathbb{C}^N)$. Choose $f_\pm \in H^1(\Omega_\pm;\mathbb{C}^N)$ such that $f_\pm|_\Sigma = \mp\frac{i}{2}(\alpha\cdot\nu)\varphi$. Then $f \in \ker \Gamma_1^{(s)}$ and $\Gamma_0^{(s)} f = \varphi$. Since this can be done for all functions $\varphi \in H_\alpha^{1/2}(\Sigma;\mathbb{C}^N)$, we have shown (5.9).

To verify (5.10) note that the inclusion "$\subset$" follows from $\ker \Gamma_0^{(s)} = H^1(\mathbb{R}^d;\mathbb{C}^N)$ and the definition of $\Gamma_1^{(s)}$. For the converse inclusion "$\supset$" let $\varphi \in H^{1/2}(\Sigma;\mathbb{C}^N)$. Choose $f \in H^1(\mathbb{R}^d;\mathbb{C}^N)$ such that $f|_\Sigma = \varphi$. Then $f \in \ker \Gamma_0^{(s)}$ and $\Gamma_1^{(s)} f = \varphi$. Since this can be done for all $\varphi \in H^{1/2}(\Sigma;\mathbb{C}^N)$, we have verified (5.10). Hence, we have shown that $\{L^2(\Sigma;\mathbb{C}^N), \Gamma_0^{(s)}, \Gamma_1^{(s)}\}$ is indeed a quasi boundary triplet for all $s \in [\frac{1}{2}, 1]$. Thus, except for formula (5.7), assertion (i) has been shown. Equation (5.7) will be proved together with items (ii) and (iii).

Next, we show that $\gamma^{(s)}(\lambda)^*$ is compact for all s. For this purpose, recall that formula (2.2) implies that $\gamma^{(s)}(\lambda)^* = \Gamma_1^{(s)}(A_0 - \overline{\lambda})^{-1}$. Since $(A_0 - \overline{\lambda})^{-1} : L^2(\mathbb{R}^d;\mathbb{C}^N) \to H^1(\mathbb{R}^d;\mathbb{C}^N)$ is bounded, we see that $\gamma^{(s)}(\lambda)^*$ is actually independent of s and, furthermore, that

$$\gamma^{(s)}(\lambda)^* : L^2(\mathbb{R}^d;\mathbb{C}^N) \to H^{1/2}(\Sigma;\mathbb{C}^N)$$

is also bounded. Since $H^{1/2}(\Sigma;\mathbb{C}^N)$ is compactly embedded in $L^2(\Sigma;\mathbb{C}^N)$, the claimed compactness of $\gamma^{(s)}(\lambda)^*$ follows.

The remaining assertions will be proved in three steps for various values of $s \in [\frac{1}{2}, 1]$.

Step 1 for $s = \frac{1}{2}$. Consider for $\varphi \in L^2(\Sigma;\mathbb{C}^N)$ the function $f_\lambda := \Phi_\lambda \varphi$, see (5.3). Then one has $f_\lambda \in H_\alpha^{1/2}(\mathbb{R}^d \setminus \Sigma;\mathbb{C}^N) = \operatorname{dom} T^{(1/2)}$ by Lemma 5.2, and by (5.5) we get $\Gamma_0^{(1/2)} f_\lambda = \varphi$. Therefore, $\operatorname{ran} \Gamma_0^{(1/2)} = L^2(\Sigma;\mathbb{C}^N)$, which is (5.7) for $s = \frac{1}{2}$. Moreover, as $G_{\lambda,d}$ in (3.8) is a fundamental solution for the Dirac equation, the definition of Φ_λ shows that

$$\big(T^{(1/2)} - \lambda\big) f_\lambda = 0 \quad \text{in } \mathbb{R}^d \setminus \Sigma.$$

Consequently, $\gamma^{(1/2)}(\lambda) = \Phi_\lambda$. Finally, using the definition of $\Gamma_1^{(1/2)}$ and (5.5), it follows that $M^{(1/2)}(\lambda) = \mathcal{C}_\lambda$ and thus, $M^{(1/2)}(\lambda)$ is bounded in $L^2(\Sigma;\mathbb{C}^N)$. This shows all claims for $s = \frac{1}{2}$.

Step 2 for $s = 1$. First note that $\operatorname{dom} T^{(1)} = H^1(\mathbb{R}^d \setminus \Sigma;\mathbb{C}^N)$, the definition of $\Gamma_0^{(1)}$, and (5.9) imply $\operatorname{ran} \Gamma_0^{(1)} = H_\alpha^{1/2}(\Sigma;\mathbb{C}^N)$. As $\{L^2(\Sigma;\mathbb{C}^N), \Gamma_0^{(1)}, \Gamma_1^{(1)}\}$ is a restriction of the triplet for $s = \frac{1}{2}$, we deduce from the already shown results that

$$\gamma^{(1)}(\lambda) = \gamma^{(1/2)}(\lambda) \upharpoonright \operatorname{ran} \Gamma_0^{(1)} = \Phi_\lambda \upharpoonright H_\alpha^{1/2}(\Sigma;\mathbb{C}^N)$$

and

$$M^{(1)}(\lambda) = M^{(1/2)}(\lambda) \upharpoonright \operatorname{ran} \Gamma_0^{(1)} = \mathcal{C}_\lambda \upharpoonright H_\alpha^{1/2}(\Sigma;\mathbb{C}^N).$$

Using the closed graph theorem and the fact that $H^{1/2}_\alpha(\Sigma;\mathbb{C}^N)$ and $H^1(\mathbb{R}^d\setminus\Sigma;\mathbb{C}^N)$ are boundedly embedded in $L^2(\Sigma;\mathbb{C}^N)$ and $L^2(\mathbb{R}^d;\mathbb{C}^N)$, respectively, one gets that

$$\gamma^{(1)}(\lambda) : \operatorname{ran}\Gamma_0^{(1)} = H^{1/2}_\alpha(\Sigma;\mathbb{C}^N) \to \operatorname{dom} T^{(1)} = H^1(\mathbb{R}^d\setminus\Sigma;\mathbb{C}^N)$$

is bounded as well. The mapping properties of the trace map yield that also

$$M^{(1)}(\lambda) : \operatorname{ran}\Gamma_0^{(1)} = H^{1/2}_\alpha(\Sigma;\mathbb{C}^N) \to \operatorname{ran}\Gamma_1^{(1)} = H^{1/2}(\Sigma;\mathbb{C}^N)$$

is bounded. Hence, all claimed statements for $s=1$ have been shown.

Step 3 for $s\in(\frac{1}{2},1)$. First we note that an interpolation argument shows that

$$\Phi_\lambda : H^{s-1/2}(\Sigma;\mathbb{C}^N) \to H^s_\alpha(\mathbb{R}^d\setminus\Sigma;\mathbb{C}^N) = \operatorname{dom} T^{(s)}$$

is bounded. Together with (5.5) this implies that $\operatorname{ran}\Gamma_0^{(s)} = H^{s-1/2}_\alpha(\Sigma;\mathbb{C}^N)$, i.e., (5.7), holds for $s\in(\frac{1}{2},1)$. Hence, we have $\gamma^{(s)}(\lambda)=\Phi_\lambda \restriction H^{s-1/2}_\alpha(\Sigma;\mathbb{C}^N)$ and the trace theorem shows that

$$M^{(s)}(\lambda)=\Gamma_1^{(s)}\gamma^{(s)}(\lambda) : H^{s-1/2}_\alpha(\Sigma;\mathbb{C}^N)\to H^{s-1/2}(\Sigma;\mathbb{C}^N)$$

is bounded. Thus, all claims are proved. □

In the next theorem we study the self-adjointness of a Dirac operator A_τ with a Lorentz scalar δ-shell interaction of strength $\tau\in\mathbb{R}\setminus\{0\}$, which is formally given by $-i(\alpha\cdot\nabla)+m\alpha_0+\tau\alpha_0\delta_\Sigma$. In a similar way as in (4.2) we define the operator A_τ by

$$A_\tau := T^{(1/2)}\restriction \ker\big(\Gamma_0^{(1/2)}+\tau\alpha_0\Gamma_1^{(1/2)}\big).$$

The operator A_τ is given more explicitly by

$$\begin{aligned} A_\tau f &= (-i(\alpha\cdot\nabla)+m\alpha_0)f_+ \oplus (-i(\alpha\cdot\nabla)+m\alpha_0)f_-,\\ \operatorname{dom} A_\tau &= \Big\{f\in H^{1/2}_\alpha(\mathbb{R}^d\setminus\Sigma;\mathbb{C}^N) : i(\alpha\cdot\nu)(f_+|_\Sigma - f_-|_\Sigma)+\frac{\tau}{2}\alpha_0(f_+|_\Sigma+f_-|_\Sigma)=0\Big\}. \end{aligned}$$

This operator was investigated under various assumptions in Behrndt et al. (2019; 2020); Holzmann, Ourmières-Bonafos & Pankrashkin (2018); Pizzichillo & Van Den Bosch (2019). In the following theorem it is shown, for the first time, that A_τ is self-adjoint when the interaction support $\Sigma\subset\mathbb{R}^d$ is an arbitrary closed bounded Lipschitz smooth hypersurface.

Theorem 5.4. *For any $\tau\in\mathbb{R}\setminus\{0\}$ the operator A_τ is self-adjoint in $L^2(\mathbb{R}^d;\mathbb{C}^N)$ and the following statements hold:*

(i) *For $\lambda\in\rho(A_\tau)$ the resolvent of A_τ is given by*

$$(A_\tau-\lambda)^{-1}=(A_0-\lambda)^{-1}-\Phi_\lambda\left(\frac{1}{\tau}\alpha_0+\mathcal{C}_\lambda\right)^{-1}\Phi_{\bar\lambda}^*.$$

(ii) $\sigma_{\text{ess}}(A_\tau)=\sigma_{\text{ess}}(A_0)=(-\infty,-m]\cup[m,\infty)$.

(iii) *$\sigma_{\text{disc}}(A_\tau)$ is finite and $\lambda\in\sigma_{\text{disc}}(A_\tau)$ if and only if $0\in\sigma_{\text{p}}(\frac{1}{\tau}\alpha_0+\mathcal{C}_\lambda)$.*

Remark 5.5. By Theorem 5.4 the operator A_τ is self-adjoint defined on a subset of

$$H^{1/2}_\alpha(\mathbb{R}^d \setminus \Sigma; \mathbb{C}^N).$$

If Σ is a smooth hypersurface, then it is known that A_τ is self-adjoint and

$$\text{dom } A_\tau \subset H^1(\mathbb{R}^d \setminus \Sigma; \mathbb{C}^N),$$

see (Behrndt et al., 2020; Holzmann, Ourmières-Bonafos & Pankrashkin, 2018). However, for more general Lipschitz smooth hypersurfaces this smoothness in the operator domain can not be expected, as was shown explicitly in (Le Treust & Ourmières-Bonafos, 2018: Remark 1.9) in the two-dimensional setting for polygonal domains.

Proof of Theorem 5.4. In order to show the self-adjointness of A_τ, it suffices, according to Theorem 2.3 and the discussion following it, to verify that

$$\begin{aligned}\text{ran } \big(\Gamma_1^{(1/2)}(A_0 \pm i)^{-1}\big) = H^{1/2}(\Sigma; \mathbb{C}^N) &\subset \text{ran } \left(\frac{1}{\tau}\alpha_0 + M^{(1/2)}_{\pm i}\right) \\ &= \text{ran } \left(\frac{1}{\tau}\alpha_0 + \mathcal{C}_{\pm i}\right).\end{aligned} \tag{5.11}$$

In order to see this, we prove that $\frac{1}{\tau}\alpha_0 + \mathcal{C}_{\pm i}$ is bijective in $L^2(\Sigma; \mathbb{C}^N)$. First, we note that $\frac{1}{\tau}\alpha_0 + \mathcal{C}_{\pm i}$ is injective, as otherwise the symmetric operator A_τ would have the non-real eigenvalue $\pm i$ by Theorem 2.3. Next, by (2.3) we have that

$$\mathcal{C}_{\pm i} = M^{(1/2)}(\pm i) = M^{(1/2)}(0) \pm i\gamma^{(1/2)}(0)^*\gamma^{(1/2)}(\pm i) = \mathcal{C}_0 + \mathcal{K}_{\pm i}$$

and note that $\mathcal{K}_{\pm i} = \pm i\gamma^{(1/2)}(0)^*\gamma^{(1/2)}(\pm i)$ is compact in $L^2(\Sigma; \mathbb{C}^N)$ due to Theorem 5.3 (ii). Next, we compute

$$\left(\frac{1}{\tau}\alpha_0 + \mathcal{C}_{\pm i}\right)^2 = \frac{1}{\tau^2} I_N + \mathcal{C}_0^2 + \frac{1}{\tau}\big(\alpha_0 \mathcal{C}_{\pm i} + \mathcal{C}_{\pm i}\alpha_0\big) + \mathcal{K}_{\pm i}^2 + \mathcal{C}_0\mathcal{K}_{\pm i} + \mathcal{K}_{\pm i}\mathcal{C}_0.$$

Since $\mathcal{C}_0$ is self-adjoint, the operator $\frac{1}{\tau^2} I_N + \mathcal{C}_0^2$ is a strictly positive self-adjoint operator and, hence, it is a Fredholm operator with index zero. Next, due to the anti-commutation relation (3.3) it is not difficult to see that

$$\frac{1}{\tau}\big(\alpha_0 \mathcal{C}_{\pm i} + \mathcal{C}_{\pm i}\alpha_0\big) = \frac{1}{\tau}(2m \pm 2i\alpha_0)\mathcal{S}(-m^2 - 1),$$

where $\mathcal{S}(\nu)$ is the single layer boundary integral operator for $-\Delta - \nu$. According to (Holzmann & Unger, 2020: Lemma 3.4) the latter operator is compact. Since also $\mathcal{K}_{\pm i}$ is compact, we conclude that $\big(\frac{1}{\tau}\alpha_0 + \mathcal{C}_{\pm i}\big)^2$ must be a Fredholm operator with index zero. Since $\frac{1}{\tau}\alpha_0 + \mathcal{C}_{\pm i}$ is injective, we conclude that $\big(\frac{1}{\tau}\alpha_0 + \mathcal{C}_{\pm i}\big)^2$ is also injective and hence, as it has Fredholm index zero, it must be surjective. Therefore $\frac{1}{\tau}\alpha_0 + \mathcal{C}_{\pm i}$ is also bijective. This shows that (5.11) holds and thus, A_τ is self-adjoint.

Next, by Theorem 2.3 the claimed resolvent formula in (i) holds for $\lambda = \pm i$. The map $\frac{1}{\tau}\alpha_0 + \mathcal{C}_{\pm i}$ is bijective and, hence, boundedly invertible. This, the mapping properties of $\Phi_{\pm i}$ and $\Phi^*_{\mp i}$ from Theorem 5.3, and Kreĭn's resolvent formula imply assertion (ii). The resolvent formula in item (i) for $\lambda \in \rho(A_\tau)$ is now a direct consequence of Theorem 2.3. The fact that $\sigma_{\text{disc}}(A_\tau)$ is finite can be shown in the same way as in (Behrndt et al., 2020: Proposition 3.8), while the Birman-Schwinger principle in (iii) follows again directly from Theorem 2.3 and the representation of $M^{(1/2)}(\lambda)$ from Theorem 5.3. □

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THE ORIGINAL WEYL-TITCHMARSH FUNCTIONS AND SECTORIAL SCHRÖDINGER L-SYSTEMS

Sergey Belyi and Eduard Tsekanovskiĭ

Dedicated with great pleasure to Seppo Hassi on the occasion of his 60th birthday

1 Introduction

This paper is part of an ongoing project studying the realizations of the original Weyl-Titchmarsh function $m_\infty(z)$ and its linear-fractional transformation $m_\alpha(z)$ associated with a Schrödinger operator in $L_2[\ell,+\infty)$. In this project the Herglotz-Nevanlinna functions $-m_\infty(z)$ and $1/m_\infty(z)$ as well as $-m_\alpha(z)$ and $1/m_\alpha(z)$ are being realized as impedance functions of L-systems with a dissipative Schrödinger main operator T_h, $\operatorname{Im} h > 0$. For the sake of brevity we will refer to these L-systems as *Schrödinger L-systems*. The formal definition, exposition, and discussion of general and Schrödinger L-systems are presented in Sections 2 and 4. We capitalize on the fact that all Schrödinger L-systems $\Theta_{\mu,h}$ form a two-parametric family whose members are uniquely defined by a real-valued parameter μ and a complex boundary value h of the main dissipative operator.

The focus of this paper is on the case when the realizing Schrödinger L-systems are based on a non-negative symmetric Schrödinger operator with deficiency indices $(1,1)$ that has an accretive state-space operator. It is known that in this case the impedance functions of such L-systems are Stieltjes functions, see (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011). Here we study the situation when the realizing Schrödinger L-systems are also sectorial and the Weyl-Titchmarsh functions $-m_\alpha(z)$ fall into *sectorial* classes S^β and S^{β_1,β_2} of Stieltjes functions that are discussed in detail in Section 3. Section 5 provides us with the general realization results (obtained in Belyi & Tsekanovskiĭ (2021)) for the functions $-m_\infty(z)$, $1/m_\infty(z)$, and $-m_\alpha(z)$. It is shown that $-m_\infty(z)$, $1/m_\infty(z)$, and $-m_\alpha(z)$ can be realized as impedance functions of Schrödinger L-systems $\Theta_{0,i}$, $\Theta_{\infty,i}$, and $\Theta_{\tan\alpha,i}$, respectively.

The main results of the paper are contained in Section 6. There we apply the realization theorems from Section 5 to Schrödinger L-systems that are based on a non-negative symmetric Schrödinger operator to obtain additional properties. Utilizing the results presented in Section 4, we derive some new features of Schrödinger L-systems $\Theta_{\tan\alpha,i}$ whose impedance functions fall into particular sectorial classes S^{β_1,β_2} with β_1 and β_2 explicitly described. The results are given in terms of the parameter α that appears in the definition of the function $m_\alpha(z)$. Moreover, the knowledge of the limit value $m_\infty(-0)$ and the value of α allow us to find the angle of sectoriality of the main and state-space operators of the realizing L-system. This, in turn, leads to connections to Kato's problem about sectorial extensions of sectorial forms.

The paper is concluded with an example that illustrates the main results and concepts. The present work is a further development of the theory of open physical systems conceived by M. Livšic (1966).

2 Preliminaries

For a pair of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ we denote by $[\mathcal{H}_1, \mathcal{H}_2]$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\dot{A}$ be a closed, densely defined, symmetric operator in a Hilbert space $\mathcal{H}$ with inner product (f, g), where $f, g \in \mathcal{H}$. Any non-symmetric operator T in $\mathcal{H}$ such that $\dot{A} \subset T \subset \dot{A}^*$ will be called a *quasi-self-adjoint extension* of $\dot{A}$.

Consider the rigged Hilbert space $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$, where $\mathcal{H}_+ = \mathrm{dom}\ \dot{A}^*$ and

$$(f, g)_+ = (f, g) + (\dot{A}^* f, \dot{A}^* g), \qquad f, g \in \mathrm{dom}\ A^*,$$

see (Berezansky, 1968; Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011). Let $\mathcal{R}$ be the ***Riesz-Berezansky operator*** $\mathcal{R}$ from $\mathcal{H}_-$ onto $\mathcal{H}_+$ such that $(f, g) = (f, \mathcal{R}g)_+$ and $\|\mathcal{R}g\|_+ = \|g\|_-$, where $f \in \mathcal{H}_+$, $g \in \mathcal{H}_-$, see (Berezansky, 1968; Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011). Identifying the space conjugate to $\mathcal{H}_\pm$ with $\mathcal{H}_\mp$, we get that if $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$, then $\mathbb{A}^* \in [\mathcal{H}_+, \mathcal{H}_-]$. An operator $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a *self-adjoint bi-extension* of a symmetric operator $\dot{A}$ if $\mathbb{A} = \mathbb{A}^*$ and $\mathbb{A} \supset \dot{A}$. Let $\mathbb{A}$ be a self-adjoint bi-extension of $\dot{A}$ and let the operator $\hat{A}$ in $\mathcal{H}$ be defined as follows

$$\mathrm{dom}\ \hat{A} = \{f \in \mathcal{H}_+ : \mathbb{A}f \in \mathcal{H}\}, \qquad \hat{A} = \mathbb{A}\!\upharpoonright \mathrm{dom}\ \hat{A}.$$

Then the operator $\hat{A}$ is called a *quasi-kernel* of a self-adjoint bi-extension $\mathbb{A}$, see (Tsekanovskiĭ & Šmuljan, 1977) and (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011: Section 2.1). A self-adjoint bi-extension $\mathbb{A}$ of a symmetric operator $\dot{A}$ is called *t-self-adjoint* if its quasi-kernel $\hat{A}$ is a self-adjoint operator in $\mathcal{H}$, see (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011: Definition 4.3.1). An operator $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a *quasi-self-adjoint bi-extension* of an operator T if $\mathbb{A} \supset T \supset \dot{A}$ and $\mathbb{A}^* \supset T^* \supset \dot{A}$.

We will be mostly interested in the following type of quasi-self-adjoint bi-extensions. Let T be a quasi-self-adjoint extension of $\dot{A}$ with nonempty resolvent set $\rho(T)$. A quasi-self-adjoint bi-extension $\mathbb{A}$ of an operator T is called a $(*)$*-extension* of T if $\mathrm{Re}\,\mathbb{A}$ is a t-self-adjoint bi-extension of $\dot{A}$, see (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011: Definition 3.3.5). In what follows we assume that $\dot{A}$ has deficiency indices $(1, 1)$. In this case it is known that every quasi-self-adjoint extension T of $\dot{A}$ admits $(*)$-extensions, see (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011). The description of all $(*)$-extensions via the Riesz-Berezansky operator $\mathcal{R}$ can be found in Arlinskiĭ, Belyi & Tsekanovskiĭ (2011: Section 4.3).

Recall that a linear operator T in a Hilbert space $\mathcal{H}$ is called *nonnegative*, *accretive*, or *dissipative* if $(Tf, f) \geq 0$, $\mathrm{Re}\,(Tf, f) \geq 0$, or $\mathrm{Im}\,(Tf, f) \geq 0$ for all $f \in \mathrm{dom}\ T$ respectively, see (Kato, 1966). An accretive operator T is called β*-sectorial* if there exists a value $\beta \in (0, \pi/2)$ such that

$$(\cot\beta)|\mathrm{Im}\,(Tf, f)| \leq \mathrm{Re}\,(Tf, f), \qquad f \in \mathrm{dom}\ T, \tag{2.1}$$

see (Kato, 1966). We say that the angle of sectoriality β is *exact* for a β-sectorial operator T if

$$\tan\beta = \sup_{f \in \mathrm{dom}\, T} \frac{|\mathrm{Im}\,(Tf, f)|}{\mathrm{Re}\,(Tf, f)}.$$

An accretive operator is called *extremal accretive* if it is not β-sectorial for any $\beta \in (0, \pi/2)$. A $(*)$-extension $\mathbb{A}$ of T is called *accretive* if $\mathrm{Re}\,(\mathbb{A}f, f) \geq 0$ for all $f \in \mathcal{H}_+$. This is equivalent to the real part $\mathrm{Re}\,\mathbb{A} = (\mathbb{A} + \mathbb{A}^*)/2$ being a nonnegative t-self-adjoint bi-extension of $\dot{A}$.

The following definition is a "lite" version of the definition of an L-system given for a scattering L-system with a one-dimensional input-output space. It is tailored to the case when the symmetric operator of an L-system has deficiency indices $(1,1)$. The general definition of an L-system can be found in Arlinskiĭ, Belyi & Tsekanovskiĭ (2011: Definition 6.3.4), see also (Belyi et al., 2006) for a non-canonical version.

Definition 2.1. An array

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix} \tag{2.2}$$

is called an *L-system* if:

(1) T is a dissipative quasi-self-adjoint extension of a symmetric operator $\dot{A}$ with deficiency indices $(1,1)$;

(2) $\mathbb{A}$ is a $(*)$-extension of T;

(3) $\operatorname{Im}\mathbb{A} = KK^*$, where $K \in [\mathbb{C}, \mathcal{H}_-]$ and $K^* \in [\mathcal{H}_+, \mathbb{C}]$.

The operators T and $\mathbb{A}$ in the above definition are called the *main* and *state-space operator*, respectively, of the L-system Θ, and K is called a *channel operator*. It is easy to see that the operator $\mathbb{A}$ of the system (2.2) is such that $\operatorname{Im}\mathbb{A} = (\cdot,\chi)\chi$, $\chi \in \mathcal{H}_-$ and pick $Kc = c\cdot\chi$, $c \in \mathbb{C}$, see (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011). The L-system Θ in (2.2) is called *minimal* if the operator $\dot{A}$ is a prime operator in $\mathcal{H}$, i.e., if there does not exist a non-trivial reducing invariant subspace of $\mathcal{H}$ on which it induces a self-adjoint operator. Minimal L-systems of the form (2.2) with a one-dimensional input-output space were also considered in Belyi, Makarov & Tsekanovskiĭ (2015).

We associate with an L-system Θ the function

$$W_\Theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}K, \qquad z \in \rho(T),$$

which is called the *transfer function* of the L-system Θ. We also consider the function

$$V_\Theta(z) = K^*(\operatorname{Re}\mathbb{A} - zI)^{-1}K, \tag{2.3}$$

which is called the *impedance function* of an L-system Θ of the form (2.2). The transfer function $W_\Theta(z)$ of the L-system Θ and the impedance function $V_\Theta(z)$ of the form (2.3) are connected by the following relations valid for $\operatorname{Im} z \neq 0$ and $z \in \rho(T)$:

$$\begin{aligned} V_\Theta(z) &= i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I], \\ W_\Theta(z) &= (I + iV_\Theta(z))^{-1}(I - iV_\Theta(z)). \end{aligned}$$

An L-system Θ of the form (2.2) is called an *accretive L-system* if its state-space operator $\mathbb{A}$ is accretive, that is $\operatorname{Re}(\mathbb{A}f, f) \geq 0$ for all $f \in \mathcal{H}_+$, see (Belyi & Tsekanovskiĭ, 2008; Dovzhenko & Tsekanovskiĭ, 1990). An accretive L-system is called *sectorial* if the operator $\mathbb{A}$ is sectorial, i.e., if it satisfies (2.1) for some $\beta \in (0, \pi/2)$ and all $f \in \mathcal{H}_+$.

3 Sectorial classes and their realizations

A scalar function $V(z)$ is called a *Herglotz-Nevanlinna function* if it is holomorphic on $\mathbb{C}\setminus\mathbb{R}$, symmetric with respect to the real axis, i.e., $V(z)^* = V(\bar{z})$, $z\in\mathbb{C}\setminus\mathbb{R}$, and satisfies the positivity condition $\operatorname{Im} V(z)\geq 0$, $z\in\mathbb{C}_+$. The class of all Herglotz-Nevanlinna functions that can be realized as impedance functions of L-systems and connections with Weyl-Titchmarsh functions can be found in Arlinskiĭ, Belyi & Tsekanovskiĭ (2011), Belyi, Makarov & Tsekanovskiĭ (2015), Derkach, Malamud & Tsekanovskiĭ (1989), Gesztesy & Tsekanovskiĭ (2000), and references therein. The following definition can be found in Kac & Krein (1974): a scalar Herglotz-Nevanlinna function $V(z)$ is a *Stieltjes function* if it is holomorphic in $\operatorname{Ext}[0,+\infty)$ and

$$\frac{\operatorname{Im}[zV(z)]}{\operatorname{Im} z}\geq 0.$$

It is known that a Stieltjes function $V(z)$ admits the following integral representation

$$V(z)=\gamma+\int\limits_0^\infty\frac{dG(t)}{t-z}, \tag{3.1}$$

where $\gamma\geq 0$ and $G(t)$ is a non-decreasing function on $[0,+\infty)$ with $\int_0^\infty\frac{dG(t)}{1+t}<\infty$, see (Kac & Krein, 1974). We are going to focus on the *class* $S_0(R)$ of scalar Stieltjes functions such that the measure $G(t)$ in the representation (3.1) is of unbounded variation, see (Belyi & Tsekanovskiĭ, 2008; Dovzhenko & Tsekanovskiĭ, 1990; Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011). It was shown in Arlinskiĭ, Belyi & Tsekanovskiĭ (2011), see also (Belyi & Tsekanovskiĭ, 2008), that such a function $V(z)$ can be realized as the impedance function of an accretive L-system Θ of the form (2.2) with a densely defined symmetric operator if and only if it belongs to the class $S_0(R)$.

Now we are going to consider sectorial subclasses of scalar Stieltjes functions introduced in Alpay & Tsekanovskiĭ (2000). Let $\beta\in(0,\frac{\pi}{2})$. The *sectorial subclass* S^β of Stieltjes functions consists of all scalar functions $V(z)$ for which

$$\sum_{k,l=1}^n\left[\frac{z_kV(z_k)-\bar{z}_lV(\bar{z}_l)}{z_k-\bar{z}_l}-(\cot\beta)\,V(\bar{z}_l)V(z_k)\right]h_k\bar{h}_l\geq 0,$$

for arbitrary sequences of complex numbers $\{z_k\}$, $\operatorname{Im} z_k>0$, and $\{h_k\}$. For $0<\beta_1<\beta_2<\frac{\pi}{2}$, we have

$$S^{\beta_1}\subset S^{\beta_2}\subset S,$$

where S denotes the class of all Stieltjes functions (which corresponds to the case $\beta=\frac{\pi}{2}$). Let Θ be a minimal L-system of the form (2.2) with a densely defined non-negative symmetric operator $\dot{A}$. Then the impedance function $V_\Theta(z)$ defined by (2.3) belongs to the class S^β if and only if the operator $\mathbb{A}$ of the L-system Θ is β-sectorial, see (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011).

Let $0\leq\beta_1<\frac{\pi}{2}$, $0<\beta_2\leq\frac{\pi}{2}$, and $\beta_1\leq\beta_2$. We say that a scalar Stieltjes function $V(z)$ belongs to the *class* S^{β_1,β_2} if

$$\tan\beta_1=\lim_{x\to-\infty}V(x)\quad\text{and}\quad\tan\beta_2=\lim_{x\to-0}V(x).$$

The following connection between the classes S^β and S^{β_1,β_2} can be found in Arlinskiĭ, Belyi &

Tsekanovskiĭ (2011). Let Θ be an L-system of the form (2.2) with a densely defined non-negative symmetric operator $\dot{A}$ with deficiency numbers $(1,1)$ and let $\mathbb{A}$ be a β-sectorial $(*)$-extension of T, then the impedance function $V_\Theta(z)$, defined by (2.3), belongs to the class S^{β_1,β_2}, $\tan\beta_2 \le \tan\beta$. Moreover, the main operator T is $(\beta_2-\beta_1)$-sectorial with the exact angle of sectoriality $\beta_2-\beta_1$. In the case when β is the exact angle of sectoriality of the operator T we have that $V_\Theta(z) \in S^{0,\beta}$, see (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011). It also follows that, under this set of assumptions, the impedance function $V_\Theta(z)$ is such that $\gamma = 0$ in the representation (3.1).

Now let Θ be an L-system of the form (2.2), where $\mathbb{A}$ is a $(*)$-extension of T and $\dot{A}$ is a closed densely defined non-negative symmetric operator with deficiency numbers $(1,1)$. It was proved in Arlinskiĭ, Belyi & Tsekanovskiĭ (2011) that if the impedance function $V_\Theta(z)$ belongs to the class S^{β_1,β_2} and $\beta_2 \neq \pi/2$, then $\mathbb{A}$ is β-sectorial, where

$$\tan\beta = \tan\beta_2 + 2\sqrt{\tan\beta_1(\tan\beta_2 - \tan\beta_1)}.$$

Let Θ be an L-system satisfying the above conditions. Then the operator $\mathbb{A}$ is a β-sectorial $(*)$-extension of a β-sectorial operator T with the exact angle $\beta \in (0,\pi/2)$ if and only if $V_\Theta(z) \in S^{0,\beta}$, see Arlinskiĭ, Belyi & Tsekanovskiĭ (2011). Moreover, the angle β is determined by the formula

$$\tan\beta = \int_0^\infty \frac{dG(t)}{t},$$

where $G(t)$ is the measure from integral representation (3.1) of $V_\Theta(z)$.

4 L-systems with Schrödinger operator and their impedance functions

Let $\mathcal{H} = L_2[\ell,+\infty)$, $\ell \ge 0$, and let $l(y) = -y'' + q(x)y$, where q is a real locally summable function on $[\ell,+\infty)$. Suppose that the symmetric operator

$$\begin{cases} \dot{A}y = -y'' + q(x)y, \\ y(\ell) = y'(\ell) = 0, \end{cases} \tag{4.1}$$

has deficiency indices (1,1). Let D^* be the set of functions y for which y and y' are locally absolutely continuous, and $l(y) \in L_2[\ell,+\infty)$. Provide the space $\mathcal{H}_+ = \operatorname{dom} \dot{A}^* = D^*$ with the scalar product

$$(y,z)_+ = \int_\ell^\infty \left(y(x)\overline{z(x)} + l(y)\overline{l(z)} \right) dx, \qquad y,z \in D^*.$$

Let $\mathcal{H}_+ \subset L_2[\ell,+\infty) \subset \mathcal{H}_-$ be the corresponding triplet of Hilbert spaces. Consider the operators

$$\begin{cases} T_h y = l(y) = -y'' + q(x)y, \\ hy(\ell) - y'(\ell) = 0, \end{cases} \quad \text{and} \quad \begin{cases} T_h^* y = l(y) = -y'' + q(x)y, \\ \overline{h}y(\ell) - y'(\ell) = 0, \end{cases} \tag{4.2}$$

where $\operatorname{Im} h > 0$. Let $\dot{A}$ be a symmetric operator of the form (4.1) with deficiency indices (1,1) generated by the differential expression $l(y) = -y'' + q(x)y$. Moreover, let $\varphi_k(x,\lambda)$, $k = 1,2$, be the solutions of the following Cauchy problems:

$$\begin{cases} l(\varphi_1) = \lambda\varphi_1, \\ \varphi_1(\ell,\lambda) = 0, \\ \varphi_1'(\ell,\lambda) = 1, \end{cases} \quad \text{and} \quad \begin{cases} l(\varphi_2) = \lambda\varphi_2, \\ \varphi_2(\ell,\lambda) = -1, \\ \varphi_2'(\ell,\lambda) = 0. \end{cases}$$

It is well known that there exists a function $m_\infty(\lambda)$ introduced by H. Weyl (1909; 1910) for which

$$\varphi(x,\lambda) = \varphi_2(x,\lambda) + m_\infty(\lambda)\varphi_1(x,\lambda)$$

belongs to $L_2[\ell,+\infty)$, see (Naimark, 1968; Levitan, 1987). The function $m_\infty(\lambda)$ is not a Herglotz-Nevanlinna function, but $-m_\infty(\lambda)$ and $1/m_\infty(\lambda)$ are.

Now we shall construct an L-system based on a non-self-adjoint Schrödinger operator T_h with $\operatorname{Im} h > 0$. It was shown in Arlinskiĭ & Tsekanovskiĭ (2004) and Arlinskiĭ, Belyi & Tsekanovskiĭ (2011) that the set of all ($*$)-extensions of a non-self-adjoint Schrödinger operator T_h of the form (4.2) in $L_2[\ell,+\infty)$ can be represented in the form

$$\begin{aligned} \mathbb{A}_{\mu,h}\, y &= -y'' + q(x)y - \frac{1}{\mu - h}\,[y'(\ell) - hy(\ell)]\,[\mu\delta(x-\ell) + \delta'(x-\ell)], \\ \mathbb{A}^*_{\mu,h}\, y &= -y'' + q(x)y - \frac{1}{\mu - \overline{h}}\,[y'(\ell) - \overline{h}y(\ell)]\,[\mu\delta(x-\ell) + \delta'(x-\ell)]. \end{aligned} \tag{4.3}$$

Moreover, the formulas (4.3) establish a one-to-one correspondence between the set of all ($*$)-extensions of a Schrödinger operator T_h of the form (4.2) and all $\mu \in [-\infty,+\infty]$. One can easily check that the ($*$)-extension $\mathbb{A}$ in (4.3) of the non-self-adjoint dissipative Schrödinger operator T_h, $\operatorname{Im} h > 0$, of the form (4.2) satisfies the condition

$$\operatorname{Im}\mathbb{A}_{\mu,h} = \frac{\mathbb{A}_{\mu,h} - \mathbb{A}^*_{\mu,h}}{2i} = (\cdot, g_{\mu,h})g_{\mu,h}, \tag{4.4}$$

where

$$g_{\mu,h} = \frac{(\operatorname{Im} h)^{\frac{1}{2}}}{|\mu - h|}\,[\mu\delta(x-\ell) + \delta'(x-\ell)]. \tag{4.5}$$

Here $\delta(x-\ell)$ and $\delta'(x-\ell)$ are the delta-function and its derivative at the point ℓ, respectively. Furthermore,

$$(y, g_{\mu,h}) = \frac{(\operatorname{Im} h)^{\frac{1}{2}}}{|\mu - h|}\,[\mu y(\ell) - y'(\ell)],$$

where $y \in \mathcal{H}_+$, $g \in \mathcal{H}_-$, and $\mathcal{H}_+ \subset L_2[\ell,+\infty) \subset \mathcal{H}_-$ is the triplet of Hilbert spaces discussed above.

It was also shown in Arlinskiĭ, Belyi & Tsekanovskiĭ (2011) that the quasi-kernel $\hat{A}_\xi$ of $\operatorname{Re}\mathbb{A}_{\mu,h}$ is given by

$$\begin{cases} \hat{A}_\xi y = -y'' + q(x)y, \\ y'(\ell) = \xi y(\ell), \end{cases} \quad \text{where} \quad \xi = \frac{\mu \operatorname{Re} h - |h|^2}{\mu - \operatorname{Re} h}. \tag{4.6}$$

Let $E = \mathbb{C}$ and $K_{\mu,h}c = cg_{\mu,h}$, where $c \in \mathbb{C}$. Then it is clear that

$$K^*_{\mu,h} y = (y, g_{\mu,h}), \qquad y \in \mathcal{H}_+, \tag{4.7}$$

and $\operatorname{Im}\mathbb{A}_{\mu,h} = K_{\mu,h}K^*_{\mu,h}$, see (4.4). Therefore the array

$$\Theta_{\mu,h} = \begin{pmatrix} \mathbb{A}_{\mu,h} & K_{\mu,h} & 1 \\ \mathcal{H}_+ \subset L_2[\ell,+\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix} \tag{4.8}$$

is an L-system with the main operator T_h, $\operatorname{Im} h > 0$, of the form (4.2), the state-space operator $\mathbb{A}_{\mu,h}$ of the form (4.3), and with the channel operator $K_{\mu,h}$ of the form (4.7). It was established in Arlinskiĭ & Tsekanovskiĭ (2004) and Arlinskiĭ, Belyi & Tsekanovskiĭ (2011) that the transfer and impedance functions of $\Theta_{\mu,h}$ are

$$W_{\Theta_{\mu,h}}(z) = \frac{\mu - h}{\mu - \overline{h}}\,\frac{m_\infty(z) + \overline{h}}{m_\infty(z) + h}$$

and

$$V_{\Theta_{\mu,h}}(z) = \frac{(m_\infty(z) + \mu)\operatorname{Im} h}{(\mu - \operatorname{Re} h)\, m_\infty(z) + \mu \operatorname{Re} h - |h|^2},$$

respectively. It was shown in Arlinskiĭ, Belyi & Tsekanovskiĭ (2011: Section 10.2) that if the parameters μ and ξ are related as in (4.6), then the two L-systems $\Theta_{\mu,h}$ and $\Theta_{\xi,h}$ of the form (4.8) have the following properties

$$W_{\Theta_{\mu,h}}(z) = -W_{\Theta_{\xi,h}}(z), \qquad V_{\Theta_{\mu,h}}(z) = -\frac{1}{V_{\Theta_{\xi,h}}(z)}, \qquad \text{with} \quad \xi = \frac{\mu \operatorname{Re} h - |h|^2}{\mu - \operatorname{Re} h}.$$

5 Realizations of $-m_\infty(z)$, $1/m_\infty(z)$, and $m_\alpha(z)$

It is known that the original Weyl-Titchmarsh function $m_\infty(z)$ has the property that $-m_\infty(z)$ is a Herglotz-Nevanlinna function, see (Levitan, 1987; Naimark, 1968). The question whether $-m_\infty(z)$ can be realized as the impedance function of a Schrödinger L-system is answered in the following theorem.

Theorem 5.1 (Belyi & Tsekanovskiĭ (2021)). *Let $\dot{A}$ be a symmetric Schrödinger operator of the form* (4.1) *with deficiency indices $(1,1)$ in $\mathcal{H} = L^2[\ell,\infty)$. If $m_\infty(z)$ is the Weyl-Titchmarsh function of $\dot{A}$, then the Herglotz-Nevanlinna function $-m_\infty(z)$ can be realized as the impedance function of a Schrödinger L-system $\Theta_{\mu,h}$ of the form* (4.8) *with $\mu = 0$ and $h = i$.*

Conversely, let $\Theta_{\mu,h}$ be a Schrödinger L-system of the form (4.8) *with the symmetric operator $\dot{A}$ such that $V_{\Theta_{\mu,h}}(z) = -m_\infty(z)$ for all $z \in \mathbb{C}_\pm$ and $\mu \in \mathbb{R} \cup \{\infty\}$. Then the parameters μ and h defining $\Theta_{\mu,h}$ are given by $\mu = 0$ and $h = i$.*

A similar result was proved for the function $1/m_\infty(z)$.

Theorem 5.2 (Belyi & Tsekanovskiĭ (2021)). *Let $\dot{A}$ be a symmetric Schrödinger operator of the form* (4.1) *with deficiency indices $(1,1)$ in $\mathcal{H} = L^2[\ell,\infty)$. If $m_\infty(z)$ is the Weyl-Titchmarsh function of $\dot{A}$, then the Herglotz-Nevanlinna function $1/m_\infty(z)$ can be realized as the impedance function of a Schrödinger L-system $\Theta_{\mu,h}$ of the form* (4.8) *with $\mu = \infty$ and $h = i$.*

Conversely, let $\Theta_{\mu,h}$ be a Schrödinger L-system of the form (4.8) *with the symmetric operator $\dot{A}$ such that $V_{\Theta_{\mu,h}}(z) = \frac{1}{m_\infty(z)}$ for all $z \in \mathbb{C}_\pm$ and $\mu \in \mathbb{R} \cup \{\infty\}$. Then the parameters μ and h defining $\Theta_{\mu,h}$ are given by $\mu = \infty$ and $h = i$.*

We note that both L-systems $\Theta_{0,i}$ and $\Theta_{\infty,i}$, obtained in Theorems 5.1 and 5.2, share the same main operator

$$\begin{cases} T_i\, y = -y'' + q(x)y, \\ y'(\ell) = i\, y(\ell). \end{cases} \tag{5.1}$$

Now we recall the definition of the Weyl-Titchmarsh functions $m_\alpha(z)$. Let $\dot{A}$ be a symmetric operator of the form (4.1) with deficiency indices (1,1) generated by the differential expression $l(y) = -y'' + q(x)y$. Moreover, let $\varphi_\alpha(x,z)$ and $\theta_\alpha(x,z)$ be the solutions of the following Cauchy problems:

$$\begin{cases} l(\varphi_\alpha) = z\varphi_\alpha, \\ \varphi_\alpha(\ell,z) = \sin\alpha, \\ \varphi'_\alpha(\ell,z) = -\cos\alpha, \end{cases} \quad \text{and} \quad \begin{cases} l(\theta_\alpha) = z\theta_\alpha, \\ \theta_\alpha(\ell,z) = \cos\alpha, \\ \theta'_\alpha(\ell,z) = \sin\alpha. \end{cases}$$

Then it is known that there exists a function $m_\alpha(z)$, analytic in $\mathbb{C}_\pm$, for which

$$\psi(x,z) = \theta_\alpha(x,z) + m_\alpha(z)\varphi_\alpha(x,z) \tag{5.2}$$

belongs to $L_2[\ell,+\infty)$, see (Danielyan, 1990; Naimark, 1968; Titchmarsh, 1962). It is easy to see that if $\alpha = \pi$, then $m_\pi(z) = m_\infty(z)$. The functions $m_\alpha(z)$ and $m_\infty(z)$ are connected by

$$m_\alpha(z) = \frac{\sin\alpha + m_\infty(z)\cos\alpha}{\cos\alpha - m_\infty(z)\sin\alpha}, \tag{5.3}$$

see (Danielyan, 1990; Titchmarsh, 1962). We know that for any real α the function $-m_\alpha(z)$ is a Herglotz-Nevanlinna function, see (Naimark, 1968; Titchmarsh, 1962). Also, modifying (5.3) slightly, we obtain

$$-m_\alpha(z) = \frac{\sin\alpha + m_\infty(z)\cos\alpha}{-\cos\alpha + m_\infty(z)\sin\alpha} = \frac{\cos\alpha + \frac{1}{m_\infty(z)}\sin\alpha}{\sin\alpha - \frac{1}{m_\infty(z)}\cos\alpha}. \tag{5.4}$$

The following realization theorem for the Herglotz-Nevanlinna functions $-m_\alpha(z)$ is similar to Theorem 5.1.

Theorem 5.3 (Belyi & Tsekanovskiĭ (2021))**.** *Let $\dot{A}$ be a symmetric Schrödinger operator of the form* (4.1) *with deficiency indices $(1,1)$ in $\mathcal{H} = L^2[\ell,\infty)$. If $m_\alpha(z)$ is the function of $\dot{A}$ described in* (5.2)*, then the Herglotz-Nevanlinna function $-m_\alpha(z)$ can be realized as the impedance function of a Schrödinger L-system $\Theta_{\mu,h}$ of the form* (4.8) *with*

$$\mu = \tan\alpha \quad \textit{and} \quad h = i. \tag{5.5}$$

Conversely, let $\Theta_{\mu,h}$ be a Schrödinger L-system of the form (4.8) *with the symmetric operator $\dot{A}$ such that*

$$V_{\Theta_{\mu,h}}(z) = -m_\alpha(z),$$

for all $z \in \mathbb{C}_\pm$ and $\mu \in \mathbb{R} \cup \{\infty\}$. Then the parameters μ and h defining $\Theta_{\mu,h}$ are given by (5.5).

In case $\alpha = \pi$ we obtain $\mu_\alpha = 0$, $m_\pi(z) = m_\infty(z)$, and the realizing Schrödinger L-system $\Theta_{0,i}$ is described in Belyi & Tsekanovskiĭ (2021: Section 5). In case $\alpha = \pi/2$, then we obtain $\mu_\alpha = \infty$, $-m_\alpha(z) = 1/m_\infty(z)$, and the realizing Schrödinger L-system is $\Theta_{\infty,i}$, see (Belyi & Tsekanovskiĭ, 2021: Section 5). Assuming that $\alpha \in (0,\pi]$ and neither $\alpha = \pi$ nor $\alpha = \pi/2$, we give the description of a Schrödinger L-system $\Theta_{\mu_\alpha,i}$ realizing $-m_\alpha(z)$ as follows:

$$\Theta_{\tan\alpha,i} = \begin{pmatrix} \mathbb{A}_{\tan\alpha,i} & K_{\tan\alpha,i} & 1 \\ \mathcal{H}_+ \subset L_2[\ell,+\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}, \tag{5.6}$$

where

$$\begin{aligned} \mathbb{A}_{\tan\alpha,i}\, y &= l(y) - \frac{1}{\tan\alpha - i}[y'(\ell) - iy(\ell)][(\tan\alpha)\delta(x-\ell) + \delta'(x-\ell)], \\ \mathbb{A}^*_{\tan\alpha,i}\, y &= l(y) - \frac{1}{\tan\alpha + i}\,[y'(\ell) + iy(\ell)][(\tan\alpha)\delta(x-\ell) + \delta'(x-\ell)], \end{aligned} \tag{5.7}$$

and $K_{\tan\alpha,i}\, c = c\, g_{\tan\alpha,i}$, $c \in \mathbb{C}$, with

$$g_{\tan\alpha,i} = (\tan\alpha)\delta(x-\ell) + \delta'(x-\ell).$$

Also,

$$\begin{aligned} V_{\Theta_{\tan\alpha,i}}(z) &= -m_\alpha(z), \\ W_{\Theta_{\tan\alpha,i}}(z) &= \frac{\tan\alpha - i}{\tan\alpha + i} \cdot \frac{m_\infty(z) - i}{m_\infty(z) + i} = (-e^{2\alpha i})\,\frac{m_\infty(z) - i}{m_\infty(z) + i}. \end{aligned}$$

The realization theorem for the Herglotz-Nevanlinna function $1/m_\alpha(z)$ is similar to Theorem 5.2 and can be found in Belyi & Tsekanovskiĭ (2021).

6 Non-negative Schrödinger operators and sectorial L-systems

Now let us assume that $\dot{A}$ is a densely defined non-negative symmetric operator of the form (4.1) with deficiency indices (1,1) generated by the differential expression $l(y) = -y'' + q(x)y$.

Theorem 6.1 (Tsekanovskiĭ (1980; 1981; 1987))**.** *Let $\dot{A}$ be a nonnegative symmetric Schrödinger operator of the form* (4.1) *with deficiency indices* $(1,1)$ *in* $\mathcal{H} = L^2[\ell,\infty)$ *and let the operator* T_h *be given by* (4.2)*. Then the following statements hold:*

(1) *the operator $\dot{A}$ has more than one non-negative self-adjoint extension, i.e., the Friedrichs extension A_F and the Kreĭn-von Neumann extension A_K do not coincide, if and only if $m_\infty(-0) < \infty$;*

(2) *the operator T_h, $h = \bar{h}$, coincides with the Kreĭn-von Neumann extension A_K if and only if $h = -m_\infty(-0)$;*

(3) *the operator T_h is accretive if and only if* $\operatorname{Re} h \geq -m_\infty(-0)$*;*

(4) *the operator T_h, $h \neq \bar{h}$, is β-sectorial if and only if* $\operatorname{Re} h > -m_\infty(-0)$*;*

(5) *the operator T_h, $h \neq \bar{h}$, is accretive, but not β-sectorial for any $\beta \in (0, \frac{\pi}{2})$ if and only if* $\operatorname{Re} h = -m_\infty(-0)$*;*

(6) *if the operator T_h,* $\operatorname{Im} h > 0$*, is β-sectorial, then the exact angle β can be calculated via*

$$\tan\beta = \frac{\operatorname{Im} h}{\operatorname{Re} h + m_\infty(-0)}. \tag{6.1}$$

For the remainder of this paper we assume that $m_\infty(-0) < \infty$. Then, according to Theorem 6.1 above, the operator T_h, $\operatorname{Im} h > 0$, is accretive and/or sectorial, see also (Arlinskiĭ & Tsekanovskiĭ, 2009; Tsekanovskiĭ, 1980; 1992). It was shown in Arlinskiĭ, Belyi & Tsekanovskiĭ (2011) that if T_h, $\operatorname{Im} h > 0$, is an accretive Schrödinger operator of the form (4.2), then for all real μ satisfying the following inequality

$$\mu \geq \frac{(\operatorname{Im} h)^2}{m_\infty(-0) + \operatorname{Re} h} + \operatorname{Re} h, \tag{6.2}$$

the formulas (4.3) define the set of all accretive $(*)$-extensions $\mathbb{A}_{\mu,h}$ of the operator T_h. Moreover, an accretive $(*)$-extension $\mathbb{A}_{\mu,h}$ of a sectorial operator T_h with exact angle of sectoriality $\beta \in (0, \pi/2)$ also preserves the same exact angle of sectoriality if and only if $\mu = +\infty$ in (4.3), see (Belyi & Tsekanovskiĭ, 2019: Theorem 3). Also, $\mathbb{A}_{\mu,h}$ is an accretive $(*)$-extension of T_h that is not β-sectorial for any $\beta \in (0, \pi/2)$ if and only if in (4.3)

$$\mu = \frac{(\operatorname{Im} h)^2}{m_\infty(-0) + \operatorname{Re} h} + \operatorname{Re} h, \tag{6.3}$$

see (Belyi & Tsekanovskiĭ, 2019: Theorem 4). An accretive operator T_h has a unique accretive $(*)$-extension $\mathbb{A}_{\infty,h}$ if and only if $\operatorname{Re} h = -m_\infty(-0)$. Then this unique $(*)$-extension has the form

$$\begin{aligned} \mathbb{A}_{\infty,h} y &= -y'' + q(x)y + [hy(\ell) - y'(\ell)]\,\delta(x-\ell), \\ \mathbb{A}^*_{\infty,h} y &= -y'' + q(x)y + [\bar{h}y(\ell) - y'(\ell)]\,\delta(x-\ell). \end{aligned} \tag{6.4}$$

Now consider the functions $m_\alpha(z)$ described by (5.2)-(5.3) and associated with the non-negative operator $\dot{A}$ above. The parameter α in the definition of $m_\alpha(z)$ affects the L-system realizing $-m_\alpha(z)$ as follows: if the non-negative symmetric Schrödinger operator satisfies $m_\infty(-0) \geq 0$, then the L-system $\Theta_{\tan\alpha,i}$ of the form (5.6) realizing the function $-m_\alpha(z)$ is accretive if and only if

$$\tan\alpha \geq (m_\infty(-0))^{-1}, \tag{6.5}$$

see (Belyi & Tsekanovskiĭ, 2021: Theorem 6.3). Note that if $m_\infty(-0) = 0$ in (6.5), then $\alpha = \pi/2$ and $-m_{\frac{\pi}{2}}(z) = 1/m_\infty(z)$. Also, from Belyi & Tsekanovskiĭ (2021: Theorem 6.2) we know that if $m_\infty(-0) \geq 0$, then $1/m_\infty(z)$ is realized by an accretive system $\Theta_{\infty,i}$.

Having established criteria for an L-system realizing $-m_\alpha(z)$ to be accretive, we can look into more of its properties. There are two choices for an accretive L-system $\Theta_{\tan\alpha,i}$: it is either (1) *accretive sectorial* or (2) *accretive extremal*. In the case (1) the operator $\mathbb{A}_{\tan\alpha,i}$ of the form (5.7) is β_1-sectorial with some angle of sectoriality β_1 that can only exceed the exact angle of sectoriality β of T_i. In the case (2) the state-space operator $\mathbb{A}_{\tan\alpha,i}$ is extremal (not sectorial for any $\beta \in (0, \pi/2)$) and is a $(*)$-extension of T_i that itself can be either β-sectorial or extremal. These possibilities were described in detail in Belyi & Tsekanovskiĭ (2021: Theorem 6.4). In particular, it was shown that for the accretive L-system $\Theta_{\tan\alpha,i}$ realizing the function $-m_\alpha(z)$ the following is true:

(1) If $m_\infty(-0) = 0$, then there exists only one accretive L-system $\Theta_{\infty,i}$ realizing $-m_\alpha(z)$. This L-system is extremal and its main operator T_i is extremal as well.

(2) If $m_\infty(-0) > 0$, then T_i is β-sectorial for $\beta \in (0, \pi/2)$ and

   (a) if $\tan\alpha = 1/m_\infty(-0)$, then $\Theta_{\tan\alpha,i}$ is extremal;

   (b) if $\frac{1}{m_\infty(-0)} < \tan\alpha < +\infty$, then $\Theta_{\tan\alpha,i}$ is β_1-sectorial with $\beta_1 > \beta$;

   (c) if $\tan\alpha = +\infty$, then $\Theta_{\infty,i}$ is β-sectorial.

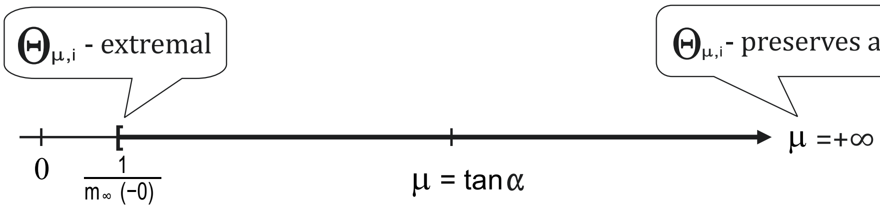

Figure 1. Accretive L-systems $\Theta_{\mu,i}$.

Figure 1 above describes the dependence of the properties of L-systems realizing $-m_\alpha(z)$ on the value of μ and, hence, on α. The bold part of the real line depicts values of $\mu = \tan\alpha$ that produce accretive L-systems $\Theta_{\mu,i}$.

Additional analytic properties of the functions $-m_\infty(z)$, $1/m_\infty(z)$, and $-m_\alpha(z)$ were described in Belyi & Tsekanovskiĭ (2021: Theorem 6.5). It was proved there that under the current set of assumptions we have:

(1) The function $1/m_\infty(z)$ is a Stieltjes function if and only if $m_\infty(-0) \geq 0$.

(2) The function $-m_\infty(z)$ is never a Stieltjes function.[†]

(3) The function $-m_\alpha(z)$ given by (5.3) is a Stieltjes function if and only if

$$0 < \frac{1}{m_\infty(-0)} \leq \tan\alpha.$$

As the case that the realizing L-system $\Theta_{\tan\alpha,i}$ is accretive maximal does not require any further elaboration, we will now restrict ourselves to the case when it is accretive sectorial. To begin with, let Θ be an L-system of the form (4.8), where $\mathbb{A}$ is a $(*)$-extension (4.3) of the accretive Schrödinger operator T_h. Here we summarize and list some known facts about possible accretivity and sectoriality of Θ:

- The operator $\mathbb{A}_{\mu,h}$ of $\Theta_{\mu,h}$ is accretive if and only if (6.2) holds, see (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011).
- If an accretive operator T_h, $\operatorname{Im} h > 0$, is β-sectorial, then (6.1) holds, see Theorem 5.1. Conversely, if $\operatorname{Im} h > 0$ and $\operatorname{Re} h > -m_\infty(-0)$, then the operator T_h of the form (4.2) is β-sectorial and β is given by (6.1).
- The operator T_h is accretive but not β-sectorial for any $\beta \in (0, \pi/2)$ if and only if the equality $\operatorname{Re} h = -m_\infty(-0)$ holds.
- If $\Theta_{\mu,h}$ is such that $\mu = +\infty$, then $V_{\Theta_{\infty,h}}(z)$ belongs to the class $S^{0,\beta}$. In the case when $\mu \neq +\infty$ we have $V_{\Theta_{\mu,h}}(z) \in S^{\beta_1,\beta_2}$, see (Belyi, 2011).
- The operator $\mathbb{A}_{\mu,h}$ is a β-sectorial $(*)$-extension of the operator T_h (with the same angle of sectoriality) if and only if $\mu = +\infty$ in (4.3), see (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011; Belyi & Tsekanovskiĭ, 2019).

[†]It will be shown in an forthcoming paper that if $m_\infty(-0) \geq 0$, then the function $-m_\infty(z)$ is actually an inverse Stieltjes function.

- If the operator T_h is β-sectorial with the exact angle of sectoriality β, then it admits only one β-sectorial $(*)$-extension $\mathbb{A}_{\mu,h}$ with the same angle of sectoriality β. Consequently, $\mu = +\infty$ and $\mathbb{A}_{\mu,h} = \mathbb{A}_{\infty,h}$ has the form (6.4).
- A $(*)$-extension $\mathbb{A}_{\mu,h}$ of the operator T_h is accretive but not β-sectorial for any $\beta \in (0, \pi/2)$ if and only if the value of μ in (4.3) is defined by (6.3).

Note that it follows from the above that any β-sectorial operator T_h with the exact angle of sectoriality $\beta \in (0, \pi/2)$ admits only one accretive $(*)$-extension $\mathbb{A}_{\mu,h}$ that is not β-sectorial for any $\beta \in (0, \pi/2)$. This extension takes the form (4.3) with μ given by (6.3).

Now let us consider a function $-m_\alpha(z)$ and a Schrödinger L-system $\Theta_{\tan\alpha,i}$ of the form (5.6) that realizes it. According to Belyi & Tsekanovskiĭ (2021: Theorem 6.4 & Theorem 6.5) this L-system $\Theta_{\tan\alpha,i}$ is sectorial if and only if

$$\tan\alpha > \frac{1}{m_\infty(-0)}. \tag{6.6}$$

Furthermore, if the L-system $\Theta_{\tan\alpha,i}$ is assumed to be β-sectorial, then its impedance function $V_{\Theta_{\tan\alpha,i}}(z) = -m_\alpha(z)$ belongs to S^β. The following theorem provides more refined properties of $-m_\alpha(z)$ for this case.

Theorem 6.2. *Let $\Theta_{\tan\alpha,i}$ be the accretive L-system of the form* (5.6) *realizing the function $-m_\alpha(z)$ associated with the non-negative operator $\dot{A}$, and let $\mathbb{A}_{\tan\alpha,i}$ be a β-sectorial $(*)$-extension of T_i defined by* (5.1). *Then the function $-m_\alpha(z)$ belongs to the class S^{β_1,β_2}, where* $\tan\beta_2 \le \tan\beta$, *and*

$$\tan\beta_1 = \cot\alpha \quad \textit{and} \quad \tan\beta_2 = \frac{\tan\alpha + m_\infty(-0)}{(\tan\alpha)m_\infty(-0) - 1}. \tag{6.7}$$

Moreover, the operator T_i is $(\beta_2 - \beta_1)$-sectorial with the exact angle of sectoriality $\beta_2 - \beta_1$.

Proof. First recall from (5.3) that

$$-m_\alpha(z) = \frac{\sin\alpha + m_\infty(z)\cos\alpha}{-\cos\alpha + m_\infty(z)\sin\alpha} = \frac{\tan\alpha + m_\infty(z)}{(\tan\alpha)m_\infty(z) - 1}. \tag{6.8}$$

Since $\Theta_{\tan\alpha,i}$ is β-sectorial, (6.6) holds and its impedance function $V_{\Theta_{\tan\alpha,i}}(z) = -m_\alpha(z)$ belongs to S^β and to S^{β_1,β_2}. In order to prove (6.7), pass to the limits in (6.8)

$$\tan\beta_1 = \lim_{x\to-\infty}(-m_\alpha(x)) = \lim_{x\to-\infty}\frac{\frac{\tan\alpha}{m_\infty(x)} + 1}{\tan\alpha - \frac{1}{m_\infty(x)}} = \cot\alpha,$$

where we used $\lim_{x\to-\infty} m_\infty(x) = +\infty$, see (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011: Section 10.3), and

$$\tan\beta_2 = \lim_{x\to-0}(-m_\alpha(x)) = \frac{\tan\alpha + m_\infty(-0)}{(\tan\alpha)m_\infty(-0) - 1}.$$

In order to show the rest, we apply (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011: Theorem 9.8.4): if $\mathbb{A}$ is a β-sectorial $(*)$-extension of a main operator T of an L-system Θ, then the impedance function $V_\Theta(z)$ belongs to the class S^{β_1,β_2}, $\tan\beta_2 \le \tan\beta$, and T is $(\beta_2-\beta_1)$-sectorial with the exact angle of sectoriality $\beta_2 - \beta_1$. It can also be checked directly that (6.7) (under condition (6.6)) implies $0 < \beta_2 - \beta_1 < \pi/2$ and, hence, the definition of $(\beta_2 - \beta_1)$-sectoriality applies correctly. $\square$

Now we state and prove the following.

Theorem 6.3. *Let $\Theta_{\tan\alpha,i}$ be an accretive L-system of the form* (5.6) *that realizes $-m_\alpha(z)$, where $\mathbb{A}_{\tan\alpha,i}$ is a $(*)$-extension of a θ-sectorial operator T_i with exact angle of sectoriality θ. Moreover, let $\alpha_* \in \left(\arctan\left(\frac{1}{m_\infty(-0)}\right), \frac{\pi}{2}\right)$ be a fixed value that defines $\mathbb{A}_{\tan\alpha_*,i}$ via* (4.3)*, and assume that $-m_\alpha(z) \in S^{\beta_1,\beta_2}$. Then a $(*)$-extension $\mathbb{A}_{\tan\alpha,i}$ of T_i is β-sectorial for any $\alpha \in [\alpha_*, \pi/2)$ with*

$$\tan\beta = \tan\beta_1 + 2\sqrt{\tan\beta_1\,\tan\beta_2}, \qquad \tan\beta > \tan\theta. \tag{6.9}$$

Moreover, if $\alpha = \pi/2$, then

$$\beta = \beta_2 - \beta_1 = \theta = \arctan\left(\frac{1}{m_\infty(-0)}\right).$$

Proof. First note that the conditions imply that $\tan\alpha_* \in \left(\frac{1}{m_\infty(-0)}, +\infty\right)$. Thus, according to Belyi & Tsekanovskiĭ (2021: Theorem 6.4; part 2(c)) a $(*)$-extension $\mathbb{A}_{\tan\alpha,i}$ is β-sectorial for some $\beta \in (0, \pi/2)$. Then we can apply Theorem 6.2 to confirm that $-m_\alpha(z) \in S^{\beta_1,\beta_2}$, where β_1 and β_2 are described by (6.7). The equality in (6.9) follows from Belyi & Tsekanovskiĭ (2019: Theorem 8) applied to the L-system $\Theta_{\tan\alpha,i}$ with $\mu = \tan\alpha$, see also (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011: Theorem 9.8.7). Since $\mathbb{A}_{\tan\alpha,i}$ is a β-sectorial extension of a θ-sectorial operator T_i, we have $\tan\beta \geq \tan\theta$ with equality possible only when $\mu = \tan\alpha = \infty$, see (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011; Belyi & Tsekanovskiĭ, 2019). Since we chose $\alpha \in [\alpha_*, \pi/2)$, it follows that $\tan\alpha \neq \infty$ and, hence, $\tan\beta > \tan\theta$, which confirms the second part of (6.9).

If we assume that $\alpha = \pi/2$, then $-m_\alpha(z) = 1/m_\infty(z)$ is realized by the L-system $\Theta_{\infty,i}$ (see Theorem 5.2) that preserves the angle of sectoriality of its main operator T_i, see (Belyi & Tsekanovskiĭ, 2021: Theorem 6.4) and Figure 1. Therefore, $\beta = \theta$. If we combine this fact with $-m_\alpha(z) \in S^{\beta_1,\beta_2}$ and apply Theorem 6.2 we also obtain that $\beta = \beta_2 - \beta_1$. Finally, since T_i is θ-sectorial, (6.1) yields $\tan\theta = \frac{1}{m_\infty(-0)}$. □

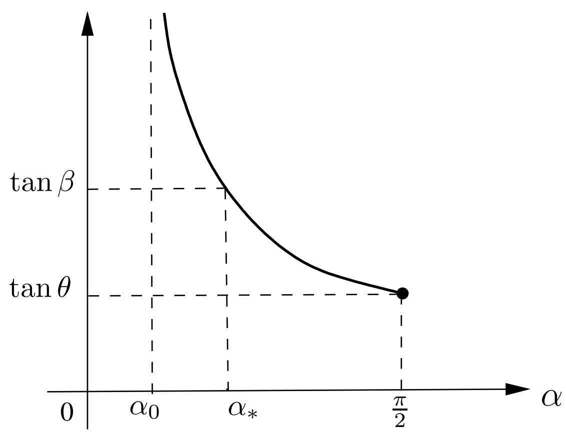

Figure 2. Angle of sectoriality β. Here $\alpha_0 = \arctan\left(\frac{1}{m_\infty(-0)}\right)$.

Note that Theorem 6.3 provides us with a value β which serves as a universal angle of sectoriality for the entire indexed family of $(*)$-extensions $\mathbb{A}_{\tan\alpha,i}$ of the form (5.6) as depicted in Figure 2. That figure clearly shows that if $\alpha = \pi/2$, then $\tan\beta = \tan\theta$.

7 Example

We conclude this paper with a simple illustration. Consider the differential expression with the Bessel potential

$$l_\nu = -\frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{x^2}, \qquad x \in [1, \infty),$$

in the Hilbert space $\mathcal{H} = L^2[1, \infty)$ and assume that $\nu > 0$. The minimal symmetric operator $\dot{A}$ in

$$\begin{cases} \dot{A}\, y = -y'' + \frac{\nu^2 - 1/4}{x^2} y, \\ y(1) = y'(1) = 0, \end{cases}$$

has defect numbers $(1, 1)$. Let $\nu = 3/2$. It is known that in this case

$$m_\infty(z) = -\frac{iz - \frac{3}{2}\sqrt{z} - \frac{3}{2}i}{\sqrt{z} + i} - \frac{1}{2} = \frac{\sqrt{z} - iz + i}{\sqrt{z} + i} = 1 - \frac{iz}{\sqrt{z} + i}$$

and $m_\infty(-0) = 1$, see (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011). The minimal symmetric operator then becomes

$$\begin{cases} \dot{A}\, y = -y'' + \frac{2}{x^2} y, \\ y(1) = y'(1) = 0. \end{cases} \tag{7.1}$$

The main operator T_h of the form (4.2) is written for $h = i$ as

$$\begin{cases} T_i\, y = -y'' + \frac{2}{x^2} y, \\ y'(1) = i\, y(1). \end{cases} \tag{7.2}$$

It will be shared by the whole family of L-systems realizing the functions $-m_\alpha(z)$ described by (5.2)-(5.3). This operator is accretive and β-sectorial due to $\mathrm{Re}\, h = 0 > -m_\infty(-0) = -1$. Its exact angle of sectoriality is given by

$$\tan\beta = \frac{\mathrm{Im}\, h}{\mathrm{Re}\, h + m_\infty(-0)} = \frac{1}{0+1} = 1 \quad \text{or} \quad \beta = \frac{\pi}{4},$$

see (6.1). A family of L-systems $\Theta_{\tan\alpha, i}$ of the form (5.6) that realizes $-m_\alpha(z)$ described by (5.2)–(5.4) as

$$-m_\alpha(z) = \frac{(\sqrt{z} - iz + i)\cos\alpha + (\sqrt{z} + i)\sin\alpha}{(\sqrt{z} - iz + i)\sin\alpha - (\sqrt{z} + i)\cos\alpha},$$

was constructed in Belyi & Tsekanovskiĭ (2021). According to Belyi & Tsekanovskiĭ (2021: Theorem 6.3) the L-systems $\Theta_{\tan\alpha, i}$ in (5.6) are accretive if

$$1 = \frac{1}{m_\infty(-0)} \le \tan\alpha < +\infty.$$

Using Belyi & Tsekanovskiĭ (2021: Theorem 6.4; part (2c)), we get that the realizing L-system $\Theta_{\tan\alpha, i}$ in (5.6) preserves the angle of sectoriality and becomes $\frac{\pi}{4}$-sectorial if $\mu = \tan\alpha = +\infty$ or $\alpha = \pi/2$. Therefore the L-system

$$\Theta_{\infty, i} = \begin{pmatrix} \mathbb{A}_{\infty, i} & K_{\infty, i} & 1 \\ \mathcal{H}_+ \subset L_2[1, +\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix},$$

where

$$\mathbb{A}_{\infty,i}\, y = -y'' + \frac{2}{x^2}y - \, [y'(1) - iy(1)]\, \delta(x-1),$$
$$\mathbb{A}^*_{\infty,i}\, y = -y'' + \frac{2}{x^2}y - \, [y'(1) + iy(1)]\, \delta(x-1),$$

and $K_{\infty,i}c = cg_{\infty,i}$, $c \in \mathbb{C}$, with $g_{\infty,i} = \delta(x-1)$, realizes $-m_{\frac{\pi}{2}}(z) = 1/m_\infty(z)$. Furthermore,

$$\begin{aligned} V_{\Theta_{\infty,i}}(z) &= -m_{\frac{\pi}{2}}(z) = \frac{1}{m_\infty(z)} = \frac{\sqrt{z}+i}{\sqrt{z}-iz+i}, \\ W_{\Theta_{\infty,i}}(z) &= (-e^{\pi i})\,\frac{m_\infty(z)-i}{m_\infty(z)+i} = \frac{(1-i)\sqrt{z}-iz+1+i}{(1+i)\sqrt{z}-iz-1+i}. \end{aligned} \tag{7.3}$$

This L-system $\Theta_{\infty,i}$ is clearly accretive according to Belyi & Tsekanovskiĭ (2021: Theorem 6.2), which can also be independently confirmed by direct evaluation

$$(\mathrm{Re}\,\mathbb{A}_{\infty,i}\, y, y) = \|y'(x)\|^2_{L^2} + 2\|y(x)/x\|^2_{L^2} \geq 0.$$

Moreover, according to Belyi & Tsekanovskiĭ (2021: Theorem 6.4), see also (Arlinskiĭ, Belyi & Tsekanovskiĭ, 2011: Theorem 9.8.7), the L-system $\Theta_{\infty,i}$ is $\frac{\pi}{4}$-sectorial. Taking into account that $(\mathrm{Im}\,\mathbb{A}_{\infty,i}\, y, y) = |y(1)|^2$, see formula (4.5), we obtain inequality (2.1) with $\beta = \frac{\pi}{4}$, that is $(\mathrm{Re}\,\mathbb{A}_{\infty,i}\, y, y) \geq |(\mathrm{Im}\,\mathbb{A}_{\infty,i}\, y, y)|$, or

$$\|y'(x)\|^2_{L^2} + 2\|y(x)/x\|^2_{L^2} \geq |y(1)|^2.$$

In addition, we have shown that the β-sectorial form $(T_i y, y)$ defined on $\mathrm{dom}\; T_i$ can be extended to the β-sectorial form $(\mathbb{A}_{\infty,i}\, y, y)$ defined on $\mathcal{H}_+ = \mathrm{dom}\; \dot{A}^*$, see (7.1)–(7.2), having the exact angle of sectoriality $\beta = \pi/4$ (i.e., exact for both forms). A general problem of extending sectorial sesquilinear forms to sectorial ones was mentioned by T. Kato (1966). It can be easily seen that the function $-m_{\frac{\pi}{2}}(z)$ in (7.3) belongs to the sectorial class $S^{0,\frac{\pi}{4}}$ of Stieltjes functions.

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$\mathcal{PT}$-SYMMETRIC HAMILTONIANS AS COUPLINGS OF DUAL PAIRS

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Dedicated to our friend and colleague Seppo Hassi on the occasion of his 60th birthday

1 Introduction

In the seminal paper (Bender & Boettcher, 1998) a new view of quantum mechanics was proposed. This new view differs from the old one in that the restriction on the Hamiltonian to be Hermitian is relaxed: now the Hamiltonian is $\mathcal{PT}$-symmetric. Here $\mathcal{P}$ is parity and $\mathcal{T}$ is time reversal. Since 1998, $\mathcal{PT}$-symmetric Hamiltonians have been analyzed intensively by many authors. In Mostafazadeh (2002) $\mathcal{PT}$-symmetry was embedded into the more general mathematical framework of pseudo-Hermiticity or, what is the same, self-adjoint operators in Kreĭn spaces, see (Langer & Tretter, 2004; Azizov & Trunk, 2012; Hassi & Kuzhel, 2013; Leben & Trunk, 2019). For a general introduction to $\mathcal{PT}$-symmetric quantum mechanics we refer to the overview paper of Mostafazadeh (2010) and to the books of Moiseyev (2011) and Bender (2019).

A prominent class consists of the $\mathcal{PT}$-symmetric Hamiltonians

$$H := \frac{1}{2}p^2 - (iz)^{N+2},$$

where N is a positive integer, see (Bender, Brody & Jones, 2002). The associated eigenvalue problem is defined on a contour Γ in the complex plane which is contained in a specific area in the complex plane, the so-called Stokes wedges, see (Bender & Boettcher, 1998),

$$-y''(z) - (iz)^{N+2}y(z) = \lambda y(z), \quad z \in \Gamma, \tag{1.1}$$

where $\lambda \in \mathbb{C}$ is the eigenvalue parameter. Recall that a *Stokes wedge* S_k, $k = 0, \dots, N+3$, is an open sector in the plane with vertex zero,

$$S_k := \left\{ z \in \mathbb{C} : -\frac{N+2}{2N+8}\pi + \frac{2k-2}{4+N}\pi < \arg(z) < -\frac{N+2}{2N+8}\pi + \frac{2k}{4+N}\pi \right\},$$

see (Bender et al., 2006). The boundary of S_k consists of two rays from the origin, the so-called *Stokes lines*. $\mathcal{PT}$-symmetry forces Γ to lie in two Stokes wedges, which are symmetric with respect to the imaginary axis.

In Mostafazadeh (2005) the contour Γ in equation (1.1) was parameterized by a real parameter. In Bender et al. (2006) and in Jones & Mateo (2006) this approach was extended to different parameterizations and contours. Here we choose, for simplicity, Γ to be a wedge-shaped contour,

$$\Gamma := \{xe^{i\phi \operatorname{sgn} x} : x \in \mathbb{R}\}, \tag{1.2}$$

for some angle $\phi \in (-\pi/2, \pi/2)$, see Figure 1.

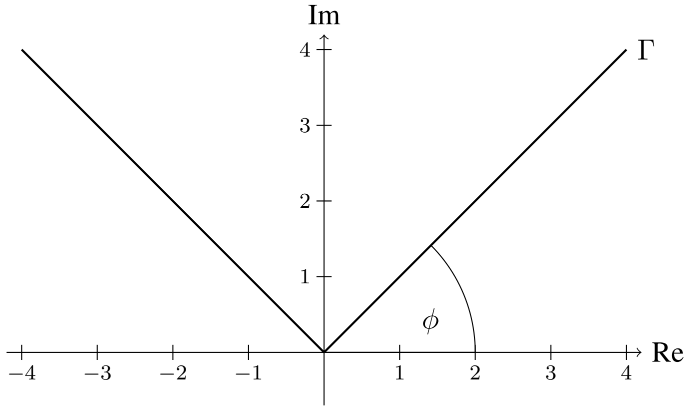

Figure 1. The complex contour Γ.

Let $z : \mathbb{R} \to \mathbb{C}$ parameterize Γ via $z(x) := xe^{i\phi \operatorname{sgn} x}$. Then y solves (1.1) for $z \neq 0$ if and only if the pair of functions u_+ and u_-, given by $u_\pm(x) := y(z(x))$, $x \in \mathbb{R}_\pm$, solves

$$\mathfrak{a}_-[u_-] = \lambda u_-, \quad x \in \mathbb{R}_-, \qquad \mathfrak{a}_+[u_+] = \lambda u_+, \quad x \in \mathbb{R}_+, \tag{1.3}$$

where the differential expressions $\mathfrak{a}_\pm$ are given by

$$\mathfrak{a}_\pm[u_+] = -e^{\mp 2i\phi} u_\pm'' - (ix)^{N+2} e^{\pm i(N+2)\phi} u_\pm. \tag{1.4}$$

In what follows we assume that Γ lies in Stokes wedges and then, by Leben & Trunk (2019), the differential expressions $\mathfrak{a}_\pm$ are in the limit-point case at $\pm\infty$ according to the classification in Brown et al. (1999), which is a refinement of the classification in Sims (1957). We mention, that the limit-circle case can be treated in a similar way as in Azizov & Trunk (2010; 2012).

The theory of $\mathcal{PT}$-symmetry claims that the main object, the Hamiltonian, commutes under the joint action of the parity $\mathcal{P}$ and the time reversal $\mathcal{T}$,

$$(\mathcal{P}f)(x) := f(-x), \qquad (\mathcal{T}f)(x) := \overline{f(x)}. \tag{1.5}$$

The time reversal $\mathcal{T}$ applied to the differential expressions $\mathfrak{a}_\pm$ gives rise to new differential expressions $\mathfrak{b}_\pm = \mathcal{T}\mathfrak{a}_\pm\mathcal{T}$ defined on $\mathbb{R}_\pm$

$$\mathfrak{b}_\pm[v_\pm] = -e^{\pm 2i\phi} v_\pm'' - (-ix)^{N+2} e^{\mp i(N+2)\phi} v_\pm. \tag{1.6}$$

In Section 3 we introduce the minimal operators $A_\pm$ and $B_\pm$ associated with $\mathfrak{a}_\pm$ and $\mathfrak{b}_\pm$ in $L^2(\mathbb{R}_\pm)$ and show that

$$\langle A_\pm f, g\rangle_\pm = \langle f, B_\pm g\rangle_\pm, \qquad \text{for all } f \in \operatorname{dom} A_\pm,\ g \in \operatorname{dom} B_\pm. \tag{1.7}$$

Here $\langle\cdot,\cdot\rangle_\pm$ stands for the usual inner products in the Hilbert spaces $L^2(\mathbb{R}_\pm)$. Condition (1.7) shows that the pairs (A_+, B_+) and (A_-, B_-) form dual pairs, see Section 2.1 for details. An extension theory for dual pairs based on the boundary triple technique was developed by Malamud & Mogilevskiĭ (2002). This is a generalization of the boundary triple approach to the extension theory of symmetric operators which was developed by Calkin (1939); Kočhubeĭ (1975); Gorbachuk & Gorbachuk (1991); Derkach & Malamud (1991), and others. For recent developments of the method of boundary triples and its application to the extension theory of differential operators, see the monographs by Derkach & Malamud (2017) and by Behrndt, Hassi & de Snoo (2020).

Following this approach, we construct in Theorem 3.1 boundary triples for dual pairs (A_+, B_+) and (A_-, B_-). As our interest is focused on the Hamiltonian in $L^2(\mathbb{R})$ and not on the differential expressions $\mathfrak{a}_\pm$ and $\mathfrak{b}_\pm$, which are defined on the semi-axes, we extend the coupling method for symmetric operators from Derkach et al. (2000) to the case of dual pairs and create a new dual pair (A, B) of operators defined on $\mathbb{R}$. This dual pair (A, B) is called the coupling of the dual pairs (A_+, B_+) and (A_-, B_-), see Theorem 2.5 and Definition 2.6 below.

We show that the operator $\mathcal{PT}$ intertwines the dual pairs (A_+, B_+) and (A_-, B_-), i.e.,

$$\mathcal{PT}A_+ = A_-\mathcal{PT} \quad \text{and} \quad \mathcal{PT}B_+ = B_-\mathcal{PT}.$$

Due to our construction of the coupling, these relations imply that the operator A is $\mathcal{PT}$-*symmetric*

$$\mathcal{PT}A = A\mathcal{PT}.$$

Moreover, the operator A turns out to be $\mathcal{P}$-symmetric in the Kreĭn space $(\mathfrak{H}, [\cdot,\cdot])$ with the fundamental symmetry $\mathcal{P}$ in $\mathfrak{H} = L^2(\mathbb{R})$. In Leben & Trunk (2019) it was shown that the extension H_0 of A, defined as a restriction of the adjoint A^+ to the domain

$$\text{dom } H_0 = \left\{u_+ \oplus u_- \in \text{dom } A^+ : \, u_+(0) - u_-(0) = e^{-2i\phi}u'_+(0) - e^{2i\phi}u'_-(0) = 0\right\},$$

is a $\mathcal{PT}$-symmetric and $\mathcal{P}$-selfadjoint operator in the Kreĭn space $(\mathfrak{H}, [\cdot,\cdot])$. Here A^+ stands for the adjoint with respect to the Kreĭn space inner product $[\cdot,\cdot]$. In Theorem 3.2 below, which is the main result of this note, we find a one-parameter family $\{H_\alpha\}_{\alpha\in\mathbb{R}}$ of $\mathcal{PT}$-symmetric and $\mathcal{P}$-selfadjoint extensions of A in the Kreĭn space $(\mathfrak{H}, [\cdot,\cdot])$ with domain

$$\text{dom } H_\alpha = \left\{u_+ \oplus u_- \in \text{dom } A^+ : \, u_+(0) - u_-(0) = 0, \, e^{-2i\phi}u'_+(0) - e^{2i\phi}u'_-(0) = \alpha u_+(0)\right\}.$$

Theorem 3.2 is based on the abstract construction of the coupling (A, B) of two dual pairs (A_+, B_+) and (A_-, B_-) in Theorem 2.5 and the description of all $\mathcal{PT}$-symmetric and $\mathcal{P}$-selfadjoint extensions of A given in Theorem 2.14.

Summing up, the results presented here promote the use of boundary triple techniques for dual pairs and techniques from Sturm–Liouville theory for complex potentials in the study of $\mathcal{PT}$-symmetric quantum mechanics. This is in line with Leben & Trunk (2019) and it is, to some extent, a surprise that in the physical literature the techniques presented here were never exploited. It is the aim of this paper to recall those techniques and, hence, provide a mathematically sound setting of the (nowadays) classical Bender–Boettcher-theory.

2 Coupling of dual pairs and parity

In this section we recall known facts about dual pairs of linear operators, their boundary triples and corresponding Weyl functions, and coupling from Malamud & Mogilevskiĭ (2002). However, our notations differ slightly from that paper; we mainly follow the notations of Baidiuk, Derkach & Hassi (2021).

Moreover, throughout this paper we use the following notations. By $\mathbb{R}_+$ and $\mathbb{R}_-$ we denote the set of all positive and negative reals, respectively. For $z \in \mathbb{C}$, $\overline{z}$ denotes the complex conjugate of z. All operators in this paper are densely defined linear operators in some Hilbert spaces. For such operators T, we use the common notation $\mathrm{dom}\ T$, $\mathrm{ran}\ T$, and $\ker T$ for the domain, the range, and the null-space, respectively, of T. Moreover, as usual, $\rho(T)$, $\sigma(T)$, and $\sigma_p(T)$ stand for the resolvent set, the spectrum, and the point spectrum, respectively, of T. The inner product in a Hilbert space is usually denoted by $\langle \cdot, \cdot \rangle$ and the adjoint of the operator T by T^*. The set of all bounded and everywhere defined operators in a Hilbert space $\mathfrak{H}$ is denoted by $\mathcal{L}(\mathfrak{H})$.

2.1 Dual pairs of linear operators and Weyl functions

Definition 2.1. A pair (A, B) of densely defined closed linear operators A and B in a Hilbert space $(\mathfrak{H}, \langle \cdot, \cdot \rangle)$ is called a *dual pair*, if

$$\langle Af, g \rangle - \langle f, Bg \rangle = 0 \qquad \text{for all } f \in \mathrm{dom}\ A,\ g \in \mathrm{dom}\ B. \tag{2.1}$$

The equality (2.1) means that

$$A \subset B^* \quad \text{and} \quad B \subset A^*.$$

Clearly, if (A, B) is a dual pair, then (B, A) is also a dual pair.

Definition 2.2. Let (A, B) be a dual pair in a Hilbert space $\mathfrak{H}$, let $\mathcal{H}_1$, $\mathcal{H}_2$ be auxiliary Hilbert spaces, and let

$$\Gamma^B = \begin{pmatrix} \Gamma_1^B \\ \Gamma_2^B \end{pmatrix} : \mathrm{dom}\ B^* \to \mathcal{H}_1 \times \mathcal{H}_2 \quad \text{and} \quad \Gamma^A = \begin{pmatrix} \Gamma_1^A \\ \Gamma_2^A \end{pmatrix} : \mathrm{dom}\ A^* \to \mathcal{H}_1 \times \mathcal{H}_2 \tag{2.2}$$

be linear operators. Then the triple $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$ is called a *boundary triple for the dual pair* (A, B) if:

(1) the mappings Γ^B and Γ^A in (2.2) are surjective;

(2) the following identity holds for every $f \in \mathrm{dom}\ B^*$, $g \in \mathrm{dom}\ A^*$,

$$\langle B^* f, g \rangle - \langle f, A^* g \rangle = \langle \Gamma_1^B f, \Gamma_1^A g \rangle_{\mathcal{H}_1} - \langle \Gamma_2^B f, \Gamma_2^A g \rangle_{\mathcal{H}_2}.$$

It is easily seen that if a triple $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$ is a boundary triple for a dual pair (A, B), then the following identity also holds

$$\langle A^* g, f \rangle - \langle g, B^* f \rangle = \langle \Gamma_2^A g, \Gamma_2^B f \rangle_{\mathcal{H}_2} - \langle \Gamma_1^A g, \Gamma_1^B f \rangle_{\mathcal{H}_1}, \quad f \in \mathrm{dom}\ B^*,\ g \in \mathrm{dom}\ A^*, \tag{2.3}$$

and, hence, the triple

$$(\mathcal{H}_2 \times \mathcal{H}_1, (\Gamma^B)^T, (\Gamma^A)^T) := \left(\mathcal{H}_2 \times \mathcal{H}_1, \begin{pmatrix} \Gamma_2^B \\ \Gamma_1^B \end{pmatrix}, \begin{pmatrix} \Gamma_2^A \\ \Gamma_1^A \end{pmatrix} \right) \tag{2.4}$$

is a boundary triple for the dual pair (B, A). The boundary triple (2.4) is called *transposed* with respect to the boundary triple $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$.

A linear operator $\widetilde{A}$ is called a *proper extension* of a dual pair (A, B) if

$$A \subset \widetilde{A} \subset B^*.$$

The proper extension A_2 of A is defined as the restriction of B^* to the set

$$\mathrm{dom}\; A_2 = \{f \in \mathrm{dom}\; B^* : \Gamma_2^B f = 0\}. \tag{2.5}$$

Similarly, the proper extension B_1 of B is defined as the restriction of A^* to the set

$$\mathrm{dom}\; B_1 = \{f \in \mathrm{dom}\; A^* : \Gamma_1^A f = 0\}. \tag{2.6}$$

For every $z \in \rho(A_2)$ the following decomposition holds

$$\mathrm{dom}\; B^* = \mathrm{dom}\; A_2 \dotplus \mathfrak{N}_z(B^*), \qquad \text{where} \quad \mathfrak{N}_z(B^*) := \ker(B^* - zI),$$

and, consequently, the mapping $\Gamma_2^B|_{\mathfrak{N}_z(B^*)} : \mathfrak{N}_z(B^*) \to \mathcal{H}_2$ is boundedly invertible, see (Malamud & Mogilevskiĭ, 2002) for details. In a similar way, for every $z \in \rho(B_1)$ the following decomposition holds

$$\mathrm{dom}\; A^* = \mathrm{dom}\; B_1 \dotplus \mathfrak{N}_z(A^*), \qquad \text{where} \quad \mathfrak{N}_z(A^*) := \ker(A^* - zI),$$

and, hence, the mapping $\Gamma_1^A|_{\mathfrak{N}_z(A^*)} : \mathfrak{N}_z(A^*) \to \mathcal{H}_1$ is boundedly invertible for $z \in \rho(B_1)$.

Moreover, in light of (2.3), (2.5), and (2.6), one has that $B_1 = A_2^*$ and, hence, in particular the following identity holds

$$\rho(B_1) = \overline{\rho(A_2)}.$$

Definition 2.3. The operator functions

$$\gamma(z) := (\Gamma_2^B|_{\mathfrak{N}_z(B^*)})^{-1} \quad \text{and} \quad M(z) := \Gamma_1^B(\Gamma_2^B|_{\mathfrak{N}_z(B^*)})^{-1}, \qquad z \in \rho(A_2),$$

are called the γ*-field* and the *Weyl function*, respectively, of the dual pair (A, B), corresponding to the boundary triple $\Pi = (\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$.

Clearly, the operator functions

$$\gamma^T(z) := (\Gamma_1^A|_{\mathfrak{N}_z(A^*)})^{-1} \quad \text{and} \quad M^T(z) := \Gamma_2^A(\Gamma_1^A|_{\mathfrak{N}_z(A^*)})^{-1}, \qquad z \in \rho(B_1),$$

are the γ-field and the Weyl function, respectively, of the dual pair (B, A), corresponding to the transposed boundary triple $(\mathcal{H}_2 \times \mathcal{H}_1, (\Gamma^B)^T, (\Gamma^A)^T)$, cf. (2.4). Notice that

$$M^T(z) = M(\bar{z})^*, \qquad z \in \rho(B_1) = \overline{\rho(A_2)}.$$

Let Θ be a linear relation from $\mathcal{H}_1$ to $\mathcal{H}_2$, i.e., a subspace of $\mathcal{H}_1 \times \mathcal{H}_2$, see, e.g., Arens (1961). Consider the restriction A_Θ of B^* to the subspace

$$\mathrm{dom}\; A_\Theta = \{f \in \mathrm{dom}\; B^* : \Gamma^B f \in \Theta\}.$$

The following statement describes some spectral properties of the extension A_Θ.

Lemma 2.4. *Let (A, B) be a dual pair in a Hilbert space $\mathfrak{H}$, let $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$ be a boundary triple for the dual pair (A, B), let M be the corresponding Weyl function, let Θ be a linear relation from $\mathcal{H}_1$ to $\mathcal{H}_2$, and let $z \in \rho(A_2)$. Then the following statements hold:*

(i) A^*_Θ *is the restriction of* A^* *to*

$$\text{dom } A^*_\Theta = \{f \in \text{dom } A^* : \Gamma^A f \in \Theta^*\}.$$

(ii) $z \in \sigma_p(A_\Theta)$ *if and only if* $0 \in \sigma_p(I_{\mathcal{H}_2} - \Theta M(z))$. *In this case*

$$\ker(A_\Theta - zI) = \gamma(z)\ker(I_{\mathcal{H}_2} - \Theta M(z)).$$

(iii) $z \in \rho(A_\Theta)$ *if and only if* $0 \in \rho(I_{\mathcal{H}_2} - \Theta M(z))$.

2.2 Coupling of dual pairs

Theorem 2.5. *Let* (A_+, B_+) *and* (A_-, B_-) *be dual pairs in Hilbert spaces* $\mathfrak{H}_+$ *and* $\mathfrak{H}_-$, *respectively, let* $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^{A_\pm}, \Gamma^{B_\pm})$ *be a boundary triple for the dual pair* $(A_\pm, B_\pm)$, *and let* $M_\pm$ *be the corresponding Weyl function. Denote by* A^* *and* B^* *the restrictions of the operators* $A_+^* \oplus A_-^*$ *and* $B_+^* \oplus B_-^*$ *to the domains*

$$\text{dom } A^* = \{g_+ \oplus g_- : g_\pm \in \text{dom } A_\pm^*,\ \Gamma_1^{A_+} g_+ = \Gamma_1^{A_-} g_-\} \tag{2.7}$$

and

$$\text{dom } B^* = \{f_+ \oplus f_- : f_\pm \in \text{dom } B_\pm^*,\ \Gamma_2^{B_+} f_+ = \Gamma_2^{B_-} f_-\}, \tag{2.8}$$

respectively. Then the following statements hold:

(i) *The operators* $A := (A^*)^*$ *and* $B := (B^*)^*$ *are restrictions of the operators* B^* *and* A^*, *respectively, to the domains*

$$\text{dom } A = \{f_+ \oplus f_- : f_\pm \in \text{dom } B_\pm^*, \Gamma_2^{B_+} f_+ = \Gamma_2^{B_-} f_- = \Gamma_1^{B_+} f_+ + \Gamma_1^{B_-} f_- = 0\}, \tag{2.9}$$

$$\text{dom } B = \{g_+ \oplus g_- : g_\pm \in \text{dom } A_\pm^*, \Gamma_1^{A_+} g_+ = \Gamma_1^{A_-} g_- = \Gamma_2^{A_+} g_+ + \Gamma_2^{A_-} g_- = 0\}, \tag{2.10}$$

and (A, B) *is a dual pair in* $\mathfrak{H}_+ \oplus \mathfrak{H}_-$.

(ii) *The triple* $\Pi = (\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$ *with*

$$\Gamma^A g = \begin{pmatrix} \Gamma_1^{A_+} g_+ \\ \Gamma_2^{A_+} g_+ + \Gamma_2^{A_-} g_- \end{pmatrix} \quad \text{and} \quad \Gamma^B f = \begin{pmatrix} \Gamma_1^{B_+} f_+ + \Gamma_1^{B_-} f_- \\ \Gamma_2^{B_+} f_+ \end{pmatrix}, \qquad \begin{matrix} f \in \text{dom } B^*, \\ g \in \text{dom } A^*, \end{matrix}$$

is a boundary triple for the dual pair (A, B).

(iii) *The Weyl function* $M(z)$ *corresponding to the boundary triple* $\Pi = (\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$ *is given by*

$$M(z) = M_+(z) + M_-(z), \qquad z \in \rho(A_2), \tag{2.11}$$

where A_2 *is defined by* (2.5).

Proof. The proof of this theorem consists of three parts: (i) and (ii) are established in (a) and (b), and (iii) is proven in (c).

(a) Let $f = f_+ \oplus f_- \in \mathrm{dom}\,(B_+^* \oplus B_-^*)$, $g = g_+ \oplus g_- \in \mathrm{dom}\,(A_+^* \oplus A_-^*)$. Then it follows from the equalities

$$\begin{aligned}\langle B_+^* f_+, g_+\rangle - \langle f_+, A_+^* g_+\rangle &= \langle \Gamma_1^{B+} f_+, \Gamma_1^{A+} g_+\rangle_{\mathcal{H}_1} - \langle \Gamma_2^{B+} f_+, \Gamma_2^{A+} g_+\rangle_{\mathcal{H}_2},\\ \langle B_-^* f_-, g_-\rangle - \langle f_-, A_-^* g_-\rangle &= \langle \Gamma_1^{B-} f_-, \Gamma_1^{A-} g_-\rangle_{\mathcal{H}_1} - \langle \Gamma_2^{B-} f_-, \Gamma_2^{A-} g_-\rangle_{\mathcal{H}_2},\end{aligned}$$

that

$$\begin{aligned}\langle (B_+^* \oplus B_-^*) f, g\rangle - \langle f, (A_+^* \oplus A_-^*) g\rangle = \langle \Gamma_1^{B+} f_+, \Gamma_1^{A+} g_+\rangle_{\mathcal{H}_1} - \langle \Gamma_2^{B+} f_+, \Gamma_2^{A+} g_+\rangle_{\mathcal{H}_2}\\ + \langle \Gamma_1^{B-} f_-, \Gamma_1^{A-} g_-\rangle_{\mathcal{H}_1} - \langle \Gamma_2^{B-} f_-, \Gamma_2^{A-} g_-\rangle_{\mathcal{H}_2}.\end{aligned} \tag{2.12}$$

The equality (2.9) follows from (2.12) since the mappings $\Gamma^{A\pm} : \mathrm{dom}\,A_\pm^* \to \mathcal{H}_1 \times \mathcal{H}_2$ are surjective. Similarly, (2.10) follows from (2.12) since the mappings $\Gamma^{B\pm} : \mathrm{dom}\,B_\pm^* \to \mathcal{H}_1 \times \mathcal{H}_2$ are surjective.

(b) Next, for $f \in \mathrm{dom}\,B^*$ and $g \in \mathrm{dom}\,A^*$ the equation (2.12) takes the form

$$\langle B^* f, g\rangle - \langle f, (A^*) g\rangle = \langle \Gamma_1^{B+} f_+ + \Gamma_1^{B-} f_-, \Gamma_1^{A+} g_+\rangle_{\mathcal{H}_1} - \langle \Gamma_2^{B+} f_+, \Gamma_2^{A+} g_+ + \Gamma_2^{A-} g_-\rangle_{\mathcal{H}_2}.$$

This proves that (A, B) is a dual pair in $\mathfrak{H}_+ \oplus \mathfrak{H}_-$ and that (ii) holds.

(c) It follows from (2.8) that the γ-field of (A, B) corresponding to the boundary triple Π takes the form

$$\gamma(z) = \gamma_+(z) \oplus \gamma_-(z),$$

where $\gamma_\pm(z)$ are γ-fields of $(A_\pm, B_\pm)$ corresponding to the boundary triples $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^{A\pm}, \Gamma^{B\pm})$. Now formula (2.11) follows from the definition of the Weyl function, see Definition 2.3. $\square$

Definition 2.6. The dual pair (A, B) constructed in (2.9) and (2.10) is called the *coupling of the dual pairs* (A_+, B_+) *and* (A_-, B_-) relative to the triples

$$(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^{A_+}, \Gamma^{B_+}) \quad \text{and} \quad (\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^{A_-}, \Gamma^{B_-}).$$

2.3 Real dual pairs and real boundary triples

Let $\mathcal{T}$ be a *conjugation* (time reversal) operator in a Hilbert space $(\mathfrak{H}, \langle\cdot,\cdot\rangle)$, i.e., $\mathcal{T}$ is antilinear, $\mathcal{T}^2 = I_{\mathfrak{H}}$, and

$$\langle \mathcal{T} f, \mathcal{T} g\rangle = \langle g, f\rangle \quad \text{for all} \quad f, g \in \mathfrak{H}.$$

In what follows, we suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ coincide: $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$.

Definition 2.7. Let $\mathcal{T}$ and $j_{\mathcal{H}}$ be conjugations in $\mathfrak{H}$ and $\mathcal{H}$, respectively. A dual pair (A, B) in $\mathfrak{H}$ is called $\mathcal{T}$*-real* if

$$\mathcal{T}\,\mathrm{dom}\,A = \mathrm{dom}\,B \quad \text{and} \quad \mathcal{T} A = B \mathcal{T}. \tag{2.13}$$

A boundary triple $(\mathcal{H}^2, \Gamma^A, \Gamma^B)$ for (A, B) is called $(j_{\mathcal{H}}, \mathcal{T})$*-real* if

$$j_{\mathcal{H}} \Gamma_1^B = \Gamma_2^A \mathcal{T} \quad \text{and} \quad j_{\mathcal{H}} \Gamma_2^B = \Gamma_1^A \mathcal{T}.$$

Observe that the conditions (2.13) are clearly equivalent to

$$\mathcal{T}\,\mathrm{dom}\,A^* = \mathrm{dom}\,B^* \quad \text{and} \quad \mathcal{T}A^* = B^*\mathcal{T}.$$

Lemma 2.8. *Let (A, B) be a $\mathcal{T}$-real dual pair and let $(\mathcal{H}^2, \Gamma^A, \Gamma^B)$ be a $(j_{\mathcal{H}}, \mathcal{T})$-real boundary triple for (A, B). Then the corresponding Weyl function $M(z)$ satisfies the condition*

$$M(z) = j_{\mathcal{H}} M(z)^* j_{\mathcal{H}}, \qquad z \in \rho(A_2).$$

In what follows we consider a Hilbert space $\mathfrak{H}$ decomposed into an orthogonal sum

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_- \tag{2.14}$$

of two subspaces $\mathfrak{H}_\pm$ with conjugations $\mathcal{T}_\pm \in \mathcal{L}(\mathfrak{H}_\pm)$. Then the orthogonal sum

$$\mathcal{T} = \mathcal{T}_+ \oplus \mathcal{T}_- \tag{2.15}$$

is a conjugation in $\mathfrak{H}$.

Theorem 2.9. *Let a Hilbert space $\mathfrak{H}$ and a conjugation $\mathcal{T}$ in $\mathfrak{H}$ be such that* (2.14) *and* (2.15) *hold. Moreover, let $(A_\pm, B_\pm)$ be $\mathcal{T}_\pm$-real dual pairs in the Hilbert spaces $\mathfrak{H}_\pm$. Finally, with $j_{\mathcal{H}}$ a conjugation in $\mathcal{H}$, let $(\mathcal{H}^2, \Gamma^{A_\pm}, \Gamma^{B_\pm})$ be $(j_{\mathcal{H}}, \mathcal{T})$-real boundary triples for $(A_\pm, B_\pm)$, and let*

$$A_0 := A_+ \oplus A_- \quad \text{and} \qquad B_0 := B_+ \oplus B_-.$$

Then the following statements hold:

(i) *The dual pair (A_0, B_0) is $\mathcal{T}$-real and the boundary triple $((\mathcal{H} \oplus \mathcal{H})^2, \Gamma^{A_0}, \Gamma^{B_0})$ with*

$$\Gamma^{A_0} = \Gamma^{A_+} \oplus \Gamma^{A_-} \quad \text{and} \quad \Gamma^{B_0} = \Gamma^{B_+} \oplus \Gamma^{B_-}$$

is $(j_{\mathcal{H}\oplus\mathcal{H}}, \mathcal{T})$-real, where $j_{\mathcal{H}\oplus\mathcal{H}} := j_{\mathcal{H}} \oplus j_{\mathcal{H}}$.

(ii) *The coupling (A, B) of the dual pairs (A_+, B_+) and (A_-, B_-), constructed in* (2.9) *and* (2.10) *is $\mathcal{T}$-real.*

(iii) *The boundary triple $(\mathcal{H}^2, \Gamma^A, \Gamma^B)$ from Theorem* 2.5 *is $(j_{\mathcal{H}}, \mathcal{T})$-real.*

2.4 Parity and $\mathcal{P}$-selfadjoint operators

Definition 2.10. Let $\mathcal{H}_\pm$ be Hilbert spaces and $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$. An operator $\mathcal{P} \in \mathcal{L}(\mathfrak{H})$ will be called an (abstract) *parity* operator if

$$\mathcal{P} = \mathcal{P}^*, \quad \mathcal{P}^2 = I_{\mathfrak{H}}, \quad \text{and} \quad \mathcal{P}\mathfrak{H}_\pm = \mathfrak{H}_\mp.$$

Now consider a Hilbert space $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ with a parity operator $\mathcal{P}$ and a conjugation $\mathcal{T} \in \mathcal{L}(\mathfrak{H})$, such that

$$\mathcal{T}\mathcal{P} = \mathcal{P}\mathcal{T} \quad \text{and} \quad \mathcal{T}\mathfrak{H}_\pm = \mathfrak{H}_\pm. \tag{2.16}$$

The conditions (2.16) mean that the operator $\mathcal{T}$ admits the representation as an orthogonal sum $\mathcal{T} = \mathcal{T}_+ \oplus \mathcal{T}_-$ of two conjugations $\mathcal{T}_+$ and $\mathcal{T}_-$ in Hilbert spaces $\mathfrak{H}_+$ and $\mathfrak{H}_-$, respectively.

Lemma 2.11. *Let $\mathcal{P}$ be a parity operator in $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ and let $\mathcal{T}$ be a conjugation in $\mathfrak{H}$ such that* (2.16) *holds. Let $(A_\pm, B_\pm)$ be $\mathcal{T}_\pm$-real dual pairs in the Hilbert spaces $\mathfrak{H}_\pm$, such that*

$$\mathcal{P}A_+ = B_-\mathcal{P} \quad \text{and} \quad \mathcal{P}B_+ = A_-\mathcal{P}. \tag{2.17}$$

Then the following statements hold:

(i) $\mathcal{PT}\,\mathrm{dom}\,A_+ = \mathrm{dom}\,A_-$, $\mathcal{PT}\,\mathrm{dom}\,B_+ = \mathrm{dom}\,B_-$, *and*

$$\mathcal{PT}A_+ = A_-\mathcal{PT}, \qquad \mathcal{PT}B_+ = B_-\mathcal{PT}; \tag{2.18}$$

(ii) $\mathcal{P}\,\mathrm{dom}\,A_+^* = \mathrm{dom}\,B_-^*$, $\mathcal{P}\,\mathrm{dom}\,B_+^* = \mathrm{dom}\,A_-^*$, *and*

$$\mathcal{P}A_+^* = B_-^*\mathcal{P}, \qquad \mathcal{P}B_+^* = A_-^*\mathcal{P}.$$

Proof. (i) Since the dual pairs $(A_\pm, B_\pm)$ are real with respect to $\mathcal{T}_\pm$, one has

$$\mathcal{T}_+A_+ = B_+\mathcal{T}_+, \qquad \mathcal{T}_-A_- = B_-\mathcal{T}_-. \tag{2.19}$$

Let $f_+ \in \mathrm{dom}\,A_+$. Then by (2.19) $\mathcal{T}f_+ \in \mathrm{dom}\,B_+$ and $B_+\mathcal{T}f_+ = \mathcal{T}A_+f_+$. Next by (2.17)

$$\mathcal{PT}f_+ \in \mathrm{dom}\,A_- \quad \text{and} \quad A_-\mathcal{PT}f_+ = \mathcal{P}B_+\mathcal{T}f_+ = \mathcal{PT}A_+f_+.$$

The proofs of the inclusion $\mathcal{PT}\,\mathrm{dom}\,A_- \subseteq \mathrm{dom}\,A_+$ and of the second equality in (2.18) are similar.

(ii) Applying $\mathcal{P}$ to the left and right of the equalities in (2.17) and using the identity $\mathcal{P}^2 = I_{\mathfrak{H}}$ yields $A_+\mathcal{P} = \mathcal{P}B_-$ and $B_+\mathcal{P} = \mathcal{P}A_-$. From these identities the assertions in (ii) are immediate. □

Definition 2.12. A closed linear operator A in $\mathfrak{H}$ is said to be $\mathcal{PT}$*-symmetric* if for all $f \in \mathrm{dom}\,A$ we have

$$\mathcal{PT}f \in \mathrm{dom}\,A \quad \text{and} \quad \mathcal{PT}Af = A\mathcal{PT}f.$$

Consider the Kreĭn space $(\mathfrak{H}, [\cdot,\cdot])$ with an indefinite inner product given by

$$[f, g] := \langle \mathcal{P}f, g\rangle_{\mathfrak{H}}. \tag{2.20}$$

For the definition of a Kreĭn space we refer to the books of Azizov & Iokhvidov (1989) and Bognar (1974). Recall that a densely defined linear operator A in $\mathfrak{H}$ is called $\mathcal{P}$*-symmetric* if

$$[Af, g] = [f, Ag] \qquad \text{for all } f,\ g \in \mathrm{dom}\,A.$$

Denote by A^+ the adjoint operator in $(\mathfrak{H}, [\cdot,\cdot])$, i.e., $A^+ = \mathcal{P}A^*\mathcal{P}$. For a $\mathcal{P}$-symmetric operator A one has $A \subseteq A^+$. The operator A is called $\mathcal{P}$*-selfadjoint* if $A = A^+$. The following definition of a boundary triple for the $\mathcal{P}$-symmetric operator A was presented in Derkach (1995).

Definition 2.13. Let $\mathcal{H}$ be an auxiliary Hilbert space and let Γ_1, Γ_2 be linear operators from $\mathrm{dom}\,A^+$ to $\mathcal{H}$. The triple $(\mathcal{H}, \Gamma_1, \Gamma_2)$ is called a *boundary triple for the $\mathcal{P}$-symmetric operator A* if the following conditions are satisfied:

(i) the mapping $\Gamma := \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}$ from $\mathrm{dom}\ A^+$ to $\mathcal{H}^2$ is surjective;

(ii) the following identity holds for every $f, g \in \mathrm{dom}\ A^+$

$$[A^+ f, g] - [f, A^+ g] = \langle \Gamma_1 f, \Gamma_2 g \rangle_{\mathcal{H}} - \langle \Gamma_2 f, \Gamma_1 g \rangle_{\mathcal{H}}.$$

In the next theorem we show that the coupling operator A is $\mathcal{P}$-symmetric and $\mathcal{PT}$-symmetric, and describe the set of all $\mathcal{P}$-selfadjoint and $\mathcal{PT}$-symmetric extensions of the operator A.

Theorem 2.14. *Let $\mathcal{P}$ be a parity operator in $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$, let $\mathcal{T}$ be a conjugation in $\mathfrak{H}$ such that* (2.16) *holds, and let $(A_\pm, B_\pm)$ be $\mathcal{T}_\pm$-real dual pairs in the Hilbert spaces $\mathfrak{H}_\pm$ such that* (2.17) *holds. With $j_{\mathcal{H}}$ a conjugation in $\mathcal{H}$, let $(\mathcal{H}^2, \Gamma^{A_\pm}, \Gamma^{B_\pm})$ be $(j_{\mathcal{H}}, \mathcal{T})$-real boundary triples for $(A_\pm, B_\pm)$, such that*

$$\begin{pmatrix} \Gamma_1^{B_+} \\ \Gamma_2^{B_+} \end{pmatrix} f_+ = \begin{pmatrix} \Gamma_2^{A_-} \\ \Gamma_1^{A_-} \end{pmatrix} \mathcal{P} f_+ \quad \textit{and} \quad \begin{pmatrix} \Gamma_1^{B_-} \\ \Gamma_2^{B_-} \end{pmatrix} f_- = \begin{pmatrix} \Gamma_2^{A_+} \\ \Gamma_1^{A_+} \end{pmatrix} \mathcal{P} f_-, \quad f_\pm \in \mathrm{dom}\ B_\pm^*. \tag{2.21}$$

Moreover, let (A, B) be the coupling of the dual pairs (A_+, B_+) and (A_-, B_-) given by (2.9), (2.10)*, and let Θ be a linear relation in $\mathcal{H}$. Then the following statements hold:*

(i) *The operator A is $\mathcal{PT}$-symmetric, $\mathcal{P}$-symmetric, and $A^+ = B^*$.*

(ii) *The triple $(\mathcal{H}, \Gamma_1^B, \Gamma_2^B)$ is a boundary triple for the $\mathcal{P}$-symmetric operator A.*

(iii) *The extension A_Θ of the operator A, given by*

$$\mathrm{dom}\ A_\Theta = \left\{ f \in \mathrm{dom}\ B^* : \begin{pmatrix} \Gamma_1 f \\ \Gamma_2 f \end{pmatrix} \in \Theta \right\}, \quad A_\Theta = B^*|_{\mathrm{dom}\, A_\Theta},$$

is $\mathcal{P}$-selfadjoint if and only if $\Theta = \Theta^*$.

(iv) *A_Θ is $\mathcal{PT}$-symmetric if and only if $\Theta = j_{\mathcal{H}} \Theta j_{\mathcal{H}}$.*

3 $\mathcal{PT}$-symmetric Hamiltonians

Here we return to the investigation of the non-Hermitian $\mathcal{PT}$-invariant Hamiltonians presented in the introduction, that is, we study equation (1.1) on the wedge shaped contour Γ, cf. (1.2). By substituting $z(x) := x e^{i\phi\, \mathrm{sgn}\, x}$ into (1.1) one obtains the two differential expressions given by (1.3) and (1.4). Assume that the differential expressions $\mathfrak{a}_\pm$ in (1.4) are in the limit point case at $\pm\infty$. As presented in Section 1, this is the case if and only if the angle ϕ of the wedge satisfies

$$\phi \neq -\frac{N+2}{2N+8}\pi + \frac{2k}{4+N}\pi \quad \text{for } k = 0, \ldots, N+3. \tag{3.1}$$

Then by Leben & Trunk (2019: Lemma 1) the differential expressions $\mathfrak{b}_\pm$ in (1.6) are also in the limit point case at $\pm\infty$. Define the operators $A_\pm$ and $B_\pm$ associated with $\mathfrak{a}_\pm$ and $\mathfrak{b}_\pm$ in $L^2(\mathbb{R}_\pm)$ as

$$A_\pm f_\pm := \mathfrak{a}_\pm[f_\pm] \quad \text{and} \quad B_\pm g_\pm := \mathfrak{b}_\pm[g_\pm] \quad \text{for } f_\pm \in \mathrm{dom}\ A_\pm,\ g_\pm \in \mathrm{dom}\ B_\pm,$$

respectively, with the domains

$$\text{dom } A_\pm := \{u_\pm \in L^2(\mathbb{R}_\pm) : \mathfrak{a}_\pm[u_\pm] \in L^2(\mathbb{R}_\pm),\, u'_\pm \in AC_{loc}(\mathbb{R}_\pm),\, u_\pm(0_\pm) = u'_\pm(0_\pm) = 0\},$$

$$\text{dom } B_\pm := \{v_\pm \in L^2(\mathbb{R}_\pm) : \mathfrak{b}_\pm[v_\pm] \in L^2(\mathbb{R}_\pm),\, v'_\pm \in AC_{loc}(\mathbb{R}_\pm),\, v_\pm(0_\pm) = v'_\pm(0_\pm) = 0\}.$$

These operators are in some sense the minimal operators. It follows from Leben & Trunk (2019: Proposition 1 & Theorem 3) that the (maximal) operators $A_\pm^*$ and $B_\pm^*$ are generated by differential expressions in $L^2(\mathbb{R}_\pm)$ where the roles of $\mathfrak{a}_\pm$ and $\mathfrak{b}_\pm$ are switched in the sense that the differential expressions $\mathfrak{a}_\pm$ are now related to $B_\pm^*$ and the differential expressions $\mathfrak{b}_\pm$ are related to $A_\pm^*$:

$$B_\pm^* f_\pm := \mathfrak{a}_\pm[f_\pm] \quad \text{and} \quad A_\pm^* g_\pm := \mathfrak{b}_\pm[g_\pm] \quad \text{for } f_\pm \in \text{dom } B_\pm^*,\ g_\pm \in \text{dom } A_\pm^*,$$

with

$$\begin{aligned} \text{dom } B_\pm^* &:= \{u_\pm \in L^2(\mathbb{R}_\pm) : \mathfrak{a}_\pm[u_\pm] \in L^2(\mathbb{R}_\pm),\, u'_\pm \in AC_{loc}(\mathbb{R}_\pm)\}, \\ \text{dom } A_\pm^* &:= \{v_\pm \in L^2(\mathbb{R}_\pm) : \mathfrak{b}_\pm[v_\pm] \in L^2(\mathbb{R}_\pm),\, v'_\pm \in AC_{loc}(\mathbb{R}_\pm)\}. \end{aligned}$$

Theorem 3.1. *The pairs (A_-, B_-) and (A_+, B_+) are dual pairs. The triple $(\mathbb{C}^2, \Gamma^{A_+}, \Gamma^{B_+})$,*

$$\Gamma^{B_+} u_+ = \begin{pmatrix} e^{-2i\phi} u'_+(0) \\ u_+(0) \end{pmatrix} \quad \textit{and} \quad \Gamma^{A_+} v_+ = \begin{pmatrix} v_+(0) \\ e^{2i\phi} v'_+(0) \end{pmatrix}, \qquad \begin{matrix} u_+ \in \text{dom } B_+^*, \\ v_+ \in \text{dom } A_+^*, \end{matrix}$$

is a boundary triple for the dual pair (A_+, B_+). The triple $(\mathbb{C}^2, \Gamma^{A_-}, \Gamma^{B_-})$,

$$\Gamma^{B_-} u_- = \begin{pmatrix} -e^{2i\phi} u'_-(0) \\ u_-(0) \end{pmatrix} \quad \textit{and} \quad \Gamma^{A_-} v_- = \begin{pmatrix} v_-(0) \\ -e^{-2i\phi} v'_-(0) \end{pmatrix}, \qquad \begin{matrix} u_- \in \text{dom } B_-^*, \\ v_- \in \text{dom } A_-^*, \end{matrix}$$

is a boundary triple for the dual pair (A_-, B_-).

Proof. Integration by parts and (Leben & Trunk, 2019: Proposition 1) show

$$\langle A_\pm u_\pm, v_\pm \rangle = \langle u_\pm, B_\pm v_\pm \rangle, \qquad u_\pm \in \text{dom } A_\pm, v_\pm \in \text{dom } B_\pm.$$

This proves the first statement. It follows from (Leben & Trunk, 2019: Proposition 1) that for $u_+ \in \text{dom } B_+^*$ and $v_+ \in \text{dom } A_+^*$

$$\begin{aligned} \langle B_+^* u_+, v_+ \rangle - \langle u_+, A_+^* v_+ \rangle &= -e^{-2i\phi} \int_0^\infty u''_+(x) \overline{v_+(x)}\, dx + e^{-2i\phi} \int_0^\infty u_+(x) \overline{v''_+(x)}\, dx \\ &= e^{-2i\phi} (u'_+(0) \overline{v_+(0)} - u_+(0) \overline{v'_+(0)}). \end{aligned}$$

Hence, $(\mathbb{C}^2, \Gamma^{A_+}, \Gamma^{B_+})$ is a boundary triple for the dual pair (A_+, B_+). The statement for the dual pair (A_-, B_-) is shown in the same way. □

Recall that the coupling (A, B) of the dual pairs (A_+, B_+) and (A_-, B_-) consists of a pair of operators $A = (B_+^* \oplus B_-^*)|_{\text{dom } A}$ and $B = (A_+^* \oplus A_-^*)|_{\text{dom } B}$ with the domains

$$\text{dom } A = \{u_+ \oplus u_- : u_\pm \in \text{dom } B_\pm^*,\ u_+(0) = u_-(0) = e^{-2i\phi} u'_+(0) - e^{2i\phi} u'_-(0) = 0\}, \tag{3.2}$$

$$\text{dom } B = \{u_+ \oplus u_- : u_\pm \in \text{dom } A_\pm^*,\ u_+(0) = u_-(0) = e^{2i\phi} u'_+(0) - e^{-2i\phi} u'_-(0) = 0\}, \tag{3.3}$$

see Theorem 2.5.

We define the parity $\mathcal{P}$ and time reversal $\mathcal{T}$ as in (1.5). The parity $\mathcal{P}$ gives rise to a new inner product $[\cdot,\cdot] = \langle \mathcal{P}\cdot,\cdot\rangle$ (see also (2.20)), which was considered in many papers, we mention only (Mostafazadeh, 2010). It is easy to see that the parity $\mathcal{P}$ and the time reversal $\mathcal{T}$ satisfy (2.16), where $\mathfrak{H}_\pm := L^2(\mathbb{R}_\pm)$. Due to Theorem 2.14, the operator A is $\mathcal{PT}$-symmetric and $\mathcal{P}$-symmetric in the Kreĭn space $(L^2(\mathbb{R}),[\cdot,\cdot]) = (L^2(\mathbb{R}_-)\oplus L^2(\mathbb{R}_+),[\cdot,\cdot])$. The (Kreĭn space) adjoint A^+ of A coincides with $B^* = (B_+^* \oplus B_-^*)|_{\mathrm{dom}\, B^*}$, where

$$\mathrm{dom}\; B^* = \{u_+ \oplus u_- : \, u_\pm \in \mathrm{dom}\; B_\pm^*, \, u_+(0) = u_-(0)\}.$$

An application of Theorem 2.14 gives a one-parameter family $\{H_\alpha\}_{\alpha\in\mathbb{R}}$ of $\mathcal{PT}$-symmetric and $\mathcal{P}$-selfadjoint extensions of A in the Kreĭn space $(L^2(\mathbb{R}),[\cdot,\cdot])$. This is the main result of this note.

Theorem 3.2. *Let the angle ϕ satisfies* (3.1) *and let A be the coupling operator constructed in* (3.2)*. Then the following statements are true:*

(i) *A boundary triple $(\mathbb{C},\Gamma_1,\Gamma_2)$ for the $\mathcal{P}$-symmetric operator A is given by*

$$\Gamma_1 u = e^{-2i\phi}u_+'(0) - e^{2i\phi}u_-'(0) \quad \textit{and} \quad \Gamma_2 u = u_+(0), \quad u = u_+ \oplus u_- \in \mathrm{dom}\; B^*.$$

(ii) *The extension H_α of the operator A, defined as a restriction of A^+ to the domain*

$$\mathrm{dom}\; H_\alpha = \left\{u_+ \oplus u_- \in \mathrm{dom}\; B^* : \, e^{-2i\phi}u_+'(0) - e^{2i\phi}u_-'(0) = \alpha u_+(0)\right\},$$

is $\mathcal{P}$-selfadjoint if and only if $\alpha \in \mathbb{R}$.

(iii) *H_α is $\mathcal{PT}$-symmetric if and only if $\alpha \in \mathbb{R}$.*

Proof. By construction the dual pairs (A_+,B_+) and (A_-,B_-) are $\mathcal{T}_\pm$-real and the parity operator $\mathcal{P}$ intertwines the operators A_+, B_- and A_-, B_+, that is, (2.17) holds. Moreover, the boundary triples $(\mathbb{C}^2,\Gamma^{A_+},\Gamma^{B_+})$ and $(\mathbb{C}^2,\Gamma^{A_-},\Gamma^{B_-})$ are also $(j_\mathbb{C},\mathcal{T})$-real and satisfy the condition (2.21). Here $j_\mathbb{C}$ stands for the usual complex conjugation in $\mathbb{C}$. Hence, all assumptions in Theorem 2.14 are satisfied and the statements in Theorem 3.2 follow directly from Theorem 2.14. □

In Leben & Trunk (2019) only the extension for the parameter value $\alpha = 0$ was considered. More precisely, there it was shown that H_0 is an extension of A with domain

$$\mathrm{dom}\; H_0 = \left\{u_+ \oplus u_- : \, u_\pm \in \mathrm{dom}\; B_\pm^*, \, u_+(0) - u_-(0) = e^{-2i\phi}u_+'(0) - e^{2i\phi}u_-'(0) = 0\right\}$$

which is $\mathcal{PT}$-symmetric and $\mathcal{P}$-selfadjoint. The family H_α, $\alpha \in \mathbb{R}$, of extensions obtained in Theorem 3.2 is in some sense an analogue of the δ-interaction for the differential operation $\mathfrak{a}$.

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POSITIVE AND NEGATIVE EXAMPLES FOR THE RIESZ BASIS PROPERTY OF INDEFINITE STURM-LIOUVILLE PROBLEMS

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Dedicated to Seppo Hassi on the occasion of his 60th birthday

1 Introduction

We consider the indefinite Sturm-Liouville eigenvalue problem

$$-f'' = \lambda r f \quad \text{on} \quad [-1, 1],$$

with a weight function r changing its sign at 0. It is known that, depending on the behavior of r and on the type of the imposed self-adjoint boundary conditions, the so-called Riesz basis property of the eigenfunctions in the Hilbert space $L^2_{|r|}[-1, 1]$ can be valid or not. More precisely, let $r \in L^1[-1, 1]$ be a real function with a single so-called *turning point*, i.e., with a sign change, at 0, say

$$r(x) < 0 \quad \text{a.e. on} \quad [-1, 0), \qquad r(x) > 0 \quad \text{a.e. on} \quad (0, 1].$$

Due to the sign change of r we cannot expect that the eigenfunctions of the corresponding Sturm-Liouville eigenvalue problem with self-adjoint boundary conditions form an orthonormal basis in the Hilbert space $L^2_{|r|}[-1, 1]$ with the inner product

$$(f, g) = \int_{-1}^{1} f\bar{g}|r|\, dx, \qquad f, g \in L^2_{|r|}[-1, 1].$$

However, the eigenfunctions may form a *Riesz basis* in this space, i.e., an orthonormal basis with respect to some inner product equivalent to $(\cdot, \cdot)$. In this case, we say that the eigenvalue problem has the *Riesz basis property*.

This Riesz basis property has been intensively studied during the last decades, see the overview paper (Fleige, 2015). It was first observed by Binding and Ćurgus (2004) that the type of the boundary conditions plays an important role. In fact, for the same weight function r, the eigenvalue problem with Dirichlet boundary conditions

$$-f'' = \lambda r f \quad \text{on} \quad [-1, 1], \qquad f(-1) = f(1) = 0, \tag{1.1}$$

can have the Riesz basis property, while the eigenvalue problem with antiperiodic boundary conditions

$$-f'' = \lambda r f \quad \text{on} \quad [-1, 1], \qquad f(-1) + f(1) = 0, \quad f'(-1) + f'(1) = 0. \tag{1.2}$$

does not have the Riesz basis property, This result was sharpened in Ćurgus, Fleige & Kostenko (2013: Theorem 4.10). Under a certain oddness condition a necessary and sufficient criterion on the weight function r was presented for the Riesz basis property of an eigenvalue problem with quite general boundary conditions. In case of the eigenvalue problem (1.1) this criterion only involves the behavior of r at the turning point, whereas in case of the eigenvalue problem (1.2) also the boundary comes into play, see Theorem 2.4 below. In some sense the paper (Ćurgus, Fleige & Kostenko, 2013) was the endpoint of a number of papers improving the conditions for the Riesz basis property, at least for the case of a regular differential expression, e.g., (Ćurgus & Langer, 1989; Parfenov, 2003; 2005; Pyatkov, 2005). Note that further developments were mainly concerned with singular differential expressions and criteria involving the Titchmarsh-Weyl function, see e.g., (Kostenko, 2013; Ćurgus, Derkach & Trunk, 2020).

The present paper can be regarded as a certain addition to Ćurgus, Fleige & Kostenko (2013). The main intention is a presentation of examples for the validity and non-validity of the Riesz basis property of eigenvalue problems (1.1) and (1.2), illustrating the different arguments for various settings.

Finally, an outline of the paper is given. In Section 2 we recall some conditions concerning the weight function r appearing in the eigenvalue problems (1.1) and (1.2) from the paper (Ćurgus, Fleige & Kostenko, 2013; Parfenov, 2003; 2005; Pyatkov, 2005). In Section 3 all possible, positive and negative, cases are obtained by certain modifications of the same "bad" weight function. Nevertheless, on the left of the turning point this "bad" weight always remains unchanged. Note that for the proof of these examples also a new result had to be added to the general theory in Section 2.

2 Some known conditions and a slight extension of these results

First of all, it should be mentioned that for the eigenvalue problems (1.1) and (1.2) there cannot appear any root functions except the eigenfunctions. This follows, for example, by means of Kreĭn space methods. To see this let A_D and A_a denote the operators in $L^2_{|r|}[-1,1]$ associated with (1.1) and (1.2). Thus, in other words, $A_D f = -\frac{1}{r} f''$, when f belongs to $\operatorname{dom} A_D$, defined by

$$\operatorname{dom} A_D = \left\{ f \in L^2_{|r|}[-1,1] : \frac{1}{r} f'' \in L^2_{|r|}[-1,1],\ f(-1) = f(1) = 0 \right\},$$

and, likewise, $A_a f = -\frac{1}{r} f''$, when f belongs to $\operatorname{dom} A_a$, defined by

$$\operatorname{dom} A_a = \{ f \in L^2_{|r|}[-1,1] : \frac{1}{r} f'' \in L^2_{|r|}[-1,1],\ f(-1) + f(1) = 0,\ f'(-1) + f'(1) = 0 \}.$$

Here is the above mentioned observation concerning the eigenvalue problems (1.1) and (1.2).

Lemma 2.1. *Let $\lambda \in \mathbb{C}$ and assume that $f \in \operatorname{dom} A_D^2$ and $g \in \operatorname{dom} A_a^2$ satisfy*

$$(A_D - \lambda)^2 f = 0 \quad \textit{and} \quad (A_a - \lambda)^2 g = 0.$$

Then f and g already satisfy

$$(A_D - \lambda) f = 0 \quad \textit{and} \quad (A_a - \lambda) g = 0.$$

Proof. Note that $L^2_{|r|}[-1,1]$ is a Kreĭn space with the inner product

$$[f, g] = \int_{-1}^{1} f \bar{g} r \, dx, \qquad f, g \in L^2_{|r|}[-1,1],$$

and that A_D and A_a are self-adjoint and definitizable operators in this space; see (Ćurgus & Langer, 1989). Furthermore, observe that the operator A_D is nonnegative, since

$$[A_D f, f] = -\int_{-1}^{1} f'' \bar{f}\, dx = \int_{-1}^{1} |f'|^2\, dx \geq 0, \qquad f \in \operatorname{dom} A_D,$$

due to the boundary conditions. Likewise, one sees that A_a is nonnegative. Therefore, in both cases, $p(\lambda) = \lambda$ is a definitizing polynomial. By (Langer, 1982: Proposition II, 2.1) a Jordan chain of length at least 2 can only appear for the zeros of the definitizing polynomial, i.e., for $\lambda = 0$. However, $\lambda = 0$ is neither an eigenvalue for A_D nor for A_a. $\square$

Note that the situation sketched in Lemma 2.1 is different in the case of periodic boundary conditions; see (Ćurgus, Fleige & Kostenko, 2013: Example 4.12).

For convenience we now recall some definitions and known results about the Riesz basis property which will be used in the subsequent examples. We start with some properties from the theory of regularly varying functions studied at 0 rather than at ∞, see, e.g., (Bingham, Goldie & Teugels, 1987; Kostenko, 2013; Ćurgus, Derkach & Trunk, 2020). Let I be a nondecreasing function defined on $[0, b]$ with $b > 0$ such that

$$I(x) > 0, \quad x \in (0, b], \qquad 0 = I(0) = \lim_{x \searrow 0} I(x). \tag{2.1}$$

Then the function I is said to be *slowly varying* if

$$\lim_{x \searrow 0} \frac{I(tx)}{I(x)} = 1, \qquad \text{for all } t > 0,$$

and is said to be *positively increasing* if

$$\limsup_{x \searrow 0} \frac{I(t_0 x)}{I(x)} < 1, \qquad \text{for some } t_0 \in (0, 1).$$

Of course, these properties exclude each other: a slowly varying function cannot be positively increasing, and vice versa. Below, these properties will be checked for the functions

$$I_0^+(x) := \int_0^x r\, dt \quad \text{and} \quad I_1^-(x) := \int_{1-x}^{1} r\, dt, \qquad x \in [0, 1], \tag{2.2}$$

obviously satisfying (2.1). Furthermore, some local oddness conditions on the weight function will be used. A function r is called *locally odd at the turning point* if there exists $\varepsilon > 0$ such that the restriction of r to $(-\varepsilon, \varepsilon)$ is odd. Similarly, a function r is called *locally odd at the boundary* if there exists $\varepsilon > 0$ such that $r(-1 + x) = -r(1 - x)$ for a.a. $x \in (0, \varepsilon)$.

The following result, here formulated in the terminology of positively increasing functions as in Ćurgus, Fleige & Kostenko (2013: Corollary 3.6), goes back to Parfenov (2003; 2005).

Theorem 2.2 (Parfenov (2003; 2005))**.** *The following statements hold:*

(i) *If I_0^+ is positively increasing, then the eigenvalue problem* (1.1) *has the Riesz basis property.*

(ii) *If r is locally odd at the turning point, then the eigenvalue problem* (1.1) *has the Riesz basis property if and only if I_0^+ is positively increasing.*

Originally, (ii) was formulated for the stronger case of odd weights. However, Pyatkov (2005: Theorem 4.2) observed that only the local behaviour at the turning point is relevant for the Riesz basis property of (1.1). Next we recall another aspect of Pyatkov's result (Pyatkov, 2005: Theorem 4.2), again in the terminology of positively increasing functions as in Ćurgus, Fleige & Kostenko (2013: Theorem 4.1).

Theorem 2.3 (Pyatkov (2005))**.** *If I_1^- is positively increasing, then the eigenvalue problem* (1.2) *has the Riesz basis property if and only if the eigenvalue problem* (1.1) *has the Riesz basis property.*

This result was improved in Ćurgus, Fleige & Kostenko (2013: Theorem 4.10) in the case when certain oddness conditions are satisfied.

Theorem 2.4 (Ćurgus, Fleige & Kostenko (2013))**.** *If r is locally odd at the turning point and at the boundary, then the eigenvalue problem* (1.2) *has the Riesz basis property if and only if I_0^+ and I_1^- are both positively increasing.*

Note that originally the last two results were formulated in a more general setting and they are now applied in the present situation. Finally, we present a slight extension of these results, which will be proved in a similar way as (Ćurgus, Fleige & Kostenko, 2013: Theorem 4.10).

Theorem 2.5. *If r is locally odd at the boundary, then the eigenvalue problem* (1.2) *has the Riesz basis property if and only if the eigenvalue problem* (1.1) *has the Riesz basis property and I_1^- is positively increasing.*

Proof. By (Ćurgus, Fleige & Kostenko, 2013: Lemma 4.7) the Riesz basis property of (1.2) is equivalent to the Riesz basis property of the shifted problem

$$-f'' = \lambda \widetilde{r} f \quad \text{on} \quad [\widetilde{a}, \widetilde{b}], \qquad f(\widetilde{a}) + f(\widetilde{b}) = 0, \quad f'(\widetilde{a}) + f'(\widetilde{b}) = 0, \tag{2.3}$$

in $L^2_{|\widetilde{r}|}[\widetilde{a}, \widetilde{b}]$, where $\widetilde{a} := -1 - \varepsilon$ and $\widetilde{b} := 1 - \varepsilon$ for some $\varepsilon \in (0, \frac{1}{2})$. Here the weight function $\widetilde{r}$ is given by

$$\widetilde{r}(x) := \begin{cases} r(x+2), & x \in [\widetilde{a}, -1), \\ r(x), & x \in [-1, \widetilde{b}]. \end{cases}$$

This function has two turning points, i.e., sign changes, at 0 and at -1. Thus we can apply (Pyatkov, 2005: Theorem 4.2) (using a criterion for an even number of turning points, see also (Ćurgus, Fleige & Kostenko, 2013: Theorem 4.1)) and obtain the equivalence of the Riesz basis property of (2.3) to the Riesz basis properties of two local problems with $\widetilde{r}$ on $[-\delta, \delta]$ and $\widetilde{r}$ on $[-1-\delta, -1+\delta]$, both with Dirichlet boundary conditions. Here, $\delta \in (0, \varepsilon)$ can be chosen arbitrarily. By assumption, the weight $\widetilde{r}$ is locally odd at the turning point -1; i.e., $\widetilde{r}(-1-x) = -\widetilde{r}(-1+x)$ for a.a. x in a neighborhood of 0. Therefore, by Parfenov's result, see Theorem 2.2 (ii), the Riesz basis property for the problem on $[-1-\delta, -1+\delta]$ is equivalent to the condition that the function

$$\int_{-1-x}^{-1} \widetilde{r}\, dt = I_1^-(x)$$

is positively increasing. Furthermore, again by (Pyatkov, 2005: Theorem 4.2), the Riesz basis property for the problem on $[-\delta, \delta]$ is equivalent to the Riesz basis property of (1.1). $\square$

Note that the arguments above also apply in a more general setting: Theorem 2.5 remains true with (1.2) replaced by the eigenvalue problem

$$-f'' + qf = \lambda r f, \qquad e^{it} f(-1) = f(1), \quad f'(-1) = e^{-it} f'(1) + d\, f(-1),$$

with a real potential $q \in L^1[-1,1]$, $t \in [0, 2\pi)$, and $d \in \mathbb{R}$, see also the proof of (Ćurgus, Fleige & Kostenko, 2013: Proposition 4.9). However, in this case there may appear a Jordan chain as in Ćurgus, Fleige & Kostenko (2013: Example 4.12). Therefore, as in Ćurgus, Fleige & Kostenko (2013), for the Riesz basis property not only eigenfunctions but also root functions must be allowed.

3 Examples

We start with an example which combines the negative properties of (Ćurgus, Fleige & Kostenko, 2013: Example 3.17) (going back to (Parfenov, 2003)) and (Ćurgus, Fleige & Kostenko, 2013: Example 4.12 (ii)). It is worse than each of these examples in the sense that the Riesz basis criteria at the turning point and at the boundary are both violated simultaneously.

3.1 A weight with "bad" behavior at the turning point and at the boundary

First, consider the functions r_0 and r_1 belonging to $L^1[-1,1]$ defined as

$$r_0(x) := \frac{1}{x(1 - \log|x|)^2} \quad \text{and} \quad r_1(x) := \frac{\operatorname{sgn}(x)}{(1 - |x|)(1 - \log(1 - |x|))^2}, \tag{3.1}$$

for $x \in (-1, 1) \setminus \{0\}$. Furthermore, define the function r as

$$r := r_0 + r_1. \tag{3.2}$$

Lemma 3.1. *For the weight function r from* (3.2)*, the functions I_0^+ and I_1^-, as defined in* (2.2)*, coincide: $I_0^+ = I_1^-$. Moreover, this function $I_0^+ = I_1^-$ is slowly varying.*

Proof. First, note that for $x \in (0, 1)$ we have that $r_1(x) = r_0(1 - x)$ and, hence,

$$I_1^-(x) = \int_{1-x}^{1} (r_0 + r_1)\, dt = \int_0^x (r_0(1-s) + r_1(1-s))\, ds = \int_0^x (r_1(s) + r_0(s))\, ds = I_0^+(x).$$

Now, observe that for $t > 0$ the following limits exist

$$\lim_{x \searrow 0} \frac{t\, r_0(tx)}{r_0(x)} = \lim_{x \searrow 0} \frac{(1 - \log x)^2}{(1 - \log x - \log t)^2} = 1 \quad \text{and} \quad \lim_{x \searrow 0} \frac{t\, r_1(tx)}{r_0(x)} = 0.$$

In order to see that I_0^+ is slowly varying use l'Hôpital's rule:

$$\lim_{x \searrow 0} \frac{I_0^+(tx)}{I_0^+(x)} = \lim_{x \searrow 0} \frac{t(r_0(tx) + r_1(tx))}{r_0(x) + r_1(x)} = \lim_{x \searrow 0} \frac{t\left(\frac{r_0(tx)}{r_0(x)} + \frac{r_1(tx)}{r_0(x)}\right)}{1 + \frac{r_1(x)}{r_0(x)}} = 1. \qquad \square$$

Since r is odd here, we can now conclude the following result from Lemma 3.1, Theorem 2.2 (ii), and Theorem 2.4.

Proposition 3.2. *For the weight function r as in* (3.2) *neither the eigenvalue problem* (1.1) *nor the eigenvalue problem* (1.2) *has the Riesz basis property.*

In the following we "smoothen" the weight function in (3.2) in some sense.

3.2 "Relaxing" the weight on the right of the turning point

Using again the functions r_0 and r_1 from (3.1), we now consider the weight function

$$r(x) := \begin{cases} 1, & x \in [c,d], \\ r_0(x) + r_1(x), & x \in [-1,1] \setminus [c,d], \end{cases} \tag{3.3}$$

for some numbers $0 \le c < d \le 1$. Obviously, this weight is not odd any more, but it is locally odd at the turning point if $c > 0$, and locally odd at the boundary if $d < 1$.

Lemma 3.3. *Let the weight function r be given by* (3.3). *Then the following statements hold:*

(i) *If $0 < c < d < 1$, then the functions I_0^+ and I_1^- are slowly varying.*

(ii) *If $0 < c < d = 1$, then I_0^+ is slowly varying and I_1^- is positively increasing.*

(iii) *If $0 = c < d < 1$, then I_0^+ is positively increasing and I_1^- is slowly varying.*

(iv) *If $c = 0$ and $d = 1$, then the functions I_0^+ and I_1^- are positively increasing.*

Proof. In some cases the functions I_0^+ and I_1^- coincide locally at 0 with the corresponding function in Lemma 3.1. In these cases the functions are slowly varying by Lemma 3.1. In all other cases these functions are linear near 0 and, hence, positively increasing. □

Finally, the theorems from Section 2 lead to the following three different Riesz basis results in the four cases of Lemma 3.3.

Proposition 3.4. *Let the weight function r be given by* (3.3). *Then the following statements hold:*

(i) *If $0 < c < d < 1$, then neither the eigenvalue problem* (1.1) *nor the eigenvalue problem* (1.2) *has the Riesz basis property.*

(ii) *If $0 < c < d = 1$, then neither the eigenvalue problem* (1.1) *nor the eigenvalue problem* (1.2) *has the Riesz basis property.*

(iii) *If $0 = c < d < 1$, then the eigenvalue problem* (1.1) *has the Riesz basis property but the eigenvalue problem* (1.2) *does not.*

(iv) *If $c = 0$ and $d = 1$, then both eigenvalue problems* (1.1) *and* (1.2) *have the Riesz basis property.*

Proof. As in Proposition 3.2, statement (i) follows from Theorem 2.2 (ii) and Theorem 2.4, since r is in this case locally odd at the turning point and at the boundary. Similarly, we obtain statement (ii) for problem (1.1) from Theorem 2.2 (ii), and then also for problem (1.2) from Theorem 2.3. (Note that here we cannot apply Theorem 2.4, since r is no longer locally odd at the boundary.) Statement (iii) follows from Theorem 2.2 (i) and Theorem 2.5. Here, we use the fact that r is in this case locally odd at the boundary. Finally, statement (iv) can be obtained from Theorem 2.2 (i) and Theorem 2.3. □

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ON PARSEVAL J-FRAMES

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This paper is dedicated to Seppo Hassi, to whom we would like to send our warmest wishes for his 60th birthday. All the best!

1 Introduction

Frame theory finds many applications in engineering, applied mathematics, and computer sciences. It turns out to be useful because of its properties, which are unavailable for bases, e.g., non-unique decompositions. For instance, frames have been shown to be useful in signal processing applications when noisy channels are involved, because a frame allows one to reconstruct vectors (signals) even if some of the frame coefficients are missing (or corrupted), see, e.g., (Christensen, 2016; Heil, 2011; Mallat, 1999).

Usually, frames are defined in a Hilbert space setting. Let $\mathcal{H}$ be a Hilbert space with an inner product $(\cdot,\cdot)$ and let $\mathbb{J}$ be the index set (countable or finite). Then a set of vectors

$$\mathcal{F}_\varphi = \{\varphi_j : j \in \mathbb{J}\}$$

is called a *frame* for $\mathcal{H}$ if there exist constants (frame bounds) $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{j\in\mathbb{J}} |(f,\varphi_j)|^2 \le B\|f\|^2, \qquad f \in \mathcal{H}. \tag{1.1}$$

Furthermore, the set $\mathcal{F}_\varphi$ is called a *Parseval frame* if the inequalities (1.1) hold for the constants $A = B = 1$.

A frame $\mathcal{F}_\psi = \{\psi_j : j \in \mathbb{J}\}$ satisfying the condition

$$f = \sum_{j\in\mathbb{J}} (f,\psi_j)\varphi_j, \qquad f \in \mathcal{H}, \tag{1.2}$$

is called a *dual frame* of $\mathcal{F}_\varphi$. In general, dual frames are not determined uniquely and their proper choice (i.e., one that fits the specific problem well) is of great importance. A description of dual frames based on the Naimark dilation theorem is discussed in Kamuda & Kużel (2020).

Each Parseval frame $\mathcal{F}_\varphi$ is dual to itself and the reconstruction formula (1.2) can be simplified

$$f = \sum_{j\in\mathbb{J}} (f,\varphi_j)\varphi_j, \qquad f \in \mathcal{H}. \tag{1.3}$$

There are several advantages in considering frames in the setting of Kreĭn spaces, instead of Hilbert spaces. For example, noise can be extracted from the signal in an easy way. Indeed, consider a signal with two dominants (e.g., given by high and low frequency signals). Let the projection P in $\mathcal{H}$ be an ideal low pass filter and let the projection $I - P$ in $\mathcal{H}$ be an ideal high pass filter. A signal (vector) $\varphi \in \mathcal{H}$, for which the equality $\|P\varphi\| = \|(I-P)\varphi\|$ holds, is considered as a noise. In other words,

φ is considered to be a noise if its high frequency level is equal to its low frequency level. From a practical perspective, it is convenient to fix $\varepsilon > 0$, and to say that φ is considered to be a noise if

$$\left| \|P\varphi\|^2 - \|(I-P)\varphi\|^2 \right| < \varepsilon.$$

Every difference between the norms for φ below ε implies that φ is a noise. From a Hilbert space view, one has to look at the decomposition $\varphi = P\varphi + (I-P)\varphi$. Hence, it is necessary to consider two frames $\{P\varphi\} \cup \{(I-P)\varphi\}$. From a Kreĭn space perspective one can, using P, introduce a fundamental symmetry

$$J = P - (I-P) = 2P - I,$$

and define an indefinite inner product

$$[f,g] = (Jf,g) = (Pf,Pg) - ((I-P)f,(I-Pg)), \qquad f,g \in \mathcal{H}.$$

Thus φ is considered as a noise when $|[\varphi,\varphi]| < \varepsilon$ (for a fixed ε). For more details about this example, we encourage the reader to look at Giribet et al. (2012: Section 3).

In the present paper, we extend the concept of Parseval frames to the setting of Kreĭn spaces, combining the simplicity of the reconstruction formula (1.3) with the additional possibilities for the reduction of noises offered by Kreĭn spaces. In the last few years, several papers devoted to the development of frame theory in Kreĭn space have been published, e.g., (Acosta-Humánez, Esmeral & Ferrer, 2015; Escobar, Esmeral & Ferrer, 2016; Esmeral, Ferrer & Wagner, 2015; Esmeral, Ferrer & Lora, 2016; Giribet et al., 2012; Giribet, Maestripieri & Martínez Pería, 2018). A review of the definitions for frames in Kreĭn spaces is given in Kamuda & Kużel (2019).

Our definition of Parseval J-frames is based on the concept of dual quasi-maximal subspaces introduced in Kamuda, Kuzhel & Sudilovskaja (2019) and discussed in Section 2. In Section 3 we define Parseval J-frames and establish their principal properties. In Section 4 we show that eigenfunctions of the harmonic oscillator $H = -\frac{d^2}{dx^2} + x^2 + 2iax$, associated with a $\mathcal{PT}$-symmetric potential, constitute a Parseval J-frame, where J is the space parity operator.

2 Dual quasi-maximal subspaces

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and let J be a non-trivial fundamental symmetry, i.e., $J = J^*$, $J^2 = I$, and $J \neq \pm I$. The space $\mathcal{H}$ endowed with the indefinite inner product (indefinite metric)

$$[f,g] = (Jf,g), \qquad f,g \in \mathcal{H},$$

is called a *Kreĭn space;* it will be denoted by $(\mathcal{H},[\cdot,\cdot])$. All of the following topological notions are considered in the Hilbert space topology.

A closed subspace $\mathcal{L}$ of the Kreĭn space $(\mathcal{H},[\cdot,\cdot])$ is called *neutral, negative, or positive* if all nonzero elements $f \in \mathcal{L}$ are, respectively, neutral $[f,f] = 0$, negative $[f,f] < 0$, or positive $[f,f] > 0$. A subspace $\mathcal{L}$ of $\mathcal{H}$ is called *definite* if it is either positive or negative. Subspaces $\mathcal{L}_\pm$ of $\mathcal{H}$ are called *dual* if $\mathcal{L}_+$ is positive, $\mathcal{L}_-$ is negative, and they are orthogonal with respect to the indefinite metric (*J-orthogonal*), the latter meaning that $[f_+,f_-] = 0$ for all $f_\pm \in \mathcal{L}_\pm$.

In each of the above mentioned classes we can define maximal subspaces. For instance, a closed positive subspace $\mathcal{L}^{\max}$ is called *maximal positive* if $\mathcal{L}^{\max}$ is not a proper subspace of a positive subspace in the Kreĭn space $(\mathcal{H}, [\cdot,\cdot])$. Similarly, subspaces $\mathcal{L}_{\pm}^{\max}$ are called *dual maximal definite* if they are dual, $\mathcal{L}_{+}^{\max}$ is maximal positive, and $\mathcal{L}_{-}^{\max}$ is maximal negative.

The pair of subspaces $\mathcal{H}_{+} = \ker(I-J)$ and $\mathcal{H}_{-} = \ker(I+J)$ in the fundamental decomposition of the Kreĭn space

$$\mathcal{H} = \mathcal{H}_{+} [\oplus] \mathcal{H}_{-} \tag{2.1}$$

is an example of dual maximal definite subspaces; the brackets in (2.1) mean that $\mathcal{H}_{\pm}$ are orthogonal with respect to $[\cdot,\cdot]$.

The next result follows from (Albeverio & Kuzhel, 2015; Kamuda, Kuzhel & Sudilovskaja, 2019).

Lemma 2.1. *The subspaces $\mathcal{L}_{\pm}^{\max}$ are dual maximal definite if and only if there exists a self-adjoint operator Q in $\mathcal{H}$ that anticommutes with J, i.e.,*

$$QJf = -JQf, \qquad f \in \mathcal{D}(Q),$$

such that

$$\mathcal{L}_{+}^{\max} = (I + \tanh Q/2)\mathcal{H}_{+} \quad \textit{and} \quad \mathcal{L}_{-}^{\max} = (I + \tanh Q/2)\mathcal{H}_{-}. \tag{2.2}$$

The self-adjoint strong contraction $\tanh Q/2$ in (2.2) anticommutes with J and it characterizes the "deviation" of the dual maximal definite subspaces $\mathcal{L}_{\pm}^{\max}$ with respect to $\mathcal{H}_{\pm}$. By a *strong contraction* we mean an operator T such that $\|Tf\| < \|f\|$ for nonzero f.

In view of Lemma 2.1, a self-adjoint operator Q can be considered as a parameter describing all possible pairs of dual maximal definite subspaces $\mathcal{L}_{\pm}^{\max}$. This fact allows one to associate with $\mathcal{L}_{\pm}^{\max}$ a new Hilbert space $\mathcal{H}_{-Q}$, which is determined as the completion of the direct sum

$$\mathcal{D}_{\max} = \mathcal{L}_{+}^{\max} [\dot{+}] \mathcal{L}_{-}^{\max} \tag{2.3}$$

with respect to the norm $\|\cdot\|_{-Q} = \sqrt{(\cdot,\cdot)}$, generated by the inner product

$$(f,g)_{-Q} = (e^{-Q}f, g), \qquad f, g \in \mathcal{D}_{\max}.$$

If Q is bounded, then $\mathcal{D}_{\max} = \mathcal{H}$, the space $\mathcal{H}_{-Q}$ coincides with $\mathcal{H}$ (as the set of elements), and $(\cdot,\cdot)_{-Q}$ is equivalent to the initial inner product $(\cdot,\cdot)$.

Let $\mathcal{L}_{\pm}$ be a pair of dual definite subspaces such that their J-orthogonal sum

$$\mathcal{L}_{+} [\dot{+}] \mathcal{L}_{-} \tag{2.4}$$

is dense in $\mathcal{H}$. Due to the Phillips result (Phillips, 1961: Theorem 2.1), the pair $\mathcal{L}_{\pm}$ can be extended to dual maximal definite subspaces $\mathcal{L}_{\pm}^{\max}$. In general, this extension is not determined uniquely, see (Kamuda, Kuzhel & Sudilovskaja, 2019; Langer, 1970).

Definition 2.2. The dual definite subspaces $\mathcal{L}_{\pm}$ are called *quasi-maximal* if their direct sum (2.4) is dense in $\mathcal{H}$ and there exists extensions $\mathcal{L}_{\pm} \to \mathcal{L}_{\pm}^{\max}$ to dual maximal definite subspaces $\mathcal{L}_{\pm}^{\max}$ such that (2.4) is a dense set in the Hilbert space $\mathcal{H}_{-Q}$ associated with $\mathcal{L}_{\pm}^{\max}$.

The next technical result was proved in Kamuda, Kuzhel & Sudilovskaja (2019: Lemma 4.7).

Lemma 2.3. *Let $\mathcal{L}_+$ and $\mathcal{L}_-$ be a pair of dual definite subspaces and let $\mathcal{L}_+^{\max}$ and $\mathcal{L}_-^{\max}$ be their extensions to the dual maximal definite subspaces, respectively. Moreover, let $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-$ be a subspace of $\mathcal{H}$, where $\mathcal{M}_\pm$ determine $\mathcal{L}_\pm \subset \mathcal{L}_\pm^{\max}$ in* (2.2)*, i.e.,*

$$\mathcal{L}_+ = (I + \tanh Q/2)\mathcal{M}_+, \quad \mathcal{L}_- = (I + \tanh Q/2)\mathcal{M}_-, \quad \mathcal{M}_+ \subset \mathcal{H}_+, \quad \mathcal{M}_- \subset \mathcal{H}_-. \tag{2.5}$$

Then the direct sum (2.4) *is dense in the Hilbert space $\mathcal{H}_{-Q}$, constructed by $\mathcal{L}_\pm^{\max}$, if and only if*

$$\mathcal{R}(\cosh^{-1} Q/2) \cap (\mathcal{H} \ominus \mathcal{M}) = \{0\}.$$

The direct sum (2.4) allows one to define the operator

$$Sf = S(f_+ + f_-) = Jf_+ - Jf_-, \qquad f_\pm \in \mathcal{L}_\pm.$$

with the domain $\mathcal{D}(S) = \mathcal{L}_+ [\dot{+}] \mathcal{L}_-$. It follows from (Kamuda, Kuzhel & Sudilovskaja, 2019) that S is a densely defined symmetric operator in $\mathcal{H}$ and

$$(S(f_+ + f_-), f_+ + f_-) = [f_+, f_+] - [f_-, f_-] > 0.$$

Similarly, the direct sum (2.3) of dual maximal definite subspaces $\mathcal{L}_\pm^{\max}$ determines a positive self-adjoint operator

$$Af = A(f_+ + f_-) = Jf_+ - Jf_-, \quad f_\pm \in \mathcal{L}_\pm^{\max}, \quad \mathcal{D}(A) = \mathcal{D}_{\max}, \tag{2.6}$$

which is an extension of S when the pair $\mathcal{L}_\pm^{\max}$ is an extension of $\mathcal{L}_\pm$, see (Albeverio & Kuzhel, 2015; Kamuda, Kuzhel & Sudilovskaja, 2019).

From the construction it follows that $\mathcal{L}_\pm = (I \pm JS)\mathcal{D}(S)$. Therefore, S determines the subspaces $\mathcal{L}_\pm$ and one can expect that the quasi-maximality of $\mathcal{L}_\pm$ can be characterized in terms of self-adjoint extensions of S. For this reason, we recall from Arlinskiĭ et al. (2001) that a nonnegative self-adjoint extension A of S is called *an extremal extension* if

$$\inf_{f \in \mathcal{D}(S)} (A(\phi - f), (\phi - f)) = 0 \qquad \text{for all } \phi \in \mathcal{D}(A). \tag{2.7}$$

The Friedrichs extension and the Kreĭn-von Neumann extension are examples of extremal extensions of S.

Theorem 2.4. *The dual definite subspaces $\mathcal{L}_\pm$ are quasi-maximal if and only if there exists an extremal extension $A = e^{-Q}$ of S, where Q anticommutes with J.*

Proof. Let $\mathcal{L}_\pm$ be quasi-maximal subspaces and let $\mathcal{L}_\pm^{\max}$ be the corresponding dual maximal definite subspaces as in Definition 2.2. Due to Lemma 2.1, the spaces $\mathcal{L}_\pm^{\max}$ are defined by (2.2). Therefore, vectors $f_\pm \in \mathcal{L}_\pm^{\max}$ have the form $f_\pm = (I + \tanh Q/2)x_\pm$, with $x_\pm \in \mathcal{H}_\pm$, and the operator A defined by (2.6) acts as follows:

$$A(f_+ + f_-) = (I - \tanh Q/2)x_+ + (I - \tanh Q/2)x_- = e^{-Q}(f_+ + f_-). \tag{2.8}$$

Here we used the relations $Jx_\pm = \pm x_\pm$, $e^{-Q}(I + \tanh Q/2) = I - \tanh Q/2$, and the fact that $\tanh Q/2$ anticommutes with J. Therefore $A = e^{-Q}$ is a self-adjoint extension of S. Now we should prove that A is an extremal extension. To do that, we rewrite (2.7) with the use of $\|\cdot\|_{-Q}$,

$$\inf_{f\in\mathcal{D}(S)}(A(\phi-f),(\phi-f))=\inf_{f\in\mathcal{D}(S)}\|\phi-f\|^2_{-Q}=0 \qquad \text{for all } \phi\in\mathcal{D}(e^{-Q}),$$

and take into account that $\mathcal{D}(S)$ is dense in the Hilbert space $\mathcal{H}_{-Q}$.

The converse statement is obvious: the required dual maximal definite subspaces $\mathcal{L}_{\pm}^{\max}$ in Definition 2.2 are determined by the formula $\mathcal{L}_{\pm}^{\max}=(I\pm JA)\mathcal{D}(A)$, where $A=e^{-Q}$ and Q anticommutes with J. □

3 Parseval J-frames

Definition 3.1. A set of vectors $\mathcal{F}_{\varphi}=\{\varphi_j : j\in\mathbb{J}\}$ is called a *Parseval J-frame* if there exists a pair of dual quasi-maximal subspaces $\mathcal{L}_{\pm}$ such that each vector $\varphi_j\in\mathcal{F}_{\varphi}$ belongs either to $\mathcal{L}_+$ or $\mathcal{L}_-$, and

$$|[f_{\pm},f_{\pm}]|=\sum_{j\in\mathbb{J}}|[f_{\pm},\varphi_j]|^2 \qquad \text{for all } f_{\pm}\in\mathcal{L}_{\pm}. \tag{3.1}$$

Theorem 3.2. *For each Parseval J-frame $\mathcal{F}_{\varphi}$ there exists a self-adjoint operator Q which anticommutes with J, such that $\mathcal{F}_{\varphi}$ is a Parseval frame in a new Hilbert space $\mathcal{H}_{-Q}$.*

Proof. Each Parseval J-frame is associated with certain dual quasi-maximal subspaces $\mathcal{L}_{\pm}$. According to Definition 2.2, there exists a self-adjoint operator Q which anticommutes with J and which is such that the direct sum $\mathcal{L}_+ + \mathcal{L}_-$ is dense in the Hilbert space $\mathcal{H}_{-Q}$. In view of (2.6) and (2.8),

$$e^{-Q}f=e^{-Q}(f_+ + f_-)=Jf_+ - Jf_-, \qquad f_{\pm}\in\mathcal{L}_{\pm}^{\max}.$$

This means that the subspaces $\mathcal{L}_{\pm}^{\max}$ are orthogonal with respect to the inner product $(\cdot,\cdot)_{-Q}$ of $\mathcal{H}_{-Q}$. Moreover,

$$\|f_{\pm}\|^2_{-Q}=|[f_{\pm},f_{\pm}]|, \quad (f_{\pm},g_{\pm})_{-Q}=\pm[f_{\pm},g_{\pm}], \quad f_{\pm},g_{\pm}\in\mathcal{L}_{\pm}. \tag{3.2}$$

Taking these relations into account, we rewrite (3.1) as

$$\|f\|^2_{-Q}=\|f_+\|^2_{-Q}+\|f_-\|^2_{-Q}=\sum_{j\in\mathbb{J}}|(f_{\pm},\varphi_j)_{-Q}|^2 \tag{3.3}$$

for $f=f_+ + f_- \in \mathcal{L}_+[\dot{+}]\mathcal{L}_-$. The relation (3.3) is extended to all $f\in\mathcal{H}_{-Q}$ with the use of (Christensen, 2016: Lemma 5.1.7). □

By Theorem 3.2 every Parseval J-frame $\mathcal{F}_{\varphi}$ turns out to be a Parseval frame in a suitably chosen Hilbert space $\mathcal{H}_{-Q}$ and the following reconstruction formula holds

$$f=\sum_{j\in\mathbb{J}}(f,\varphi_j)_{-Q}\,\varphi_j, \qquad f\in\mathcal{H}_{-Q}, \tag{3.4}$$

see (1.3), where the series converges in $\mathcal{H}_{-Q}$. Assuming that $f=f_+ + f_- \in\mathcal{L}_+[\dot{+}]\mathcal{L}_-$ and using the second relation in (3.2) we obtain

$$\begin{aligned}(f,\varphi_j)_{-Q} &= (f_+,\varphi_j)_{-Q} = [f_+,\varphi_j] = [f,\varphi_j], && \varphi_j\in\mathcal{L}_+;\\ (f,\varphi_j)_{-Q} &= (f_-,\varphi_j)_{-Q} = -[f_-,\varphi_j] = -[f,\varphi_j], && \varphi_j\in\mathcal{L}_-.\end{aligned}$$

Therefore, for $f\in\mathcal{L}_+\,[\dot{+}]\,\mathcal{L}_-$, the series (3.4) takes the form

$$f=\sum_{j\in\mathbb{J}}\delta_j[f,\varphi_j]\varphi_j,\qquad \delta_j=\mathrm{sgn}([\varphi_j,\varphi_j]).$$

The obtained relation leads to the conclusion that

$$[f,f]=\sum_{j\in\mathbb{J}}\delta_j|[f,\varphi_j]|^2.$$

As was mentioned above, one of the characteristic properties of J-frames is the possibility to interpret the signal f as a noise if its indefinite metric $[f,f]$ is close to 0. Given that fact, a Parseval J-frame $\mathcal{F}_\varphi$ turns out to be useful. The above formula allows one to extract vectors f, which represent a noise by evaluation of the indefinite metric coefficients $[f,\varphi_j]$. The next statement shows that Parseval J-frames can be easily constructed.

Theorem 3.3. *Let $\mathcal{F}_\varphi=\{\varphi_j : j\in\mathbb{J}\}$ be a complete set in a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(i) *$\mathcal{F}_\varphi$ is a Parseval J-frame.*

(ii) *There exist a, not necessarily bounded, self-adjoint operator Q in $\mathcal{H}$, which anticommutes with J, and a Parseval frame $\mathcal{F}_e=\{e_j : j\in\mathbb{J}\}$ consisting of eigenfunctions of the operator J, i.e., $Je_j=e_j$ or $Je_j=-e_j$, such that*

$$\varphi_j=e^{Q/2}e_j,\qquad j\in\mathbb{J}. \tag{3.5}$$

Proof. (i) $\Rightarrow$ (ii) In view of Theorem 3.2, there exists a self-adjoint operator Q in $\mathcal{H}$ that anticommutes with J and $\mathcal{F}_\varphi$ is a Parseval frame in the Hilbert space $\mathcal{H}_{-Q}$. Setting $f\in\mathcal{D}(e^{-Q/2})\subset\mathcal{H}_{-Q}$ in (3.4) and taking into account that $\varphi_j\in\mathfrak{L}_+\,[\dot{+}]\,\mathfrak{L}_-\subset\mathcal{D}(e^{-Q})$ we arrive at the conclusion that

$$\|\gamma\|^2=\sum_{j\in\mathbb{J}}|(\gamma,e^{-Q/2}\varphi_j)|^2,$$

for all $\gamma=e^{-Q/2}f$ in the set $\mathcal{R}(e^{-Q/2})$ which is dense in $\mathcal{H}$. By Christensen (2016: Lemma 5.1.9) the obtained equality can be extended to the whole space $\mathcal{H}$. Hence, $\mathcal{F}_e=\{e_j=e^{-Q/2}\varphi_j : j\in\mathbb{J}\}$ is a Parseval frame in $\mathcal{H}$ and the relation (3.5) holds.

The definition of e_j gives us that $e_j\in\mathcal{D}(e^{Q/2})\cap\mathcal{D}(e^{-Q/2})$, since $\varphi_j\in\mathcal{D}(e^{-Q})$. Therefore, it belongs to the domain of definition of the operator $\cosh Q/2=\frac{1}{2}(e^{Q/2}+e^{-Q/2})$. Moreover,

$$(I+\tanh Q/2)\cosh Q/2\,e_j=(\cosh Q/2+\sinh Q/2)\,e_j=e^{Q/2}\,e_j=\varphi_j\in\mathcal{L}_\pm. \tag{3.6}$$

Comparing the above relation with (2.2) and taking into account that $\ker(I+\tanh Q/2)=\{0\}$ (since $\tanh Q/2$ is a strong contraction) we arrive at the conclusion that

$$\cosh Q/2\,e_j\in\mathcal{H}_+\quad\text{or}\quad\cosh Q/2\,e_j\in\mathcal{H}_-. \tag{3.7}$$

Since $\cosh Q/2$ commutes with J, one has

$$J\cosh Q/2 = J\frac{1}{2}\big(e^{Q/2}+e^{-Q/2}\big) = \frac{1}{2}\big(e^{-Q/2}+e^{Q/2}\big)J = \cosh Q/2\, J.$$

Therefore formula (3.7) implies that $e_j \in \mathcal{H}_+ = \ker(I-J)$ or $e_j \in \mathcal{H}_- = \ker(I+J)$.

(ii) $\Rightarrow$ (i) In view of (3.5),

$$\begin{aligned}[][\varphi_j,\varphi_i] &= (Je^{Q/2}e_j, e^{Q/2}e_i) = (e^{-Q/2}Je_j, e^{Q/2}e_i)\\ &= (Je_j,e_i) = \begin{cases} (e_j,e_i), & e_j,e_i\in\mathcal{H}_+,\\ -(e_j,e_i), & e_j,e_i\in\mathcal{H}_-,\\ 0, & e_j\in\mathcal{H}_\pm, e_i\in\mathcal{H}_\mp.\end{cases}\end{aligned}$$

This means that each non-zero vector $f\in \operatorname{span}\{\varphi_n = e^{Q/2}e_n \,:\, e_n\in\mathcal{H}_+\}$ is positive, because

$$[f,f] = \Big[\sum_j c_j\varphi_j, \sum_k c_k\varphi_k\Big] = \sum_{j,k} c_j\bar{c}_k(e_j,e_k) = \Big(\sum_j c_je_j, \sum_k c_ke_k\Big) = \|e^{-Q/2}f\|^2.$$

Similarly, vectors $g\in\operatorname{span}\{\varphi_j = e^{Q/2}e_j \,:\, e_j\in\mathcal{H}_-\}$ are negative. Obviously, $[f,g]=0$.

Denote by $\mathcal{L}_\pm$ the completion of $\operatorname{span}\{\varphi_j = e^{Q/2}e_j : e_j\in\mathcal{H}_\pm\}$ in $\mathcal{H}$. By the construction, $\mathcal{L}_\pm$ are dual definite subspaces. Furthermore, for each $f\in\mathcal{L}_+$,

$$[f,\varphi_j] = (Jf, e^{Q/2}e_j) = (f, Je^{Q/2}e_j) = (f, e^{-Q/2}Je_j) = (e^{-Q/2}f, e_j).$$

Taking into account that $\mathcal{F}_e = \{e_j : j\in\mathbb{J}\}$ is a Parseval frame, we get

$$\sum_{j\in\mathbb{J}} |[f,\varphi_j]|^2 = \sum_{j\in\mathbb{J}} |(e^{-Q/2}f,e_j)|^2 = \|e^{-Q/2}f\|^2 = (f,f)_{-Q} = [f,f], \qquad f\in\mathcal{L}_+.$$

Similarly, for $f\in\mathcal{L}_-$, $[f,\varphi_j] = -(e^{-Q/2}f,e_j)$ and

$$\sum_{j\in\mathbb{J}} |[f,\varphi_j]|^2 = \sum_{j\in\mathbb{J}} |(e^{-Q/2}f,e_j)|^2 = \|e^{-Q/2}f\|^2 = (f,f)_{-Q} = -[f,f].$$

This analysis leads to the conclusion that (3.1) holds. To complete the proof, it suffices to verify that the spaces $\mathcal{L}_\pm$ are quasi-maximal. To this end, we show that $\mathcal{L}_+\,[\dot{+}]\,\mathcal{L}_-$ is a dense set in $\mathcal{H}_{-Q}$.

Taking into account that the spaces $\mathcal{L}_\pm$ are defined as completions of span$\{\varphi_j\}$, as above, and comparing the formulas (2.5) and (3.6) we arrive at the conclusion that $\mathcal{M}$ coincides with the completion of $\operatorname{span}\{\cosh Q/2\, e_j \,:\, j\in\mathbb{J}\}$. Hence,

$$\mathcal{H}\ominus\mathcal{M} = \mathcal{H}\ominus\operatorname{span}\{\cosh Q/2\, e_j\}.$$

Assume that $\mathcal{L}_+\,[\dot{+}]\,\mathcal{L}_-$ is not dense in $\mathcal{H}_{-Q}$. Then, in view of Lemma 2.3, there exists a vector $p = \cosh^{-1} Q/2\, u \neq 0$, such that $p\in\mathcal{H}\ominus\mathcal{M}$, i.e.,

$$0 = (p, \cosh Q/2\, e_j) = (\cosh^{-1} Q/2\, u, \cosh Q/2\, e_j) = (u, e_j) \qquad \text{for all } e_j\in\mathcal{F}_e.$$

Since $\mathcal{F}_e$ is a Parseval frame in $\mathcal{H}$, this means that $u=0$ and ,hence, $p=0$; a contradiction. Consequently, $\mathcal{L}_+\,[\dot{+}]\,\mathcal{L}_-$ is a dense set in $\mathcal{H}_{-Q}$. $\square$

4 Harmonic oscillator with $\mathcal{PT}$-symmetric potential

In the space $\mathcal{H} = L_2(\mathbb{R})$ we consider the fundamental symmetry $J = \mathcal{P}$, where $\mathcal{P}f(x) = f(-x)$ is the space parity operator. The subspaces $\mathcal{H}_\pm$ of the fundamental decomposition (2.1) coincide with the subspaces of even and odd functions of $L_2(\mathbb{R})$.

The Hermite functions

$$e_j(x) = \frac{1}{\sqrt{2^j j! \sqrt{\pi}}} H_j(x) e^{-x^2/2}, \quad H_j(x) = e^{x^2/2}\left(x - \frac{d}{dx}\right)^j e^{-x^2/2}, \qquad j \in \mathbb{J} = \mathbb{N} \cup \{0\},$$

are eigenfunctions, $H_0 e_j = (2j+1)e_j$, of the harmonic oscillator

$$H_0 = -\frac{d^2}{dx^2} + x^2, \qquad \mathcal{D}(H_0) = \{f \in W_2^2(\mathbb{R}) : x^2 f \in L_2(\mathbb{R})\}$$

and they form an orthonormal basis $\mathcal{F}_e = \{e_j : j \in \mathbb{J}\}$ of $L_2(\mathbb{R})$. The functions e_j are even or odd for j being even or odd, respectively. This means that $e_j \in \mathcal{H}_+$ or $e_j \in \mathcal{H}_-$. Since the functions $e_j(x)$ are entire functions on $\mathbb{C}$, their complex shift can be defined:

$$\varphi_j(x) := e_j(x + ia), \qquad a \in \mathbb{R} \setminus \{0\}, \quad j = 0, 1, 2, \ldots$$

The set $\mathcal{F}_\varphi = \{\varphi_j : j \in \mathbb{J}\}$ is complete in $L_2(\mathbb{R})$, see (Mityagin, Siegl & Viola, 2017: Lemma 2.5); in the following the dependence on a will not be explicitly indicated. Applying the Fourier transform

$$Ff = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx$$

to φ_j, we obtain $F\varphi_j = e^{-a\xi} F e_j$. Therefore, $\varphi_j = F^{-1} e^{-a\xi} F e_j$. The last relation can be rewritten as

$$\varphi_j = e^{Q/2} e_j,$$

where $Q = 2ai\frac{d}{dx}$, $\mathcal{D}(Q) = W_2^1(\mathbb{R})$, is a self-adjoint operator in $L_2(\mathbb{R})$ that anticommutes with $J = \mathcal{P}$ in $L_2(\mathbb{R})$. By virtue of Theorem 3.3, the set $\mathcal{F}_\varphi$ is a Parseval J-frame.

The set $\mathcal{F}_\varphi$ cannot be a Schauder basis in $L_2(\mathbb{R})$. Indeed, assume that $\mathcal{F}_\varphi$ is a Schauder basis. Then, by Heil (2011: Theorem 4.13),

$$1 \leq \|\varphi_j\|\, \|\psi_j\| \leq C, \qquad j \in \mathbb{J},$$

where $\mathcal{F}_\psi = \{\psi_j : j \in \mathbb{J}\}$ is the bi-orthogonal sequence for $\mathcal{F}_\varphi$. It is easy to see that

$$\psi_j = \operatorname{sgn}([\varphi_j, \varphi_j]) J\varphi_j.$$

Hence, the last inequalities take the form $1 \leq \|\varphi_j\|^2 \leq C$. On the other hand, by (Mityagin, Siegl & Viola, 2017: Theorem 2.6)

$$\lim_{j \to \infty} \frac{1}{\sqrt{j}} \log \|\varphi_j\|^2 = 2^{3/2}|a|,$$

which contradicts the inequality $\|\varphi_j\|^2 \leq C$. Hence, $\mathcal{F}_\varphi$ cannot be a Schauder basis.

The functions of the Parseval J-frame $\mathcal{F}_\varphi$ are simple eigenfunctions, $H\varphi_j = (2j + 1 + a^2)\varphi_j$, of the non-self-adjoint operator:

$$H = -\frac{d^2}{dx^2} + x^2 + 2iax, \qquad \mathcal{D}(H) = \mathcal{D}(H_0),$$

see (Mityagin, Siegl & Viola, 2017: Lemma 2.4). The operator H can be considered as a perturbation of the harmonic oscillator H_0 by a $\mathcal{PT}$-symmetric potential $V(x) = 2iax$, i.e., $H = H_0 + V$. The $\mathcal{PT}$-symmetry of $V(x)$ means that $\mathcal{PT}V(x) = V(x)\mathcal{PT}$, where $\mathcal{T}$ is the complex conjugation operator, i.e., $\mathcal{T}f = \overline{f}$.

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IDEMPOTENT RELATIONS, SEMI-PROJECTIONS, AND GENERALIZED INVERSES

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Dedicated to our friend Seppo Hassi on the occasion of his sixtieth birthday

1 Introduction

As a motivation for this paper, consider the identities

$$ABA = A \quad \text{and} \quad BAB = B, \tag{1.1}$$

where $A \in \mathbf{B}(\mathfrak{H}, \mathfrak{K})$ and $B \in \mathbf{B}(\mathfrak{K}, \mathfrak{H})$ for Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$, which have been studied in the literature in much detail. Clearly, if (1.1) holds, then the operators $BA \in \mathbf{B}(\mathfrak{H})$ and $AB \in \mathbf{B}(\mathfrak{K})$ are both idempotent. If A is invertible, i.e., $A^{-1} \in \mathbf{B}(\mathfrak{K}, \mathfrak{H})$, then $B = A^{-1}$ makes (1.1) valid. However, given any $A \in \mathbf{B}(\mathfrak{H}, \mathfrak{K})$ with $\operatorname{ran} A$ closed in $\mathfrak{K}$, there are also candidates for $B \in \mathbf{B}(\mathfrak{K}, \mathfrak{H})$, so that (1.1) is satisfied and so that, in addition,

$$(BA)^* = BA \quad \text{and} \quad (AB)^* = AB; \tag{1.2}$$

thus BA and AB are orthogonal projections. In the matrix case, all this goes back to E.H. Moore (1920), A. Bjerhammar (1951), and R. Penrose (1955); see also, for instance, (Ben-Israel & Greville, 2003; Campbell & Meyer, 1991; Nashed, 1976; Rao & Mitra, 1971). An extension to the case that the operators A and B are unbounded can be found in Labrousse & M'Bekhta (1992) and Labrousse (1992). The purpose of the present paper is to look at the formal aspects of the identities in (1.1) and (1.2) in the wider context of linear relations, but in the absence of any topology, and to give a survey of the characteristic results.

In order to consider the identities (1.1) in an algebraic setting, let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let B be a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. Products of linear relations will be in the sense of relations. When the identities (1.1) are satisfied by linear relations A and B, then it is clear that the products BA and AB are idempotent relations in $\mathfrak{H}$ and $\mathfrak{K}$, respectively; cf. Definition 5.1. In the present general context the conditions in (1.2) are replaced by

$$\operatorname{ran} BA \subset \operatorname{dom} BA \quad \text{and} \quad \operatorname{ran} AB \subset \operatorname{dom} AB, \tag{1.3}$$

i.e., BA and AB are semi-projections; cf. Definition 5.3. The notions of idempotent relation and semi-projection go back to Labrousse (2003). Note, in particular, that if $B = A^{-1}$, the formal inverse of A, then the linear relations $A^{-1}A$ and AA^{-1} are indeed semi-projections; cf. (2.1). There are various other possibilities to satisfy (1.1) and (1.3) by considering, instead of A^{-1}, specific choices for an algebraic operator part of A^{-1}.

Here is an outline of the contents of this paper. Some useful properties are recalled in Section 2. As a preparation there are several modifications of the notion of idempotent relation that will be considered in Section 3 and Section 4. Idempotent relations and semi-projections are discussed in Section 5. Also the geometric meaning of semi-projections is explained there. Section 6 contains a resolvent identity for semi-projections. In Sections 7, 8, and 9 the cases $A \subset ABA$, $ABA \subset A$, and $A = ABA$ are considered, respectively; they serve as illustrations for the earlier sections. In Section 10 the previous cases are characterized in terms of set inclusions between A and B^{-1}. The notion of generalized inverse for linear relations is introduced in Section 11. It is shown in Section 12 that one may choose an operator part for A^{-1}, so that it serves as a generalized inverse. In Section 13 the results from Sections 3 and Section 4 are augmented.

2 Preliminaries

Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces and let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$. Recall that a linear relation A is defined as a linear subspace of the product space $\mathfrak{H} \times \mathfrak{K}$, with $\mathrm{dom}\ A$, $\mathrm{ran}\ A$, $\ker A$, and $\mathrm{mul}\ A$ being the domain, range, kernel, and multivalued part of A; cf. (Arens, 1961; Sandovici, de Snoo & Winkler, 2007); see also (Behrndt, Hassi & de Snoo, 2020). The inverse of a relation A is given by

$$A^{-1} := \{\{g, f\} : \{f, g\} \in A\},$$

which is a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. It is not difficult to see that the following componentwise sum decompositions hold

$$\begin{aligned} A^{-1}A &= I_{\mathrm{dom}\,A} \,\widehat{+}\, (\{0\} \times \ker A) = I_{\mathrm{dom}\,A} \,\widehat{+}\, (\ker A \times \{0\}), \\ AA^{-1} &= I_{\mathrm{ran}\,A} \,\widehat{+}\, (\mathrm{mul}\ A \times \{0\}) = I_{\mathrm{ran}\,A} \,\widehat{+}\, (\{0\} \times \mathrm{mul}\ A), \end{aligned} \tag{2.1}$$

cf. (Behrndt, Hassi & de Snoo, 2020). Here, and in the following, all such products are meant in the sense of linear relations. In particular, it is useful to observe that for any linear relation A one has the identities

$$AA^{-1}A = A \quad \text{and} \quad A^{-1}AA^{-1} = A^{-1}, \tag{2.2}$$

as can be easily verified by means of the identities in (2.1). Note that (2.1) shows that the terminology of inverse relation is a rather formal way of speaking.

Let L and R be linear relations from $\mathfrak{H}$ to $\mathfrak{K}$ such that $L \subset R$. If C is a linear relation from $\mathfrak{K}$ to $\mathfrak{K}'$, then $CL \subset CR$, while if C is a linear relation from $\mathfrak{H}'$ to $\mathfrak{H}$, then $LC \subset RC$; cf. (Arens, 1961).

Next a number of useful statements will follow.

Lemma 2.1. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, and let L and R be linear relations from $\mathfrak{H}$ to $\mathfrak{K}$. Then the following statements are equivalent:*

(i) $L \subset R$ *and* $\mathrm{dom}\ L \supset \mathrm{dom}\ R$;

(ii) $R = L \,\widehat{+}\, (\{0\} \times \mathrm{mul}\ R)$.

Moreover, the following statements are equivalent:

(iii) $L \subset R$ *and* $\mathrm{ran}\ L \supset \mathrm{ran}\ R$;

(iv) $R = L \,\widehat{+}\, (\ker R \times \{0\})$.

Proof. By symmetry it suffices to show the equivalence between (i) and (ii).

(i) $\Rightarrow$ (ii) It suffices to show that $R \subset L \,\widehat{+}\, (\{0\} \times \text{mul } R)$. For this purpose, let $\{h, h'\} \in R$. Since $h \in \text{dom } R \subset \text{dom } L$, there exists an element $k' \in \mathfrak{K}$ such that $\{h, k'\} \in L$. Hence, with $\varphi' = h' - k'$, it follows that

$$\{h, h'\} = \{h, k'\} + \{0, \varphi'\},$$

and thus $\{0, \varphi'\} \in R$ or $\varphi' \in \text{mul } R$. Hence (ii) follows.

(ii) $\Rightarrow$ (i) This implication is trivial. □

Corollary 2.2. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, and let L and R be linear relations from $\mathfrak{H}$ to $\mathfrak{K}$. Then the following statements are equivalent:*

(i) $L = R$*;*

(ii) $L \subset R$*,* $\text{dom } L \supset \text{dom } R$*, and* $\text{mul } L \supset \text{mul } R$*;*

(iii) $L \subset R$*,* $\text{ran } L \supset \text{ran } R$*, and* $\ker L \supset \ker R$*.*

Corollary 2.3. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, and let L and R be linear relations from $\mathfrak{H}$ to $\mathfrak{K}$. Then the following statements hold:*

(a) *If* $L \subset R$*,* $\text{dom } L = \mathfrak{H}$*, and* $\text{mul } R = \{0\}$*, then* $L = R$*.*

(b) *If* $L \subset R$*,* $\text{ran } L = \mathfrak{K}$*, and* $\ker R = \{0\}$*, then* $L = R$*.*

Definition 2.4. Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, and let L and R be linear relations from $\mathfrak{H}$ to $\mathfrak{K}$. The relation L is said to be an *algebraic operator part* of R if

(a) $L \subset R$;

(b) $\text{dom } L = \text{dom } R$;

(c) $\text{mul } L = \{0\}$.

As a consequence of Lemma 2.1 (ii), the following characterization of the algebraic operator part holds.

Corollary 2.5. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, and let L and R be linear relations from $\mathfrak{H}$ to $\mathfrak{K}$. Then L is an algebraic operator part of R if and only if*

$$R = L \,\widehat{+}\, (\{0\} \times \text{mul } R), \;\; \textit{direct sum}.$$

Definition 2.6. Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, and let R be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$. The relation R is said to be *decomposable* if there exists an algebraic operator part L of R.

In order to characterize the notion of a decomposable relation, it is helpful to introduce projections.

Definition 2.7. Let $\mathfrak{H}$ be a linear space with linear subspaces $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$, and assume the direct sum decomposition

$$\mathfrak{X} = \mathfrak{Y} + \mathfrak{Z}.$$

The *projection* P from $\mathfrak{X}$ onto $\mathfrak{Z}$, parallel to $\mathfrak{Y}$, associated with this decomposition, is defined by $Px := z$, when $x = y + z$, $x \in \mathfrak{X}$, $y \in \mathfrak{Y}$, and $z \in \mathfrak{Z}$.

Clearly, P is a well-defined linear operator in $\mathfrak{H}$, which is idempotent. Moreover, one sees that

$$\mathrm{dom}\ P = \mathfrak{X}, \quad \mathrm{ran}\ P = \mathfrak{Z}, \quad \text{and} \quad \ker P = \mathfrak{Y}.$$

It is clear that a relation R is decomposable if and only if there exists a projection P such that

$$P : \mathrm{ran}\ R \to \mathrm{mul}\ R.$$

Let L be an algebraic operator part of R, in other words, $R = L \mathbin{\widehat{+}} (\{0\} \times \mathrm{mul}\ R)$, direct sum. Then the projection $\mathbb{P}$ from R onto $\{0\} \times \mathrm{mul}\ R$ has the property $\ker \mathbb{P} = L$. Conversely, any projection $\mathbb{P}$ from R onto $\{0\} \times \mathrm{mul}\ R$ leads to an algebraic operator part of R.

Lemma 2.8. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let R be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let P be a projection from* $\mathrm{ran}\ R$ *onto* $\mathrm{mul}\ R$. *Then*

$$\mathbb{P}\{x, y\} = \{0, Py\}, \qquad \{x, y\} \in R, \tag{2.3}$$

defines a projection $\mathbb{P}$ from R onto $\{0\} \times \mathrm{mul}\ R$.

Lemma 2.9. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, and let $\mathbb{P}$ be a projection from R onto* $\{0\} \times \mathrm{mul}\ R$. *Then $\mathbb{P}$ is of the form* (2.3)*, for some projection P from* $\mathrm{ran}\ R$ *to* $\mathrm{mul}\ R$*, if and only if*

$$\mathbb{P}\{x, 0\} = \{0, 0\}, \qquad \{x, 0\} \in R. \tag{2.4}$$

Proof. Assume that the projection $\mathbb{P}$ is of the form (2.3). Then the property (2.4) follows from property (2.3).

For the converse, let $\mathbb{P}$ be the projection from R onto $\{0\} \times \mathrm{mul}\ R$. Then for $\{x, y\} \in R$ one has

$$\{x, y\} = \{u, v\} + \{0, \varphi\} \quad \text{with} \quad \{u, v\} \in \ker \mathbb{P} \quad \text{and} \quad \{0, \varphi\} = \mathbb{P}\{x, y\}.$$

It is clear that with this decomposition

$$P = \{\{y, \varphi\} : \{u, v\} \in R,\ \{0, \varphi\} \in \{0\} \times \mathrm{mul}\ R\}$$

defines a linear relation from $\mathrm{ran}\ R$ onto $\mathrm{mul}\ R$. Moreover, P is the graph of a linear operator, in fact of a projection, if the projection $\mathbb{P}$ satisfies (2.4). In this case $\varphi = Py$ and $\mathbb{P}$ is of the form (2.3). $\square$

The discussion of operator parts in the presence of topologies is interesting. For the case of Hilbert spaces, see, for instance, (Hassi, de Snoo & Szafraniec, 2009).

3 Linear relations included in a product

Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces and let P be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$. If Q is a linear relation in $\mathfrak{H}$, then one can consider the product relation PQ, while if Q is a linear relation in $\mathfrak{K}$ one can consider the product relation QP.

First consider the case that Q is a linear relation in $\mathfrak{H}$. Then PQ is a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$ and it is clear that $\mathrm{dom}\ PQ \subset \mathrm{dom}\ Q$ and $\mathrm{ran}\ PQ \subset \mathrm{ran}\ P$. Hence, the following implication holds

$$P \subset PQ \quad \Rightarrow \quad \begin{cases} \mathrm{dom}\ P \subset \mathrm{dom}\ PQ \subset \mathrm{dom}\ Q, \\ \mathrm{ran}\ P = \mathrm{ran}\ PQ. \end{cases} \tag{3.1}$$

Therefore, Lemma 2.1 shows that

$$P \subset PQ \quad \Leftrightarrow \quad PQ = P \,\widehat{+}\, (\ker PQ \times \{0\}). \tag{3.2}$$

Such relations have some useful properties.

Lemma 3.1. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, and let P be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$. Assume that Q is a linear relation in $\mathfrak{H}$ such that $P \subset PQ$. Then*

$$\mathrm{dom}\ P \subset \ker P + \mathrm{ran}\ Q. \tag{3.3}$$

As a consequence the following equivalences hold:

$$\ker P \subset \mathrm{ran}\ Q \quad \Leftrightarrow \quad \mathrm{dom}\ P \subset \mathrm{ran}\ Q, \tag{3.4}$$

and

$$\mathrm{dom}\ P = \ker P + \mathrm{ran}\ Q \quad \Leftrightarrow \quad \mathrm{ran}\ Q \subset \mathrm{dom}\ P. \tag{3.5}$$

Proof. To see (3.3), let $x \in \mathrm{dom}\ P$. Then $\{x, y\} \in P$ for some $y \in \mathfrak{K}$. Since $P \subset PQ$ there exists an element $z \in \mathfrak{H}$ such that $\{x, z\} \in Q$ and $\{z, y\} \in P$. Hence

$$\{x - z, 0\} = \{x, y\} - \{z, y\} \in P,$$

which shows that $x - z \in \ker P$. Thus one concludes $x = (x - z) + z \in \ker P + \mathrm{ran}\ Q$. This shows (3.3). Clearly, the equivalences in (3.4) and (3.5) are consequences of (3.3). □

Next consider the case that Q is a linear relation in $\mathfrak{K}$. Then QP is a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$ and it is clear that $\mathrm{dom}\ QP \subset \mathrm{dom}\ P$ and $\mathrm{ran}\ QP \subset \mathrm{ran}\ Q$. Hence, it follows that

$$P \subset QP \quad \Rightarrow \quad \begin{cases} \mathrm{ran}\ P \subset \mathrm{ran}\ QP \subset \mathrm{ran}\ Q, \\ \mathrm{dom}\ P = \mathrm{dom}\ QP. \end{cases}$$

Therefore, Lemma 2.1 shows that

$$P \subset QP \quad \Leftrightarrow \quad QP = P \,\widehat{+}\, (\{0\} \times \mathrm{mul}\ QP). \tag{3.6}$$

Such relations have some useful properties; cf. Lemma 3.1.

Lemma 3.2. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, and let P be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$. Assume that Q is a linear relation in $\mathfrak{K}$ such that $P \subset QP$. Then*

$$\text{ran } P \subset \text{mul } P + \text{dom } Q.$$

As a consequence the following equivalences hold:

$$\text{mul } P \subset \text{dom } Q \quad \Leftrightarrow \quad \text{ran } P \subset \text{dom } Q,$$

and

$$\text{ran } P = \text{mul } P + \text{dom } Q \quad \Leftrightarrow \quad \text{dom } Q \subset \text{ran } P.$$

4 Linear relations containing a product

Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces and let P be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$. As in the previous section the interplay between P and another linear relation Q is considered.

First consider the case that Q is a linear relation in $\mathfrak{H}$. Then PQ is a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$ and it is clear that $\ker Q \subset \ker PQ$ and $\text{mul } P \subset \text{mul } PQ$. Hence it follows that

$$PQ \subset P \quad \Rightarrow \quad \begin{cases} \ker Q \subset \ker PQ \subset \ker P, \\ \text{mul } PQ = \text{mul } P. \end{cases} \tag{4.1}$$

The relations that satisfy $PQ \subset P$ have some useful properties.

Lemma 4.1. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, and let P be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$. Assume that Q is a linear relation in $\mathfrak{H}$ such that $PQ \subset P$. Then*

$$\text{dom } P \cap \text{mul } Q \subset \ker P. \tag{4.2}$$

As a consequence the following equivalences hold:

$$\text{mul } Q \subset \text{dom } P \quad \Leftrightarrow \quad \text{mul } Q \subset \ker P, \tag{4.3}$$

and

$$\text{dom } P \cap \text{mul } Q = \ker P \quad \Leftrightarrow \quad \ker P \subset \text{mul } Q. \tag{4.4}$$

Proof. To see (4.2), let $x \in \text{dom } P \cap \text{mul } Q$. Then $\{x, y\} \in P$ for some $y \in \mathfrak{K}$, and $\{0, x\} \in Q$. Hence $\{0, y\} \in PQ \subset P$ and therefore $\{x, 0\} = \{x, y\} - \{0, y\} \in P$, which shows that $x \in \ker P$. This shows (4.2). Clearly, the equivalences in (4.3) and (4.4) are consequences of (4.2). $\square$

Next consider the case that Q is a linear relation in $\mathfrak{K}$. Then QP is a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$ and it is clear that $\ker P \subset \ker QP$ and $\text{mul } Q \subset \text{mul } QP$. Hence it follows that

$$QP \subset P \quad \Rightarrow \quad \begin{cases} \text{mul } Q \subset \text{mul } QP \subset \text{mul } P, \\ \ker QP = \ker P. \end{cases}$$

The linear relations that satisfy $QP \subset P$ have some useful properties; cf. Lemma 4.1.

Lemma 4.2. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, and let P be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$. Assume that Q is a linear relation in $\mathfrak{K}$ such that $QP \subset P$. Then*

$$\text{ran } P \cap \ker Q \subset \text{mul } P.$$

As a consequence the following equivalences hold:

$$\ker Q \subset \text{ran } P \quad \Leftrightarrow \quad \ker Q \subset \text{mul } P,$$

and

$$\text{ran } P \cap \ker Q = \text{mul } P \quad \Leftrightarrow \quad \text{mul } P \subset \ker Q.$$

5 Idempotent linear relations and semi-projections

In this section the notions of idempotent linear relation and semi-projection are introduced. These definitions and corresponding lemmas go back to Labrousse (2003). The main aim of this section is to characterize semi-projections in terms of various equivalent conditions.

Definition 5.1. Let P be a linear relation in a linear space $\mathfrak{H}$. Then P is said to be *idempotent* if $P^2 = P$.

Lemma 5.2. *Let P be a linear relation in a linear space $\mathfrak{H}$. Then P is idempotent if and only if $I - P$ is idempotent.*

Proof. It suffices to assume that P is idempotent and to prove that $I - P$ is idempotent, i.e., to show that $(I - P) = (I - P)^2$.

($\subset$) Let $\{x, y\} \in I - P$, then $\{x, x - y\} \in P = P^2$, so that $\{x, z\} \in P$ and $\{z, x - y\} \in P$ for some $z \in H$. Hence $\{x + z, x + z - y\} \in P$ or $\{x + z, y\} \in I - P$, so that

$$\{x - z, y\} = \{2x, 2y\} - \{x + z, y\} \in I - P.$$

Together with $\{x, x - z\} \in I - P$, this gives $\{x, y\} \in (I - P)^2$. Thus $I - P \subset (I - P)^2$.

($\supset$) Let $\{x, y\} \in (I - P)^2$, then $\{x, z\} \in I - P$ and $\{z, y\} \in I - P$ for some $z \in H$. Consequently, $\{x, x - z\} \in P$ and $\{z, z - y\} \in P$, so that $\{x - z, x + y - 2z\} \in P$. Thus one also has $\{x, x + y - 2z\} \in P^2 = P$, so that $\{x, 2z - y\} \in I - P$. This leads to

$$\{x, y\} = \{2x, 2z\} - \{x, 2z - y\} \in I - P.$$

Thus $(I - P)^2 \subset I - P$. □

Definition 5.3. Let P be a linear relation in a linear space $\mathfrak{H}$. Then P is said to be a *semi-projection* if P is idempotent and $\text{ran } P \subset \text{dom } P$.

Lemma 5.4. *Let P be a linear relation in a linear space $\mathfrak{H}$. Then P is a semi-projection if and only if $I - P$ is a semi-projection.*

Proof. It suffices to show that with P also $I - P$ is a semi-projection. For this purpose, assume that P is a semi-projection; thus P is idempotent and satisfies $\operatorname{ran} P \subset \operatorname{dom} P$. Consequently $I - P$ is idempotent by Lemma 5.2 and from the inclusion $\operatorname{ran} P \subset \operatorname{dom} P$ follows that

$$\operatorname{ran}(I-P) \subset \operatorname{dom} P + \operatorname{ran} P = \operatorname{dom} P = \operatorname{dom}(I-P).$$

Thus $I - P$ is a semi-projection. □

Observe that P is idempotent if and only if $P \subset P^2$ and $P^2 \subset P$. Hence the results from Section 3 and Section 4 (with $Q = P$) may be applied.

Proposition 5.5. *Let P be an idempotent relation in a linear space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $\operatorname{dom} P = \ker P + \operatorname{ran} P$;

(ii) $\operatorname{ran} P \subset \operatorname{dom} P$;

(iii) $\operatorname{mul} P \subset \ker P$;

(iv) $\ker P \cap \operatorname{ran} P = \operatorname{mul} P$.

Consequently, P is a semi-projection if and only if one of the preceding conditions holds.

Proof. (i) ⇔ (ii) This follows from Lemma 3.1 with $\mathfrak{K} = \mathfrak{H}$ and $Q = P$.

(ii) ⇒ (iii) The assumption $\operatorname{ran} P \subset \operatorname{dom} P$ implies $\operatorname{mul} P \subset \operatorname{dom} P$. Consequently, the equivalence in (4.3) of Lemma 4.1 with $\mathfrak{K} = \mathfrak{H}$ and $Q = P$ yields that $\operatorname{mul} P \subset \ker P$.

(iii) ⇒ (ii) The assumption $\operatorname{mul} P \subset \ker P$ implies $\operatorname{mul} P \subset \operatorname{dom} P$. Consequently, the first equivalence in Lemma 3.2 with $\mathfrak{K} = \mathfrak{H}$ and $Q = P$ yields that $\operatorname{ran} P \subset \operatorname{dom} P$.

(iii) ⇔ (iv) This follows from Lemma 4.2 with $\mathfrak{K} = \mathfrak{H}$ and $Q = P$. □

Furthermore, it is useful to note that with P also P^{-1} is idempotent, which leads to the following proposition.

Proposition 5.6. *Let P be an idempotent relation in a linear space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $\operatorname{ran} P = \operatorname{mul} P + \operatorname{dom} P$;

(ii) $\operatorname{dom} P \subset \operatorname{ran} P$;

(iii) $\ker P \subset \operatorname{mul} P$;

(iv) $\operatorname{dom} P \cap \operatorname{mul} P = \ker P$.

Consequently, P^{-1} is a semi-projection if and only if one of the preceding conditions holds.

Proof. Since with P also the inverse P^{-1} is idempotent, the equivalence of the items (i)–(iv) follows from applying Proposition 5.5 to the relation P^{-1}. $\square$

The notion of semi-projection has a simple geometric interpretation; cf. Definition 2.7. This geometric explanation goes back to discussions with Seppo Hassi.

Proposition 5.7. *Let $\mathfrak{H}$ be a linear space. Let $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ be linear subspaces of $\mathfrak{H}$, and assume that*

$$\mathfrak{X} = \mathfrak{Y} + \mathfrak{Z}. \tag{5.1}$$

Then the linear relation P in $\mathfrak{H}$, defined by

$$P = \{\{x, z\} : x = y + z,\ x \in \mathfrak{X},\ y \in \mathfrak{Y},\ z \in \mathfrak{Z}\}, \tag{5.2}$$

is a semi-projection with the properties

$$\mathrm{dom}\ P = \mathfrak{X}, \quad \mathrm{ran}\ P = \mathfrak{Z}, \quad \ker P = \mathfrak{Y}, \quad \textit{and} \quad \mathrm{mul}\ P = \mathfrak{Y} \cap \mathfrak{Z}. \tag{5.3}$$

Moreover, every semi-projection in $\mathfrak{H}$ is of this form.

Proof. Assume that (5.1) is satisfied. Then it is clear that P in (5.2) is a well-defined linear relation in $\mathfrak{H}$. Furthermore, P is idempotent. To see that $P \subset P^2$, observe that $\{x, z\} \in P$ implies $\{x, z\} \in P^2$, since $\{z, z\} \in P$. Likewise, to see that $P^2 \subset P$, let $\{x, z\} \in P^2$. Then $\{x, \psi\} \in P$ and $\{\psi, z\} \in P$ for some $\psi \in \mathfrak{Z}$, and consequently, with some $\varphi \in \mathfrak{Y}$ and $\rho \in \mathfrak{Y}$ one has

$$x = \varphi + \psi \quad \text{and} \quad \psi = \rho + z.$$

Therefore $x = \varphi + \rho + z$ with $\varphi + \rho \in \mathfrak{Y}$ and, hence, $\{x, z\} \in P$. Consequently, P is idempotent. In addition, one sees that (5.3) is satisfied. Thus, it follows from Proposition 5.5 that P in (5.2) is a semi-projection.

Now let P be any semi-projection in $\mathfrak{H}$, so that P is idempotent and $\mathrm{dom}\ P = \ker P + \mathrm{ran}\ P$; cf. Proposition 5.5. Let $\{x, z\} \in P$. Then, by assumption, $x = \alpha + \beta$ with $\alpha \in \ker P$ and $\beta \in \mathrm{ran}\ P$, i.e., $\{\alpha, 0\} \in P$ and $\{\gamma, \beta\} \in P$ for some $\gamma \in \mathfrak{H}$. Note that

$$\{x, z\} = \{\alpha, 0\} + \{\beta, z\},$$

which implies that $\{\beta, z\} \in P$. As $\{\gamma, \beta\} \in P$, one also sees that $\{\gamma, z\} \in P^2 = P$, so that $\beta - z \in \mathrm{mul}\ P$. Since $\mathrm{mul}\ P = \ker P \cap \mathrm{ran}\ P \subset \ker P$ by Proposition 5.5, it follows that

$$x = \alpha + \beta = \alpha + \beta - z + z, \quad \text{where} \quad \alpha + \beta - z \in \ker P.$$

Thus the assertion follows with $\mathfrak{X} = \mathrm{dom}\ P$, $\mathfrak{Y} = \ker P$, and $\mathfrak{Z} = \mathrm{ran}\ P$. $\square$

In the context of Proposition 5.7 one can view P as a multivalued projection from $\mathfrak{X}$ onto $\mathfrak{Z}$, parallel to $\mathfrak{Y}$, associated with the not necessarily direct decomposition (5.1).

6 A resolvent formula for semi-projections

This section contains a formula for the resolvent relation of a semi-projection in $\mathfrak{H}$. It may be convenient to first remember that for any linear relation P in $\mathfrak{H}$ and any $\lambda \in \mathbb{C}$ the linear relation $(\lambda I - P)^{-1}$ is called the *resolvent relation* of P

$$(\lambda I - P)^{-1} = \{\{\lambda x - y, x\} : \{x, y\} \in P\}.$$

Then it is clear that

$$\{x, y\} \in P \quad \Leftrightarrow \quad \{\lambda x - y, x\} \in (\lambda I - P)^{-1}, \tag{6.1}$$

and setting $y = \lambda x$ in (6.1) gives the equivalence

$$\{x, \lambda x\} \in P \quad \Leftrightarrow \quad \{0, x\} \in (\lambda I - P)^{-1}. \tag{6.2}$$

Now let P be a semi-projection in a linear space $\mathfrak{H}$, so that $\mathrm{ran}\, P \subset \mathrm{dom}\, P$ or, equivalently, $\mathrm{mul}\, P \subset \ker P$, see Proposition 5.5. Then the following observations are straightforward.

Lemma 6.1. *Let P be a semi-projection in a linear space $\mathfrak{H}$. Then for $\lambda \in \mathbb{C} \setminus \{0, 1\}$*

$$\{x, \lambda x\} \in P \quad \Rightarrow \quad x \in \mathrm{mul}\, P, \tag{6.3}$$

while for all $\lambda \in \mathbb{C}$

$$x \in \mathrm{mul}\, P \quad \Rightarrow \quad \{x, \lambda x\} \in P. \tag{6.4}$$

Proof. Assume that $\{x, \lambda x\} \in P$ for some $\lambda \in \mathbb{C}$. Then, since P is idempotent, $\{x, \lambda^2 x\} \in P$, and hence $\{0, \lambda(\lambda - 1)x\} \in P$. Thus (6.3) has been verified. Since $\mathrm{mul}\, P \subset \ker P$, it is clear that $x \in \mathrm{mul}\, P$ implies that $\{x, \lambda x\} \in P$. Thus (6.4) has been verified. □

As a consequence of (6.4), it can be noted that semi-projections, that are not operators, have nontrivial singular chains; cf. (Sandovici, de Snoo & Winkler, 2004) and (Berger, de Snoo, Trunk & Winkler, 2021). In the general case of semi-projections, the resolvent identity in the following proposition is an identity between linear relations.

Proposition 6.2. *Let P be a semi-projection in a linear space $\mathfrak{H}$ and let $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Then*

$$(\lambda I - P)^{-1} = \left(\frac{1}{\lambda} I + \frac{1}{\lambda(\lambda - 1)} P \right) \,\widehat{+}\, (\{0\} \times \mathrm{mul}\, P). \tag{6.5}$$

Proof. Assume that P is a semi-projection and that $\lambda \in \mathbb{C}$. If $x \in \mathrm{mul}\, P$, then $\{x, \lambda x\} \in P$ by Lemma 6.1. Hence, by (6.2), it follows that $\{0, x\} \in (\lambda I - P)^{-1}$. Consequently, one sees that

$$\mathrm{mul}\, P \subset \mathrm{mul}\, (\lambda I - P)^{-1}, \qquad \lambda \in \mathbb{C}. \tag{6.6}$$

Moreover, by (6.1), every element in $(\lambda I - P)^{-1}$ is of the form $\{\lambda x - y, x\}$ for some $\{x, y\} \in P$. Thus, thanks to $\mathrm{ran}\, P \subset \mathrm{dom}\, P$, it therefore follows that

$$\mathrm{dom}\, (\lambda I - P)^{-1} \subset \mathrm{dom}\, P, \qquad \lambda \in \mathbb{C}. \tag{6.7}$$

Now the restriction $\lambda \in \mathbb{C} \setminus \{0, 1\}$ will be assumed and the following inclusion will be established:

$$((\lambda - 1)I + P) \mathbin{\widehat{+}} (\{0\} \times \text{mul } P) \subset \lambda(\lambda - 1)(\lambda I - P)^{-1}. \tag{6.8}$$

Observe that (6.6) gives $\text{mul } P \subset \text{mul } \lambda(\lambda - 1)(\lambda I - P)^{-1}$, and thus

$$(\{0\} \times \text{mul } P) \subset \lambda(\lambda - 1)(\lambda I - P)^{-1}.$$

Hence, in order to establish (6.8), it remains to show that

$$((\lambda - 1)I + P) \subset \lambda(\lambda - 1)(\lambda I - P)^{-1}.$$

For this, observe that every element in $(\lambda - 1)I + P$ is of the form $\{x, (\lambda - 1)x + y\}$ for some $\{x, y\} \in P$. Hence, the required inclusion follows, once it is recalled that $\{x, y\} \in P$ implies that $x - y \in \ker P$. Thus the inclusion (6.8) has been established.

The inclusion in (6.7) guarantees that in (6.8) the domain of the right-hand side is contained in the domain of the left-hand side when $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Thanks to the equivalence (i) $\Leftrightarrow$ (ii) in Lemma 2.1, one concludes that there is equality in (6.8). It is clear that equality in (6.8) is equivalent to (6.5). □

7 The inclusion $A \subset ABA$

Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let B be a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. Observe that the inclusion $A \subset ABA$ can be written as

$$A \subset A(BA) \quad \text{or} \quad A \subset (AB)A. \tag{7.1}$$

Hence, the inclusion $A \subset ABA$ leads to some automatic identities.

Lemma 7.1. *Assume that $A \subset ABA$. Then*

(a) $\text{dom } A = \text{dom } ABA = \text{dom } BA$*;*

(b) $\text{ran } A = \text{ran } ABA = \text{ran } AB$*.*

Proof. To see (a) apply the first conclusion in (3.1) with $P = A$ and $Q = BA$, then

$$\text{dom } A \subset \text{dom } ABA \subset \text{dom } BA,$$

and note that $\text{dom } BA \subset \text{dom } A$. To see (b) apply the second conclusion in (3.1) with $P = A$ and $Q = BA$, then

$$\text{ran } A = \text{ran } ABA,$$

and note that $\text{ran } ABA \subset \text{ran } AB \subset \text{ran } A$. □

Due to the first inclusion in (7.1), Lemma 3.1 implies the following lemma.

Lemma 7.2. *Assume that $A \subset ABA$. Then*

$$\mathrm{dom}\ A \subset \ker A + \mathrm{ran}\ BA.$$

As a consequence the following equivalences hold:

$$\ker A \subset \mathrm{ran}\ BA \quad \Leftrightarrow \quad \mathrm{dom}\ A \subset \mathrm{ran}\ BA,$$

and

$$\mathrm{dom}\ A = \ker A + \mathrm{ran}\ BA \quad \Leftrightarrow \quad \mathrm{ran}\ BA \subset \mathrm{dom}\ A.$$

Due to the second inclusion in (7.1), Lemma 3.2 implies the following lemma.

Lemma 7.3. *Assume that $A \subset ABA$. Then*

$$\mathrm{ran}\ A \subset \mathrm{mul}\ A + \mathrm{dom}\ AB.$$

As a consequence the following equivalences hold:

$$\mathrm{mul}\ A \subset \mathrm{dom}\ AB \quad \Leftrightarrow \quad \mathrm{ran}\ A \subset \mathrm{dom}\ AB,$$

and

$$\mathrm{ran}\ A = \mathrm{mul}\ A + \mathrm{dom}\ AB \quad \Leftrightarrow \quad \mathrm{dom}\ AB \subset \mathrm{ran}\ A.$$

8 The inclusion $ABA \subset A$

Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let B be a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. Observe that the inclusion $ABA \subset A$ can be written as

$$A(BA) \subset A \quad \text{or} \quad (AB)A \subset A. \tag{8.1}$$

Hence, the inclusion $ABA \subset A$ leads to some automatic identities.

Lemma 8.1. *Assume that $ABA \subset A$. Then*

(a) $\ker BA = \ker A = \ker ABA$;

(b) $\mathrm{mul}\ AB = \mathrm{mul}\ A = \mathrm{mul}\ ABA$.

Proof. To see (a) apply the first conclusion in (4.1) with $P = A$ and $Q = BA$, then

$$\ker BA \subset \ker ABA \subset \ker A.$$

Hence, (a) holds, because $\ker A \subset \ker BA$. To see (b) apply the second conclusion in (4.1) with $P = A$ and $Q = BA$, then

$$\mathrm{mul}\ ABA = \mathrm{mul}\ A,$$

and note that $\mathrm{mul}\ A \subset \mathrm{mul}\ AB \subset \mathrm{mul}\ ABA$. □

Due to the first inclusion in (8.1), Lemma 4.1 implies the following lemma.

Lemma 8.2. *Assume that $ABA \subset A$. Then*

$$\text{dom } A \cap \text{mul } BA \subset \ker A.$$

As a consequence the following equivalences hold:

$$\text{mul } BA \subset \text{dom } A \quad \Leftrightarrow \quad \text{mul } BA \subset \ker A,$$

and

$$\text{dom } A \cap \text{mul } BA = \ker A \quad \Leftrightarrow \quad \ker A \subset \text{mul } BA.$$

Due to the second inclusion in (7.1), Lemma 4.2 implies the following lemma.

Lemma 8.3. *Assume that $ABA \subset A$. Then*

$$\text{ran } A \cap \ker AB \subset \text{mul } A.$$

As a consequence the following equivalences hold:

$$\ker AB \subset \text{ran } A \quad \Leftrightarrow \quad \ker AB \subset \text{mul } A,$$

and

$$\text{ran } A \cap \ker AB = \text{mul } A \quad \Leftrightarrow \quad \text{mul } A \subset \ker AB.$$

9 The identity $A = ABA$

Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let B be a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. The identity $A = ABA$ leads to some automatic identities, which can be seen by combining Lemma 7.1 and Lemma 8.1 for the inclusions $A \subset ABA$ and $ABA \subset A$, respectively.

Lemma 9.1. *Assume that $A = ABA$. Then*

(a) $\text{dom } BA = \text{dom } A$*;*

(b) $\text{ran } AB = \text{ran } A$*;*

(c) $\ker BA = \ker A$*;*

(d) $\text{mul } AB = \text{mul } A$*.*

Furthermore, note that $A = ABA$ implies that the products AB and BA are automatically idempotent relations in $\mathfrak{H}$ and $\mathfrak{K}$, respectively; cf. Definition 5.1. To see this, recall that an identity between relations may be multiplied from the left or from the right remaining an identity; cf. (Arens, 1961). An application of the equivalences in Proposition 5.5 and Proposition 5.6 for the products AB and BA leads to the two following propositions giving necessary and sufficient conditions for AB and BA to be semi-projections; cf. Definition 5.3. Note that the identities from Lemma 9.1 have been used in the formulation of the following descriptions.

Proposition 9.2. *Assume that $ABA = A$. Then the following statements are equivalent:*

(i) $\text{dom } AB = \ker AB + \text{ran } A$;

(ii) $\text{ran } A \subset \text{dom } AB$;

(iii) $\text{mul } A \subset \ker AB$;

(iv) $\ker AB \cap \text{ran } A = \text{mul } A$.

Consequently, the idempotent relation AB is a semi-projection if and only if one of the conditions (i)-(iv) *holds. Moreover, the following statements are equivalent:*

(v) $\text{ran } A = \text{dom } AB + \text{mul } A$;

(vi) $\text{dom } AB \subset \text{ran } A$;

(vii) $\ker AB \subset \text{mul } A$;

(viii) $\text{dom } AB \cap \text{mul } A = \ker AB$.

Consequently, the idempotent relation $(AB)^{-1}$ is a semi-projection if and only if one of the conditions (v)-(viii) *holds.*

Proposition 9.3. *Assume that $ABA = A$. Then the following statements are equivalent:*

(i) $\text{dom } A = \ker A + \text{ran } BA$;

(ii) $\text{ran } BA \subset \text{dom } A$;

(iii) $\text{mul } BA \subset \ker A$;

(iv) $\ker A \cap \text{ran } BA = \text{mul } BA$.

Consequently, the idempotent relation BA is a semi-projection if and only if one of the conditions (i)-(iv) *holds. Moreover, the following statements are equivalent:*

(v) $\text{ran } BA = \text{dom } A + \text{mul } BA$;

(vi) $\text{dom } A \subset \text{ran } BA$;

(vii) $\ker A \subset \text{mul } BA$;

(viii) $\text{dom } A \cap \text{mul } BA = \ker A$.

Consequently, the idempotent relation $(BA)^{-1}$ is a semi-projection if and only if one of the conditions (v)-(viii) *holds.*

10 Characterizations of inclusions

Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let B be a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. First a simple but useful observation is presented.

Lemma 10.1. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let B be a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. Then the following equivalences hold:*

$$\{0\} \times \mathrm{mul}\ A \subset (\ker A \times \{0\}) \,\widehat{+}\, B^{-1} \quad \Leftrightarrow \quad \mathrm{mul}\ A \subset \ker AB, \tag{10.1}$$

and

$$\ker A \times \{0\} \subset B^{-1} \,\widehat{+}\, (\{0\} \times \mathrm{mul}\ A) \quad \Leftrightarrow \quad \ker A \subset \mathrm{mul}\ BA. \tag{10.2}$$

Proof. By symmetry it suffices to show (10.1).

($\Rightarrow$) Let $\rho \in \mathrm{mul}\ A$. Then $\{0, \rho\} = \{x, 0\} + \{-x, \rho\}$ with $x \in \ker A$ and $\{\rho, -x\} \in B$. Hence, it follows that $\rho \in \ker AB$.

($\Leftarrow$) Let $\rho \in \mathrm{mul}\ A$. Then $\rho \in \ker AB$ implies $\{\rho, x\} \in B$ and $\{x, 0\} \in A$, which gives that $\{0, \rho\} = -\{x, 0\} + \{x, \rho\}$. Therefore $\{0\} \times \mathrm{mul}\ A \subset (\ker A \times \{0\}) \,\widehat{+}\, B^{-1}$. $\square$

Next it will be shown that each of the inclusions $A \subset ABA$ and $A \subset ABA$ gives a certain interplay between A and B^{-1}.

Lemma 10.2. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let B be a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $A \subset ABA$;

(ii) $A \subset (\ker A \times \{0\}) \,\widehat{+}\, B^{-1} \,\widehat{+}\, (\{0\} \times \mathrm{mul}\ A)$.

Moreover, the following statements are equivalent:

(iii) $A \subset ABA$ *and* $\mathrm{mul}\ A \subset \ker AB$;

(iv) $A \subset (\ker A \times \{0\}) \,\widehat{+}\, B^{-1}$.

Finally, the following statements are equivalent:

(v) $A \subset ABA$ *and* $\ker A \subset \mathrm{mul}\ BA$;

(vi) $A \subset B^{-1} \,\widehat{+}\, (\{0\} \times \mathrm{mul}\ A)$.

Proof. (i) $\Rightarrow$ (ii) Let $\{u, v\} \in A$. Then, by assumption, $\{u, v\} \in ABA$, and there exist elements $s \in \mathfrak{K}$ and $t \in \mathfrak{H}$ such that

$$\{u, s\} \in A, \qquad \{s, t\} \in B, \qquad \{t, v\} \in A,$$

which, since $\{u,v\} \in A$, implies that

$$\{u-t,0\} \in A \quad \text{and} \quad \{0,v-s\} \in A.$$

Hence, it follows that

$$\{u,v\} = \{u-t,0\} + \{t,s\} + \{0,v-s\} \in (\ker A \times \{0\}) \,\widehat{+}\, B^{-1} \,\widehat{+}\, (\{0\} \times \text{mul } A).$$

Thus (ii) has been shown.

(ii) $\Rightarrow$ (i) Let $\{u,v\} \in A$. Then, by assumption, there exist elements $\alpha \in \ker A$ and $\beta \in \text{mul } A$, such that

$$\{u,v\} = \{\alpha,0\} + \{u-\alpha,v-\beta\} + \{0,\beta\},$$

where $\{v-\beta,u-\alpha\} \in B$. Since $\{u,v-\beta\} \in A$ and $\{u-\alpha,v\} \in A$, it follows that $\{u,v\} \in ABA$.

(iii) $\Leftrightarrow$ (iv) This follows from the equivalence (i) $\Leftrightarrow$ (ii) and the equivalence (10.1) contained in Lemma 10.1.

(v) $\Leftrightarrow$ (vi) This follows from the equivalence (i) $\Leftrightarrow$ (ii) and the equivalence (10.2) contained in Lemma 10.1. $\square$

As a direct corollary of the equivalences of (i) and (ii) in Lemma 10.2 one obtains the following characterization.

Corollary 10.3. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let B be a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $A \subset ABA$, $\text{mul } A \subset \ker B$, *and* $\ker A \subset \text{mul } B$*;*

(ii) $A \subset B^{-1}$.

The converse inclusions lead to a similar result.

Lemma 10.4. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let B be a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $ABA \subset A$, $\text{ran } B \subset \text{dom } A$, *and* $\text{dom } B \subset \text{ran } A$*;*

(ii) $B^{-1} \subset A$.

Proof. (i) $\Rightarrow$ (ii) Assume the inclusion in (i) and let $\{u,v\} \in B$. Then it follows by assumption that there exist elements $\varphi \in \mathfrak{H}$ and $\psi \in \mathfrak{K}$ such that $\{\varphi,u\} \in A$ and $\{v,\psi\} \in A$. Therefore one sees that $\{\varphi,\psi\} \in ABA \subset A$. As a consequence it follows that $\{\psi,\varphi\} \in A^{-1}$, which leads to $\{v,u\} \in AA^{-1}A \subset A$ by (2.2), so that $\{u,v\} \in A^{-1}$. Thus $B \subset A^{-1}$ or $B^{-1} \subset A$.

(ii) $\Rightarrow$ (i) Assume the inclusion in (ii). Then also $B \subset A^{-1}$ which implies that

$$ABA \subset AA^{-1}A = A,$$

by means of (2.1). $\square$

Combining Corollary 10.3 and Lemma 10.4 gives the following result.

Corollary 10.5. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let B be a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $A = ABA$, $\operatorname{mul} A \subset \ker B$, $\ker A \subset \operatorname{mul} B$, $\operatorname{ran} B \subset \operatorname{dom} A$, *and* $\operatorname{dom} B \subset \operatorname{ran} A$*;*

(ii) $A = B^{-1}$.

11 Characterization of generalized inverses

Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let B be a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. Recall from Lemma 10.2 that

$$A \subset B^{-1} \,\widehat{+}\, (\ker A \times \{0\}) \quad \Leftrightarrow \quad A \subset ABA \quad \text{and} \quad \operatorname{mul} A \subset \ker AB, \tag{11.1}$$

and, by symmetry, one obtains

$$B \subset A^{-1} \,\widehat{+}\, (\ker B \times \{0\}) \quad \Leftrightarrow \quad B \subset BAB \quad \text{and} \quad \operatorname{mul} B \subset \ker BA. \tag{11.2}$$

It is convenient to have extra conditions in (11.1) and (11.2) that guarantee identities $A = ABA$ and $B = BAB$, respectively, instead of inclusions.

Lemma 11.1. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let B be a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $A \subset B^{-1} \,\widehat{+}\, (\ker A \times \{0\})$ *and* $\operatorname{dom} A \cap \operatorname{mul} B \subset \ker A$*;*

(ii) $A = ABA$ *and* $\operatorname{mul} A \subset \ker AB$.

Similarly, the following statements are equivalent:

(iii) $B \subset A^{-1} \,\widehat{+}\, (\ker B \times \{0\})$ *and* $\operatorname{dom} B \cap \operatorname{mul} A \subset \ker B$*;*

(iv) $B = BAB$ *and* $\operatorname{mul} B \subset \ker BA$.

Proof. By symmetry, only the first equivalence needs to be verified.

(i) $\Rightarrow$ (ii) Due to the equivalence in (11.1), the first assumption in (i) implies that $A \subset ABA$ and $\operatorname{mul} A \subset \ker AB$. To conclude $A = ABA$, it suffices by Corollary 2.2 to show that

$$\operatorname{dom} ABA \subset \operatorname{dom} A \quad \text{and} \quad \operatorname{mul} ABA \subset \operatorname{mul} A.$$

The first inclusion follows from Lemma 7.1. For the second inclusion note that the established inclusion $\operatorname{mul} A \subset \ker AB$ implies the inclusion $\operatorname{mul} ABA \subset \operatorname{mul} AB$, while the assumption $\operatorname{dom} A \cap \operatorname{mul} B \subset \ker A$ implies the inclusion $\operatorname{mul} AB \subset \operatorname{mul} A$.

(ii) $\Rightarrow$ (i) The first statement in (i) is a consequence of $A = ABA$ and $\operatorname{mul} A \subset \ker AB$ by the equivalence in (11.1). The assumption $A = ABA$ implies by Lemma 8.2 the following inclusion $\operatorname{dom} A \cap \operatorname{mul} BA \subset \ker A$. This gives the second statement since $\operatorname{mul} B \subset \operatorname{mul} BA$. $\square$

Note that the consequences of the identity $A = ABA$ can be found in Section 9. In case $B = BAB$ one gets similar results by interchanging A and B. Let it suffice to mention that if $A = ABA$ and $B = BAB$, then

$$\begin{aligned} &\mathrm{dom}\ BA = \mathrm{dom}\ A, \quad \mathrm{ran}\ AB = \mathrm{ran}\ A, \quad \ker BA = \ker A, \quad \mathrm{mul}\ AB = \mathrm{mul}\ A,\\ &\mathrm{dom}\ AB = \mathrm{dom}\ B, \quad \mathrm{ran}\ BA = \mathrm{ran}\ B, \quad \ker AB = \ker B, \quad \mathrm{mul}\ BA = \mathrm{mul}\ B, \end{aligned} \tag{11.3}$$

as follows from Lemma 9.1; cf. (Labrousse, 1992). One can proceed with Lemma 9.2 and Lemma 9.3 in a similar way. Also recall that both AB and BA are idempotent when $A = ABA$ and $B = BAB$. This leads to the following definition.

Definition 11.2. Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let B be a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. Then A and B are said to be *generalized inverses (of each other)* if

$$ABA = A, \qquad BAB = B,$$

and, in addition, BA and AB are semi-projections in $\mathfrak{H}$ and $\mathfrak{K}$, respectively.

In the above definition the linear relations A and B play a symmetric role. One also uses the terminology that B is a generalized inverse of A (or vice versa). Note that if $B = A^{-1}$, then A and B are generalized inverses. To see this, recall that $AA^{-1}A = A$ and $A^{-1}AA^{-1} = A^{-1}$, while $A^{-1}A$ and AA^{-1} are semi-projections; cf. (2.1) and (2.2). In the above mentioned terminology one can say that $B = A^{-1}$ is a generalized inverse of A. The following theorem incorporates this special situation; see also (Labrousse, 2021).

Theorem 11.3. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let B be a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. Then A and B are generalized inverses of each other if and only if the following statements hold:*

$$A \,\widehat{+}\, (\{0\} \times \ker B) = B^{-1} \,\widehat{+}\, (\ker A \times \{0\}), \tag{11.4}$$

$$\mathrm{mul}\ B \subset \ker A \quad \textit{and} \quad \mathrm{mul}\ A \subset \ker B. \tag{11.5}$$

Proof. Assume that A and B are generalized inverses of each other. Then the assumptions that $ABA = A$ and that BA is a semi-projection imply by Proposition 9.3 (iii) that $\mathrm{mul}\ BA \subset \ker A$. Likewise, the assumptions that $BAB = B$ and that AB is a semi-projection give $\mathrm{mul}\ AB \subset \ker B$. Hence by (11.3) one sees that (11.5) is satisfied, which can also be written as

$$\mathrm{mul}\ B \subset \ker AB \quad \text{and} \quad \mathrm{mul}\ A \subset \ker AB.$$

Together with the assumption $ABA = A$ this gives via Lemma 11.1 that

$$A \subset B^{-1} \,\widehat{+}\, (\ker A \times \{0\}) \quad \text{and} \quad B \subset A^{-1} \,\widehat{+}\, (\ker B \times \{0\}),$$

so that also

$$A \,\widehat{+}\, (\{0\} \times \ker B) \subset B^{-1} \,\widehat{+}\, (\ker A \times \{0\}) \text{ and } B \,\widehat{+}\, (\{0\} \times \ker A) \subset A^{-1} \,\widehat{+}\, (\ker B \times \{0\}),$$

which gives (11.4).

Conversely, assume that (11.4) and (11.5) hold. Then

$$A \subset B^{-1} \,\widehat{+}\, (\ker A \times \{0\}) \quad \text{and} \quad \mathrm{mul}\, B \subset \ker A,$$

so that by Lemma 11.1 one has $A = ABA$ and $\mathrm{mul}\, A \subset \ker AB$. Likewise, one concludes that $B = BAB$ and $\mathrm{mul}\, B \subset \ker BA$. Finally, Proposition 9.2 (iii) implies that AB and BA are semi-projections. □

As to Definition 11.2, the question arises if for a linear relation A from $\mathfrak{H}$ to $\mathfrak{K}$ one can choose a generalized inverse with special properties. For instance, does there exist a generalized inverse of A which is an operator?

12 Special generalized inverses

Let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$. Then, in general, its (formal) inverse relation A^{-1} is not the graph of an operator, since $\mathrm{mul}\, A^{-1} = \ker A$. However, any projection from $\mathrm{dom}\, A$ onto $\ker A$ leads to an algebraic operator part of the relation A^{-1}; cf. Section 2. It will be shown that any such algebraic operator part will serve as a generalized inverse.

Let Q be a projection from $\mathrm{dom}\, A$ onto $\ker A$. Then the following identity holds

$$A(I - Q) = A. \tag{12.1}$$

To see this, observe that $A(I-Q) = \{\{(I-Q)f, g\} : \{f,g\} \in A\}$ and that $\{Qf, 0\} \in A$ when $\{f,g\} \in A$. As a consequence of (12.1) one obtains

$$A^{-1} = \{\{g, (I-Q)f\} : \{f,g\} \in A\}. \tag{12.2}$$

Note that if $g = 0$ in (12.2), then $f = Qf \in \ker A$ and $(I-Q)f = 0$. In light of these facts, the following definition is natural.

Definition 12.1. Let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$ and let Q be a projection from $\mathrm{dom}\, A$ onto $\ker A$. Then the linear relation

$$(A^{-1})_s = \{\{g, (I-Q)f\} : \{f,g\} \in A\}$$

is called the *algebraic operator part* of A^{-1} (*relative to the projection* Q).

Lemma 12.2. *Let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$ and let $(A^{-1})_s$ be the algebraic operator part of A^{-1}. Then*

$$(A^{-1})_s A = I - Q \tag{12.3}$$

and

$$A(A^{-1})_s = I_{\mathrm{ran}\, A} \,\widehat{+}\, (\{0\} \times \mathrm{mul}\, A). \tag{12.4}$$

Proof. In order to verify (12.3) two inclusions will be shown.

($\subset$) Let $\{h,l\} \in (A^{-1})_s A$. Then $\{h,\varphi\} \in A$ and $\{\varphi,l\} \in (A^{-1})_s$ for some $\varphi \in \mathfrak{K}$. Note that

$$\{\varphi,l\} = \{g,(I-Q)f\} \quad \text{for some} \quad \{f,g\} \in A,$$

so that $\varphi = g$ and $l = (I-Q)f$. In particular, it follows that $\{h,g\} = \{h,\varphi\} \in A$ and thus $Q(f-h) = 0$. Since $l = (I-Q)f$, one obtains $l = (I-Q)h$. Therefore it follows that $\{h,l\} = \{h,(I-Q)h\} \in I-Q$. This shows $(A^{-1})_s A \subset I-Q$.

($\supset$) Let $\{h,l\} \in I-Q$, then $h \in \mathrm{dom}\, A$ and $l = (I-Q)h$. Thus there exists some $g \in \mathfrak{K}$ so that $\{h,g\} \in A$ and also $\{(I-Q)h,g\} \in A$, i.e., $\{g,(I-Q)h\} \in (A^{-1})_s$. Since $l = (I-Q)h$ this implies that $\{h,l\} \in (A^{-1})_s A$. This shows the inclusion $I-Q \subset (A^{-1})_s A$.

In order to verify (12.4) two inclusions will be shown.

($\subset$) It follows from (12.2) and (2.1) that

$$A(A^{-1})_s \subset AA^{-1} = I_{\mathrm{ran}\, A} \,\widehat{+}\, (\{0\} \times \mathrm{mul}\, A).$$

($\supset$) First observe that $\{0\} \times \mathrm{mul}\, A \subset A(A^{-1})_s$. Next it will be shown that $I_{\mathrm{ran}\, A} \subset A(A^{-1})_s$. To see this, let $k \in \mathrm{ran}\, A$. Then there exists $h \in \mathfrak{H}$ such that $\{h,k\} \in A$ and also $\{(I-Q)h,k\} \in A$. Since $\{k,(I-Q)h\} \in (A^{-1})_s$, it follows that $\{k,k\} \in A(A^{-1})_s$. Hence $I_{\mathrm{ran}\, A} \subset A(A^{-1})_s$. □

Corollary 12.3. *Let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$ and let $(A^{-1})_s$ be the algebraic operator part of A^{-1}. Then*

$$A(A^{-1})_s A = A, \tag{12.5}$$

and

$$(A^{-1})_s A (A^{-1})_s = (A^{-1})_s. \tag{12.6}$$

Proof. It follows from Lemma 12.2 that $A(A^{-1})_s A = A(I-Q)$. The statement in (12.5) now follows from (12.1).

Likewise, it follows from Lemma 12.2 that $(A^{-1})_s A (A^{-1})_s = (I-Q)(A^{-1})_s$. In order to show (12.6), it suffices to show that

$$(I-Q)(A^{-1})_s = (A^{-1})_s. \tag{12.7}$$

However, the identity (12.7) is clear, as it is a direct consequence of Definition 12.1 and the fact that $I-Q$ is an idempotent operator. □

A combination of Definition 12.1, Lemma 12.2, and Corollary 12.3 leads to the following theorem.

Theorem 12.4. *Let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$ and let $B = (A^{-1})_s$ be the algebraic operator part of its inverse. Then A and the operator B are generalized inverses.*

In the presence of topologies and under additional condition there exist generalized inverses as in Definition 12.1; this goes beyond the context of this survey.

13 Further characterizations

Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, and let P be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$. In Section 3 and Section 4 above, one can find results for the interplay with a second relation Q, either in $\mathfrak{H}$ or $\mathfrak{K}$. Recall, in particular, that the inclusions $P \subset PQ$ and $P \subset QP$ were characterized in terms of identities in (3.2) and (3.6). In this section the various inclusions $P \subset PQ$, $P \subset QP$, $PQ \subset P$, and $QP \subset P$ will be characterized in terms of inclusions.

Lemma 13.1. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let P be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let Q be a linear relation in $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $P \subset PQ$;

(ii) $\mathrm{dom}\, P \subset \mathrm{dom}\, Q$ *and*

$$Q\restriction_{\mathrm{dom}\, P} \subset I_{\mathfrak{H}} \,\widehat{+}\, (\ker P \times \{0\}) \,\widehat{+}\, (\{0\} \times \mathrm{mul}\, Q)\,.$$

Proof. (i) $\Rightarrow$ (ii) Assume that (i) holds. Then the first inclusion in (ii) is clear. To prove the second inclusion let $\{x,t\} \in Q\restriction_{\mathrm{dom}\, P}$. Since $x \in \mathrm{dom}\, P$ it follows that $\{x,y\} \in P$ for some $y \in \mathrm{ran}\, P$ and thus $\{x,y\} \in PQ$ by (i). Hence $\{x,z\} \in Q$ and $\{z,y\} \in P$ for some $z \in \mathfrak{H}$. Consequently, $x - z \in \ker P$ and $t - z \in \mathrm{mul}\, Q$. Therefore one sees that

$$\{x,t\} = \{z,z\} + \{x-z,0\} + \{0,t-z\} \in I_{\mathfrak{H}} \,\widehat{+}\, (\ker P \times \{0\}) \,\widehat{+}\, (\{0\} \times \mathrm{mul}\, Q)\,.$$

Hence the second inclusion in (ii) has been shown.

(ii) $\Rightarrow$ (i) Assume that (ii) holds and let $\{x,y\} \in P$. Since $\mathrm{dom}\, P \subset \mathrm{dom}\, Q$ one observes that $\{x,t\} \in Q\restriction_{\mathrm{dom}\, P}$ for some $t \in \mathfrak{H}$. Then

$$\{x,t\} = \{v,v\} + \{u,0\} + \{0,m\},$$

for some $v \in \mathfrak{H}$, $u \in \ker P$, and $m \in \mathrm{mul}\, Q$. Then $x = u + v$, $t = v + m$ so that

$$\{v,y\} = \{x-u,y\} = \{x,y\} - \{u,0\} \in P,$$

and also,

$$\{x,v\} = \{x,t-m\} = \{x,t\} - \{0,m\} \in Q.$$

Consequently, $\{x,y\} \in PQ$. Hence (i) has been shown. $\square$

Lemma 13.2. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let P be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let Q be a linear relation in $\mathfrak{K}$. Then the following statements are equivalent:*

(i) $P \subset QP$;

(ii) $\mathrm{dom}\, P = \mathrm{dom}\, QP$, $\mathrm{mul}\, P \subset \mathrm{mul}\, QP$, *and*

$$Q\restriction_{\mathrm{ran}\, P} \subset I_{\mathfrak{K}} \,\widehat{+}\, (\{0\} \times \mathrm{mul}\, QP)\,.$$

Proof. (i) $\Rightarrow$ (ii) Assume that (i) holds. Then, clearly, $\mathrm{dom}\ P = \mathrm{dom}\ QP$ and $\mathrm{mul}\ P \subset \mathrm{mul}\ QP$. It remains to show the last inclusion in (ii). Let

$$\{x,y\} \in Q \restriction_{\mathrm{ran}\,P \cap \mathrm{dom}\,Q},$$

so that $x \in \mathrm{ran}\ P \cap \mathrm{dom}\ Q$. Hence $\{z,x\} \in P$ for some $z \in \mathrm{dom}\ P$ and thus $\{z,y\} \in QP$. Furthermore observe that $\{z,x\} \in P \subset QP$ by (i). Therefore, $\{0, y-x\} = \{z,y\} - \{z,x\} \in QP$, which further shows that

$$\{x,y\} = \{x,x\} + \{0, y-x\} \in I_{\mathfrak{K}} \mathbin{\widehat{+}} (\{0\} \times \mathrm{mul}\ QP)\,.$$

Hence the last inclusion in (ii) holds.

(ii) $\Rightarrow$ (i) Assume that (ii) holds. Let $\{x,y\} \in P$ so that $x \in \mathrm{dom}\ P = \mathrm{dom}\ QP$ by the identity in (ii). Then $\{x,z\} \in QP$ for some $z \in \mathrm{ran}\ QP$; and thus $\{x,t\} \in P$ and $\{t,z\} \in Q$ for some $t \in \mathrm{ran}\ P \cap \mathrm{dom}\ Q$. It follows from $\{t,z\} \in Q$ and the last inclusion in (ii) that $t - z \in \mathrm{mul}\ QP$. This leads to

$$\{x,t\} = \{x,z\} + \{0, t-z\} \in QP.$$

It follows from $\{x,y\}$, $\{x,t\} \in P$ that $y - t \in \mathrm{mul}\ P \subset \mathrm{mul}\ QP$, due to the first inclusion in (ii). Consequently,

$$\{x,y\} = \{x,t\} + \{0, y-t\} \in QP.$$

Thus $P \subset QP$ and (i) has been shown. $\square$

Lemma 13.3. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let P be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let Q be a linear relation in $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $PQ \subset P$,

(ii) $\mathrm{mul}\ PQ = \mathrm{mul}\ P$, $\mathrm{dom}\ PQ \subset \mathrm{dom}\ P$, *and*

$$Q \restriction_{\mathrm{dom}\,PQ} \subset I_{\mathfrak{H}} \mathbin{\widehat{+}} (\ker P \times \{0\}) \mathbin{\widehat{+}} (\{0\} \times \mathrm{mul}\ Q)\,.$$

Proof. (i) $\Rightarrow$ (ii) Assume that (i) holds. The identity and the first inclusion in (ii) are clear. Now let $\{x,y\} \in Q \restriction_{\mathrm{dom}\,PQ}$, so that $\{x,t\} \in PQ$ for some $t \in \mathrm{ran}\ PQ$ and $\{x,t\} \in P$ by (i). Note that $\{x,z\} \in Q$ and $\{z,t\} \in P$ for some $z \in \mathfrak{H}$. Consequently, $x - z \in \ker P$ and $y - z \in \mathrm{mul}\ Q$. Therefore one sees that

$$\{x,y\} = \{z,z\} + \{x-z, 0\} + \{0, y-z\} \in I_{\mathfrak{H}} \mathbin{\widehat{+}} (\ker P \times \{0\}) \mathbin{\widehat{+}} (\{0\} \times \mathrm{mul}\ Q)\,.$$

Hence (ii) has been shown.

(ii) $\Rightarrow$ (i) Assume that (ii) holds and let $\{x,y\} \in PQ$. Then $x \in \mathrm{dom}\ PQ$, $\{x,\alpha\} \in Q$, and $\{\alpha, y\} \in P$ for some $\alpha \in \mathfrak{H}$. Since $\mathrm{dom}\ PQ \subset \mathrm{dom}\ P$ it follows that $\{x,\beta\} \in P$ for some $\beta \in \mathfrak{K}$. By the second inclusion in (ii) one sees that

$$\{x,\alpha\} = \{u,u\} + \{p,0\} + \{0,m\},$$

for some $u \in \mathfrak{H}$, $p \in \ker P$ and $m \in \mathrm{mul}\ Q$. Then $x = p + u$, $\alpha = u + m$, so that

$$\{m, y-\beta\} = -\{p+u, \beta\} + \{u+m, y\} + \{p, 0\} \in P.$$

Since $\{0, m\} \in Q$ it follows that $\{0, y - \beta\} \in PQ$, so that $y - \beta \in \text{mul } PQ = \text{mul } P$ by the identity in (ii). This implies that

$$\{x, y\} = \{x, \beta\} + \{0, y - \beta\} \in P.$$

Hence (i) has been shown. □

Lemma 13.4. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces, let P be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$, and let Q be a linear relation in $\mathfrak{K}$. Then the following statements are equivalent:*

(i) $QP \subset P$;

(ii) $Q \upharpoonright_{\text{ran } P} \subset I_{\mathfrak{K}} \widehat{+} (\{0\} \times \text{mul } P)$.

Proof. (i) ⇒ (ii) Assume that (i) holds. Let $\{x, y\} \in Q \upharpoonright_{\text{ran } P}$. Since $x \in \text{ran } P$ it follows that $\{z, x\} \in P$ for some $z \in \text{dom } P$. Thus, $\{z, y\} \in QP \subset P$ by (i). This implies that

$$\{0, y - x\} = \{z, y\} - \{z, x\} \in P,$$

so that $y - x \in \text{mul } P$. Consequently,

$$\{x, y\} = \{x, x\} + \{0, y - x\} \in I_{\mathfrak{K}} \widehat{+} (\{0\} \times \text{mul } P)\,.$$

Hence (ii) has been shown.

(ii) ⇒ (i) Now assume that (ii) holds. Let $\{x, y\} \in QP$ so that $\{x, z\} \in P$ and $\{z, y\} \in Q$ for some $z \in \text{ran } P \cap \text{dom } Q$. It follows from (ii) that

$$\{z, y\} = \{z, z\} + \{0, y - z\},$$

with $y - z \in \text{mul } P$, which further leads to

$$\{x, y\} = \{x, z\} + \{0, y - z\} \in P.$$

Hence $QP \subset P$ which shows that (i) □

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LIPSCHITZ PROPERTY OF EIGENVALUES AND EIGENVECTORS OF 2×2 DIRAC-TYPE OPERATORS

Anton Lunyov and Mark Malamud

Dedicated to our friend and colleague Seppo Hassi on the occasion of his sixtieth birthday

1 Introduction

Continuing our investigation (Lunyov & Malamud, 2016), this paper is concerned with the stability properties of different spectral characteristics of a boundary value problem associated in $L^2([0,1];\mathbb{C}^2)$ with the following first order system of differential equations

$$\mathcal{L}y = -iB^{-1}y' + Q(x)y = \lambda y, \qquad y = \operatorname{col}(y_1, y_2), \qquad x \in [0,1], \tag{1.1}$$

where

$$B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad b_1 < 0 < b_2, \quad \text{and} \quad Q = \begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{pmatrix} \in L^1([0,1];\mathbb{C}^{2\times 2}).$$

If $B = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then the system (1.1) is equivalent to the Dirac system, see the classical monographs (Levitan & Sargsyan, 1991; Marchenko, 1986). Let us associate with the system (1.1) the following linearly independent boundary conditions (BCs)

$$U_j(y) := a_{j1}y_1(0) + a_{j2}y_2(0) + a_{j3}y_1(1) + a_{j4}y_2(1) = 0, \qquad j \in \{1,2\}. \tag{1.2}$$

Moreover, denote by $L(Q) := L_U(Q)$ the operator in $L^2([0,1];\mathbb{C}^2)$ associated with the boundary value problem (BVP) (1.1)–(1.2); its action is defined by the differential expression $\mathcal{L}$ in (1.1) and its domain is given by

$$\operatorname{dom}(L_U(Q)) = \{f \in AC([0,1];\mathbb{C}^2) : \ \mathcal{L}f \in L^2([0,1];\mathbb{C}^2),\ U_1(f) = U_2(f) = 0\}. \tag{1.3}$$

The above-mentioned stability properties refer to a perturbation of the potential $Q \to \widetilde{Q}$.

The completeness property of the system of root vectors (SRV) of BVPs for general $n \times n$ systems of the form (1.1) with a nonsingular diagonal $n \times n$ matrix B with complex entries and a potential matrix $Q(\cdot)$ of the form

$$B = \operatorname{diag}(b_1, b_2, \ldots, b_n) \in \mathbb{C}^{n\times n} \quad \text{and} \quad Q(\cdot) =: (q_{jk}(\cdot))_{j,k=1}^n \in L^1([0,1];\mathbb{C}^{n\times n})$$

was established in Malamud & Oridoroga (2012) for a wide class of BVPs; note that for 2×2 Dirac systems with $Q \in C([0,1];\mathbb{C}^{2\times 2})$ it was proved earlier in Marchenko (1986: Chapter 1.3). In Malamud & Oridoroga (2012); Lunyov & Malamud (2014a; 2015) the authors also found completeness conditions for non-regular and even degenerate BCs. In Lunyov & Malamud (2015) the Riesz basis property (with and without parentheses) of SRV was also established for different classes of BVPs for $n \times n$ systems with arbitrary B and $Q \in L^\infty([0,1];\mathbb{C}^{n\times n})$. Note also that BVPs for the

$2m \times 2m$ Dirac equation, i.e., the case that $B = \mathrm{diag}(-I_m, I_m)$, were investigated in Mykytyuk & Puyda (2013) (Bari-Markus property for Dirichlet BVP with $Q \in L^2([0,1]; \mathbb{C}^{2m\times 2m})$ and in Kurbanov & Abdullayeva (2018); Kurbanov & Gadzhieva (2020) (Bessel and Riesz basis properties on an abstract level).

The Riesz basis property in $L^2([0,1]; \mathbb{C}^2)$ of BVP (1.1)–(1.2), i.e., of the operator $L_U(Q)$ defined above, was investigated with various assumptions on the potential matrix Q in numerous papers, see (Trooshin & Yamamoto, 2001; 2002; Hassi & Oridoroga, 2009; Djakov & Mityagin, 2010; Baskakov, Derbushev & Shcherbakov, 2011; Djakov & Mityagin, 2012a;b;c; 2013; Lunyov & Malamud, 2014b; Savchuk & Shkalikov, 2014; Lunyov & Malamud, 2016; Uskova, 2019) and references therein. At that time the strongest result was obtained by P. Djakov and B. Mityagin (2010; 2012c), and A. Baskakov, A. Derbushev, and A. Shcherbakov (2011). They proved under the assumption $Q \in L^2([0,1]; \mathbb{C}^{2\times 2})$ that SRV of the BVP (1.1)–(1.2) with strictly regular BCs forms a Riesz basis, and with BCs that are only regular forms a Riesz basis with parentheses. Note, however, that the methods of these papers substantially rely on L^2-techniques (such as Parseval's equality, Hilbert-Schmidt operators, etc.) and cannot be applied to L^1-potentials.

Later the case $Q \in L^1([0,1]; \mathbb{C}^{2\times 2})$ was treated independently and with different methods by the authors (Lunyov & Malamud, 2014b; 2016) on the one hand, and by A.M. Savchuk and A.A. Shkalikov (2014) on the other hand. It was proved that a BVP (1.1)–(1.2) with $Q \in L^1([0,1]; \mathbb{C}^{2\times 2})$ and strictly regular boundary conditions has the Riesz basis property, while a BVP whose BCs are only regular has the property of Riesz basis with parentheses.

Recall in this connection that the boundary conditions (1.2) are called *regular* if and only if they are equivalent to the following conditions

$$\widehat{U}_1(y) = y_1(0) + by_2(0) + ay_1(1) = 0, \qquad \widehat{U}_2(y) = dy_2(0) + cy_1(1) + y_2(1) = 0, \tag{1.4}$$

for certain $a, b, c, d \in \mathbb{C}$ satisfying $ad - bc \neq 0$. Recall also that regular BCs (1.2) are called *strictly regular* if the sequence $\lambda_0 = \{\lambda_n^0\}_{n\in\mathbb{Z}}$ of the eigenvalues of the unperturbed BVP (1.1)–(1.2) (of the operator $L_U(0)$, i.e., $Q = 0$) is asymptotically separated. In particular, the eigenvalues $\{\lambda_n^0\}_{|n|>n_0}$ are geometrically and algebraically simple.

It is well known that non-degenerate separated BCs are always strictly regular. Moreover, the conditions (1.4) are strictly regular for the Dirac operator if and only if $(a-d)^2 \neq -4bc$. In particular, antiperiodic (periodic) BC are regular but not strictly regular for Dirac systems, while they become strictly regular for Dirac-type systems if $-b_1, b_2 \in \mathbb{N}$ and $b_2 - b_1$ is odd.

To describe our approach to the Riesz basis property used in Lunyov & Malamud (2014b; 2016), let us denote by $e_\pm(\cdot, \lambda)$ the solutions of the system (1.1) satisfying the initial conditions

$$e_\pm(0, \lambda) = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}.$$

Our investigation in Lunyov & Malamud (2014b; 2016) substantially relies on the following representation of the solutions $e_\pm(\cdot, \lambda)$ by means of triangular transformation operators:

$$e_\pm(x, \lambda) = (I + \mathcal{K}_Q^\pm) e_\pm^0(x, \lambda) = e_\pm^0(x, \lambda) + \int_0^x K_Q^\pm(x,t) e_\pm^0(t, \lambda)\, dt, \tag{1.5}$$

where $e^0_\pm(x,\lambda) = \text{col}\left(e^{ib_1\lambda x}, \pm e^{ib_2\lambda x}\right)$ and $K^\pm_Q = \left(K^\pm_{jk}\right)^2_{j,k=1} \in X^{0,2}_{1,1} \cap X^{0,2}_{\infty,1}$; see (2.1) and (2.2) for the definitions of these spaces.

Let us denote by $\Lambda_Q := \Lambda_{U,Q} = \{\lambda_{Q,n}\}_{n\in\mathbb{Z}} := \{\lambda_n\}_{n\in\mathbb{Z}}$ the spectrum of the operator $L_U(Q)$. Our main tool in the investigation of the asymptotic behavior of the eigenvalues is the characteristic determinant $\Delta_Q(\cdot) = \Delta_{Q,U}(\cdot)$. This function is an entire function whose zeros coincide with the sequence Λ_Q of eigenvalues counting multiplicities, see the formulas (4.1)–(4.4). The representation (1.5) immediately leads to the following key formula for the characteristic determinant $\Delta_Q(\cdot)$ of the problem (1.1)–(1.2):

$$\Delta_Q(\lambda) = \Delta_0(\lambda) + \int_0^1 g_{1,Q}(t) e^{ib_1\lambda t} dt + \int_0^1 g_{2,Q}(t) e^{ib_2\lambda t}\, dt, \tag{1.6}$$

where $g_{k,Q}(\cdot) \in L^1[0,1],\ k \in \{1,2\}$, are expressed via $K^\pm_{jk}(1,\cdot)$, see (4.12) and (4.7). Recall that $\Delta_0(\cdot) = \Delta_{0,U}(\cdot)$ is the characteristic determinant of the problem (1.1)–(1.2) with $Q = 0$, see the identity (4.5).

Formula (1.6) immediately yields an estimate of the difference $\Delta_Q(\lambda) - \Delta_0(\lambda)$ from above. Combining this estimate with the classical estimate of $\Delta_0(\cdot)$ from below and applying the Rouché theorem one arrives at the asymptotic formula

$$\lambda_n = \lambda_n^0 + o(1), \quad \text{as} \quad n \to \infty, \tag{1.7}$$

relating the eigenvalues $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ and $\Lambda_0 = \{\lambda_n^0\}_{n\in\mathbb{Z}}$ of the operators $L_U(Q)$ and $L_U(0)$ (with regular BCs), respectively; see (Lunyov & Malamud, 2014b; 2016) for details and also (Savchuk & Shkalikov, 2014), where the formula (1.7) was obtained by another method. Note also that the representation (1.6) for the determinant $\Delta_Q(\cdot)$ was substantially used in papers by A.S. Makin (2020; 2021).

In Lunyov & Malamud (2014b; 2016) we also applied the representation (1.5) to obtain asymptotic formulas for the solutions of the equation (1.1) as well as for eigenfunctions of the BVP (1.1)–(1.2). In the present paper we continue the investigation from (Lunyov & Malamud, 2014b; 2016) of the BVP (1.1)–(1.2) and the transformation operators for the system (1.1). In Section 2 we prove the Lipschitz property of the mappings $Q \to K^\pm$ on the balls

$$\mathbb{U}^{2\times 2}_{p,r} := \left\{F \in L^p([0,1];\mathbb{C}^{2\times 2}) : \|F\|_p := \|F\|_{L^p([0,1];\mathbb{C}^{2\times 2})} \le r\right\}, \qquad r > 0, \tag{1.8}$$

in $L^p([0,1];\mathbb{C}^{2\times 2})$. Namely, our first main result reads as follows.

Theorem 1.1. *For any $p \in [1,\infty)$ and $r > 0$, there exists $C = C(B,p,r) > 0$ such that the following uniform estimate holds*

$$\|K^\pm_Q - K^\pm_{\widetilde{Q}}\|_{X^2_{\infty,p}} + \|K^\pm_Q - K^\pm_{\widetilde{Q}}\|_{X^2_{1,p}} \le C\,\|Q - \widetilde{Q}\|_p, \qquad Q, \widetilde{Q} \in \mathbb{U}^{2\times 2}_{p,r}. \tag{1.9}$$

Here $K^\pm_{\widetilde{Q}}$ are the kernels from the representation (1.5) *for the solutions $\widetilde{e}_\pm$ of* (1.1)*, with $\widetilde{Q}$ in place of Q, and the spaces $X^2_{\infty,p}$ and $X^2_{1,p}$ are as introduced in* (2.1) *and* (2.2)*, respectively.*

Combining the uniform estimate (1.9) with the representation (1.6) we obtain the following statement concerning the *Lipschitz property* of the map $Q \to g_{l,Q}$ on L^p-balls. It will play a crucial role in our approach to subsequent estimates.

Proposition 1.2. *Let $Q, \widetilde{Q} \in \mathbb{U}^{2\times 2}_{p,r}$ with $p \in [1,\infty]$, and let $g_k := g_{k,Q} - g_{k,\widetilde{Q}}$, $k \in \{1,2\}$, see* (4.12)*. Then $g_k \in L^p[0,1]$, $k \in \{1,2\}$, and the difference of characteristic determinants of the problem* (1.1)–(1.2) *admits the following representation*

$$\Delta_Q(\lambda) - \Delta_{\widetilde{Q}}(\lambda) = \int_0^1 g_1(t)e^{ib_1\lambda t}\,dt + \int_0^1 g_2(t)e^{ib_2\lambda t}\,dt. \tag{1.10}$$

Moreover, there exists a constant $C = C(B,p,r) > 0$ such that

$$\|g_1\|_p + \|g_2\|_p = \|g_{1,Q} - g_{1,\widetilde{Q}}\|_p + \|g_{2,Q} - g_{2,\widetilde{Q}}\|_p \le C\,\|Q - \widetilde{Q}\|_p. \tag{1.11}$$

As an immediate application of Proposition 1.2 we complete the formula (1.7) by establishing the c_0-Lipschitz property of the spectrum $\Lambda_Q = \{\lambda_{Q,n}\}_{n\in\mathbb{Z}}$ on compact sets: if the boundary conditions (1.2) are regular, then for each compact $\mathcal{K}\,(\subset \mathbb{U}^{2\times 2}_{1,r})$ and any $\varepsilon > 0$ there exists $N_\varepsilon > 0$, not dependent on $Q \in \mathcal{K}$, such that the following uniform relation holds

$$\sup_{|n|>N_\varepsilon} \left|\lambda_{Q,n} - \lambda_n^0\right| \le \varepsilon, \qquad Q \in \mathcal{K}. \tag{1.12}$$

In the case of Dirac systems this result was established by Sadovnichaya (2016: Theorem 3).

As is evident from the representations (1.5) and (1.10), the stability of eigenvalues and eigenvectors of the operator $L(Q)$ reduces to certain properties of the Fourier transform and the "maximal" Fourier transform

$$F_g(\lambda) := \int_0^1 g(t)e^{i\lambda t}\,dt \quad \text{and} \quad \mathscr{F}_g(\lambda) := \sup_{x\in[0,1]} \left|\int_0^x g(t)e^{i\lambda t}\,dt\right|, \qquad \lambda \in \mathbb{C}.$$

To this end, we generalize the classical Hausdorff-Young and Hardy-Littlewood theorems for Fourier coefficients, see (Zigmund, 1959: Theorems XII.2.3 & XII.3.19). Throughout the paper p' will denote $p/(p-1)$, and the following notations with $h > 0$ and $n \in \mathbb{Z}$ will be used for strips in the complex plane:

$$\Pi_h := \{z \in \mathbb{C} : |\mathrm{Im}\, z| \le h\}, \qquad \Pi_{h,n} := \{z \in \mathbb{C} : n \le \mathrm{Re}\, z \le n+1,\ |\mathrm{Im}\, z| \le h\}. \tag{1.13}$$

Moreover, note that the concept of an incompressible sequence with density d will be defined in Definition 4.6.

Proposition 1.3. *Let $p \in (1,2]$. Then there exists a constant $C = C(p,h,d) > 0$ such that the following estimates hold uniformly for $g \in L^p[0,1]$ and for incompressible sequences $\Lambda = \{\mu_n\}_{n\in\mathbb{Z}}$ with density d contained in the strip Π_h with $h > 0$:*

$$\sum_{n\in\mathbb{Z}} |F_g(\mu_n)|^{p'} \le \sum_{n\in\mathbb{Z}} \mathscr{F}_g^{p'}(\mu_n) \le C\,\|g\|_p^{p'}, \tag{1.14}$$

$$\sum_{n\in\mathbb{Z}} (1+|n|)^{p-2}|F_g(\mu_n)|^p \le \sum_{n\in\mathbb{Z}} (1+|n|)^{p-2}\mathscr{F}_g^p(\mu_n) \le C\,\|g\|_p^p. \tag{1.15}$$

The proof of the inequalities in (1.14)–(1.15) involving the "maximal" Fourier transform $\mathscr{F}_g$ relies on the deep Carleson-Hunt theorem, while the estimates of ordinary Fourier transforms are elementary in character. Inequality (1.15) generalizes the Hardy-Littlewood theorem and coincides with it for the ordinary Fourier transform when $\mu_n = 2\pi n$. In turn, this inequality is an important ingredient in proving the estimate (1.19) below. By combining Propositions 1.2 and 1.3 we establish the Lipschitz property of the mapping $Q \to \Lambda_Q$ in different norms.

Theorem 1.4. *Let $Q, \widetilde{Q} \in \mathbb{U}^{2\times 2}_{p,r}$ for some $p \in (1,2]$ and $r > 0$, and let the boundary conditions* (1.2) *be strictly regular. Then there exists an enumeration of the spectra $\{\lambda_{Q,n}\}_{n\in\mathbb{Z}}$ and $\{\lambda_{\widetilde{Q},n}\}_{n\in\mathbb{Z}}$ of the operators $L_U(Q)$ and $L_U(\widetilde{Q})$, respectively, and a set $\mathcal{I}_{Q,\widetilde{Q}} \subset \mathbb{Z}$, such that with certain constants $C, C_1, C_2, N > 0$, not dependent on Q and $\widetilde{Q}$, the following uniform estimates hold:*

$$\operatorname{card}\left(\mathbb{Z} \setminus \mathcal{I}_{Q,\widetilde{Q}}\right) \le N, \tag{1.16}$$

$$C_1 \left|\Delta_{\widetilde{Q}}(\lambda_{Q,n})\right| \le |\lambda_{Q,n} - \lambda_{\widetilde{Q},n}| \le C_2 \left|\Delta_{\widetilde{Q}}(\lambda_{Q,n})\right|, \quad n \in \mathcal{I}_{Q,\widetilde{Q}}, \tag{1.17}$$

$$\sum_{n\in\mathcal{I}_{Q,\widetilde{Q}}} \left|\lambda_{Q,n} - \lambda_{\widetilde{Q},n}\right|^{p'} \le C\,\|Q - \widetilde{Q}\|_p^{p'}, \tag{1.18}$$

$$\sum_{n\in\mathcal{I}_{Q,\widetilde{Q}}} (1+|n|)^{p-2} \left|\lambda_{Q,n} - \lambda_{\widetilde{Q},n}\right|^{p} \le C\,\|Q - \widetilde{Q}\|_p^{p}. \tag{1.19}$$

On a compact set $\mathcal{K}$ in $L^p([0,1];\mathbb{C}^{2\times 2})$ the subsets $\mathcal{I}_{Q,\widetilde{Q}} \subset \mathbb{Z}$ can be chosen independent of the pair $\{Q, \widetilde{Q}\}$ and, in view of (1.16), the summation in (1.18)–(1.19) takes the form $\sum_{|n|\ge N_1}$. Here $N_1 \in \mathbb{N}$ does not depend on $Q, \widetilde{Q} \in \mathcal{K}$.

Note also that the two-sided estimate (1.17) plays a crucial role in the proof of the estimates in (1.18)–(1.19). On account of the representation (1.10), it reduces the Lipschitz property of the map $Q \to \Lambda_Q$ to the property that the generalized Fourier coefficients of g_1 and g_2 belong to certain weighted ℓ^p-spaces.

Observe that in proving (1.18)–(1.19) we use only the evaluation of the ordinary Fourier transform and we do not use the deep Carleson-Hunt result. In particular, the proof of (1.19) relies only on the uniform estimate between the first and third terms in (1.15), i.e., it concerns only the ordinary F and not the "maximal" $\mathscr{F}$. This fact makes the proof of the estimates (1.18)–(1.19) elementary in character.

Relation (1.12) is also valid for regular BCs and extends Theorem 3 from Sadovnichaya (2016) to the case of Dirac-type systems ($b_1 \ne -b_2$). When $\widetilde{Q} = 0$, then the estimates (1.18)–(1.19) give ℓ^p-estimates (uniform on balls) of the remainder in the asymptotic formula (1.7) for the eigenvalues of the strictly regular problem (1.2) for Dirac-type systems. For the Dirac operator ($-b_1 = b_2 = 1$) the estimate (1.18) with $\widetilde{Q} = 0$ generalizes the corresponding result obtained first by Savchuk & Shkalikov (2014: Theorem 4.3 & Theorem 4.5) with a constant C that depends on Q (i.e., for the two points compact set $\mathcal{K} = \{Q, 0\}$) and later in Savchuk & Sadovnichaya (2018) for arbitrary compact sets $\mathcal{K}$ in $L^1([0,1];\mathbb{C}^{2\times 2})$.

Note in this connection that A. Gomilko and L. Rzepnicki (2020), and A. Gomilko (2020) obtained new, sharp, asymptotic formulas for eigenfunctions of Sturm–Liouville operators with singular potentials, and for eigenvalues and eigenfunctions of Dirichlet BVPs for the Dirac system with $Q \in L^p([0,1];\mathbb{C}^{2\times 2})$, $1 \le p < 2$.

The weighted estimate (1.19) is new even for the Dirac system with $Q \in \mathbb{U}^{2\times 2}_{p,r}$ and $\widetilde{Q} = 0$, and even for the trivial compact set $\mathcal{K} = \{Q, 0\}$.

Turning to the stability of the eigenvectors of the operator $L(Q)$, we first investigate the Fourier transform of the kernels $K_Q^{\pm}$ of the transformation operators in the representation (1.5). Namely, in Theorem 7.3 we estimate deviations

$$\int_0^x \big(K_Q^{\pm} - K_{\widetilde{Q}}^{\pm}\big)_{jk}(x,t)e^{ib_k\lambda t}\,dt$$

via "maximal" Fourier transforms of the deviations $Q - \widetilde{Q}$ and $\|Q - \widetilde{Q}\|_1\,\widetilde{Q}$. Furthermore, the representation (1.5) leads to similar estimates for the fundamental matrix solution $\Phi_Q(x,\lambda)$ of the system (1.1), which in the case of $\widetilde{Q} = 0$ reads as follows.

Proposition 1.5. *Let $Q \in \mathbb{U}_{1,r}^{2\times 2}$ for some $r > 0$. Then there exists $C = C(B,r) > 0$, not dependent on Q, such that the following uniform estimate holds for $x \in [0,1]$ and $\lambda \in \mathbb{C}$,*

$$\begin{aligned}
&|\Phi_Q(x,\lambda) - \Phi_0(x,\lambda)|_{\mathbb{C}^{2\times 2}} \\
&\quad \le 2\sum_{j,k=1}^{2}\left|\int_0^x K_{jk}^{+}(x,t)e^{ib_k\lambda t}\,dt\right| + 2\sum_{j,k=1}^{2}\left|\int_0^x K_{jk}^{-}(x,t)e^{ib_k\lambda t}\,dt\right| \\
&\quad \le C\,e^{2(b_2-b_1)|\operatorname{Im}\lambda|x}\sum_{j\ne k}\sup\nolimits_{s\in[0,x]}\left|\int_0^s Q_{jk}(t)e^{i(b_k-b_j)\lambda t}\,dt\right|.
\end{aligned}$$

Now we are ready to state ℓ^p-stability properties of eigenfunctions of the operators $L_U(Q)$. Assume the spectrum $\Lambda_{U,Q} = \{\lambda_{Q,n}\}_{n\in\mathbb{Z}}$ of $L_U(Q)$ to be asymptotically simple, and introduce a sequence $\{f_{Q,n}\}_{|n|>N}$ of the corresponding normalized eigenfunctions: $L_U(Q)f_{Q,n} = \lambda_{Q,n}f_{Q,n}$.

Theorem 1.6. *Let $Q, \widetilde{Q} \in \mathbb{U}_{p,r}^{2\times 2}$, $p \in (1,2]$, $p' := p/(p-1)$, and $r > 0$. Moreover, assume the BCs $\{U_j\}_1^2$ of the form* (1.2) *to be strictly regular. Then there exist enumerations of the spectra $\{\lambda_{Q,n}\}_{n\in\mathbb{Z}}$ and $\{\lambda_{\widetilde{Q},n}\}_{n\in\mathbb{Z}}$ of the operators $L_U(Q)$ and $L_U(\widetilde{Q})$, respectively, and a set $\mathcal{I}_{Q,\widetilde{Q}} \subset \mathbb{Z}$, such that for some constants $C, N > 0$, not dependent on Q and $\widetilde{Q}$, the following estimates hold*

$$\sum_{n\in\mathcal{I}_{Q,\widetilde{Q}}}\big\|f_{Q,n} - f_{\widetilde{Q},n}\big\|_\infty^{p'} \le C\,\|Q-\widetilde{Q}\|_p^{p'}, \tag{1.20}$$

$$\sum_{n\in\mathcal{I}_{Q,\widetilde{Q}}}(1+|n|)^{p-2}\big\|f_{Q,n} - f_{\widetilde{Q},n}\big\|_\infty^{p} \le C\,\|Q-\widetilde{Q}\|_p^{p}. \tag{1.21}$$

On compact sets $\mathcal{K}$ in L^p the estimates (1.20)–(1.21) are simplified, since the subsets $\mathcal{I}_{Q,\widetilde{Q}} \subset \mathbb{Z}$ can then be chosen to be independent of the pair $\{Q, \widetilde{Q}\}$. Moreover, in view of (1.16), the summation in (1.18)–(1.21) can in that case be replaced by $\sum_{|n|\ge N_1}$. Here $N_1 \in \mathbb{N}$ does not depend on Q and $\widetilde{Q}$. Inequality (1.21) generalizes the classical Hardy-Littlewood inequality for Fourier coefficients (Zigmund, 1959: Theorem XII.3.19), see Remark 7.9.

Recall that antiperiodic boundary conditions could be strictly regular for Dirac-type operators as opposed to the Dirac case. Therefore, all the previous results imply the following surprising statement.

Corollary 1.7. *Let $Q, \widetilde{Q} \in \mathbb{U}_{p,r}^{2\times 2}$, $p \in (1,2]$, and let $-b_1, b_2 \in \mathbb{N}$ and $b_2 - b_1$ be odd. Then antiperiodic BCs are strictly regular and, hence, the operator $L_U(Q)$ has the Riesz basis property. Moreover, the corresponding eigenvalues and eigenvectors satisfy the uniform Lipschitz type estimates* (1.18)–(1.19) *and* (1.20)–(1.21).

This result demonstrates a substantial difference between Dirac and Dirac-type operators.

Observe in conclusion that periodic and antiperiodic (necessarily non-strictly regular) BVPs for 2×2 Dirac and Sturm-Liouville equations have also attracted attention during the last decade. For instance, a criterion for SRV of the periodic BVP for 2×2 Dirac equation to contain a Riesz basis (without parentheses!) was obtained by P. Djakov and B. Mityagin in (2012b), see also the recent survey (Djakov & Mityagin, 2020) and the recent papers by A.S. Makin (2021; 2020), and the references therein. It is also worth mentioning that F. Gesztesy and V.A. Tkachenko (2009; 2012), for $q \in L^2[0, \pi]$, and P. Djakov and B.S. Mityagin (2012b), for $q \in W^{-1,2}[0, \pi]$, established by different methods a criterion for SRV to contain a Riesz basis for the Sturm-Liouville operator $-\frac{d^2}{dx^2} + q(x)$ on $[0, \pi]$, see also the survey (Makin, 2012).

The contents of the paper will now be briefly described. In Section 2 the Banach spaces $X_{1,p}$ and $X_{\infty,p}$ are studied. Section 3 is concerned with triangular transformation operators. The general properties of a 2×2 Dirac-type BVP are discussed in Section 4. In Section 5 one can find Fourier transform estimates. The stability properties of eigenvalues are discussed in Section 6 and the stability properties of eigenfunctions are discussed in Section 7.

2 The Banach spaces $X_{1,p}$ and $X_{\infty,p}$

Let $p \in [1, \infty]$. Following (Malamud, 1994) denote by $X_{1,p} := X_{1,p}(\Omega)$ and $X_{\infty,p} := X_{\infty,p}(\Omega)$ the linear spaces composed of (equivalent classes of) measurable functions defined on the triangular set $\Omega := \{(x, t) : 0 \le t \le x \le 1\}$ satisfying

$$\|f\|^p_{X_{1,p}} := \operatorname*{ess\,sup}_{t\in[0,1]} \int_t^1 |f(x, t)|^p \, dx < \infty, \qquad p < \infty, \tag{2.1}$$

$$\|f\|^p_{X_{\infty,p}} := \operatorname*{ess\,sup}_{x\in[0,1]} \int_0^x |f(x, t)|^p \, dt < \infty, \qquad p < \infty, \tag{2.2}$$

respectively, and $\|f\|_{X_{1,\infty}} = \|f\|_{X_{\infty,\infty}} := \operatorname{ess\,sup}_{(x,t)\in\Omega} |f(x, t)|$. It can easily be shown that the spaces $X_{1,p}$ and $X_{\infty,p}$ equipped with the norms (2.1) and (2.2) form Banach spaces that are not separable. Denote by $X^0_{1,p} := X^0_{1,p}(\Omega)$ and $X^0_{\infty,p} := X^0_{\infty,p}(\Omega)$ the closures of the subspace of continuous functions $C(\Omega)$ in $X_{1,p}(\Omega)$ and $X_{\infty,p}(\Omega)$, respectively. Clearly, the set $C^1(\Omega)$ of smooth functions is also dense in both spaces $X^0_{1,p}$ and $X^0_{\infty,p}$. Note also that the following embeddings hold and are continuous

$$X_{1,p_1} \subset X_{1,p_2} \subset X_{1,1} \qquad \text{and} \qquad X_{\infty,p_1} \subset X_{\infty,p_2} \subset X_{\infty,1}, \qquad p_1 > p_2 \ge 1.$$

The following simple property of the spaces $X^0_{1,p}$ and $X^0_{\infty,p}$ will be important in the sequel.

Lemma 2.1. *Let $p \ge 1$. For each $a \in [0, 1]$ the trace mappings*

$$\begin{aligned} i_{a,\infty} :\ C(\Omega) \to C[0, a], \qquad & i_{a,\infty}\big(N(x, t)\big) := N(a, t), \\ i_{a,1} :\ C(\Omega) \to C[a, 1], \qquad & i_{a,1}\big(N(x, t)\big) := N(x, a), \end{aligned}$$

admit continuous extensions, which are also denoted by $i_{a,\infty}$ and $i_{a,1}$, to mappings from $X^0_{\infty,p}(\Omega)$ onto $L^p[0, a]$ and $X^0_{1,p}(\Omega)$ onto $L^p[a, 1]$, respectively.

Going over to the vector case we denote for $u = \mathrm{col}(u_1, \ldots, u_n) \in \mathbb{C}^n$

$$|u|_\alpha^\alpha := |u_1|^\alpha + \ldots + |u_n|^\alpha, \quad 0 < \alpha < \infty, \qquad |u|_\infty = \max\{|u_1|, \ldots, |u_n|\}.$$

Furthermore, for $A = (a_{jk})_{j,k=1}^n \in \mathbb{C}^{n\times n}$ we define

$$|A|_{\alpha\to\beta} := \sup\{|Au|_\beta : u \in \mathbb{C}^n, |u|_\alpha = 1\}, \qquad \alpha, \beta \in (0, \infty].$$

Now we are ready to introduce the Banach spaces

$$X_{1,p}^n := X_{1,p}(\Omega; \mathbb{C}^{n\times n}) \qquad \text{and} \qquad X_{\infty,p}^n := X_{\infty,p}(\Omega; \mathbb{C}^{n\times n}),$$

consisting of $n \times n$ matrix-functions $F = (F_{jk})_{j,k=1}^n$ with $X_{1,p}$- and $X_{\infty,p}$-entries, respectively, equipped with the norms

$$\|F\|_{X_{1,p}^n}^p := \operatorname*{ess\,sup}_{t\in[0,1]} \int_t^1 |F(x,t)|_{1\to p}^p \, dx < \infty, \qquad p \in [1, \infty),$$

$$\|F\|_{X_{\infty,p}^n}^p := \operatorname*{ess\,sup}_{x\in[0,1]} \int_0^x |F(x,t)|_{p'\to\infty}^p \, dt < \infty, \qquad p \in [1, \infty).$$

Moreover, $\|F\|_{X_{1,\infty}^n} = \|F\|_{X_{\infty,\infty}^n} := \operatorname{ess\,sup}_{(x,t)\in\Omega} |F(x,t)|_{1\to\infty}$. Besides, we introduce the subspaces

$$X_{1,p}^{0,n} := X_{1,p}^0(\Omega; \mathbb{C}^{n\times n}) \qquad \text{and} \qquad X_{\infty,p}^{0,n} := X_{\infty,p}^0(\Omega; \mathbb{C}^{n\times n}),$$

which are separable parts of $X_{1,p}^n$ and $X_{\infty,p}^n$, respectively.

Furthermore, for brevity, throughout the section we use the following notation

$$L^s := L^s([0,1]; \mathbb{C}^n), \qquad s \in [1, \infty].$$

With each measurable matrix kernel $N(\cdot,\cdot) = \big(N_{jk}(\cdot,\cdot)\big)_{j,k=1}^n$ on Ω one associates a Volterra type operator $\mathcal{N}$ as follows

$$\mathcal{N} : f \mapsto \int_0^x N(x,t) f(t) \, dt. \tag{2.3}$$

Denote by $\|\mathcal{N}\|_{\alpha\to\beta} := \|N\|_{L^\alpha\to L^\beta}$, $\alpha, \beta \in [1, \infty]$, the norm for bounded operators $\mathcal{N}$ acting from L^α to L^β.

The following result demonstrates the natural occurrence of the spaces $X_{1,p}^n$ and $X_{\infty,p}^n$ in the study of the integral operators acting from L^α to L^β for special α and β. In particular, the third statement sheds light on the interpolation role of these spaces, cf. (Malamud, 1994). This result substantially complements Lemma 2.3 from (Lunyov & Malamud, 2016).

Recall that a Volterra operator on a Banach space is a compact operator with zero spectrum.

Proposition 2.2. *Let $\mathcal{N}$ be a Volterra type operator given by* (2.3) *for a measurable matrix-function $N(\cdot,\cdot)$ and let $p \in [1, \infty]$. Then the following statements hold:*

(i) *The inclusion $\mathcal{N} \in \mathcal{B}(L^1, L^p)$ holds if and only if $N \in X_{1,p}^n$, in which case*

$$\|\mathcal{N}\|_{1\to p} = \|N\|_{X_{1,p}^n}.$$

Moreover, if $N \in X^{0,n}_{1,p}$*, then the operator* $\mathcal{N}$ *is compact from* L^1 *to* L^p *and the following relation holds*

$$-\mathcal{N}\,(I+\mathcal{N})^{-1} =: \mathcal{S} \in \mathcal{B}(L^1, L^p), \quad \textit{where } \mathcal{S} : f \mapsto \int_0^x S(x,t)f(t)\,dt \textit{ with } S \in X^{0,n}_{1,p};$$

here $(I+\mathcal{N})^{-1}$ *is treated as an operator from* $\mathcal{B}(L^1, L^1)$.

(ii) *The inclusion* $\mathcal{N} \in \mathcal{B}(L^{p'}, L^\infty)$ *holds if and only if* $N \in X^n_{\infty,p}$*, in which case*

$$\|\mathcal{N}\|_{p'\to\infty} = \|N\|_{X^n_{\infty,p}}.$$

Moreover, if $N \in X^{0,n}_{\infty,p}$*, then* $\mathcal{N}$ *maps* $L^{p'}$ *to* $\mathcal{C} := C([0,1];\mathbb{C}^n)$ *and is compact. Let* $\mathcal{N}_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ *be a restriction of* $\mathcal{N}$ *to* $\mathcal{C}$*, then* $(I+\mathcal{N}_\mathcal{C})^{-1} \in \mathcal{B}(\mathcal{C},\mathcal{C})$ *and the following relation holds*

$$-(I+\mathcal{N}_\mathcal{C})^{-1}\,\mathcal{N} =: \mathcal{S} \in \mathcal{B}(L^{p'}, \mathcal{C}), \quad \textit{where } \mathcal{S} : f \mapsto \int_0^x S(x,t)f(t)\,dt \textit{ with } S \in X^{0,n}_{\infty,p}.$$

(iii) *Let* $N \in X^n_{1,1} \cap X^n_{\infty,1} \subset X^n_{1,p} \cap X^n_{\infty,p}$*. Then* $\mathcal{N} \in \mathcal{B}(L^s, L^s)$ *for each* $s \in [1,\infty]$*, and*

$$\|\mathcal{N}\|_{s\to s} \le \|N\|^{1/s}_{X^n_{1,1}}\,\|N\|^{1-1/s}_{X^n_{\infty,1}}.$$

(iv) *Let* $N \in X^{0,n}_{1,p} \cap X^{0,n}_{\infty,p}$*. Then* $\mathcal{N}$ *is a Volterra operator in* L^s *for each* $s \in [1,\infty]$*, and the inverse operator* $(I+\mathcal{N})^{-1}$ *is given by*

$$(I+\mathcal{N})^{-1} =: I + \mathcal{S}, \quad \textit{where } \mathcal{S} : f \mapsto f + \int_0^x S(x,t)f(t)\,dt \textit{ with } S \in X^{0,n}_{1,p} \cap X^{0,n}_{\infty,p}.$$

Remark 2.3. In connection with Proposition 2.2, let us recall Theorems XI.1.5 and XI.1.6 from Kantorovich & Akilov (1977) concerning integral representations of bounded linear operators. Namely, let $p \in (1,\infty]$, and let $\mathcal{R}$ and $\mathcal{S}$ be bounded linear operators from $L^1[0,1]$ to $L^p[0,1]$ and $L^{p'}[0,1]$ to $C[0,1]$, respectively. Then they admit the following integral representations:

$$(\mathcal{R}f)(x) = \int_0^1 R(x,t)f(t)\,dt, \qquad \|\mathcal{R}\|^p_{1\to p} = \operatorname*{ess\,sup}_{t\in[0,1]} \int_0^1 |R(x,t)|^p\,dx < \infty,$$
$$(\mathcal{S}f)(x) = \int_0^1 S(x,t)f(t)\,dt, \qquad \|\mathcal{S}\|^p_{p'\to\infty} = \operatorname*{ess\,sup}_{x\in[0,1]} \int_0^1 |S(x,t)|^p\,dt < \infty.$$

3 Triangular transformation operators

The existence of a triangular transformation operator for the system (1.1) with summable potential matrix $Q \in L^1([0,1];\mathbb{C}^{2\times 2})$ was established in our previous paper (Lunyov & Malamud, 2016). Moreover, the case $B = B^* \in \mathbb{C}^{n\times n}$ and $Q \in L^\infty([0,1];\mathbb{C}^{n\times n})$ was treated earlier in Malamud (1999).

The purpose of this section is to prove the Lipschitz property (in respective norms) for the kernels of the transformation operators of $Q \in L^p([0,1];\mathbb{C}^{2\times 2})$. We start with the following result from Lunyov & Malamud (2016).

Theorem 3.1 (Lunyov & Malamud (2016: Theorem 2.5)). *Let* $Q = \operatorname{codiag}(Q_{12}, Q_{21})$ *belong to* $L^1([0,1]; \mathbb{C}^{2\times 2})$. *Assume that* $e_\pm(\cdot, \lambda)$ *are the solutions of the system* (1.1) *satisfying the initial conditions* $e_\pm(0,\lambda) = \binom{1}{\pm 1}$. *Then the solutions* $e_\pm(\cdot,\lambda)$ *admit the following representation by means of the triangular transformation operators* $K_Q^\pm = \big(K_{jk}^\pm\big)_{j,k=1}^2 \in X_{1,1}^{0,2} \cap X_{\infty,1}^{0,2}$

$$e_\pm(x,\lambda) = e_\pm^0(x,\lambda) + \int_0^x K_Q^\pm(x,t) e_\pm^0(t,\lambda)\,dt, \qquad e_\pm^0(x,\lambda) = \begin{pmatrix} e^{ib_1\lambda x} \\ \pm e^{ib_2 \lambda x}\end{pmatrix}. \tag{3.1}$$

It was shown in Malamud (1999) that if $Q = \operatorname{codiag}(Q_{12}, Q_{21}) \in C^1([0,1]; \mathbb{C}^{2\times 2})$, then the matrix kernel in the triangular representation (3.1) is smooth, $K^\pm = K_Q^\pm = \big(K_{jk}^\pm\big)_{j,k=1}^2 \in C^1(\Omega, \mathbb{C}^{2\times 2})$, and it is the unique solution of the following boundary value problem

$$B^{-1} D_x K^\pm(x,t) + D_t K^\pm(x,t) B^{-1} + iQ(x) K^\pm(x,t) = 0, \tag{3.2}$$

$$K^\pm(x,x)B^{-1} - B^{-1}K^\pm(x,x) = iQ(x), \qquad x \in [0,1], \tag{3.3}$$

$$K^\pm(x,0)B^{-1}\begin{pmatrix} 1 \\ \pm 1\end{pmatrix} = 0, \qquad x \in [0,1]. \tag{3.4}$$

The proof of this result in Malamud (1999) was divided into two steps. First it was proved that there exists the smooth unique solution $R_Q = \big(R_{jk}\big)_{j,k=1}^2 \in C^1(\Omega, \mathbb{C}^{2\times 2})$ of the problem (3.2)–(3.3), satisfying, instead of (3.4), the following conditions:

$$R_{11}(x,0) = R_{22}(x,0) = 0, \qquad x \in [0,1]. \tag{3.5}$$

As the second step we defined the kernels $K_Q^\pm$ via the auxiliary matrix function R_Q by formula (3.7) and showed that they have the required properties. By means of this result, the following proposition was proved in Lunyov & Malamud (2016); it is the starting point of our investigation here. Note also that for smooth kernels relation (3.7) was already exploited in Malamud (1999).

Proposition 3.2 (Lunyov & Malamud (2016)). *Let* $Q \in L^1([0,1]; \mathbb{C}^{2\times 2})$ *and let* $K_Q^\pm$ *be the kernels of the corresponding transformation operators from the representation* (3.1). *Then there exist*

$$R_Q = (R_{jk})_{j,k=1}^2 \in X_{1,1}^{0,2} \cap X_{\infty,1}^{0,2} \quad \textit{and} \quad P_Q^\pm = \operatorname{diag}(P_1^\pm, P_2^\pm) \in L^1([0,1]; \mathbb{C}^{2\times 2}), \tag{3.6}$$

such that

$$K_Q^\pm(x,t) = R_Q(x,t) + P_Q^\pm(x-t) + \int_t^x R_Q(x,s) P_Q^\pm(s-t)\,ds, \qquad 0 \le t \le x \le 1. \tag{3.7}$$

Moreover, $R_Q(\cdot,\cdot)$ *is the unique solution of the following system for* $0 \le t \le x \le 1$,

$$R_{kk}(x,t) = -ib_k \int_{x-t}^x Q_{kj}(\xi) R_{jk}\big(\xi, \xi - x + t\big)\,d\xi, \tag{3.8}$$

$$R_{jk}(x,t) = -ib_j \alpha_j Q_{jk}(\alpha_k x + \alpha_j t) - ib_j \int_{\alpha_k x + \alpha_j t}^x Q_{jk}(\xi) R_{kk}\Big(\xi, \frac{b_j}{b_k}(\xi - x) + t\Big)\,d\xi, \tag{3.9}$$

where $\alpha_k := \frac{b_j}{b_j - b_k}$ *with* $j = 2/k$ *for* $k \in \{1,2\}$.

Note that for smooth Q, i.e., for $Q \in C^1([0,1]; \mathbb{C}^{2\times 2})$, the system (3.8)–(3.9) is equivalent to the system (3.2)–(3.3), (3.5). To refine Theorem 3.1 in the L^p-case, we start by refining properties of the auxiliary kernel R_Q appearing in Proposition 3.2. In the following result we show that the (nonlinear) mapping $Q \to R = R_Q$ is Lipschitz in $X^2_{1,p}$ and $X^2_{\infty,p}$ on each ball of radius r in L^p. Proposition 2.2 can be used to establish part (ii) thereof.

Proposition 3.3. *Let $Q, \widetilde{Q} \in \mathbb{U}^{2\times 2}_{p,r}$ for some $p \geq 1$ and $r > 0$. Then the following statements hold:*

(i) *The solutions R_Q and $R_{\widetilde{Q}}$ of the system of integral equations* (3.8)–(3.9) *for Q and $\widetilde{Q}$ respectively, unique in $X^{0,2}_{1,1} \cap X^{0,2}_{\infty,1}$, belong to $X^{0,2}_{1,p} \cap X^{0,2}_{\infty,p}$ and the following uniform estimate holds*

$$\|R_Q - R_{\widetilde{Q}}\|_{X^2_{1,p}} + \|R_Q - R_{\widetilde{Q}}\|_{X^2_{\infty,p}} \leq C_0 \,\|Q - \widetilde{Q}\|_p,$$

where the constant $C_0 > 0$ does not depend on $Q, \widetilde{Q} \in \mathbb{U}^{2\times 2}_{p,r}$.

(ii) *The operator*

$$\mathcal{R}_Q : f \mapsto \int_0^x R_Q(x,t) f(t)\, dt, \qquad f \in L^s([0,1], \mathbb{C}^2),$$

is a Volterra operator in every space $L^s([0,1], \mathbb{C}^2)$, $s \in [1,\infty]$. Moreover, there exists a constant $C_1 = C_1(B,p,r) > 0$, not dependent on $s \in [1,\infty]$ and $Q, \widetilde{Q} \in \mathbb{U}^{2\times 2}_{p,r}$, such that the following uniform estimate holds

$$\|(I + \mathcal{R}_Q)^{-1} - (I + \mathcal{R}_{\widetilde{Q}})^{-1}\|_{s\to s} \leq C_1 \,\|Q - \widetilde{Q}\|_p,$$

Combining these properties of the kernel $R_Q(\cdot,\cdot)$ with the convolution identity (3.7) allows us to prove the main result of this section: the Lipschitz property of the mapping $Q \mapsto K^{\pm}_Q$ on the balls in $L^p\big([0,1]; \mathbb{C}^{2\times 2}\big)$.

Theorem 3.4. *Let $Q, \widetilde{Q} \in \mathbb{U}^{2\times 2}_{p,r}$ for some $p \in [1,\infty)$ and $r > 0$. Moreover, let $K^{\pm}_Q$ and $K^{\pm}_{\widetilde{Q}}$ be the kernels of the corresponding transformation operators from the representation* (3.1) *for Q and $\widetilde{Q}$, respectively. Then*

$$K^{\pm}_Q, K^{\pm}_{\widetilde{Q}} \in X^{0,2}_{1,p} \cap X^{0,2}_{\infty,p},$$

and there exists a constant $C = C(B,p,r)$ that does not depend on Q and $\widetilde{Q}$, such that the following estimate holds

$$\|K^{\pm}_Q - K^{\pm}_{\widetilde{Q}}\|_{X^2_{\infty,p}} + \|K^{\pm}_Q - K^{\pm}_{\widetilde{Q}}\|_{X^2_{1,p}} \leq C \,\|Q - \widetilde{Q}\|_p.$$

Remark 3.5. (i) For 2×2 Dirac systems ($B = \mathrm{diag}(-1,1)$) with continuous potential Q the triangular transformation operators have been constructed in Levitan & Sargsyan (1991: Chapter 10.3) and Marchenko (1986: Chapter 1.2). For $Q \in L^1$ these transformation operators were constructed in Albeverio, Hryniv & Mykytyuk (2005) by an appropriate generalization of Marchenko's method.

(ii) Let $J : f \to \int_0^x f(t)\, dt$ denote the Volterra integration operator on $L^p[0,1]$. Note that the similarity of the integral Volterra operators given by (2.3) to the simplest Volterra operator of the form $B \otimes J$ acting in the spaces $L^p([0,1]; \mathbb{C}^2)$ has been investigated in Malamud (1999); Romaschenko (2008). The technique of investigating integral equations for the kernels of the transformation operators in the spaces $X_{\infty,1}(\Omega)$ and $X_{1,1}(\Omega)$ goes back to the paper (Malamud, 1994).

4 General properties of a 2×2 Dirac-type BVP

Consider the 2×2 Dirac-type equation (1.1) subject to the general boundary conditions (1.2) and the corresponding operator $L(Q)$ defined in (1.3). In this section we recall and extend some properties of this BVP from (Lunyov & Malamud, 2016). Let us set

$$A := \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}, \qquad A_{jk} := \begin{pmatrix} a_{1j} & a_{1k} \\ a_{2j} & a_{2k} \end{pmatrix}, \qquad J_{jk} := \det(A_{jk}), \tag{4.1}$$

where $j, k \in \{1, \dots, 4\}$. Moreover, let

$$\Phi(\cdot,\lambda) = \begin{pmatrix} \varphi_{11}(\cdot,\lambda) & \varphi_{12}(\cdot,\lambda) \\ \varphi_{21}(\cdot,\lambda) & \varphi_{22}(\cdot,\lambda) \end{pmatrix} =: \begin{pmatrix} \Phi_1(\cdot,\lambda) & \Phi_2(\cdot,\lambda) \end{pmatrix}, \qquad \Phi(0,\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.2}$$

be a fundamental matrix solution of the system (1.1). The eigenvalues of the problem (1.1)–(1.2) (counting multiplicity) are the zeros (counting multiplicity) of the characteristic determinant

$$\Delta_Q(\lambda) := \det \begin{pmatrix} U_1(\Phi_1(\cdot,\lambda)) & U_1(\Phi_2(\cdot,\lambda)) \\ U_2(\Phi_1(\cdot,\lambda)) & U_2(\Phi_2(\cdot,\lambda)) \end{pmatrix}. \tag{4.3}$$

Inserting (4.2) and (1.2) into (4.3), setting $\varphi_{jk}(\lambda) := \varphi_{jk}(1,\lambda)$, and taking the notations in (4.1) into account, we arrive at the following expression for the characteristic determinant

$$\Delta_Q(\lambda) = J_{12} + J_{34}e^{i(b_1+b_2)\lambda} + J_{32}\varphi_{11}(\lambda) + J_{13}\varphi_{12}(\lambda) + J_{42}\varphi_{21}(\lambda) + J_{14}\varphi_{22}(\lambda). \tag{4.4}$$

Alongside the problem (1.1)–(1.2) we consider the same problem with $\widetilde{Q}$ in place of Q. Denote the corresponding fundamental matrix solution, its entries, and the corresponding characteristic determinant as $\widetilde{\Phi}(\cdot,\lambda)$, $\widetilde{\varphi}_{jk}(\cdot,\lambda)$, $j,k \in \{1,2\}$, and $\widetilde{\Delta}(\lambda)$, respectively. If $Q = 0$, then we denote a fundamental matrix solution as $\Phi^0(\cdot,\lambda)$. Clearly

$$\Phi^0(x,\lambda) = \begin{pmatrix} e^{ib_1x\lambda} & 0 \\ 0 & e^{ib_2x\lambda} \end{pmatrix} =: \begin{pmatrix} \Phi_1^0(x,\lambda) & \Phi_2^0(x,\lambda) \end{pmatrix}, \qquad x \in [0,1], \quad \lambda \in \mathbb{C}.$$

Here $\Phi_k^0(\cdot,\lambda)$ is the kth-column of $\Phi^0(\cdot,\lambda)$. In particular, the characteristic determinant $\Delta_0(\cdot)$ becomes

$$\Delta_0(\lambda) = J_{12} + J_{34}e^{i(b_1+b_2)\lambda} + J_{32}e^{ib_1\lambda} + J_{14}e^{ib_2\lambda}. \tag{4.5}$$

In the case of Dirac systems, i.e., when $B = \operatorname{diag}(-1,1)$, this formula simplifies to

$$\Delta_0(\lambda) = J_{12} + J_{34} + J_{32}e^{-i\lambda} + J_{14}e^{i\lambda}.$$

4.1 Representation of the characteristic determinant

Our investigation of the perturbation determinant relies on the following result, clarifying our Proposition 3.1 from Lunyov & Malamud (2016) and coinciding with it for $Q \in L^1([0,1];\mathbb{C}^{2\times2})$.

Lemma 4.1. *Let $Q \in L^p([0,1];\mathbb{C}^{2\times2})$ for some $p \in [1,\infty)$. Then the functions $\varphi_{jk}(\cdot,\lambda)$, with $j,k \in \{1,2\}$, admit the following representations for $x \in [0,1]$ and $\lambda \in \mathbb{C}$*

$$\varphi_{jk}(x,\lambda) = \delta_{jk}e^{ib_k\lambda x} + \int_0^x K_{j1,k}(x,t)e^{ib_1\lambda t}\,dt + \int_0^x K_{j2,k}(x,t)e^{ib_2\lambda t}\,dt, \tag{4.6}$$

where

$$K_{jl,k} := 2^{-1}\left(K_{jl}^+ + (-1)^{l+k}K_{jl}^-\right) \in X_{1,p}^0(\Omega) \cap X_{\infty,p}^0(\Omega), \qquad j,k,l \in \{1,2\}. \tag{4.7}$$

Our study of the Lipschitz property of the eigenvalues and eigenfunctions is based on the following simple corollary of Theorem 3.4.

Lemma 4.2. *Let $Q, \widetilde{Q} \in \mathbb{U}^{2\times 2}_{p,r}$ for some $p \in [1,\infty)$ and $r > 0$. Then the following representation holds for $x \in [0,1]$ and $\lambda \in \mathbb{C}$*

$$\varphi_{jk}(x,\lambda) - \widetilde{\varphi}_{jk}(x,\lambda) = \int_0^x \widehat{K}_{j1,k}(x,t)e^{ib_1\lambda t}\,dt + \int_0^x \widehat{K}_{j2,k}(x,t)e^{ib_2\lambda t}\,dt, \tag{4.8}$$

where

$$\widehat{K}_{jl,k} := K_{jl,k} - \widetilde{K}_{jl,k} \in X^0_{1,p}(\Omega) \cap X^0_{\infty,p}(\Omega), \qquad j,k,l \in \{1,2\}. \tag{4.9}$$

Moreover, for some $C = C(p,r,B)$ the following uniform estimate holds

$$\|\widehat{K}_{jl,k}\|_{X_{\infty,p}(\Omega)} + \|\widehat{K}_{jl,k}\|_{X_{1,p}(\Omega)} \le C\,\|Q - \widetilde{Q}\|_p, \qquad j,k,l \in \{1,2\}. \tag{4.10}$$

Considering next the properties of the characteristic determinant, we first refine (Lunyov & Malamud, 2016: Lemma 4.1) in the L^p case. Note that the existence of trace values $K^{\pm}_{jk}(1,\cdot)$ is implied by Lemma 2.1 and the inclusions

$$K^{\pm}_{jk} \in X^0_{1,p} \cap X^0_{\infty,p}.$$

The later inclusions are important, because the weaker inclusions

$$K^{\pm}_{jk} \in (X_{1,p} \cap X_{\infty,p}) \setminus (X^0_{1,p} \cap X^0_{\infty,p})$$

do not ensure the existence of such traces.

Lemma 4.3. *Let $Q \in L^p([0,1];\mathbb{C}^{2\times 2})$ for some $p \in [1,\infty)$. Then the characteristic determinant $\Delta_Q(\cdot)$ of the problem* (1.1)–(1.2) *is an entire function of exponential type and admits the following representation*

$$\Delta_Q(\lambda) = \Delta_0(\lambda) + \int_0^1 g_{1,Q}(t)e^{ib_1\lambda t}\,dt + \int_0^1 g_{2,Q}(t)e^{ib_2\lambda t}\,dt, \tag{4.11}$$

where for $l \in \{1,2\}$

$$g_{l,Q}(\cdot) = J_{32}K_{1l,1}(1,\cdot) + J_{42}K_{2l,1}(1,\cdot) + J_{13}K_{1l,2}(1,\cdot) + J_{14}K_{2l,2}(1,\cdot) \in L^p[0,1]. \tag{4.12}$$

The next result is immediate by combining Lemma 4.3 with the estimate (4.10).

Lemma 4.4. *Let $Q, \widetilde{Q} \in \mathbb{U}^{2\times 2}_{p,r}$ for some $p \in [1,\infty)$ and $r > 0$. Then the following representation holds*

$$\Delta_Q(\lambda) - \Delta_{\widetilde{Q}}(\lambda) = \int_0^1 \widehat{g}_1(t)e^{ib_1\lambda t}\,dt + \int_0^1 \widehat{g}_2(t)e^{ib_2\lambda t}\,dt, \tag{4.13}$$

where $\widehat{g}_l := g_{Q,l} - g_{\widetilde{Q},l} \in L^p[0,1]$, $l \in \{1,2\}$. Moreover, for some $\widehat{C} = \widehat{C}(p,r,B,A)$, the following uniform estimate holds

$$\|\widehat{g}_1\|_p + \|\widehat{g}_2\|_p = \|g_{Q,1} - g_{\widetilde{Q},1}\|_p + \|g_{Q,2} - g_{\widetilde{Q},2}\|_p \le \widehat{C}\,\|Q - \widetilde{Q}\|_p, \qquad Q, \widetilde{Q} \in \mathbb{U}^{2\times 2}_{p,r}.$$

4.2 Regular and strictly regular boundary conditions

Recall that $J_{jk} = \det A_{jk}$, see (4.1), and recall the following definitions.

Definition 4.5. The boundary conditions (1.2) are called *regular* if

$$J_{14}J_{32} \neq 0.$$

Definition 4.6 (cf. (Katsnel'son, 1971))**.** The sequence $\Lambda := \{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$ is called an *incompressible sequence of density* $d \in \mathbb{N}$, if every rectangle $[t-1, t+1] \times \mathbb{R} \subset \mathbb{C}$ contains at most d entries of the sequence, i.e., if

$$\operatorname{card}\{n \in \mathbb{Z} : |\operatorname{Re}\lambda_n - t| \leq 1\} \leq d, \qquad t \in \mathbb{R}.$$

Let us recall certain important properties of the characteristic determinant $\Delta(\cdot)$ in the case of regular boundary conditions from Lunyov & Malamud (2016). Recall that $\mathbb{D}_r(z) \subset \mathbb{C}$ denotes the disc of radius r with center z.

Proposition 4.7 (Lunyov & Malamud (2016: Proposition 4.6))**.** *Let the boundary conditions* (1.2) *be regular and let* $\Delta_Q(\cdot)$ *be the characteristic determinant of the problem* (1.1)–(1.2)*, given by* (4.4)*. Then the following statements hold:*

(i) *The characteristic determinant* $\Delta_Q(\cdot)$ *is a sine-type function with* $h_\Delta(\pi/2) = -b_1$ *and* $h_\Delta(-\pi/2) = b_2$*. In particular, the function* $\Delta_Q(\cdot)$ *has infinitely many zeros* $\Lambda := \{\lambda_n\}_{n\in\mathbb{Z}}$ *counting multiplicities and* $\Lambda \subset \Pi_h$ *for some* $h \geq 0$*, see* (1.13)*.*

(ii) *The sequence* Λ *is incompressible.*

(iii) *For any* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *such that the determinant* $\Delta_Q(\cdot)$ *admits the following estimate from below*

$$|\Delta_Q(\lambda)| \geq C_\varepsilon(e^{-b_1 \operatorname{Im}\lambda} + e^{-b_2 \operatorname{Im}\lambda}), \qquad \lambda \in \mathbb{C} \setminus \bigcup_{n\in\mathbb{Z}} \mathbb{D}_\varepsilon(\lambda_n).$$

Clearly, the conclusions of Proposition 4.7 are valid for the characteristic determinant $\Delta_0(\cdot)$ given by (4.5). Let $\Lambda_0 = \{\lambda_n^0\}_{n\in\mathbb{Z}}$ be the sequence of its zeros counting multiplicity. From now on, let us order Λ_0 in a (possibly non-unique) way such that

$$\operatorname{Re}\lambda_n^0 \leq \operatorname{Re}\lambda_{n+1}^0, \qquad n \in \mathbb{Z}.$$

Let us recall an important result from Lunyov & Malamud (2014b; 2016) and Savchuk & Shkalikov (2014) concerning the asymptotic behavior of eigenvalues.

Proposition 4.8 (Lunyov & Malamud (2016: Proposition 4.7))**.** *Let* $Q \in L^1([0,1];\mathbb{C}^{2\times 2})$ *and let the boundary conditions* (1.2) *be regular. Then the sequence* $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ *of zeros of* $\Delta_Q(\cdot)$ *can be ordered in such a way that the following asymptotic formula holds*

$$\lambda_n = \lambda_n^0 + o(1), \quad \text{as} \quad |n| \to \infty, \quad n \in \mathbb{Z}. \tag{4.14}$$

Let us refine this ordering to have some additional important properties.

Proposition 4.9. *Let $Q \in L^1([0,1];\mathbb{C}^{2\times 2})$ and let the boundary conditions* (1.2) *be regular. Then the following statements hold:*

(i) *For any $\varepsilon > 0$ there exist $M_\varepsilon = M_\varepsilon(Q,B,A) > 0$ and $C_\varepsilon = C_\varepsilon(B,A) > 0$, such that*

$$|\Delta_Q(\lambda) - \Delta_0(\lambda)| < |\Delta_0(\lambda)|, \qquad \lambda \notin \widetilde{\Omega}_\varepsilon, \tag{4.15}$$

$$|\Delta_Q(\lambda)| > C_\varepsilon \left(e^{-b_1 \operatorname{Im}\lambda} + e^{-b_2 \operatorname{Im}\lambda}\right), \qquad \lambda \notin \widetilde{\Omega}_\varepsilon, \tag{4.16}$$

where

$$\widetilde{\Omega}_\varepsilon := \mathbb{D}_{M_\varepsilon}(0) \cup \Omega_\varepsilon^0, \qquad \Omega_\varepsilon^0 := \bigcup_{n\in\mathbb{Z}} \mathbb{D}_\varepsilon(\lambda_n^0). \tag{4.17}$$

(ii) *The sequence $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ can be ordered such that for any $\varepsilon > 0$ and $n \in \mathbb{Z}$ the values λ_n and λ_n^0 belong to the same connected component of $\widetilde{\Omega}_\varepsilon$. In addition, the relation* (4.14) *also holds for this ordering.*

Definition 4.10. Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be the sequence of zeros of the characteristic determinant $\Delta_Q(\cdot)$ of the Dirac-type operator $L_U(Q)$ with summable potential and regular boundary conditions. Let $\widetilde{\Omega}_\varepsilon$ be defined in (4.17). The ordering of Λ for which λ_n and λ_n^0 belong to the same connected component of $\widetilde{\Omega}_\varepsilon$ for all $\varepsilon > 0$ and $n \in \mathbb{Z}$, is called a *canonical ordering*.

Observe that Proposition 4.9 implies the existence of a canonical ordering for each sequence of zeros of the characteristic determinant $\Delta_Q(\cdot)$ of the Dirac-type operator $L_U(Q)$ with summable potential and regular boundary conditions.

In the sequel we need the following definitions.

Definition 4.11. A sequence $\Lambda := \{\lambda_n\}_{n\in\mathbb{Z}}$ of complex numbers is said to be *separated* if for some $\tau > 0$ the inequality $|\lambda_j - \lambda_k| > 2\tau$ holds whenever $j \neq k$. In particular, all entries of a separated sequence are distinct. Furthermore, the sequence Λ is said to be *asymptotically separated* if for some $N \in \mathbb{N}$ the subsequence $\{\lambda_n\}_{|n|>N}$ is separated.

Definition 4.12. The boundary conditions (1.2) are called *strictly regular* if they are regular, i.e., $J_{14}J_{32} \neq 0$, and the sequence of zeros $\lambda_0 = \{\lambda_n^0\}_{n\in\mathbb{Z}}$ of the characteristic determinant $\Delta_0(\cdot)$ is asymptotically separated.

In particular, if the boundary conditions (1.2) are strictly regular, then there exists $n_0 \in \mathbb{N}$ such that zeros $\{\lambda_n^0\}_{|n|>n_0}$ of its characteristic determinant are geometrically and algebraically simple.

Observe that it follows from Proposition 4.8 that the sequence $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ of zeros of $\Delta_Q(\cdot)$ is asymptotically separated if the boundary conditions are strictly regular.

Assuming the boundary conditions (1.2) to be regular, let us rewrite them in a more convenient form. Since $J_{14} \neq 0$, the inverse matrix A_{14}^{-1} exists. Therefore writing down the boundary conditions (1.2) as the vector equation $\binom{U_1(y)}{U_2(y)} = 0$ and multiplying it by the matrix A_{14}^{-1}, they are transformed into the following conditions

$$\begin{cases} \widehat{U}_1(y) = y_1(0) + by_2(0) + ay_1(1) = 0, \\ \widehat{U}_2(y) = dy_2(0) + cy_1(1) + y_2(1) = 0, \end{cases} \tag{4.18}$$

for some $a, b, c, d \in \mathbb{C}$. Now $J_{14} = 1$ and the boundary conditions (4.18) are regular if and only if $J_{32} = ad - bc \neq 0$. Thus, the characteristic determinants $\Delta_0(\cdot)$ and $\Delta(\cdot)$ take the form

$$\Delta_0(\lambda) = d + ae^{i(b_1+b_2)\lambda} + (ad - bc)e^{ib_1\lambda} + e^{ib_2\lambda},$$

$$\Delta(\lambda) = d + ae^{i(b_1+b_2)\lambda} + (ad - bc)\varphi_{11}(\lambda) + \varphi_{22}(\lambda) + c\varphi_{12}(\lambda) + b\varphi_{21}(\lambda).$$

Remark 4.13. Let us list some types of strictly regular boundary conditions (4.18). In all of these cases, except 4 (b), the set of zeros of Δ_0 is a union of a finite number of arithmetic progressions.

1. Regular BCs (4.18) for the Dirac operator ($-b_1 = b_2 = 1$) are strictly regular if and only if $(a - d)^2 \neq -4bc$.

2. Separated BCs ($a = d = 0$, $bc \neq 0$) are always strictly regular.

3. Let $b_1/b_2 \in \mathbb{Q}$. Without loss of generality we can assume that $-b_1, b_2 \in \mathbb{N}$ and that $\gcd(-b_1, b_2) = 1$. It is clear that the BCs (4.18) are strictly regular if and only if a certain polynomial of degree $b_2 - b_1$ does not have multiple roots. In addition, if $ad \neq 0$ and $bc = 0$, then the BCs (4.18) are strictly regular if and only if

$$b_1 \ln|d| + b_2 \ln|a| \neq 0 \quad \text{or} \quad b_1 \arg(-d) + b_2 \arg(-a) \notin 2\pi\mathbb{Z}.$$

   In particular, antiperiodic BCs ($a = d = 1$, $b = c = 0$) are strictly regular if and only if $b_1 - b_2$ is odd. Note that these BCs are not strictly regular in the case of a Dirac system.

4. Let $\alpha := -b_1/b_2 \notin \mathbb{Q}$. Then the problem of strict regularity of BCs is generally much more complicated. Let us list some known cases:

   (a) Let $ad \neq 0$ and $bc = 0$. Then the BCs (4.18) are strictly regular if and only if

$$b_1 \ln|d| + b_2 \ln|a| \neq 0.$$

   (b) Let $a = 0$ and $bc, d \in \mathbb{R} \setminus \{0\}$. Then the BCs (4.18) are strictly regular if and only if

$$d \neq -(\alpha + 1) \left(|bc|\alpha^{-\alpha}\right)^{\frac{1}{\alpha+1}},$$

   see (Lunyov & Malamud, 2016: Proposition 5.6).

5 Fourier transform estimates

5.1 Generalizations of the Hausdorff-Young and Hardy-Littlewood theorems

To evaluate deviations of eigenvalues of the operators $L(Q)$ and $L(\widetilde{Q})$, we extend here the classical Hausdorff-Young and Hardy-Littlewood interpolation theorems for Fourier coefficients (see (Zigmund, 1959: Theorem XII.2.3) and (Zigmund, 1959: Theorem XII.3.19), respectively) to the case of arbitrary incompressible sequences $\Lambda = \{\mu_n\}_{n\in\mathbb{Z}}$ instead of $\Lambda = \{2\pi n\}_{n\in\mathbb{Z}}$.

For an efficient estimate of eigenvectors deviations in Section 7 we will use the following (sublinear) Carleson transform (the maximal version of the classical Fourier transform)

$$\mathcal{E}_f(\lambda) := \sup_{N>0} \left| \int_{-N}^{N} F_f(t) e^{-i\lambda t}\, dt \right|, \qquad \lambda \in \mathbb{R},$$

where F_f denotes the classical Fourier transform,

$$F_f(\lambda) = \lim_{N\to\infty} \int_{-N}^{N} f(t) e^{i\lambda t}\, dt. \tag{5.1}$$

Its most important property is contained in the following Carleson-Hunt theorem, see (Grafakos, 2009: Theorems 6.2.1 & 6.3.3).

Theorem 5.1. *For any $p \in (1,\infty)$ the Carleson operator $\mathcal{E}$ is a bounded operator from $L^p(\mathbb{R})$ to itself, i.e., there exists a constant $C_p > 0$ such that*

$$\|\mathcal{E}_f\|_{L^p} \le C_p \|f\|_{L^p}, \qquad f \in L^p(\mathbb{R}).$$

For our considerations it is more convenient to consider the following version of $\mathcal{E}$

$$\mathscr{F}_g(\lambda) := \sup_{x\in[0,1]} \left| \int_0^x g(t) e^{i\lambda t}\, dt \right|, \qquad g \in L^p[0,1], \quad \lambda \in \mathbb{C}.$$

For brevity we put $\mathscr{F}_g^\theta(\lambda) := (\mathscr{F}_g(\lambda))^\theta$. Also recall that $p' = p/(p-1)$.

Combining the Carleson-Hunt theorem (Theorem 5.1) and the Hausdorff-Young theorem leads to the following result, see, e.g., (Savchuk, 2019) for details.

Proposition 5.2. *For any $p \in (1,2]$ the maximal Fourier transform $\mathscr{F}$ maps $L^p[0,1]$ boundedly into $L^{p'}[0,1]$, i.e., the following estimate holds*

$$\int_{-\infty}^{\infty} \mathscr{F}_g^{p'}(x)\, dx \le \gamma_p \, \|g\|_p^{p'}, \qquad g \in L^p[0,1], \tag{5.2}$$

where $\gamma_p > 0$ does not depend on $g \in L^p[0,1]$.

In the sequel we will need the following lemma whose proof substantially relies on the estimate (5.2). Recall the definition of $\Pi_{h,n}$ in (1.13).

Lemma 5.3. *Let $g \in L^p[0,1]$ for some $p \in (1,2]$ and $h \ge 0$, and let*

$$g_n := \sup\left\{\mathscr{F}_g(\lambda) : \lambda \in \Pi_{h,n}\right\}.$$

Then the following inequality holds

$$\sum_{n\in\mathbb{Z}} g_n^{p'} \le C_{p,h} \, \|g\|_p^{p'}, \qquad C_{p,h} := \gamma_p \, e^{p'(h+1)}.$$

The proof of Lemma 5.3 extends the classical reasoning about estimates of Hardy space functions and L^p_σ-classes of entire functions, see (Levin, 1996: Lectures 20-21) and (Katsnel'son, 1971: Lemma 2) for the case of the maximal Fourier transform.

Now we are ready to state the main result of this section which is a generalization of the Hausdorff-Young and Hardy-Littlewood theorems to the case of non-harmonic series with exponents forming an incompressible sequence $\Lambda = \{\mu_n\}_{n\in\mathbb{Z}}$ instead of $\Lambda = \{2\pi n\}_{n\in\mathbb{Z}}$.

Theorem 5.4. *Let $p \in (1,2]$. Let $\Lambda = \{\mu_n\}_{n\in\mathbb{Z}}$ be an incompressible sequence of density $d \in \mathbb{N}$ lying in the strip Π_h, and let $g \in L^p[0,1]$. Then there exists $C = C(p,h,d) > 0$ that does not depend on Λ and g, such that the following estimates hold uniformly with respect to g and Λ:*

$$\sum_{n\in\mathbb{Z}} |F_g(\mu_n)|^{p'} \le \sum_{n\in\mathbb{Z}} \mathscr{F}_g^{p'}(\mu_n) \le C\,\|g\|_p^{p'}, \tag{5.3}$$

$$\sum_{n\in\mathbb{Z}} (1+|n|)^{p-2}|F_g(\mu_n)|^p \le \sum_{n\in\mathbb{Z}} (1+|n|)^{p-2}\mathscr{F}_g^p(\mu_n) \le C\,\|g\|_p^p. \tag{5.4}$$

Estimate (5.3) is an immediate consequence of Lemma 5.3. The proof of (5.4) is based on the Marcinkiewicz theorem (Zigmund, 1959: Theorem XII.4.6). Note also that the parts of the inequalities (5.3)–(5.4) involving the classical Fourier transform F_g defined in (5.1) can be proved in a direct way, which is elementary in character, because it does not involve the Carleson-Hunt theorem.

Corollary 5.5. *Let $\Lambda = \{\mu_n\}_{n\in\mathbb{Z}}$ be a sequence of zeros of a sine-type function $\Phi(\cdot)$. Then for any $p \in (1,2]$ the estimates* (5.3) *and* (5.4) *hold uniformly in $g \in L^p[0,1]$.*

Proof. The proof is immediate from Theorem 5.4 if one notes that the null set of a sine-type function $\Phi(\cdot)$ is always incompressible, see (Levin, 1961), (Katsnel'son, 1971), and Proposition 4.7(ii). ☐

Inverse statements for the Hausdorff-Young and Hardy-Littlewood theorems also hold in the case of non-harmonic exponential series with exponents $\Lambda = \{\mu_n\}_{n\in\mathbb{Z}}$ forming the null set of a sine-type entire function instead of $\Lambda = \{2\pi n\}_{n\in\mathbb{Z}}$.

5.2 Uniform versions of the Riemann-Lebesgue lemma

Lemma 5.6, needed in the sequel, easily follows by combining Lemma 5.3 with Chebyshev's inequality. It can be understood as a uniform version of the classical Riemann-Lebesgue lemma.

Lemma 5.6. *Let $g \in \mathbb{U}_{p,r}$ for some $p \in (1,2]$ and $r > 0$. Moreover, let $b \in \mathbb{R}\setminus\{0\}$, let $h \ge 0$, and let p' be such that $1/p' + 1/p = 1$. Then for any $\delta > 0$ there exists a set $\mathcal{I}_{g,\delta} \subset \mathbb{Z}$ such that the following inequalities hold uniformly with respect to $g \in \mathbb{U}_{p,r}$*

$$\operatorname{card}(\mathbb{Z}\setminus\mathcal{I}_{g,\delta}) \le N_\delta := C\,(r/\delta)^{p'}, \tag{5.5}$$

$$\left|\int_0^1 g(t)e^{ib\lambda t}\,dt\right| \le \sup\nolimits_{x\in[0,1]} \left|\int_0^x g(t)e^{ib\lambda t}\,dt\right| < \delta, \qquad \lambda \in \bigcup_{n\in\mathcal{I}_{g,\delta}} \Pi_{h,n}, \tag{5.6}$$

where $\Pi_{h,n}$ is given by (1.13). *Here $C = C(p,h,b) > 0$ does not depend on g, r, and δ.*

Let us emphasize that "uniformity" in Lemma 5.6 does not relate to the set $\mathcal{I}_{g,\delta}$, but only to the "size" of its complement, see (5.5). Note also that the part of estimate (5.6) involving the regular Fourier transform $\int_0^1 g(t)e^{ib\lambda t}\,dt$ can be proved in an easier way without using the Carleson-Hunt theorem.

Next we investigate the "maximal" Fourier transform defined on the space $X_{\infty,1}(\Omega)$ by

$$\mathcal{F}[G](\lambda) := \sup_{x\in[0,1]} \left| \int_0^x G(x,t) e^{ib\lambda t}\, dt \right|, \qquad \lambda \in \mathbb{C}, \qquad G \in X_{\infty,1}(\Omega). \tag{5.7}$$

The results in the rest of this section do not use the deep Carleson-Hunt theorem. First we present the following "uniform" version of the Riemann-Lebesgue lemma for the space $X^0_{\infty,1}(\Omega)$. To this end for any $h \geq 0$ we set

$$C_0(\Pi_h) := \{\varphi \in C(\Pi_h) : \lim_{t\to\pm\infty} \varphi(t \pm iy) = 0 \text{ uniformly in } \ y \in [-h,h]\}.$$

Proposition 5.7. *Let $h \geq 0$ and let $\mathcal{F}$ be given by* (5.7). *Then the following statements hold:*

(i) *The nonlinear mapping $\mathcal{F} : X^0_{\infty,1}(\Omega) \to C(\Pi_h)$ is well-defined and is Lipschitz*

$$\|\mathcal{F}[G] - \mathcal{F}[\widetilde{G}]\|_{C(\Pi_h)} \leq e^{|b|h}\, \|G - \widetilde{G}\|_{X_{\infty,1}(\Omega)}, \qquad G, \widetilde{G} \in X^0_{\infty,1}(\Omega).$$

(ii) *For any $h \geq 0$ the mapping $\mathcal{F}$ maps $X^0_{\infty,1}(\Omega)$ continuously into $C_0(\Pi_h)$.*

(iii) *For any compact set $\mathcal{X}$ in $X^0_{\infty,1}(\Omega)$ the following relation holds*

$$\lim_{\lambda\to\infty} \mathcal{F}[G](\lambda) = 0, \tag{5.8}$$

uniformly in $G \in \mathcal{X}$ and $\lambda \in \Pi_h$.

Proposition 5.7 (iii) contains as a special case the following "uniform" version of the classical Riemann-Lebesgue lemma: for any compact set $\mathcal{K}$ in $L^1[0,1]$ one has

$$\sup_{g\in\mathcal{K}} \left| \int_0^1 g(t) e^{i\lambda t}\, dt \right| = o(1) \quad \text{as} \quad \lambda \to \infty, \quad \text{uniformly in } g \in \mathcal{K} \text{ and } \lambda \in \Pi_h.$$

Next we complete Proposition 5.7 by evaluating the "maximal" Fourier transform $\mathcal{F}[G](\cdot)$ in the plane instead of a strip.

Lemma 5.8. *Let $\mathcal{X}$ be a compact set in $X^0_{\infty,1}(\Omega)$, let $b \in \mathbb{R} \setminus \{0\}$, and let $\delta > 0$. Then there exists a constant $C = C(\mathcal{X}, b, \delta) > 0$, such that the following estimate holds*

$$\mathcal{F}[G](\lambda) \leq \delta(e^{-b\operatorname{Im}\lambda} + 1), \qquad |\lambda| > C, \quad G \in \mathcal{X},$$

uniformly in $G \in \mathcal{X}$.

Finally, we apply Proposition 5.7 (i), Theorem 3.4, and Lemma 5.8 to transformation operators.

Corollary 5.9. *Let $K^\pm_Q$ be the kernel of the transformation operator from representation* (3.1). *Then the composition*

$$Q \to K^\pm_Q \to \mathcal{F}[K^\pm_Q]$$

maps $L^p([0,1];\mathbb{C}^{2\times2})$ continuously into $C_0(\Pi_h;\mathbb{C}^{2\times2})$, $h \geq 0$, and it is a Lipschitz mapping on balls in $L^p([0,1];\mathbb{C}^{2\times2})$, $p \in [1,\infty)$

$$\|\mathcal{F}[K^\pm_Q] - \mathcal{F}[K^\pm_{\widetilde{Q}}]\|_{C(\Pi_h)} \leq e^{|b|h}\, C(B,p,r)\, \|Q - \widetilde{Q}\|_{L^p}, \qquad Q, \widetilde{Q} \in \mathbb{U}^{2\times2}_{p,r}. \tag{5.9}$$

The following statement will be useful in Section 6 when applying Rouché's theorem; it is an immediate consequence of Theorem 3.4 and Lemma 5.8.

Lemma 5.10. *Let $\mathcal{K}$ be a compact set in $L^1([0,1];\mathbb{C}^{2\times 2})$, let $Q \in \mathcal{K}$, and let $K_Q^{\pm} = (K_{jk}^{\pm})_{j,k=1}^2$ be the kernel of the transformation operator from representation* (3.1)*. Then for any $\delta > 0$ there exists a constant $M = M(\mathcal{K}, B, \delta) > 0$ such that the following estimate holds uniformly in $Q \in \mathcal{K}$*

$$\mathcal{F}[K_{jk}^{\pm}](\lambda) = \sup_{x\in[0,1]} \left| \int_0^x K_{jk}^{\pm}(x,t) e^{ib_k\lambda t}\, dt \right| \le \delta(e^{-b_k \operatorname{Im}\lambda} + 1), \qquad |\lambda| > M, \tag{5.10}$$

where $j, k \in \{1,2\}$. In particular, for any $h \ge 0$, one has

$$\sup_{Q\in\mathcal{K}} \mathcal{F}[K_{jk}^{\pm}](\lambda) \to 0 \qquad \textit{as} \quad |\lambda| \to \infty \quad \textit{and} \quad \lambda \in \Pi_h.$$

Let us demonstrate Corollary 5.9 and Lemma 5.10 for concrete examples of compacts.

Corollary 5.11. *Let $\mathcal{K}$ be a ball either in the Sobolev spaces $W_1^s[0,1]$ with $s \in \mathbb{R}_+$, in the Lipschitz space $\Lambda_\alpha[0,1]$ with $\alpha \in (0,1]$, or in the space $V[0,1]$ of functions of bounded variation. Then the relations* (5.9) *and* (5.10) *hold true uniformly in $Q \in \mathcal{K}$.*

Remark 5.12. Let us present a simple example of a non-compact set in $L^p[0,1]$ for which the uniform relation (5.8) is violated. Define the following set of functions

$$\mathcal{G} := \{g_\mu(x) := g_0(x)e^{-i\mu x} : \ \mu \in \mathbb{R}\},$$

where $g_0 \in L^p[0,1]$ is such that $c_0 := \int_0^1 g_0(t)\,dt > 0$. It is clear that

$$\mathcal{F}[g_\mu](\mu) \ge \left| \int_0^1 g_0(t) e^{-i\mu t} e^{i\mu t}\, dt \right| = c_0 \ne 0 \qquad \text{and} \qquad \lim_{|\lambda|\to\infty} \mathcal{F}[g_\mu](\lambda) = 0. \tag{5.11}$$

The last relation in (5.11) is satisfied not uniformly on $\mathcal{G}$. Moreover, inequality (5.6) holds on sets $\mathcal{I}_{g_\mu,\delta} = \mathbb{Z} \setminus (\mu - N_\delta, \mu + N_\delta)$ that depend on g_μ, and their complements "tend to infinity" when $\mu \to \infty$, but have uniformly bounded "sizes": $\operatorname{card}(\mathbb{Z} \setminus \mathcal{I}_{g_\mu,\delta}) \le 2N_\delta$. We are indebted to V.P. Zastavnyi who informed us about this example.

6 Stability property of eigenvalues

6.1 Uniform localization of spectrum

In this subsection we will obtain a version of the asymptotic formula (4.14) which is uniform with respect to $Q \in \mathcal{K}$, where $\mathcal{K}$ is either a compact set in $L^1([0,1];\mathbb{C}^{2\times 2})$ or $\mathcal{K} = \mathbb{U}_{p,r}^{2\times 2}$ for $p \in (1,2]$. Recall that A is the matrix defined in (4.1) and composed from the coefficients of the linear forms U_1 and U_2 as in (1.2) and that $B = \operatorname{diag}(b_1, b_2)$.

First, we enhance Proposition 4.8 to obtain uniform estimates for $Q \in \mathcal{K}$, where $\mathcal{K}$ is compact in $L^1([0,1];\mathbb{C}^{2\times 2})$. The following result generalizes (Sadovnichaya, 2016: Theorem 3) to the case of Dirac-type systems with regular boundary conditions. Its proof is substantially based on the representation (4.11), Lemma 5.10, and Rouché's theorem.

Proposition 6.1. *Let $\mathcal{K}$ be compact in $L^1([0,1];\mathbb{C}^{2\times 2})$ and let $Q \in \mathcal{K}$. Let the boundary conditions* (1.2) *be regular, let $\Delta(\cdot) := \Delta_Q(\cdot)$ be the corresponding characteristic determinant, and let $\Lambda := \Lambda_Q := \{\lambda_{Q,n}\}_{n\in\mathbb{Z}}$ be the canonically ordered sequence of its zeros. Moreover, let $\Lambda_0 = \{\lambda_n^0\}_{n\in\mathbb{Z}}$ be the sequence of zeros of Δ_0. Then the following estimates hold:*

(i) *There exists $M = M(\mathcal{K}, B, A) > 0$, that does not depend on Q, such that*

$$\sup_{n\in\mathbb{Z}} \left|\lambda_{Q,n} - \lambda_n^0\right| \le M, \qquad Q \in \mathcal{K}.$$

In particular, there exist $h = h(\mathcal{K}, B, A) > 0$ and $d = d(\mathcal{K}, B, A)$, that do not depend on Q, such that Λ_Q is an incompressible sequence of density d and lying in the strip Π_h.

(ii) *For any $\varepsilon > 0$ there exists a constant $N_\varepsilon = N_\varepsilon(\mathcal{K}, B, A) \in \mathbb{N}$, such that*

$$\sup_{|n|>N_\varepsilon} \left|\lambda_{Q,n} - \lambda_n^0\right| \le \varepsilon, \qquad Q \in \mathcal{K}.$$

If, in addition, the boundary conditions (1.2) *are strictly regular, then there exists a constant $\varepsilon_0 = \varepsilon_0(B, A)$, such that for any $\varepsilon \in (0, \varepsilon_0]$ the discs $\mathbb{D}_{2\varepsilon}(\lambda_{Q,n})$, $|n| > N_\varepsilon$, are disjoint and there exists a constant $\widetilde{C}_\varepsilon = \widetilde{C}_\varepsilon(B, A) > 0$, such that*

$$\min_{|\lambda-\lambda_{Q,n}|=2\varepsilon} |\Delta_Q(\lambda)| \ge \widetilde{C}_\varepsilon, \qquad |n| > N_\varepsilon, \quad Q \in \mathcal{K}.$$

Next we extend Proposition 6.1 to the case $\mathcal{K} = \mathbb{U}_{p,r}^{2\times 2}$, $p \in (1,2]$. Part (i) remains valid but the assumption $p > 1$ is important. Part (ii) is based on Lemma 5.6 that involves only the classical Fourier transform F_g without the use of the deep Carleson-Hunt theorem. It remains valid if we replace the inequality $|n| > N_\varepsilon$ by an inclusion $n \in \mathcal{I}_{Q,\varepsilon}$, assuming that the complements of the sets $\mathcal{I}_{Q,\varepsilon}$ have uniformly bounded cardinalities for $Q \in \mathbb{U}_{p,r}^{2\times 2}$.

Proposition 6.2. *Let $Q \in \mathbb{U}_{p,r}^{2\times 2}$ for some $p \in (1,2]$ and some $r > 0$. Moreover, let the boundary conditions* (1.2) *be regular, let $\Delta(\cdot) := \Delta_Q(\cdot)$ be the corresponding characteristic determinant, and let $\Lambda := \Lambda_Q = \{\lambda_{Q,n}\}_{n\in\mathbb{Z}}$ be a canonically ordered sequence of its zeros. Then the following statements hold:*

(i) *There exists a constant $M = M(p, r, B, A) > 0$, not dependent on Q, such that*

$$\sup_{n\in\mathbb{Z}} \left|\lambda_{Q,n} - \lambda_n^0\right| \le M, \qquad Q \in \mathbb{U}_{p,r}^{2\times 2}.$$

In particular, there exist $h = h(p, r, B, A) \ge 0$ and $d = d(p, r, B, A) > 0$, not dependent on Q, such that Λ_Q is an incompressible sequence of density d lying in the strip Π_h.

(ii) *For any $\varepsilon > 0$ there exists $N_\varepsilon = N_\varepsilon(p, r, B, A) \in \mathbb{N}$, that does not depend on Q, and a set $\mathcal{I}_{Q,\varepsilon} \subset \mathbb{Z}$, such that*

$$|\lambda_n - \lambda_n^0| < \varepsilon, \quad n \in \mathcal{I}_{Q,\varepsilon}, \quad \textit{and} \quad \operatorname{card}(\mathbb{Z} \setminus \mathcal{I}_{Q,\varepsilon}) \le N_\varepsilon.$$

Proposition 6.1 combined with the maximum and minimum principles imply the following uniform version of the relation $|\lambda_n - \widetilde{\lambda}_n| \asymp |\widetilde{\Delta}(\lambda_n)|$ that is pivotal for establishing the stability property of the mapping $Q \to \Lambda_Q := \{\lambda_{Q,n}\}_{n\in\mathbb{Z}}$.

Proposition 6.3. *Let $\mathcal{K}$ be a compact set in $L^1([0,1];\mathbb{C}^{2\times 2})$ and let $Q, \widetilde{Q} \in \mathcal{K}$. Moreover, let the boundary conditions* (1.2) *be strictly regular, and let $\Lambda_Q = \{\lambda_{Q,n}\}_{n\in\mathbb{Z}}$ and $\Lambda_{\widetilde{Q}} = \{\lambda_{\widetilde{Q},n}\}_{n\in\mathbb{Z}}$ be canonically ordered sequences of zeros of characteristic determinants $\Delta := \Delta_Q$ and $\widetilde{\Delta} := \Delta_{\widetilde{Q}}$, respectively. Then there exist constants $N = N(\mathcal{K}, A, B) \in \mathbb{N}$ and $C = C(\mathcal{K}, A, B) \geq 1$, that do not depend on Q and $\widetilde{Q}$, such that the following uniform estimate holds*

$$C^{-1}\,|\Delta_{\widetilde{Q}}(\lambda_{Q,n})| \leq |\lambda_{Q,n} - \lambda_{\widetilde{Q},n}| \leq C\,|\Delta_{\widetilde{Q}}(\lambda_{Q,n})|, \qquad |n| > N.$$

6.2 Stability property of eigenvalues for $Q \in L^p$

In this section we apply the abstract results from Section 5 to establish the stability of the mapping $Q \to \Lambda_Q := \{\lambda_{Q,n}\}_{n\in\mathbb{Z}}$ in different norms. Proposition 6.3 shows that to this end, it suffices to evaluate the sequences $\{\widetilde{\Delta}(\lambda_n)\}_{n\in\mathbb{Z}} = \{\Delta_{\widetilde{Q}}(\lambda_{Q,n})\}_{n\in\mathbb{Z}}$ when Q runs through either the ball $\mathbb{U}^{2\times 2}_{p,r}$ or a compact $\mathcal{K}$ in $L^1([0,1];\mathbb{C}^{2\times 2})$. In turn, these sequences can be easily evaluated by combining the representation (4.13) and the results of Section 5. For example, the estimate (5.3) implies that

$$\sum_{n\in\mathbb{Z}} \left|\Delta_{\widetilde{Q}}(\lambda_{Q,n})\right|^{p'} \leq C_{p,r,B}\,\|Q - \widetilde{Q}\|_p^{p'}, \qquad Q, \widetilde{Q} \in \mathbb{U}^{2\times 2}_{p,r}.$$

Next we enhance and complete Proposition 4.8 in the case of $Q \in L^p([0,1];\mathbb{C}^{2\times 2})$ with $p \in [1,2]$. Our first result restricts the set $\mathcal{K}$ of potential matrices to be a compact.

Theorem 6.4. *Let $\mathcal{K}$ be compact in $L^p([0,1];\mathbb{C}^{2\times 2})$ for some $p \in [1,2]$, and let $Q, \widetilde{Q} \in \mathcal{K}$. Moreover, let the boundary conditions* (1.2) *be strictly regular, and let $\Lambda_Q := \{\lambda_{Q,n}\}_{n\in\mathbb{Z}}$ and $\Lambda_{\widetilde{Q}} := \{\lambda_{\widetilde{Q},n}\}_{n\in\mathbb{Z}}$ be canonically ordered sequences of zeros of the characteristic determinants $\Delta(\cdot) := \Delta_Q(\cdot)$ and $\widetilde{\Delta}(\cdot) := \Delta_{\widetilde{Q}}(\cdot)$, respectively. Then there exist constants $N = N(\mathcal{K}, A, B) \in \mathbb{N}$ and $C = C(p, \mathcal{K}, A, B) > 0$, not dependent on Q and $\widetilde{Q}$, such that the following estimates hold:*

$$\sum_{|n|>N} |\lambda_{Q,n} - \lambda_{\widetilde{Q},n}|^{p'} \leq C\,\|Q - \widetilde{Q}\|_p^{p'}, \qquad p \in (1,2], \tag{6.1}$$

$$\sum_{|n|>N} (1+|n|)^{p-2}\,|\lambda_{Q,n} - \lambda_{\widetilde{Q},n}|^{p} \leq C\,\|Q - \widetilde{Q}\|_p^{p}, \qquad p \in (1,2]. \tag{6.2}$$

If $p = 1$, then

$$\sup\nolimits_{Q,\widetilde{Q}\in\mathcal{K}} |\lambda_{Q,n} - \lambda_{\widetilde{Q},n}| \to 0 \quad \text{as} \quad n \to \infty.$$

In other words, the set of sequences $\big\{\{|\lambda_{Q,n} - \lambda_{\widetilde{Q},n}|\}_{n\in\mathbb{Z}}\big\}_{Q,\widetilde{Q}\in\mathcal{K}}$ forms a compact set in $c_0(\mathbb{Z})$.

Applying Theorem 6.4 with a compact set $\mathcal{K} = \{Q, 0\}$, we can complete Proposition 4.8.

Corollary 6.5. *Let $Q \in L^p([0,1];\mathbb{C}^{2\times 2})$ for some $p \in (1,2]$. Moreover, let the boundary conditions* (1.2) *be strictly regular and let $\Delta(\cdot)$ be the corresponding characteristic determinant. Then the sequence $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ of its zeros can be ordered such that the following inequality holds*

$$\sum_{n\in\mathbb{Z}} \left|\lambda_n - \lambda_n^0\right|^{p'} + \sum_{n\in\mathbb{Z}} (1+|n|)^{p-2}\left|\lambda_n - \lambda_n^0\right|^p < \infty.$$

Note that the inclusion $\{\lambda_n - \lambda_n^0\}_{n\in\mathbb{Z}} \in \ell^{p'}(\mathbb{Z})$ in the case of 2×2 Dirac systems ($-b_1 = b_2 = 1$) was first obtained in Savchuk & Shkalikov (2014: Theorems 4.3 & 4.5).

Remark 6.6. Let $p = 3/2$ and $p' = 3$. Moreover, assume that $\widetilde{Q} = 0$ and let $Q \in L^p$ be fixed. Note that

$$\alpha_n := \lambda_{Q,n} - \lambda_{\widetilde{Q},n} = \left((1+|n|)\ln^2(1+|n|)\right)^{-1/3}, \qquad n \in \mathbb{Z},$$

satisfies (6.1), but does not satisfy (6.2). Hence, this sequence cannot be a sequence of eigenvalue deviations of operators $L(Q)$ and $L(0)$ for some $Q \in L^p$. Note that the results of Savchuk & Shkalikov (2014) do allow it. In fact, inequality (6.2) is generally more restrictive and, hence, leads to a sharper estimate of the sequence $\{\lambda_{Q,n} - \lambda_{\widetilde{Q},n}\}_{n\in\mathbb{Z}}$ than (6.1). For example, inequality (6.2) implies (6.1) under the general assumption $\alpha_n = o(n^{-1/p'})$ as $n \to \infty$.

Next we extend Theorem 6.4 to the case $\mathcal{K} = \mathbb{U}^{2\times 2}_{p,r}$. As in Proposition 6.2, we cannot select a universal constant N serving all potentials. Instead, we need to sum over the sets of integers whose complements have uniformly bounded cardinality.

Theorem 6.7. *Let $Q, \widetilde{Q} \in \mathbb{U}^{2\times 2}_{p,r}$ for some $p \in (1,2]$ and $r > 0$. Let the boundary conditions* (1.2) *be strictly regular, and let $\Lambda_Q = \{\lambda_{Q,n}\}_{n\in\mathbb{Z}}$ and $\Lambda_{\widetilde{Q}} = \{\lambda_{\widetilde{Q},n}\}_{n\in\mathbb{Z}}$ be canonically ordered sequences of zeros of characteristic determinants $\Delta := \Delta_Q$ and $\widetilde{\Delta} := \Delta_{\widetilde{Q}}$, respectively. Then there exist constants $N \in \mathbb{N}$, $C_1, C_2, C > 0$, not dependent on Q and $\widetilde{Q}$, and a set $\mathcal{I} := \mathcal{I}_{Q,\widetilde{Q}} \subset \mathbb{Z}$, such that the following estimates hold:*

$$\operatorname{card}\left(\mathbb{Z} \setminus \mathcal{I}_{Q,\widetilde{Q}}\right) \le N, \tag{6.3}$$

$$C_1 \left|\Delta_{\widetilde{Q}}(\lambda_{Q,n})\right| \le |\lambda_{Q,n} - \lambda_{\widetilde{Q},n}| \le C_2 \left|\Delta_{\widetilde{Q}}(\lambda_{Q,n})\right|, \qquad n \in \mathcal{I}_{Q,\widetilde{Q}}, \tag{6.4}$$

$$\sum_{n\in\mathcal{I}_{Q,\widetilde{Q}}} \left|\lambda_{Q,n} - \lambda_{\widetilde{Q},n}\right|^{p'} \le C\,\|Q - \widetilde{Q}\|_p^{p'}, \tag{6.5}$$

$$\sum_{n\in\mathcal{I}_{Q,\widetilde{Q}}} (1+|n|)^{p-2} \left|\lambda_{Q,n} - \lambda_{\widetilde{Q},n}\right|^{p} \le C\,\|Q - \widetilde{Q}\|_p^{p}. \tag{6.6}$$

Remark 6.8. (i) Observe that the proofs of all results in this section, including the proofs of Theorems 6.4 and 6.7, rely on the Bessel type inequalities (5.3)–(5.4) for the ordinary Fourier transform, not for its maximal version described in Theorem 5.4, whose proof relies on Theorem 5.1.

(ii) The case of Dirac systems ($b_1 = -b_2 = 1$) and $\widetilde{Q} = 0$ has extensively been studied in many recent papers of Sadovnichaya, Savchuk, and Shkalikov by applying a different method. In particular, the estimate (6.1) was established earlier in Savchuk & Shkalikov (2014: Theorems 4.3 & 4.5) with a constant C that depends on Q, while estimate (6.5) of Theorem 6.4 with $\widetilde{Q} = 0$ was established in Savchuk & Sadovnichaya (2018).

(iii) The weighted estimates (6.2) and (6.6), as well as the estimate (6.1), which establish stability properties of the spectrum under the perturbation $Q \to \widetilde{Q}$, are new even for Dirac system.

(iv) L. Rzepnicki (2020) obtained sharp asymptotic formulas for deviations $\lambda_n - \lambda_n^0 = \delta_n + \rho_n$ in the case of Dirichlet BVPs for the Dirac system with $Q \in L^p([0,1];\mathbb{C}^{2\times 2})$, $1 \le p < 2$. Namely, δ_n is explicitly expressed via Fourier coefficients and Fourier transforms of Q_{12} and Q_{21}, while $\{\rho_n\}_{n\in\mathbb{Z}} \in \ell^{p'/2}(\mathbb{Z})$, i.e., the convergence to zero is "twice" better than what formula (6.1) guarantees for $\lambda_n - \lambda_n^0$. A similar result was obtained for eigenfunctions.

(v) We mention also the papers (Cascaval et al., 2004), (Clark & Gesztesy, 2006), and (Brown et al., 2019), where different spectral properties of j-selfadjoint Dirac operators were investigated.

7 Stability property of eigenfunctions

7.1 Estimates of Fourier transforms of transformation operators

In this subsection we study "Fourier" transforms of the kernels of the corresponding transformation operators from the representation (3.1) of the form

$$\int_0^x K_{jk}^{\pm}(x,t)e^{i\lambda b_k t}\,dt.$$

Our investigation is motivated by the representation (4.6) for the entries of the fundamental matrix $\Phi(\cdot,\lambda)$. As distinguished from the considerations of Section 6, here our proofs substantially involve the deep Carleson-Hunt theorem via the corresponding results of Section 5.

As a first step we study "Fourier" transforms of the auxiliary kernels R_Q from the representation (3.7) for the kernels of the transformation operators $K_Q^{\pm}$. Below, we first estimate generalized "Fourier" transforms with an arbitrary bounded function, instead of the exponential function in the integral. Recall that $\alpha_k := \frac{b_j}{b_j - b_k}$, $j = 2/k$, $k \in \{1,2\}$.

Proposition 7.1. *Let $Q \in L^1([0,1];\mathbb{C}^{2\times 2})$ and let*

$$R_Q = (R_{jk})_{j,k=1}^2 \in \big(X_{1,1}^0(\Omega) \cap X_{\infty,1}^0(\Omega)\big) \otimes \mathbb{C}^{2\times 2}$$

be the (unique) solution of the system of integral equations (3.8)–(3.9)*. Moreover, let $x \in [0,1]$ be fixed, let $f \in L^\infty(\mathbb{R})$ be such that $f(t) = 0$ for $t \notin [0,x]$, and set*

$$F_{jk}(s;f) := \sup_{\substack{u\in[0,s]\\ v\in[-u,x]}} \left| \int_0^u R_{jk}(s,t) f(t+v)\,dt \right|, \qquad s \in [0,x], \quad j,k \in \{1,2\}.$$

Then the following estimates hold for $s \in [0,x]$, $k \in \{1,2\}$, and $j = 2/k$:

$$\begin{aligned}
F_{kk}(s;f) &\le |b_k| \int_0^s |Q_{kj}(t)|\, F_{jk}(t;f)\,dt, \\
F_{jk}(s;f) &\le |b_j| \sup_{\substack{u\in[0,s]\\ v\in[-u,x]}} \left| \alpha_j \int_0^u Q_{jk}(\alpha_k s + \alpha_j t)\, f(t+v)\,dt \right| \\
&\qquad + 2|b_j b_k|\, \|Q_{jk}\|_{L^1[0,s]} \int_0^s |Q_{kj}(t)|\, F_{jk}(t;f)\,dt.
\end{aligned}$$

In particular, one has that the following uniform estimate holds for $Q \in \mathbb{U}_{1,r}^{2\times 2}$, $x \in [0,1]$, $\lambda \in \mathbb{C}$, and $j,k \in \{1,2\}$

$$\sup\nolimits_{s\in[0,x]} \left| \int_0^s R_{jk}(x,t) e^{i\lambda b_k t}\,dt \right| \le C e^{(b_2-b_1)|\operatorname{Im}\lambda| x} \sup\nolimits_{s\in[0,x]} \left| \int_0^s Q_{jk}(t) e^{i(b_k-b_j)\lambda t}\,dt \right|, \quad (7.1)$$

where $j = 2/k$ and $C = C(B,r) > 0$ does not depend on Q, x, and λ.

Note that (7.1) follows by taking $f(t) = e^{ib_k\lambda t}$, $t \in [0,x]$, and $f(t) = 0$, $t \notin [0,x]$, in the formulas preceding (7.1). The estimate (7.1) allows us to obtain a similar estimate for the Fourier transforms of the auxiliary functions $P_k^{\pm}$ from the representation (3.6)–(3.7). Combining them, we arrive at the following important estimate of the Fourier transforms of the kernels $K_Q^{\pm}$.

Theorem 7.2. *Let $Q \in \mathbb{U}_{1,r}^{2\times 2}$ for some $r > 0$ and let $K_Q^{\pm}$ be the kernels of the corresponding transformation operators from representation* (3.1). *Then the following uniform estimate holds for $x \in [0,1]$ and $\lambda \in \mathbb{C}$*

$$\sum_{j,k=1}^{2} \left| \int_0^x K_{jk}^{\pm}(x,t) e^{ib_k \lambda t}\, dt \right| \le C\, e^{2(b_2-b_1)|\mathrm{Im}\,\lambda| x} \sum_{j\neq k} \sup_{s\in[0,x]} \left| \int_0^s Q_{jk}(t) e^{i(b_k-b_j)\lambda t}\, dt \right|.$$

This estimate is uniform in the sense that $C = C(B,r) > 0$ does not depend on Q, x, and λ.

7.2 Stability property of the fundamental matrix

Alongside equation (1.1) we consider similar Dirac-type equations with the same matrix B but with a different potential matrix $\widetilde{Q} \in L^1([0,1];\mathbb{C}^{2\times 2})$. Recall that $\Phi_Q(x,\lambda)$ and $\Phi_{\widetilde{Q}}(x,\lambda)$ denote the fundamental matrix solutions of the system (1.1) for Q and $\widetilde{Q}$, that satisfy the initial conditions $\Phi_Q(0,\lambda) = \Phi_{\widetilde{Q}}(0,\lambda) = I$.

We can extend Theorem 7.2 to obtain the stability of "Fourier" transforms of the kernel differences of the corresponding transformation operators from the representation (3.1). Combining it with the representations (4.8)–(4.9) in Lemma 4.2 for entries of the deviation $\Phi_Q(\cdot,\lambda) - \Phi_{\widetilde{Q}}(\cdot,\lambda)$ of the fundamental matrices, we obtain the following uniform estimate that plays an important role in studying deviations of root vectors and which is of independent interest.

Theorem 7.3. *Let $Q, \widetilde{Q} \in \mathbb{U}_{1,r}^{2\times 2}$ for some $r > 0$, and let $K_Q^{\pm}$ and $K_{\widetilde{Q}}^{\pm}$ be the kernels of the corresponding transformation operators from representation* (3.1) *for Q and $\widetilde{Q}$, respectively. Then with some $C = C(B,r) > 0$ the following uniform estimate holds for $x \in [0,1]$ and $\lambda \in \mathbb{C}$*

$$\begin{aligned}\left|\Phi_Q(x,\lambda) - \Phi_{\widetilde{Q}}(x,\lambda)\right| &\le 2 \sum_{j,k=1}^{2} \sum_{\pm} \left| \int_0^x \left(K_Q^{\pm} - K_{\widetilde{Q}}^{\pm}\right)_{jk}(x,t) e^{ib_k\lambda t}\, dt \right| \\ &\le C\, e^{2(b_2-b_1)|\mathrm{Im}\,\lambda| x} \sum_{j\neq k} \left(\sup_{s\in[0,x]} \left| \int_0^s (Q_{jk}(t) - \widetilde{Q}_{jk}(t)) e^{i(b_k-b_j)\lambda t}\, dt \right| \right. \\ &\qquad \left. + \|Q - \widetilde{Q}\|_1 \sup_{s\in[0,x]} \left| \int_0^s \widetilde{Q}_{jk}(t) e^{i(b_k-b_j)\lambda t}\, dt \right| \right).\end{aligned}$$

Combining Theorems 7.3 and 5.4, we arrive at an important stability (Lipschitz) property of the fundamental matrix.

Proposition 7.4. *Let $Q, \widetilde{Q} \in \mathbb{U}_{p,r}^{2\times 2}$ for some $p \in (1,2]$ and $r > 0$, and let $\Lambda = \{\mu_n\}_{n\in\mathbb{Z}}$ be an incompressible sequence of density d lying in the strip Π_h. Then for some $C = C(p,r,B,h,d) > 0$, not dependent on Q, $\widetilde{Q}$, and Λ, the following uniform estimates hold:*

$$\sum_{n\in\mathbb{Z}} \left\| \Phi_Q(\cdot,\mu_n) - \Phi_{\widetilde{Q}}(\cdot,\mu_n) \right\|_{\infty}^{p'} \le C\, \|Q - \widetilde{Q}\|_p^{p'},$$

$$\sum_{n\in\mathbb{Z}} (1+|n|)^{p-2} \left\| \Phi_Q(\cdot,\mu_n) - \Phi_{\widetilde{Q}}(\cdot,\mu_n) \right\|_{\infty}^{p} \le C\, \|Q - \widetilde{Q}\|_p^{p}.$$

7.3 Stability property of the eigenfunctions

Now the main results of this section are formulated. The following result for $p = 1$ generalizes (Sadovnichaya, 2016: Theorem 4) to the case of Dirac-type systems and extends it for $p \in (1, 2]$. It can be proved by combining results of the previous subsection with Theorem 6.4.

Theorem 7.5. *Let $\mathcal{K}$ be compact in $L^p([0,1];\mathbb{C}^{2\times 2})$ for some $p \in [1,2]$ and let $Q, \widetilde{Q} \in \mathcal{K}$. Let the BCs* (1.2) *be strictly regular and let $s \in (0, \infty]$. Then there exist SRVs $\{f_{Q,n}\}_{n\in\mathbb{Z}}$ and $\{f_{\widetilde{Q},n}\}_{n\in\mathbb{Z}}$ of the operators $L(Q)$ and $L(\widetilde{Q})$, respectively, such that $\|f_{Q,n}\|_s = \|f_{\widetilde{Q},n}\|_s = 1$, $|n| > N$, and that the following relations hold uniformly for $Q, \widetilde{Q} \in \mathcal{K}$:*

$$\sup_{Q,\widetilde{Q}\in\mathcal{K}} \left\|f_{Q,n} - f_{\widetilde{Q},n}\right\|_\infty \to 0 \quad as \quad |n| \to \infty, \tag{7.2}$$

$$\sum_{|n|>N} \left\|f_{Q,n} - f_{\widetilde{Q},n}\right\|_\infty^{p'} \le C \|Q - \widetilde{Q}\|_p^{p'}, \qquad p \in (1,2], \tag{7.3}$$

$$\sum_{|n|>N} (1+|n|)^{p-2} \left\|f_{Q,n} - f_{\widetilde{Q},n}\right\|_\infty^{p} \le C \|Q - \widetilde{Q}\|_p^{p}, \qquad p \in (1,2]. \tag{7.4}$$

Here the constants $N \in \mathbb{N}$ and $C > 0$ do not depend on Q, $\widetilde{Q}$, and s.

Next we extend Theorem 7.5 to the case $\mathcal{K} = \mathbb{U}_{p,r}^{2\times 2}$. As in Theorem 6.7, we cannot select a universal constant N serving all potentials. Instead, we need to sum over the sets of integers, whose complements have uniformly bounded cardinality.

Theorem 7.6. *Let $Q, \widetilde{Q} \in \mathbb{U}_{p,r}^{2\times 2}$ for some $p \in (1,2]$ and some $r > 0$. Let the BCs* (1.2) *be strictly regular and let $s \in (0,\infty]$. Then there exist SRVs $\{f_{Q,n}\}_{n\in\mathbb{Z}}$ and $\{f_{\widetilde{Q},n}\}_{n\in\mathbb{Z}}$ of the operators $L(Q)$ and $L(\widetilde{Q})$, respectively, and a set $\mathcal{I}_{Q,\widetilde{Q}} \subset \mathbb{Z}$, such that the following uniform relations hold for $Q, \widetilde{Q} \in \mathbb{U}_{p,r}^{2\times 2}$:*

$$\|f_{Q,n}\|_s = \|f_{\widetilde{Q},n}\|_s = 1, \quad n \in \mathcal{I}_{Q,\widetilde{Q}}, \qquad and \qquad \operatorname{card}\left(\mathbb{Z} \setminus \mathcal{I}_{Q,\widetilde{Q}}\right) \le N,$$

$$\sum_{n\in\mathcal{I}_{Q,\widetilde{Q}}} \left\|f_{Q,n} - f_{\widetilde{Q},n}\right\|_\infty^{p'} \le C \|Q - \widetilde{Q}\|_p^{p'},$$

$$\sum_{n\in\mathcal{I}_{Q,\widetilde{Q}}} (1+|n|)^{p-2} \left\|f_{Q,n} - f_{\widetilde{Q},n}\right\|_\infty^{p} \le C \|Q - \widetilde{Q}\|_p^{p}.$$

Here the constants $N \in \mathbb{N}$ and $C > 0$ do not depend on Q, $\widetilde{Q}$, and s.

Applying Theorem 7.5 with a two-point compact $\mathcal{K} = \{Q, 0\}$, we arrive at the following stability property of eigenfunctions demonstrating the core of both Theorems 7.5 and 7.6.

Corollary 7.7. *Let $Q \in L^p([0,1];\mathbb{C}^{2\times 2})$, $p \in (1,2]$, and let the BCs* (1.2) *be strictly regular. Then SRVs $\{f_n\}_{n\in\mathbb{Z}}$ and $\{f_n^0\}_{n\in\mathbb{Z}}$ of the operators $L(Q)$ and $L(0)$ can be chosen asymptotically normalized in $L^{p'}([0,1];\mathbb{C}^2)$ and satisfying the following uniform estimates*

$$\sum_{n\in\mathbb{Z}} \|f_n - f_n^0\|_\infty^{p'} + \sum_{n\in\mathbb{Z}} (1+|n|)^{p-2} \|f_n - f_n^0\|_\infty^{p} < \infty.$$

The following case shows that in some cases we can relax the compactness condition and even boundedness of $\mathcal{K}$ and sum over all $n \in \mathbb{Z}$ in (7.3)–(7.4).

Proposition 7.8. *Let* $Q_{12} = \widetilde{Q}_{12} = 0$ *and* $Q_{21}, \widetilde{Q}_{21} \in L^p[0,1]$ *for some* $p \in (1,2]$. *Let the BCs* (4.18) *be strictly regular with* $b = 0$. *Then the eigenvalues of the operators* $L(Q)$ *and* $L(\widetilde{Q})$ *are simple and separated, and there exist systems* $\{f_n\}_{n\in\mathbb{Z}}$ *and* $\{\widetilde{f}_n\}_{n\in\mathbb{Z}}$ *of their eigenfunctions, both normalized in* $C([0,1];\mathbb{C}^2)$*, such that the following uniform estimates hold:*

$$\sum_{n\in\mathbb{Z}} \left\| f_n - \widetilde{f}_n \right\|_\infty^{p'} \le C \, \|Q - \widetilde{Q}\|_p^p, \tag{7.5}$$

$$\sum_{n\in\mathbb{Z}} (1 + |n|)^{p-2} \left\| f_n - \widetilde{f}_n \right\|_\infty^{p} \le C \, \|Q - \widetilde{Q}\|_p^{p'}. \tag{7.6}$$

These estimates are uniform in the sense that $C = C(p, B, A) > 0$ *does not depend on* Q *and* $\widetilde{Q}$.

Remark 7.9. If $Q_{12} = 0$, $b = 0$, and $a = 1$ in the BCs (4.18), then the sequence of eigenvalues of the operator $L(Q)$ is the union of two arithmetic progressions with one of them being the sequence $\mu_n = 2(1 - b_2/b_1)\pi n$, $n \in \mathbb{Z}$. The corresponding eigenfunctions can be expressed explicitly via the Fourier coefficients

$$\int_0^x Q_{21}(t) e^{i\mu_n t} \, dt \quad \text{and} \quad \int_0^1 Q_{21}(t) e^{i\mu_n t} \, dt.$$

Hence Proposition 7.8 shows that the stability properties (7.5)–(7.6) of the eigenfunctions of the operator $L(Q)$ are equivalent to the abstract inequalities (5.3)–(5.4) from Theorem 5.4 with the sequence $\{\mu_n\}_{n\in\mathbb{Z}}$ being an arithmetic progression.

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COMPLETENESS AND MINIMALITY OF EIGENFUNCTIONS AND ASSOCIATED FUNCTIONS OF ORDINARY DIFFERENTIAL OPERATORS

Manfred Möller

Dedicated to Seppo Hassi on the occasion of his 60th birthday

1 Introduction

One of the oldest and most important results in operator theory is that a selfadjoint operator with compact resolvent in a Hilbert gives rise to an orthogonal basis of eigenvectors. This property is extensively used in the Sturm-Liouville theory. If the operator is not selfadjoint, then there may be biorthogonal bases of the operator and its adjoint, but such bases are not guaranteed. There is some substantial literature especially regarding Sturm-Liouville problems, including operators with eigenvalue parameter dependent boundary conditions.

Often, properties of the spectra of quite special cases are considered in detail, see, e.g., (Aliyev & Guliyeva, 2018; Aliyev & Namazov, 2017; Binding & Browne, 1995; Guliyev, 2019). Such properties of spectra are then used to prove minimality or basisness of a subsystem of eigenfunctions and associated functions, see, e.g., (Aliyev, 2007; Aliev & Dun'yamalieva, 2015; Kerimov & Mirzoev, 2003; Namazov, 2017). Expansion theorems predate these results, see Schneider (1974), Walter (1973), or the more recent result on completeness in Allahverdiev (2005). Shkalikov (1983) devised a linearization method for differential operator polynomials in the eigenvalue parameter and he proved completeness and minimality of the eigenvectors and associated vectors of the linearized system. However, the degree of this operator polynomial equals the order of the differential equation and his method is therefore not applicable to the differential operators studied in the current note.

For a Banach space H, Shkalikov (2019) starts with bases or complete and minimal systems in $H \oplus \mathbb{C}^N$ to give conditions when co-finite subsystems of projections onto H are bases or complete and minimal systems in H. In Möller (2020) and in this note B-biorthogonality for a bounded linear operator from a Banach space E to a Banach space F will be used, where E is densely and continuously embedded in the space H, see Section 2 for more details. This has the advantage that the general results of Mennicken & Möller (2003) for arbitrary Birkhoff regular n-th order differential operators can be applied. In particular, Birkhoff regularity guarantees the existence of co-finite systems which are minimal and complete in H, and more or less explicit criteria for the choice of such systems can be given.

For the sake of completeness, the preparation needed and presented in Möller (2020) will be repeated here. The outline of the contents is as follows. In Section 2 notation is introduced and the abstract functional analytic theorem on completeness and minimality is proved. In Section 3 it is shown how the main result, Theorem 2.6, can be applied to an n-th order ordinary Birkhoff regular differential operator with boundary conditions which may depend on λ linearly. In Section 4 the case of second order differential equations with separable boundary conditions is discussed. A general result is obtained when exactly one boundary condition depends on λ, whereas a positive result is also provided for a special case when both boundary conditions depend on λ.

2 Complete and minimal systems

2.1 Minimality

Let E, F, and H be infinite-dimensional Banach spaces such that E is a dense subset of H with continuous embedding $E \hookrightarrow H$. The dual spaces of E, F, H are denoted by E', F', H' with corresponding bilinear forms $\langle\ ,\ \rangle$. The notation $\langle\ ,\ \rangle_E$ for the bilinear form in $E \times E'$, for example, may also be used.

Definition 2.1. A sequence $(w_i)_{i=1}^{\infty}$ in the Banach space H is called *minimal* in H if

$$w_j \notin \overline{\operatorname{span}\{w_i : i \in \mathbb{N},\, i \neq j\}} \qquad \text{for all } j \in \mathbb{N}.$$

The sequence $(w_i)_{i=1}^{n}$ in H is called *complete* in H if

$$\overline{\operatorname{span}\ \{w_i : i \in \mathbb{N}\}} = H.$$

Definition 2.2. Let $B \in L(E, F)$. Two sequences $(y_i)_{i=1}^{\infty}$ in E and $(v_i)_{i=1}^{\infty}$ in F' are called B-*biorthogonal* if

$$\langle By_i, v_j \rangle_F = \delta_{ij}, \qquad i, j \in \mathbb{N}. \tag{2.1}$$

When the operator B is clear from the context, the notion B-biorthogonal may be shortened to biorthogonal.

Let F be decomposed as $F_1 \times F_2$ where F_1 is a Banach space and F_2 is a finite-dimensional space of dimension κ, where $\kappa = 0$ is allowed. Then the operator B from E to F can be decomposed as

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

where $B_l \in L(E, F_l)$, $l = 1, 2$. Furthermore it is assumed that B_1 has a continuous extension $B_0 \in L(H, F_1)$. With respect to the decomposition $F' = F_1' \times F_2'$, let $v_i =: (v_{i,1}, v_{i,2})$ for $i \in \mathbb{N}$.

Theorem 2.3 (Möller (2020: Theorem 1))**.** *Assume that the sequences $(y_i)_{i=1}^{\infty}$, $y_i \in E$, and $(v_i)_{i=1}^{\infty}$, $v_i \in F'$, are B-biorthogonal. If* $\operatorname{span}\{v_{1,2}, \ldots, v_{\kappa,2}\} = F_2'$, *then $(y_i)_{i=\kappa+1}^{\infty}$ is minimal in H.*

2.2 Completeness

For an operator S in a Banach space, its adjoint will be denoted by S^*.

Proposition 2.4. *Assume that* $\operatorname{span}\{v_{1,2}, \ldots, v_{\kappa,2}\} = F_2'$ *and that $(y_i)_{i=1}^{\infty}$ is complete in H. Then there is an integer N with $1 \leq N \leq \kappa + 1$ such that, after possibly permutating the indices $1, \ldots, \kappa$, the system $(y_i)_{i=N}^{\infty}$ is minimal and complete in H.*

Proof. If $(y_i)_{i=\kappa+1}^{\infty}$ is complete in H, then the statement holds for $N = \kappa + 1$, since $(y_i)_{i=\kappa+1}^{\infty}$ is also minimal in H by Theorem 2.3. If $(y_i)_{i=\kappa+1}^{\infty}$ is not complete in H, then

$$\overline{\operatorname{span}\{y_i : i \in \mathbb{N},\ i \geq \kappa + 1\}}^{H} \subsetneqq H = \overline{\operatorname{span}\{y_i : i \in \mathbb{N}\}}^{H}.$$

Therefore it may be assumed without loss of generality that

$$y_\kappa \notin \overline{\operatorname{span}\{y_i : i \in \mathbb{N},\ i \geq \kappa + 1\}}^H.$$

It is clear that also $(y_i)_{i=\kappa}^\infty$ is minimal in H. If $(y_i)_{i=\kappa}^\infty$ is not complete in H, then this procedure is repeated up to an index N with $2 \leq N \leq \kappa + 1$ which gives that $(y_i)_{i=N}^\infty$ is minimal and complete in H, or up to $N = 1$ which gives that $(y_i)_{i=N}^\infty$ is minimal in H. But for $N = 1$ the system $(y_i)_{i=1}^\infty$ is complete in H by assumption, and the proof is complete. □

Proposition 2.4 states a sufficient condition under which the removal of at most κ terms from the sequence results in the remaining sequence to be minimal and complete. However, the number of terms to be removed is not known, and therefore sufficient conditions will be found such that this number is exactly κ. Let

$$\widetilde{E} := \overline{\operatorname{span}\{y_i : i \in \mathbb{N}\}}^E, \qquad \widetilde{B}_1 := B_1|_{\widetilde{E}}, \qquad \widetilde{B}_2 := B_2|_{\widetilde{E}}, \quad \text{and} \quad \widetilde{B} := \begin{pmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \end{pmatrix}.$$

Proposition 2.5. *Let the assumptions of Proposition* 2.4 *be satisfied and let N be the number in the statement of that proposition. Then $\widetilde{B}_2^*$ is injective on* $\operatorname{span}\{v_{1,2}, \dots, v_{N-1,2}\}$.

Proof. Of course, we only need to consider the case $N > 1$. Assume the statement to be false. Replacing $y_1, \dots, y_{N-1}$ and $v_1, \dots, v_{N-1}$ with suitable linear combinations thereof such that (2.1) remains true, it may be assumed without loss of generality that $\widetilde{B}_2^* v_{1,2} = 0$. Then

$$\langle \widetilde{B}_2 y_i, v_{1,2} \rangle = \langle y_i, \widetilde{B}_2^* v_{i,2} \rangle = 0, \qquad i \in \mathbb{N},$$

and (2.1) would give

$$\delta_{i,1} = \langle \widetilde{B} y_i, v_1 \rangle = \langle y_i, \widetilde{B}_0^* v_{1,1} \rangle_{\widetilde{E}} = \langle y_i, \widetilde{B}_0^* v_{1,1} \rangle_H, \qquad i \in \mathbb{N}.$$

But, since $(y_i)_{i=N}^\infty$ is complete in H, this would lead to $\widetilde{B}_0^* v_{1,1} = 0$ and $1 = \langle y_1, \widetilde{B}_0^* v_{1,1} \rangle_H$, a contradiction. □

Consequently, under the assumptions of Proposition 2.4, for N in Proposition 2.4 to be $\kappa + 1$ it is necessary that $\widetilde{B}_2^*$ is injective or, equivalently, that $\widetilde{B}_2$ is surjective.

Since $(y_i)_{i=N}^\infty$ is minimal and complete in H, there exists a unique sequence $(w_i)_{i=N}^\infty$ in H' such that

$$\langle y_i, w_j \rangle_H = \delta_{i,j}, \qquad i, j \geq N. \tag{2.2}$$

Theorem 2.6. *Assume that* $\operatorname{span}\{v_{1,2}, \dots, v_{\kappa,2}\} = F_2'$, *that the sequence $(y_i)_{i=1}^\infty$ is complete in H, that $B_2(\operatorname{span}\{y_i : i \in \mathbb{N}\}) = F_2$, and that $R(B_2^*) \cap H' = \{0\}$. Then $(y_i)_{i=\kappa+1}^\infty$ is complete in H.*

Proof. By Proposition 2.4, the system $(y_i)_{i=N}^\infty$ is minimal and complete in H. In particular, $\widetilde{E}$ is dense in H. Then it is clear that with B_1 also $\widetilde{B}_1$ has a continuous extension $\widetilde{B}_0$ to H and that $\widetilde{B}_0 = B_0$. Furthermore, $\widetilde{B}_2$ and B_2 are surjective, which implies that $\widetilde{B}_2^*$ and B_2^* are injective. Hence there is a one-to-one identification of elements in $R(\widetilde{B}_2^*)$ with elements in $R(B_2^*)$. But since H' is a subspace of $\widetilde{E}'$ as well as E', the assumption $R(B_2^*) \cap H' = \{0\}$ gives that $R(\widetilde{B}_2^*) \cap H' = \{0\}$.

From (2.1) and (2.2) it follows that

$$\langle y_i, \widetilde{B}^* v_j - w_j \rangle = 0, \qquad i, j \geq N.$$

The completeness of $(y_i)_{i=1}^\infty$ in $\widetilde{E}$ and the biorthogonality (2.1) imply that $\widetilde{B}^* v_j - w_j$ is a linear combination of $\widetilde{B}^* v_l$, $l = 1, \ldots, N-1$. Therefore

$$\widetilde{B}^* v_j - w_j = \sum_{l=1}^{N-1} \langle y_l, \widetilde{B}^* v_j - w_j \rangle \widetilde{B}^* v_l = - \sum_{l=1}^{N-1} \langle y_l, w_j \rangle \widetilde{B}^* v_l, \qquad j \geq N,$$

which gives

$$\widetilde{B}_2^* v_{j,2} + \sum_{l=1}^{N-1} \langle y_l, w_j \rangle \widetilde{B}_2^* v_{l,2} = w_j - \widetilde{B}_0^* v_{j,1} - \sum_{l=1}^{N-1} \langle y_l, w_j \rangle \widetilde{B}_0^* v_{l,1} \in H', \qquad j \geq N.$$

Then $R(\widetilde{B}_2^*) \cap H' = \{0\}$ and the injectivity of $\widetilde{B}_2^*$ show that

$$v_{j,2} + \sum_{l=1}^{N-1} \langle y_l, w_j \rangle v_{l,2} = 0,$$

which is impossible when $N \leq j \leq \kappa$ because $v_{1,2}, \ldots, v_{\kappa,2}$ are linearly independent. Therefore $N = \kappa + 1$. $\square$

3 The differential operator

Let $a < b$ be real numbers. Recall the Sobolev space

$$W_p^k(a,b) = \{ y \in L_p(a,b) : \, y^{(i)} \in L_p(a,b), \; i = 1, \ldots, k \}, \qquad p \in (1, \infty), \; k \in \mathbb{N},$$

where the derivative is taken in the sense of distributions, see, e.g., (Mennicken & Möller, 2003: Section 2.1). The vector space $W_p^k(a,b)$ becomes a Banach space when equipped with the norm $\| \, \|_{p,k}$, defined by

$$\|y\|_{p,k} = \sum_{i=0}^{k} \|y^{(i)}\|_p, \qquad y \in W_p^k(a,b),$$

where $\| \, \|_p$ is the norm in $L_p(a,b)$. The dual of the Banach space $W_p^k(a,b)$ will be identified with a space of distributions $W_{p'}^{-k}[a,b]$, where $1/p + 1/p' = 1$, and the corresponding bilinear form will be denoted by $\langle \, , \, \rangle_{p,k}$, see, e.g., (Mennicken & Möller, 2003: Section 2.1).

Now let $n \geq 2$ and define

$$\mathbf{K} y = \sum_{i=0}^{n} k_i y^{(i)}$$

with $k_i \in W_{p'}^i(a,b)$ for $i = 1, \ldots, n-1$ and $k_0 \in L_{\min(p,p')}(a,b)$. It will always be assumed that k_n is a non-zero constant. The differential operator

$$L^D(\lambda) y := \mathbf{K} \eta - \lambda y, \qquad y \in W_p^n(a,b), \tag{3.1}$$

satisfies $L^D(\lambda) \in L(W_p^n(a,b), L_p(a,b))$. Together with (3.1) two-point boundary conditions

$$L^R(\lambda)y := \left(\sum_{i=0}^{n-1} w_{ki}^{(0)}(\lambda) y^{(i-1)}(a) + \sum_{i=0}^{n-1} w_{ki}^{(1)}(\lambda) y^{(i-1)}(b) \right)_{k=1}^{n} = 0 \tag{3.2}$$

are considered, where the $w_{ki}^{(j)}$ are polynomials of degree at most 1, i. e., they are constant or polynomials of degree 1. It is assumed that for each $k = 1, \ldots, n$ at least one of the $2n$ polynomials $w_{ki}^{(0)}$, $w_{ki}^{(1)}$, $i = 1, \ldots, n$, is not the zero polynomial. Clearly, $L^R(\lambda) \in L(W_p^n(a,b), \mathbb{C}^n)$.

For $\lambda \in \mathbb{C}$ define

$$L(\lambda) := (L^D(\lambda), L^R(\lambda)). \tag{3.3}$$

It is clear that there are bounded operators

$$A, B \in L(W_p^n(a,b), L_p(a,b) \times \mathbb{C}^n) \quad \text{such that} \quad L(\lambda) = A + \lambda B, \quad \lambda \in \mathbb{C}. \tag{3.4}$$

Since the dual spaces are defined via sesquilinear forms, the adjoint operator has the representation

$$L(\lambda)^* = A^* + \lambda B^*, \qquad \lambda \in \mathbb{C}.$$

In order to define Birkhoff regularity some notation is introduced first. For positive integers r let $M_r(\mathbb{C})$ be the set of $r \times r$ matrices with entries in $\mathbb{C}$ and put

$$\begin{aligned} W^{(j)}(\lambda) &= \big(w_{ki}^{(j)}(\lambda)\big)_{k,i=1}^{n}, \qquad j = 0, 1, \\ W(\lambda) &= (W^{(0)}(\lambda), W^{(1)}(\lambda)), \\ C_0(\mu) &= \operatorname{diag}(1, \mu, \ldots, \mu^{n-1}) \in M_n(\mathbb{C}), \\ l_\nu &= \deg\big[e_\nu^{\mathrm{T}}\big(W^{(0)}(\mu^n)C_0(\mu), W^{(1)}(\mu^n)C_0(\mu)\big)\big], \qquad \nu = 1, \ldots, n, \end{aligned} \tag{3.5}$$

where deg denotes the degree as a vector polynomial in the variable μ and e_ν is the ν-th standard basis vector in $\mathbb{C}^n$. Furthermore, let

$$C_1 = \begin{pmatrix} 1 & \ldots & 1 \\ k_n^{-1/n}\omega_1 & \ldots & k_n^{-1/n}\omega_n \\ \vdots & & \vdots \\ k_n^{(n-1)/n}\omega_1^{n-1} & \ldots & k_n^{(n-1)/n}\omega_n^{n-1} \end{pmatrix}, \tag{3.6}$$

where

$$\omega_j = \exp\left\{ \frac{2\pi i(j-1)}{n} \right\}, \quad j = 1, \ldots, n.$$

Then

$$\operatorname{diag}(\mu^{-l_1}, \ldots, \mu^{-l_n}) W^{(j)}(\mu^l) C_0(\mu) C_1 = W_0^{(j)} + O(\mu^{-1}), \qquad j = 0, 1.$$

Clearly, the constants l_ν defined in (3.5) satisfy $0 \le l_\nu \le 2n - 1$. Without loss of generality it will be assumed that $l_\nu \le l_{\nu+1}$ for $\nu = 1, \ldots, n-1$. Let

$$\kappa := \#\{\nu \in \{1, \ldots, n\} : l_\nu \ge n\}. \tag{3.7}$$

This means that for $\nu > n - \kappa$ at least one entry of the ν-th row of the $n \times 2n$ matrix $W(\lambda)$ is not constant, whereas for $\nu \le n - \kappa$, all entries of the ν-th row of $W(\lambda)$ are constant. Let

$\hat{l}_\nu \in \{0, \dots, n-1\}$ be such that

$$l_\nu = \hat{l}_\nu \mod (n), \qquad \nu = 1, \dots, n.$$

Clearly, $l_\nu = \hat{l}_\nu$ for $\nu \le n - \kappa$ and $l_\nu = \hat{l}_\nu + n$ for $\nu > n - \kappa$.

If $n = 2m$ is even, then let $\mathfrak{L}$ be the set of all $n \times n$ diagonal matrices with m consecutive entries 1 in the diagonal, in a cyclic arrangement, whereas the remaining entries are 0. If $n = 2m + 1$ is odd, then let $\mathfrak{L}$ be the set of all $n \times n$ diagonal matrices with m or $m + 1$ consecutive entries 1 in the diagonal, in a cyclic arrangement, whereas the remaining entries are 0. Let

$$\mathfrak{B} := \{W_0^{(0)}\Lambda + W_0^{(1)}(I_n - \Lambda) : \Lambda \in \mathfrak{L}\}. \tag{3.8}$$

The problem (3.1)-(3.2) is called *Birkhoff regular* if $\det \Phi \neq 0$ for all $\Phi \in \mathfrak{B}$. This definition of Birkhoff regularity is a special case of the characterization of Birkhoff regularity in Mennicken & Möller (2003: Theorem 7.3.2). Then it follows from Mennicken & Möller (2003: Theorem 4.3.9) that the resolvent set of L is not empty, which implies the following result.

Proposition 3.1. *If the problem* (3.1)-(3.2) *is Birkhoff regular, then the spectrum $\sigma(L)$ of L consists of a sequence $(\lambda_k)_{k=1}^{\infty}$ of eigenvalues of finite multiplicity.*

Definition 3.2. (i) An ordered set $\{y_0, y_1, \dots, y_h\}$ in E is called a *chain of an eigenvector and associated vectors* (CEAV) of L at $\lambda_0 \in \mathbb{C}$ if $y_0 \neq 0$ and if the vector polynomial

$$y := \sum_{l=0}^{h} (\cdot - \lambda_0)^l y_l$$

is such that the vector polynomial Ly has a zero at λ_0 of multiplicity $\nu(y) \ge h + 1$.
(ii) Let $y_0 \in N(L(\lambda_0)) \setminus \{0\}$. Then $\overline{\nu}(y_0)$ denotes the maximum of all multiplicities $\nu(y)$, where y is as in part (i) with $y(\lambda_0) = y_0$.
(iii) A system $\{y_l^{(j)} : 1 \le j \le r, 0 \le l \le \overline{m}_j - 1\}$ is called a *canonical system of eigenvectors and associated vectors* (CSEAV) of L at λ_0 if

(a) $\{y_0^{(1)}, \dots, y_0^{(r)}\}$ is a basis of $N(L(\lambda_0))$;

(b) $\{y_0^{(j)}, \dots, y_{\overline{m}_j - 1}^{(j)}\}$ is a CEAV of L at λ_0, $j = 1, \dots, r$;

(c) $\overline{m}_j = \max\{\overline{\nu}(y) : y \in N(L(\lambda_0)) \setminus \operatorname{span}\{y_0^{(k)} : k < j\}\}$, $j = 1, \dots, r$.

The following two propositions are special cases of results in Mennicken & Möller (2003: Section 1.10).

Proposition 3.3. *Assume that L is Birkhoff regular and let $\lambda_0 \in \sigma(L)$. Let $y_0, \dots, y_k$ be a CEAV of L at λ_0. Then*

$$(A + \lambda_0 B)y_0 = 0, \qquad (A + \lambda_0 B)y_{l+1} = -By_l, \qquad l = 0, \dots, k-1.$$

For each eigenvalue there exist biorthogonal canonical systems of eigenvectors and associated vectors of L and L^*.

Proposition 3.4. *Assume that the differential operator L is Birkhoff regular and let the CSEAV*

$$\{y_l^{(j)} : 1 \le j \le r,\, 0 \le l \le m_j - 1\}$$

of L at λ_0 be given. Then there exists a CSEAV

$$\{v_l^{(j)} : 1 \le j \le r,\, 0 \le l \le m_j - 1\}$$

of L^* at λ_0 such that these two systems are biorthogonal, i.e.,

$$\langle By_l^{(i)}, v_{m_j-1-k}^{(j)} \rangle = \delta_{ij}\, \delta_{lk}, \qquad 1 \le i \le r,\ 0 \le l \le m_i - 1,\ 1 \le j \le r,\ 0 \le k \le m_j - 1.$$

Before formulating the main result for differential operators, more notation is needed. Writing

$$W(\lambda) =: W_0 + \lambda W_1, \tag{3.9}$$

see (3.5), let $\widehat{W}_1$ be the $\kappa \times 2n$ submatrix of W_1 consisting of the last κ rows of W_1. Furthermore, for $y \in W_p^n(a,b)$ let $\widehat{Y}$ be the $2n$-vector

$$\widehat{Y} := (y(a), \dots, y^{(n-1)}(a), y(b), \dots, y^{(n-1)}(b)).$$

The obvious modification for indexed functions applies. A representation of the adjoint operator $L^*(\lambda)$ is given in Mennicken & Möller (2003: Theorem 6.5.1), namely,

$$L^*(\lambda)(u,d) = \sum_{i=0}^{n} (-1)^i \big(k_i u\big)_e^{(i)} - \lambda u_e + L^{R^*}(\lambda) d, \qquad u \in L_{p'}(a,b),\ d \in \mathbb{C}^n,$$

where u_e is the canonical extension of u to $\mathbb{R}$ by defining $u = 0$ on $\mathbb{R} \setminus [a,b]$, and where

$$L^{R^*}(\lambda) = \sum_{i=1}^{n} (-1)^{i-1} \Big(\big(W^{(0)}(\lambda) e_i\big)^{\mathrm{T}} \delta_a^{(i-1)} + \big(W^{(1)}(\lambda) e_i\big)^{\mathrm{T}} \delta_b^{(i-1)} \Big)$$

with the Dirac distributions δ_c at c for $c = a$ and $c = b$.

Theorem 3.5. *Assume that the differential operator L defined by* (3.3) *with the representation* (3.4) *is Birkhoff regular and that $(y_i)_{i=1}^\infty$ is a sequence of elements in $W_p^n(a,b)$ which consists of CSEAVs at all eigenvalues of L. Let $(v_i)_{i=1}^\infty$ be a corresponding sequence of CSEAVs (at all eigenvalues of L^*) such that, under suitable indexing, the biorthogonality relation* (2.1) *holds. With respect to the decomposition $L_p(a,b) \oplus \mathbb{C}^{n-\kappa} \oplus \mathbb{C}^\kappa$ write $v_i = (u_i, d_{i,1}, d_{i,2})$. Further assume that*

$$\operatorname{span}\{d_{1,2}, \dots, d_{\kappa,2}\} = \mathbb{C}^\kappa \quad \textit{and} \quad \operatorname{span}\{\widehat{W}_1 \widehat{Y}_i : i \in \mathbb{N}\} = \mathbb{C}^\kappa.$$

Then the system $(y_i)_{i=\kappa+1}^\infty$ is minimal and complete in $L_p(a,b)$.

Proof. Note first that the existence of the sequence $(v_i)_{i=1}^\infty$ in the statement of the theorem is guaranteed by Proposition 3.4.

Putting $E = W_p^k(a,b)$, $F = L_p(a,b) \oplus \mathbb{C}^n$, $F_1 = L_p(a,b) \oplus \mathbb{C}^{n-\kappa}$, $F_2 = \mathbb{C}^\kappa$, and $H = L_p(a,b)$, it follows in the notation of Section 2 that $B_1 y = (-y, 0)$, $y \in E$, has a continuous extension B_0 onto H given by $B_0 y = (-y, 0)$. By (Mennicken & Möller, 2003: Theorem 8.8.3 & Remark 8.8.4) each function in $f \in L_p(a,b)$ can be represented by a series $\sum_{j=1}^{\infty} f_j$, where each f_j is a finite linear combination of eigenfunctions and associated functions of L. Therefore the eigenfunctions and associated functions of L form a complete system in $L_p(a,b)$. The operator B_2 is represented by $B_2 y = \widehat{W}_1 \widehat{Y}$, $y \in E$. Thus the operator B_2^* is represented by the coefficient of λ in $(L^R)^*(\lambda)$ restricted to the last κ components of $\mathbb{C}^n$. Then (3.11) shows that $R(B_2^*)$ consists of linear combinations of Dirac distributions and their derivatives, and therefore $R(B_2^*) \cap H' = \{0\}$. An application of Theorems 2.3 and 2.6 completes the proof. □

The condition $\operatorname{span}\{\widehat{W}_1 \widehat{Y}_i : i \in \mathbb{N}\} = \mathbb{C}^\kappa$ is a very technical one and it will not be considered any further in this note in the general case. The case $n = 2$ will be discussed in the next section.

4 Completeness and minimality for second order differential operators

In general, the assumptions of Theorem 3.5 are not easy to verify. Here we consider the case $n = 2$ with separable boundary conditions. For convenience, and possibly deviating from the arrangement of the rows of $W(\lambda)$ in Section 3, it will be assumed that the first boundary condition is at a, whereas the second boundary condition is at b. Therefore

$$W^{(0)}(\lambda) = \begin{pmatrix} w_{11}^{(0)}(\lambda) & w_{12}^{(0)}(\lambda) \\ 0 & 0 \end{pmatrix}, \qquad W^{(1)}(\lambda) = \begin{pmatrix} 0 & 0 \\ w_{21}^{(1)}(\lambda) & w_{22}^{(1)}(\lambda) \end{pmatrix},$$

where neither $W^{(0)}$ nor $W^{(1)}$ is identically zero. If, say, $W^{(0)}(\lambda_0) = 0$ for some $\lambda_0 \in \mathbb{C}$, then one could factor out $\lambda - \lambda_0$ from $W^{(0)}$, which would be a rather artificial factor in the boundary matrix. Hence it is reasonable to require that $W^{(0)}(\lambda) \neq 0$ and $W^{(1)}(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$. In particular, the boundary condition at a is an initial condition, and therefore each eigenvalue has geometric multiplicity 1. Furthermore, with the notation as in (3.9),

$$W_0^{(0)} = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} \\ 0 & 0 \end{pmatrix} C_1, \qquad W_0^{(1)} = \begin{pmatrix} 0 & 0 \\ \alpha_{1,0} & \alpha_{1,1} \end{pmatrix} C_1,$$

for complex numbers $\alpha_{i,j}$ such that $(\alpha_{i,0}, \alpha_{i,1}) \neq (0,0)$ for $i = 0, 1$. Assuming the Sturm-Liouville equation, i.e., $k_2 = -1$, for simplicity and taking $k_2^{-1/2} = i$, it follows that

$$C_1 = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},$$

see (3.6). Therefore,

$$W_0^{(0)} C_1 = \begin{pmatrix} \alpha_{0,0} + i\alpha_{0,1} & \alpha_{0,0} - i\alpha_{0,1} \\ 0 & 0 \end{pmatrix}, \qquad W_0^{(1)} C_1 = \begin{pmatrix} 0 & 0 \\ \alpha_{1,0} + i\alpha_{1,1} & \alpha_{1,0} - i\alpha_{1,1} \end{pmatrix},$$

and $\mathcal{L} = \{\operatorname{diag}(1,0), \operatorname{diag}(0,1)\}$. Hence $\mathfrak{B}$ as defined in (3.8) consists of the two matrices

$$\begin{pmatrix} \alpha_{0,0} + i\alpha_{0,1} & 0 \\ 0 & \alpha_{1,0} - i\alpha_{1,1} \end{pmatrix}, \qquad \begin{pmatrix} 0 & \alpha_{0,0} - i\alpha_{0,1} \\ \alpha_{1,0} + i\alpha_{1,1} & 0 \end{pmatrix}.$$

Assuming further for simplicity that all coefficients of w_{ki} are real, it follows that all problems in this section are Birkhoff regular. For general k_2 and $\alpha_{i,j}$ it is obviously easy to determine when the problem is Birkhoff regular.

Now the cases $\kappa = 0$, $\kappa = 1$, and $\kappa = 2$ will be considered separately; for notation see Theorem 3.5 and (3.7). It will always be assumed that $k_2 = -1$ and that all w_{ki} have real coefficients.

4.1 The case $\kappa = 0$

This case is well known, it is just included for completeness. Here the assumption on the span is void, and therefore

Theorem 4.1. *If L is Birkhoff regular and if the boundary conditions are independent of the eigenvalue parameter, then the system $(y_i)_{i=1}^{\infty}$ is minimal and complete in $L_p(a, b)$.*

4.2 The case $\kappa = 1$

Since the geometric multiplicity of each eigenvalue is 1, each CSEAV consist of one CEAV, whose length is the algebraic multiplicity of the corresponding eigenvalue. Hence each y_i is therefore an element in such a chain, and we call y_i a *terminal function* if it is the last element of such a CEAV. Note that a terminal function y_i is an eigenfunction if the eigenvalue is simple, and an associated function otherwise.

Theorem 4.2. *Assume that $k_2 = -1$, that $k_1 = 0$, and that L is Birkhoff regular. If y_1 is a terminal function or if $d_{1,2} \neq 0$, then the system $(y_i)_{i=2}^{\infty}$ is minimal and complete in $L_p(a, b)$.*

Proof. The minimality has been shown in Möller (2020: Theorem 4). Since $\kappa = 1$, it therefore suffices to prove that $\widehat{W_1}Y \neq 0$ whenever y is an eigenfunction. Hence by proof of contradiction, assume that $\widehat{W_1}\widehat{Y} = 0$. Then $\widehat{W_1} = e_\iota^{\mathrm{T}} W_1$, where $\iota = 1$ if the λ-dependent boundary condition is at a and $\iota = 2$ otherwise. Furthermore, $W(\lambda)\widehat{Y} = 0$ for the eigenfunction y implies that also $e_\iota^{\mathrm{T}} W_0 \widehat{Y} = 0$. But $e_\iota^{\mathrm{T}} W_0$ and $e_\iota^{\mathrm{T}} W_1$ are linearly independent by the feasibility assumption, which means that $e_\iota^{\mathrm{T}} W_0 \widehat{Y}$ and $e_\iota^{\mathrm{T}} W_1 \widehat{Y}$ would be linearly independent linear combinations of y and y' at the corresponding endpoint. Hence both y and y' would be 0 at that endpoint, which contradicts $y \neq 0$. □

4.3 The case $\kappa = 2$

Since this note is a continuation of the work in Möller (2020), only the special case

$$y'' + \lambda y = 0, \tag{4.1}$$

$$\alpha y'(0) - \lambda y(0) = 0, \tag{4.2}$$

$$\beta y(1) - \lambda y'(1) = 0, \tag{4.3}$$

with $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ will be considered.

Theorem 4.3. *Let* $(y_i)_{i=1}^{\infty}$ *be a sequence of elements in* $W_p^n(a,b)$ *which consists of CSEAVs at all eigenvalues of* (4.1)–(4.3). *If* y_1 *and* y_2 *are terminal functions corresponding to sufficiently large real eigenvalues, then the system* $(y_i)_{i=3}^{\infty}$ *is minimal and complete in* $L_p(a,b)$.

Proof. The minimality statement is Möller (2020: Theorem 5). It was also shown there, and it is easy to see, that solutions of (4.1) and (4.2) are multiples of

$$u(x,\lambda) = \alpha\cos\sqrt{\lambda}x + \sqrt{\lambda}\sin\sqrt{\lambda}x$$

and that λ is an eigenvalue of (4.1)–(4.3) if and only if λ satisfies

$$(\alpha\beta - \lambda^2)\cos\sqrt{\lambda} + (\beta + \alpha\lambda)\sqrt{\lambda}\sin\sqrt{\lambda} = 0. \tag{4.4}$$

It is clear that

$$\widehat{W}_1\widehat{Y} = -\begin{pmatrix} y(0) \\ y'(1) \end{pmatrix}.$$

It has also been shown that

$$u'(1,\lambda) = -\alpha\sqrt{\lambda}\sin\sqrt{\lambda} + \lambda\cos\sqrt{\lambda} = \beta\frac{\alpha^2+\lambda}{\beta+\alpha\lambda}\cos\sqrt{\lambda},$$

whereas clearly $u(0,\lambda) = \alpha$. From (4.4) we conclude for eigenvalues λ that

$$[(\alpha\beta - \lambda^2)^2 + \lambda(\beta+\alpha\lambda)^2]\cos^2\sqrt{\lambda} = \lambda(\beta+\alpha\lambda)^2,$$

and therefore

$$(u'(1,\lambda))^2 = \frac{\beta^2\lambda(\alpha^2+\lambda)^2}{(\alpha\beta-\lambda^2)^2 + \lambda(\beta+\alpha\lambda)^2},$$

which shows that $\widehat{W_1}Y_i$ and $\widehat{W_1}Y_j$ are linearly independent for infinitely many pairs of eigenfunctions y_i and y_j. In particular, span $\{\widehat{W}_1\widehat{Y}_i : i \in \mathbb{N}\} = \mathbb{C}^2$. □

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PARTIALLY OVERLAPPING EVENT WINDOWS AND TESTING CUMULATIVE ABNORMAL RETURNS IN FINANCIAL EVENT STUDIES

Seppo Pynnönen

Dedicated to Professor Seppo Hassi on the occasion of his 60th birthday

1 Introduction

In financial event studies the interest is to evaluate the effect of an economic event on the value of a firm. For this evaluation data available from financial markets can be successfully used with appropriate statistical testing methodology. The analyses are mainly based on stock or other asset returns. Campbell, Lo & MacKinlay (1997: Chapter 4) is an excellent introduction to financial event studies and related statistical methods.

Instead of using returns as such, standardizing them by respective standard deviations homogenizes data and improves testing performance. Because of this improvement, standardized return based tests by Patel (1976) and Boehmer, Musumeci & Poulsen (1991) (BMP) have gained popularity over conventional non-standardized tests in testing event effects on mean security price performance. Harrington & Shrider (2007) found that in a short-horizon testing of abnormal returns (i.e., systematic deviation from expected behavior), one should always use methods that are robust to cross-sectional variation in the true abnormal returns; for a discussion of true abnormal returns, see (Harrington & Shrider, 2007). They found that BMP is a good candidate for a robust, parametric test in conventional event studies.[*]

However, a major problem in statistical tests of returns is that the returns are not normally distributed, see (Fama, 1976). Therefore, not surprisingly, non-parametric rank tests introduced by Corrado (1989; 2011), Corrado & Zivney (1992), Campbell & Wasley (1993), and Kolari & Pynnonen (2011), among others, dominate parametric tests both in terms of better size and power, see e.g., (Corrado, 1989; Corrado & Zivney, 1992; Campbell & Wasley, 1993; Kolari & Pynnonen, 2010; 2011; Luoma, 2011). Furthermore, the rank tests of Corrado & Zivney (1992) and Kolari & Pynnonen (2011) that utilize event period re-standardized returns have proven to be robust to event-induced volatility (Kolari & Pynnonen, 2010; 2011), cross-correlation due to event day clusterings (Kolari & Pynnonen, 2010), and autocorrelation (Kolari & Pynnonen, 2011). These are consistent with the view stated in the epilogue of Lehmann (2006: p. v): "Rank tests apply often to relatively simple solutions, such as one-, two-, and s-sample problems, and testing for independence and randomness, but for these situations they are often the method of choice." Moreover, the results of rank tests are invariant to monotone transformations of the underlying returns, that is, whether the returns are defined as simple, continuously compounded (log-returns), or gross-returns. The existing rank based tests, however, are not robust to cross-sectional correlation if the event days are partially over-

[*]We define conventional event studies as those focusing only on mean stock price effects. Other types of event studies include (for example) the examination of return variance effects (Beaver, 1968; Patel, 1976), trading volume (Beaver, 1968; Campbell & Wasley, 1996), accounting performance (Barber & Lyon, 1997), and earnings management procedures (Dechow, Sloan & Sweeney, 1995; Kothari, Leone & Wasley, 2005).

lapping. That is, when events in calendar time are scattered within an event window more or less randomly rather than clustered on the same calendar day, see (Kolari & Pynnonen, 2010). The current paper aims to fill this gap in the non-parametric event study testing. Kolari, Pape & Pynnonen (2018) have generalized existing parametric cross-sectional correlation robust testing towards this direction.

The rest of the paper is organized as follows. Section 2 reviews some related key literature. Section 3 defines the main concepts and derives some distributional properties of rank statistics. Section 4 introduces the new transformed rank test. Section 5 reports simulation results and Section 6 contains conclusions.

2 Review of related literature

Patell and BMP parametric tests apply straightforwardly for testing cumulative abnormal returns (CARs) over multiple day windows. With the correction suggested by Kolari & Pynnonen (2010) these tests are useful also in the case of completely clustered event days, and with the correction suggested by Kolari, Pape & Pynnonen (2018) when the event days are partially clustered. By construction the Corrado (1989) rank test applies for testing single day event returns. Testing for CARs with the same logic implies the need for defining multiple-day returns that match the number of days in the CARs, see Corrado (1989: p. 395) and Campbell & Wasley (1993: footnote 4). In practice this can be carried out by dividing the estimation period and event period into intervals matching the number of days in the CAR. Unfortunately, this procedure is not useful for a number of reasons. Foremost, it does not necessarily lead to a unique testing procedure. Also, the abnormal return model should be re-estimated for each multiple-day CAR definition. Furthermore, for a fixed estimation period, as the number of days accumulated in a CAR increases, the number of multiple-day estimation period observations reduces quickly to an impractically low number and thus, would weaken the abnormal return model estimation, cf. (Kolari & Pynnonen, 2010). Kolari & Pynnonen (2011) solve these issues in their generalized rank test approach.

On the other hand, for example, Campbell & Wasley (1993) suggest to use the Corrado (1989) rank test for testing cumulative abnormal returns by simply accumulating the respective ranks to constitute cumulative ranks. This is also the practice adopted in the Eventus® software (Cowan Research L.C., www.eventstudy.com) and is probably, for the time being, the most popular practice in multiple day applications of rank tests.

In spite of these attractive properties, the cumulative ranks test does not account for the cross-sectional correlation due to partially overlapping event windows. The correlation biases the standard errors downwards, leading to over-rejection of the null hypothesis of no event effect. This paper proposes an adjustment for the standard errors that corrects the bias.

3 Distributional properties of ranks

We begin by fixing some notations and an underlying assumption to facilitate our theoretical discussion.

Assumption 1. Stock returns r_{it} for firm i are weak white noise continuous random variables and are cross-sectionally independent over non-overlapping calendar days t, or,

$$\begin{array}{rcll} \mathbb{E}\left[r_{it}\right] & = & \mu_i & \text{for all } t, \\ \operatorname{var}\left[r_{it}\right] & = & \sigma_i^2 & \text{for all } t, \\ \operatorname{cov}\left[r_{it}, r_{iu}\right] & = & 0 & \text{for all } t \neq u, \\ \multicolumn{4}{l}{r_{it} \text{ and } r_{ju} \text{ are independent whenever } i \neq j \text{ and } t \neq u.} \end{array}$$

Note that it is a stylized fact that the variances of the returns are time varying and that there is mild autocorrelation. The time varying volatility problem can be partially captured in terms of generalized autoregressive conditional heteroskedasticity (GARCH) modeling. It is notable that under typical assumptions, GARCH-processes satisfy the weak stationary properties of Assumption 1.

Let $\text{AR}_{it} = r_{it} - \mathbb{E}\left[r_{it}\right]$ denote the abnormal return of security i on day t, and let day $t = 0$ indicate the event day. Days from $t = T_0 + 1$ to $t = T_1$ represent the estimation period relative to the event day, and days from $t = T_1 + 1$ to $t = T_2$ represent the event window. The cumulative abnormal return (CAR) from τ_1 to τ_2 with $T_1 < \tau_1 \leq \tau_2 \leq T_2$, is defined as

$$\text{CAR}_i(\tau_1, \tau_2) = \sum_{t=\tau_1}^{\tau_2} \text{AR}_{it}. \tag{3.1}$$

The time period from τ_1 to τ_2 is called in the following a *CAR window* or *CAR period*.

Standardized abnormal returns are defined as

$$\text{SAR}_{it} = \frac{\text{AR}_{it}}{S(\text{AR}_i)},$$

where

$$S(\text{AR}_i) = \sqrt{\frac{1}{T_1 - T_0 - 1} \sum_{t=T_0+1}^{T_1} \text{AR}_{it}^2}.$$

Moreover, for the purpose of accounting for the possible event induced volatility, the re-standardized abnormal returns are defined in the manner of Boehmer, Musumeci & Poulsen (1991), see also (Corrado & Zivney, 1992), as

$$\text{SAR}'_{it} = \begin{cases} \text{SAR}_{it}/S_{SAR_t}, & T_1 < t \leq T_2, \\ \text{SAR}_{it}, & \text{otherwise}, \end{cases}$$

where

$$S_{\text{SAR}_t} = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (\text{SAR}_{it} - \overline{\text{SAR}}_t)^2}$$

is the time t cross-sectional standard deviation of the SAR_{it}s. In the preceding formula n is the number of stocks in the portfolio and $\overline{\text{SAR}}_t = \frac{1}{n} \sum_{i=1}^{n} \text{SAR}_{it}$. Furthermore, let K_{it} denote the rank number of abnormal returns, where $K_{it} \in \{1, \ldots, T\}$, $t = T_0 + 1, \ldots, T_2$, $T = T_2 - T_0$, and $i = 1, \ldots, n$.

In particular, if the available observations on the estimation period vary from one series to another, then it is more convenient to deal with standardized ranks with zero mean and unit variance. For the purpose we use the known results of rank statistics, e.g., Lehmann (2006: Appendix, Section 1),

$$\mathbb{E}\left[K_{it}\right] = (T+1)/2, \quad \text{var}\left[K_{it}\right] = (T^2-1)/12, \quad \text{cov}\left[K_{is}, K_{it}\right] = -(T+1)/12, \quad s \neq t,$$

and define, cf. Hagnäs & Pynnonen (2014),

Definition 3.1. *Standardized ranks* are defined as

$$U_{it} = \frac{K_{it} - \frac{1}{2}(T+1)}{\sqrt{(T^2-1)/12}}.$$

Thus, $\mathbb{E}\left[U_{it}\right] = 0$, $\text{var}\left[U_{it}\right] = 1$, and $\text{cov}\left[U_{is}, U_{it}\right] = -1/(T-1)$.

Next we define cumulative standardized ranks for individual stocks.

Definition 3.2. The *cumulative standardized ranks* of a stock i over the event days window from τ_1 to τ_2 are defined as

$$U_i(\tau_1, \tau_2) = \sum_{t=\tau_1}^{\tau_2} U_{it}, \quad \text{where } T_1 < \tau_1 \leq \tau_2 \leq T_2. \tag{3.2}$$

Then immediately $\mu_i(\tau_1, \tau_2) = \mathbb{E}\left[U_i(\tau_1, \tau_2)\right] = 0$, and the variance equation

$$\text{var}\left[U_i(\tau_1, \tau_2)\right] = \sum_{t=t_1}^{\tau_2} \text{var}\left[U_{it}\right] + \sum_{s \neq t} \text{cov}\left[U_{is}, U_{it}\right]$$

implies that

$$\sigma_i^2(\tau_1, \tau_2) = \text{var}\left[U_i(\tau_1, \tau_2)\right] = \frac{\tau(T-\tau)}{T-1}, \tag{3.3}$$

where $i = 1, \ldots, n$, $T_1 < \tau_1 \leq \tau_2 \leq T_2$, and $\tau = \tau_2 - \tau_1 + 1$.

Rather than investigating individual (cumulative) returns, the practice in event studies is to aggregate individual returns into equally weighted portfolios.

Definition 3.3. The *average cumulative standardized ranks* are defined as the equally weighted portfolio of individual cumulative standardized ranks defined in (3.2), i.e.,

$$\bar{U}(\tau_1, \tau_2) = \frac{1}{n} \sum_{i=1}^{n} U_i(\tau_1, \tau_2) = \sum_{t=\tau_1}^{\tau_2} \bar{U}_t, \tag{3.4}$$

where $T_1 < \tau_1 \leq \tau_2 \leq T_2$, and

$$\bar{U}_t = \frac{1}{n} \sum_{i=1}^{n} U_{it}$$

is the time t average of standardized ranks.

The expected value of the average cumulative standardized ranks is the same as that of the cumulative ranks of individual securities, or

$$\mathbb{E}\left[\bar{U}(\tau_1, \tau_2)\right] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[U_i(\tau_1, \tau_2)\right] = 0.$$

If the event days are not clustered, then the cross-sectional correlations of the return series are zero, or at least negligible. Under the assumption of independence, the variance of $\bar{U}(\tau_1, \tau_2)$ is

$$\sigma_\tau^2 = \text{var}\left[\bar{U}(\tau_1, \tau_2)\right] = \frac{\tau(T-\tau)}{(T-1)n}$$

and by the central limit theorem asymptotically as $n \to \infty$,

$$Z = \left(\frac{(T-1)n}{\tau(T-\tau)}\right)^{\frac{1}{2}} \bar{U}(\tau_1, \tau_2) \sim N(0, 1). \tag{3.5}$$

The situation is more complicated if the event days are partially overlapping in calendar time, which implies cross-correlation. Recalling that the variances of $U_i(\tau_1, \tau_2)$ given in equation (3.3) are constants (independent of i), we can write the cross-covariance of $U_i(\tau_1, \tau_2)$ and $U_j(\tau_1, \tau_2)$ as

$$\text{cov}\left[U_i(\tau_1, \tau_2), U_j(\tau_1, \tau_2)\right] = \frac{\tau(T-\tau)}{T-1}\, \rho_{ij}(\tau_1, \tau_2), \tag{3.6}$$

where $\rho_{ij}(\tau_1, \tau_2)$ is the cross-sectional correlation of $U_i(\tau_1, \tau_2)$ and $U_j(\tau_1, \tau_2)$, $i, j = 1, \ldots, n$. Utilizing this and the variance-of-the-sum formula, the variance of $\bar{U}(\tau_1, \tau_2)$ in (3.4) becomes

$$\begin{aligned}
\text{var}\left[\bar{U}(\tau_1, \tau_2)\right] &= \frac{1}{n^2}\sum_{i=1}^{n} \text{var}\left[U_i(\tau_1, \tau_2)\right] + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} \text{cov}\left[U_i(\tau_1, \tau_2), U_j(\tau_1, \tau_2)\right] \\
&= \frac{\tau(T-\tau)}{(T-1)n}\left(1 + (n-1)\bar{\rho}_n(\tau_1, \tau_2)\right), \qquad (3.7)
\end{aligned}$$

where

$$\bar{\rho}_n(\tau_1, \tau_2) = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \rho_{ij}(\tau_1, \tau_2)$$

is the average cross-sectional correlation of cumulated ranks.

Cross-sectional dependence affects the asymptotic distribution of the statistic in (3.5). However, as discussed in Lehmann (1999: Section 2.8), it is frequently true that the asymptotic normality holds, provided that the average correlation, $\bar{\rho}_n(\tau_1, \tau_2)$, tends to zero rapidly enough such that

$$\frac{1}{n}\sum_{i\neq j}^{n} \rho_{ij}(\tau_1, \tau_2) = (n-1)\bar{\rho}_n(\tau_1, \tau_2) \to \gamma \qquad \text{as} \qquad n \to \infty,$$

where γ is some finite constant. Under this condition the limiting distribution of the Z-statistic in (3.5) becomes $N(0, 1+\gamma)$.

Otherwise, from a practical point of view, the crucial result of (3.7) is that the only unknown parameter to be estimated is the average cross-sectional correlation $\bar{\rho}_n(\tau_1, \tau_2)$. Hagnäs & Pynnonen (2014) discuss approaches to implicitly account for this average correlation in cumulated ranks tests when all events share the same calendar day, i.e., the case of complete overlapping (clustering) event periods. These implicit approaches, however, do not work when the event periods are partially overlapping. Therefore, by utilizing the procedure developed in Kolari, Pape & Pynnonen (2018), this paper proposes a method to estimate explicitly the cross-sectional correlation $\bar{\rho}_n(\tau_1, \tau_2)$, and thereby solves the cross-sectional correlation problem also in cases with partially overlapping event periods.

4 Correlation robust test for cumulated ranks

Following Kolari, Pape & Pynnonen (2018), let τ_{ij}, $0 \leq \tau_{ij} \leq \tau$, denote the number of calendar days that stocks i and j have overlapping calendar days within the event windows. By the independence in Assumption 1 the correlation $\mathrm{cor}\,[U_{iu}, U_{jv}]$ of the standardized ranks U_{iu} and U_{jv} is zero whenever the underlying calendar days of the relative event days, u and v, differ, and can be non-zero when the calendar days are the same. Denoting these non-zero correlations, which are also covariances, by ρ_{ij}, we obtain

$$\mathrm{cov}\,[U_i(\tau_1, \tau_2), U_j(\tau_1, \tau_2)] = \sum_{u=\tau_1}^{\tau_2} \sum_{v=\tau_1}^{\tau_2} \mathrm{cor}\,[U_{iu}, U_{ju}] = \tau_{ij}\rho_{ij}.$$

Combining this with (3.6) we obtain

$$\rho_{ij}(\tau_1, \tau_2) = \left(\frac{T-1}{T-\tau}\right) \frac{\tau_{ij}}{\tau} \rho_{ij}.$$

We can assume that the overlapping window lengths τ_{ij} and the cross-sectional correlations ρ_{ij} are not dependent on each other so that $\sum_{i \neq j} \tau_{ij}\rho_{ij} = n(n-1)\bar{\tau}\bar{\rho}$, where $\bar{\tau}$ is the average number of overlapping calendar days and $\bar{\rho}$ is the average cross-sectional correlation of U_i and U_j.† As a result, we can rewrite (3.7) as

$$\mathrm{var}\left[\bar{U}(\tau_1, \tau_2)\right] = \frac{\tau(T-\tau)}{(T-1)n}\,(1 + (n-1)\delta\bar{\rho}), \tag{4.1}$$

where $\delta = \bar{\tau}(T-1)/(\tau(T-\tau))$ adjusts the average correlation by the fraction of overlapping calendar days within the event window.

It is notable that even though the average cross-sectional correlation $\bar{\rho}$ in (4.1) is based on $n(n-1)/2$ correlations, it can be computed without estimating individual correlations by utilizing the method introduced by Edgerton & Toops (1928). Instead of the $n(n-1)/2$ individual correlations, it turns out that we need to compute only $n+1$ variances, which is a computational problem of order n viz-a-viz of order n^2 when averaging the correlations. To illustrate this idea, consider n random variables x_j, $j = 1, \ldots, n$, and define the standardized variables $z_j = x_j/\sigma_j$. Next let $\bar{z} = \sum_j z_j/n$ denote the average of the variables. Then the variance of $\bar{z}$ is

$$\sigma_{\bar{z}}^2 = \mathrm{var}\,[\bar{z}] = \frac{1 + (n-1)\bar{\rho}}{n},$$

because $\mathrm{var}\,[z_j] = 1$ and $\mathrm{cov}\,[z_j, z_k] = \mathrm{cor}\,[z_j, z_k] = \rho_{jk}$. From this result we obtain

$$\bar{\rho} = \frac{n\sigma_{\bar{z}}^2 - 1}{n-1}.$$

Thus, for large n, $\bar{\rho} \approx \sigma_{\bar{z}}^2$. In order to estimate the average cross-sectional correlation, all we need are estimates of n standard deviations of the x-variables and the variance of $\bar{z}$.

Since in our case the calendar days of different stocks are only partially overlapping, we estimate the variance of the average utilizing the clustering robust estimation technique, see, e.g., (Cameron,

†The equation follows by setting $\sum (x-\bar{x})(y-\bar{y}) = \sum xy - n\bar{x}\bar{y}$ to zero, so that $\sum xy = n\bar{x}\bar{y}$.

Gelbach & Miller, 2011), suggested in Kolari, Pape & Pynnonen (2018).

Following Kolari, Pape & Pynnonen (2018), denote the calendar days of the returns in the combined estimation and event window as $t = 1, \ldots, L$. In other words, L is the number of clusters which equals the number of separate calendar days on which returns are available in the combined estimation and event windows. Let n_t denote the number of stocks having returns on calendar day t and define

$$U_t = \sum_{k=1}^{n_t} U_{kt}.$$

Then

$$U_t^2 = \sum_{k=1}^{n_t} U_{kt}^2 + \sum_{i \neq j}^{n_t} U_{it} U_{jt},$$

which can be rearranged as

$$\sum_{i \neq j}^{n_t} U_{it} U_{jt} = U_t^2 - \sum_{k=1}^{n_t} U_{kt}^2.$$

Summing these up over the calendar days in the combined estimation and event window, we have

$$\sum_{t=1}^{L} \sum_{i \neq j}^{n_t} U_{it} U_{jt} = \sum_{t=1}^{L} U_t^2 - \sum_{t=1}^{L} \sum_{k=1}^{n_t} U_{kt}^2. \tag{4.2}$$

Taking the average, we get an estimator for the average correlation

$$\hat{\bar{\rho}} = \frac{1}{M} \sum_{t=1}^{L} \sum_{i \neq j}^{n_t} U_{it} U_{jt},$$

where

$$M = \sum_{t=1}^{L} n_t (n_t - 1)$$

is the number of cross-product terms. It is notable that days for which there is available only one return drop out automatically: if $n_t = 1$ for all t, then $\hat{\bar{\rho}} = 0$. The potentially tedious computation over all cross-products can be materially simplified by utilizing the right-hand side of (4.2) and observing that by rearranging the terms of the rightmost sum it becomes equal to nT, i.e., the number of stocks n multiplied by the combined estimation and event period T. The reason for this is that the sample variances of standardized ranks are all equal to one, see the discussion below Definition 3.1. Therefore, the only component we need is the first sum of squares on the right-hand side. As a result, we can estimate the average correlation efficiently by

$$\hat{\bar{\rho}} = \frac{N}{M} (s_U^2 - 1),$$

where $N = \sum_{t=1}^{L} n_t$ is the total number of returns, which equals nT if the combined estimation and event windows are of the same length T for all stocks, and s_U^2 is the clustering robust estimator of the variance of standardized ranks in the presence of intra-cluster correlation, i.e.,

$$s_U^2 = \frac{1}{N} \sum_{t=1}^{L} U_t^2.$$

Given the estimator of the average cross-sectional correlation, $\hat{\bar{\rho}}$, we can define an appropriate cross-sectional correlation robust test for the null hypothesis of zero cumulative abnormal returns

$$H_0 : \mu(\tau_1, \tau_2) = \mathbb{E}\left[\mathrm{CAR}(\tau_1, \tau_2)\right] = 0$$

in terms of the cumulated ranks using the z-ratio

$$z_\tau = \frac{\bar{U}(\tau_1, \tau_2)}{\sigma_\tau \sqrt{1 + \delta(n-1)\hat{\bar{\rho}}}}, \tag{4.3}$$

where σ_τ is the square root of the variance

$$\sigma_\tau^2 = \frac{\tau(T-\tau)}{(T-1)n}$$

of $\bar{U}(\tau_1, \tau_2)$ for completely non-overlapping event windows in calendar time (i.e., when $\bar{\rho} = 0$ in (4.1)), and $\tau = \tau_2 - \tau_1 + 1$ is the length of the window of cumulated abnormal returns.

5 Simulation results

We generate artificial returns utilizing the Fama & French (2015) five-factor model (FF5),

$$\begin{aligned}(r_{it} - r_f)_t = \alpha_i &+ \beta_{i,\text{mkt}}(r_m - r_f)_t + \beta_{i,\text{smb}}\text{SMB}_t + \beta_{i,\text{hml}}\text{HML}_t \\ &+ \beta_{\text{i,rmw}}\text{RMW}_t + \beta_{\text{i,cmw}}\text{CMW}_t + \epsilon_{it},\end{aligned} \tag{5.1}$$

where $r_m - r_f$ is the market excess return over the risk-free rate r_f, SMB, HML, RMW, and CMW are common market factors proposed by Fama and French. We utilize daily data from January 2, 1990 through October 30, 2020 (7,770 daily returns) to generate 20,000 initial daily return series for this sample period. The regression coefficients for each stock are generated from multivariate normal distribution with mean vector $(0, 1, 0.5, 0.5, 0.5, 0.5)$ and covariance matrix $\sigma_i^2(X'X)^{-1}$, in which σ_i^2 is the variance of the error term ϵ, with σ_i, the standard deviation, generated from the uniform distribution $U(1, 3)$ that corresponds to a range of annual volatilities varying roughly from 10 percent to 48 percent, and $X'X$ is the cross-product matrix of the Fama-French 5-factor regression model.[‡] The 7,770 error terms ϵ_{it} for stock i are generated independently from the normal distribution $N(0, \sigma_i^2)$.

In the simulations we define the abnormal returns with respect to the market model, that is

$$AR_{it} = (r_i - r_f)_t - (\hat{\alpha}_i + \hat{\beta}_i(r_m - r_f)_t),$$

where $\hat{\alpha}_i$ and $\hat{\beta}_i$ are ordinary least squares (OLS) estimates. Therefore missing common factors introduce cross-sectional correlation between the abnormal returns. The event period is ± 10 trading days around the event day $t = 0$ and the estimation period consist of 250 days prior the event periods, i.e., relative days $-260, \ldots, -11$.

[‡]Factor returns have been downloaded from French's data library:
http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

In the experiments we focus on the effect of cross-sectional correlation on the size of the test. Other issues, like event induced volatility, are well reported for example by Kolari & Pynnonen (2010; 2011). Utilizing the base design initiated by Brown & Warner (1985), we generate 1,000 samples of 50 randomly selected stocks (the returns of which are generated by the FF5 model in (5.1)) with four overlapping event days scenarios. In the first case of non-overlapping event days, the returns are cross-sectionally independent. In the second case of completely overlapping events, all firms share the same event day (calendar time), and in the third and fourth scenarios the event days are randomly scattered across 5 and 10 adjacent calendar days, i.e., one and two weeks of trading days, respectively.

We report two-tailed rejection rates for the null hypothesis of no event-effect across different event windows of 0, ± 1, ± 2, ± 5, and ± 10 around the event day, i.e., window lengths $\tau = 1, 3, 5, 11$, and 21 days. In addition to the statistic z_τ in (4.3), we also report results for the more traditional rank based test proposed by Campbell & Wasley (1993: p. 85)

$$z_{cw} = \frac{\sum_{t=\tau_1}^{\tau_2} \bar{k}_t}{\sqrt{\tau} s_{\bar{k}}}, \tag{5.2}$$

where, with $\mathbb{E}\left[K_{it}\right] = (T+1)/2$,

$$\bar{k}_t = \frac{1}{n} \sum_{i=1} \left(K_{it} - \mathbb{E}\left[K_{it}\right]\right) \quad \text{and} \quad s_{\bar{k}}^2 = \frac{1}{T} \sum_{t=T_0+1}^{T_2} \bar{k}_t^2.$$

Moreover, we also report results for a traditional parametric (cross-sectional correlation non-robust) t-statistics popular in event studies, see, e.g., (Campbell, Lo & MacKinlay, 1997: Chapter 4),

$$t_\tau = \frac{\overline{\mathrm{CAR}}(\tau_1, \tau_2)}{\mathrm{s.e}(\mathrm{CAR})}, \tag{5.3}$$

where $\overline{\mathrm{CAR}}(\tau_1, \tau_2)$ is the sample average of $\mathrm{CAR}_i(\tau_1, \tau_2)$ defined in (3.1) and $\mathrm{s.e}(\mathrm{CAR})$ is the related standard error. Under independence the null distribution of t_τ is asymptotically standard normal.

Table 1 summarizes the test statistics and their major features.

Table 1. Test statistics and their key features.

			Robustness due to		
			event	correlation caused by	
Statistic		Type	volatility	complete ovrl	partial ovrl
$z_\tau = \frac{\bar{U}(\tau_1,\tau_2)}{\sigma_\tau \sqrt{1+\delta(n-1)\hat{\bar{\rho}}}}$,	eq. (4.3)	non-parametric	yes	yes	yes
$z_{cw} = \frac{\sum_{t=\tau_1}^{\tau_2} \bar{k}_t}{\tau s_{\bar{k}}}$,	eq. (5.2)	non-parametric	no	yes	no
$t_\tau = \frac{\overline{\mathrm{CAR}}(\tau_1,\tau_2)}{\mathrm{s.e}(\mathrm{CAR})}$,	eq. (5.3)	parametric	yes	no	no

Table 2 reports the simulation results of the two-tailed rejection rates of the null hypothesis of no abnormal return at the 5% nominal rejection rate. The results are clear-cut. Panel A of the table reports the non-overlapping case with zero cross-sectional correlation. All statistics reject close to the nominal rate as would be expected. Panel B reports results of complete overlapping. That is, all events share the same calendar day, and, hence, returns are prone to cross-sectional correlation. The new z_τ and the more traditional cumulative ranks statistic z_{cw}, that both account for cross-sectional correlation, reject reasonably close to the nominal rate up to event windows ± 5 and exhibit some over-rejection on the longest event window ± 10, i.e., 21 days. As expected, the parametric, non cross-correlation robust statistic t_τ incrementally over-rejects as the event windows grows longer. Panel C reports partial overlapping with events clustered randomly within 5 trading days (about a week). For the event day testing also the a priori non-robust statistics in this regard are doing fine by rejecting at the nominal rate. However, they start to incrementally over-reject as the event window gets longer. The a priori partial overlapping robust statistic z_τ rejects close to the nominal rate up to the event windows of 5 days and over-rejects to some extend on the longest event windows of 11 and 21 days; albeit far less than the, in this regard, non-robust statistics z_{cw} and t_τ. The results are pretty much similar with the decreased overlapping in Panel D. Thus, we conclude that accounting for cross-sectional correlation is crucial to avoid biased inferences in statistical testing, not only in the case of complete overlapping of event windows but also in the case of partially overlapping cases. For the latter cases this paper has introduced a new test statistics to account for the dependence.

Table 2. Rejection rates of the null hypothesis of no event effect at the nominal 5% level when the events are non-overlapping, partially overlapping, and completely overlapping.

	CAR window length				
	1	3	5	11	21
	Event day	$(-1,+1)$	$(-2,+2)$	$(-5,+5)$	$(-10,+10)$
Panel A: Non-clustered events					
$z_\tau(\text{SCAR})$	0.048	0.054	0.049	0.052	0.063
$z_{cw}(\text{SCAR})$	0.052	0.050	0.051	0.052	0.063
$t_\tau(\text{CAR})$	0.045	0.035	0.049	0.052	0.048
Panel B: Events clustered on the same trading day					
$z_\tau(\text{SCAR})$	0.059	0.052	0.060	0.064	0.075
$z_{cw}(\text{SCAR})$	0.059	0.052	0.061	0.064	0.075
$t_\tau(\text{CAR})$	0.087	0.091	0.096	0.085	0.110
Panel C: Events clustered on 5 adjacent trading days					
$z_\tau(\text{SCAR})$	0.056	0.055	0.058	0.087	0.080
$z_{cw}(\text{SCAR})$	0.050	0.075	0.093	0.127	0.129
$t_\tau(\text{CAR})$	0.045	0.063	0.077	0.112	0.102
Panel D: Events clustered on 10 adjacent trading days					
$z_\tau(\text{SCAR})$	0.063	0.051	0.062	0.058	0.080
$z_{cw}(\text{SCAR})$	0.059	0.062	0.091	0.116	0.133
$t_\tau(\text{CAR})$	0.065	0.057	0.056	0.089	0.105

6 Conclusion

This paper proposes a new non-parametric rank based test statistic for testing cumulative abnormal returns in short run event studies. The statistic is robust with respect to event induced volatility and cross-sectional correlation due to complete or partially overlapping event windows. This latter source of cross-sectional correlation is not accounted for by the existing non-parametric test statistics. Simulation results indicate that, unlike typically utilized test statistics, the proposed statistic rejects the null hypothesis of no event effect close to the nominal significant level also in the partially overlapping case.

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ON THE KREĬN-VON NEUMANN AND FRIEDRICHS EXTENSION OF POSITIVE OPERATORS

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Dedicated to Seppo Hassi on the occasion of his 60th birthday

1 Introduction

In his profound paper (von Neumann, 1931), J. von Neumann introduced the concept of the adjoint of a densely defined possibly unbounded operator $J : \mathcal{K} \to \mathcal{H}$ between two Hilbert spaces as the operator $J^* : \mathcal{H} \to \mathcal{K}$, having the domain

$$\operatorname{dom} J^* = \{g \in \mathcal{H} : \exists k' \in \mathcal{K} \text{ such that } (Jk \,|\, g) = (k \,|\, k') \ \forall k \in \operatorname{dom} J\},$$

by setting

$$J^* g := k', \quad g \in \operatorname{dom} J^*.$$

Although the adjoint operator behaves nicer than the original one (because it is always closed), it is not necessarily densely defined. An essential question arises therefore: when is the domain $\operatorname{dom} J^*$ a dense subspace of $\mathcal{H}$? Von Neumann himself gave an elegant answer to that question. Namely, he proved that J^* is densely defined if and only if J is a closable operator. Moreover, in that case the second adjoint J^{**} of J exists and it is equal to the closure $\overline{J}$ of J:

$$\overline{J} = J^{**}.$$

At the same time, $J^{**}J^*$ and J^*J^{**} are positive self-adjoint operators in the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Note also that we have

$$\operatorname{dom}(J^{**}J^*)^{1/2} = \operatorname{dom} J^* \quad \text{and} \quad \operatorname{dom}(J^*J^{**})^{1/2} = \operatorname{dom} J^{**}$$

on the domains, and

$$\operatorname{ran}(J^{**}J^*)^{1/2} = \operatorname{ran} J^{**} \quad \text{and} \quad \operatorname{ran}(J^*J^{**})^{1/2} = \operatorname{ran} J^*$$

on the ranges. Here, for a given positive self-adjoint operator A, $A^{1/2}$ denotes the unique positive self-adjoint square root of A; see, e.g., (Sebestyén & Tarcsay, 2017).

However, if J is not closed, then J^*J and JJ^* are not self-adjoint operators in general. In fact, it is not even clear whether those operators are densely defined, and therefore it is also a non-trivial question whether they have any positive self-adjoint extensions at all. From classical works by Friedrichs, Kreĭn, and von Neumann, we know that a densely defined positive and symmetric operator may be extended to a positive self-adjoint operator, see, e.g., (Friedrichs, 1934; Kreĭn, 1947; von Neumann, 1931). In that case, there exist two distinguished self-adjoint extensions A_N and A_F of any positive symmetric operator A, such that

$$A_N \leq A_F,$$

and every positive self-adjoint extension $\widetilde{A}$ of A is between A_N and A_F: $A_N \leq \widetilde{A} \leq A_F$. The smallest extension A_N of A is called the *Kreĭn-von Neumann extension*, while the largest extension A_F of A is called the *Friedrichs extension*.

The problem of the existence of positive self-adjoint extensions has its relevance even in the non-densely defined case. Although the Friedrichs extension exists only for a densely defined operator, the smallest extension always exists if there exists any extension, see, e.g., (Sebestyén & Stochel, 1991) and also (Sebestyén & Stochel, 2007; Hassi, 2004).

In the present paper we revise the main result Theorem 1 of Sebestyén & Stochel (1991) and give some new characterizations for a not necessarily densely defined positive symmetric operator to admit positive self-adjoint extensions. More specifically, in Section 2 we collect some new properties for an operator to be closable. Based on this new characterization of closability, we establish in Section 3 the correct version of the "duality theorem" stated in Jorgensen, Pearse & Tian (2018: Theorem 5). In Section 4 we give a short proof of the fact that the "modulus square" operator T^*T of any densely defined operator T always has a positive self-adjoint extension, cf. (Sebestyén & Tarcsay, 2012: Theorem 2.1). At the same time, we shall see that this is not the case with TT^*; that operator might be even non-closable. However, we are going to establish necessary and sufficient conditions for the extendibility of TT^*. In particular, our construction of the Kreĭn-von Neumann extension in Section 4 will be used to exhibit a counterexample to (Jorgensen, Pearse & Tian, 2018: Theorem 5). Finally, in Section 5 we treat the problem of the existence of the Friedrichs extension of a densely defined positive symmetric operator. In particular, we discuss there the case when the Friedrichs extension of the operator T^*T is identical with T^*T^{**}.

2 Closable operators

Let J be a densely defined operator between the real or complex Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$. Note that J is *closable* if for each sequence $(g_n)_{n\in\mathbb{N}} \subset \operatorname{dom} J$, such that $g_n \to 0$ and $Jg_n \to h$, it follows that $h = 0$. On the other hand, a profound theorem by von Neumann tells us that J is closable if and only if J^* is densely defined, that is,

$$(\operatorname{dom} J^*)^\perp = \{0\}.$$

In the following theorem, we give an extension of von Neumann's result and collect some new characteristic properties for an operator J to be closable. For further characterizations of closability and closedness, see, e.g., (Popovici & Sebestyén, 2014; Sebestyén & Tarcsay, 2016; 2019; 2020).

Theorem 2.1. *Let J be a densely defined operator between the real or complex Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$. Then the following properties are equivalent:*

(i) *J is closable;*

(ii) $(\operatorname{dom} J^*)^\perp = \{0\}$*;*

(iii) $(\operatorname{dom} J^*)^\perp \cap (\operatorname{ran} J)^{\perp\perp} = \{0\}$*;*

(iv) $(\operatorname{dom} J^*)^\perp \subseteq \operatorname{ran}(I + JJ^*)$.

Proof. (i) $\Rightarrow$ (ii) Consider a vector $h \in (\mathrm{dom}\ J^*)^\perp$, then

$$(0,h) \in \big\{\{-J^*k,k\} :\ k \in \mathrm{dom}\ J^*\big\} = \overline{G(J)}.$$

Since J is closable, this implies $h = 0$.

(ii) $\Rightarrow$ (iii) This implication is trivial.

(iii) $\Rightarrow$ (i) Consider a sequence $(g_n)_{n\in\mathbb{N}} \subset \mathrm{dom}\ J$ such that $g_n \to 0$, and $Jg_n \to h$. Then $h \in \overline{\mathrm{ran}\ J} = (\mathrm{ran}\ J)^{\perp\perp}$. On the other hand, for every $f \in \mathrm{dom}\ J^*$

$$(f\,|\,h) = \lim_{n\to\infty}(f\,|\,Jg_n) = \lim_{n\to\infty}(J^*f\,|\,g_n) = 0,$$

which means that $h \in (\mathrm{dom}\ J^*)^\perp$. Consequently, $h = 0$ by (ii) and therefore J is closable.

(ii) $\Rightarrow$ (iv) This implication is clear.

(iv) $\Rightarrow$ (ii) We are going to show that $\mathrm{dom}\ J^*$ is dense in $\mathcal{H}$. To this aim, take $g \in (\mathrm{dom}\ J^*)^\perp$. By (iv), there exists $h \in \mathrm{dom}\ JJ^*$ such that $g = h + JJ^*h$. In particular, $h \in \mathrm{dom}\ J^*$ and one has

$$0 = (g\,|\,h) = (h + JJ^*h\,|\,h) = (h\,|\,h) + (JJ^*h\,|\,h) = \|h\|^2 + \|J^*h\|^2,$$

so that $h = 0$. This implies that $g = 0$ and therefore (iv) implies (ii). $\square$

3 Duality theorems

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with a common vector subspace $\mathcal{D}$. In Jorgensen, Pearse & Tian (2018: Theorem 5) a necessary and sufficient condition is stated for the existence of a positive and self-adjoint operator Δ on $\mathcal{H}_1$ with the duality property

$$(\Delta\varphi\,|\,\psi)_1 = (\varphi\,|\,\psi)_2, \qquad \varphi,\psi \in \mathcal{D},$$

cf. also (Jorgensen & Pearse, 2016: Theorem 4.1). Unfortunately, there is a simple but serious error in their proof and the statement itself is not true in that form either (a counterexample will be exhibited in Example 4.2 below). In Theorem 3.3 we are going to establish the correct form of that statement. Its proof depends on the following lemma.

Lemma 3.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $J : \mathcal{K} \to \mathcal{H}$ be a densely defined linear operator between them. Then the following three statement are equivalent:*

(i) $\mathrm{ran}\ J \subseteq \mathrm{dom}\ J^*$*;*

(ii) *J is closable and* $\mathrm{dom}\ J \subseteq \mathrm{dom}\ J^*J^{**}$*;*

(iii) *there exists a positive self-adjoint operator A in $\mathcal{K}$ such that* $\mathrm{dom}\ J \subseteq \mathrm{dom}\ A$ *and*

$$(Ag\,|\,k) = (Jg\,|\,Jk), \qquad g,k \in \mathrm{dom}\ J. \tag{3.1}$$

Proof. (i) ⇒ (ii) Since $\mathrm{ran}\ J \subseteq \mathrm{dom}\ J^*$, it follows that

$$(\mathrm{ran}\ J)^{\perp\perp} \subseteq (\mathrm{dom}\ J^*)^{\perp\perp} = \mathcal{H} \ominus (\mathrm{dom}\ J^*)^{\perp},$$

and, consequently,

$$(\mathrm{dom}\ J^*)^{\perp} \cap (\mathrm{ran}\ J)^{\perp\perp} = \{0\}.$$

Applying Theorem 2.1 we see that J is closable. On the other hand, $\mathrm{ran}\ J \subseteq \mathrm{dom}\ J^*$ implies that $\mathrm{dom}\ J = \mathrm{dom}\ J^*J \subseteq \mathrm{dom}\ J^*J^{**}$.

(ii) ⇒ (iii) If J is closable, then $A := J^*J^{**}$ is a positive self-adjoint operator in $\mathcal{H}$, and by (ii) one has $\mathrm{dom}\ J \subseteq \mathrm{dom}\ A$. On the other hand,

$$(Ag \,|\, k) = (J^*J^{**}g \,|\, k) = (J^{**}g \,|\, J^{**}k) = (Jg \,|\, Jk),$$

for every $g, k \in \mathrm{dom}\ J$.

(iii) ⇒ (i) Suppose that A is a positive operator with $\mathrm{dom}\ J \subseteq \mathrm{dom}\ A$ which satisfies (3.1). Let $k \in \mathrm{dom}\ J$ be arbitrary, then for every $g \in \mathrm{dom}\ J$

$$(Jg \,|\, Jk) = (Ag \,|\, k) = (g \,|\, Ak),$$

which implies $Jk \in \mathrm{dom}\ J^*$. Therefore, $\mathrm{ran}\ J \subseteq \mathrm{dom}\ J^*$. □

Remark 3.2. Let J be a closed operator. Then the inclusion

$$\mathrm{dom}\ J \subseteq \mathrm{dom}\ J^*J^{**} \tag{3.2}$$

is only possible if J is continuous and everywhere defined on $\mathcal{H}_1$, see, e.g., (Tarcsay, 2012: Lemma 2.1). This suggests that Lemma 3.1 is only relevant if J is a closable but not a closed operator.

The erroneous observation in the proof of (Jorgensen, Pearse & Tian, 2018: Theorem 5) is that (3.2) holds true provided that both J and J^* are densely defined. This makes it necessary to provide the following correct version of (Jorgensen, Pearse & Tian, 2018: Theorem 5), which can also be considered as a noncommutative version of the Lebesgue-Radon-Nikodym decomposition theorem.

Theorem 3.3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real or complex Hilbert spaces which contain a common linear manifold $\mathcal{D}$ as a vector space. Suppose that $\mathcal{D}$ is dense in $\mathcal{H}_1$ and set*

$$\mathcal{D}^* := \big\{h \in \mathcal{H}_2 : \exists C_h \geq 0 \text{ such that } |(\varphi \,|\, h)_2| \leq C_h \|\varphi\|_1 \ \ \forall \varphi \in \mathcal{D}\big\}.$$

Then the following two conditions are equivalent:

(i) *$\mathcal{D} \subseteq \mathcal{D}^*$ in $\mathcal{H}_2$;*

(ii) *there exists a positive self-adjoint operator Δ in $\mathcal{H}_2$ such that $\mathcal{D} \subseteq \mathrm{dom}\ \Delta$ in $\mathcal{H}_1$ and*

$$(\Delta\varphi \,|\, \psi)_1 = (\varphi \,|\, \psi)_2, \qquad \varphi, \psi \in \mathcal{D}. \tag{3.3}$$

Proof. Let J be the operator from $\mathcal{D} \subseteq \mathcal{H}_1$ to $\mathcal{H}_2$ defined by the identification $J\varphi := \varphi$, $\varphi \in \mathcal{D}$. Then J is a densely defined operator such that its adjoint J^* has domain $\mathcal{D}^*$: $\mathrm{dom}\ J^* = \mathcal{D}^*$. The desired equivalence follows now from Lemma 3.1. □

4 Von Neumann's problem on positive self-adjoint extendibility

Given a positive symmetric operator A in a real or complex Hilbert space $\mathcal{K}$, the question arises whether there exists a positive self-adjoint extension of A. If the operator in question is densely defined, then we know from classical papers by Friedrichs, Kreĭn, and von Neumann that the operator has a positive self-adjoint extension, see (Friedrichs, 1934; Kreĭn, 1947; von Neumann, 1931); cf. also (Ando & Nishio, 1970; Arlinskiĭ et al., 2001; Prokaj & Sebestyén, 1996a;b; Schmüdgen, 2012). However, uniqueness of the extension occurs only in the very special case when the operator in question is essentially self-adjoint. In all other cases, the set of positive self-adjoint extensions is an operator interval $[A_N, A_F]$, where A_N is the smallest (the so-called Kreĭn-von Neumann) extension, while A_F is the largest (the so-called Friedrichs) extension of A. Recall that the partial ordering among the set of positive self-adjoint operators is given by

$$A \leq B \qquad \Longleftrightarrow \qquad (I+B)^{-1} \leq (I+A)^{-1}.$$

Equivalently, by means of the square roots, one has $A \leq B$ if and only if

$$\operatorname{dom} B^{1/2} \subseteq \operatorname{dom} A^{1/2} \quad \text{and} \quad \|A^{1/2}k\|^2 \leq \|B^{1/2}k\|^2, \qquad \forall k \in \operatorname{dom} B^{1/2}.$$

The problem of the existence of positive self-adjoint extensions has its relevance even in the non-densely defined case, and was treated in detail by Sebestyén & Stochel (1991), see also (Sebestyén & Stochel, 2007; Hassi, 2004).

In the next result we revise (Sebestyén & Stochel, 1991: Theorem 1) on the existence of the Kreĭn-von Neumann extension of a positive and symmetric operator A. In this case it is convenient to introduce the linear space $\mathcal{D}_*(A)$ by

$$\mathcal{D}_*(A) := \{k \in \mathcal{K} : \, \sup\{|(Ag\,|\,k)| : \, g \in \operatorname{dom} A, \, (Ag\,|\,g) \leq 1\} < +\infty\}. \tag{4.1}$$

Theorem 4.1. *Let A be a positive and symmetric operator on a real or complex Hilbert space $\mathcal{K}$. Then the following statements are equivalent:*

(i) $\mathcal{D}_*(A)$ *as in* (4.1) *is dense in* $\mathcal{K}$*;*

(ii) *for every sequence* $(g_n)_{n\in\mathbb{N}} \subset \operatorname{dom} A$ *and* $k \in \mathcal{K}$ *such that*

$$(Ag_n\,|\,g_n)_{\mathcal{K}} \to 0 \quad \textit{and} \quad Ag_n \to k,$$

it follows that $k = 0$*;*

(iii) *there exist a Hilbert space* $\mathcal{E}$ *and a densely defined linear operator* $V : \mathcal{K} \to \mathcal{E}$ *such that* $\operatorname{dom} A \subseteq \operatorname{dom} V$, $(V(\operatorname{dom} A))^{\perp} = \{0\}$, *and*

$$\langle Vg, Vh\rangle_{\mathcal{E}} = (Ag\,|\,h)_{\mathcal{K}}, \qquad g \in \operatorname{dom} A, \, h \in \operatorname{dom} V; \tag{4.2}$$

(iv) *there exists a positive self-adjoint extension of* A*.*

If any, and hence all, assertions of (i)-(iv) *are satisfied, then there exists the smallest positive extension* A_N *of* A*.*

Proof. (i) $\Rightarrow$ (ii) Assume that $(Ag \,|\, g) = 0$ for some $g \in \text{dom } A$. Then $\sup |(Ag, k)| < \infty$ for all $k \in \mathcal{D}_*(A)$, which implies $(Ag, k) = 0$. Since $\mathcal{D}_*(A) \subseteq \mathcal{K}$ is dense by (i), it follows that $Ag = 0$.

This means that

$$\langle Ag, Ah \rangle_{\mathcal{E}} := (Ag \,|\, h)_{\mathcal{K}}, \qquad g, h \in \text{dom } A, \tag{4.3}$$

defines an inner product on $\text{ran } A$. Denote by $\mathcal{E}$ the completion of that space and consider the natural inclusion operator $J_A : \mathcal{E} \supseteq \text{ran } A \to \mathcal{K}$,

$$J_A(Ag) := Ag \in \mathcal{K}, \qquad g \in \text{dom } A. \tag{4.4}$$

Clearly, $\text{ran } A$ forms a dense linear manifold in $\mathcal{E}$ by definition, so that J_A is densely defined. On the other hand, one has

$$\text{dom } J_A^* = \mathcal{D}_*(A), \tag{4.5}$$

thanks to the identities

$$(J_A(Ag) \,|\, h)_{\mathcal{K}} = (Ag \,|\, h)_{\mathcal{K}}, \qquad g \in \text{dom } A,\ h \in \mathcal{K},$$

and

$$\langle Ag, Ag \rangle_{\mathcal{E}} = (Ag \,|\, g)_{\mathcal{K}}, \qquad g \in \text{dom } A.$$

From (4.5) and (i) we see that J_A^* is densely defined and therefore J_A is closable. From this it follows that A fulfills (ii).

(ii) $\Rightarrow$ (iii) Note that the condition in (ii) implies that (4.3) defines an inner product. With the notation as in the proof of the implication (i) $\Rightarrow$ (ii), (ii) expresses that the canonical inclusion operator $J_A : \mathcal{E} \to \mathcal{K}$ is closable. Its adjoint $J_A^* : \mathcal{K} \to \mathcal{E}$ is therefore a densely defined operator such that

$$\langle J_A^* g, Ah \rangle_{\mathcal{E}} = (g \,|\, J_A(Ah))_{\mathcal{K}} = (g \,|\, Ah)_{\mathcal{K}} = \langle Ag, Ah \rangle_{\mathcal{E}}, \qquad g, h \in \text{dom } A,$$

whence we conclude that

$$J_A^* g = Ag \in \mathcal{E}, \qquad g \in \text{dom } A. \tag{4.6}$$

As a conseqence, J_A^* provides a factorization for A in the sense of (iii):

$$\langle J_A^* g, J_A^* h \rangle_{\mathcal{E}} = \langle Ag, J_A^* h \rangle_{\mathcal{E}} = (Ag \,|\, h)_{\mathcal{K}}, \qquad g \in \text{dom } A,\ h \in \mathcal{D}_*(A). \tag{4.7}$$

Moreover, by (4.6) we see that

$$J_A^*(\text{dom } A) = \{Ag : \, g \in \text{dom } A\},$$

where the right-hand side is dense in $\mathcal{H}$ by definition. Hence, $V := J_A^*$ satisfies all requirements of (iii).

(iii) $\Rightarrow$ (iv) Let $V : \mathcal{K} \to \mathcal{E}$ be a densely defined closable operator satisfying the properties in (iii). By (4.2) we conclude that $Vg \in \text{dom } V^*$ for every $g \in \text{dom } A$ and that

$$V^* V g = Ag, \qquad g \in \text{dom } A. \tag{4.8}$$

This means that $\text{dom } V^*$ includes the dense set $V(\text{dom } A)$, and therefore V is closable. Moreover, by (4.8) we see that $A \subset V^* V \subset V^* V^{**}$, i.e., the positive self-adjoint operator $V^* V^{**}$ extends A.

(iv) $\Rightarrow$ (i): Let B be a positive self-adjoint extension of A. Then for every $k \in \mathrm{dom}\ B^{1/2}$ and $g \in \mathrm{dom}\ A$ with $(Ag\,|\,g) \leq 1$, we obtain that

$$\begin{aligned}|(Ag\,|\,k)| &= |(Bg\,|\,k)| = |(B^{1/2}g\,|\,B^{1/2}k)| \\ &\leq \|B^{1/2}g\|\|B^{1/2}k\| = (Ag\,|\,g)^{1/2}\|B^{1/2}k\| \leq \|B^{1/2}k\|,\end{aligned}$$

whence $k \in \mathcal{D}_*(A)$. This implies that

$$\mathrm{dom}\ B^{1/2} \subseteq \mathcal{D}_*(A), \tag{4.9}$$

where the former subspace is dense in $\mathcal{K}$. Hence, $\mathcal{D}_*(A)$ is dense in $\mathcal{K}$, i.e., (i) holds.

Finally, let any, and hence all, assertions of (i)-(iv) be satisfied. First note that the operator J_A defined in (4.4) is closable by (i). Hence, from (4.6) and (4.7) it follows that

$$A_N := J_A^{**}J_A^* \tag{4.10}$$

is a positive self-adjoint extension of A. Furthermore we have

$$\mathcal{D}_*(A) = \mathrm{dom}\ J_A^* = \mathrm{dom}\,(J_A^{**}J_A^*)^{1/2} \tag{4.11}$$

and the density of $\mathrm{ran}\ A$ in $\mathcal{H}$ implies for every $k \in \mathcal{D}_*(A)$ that

$$\begin{aligned}\|(J_A^{**}J_A^*)^{1/2}k\|_{\mathcal{K}}^2 &= \|J_A^*k\|_{\mathcal{E}}^2 \\ &= \sup\big\{|\langle Ag, J_A^*k\rangle_{\mathcal{E}}|^2 : g \in \mathrm{dom}\ A, \langle Ag, Ag\rangle_{\mathcal{E}} \leq 1\big\} \\ &= \sup\big\{|(J_A(Ag)\,|\,k)_{\mathcal{K}}|^2 : g \in \mathrm{dom}\ A,\ (Ag\,|\,g)_{\mathcal{K}} \leq 1\big\} \\ &= \sup\big\{|(Ag\,|\,k)_{\mathcal{K}}|^2 : g \in \mathrm{dom}\ A,\ (Ag\,|\,g)_{\mathcal{K}} \leq 1\big\}.\end{aligned}$$

Next we show that A_N as in (4.10) is the smallest self-adjoint extension of A. Let therefore B be any positive self-adjoint extension of A. Since the positive self-adjoint operator B has no proper self-adjoint extension, applying the above construction for B, we infer that $B = J_B^{**}J_B^*$. By the inclusion (4.9) we have $\mathrm{dom}\ B^{1/2} \subseteq \mathrm{dom}\ A_N^{1/2}$, see (4.10) and (4.11), and from the above calculation we obtain that, for every $k \in \mathrm{dom}\ B^{1/2}$,

$$\begin{aligned}\|A_N^{1/2}k\|^2 = \|(J_A^{**}J_A^*)^{1/2}k\|^2 &= \sup\big\{|(Ag\,|\,k)|^2 : g \in \mathrm{dom}\ A,\ (Ag\,|\,g) \leq 1\big\} \\ &\leq \sup\big\{|(Bg\,|\,k)|^2 : g \in \mathrm{dom}\ B,\ (Bg\,|\,g) \leq 1\big\} \\ &= \|(J_B^{**}J_B^*)^{1/2}k\|_{\mathcal{K}}^2 = \|B^{1/2}k\|_{\mathcal{K}}^2.\end{aligned}$$

Hence $A_N \leq B$, as it is stated. $\square$

As was mentioned in the previous section, the statement of (Jorgensen, Pearse & Tian, 2018: Theorem 5) is not correct, as with the notation used in Theorem 3.2, they assert that the existence of the positive self-adjoint operator Δ satisfying (3.3) is equivalent to $\mathcal{D}^*$ being dense in $\mathcal{H}_2$. Based on the preceding theorem and its proof, it will be shown by a counterexample that their assertion is not true in general.

Example 4.2. Consider an unbounded positive self-adjoint operator A in a Hilbert space $\mathcal{K}$ and set

$$\mathcal{D} := \mathrm{ran}\ A.$$

Denote by $\mathcal{E}$ the "energy space" associated with A and by J the corresponding inclusion operator $J : \mathcal{E} \supseteq \operatorname{ran} A \to \mathcal{K}$ as in the proof of Theorem 4.1. Then $\mathcal{D}$ is a common vector subspace of $\mathcal{E}$ and $\mathcal{K}$ such that $\mathcal{D} \subseteq \mathcal{E}$ is dense. Furthermore,

$$\begin{aligned}\mathcal{D}^* :=& \{k \in \mathcal{K} : \exists C_k \geq 0 \text{ such that } |(\varphi \,|\, k)_{\mathcal{K}}^2| \leq C_h \|\varphi\|_{\mathcal{E}}^2 \;\; \forall \varphi \in \mathcal{D}\} \\ =& \{k \in \mathcal{K} : \exists C_k \geq 0 \text{ such that } |(Ah \,|\, k)_{\mathcal{H}}^2| \leq C_k (Ah \,|\, h)_{\mathcal{K}} \;\; \forall h \in \operatorname{dom} A\},\end{aligned}$$

from which we conclude that

$$\mathcal{D}^* = \mathcal{D}_*(A) = \operatorname{dom} J^*,$$

so $\mathcal{D}^* \subseteq \mathcal{K}$ is dense. Suppose that the conclusion of (Jorgensen, Pearse & Tian, 2018: Theorem 5) is true, then by that theorem the density of $\mathcal{D}^*$ in $\mathcal{K}$ implies that there exists a positive self-adjoint operator $\Delta : \mathcal{E} \to \mathcal{E}$, $\mathcal{D} \subseteq \operatorname{dom} \Delta$, such that

$$\langle \Delta\varphi, \varphi \rangle_{\mathcal{E}} = (\varphi \,|\, \varphi)_{\mathcal{K}}, \qquad \varphi \in \mathcal{D}.$$

From this we conclude that

$$(J(Ah) \,|\, Ak)_{\mathcal{K}} = (Ah \,|\, Ak)_{\mathcal{K}} = \langle Ah, \Delta(Ak) \rangle_{\mathcal{E}}, \qquad h \in \operatorname{dom} A,$$

which in turn means that $Ak \in \operatorname{dom} J^*$ and $J^*(Ak) = \Delta(Ak)$. As a consequence we see that $\operatorname{ran} A \subseteq \operatorname{dom} J^*$, and since $\operatorname{dom} A \subseteq \operatorname{dom} J^*$ holds true as well, we obtain that

$$\mathcal{K} = \operatorname{dom} A + \operatorname{ran} A \subseteq \operatorname{dom} J^*.$$

So J^* is an everywhere defined bounded operator on $\mathcal{K}$, and therefore so is $A = J^{**}J^*$. This is in contradiction to the assumption that A is an unbounded operator.

Thanks to a classical result of J. von Neumann (von Neumann, 1931) we know that T^*T and TT^* are positive self-adjoint operators whenever T is densely defined and closed. In Sebestyén & Tarcsay (2014) we proved the converse of that statement: if both T^*T and TT^* are self-adjoint then T is necessarily closed, see also (Gesztesy & Schmüdgen, 2019) and (Sandovici, 2018) for the case of linear relations. This means that if T is not closed (or not even closable), then either T^*T or TT^* (or even both) fail to be self-adjoint. In fact, TT^* might even be non-closable; however, surprisingly, T^*T behaves well. Namely, it was proved in Sebestyén & Tarcsay (2012: Theorem 2.1) that T^*T always has a positive self-adjoint extension. We provide a short proof of that result.

Theorem 4.3. *Let $T : \mathcal{K} \to \mathcal{H}$ be a densely defined linear operator between the real or complex Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$. Then T^*T has a positive self-adjoint extension.*

Proof. Consider the positive symmetric operator $A := T^*T$. We are going to show that

$$\operatorname{dom} T \subseteq \mathcal{D}_*(A).$$

Indeed, for $g \in \operatorname{dom} A$ and $k \in \operatorname{dom} T$, we have

$$\begin{aligned}|(Ag \,|\, k)|^2 &= |(T^*Tg \,|\, k)|^2 = |(Tg \,|\, Tk)|^2 \leq (Tg \,|\, Tg)(Tk \,|\, Tk) \\ &= (T^*Tg \,|\, g)(Tk \,|\, Tk) = (Ag \,|\, g)(Tk \,|\, Tk).\end{aligned}$$

Hence $\mathcal{D}_*(A)$ is dense in $\mathcal{K}$. Thus A has by Theorem 4.1 a positive self-adjoint extension. $\square$

In the next result we deal with the positive extendibility of TT^*.

Theorem 4.4. *Let $T : \mathcal{K} \to \mathcal{H}$ be a densely defined operator between the real or complex Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$. Then the following two statements are equivalent:*

(i) *TT^* has a positive self-adjoint extension;*

(ii) *$T|_{\mathrm{dom}\, T \cap \mathrm{ran}\, T^*}$ is a closable operator.*

Proof. The positive symmetric operator $A \coloneqq TT^*$ has a positive self-adjoint extension if and only if it satisfies condition (ii) of Theorem 4.1. That is, according to that result, T^*T has a positive self-adjoint extension if and only if for every sequence $(h_n) \subset \mathrm{dom}\, TT^*$ and every vector $f \in \mathcal{H}$ the conditions

$$(TT^*h_n \,|\, h_n) = \|T^*h_n\|^2 \to 0 \qquad \text{and} \qquad TT^*h_n \to f,$$

imply that $f = 0$. Evidently, this is equivalent to the closability of the restriction of T to the set $\mathrm{ran}\, T^* \cap \mathrm{dom}\, T$. $\square$

In the following example, we show that TT^* may have a bounded positive self-adjoint extension in some cases even if T is not even closable.

Example 4.5. Let $\mathcal{K}$ be a separable Hilbert space and consider two orthonormal bases in it

$$\{e_{n,m} : n, m \in \mathbb{N}\} \quad \text{and} \quad \{f_n : n \in \mathbb{N}\}.$$

Let us define the operator T on the vectors $e_{n,m}$ by setting

$$Te_{n,m} \coloneqq mf_n, \qquad n, m \in \mathbb{N},$$

and then extend it by linearity to $\mathrm{dom}\, T \coloneqq \mathrm{span}\,\{e_{n,m} : n, m \in \mathbb{N}\}$. It follows from this definition that $\mathrm{dom}\, T^* = \{0\}$. In order to see this, observe that for $z \in \mathrm{dom}\, T^*$ and $n \in \mathbb{N}$ we have

$$(z \,|\, f_n) = \frac{1}{m}(z \,|\, Te_{n,m}) = \frac{1}{m}(T^*z \,|\, e_{n,m}),$$

for any $m \in \mathbb{N}$. Letting $m \to \infty$ gives that $(z \,|\, f_n) = 0$ and, hence, $z = 0$. Consequently, T is non-closable (in fact, T is a maximal singular operator), but $A = 0$ is a (bounded) positive self-adjoint extension of TT^*.

The previous example demonstrated that TT^* can behave nicely even though T is singular. However, as the following example illustrates, there exists an operator T such that TT^* is non-closable.

Example 4.6. Consider a maximal singular operator T in a Hilbert space $\mathcal{K}$, that is, an operator such that $\mathrm{dom}\, T^* = \{0\}$ (take e.g. the operator T from Example 4.5). Consider the following operator

$$J : \mathcal{K} \supseteq \mathrm{dom}\, T \to \mathcal{K} \times \mathcal{K}, \qquad Jg \coloneqq \{g, Tg\}.$$

Then it is easy to verify that $\mathrm{dom}\, J^* = \mathcal{K} \times \mathrm{dom}\, T^* = \mathcal{K} \times \{0\}$, and $J^*\{k, 0\} = k$. In particular, $\mathrm{dom}\, JJ^* = \mathrm{dom}\, T \times \{0\}$, and

$$JJ^*\{g, 0\} = \{g, Tg\}, \qquad g \in \mathrm{dom}\, T.$$

Furthermore, we claim that J is not closable. Indeed, take any nonzero $k \in \mathcal{K}$, then there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $\operatorname{dom} T$ such that $g_n \to 0$ and $Tg_n \to k$. Then

$$JJ^*\{g_n, 0\} = \{g_n, Tg_n\} \to \{0, k\},$$

which means that JJ^* may not be closable.

Theorem 4.7. *Let $T : \mathcal{K} \to \mathcal{H}$ be a densely defined closable linear operator, such that*

$$\operatorname{dom} T \subseteq \operatorname{ran} T^*. \tag{4.12}$$

Then $T^{**}T^*$ agrees with the Kreĭn-von Neumann extension of TT^*.

Proof. Denote by $\mathcal{E}$ the completion of $\operatorname{ran} TT^*$ under the inner product

$$\langle TT^*h, TT^*f\rangle := (TT^*h \,|\, f) = (T^*h \,|\, T^*f), \qquad h, f \in \operatorname{dom} TT^*.$$

By the construction of the proof of Theorem 4.1, the Kreĭn-von Neumann extension of TT^* is of the form $J^{**}J^*$, where J is the natural inclusion operator from $\mathcal{E} \supseteq \operatorname{ran} TT^*$ into $\mathcal{H}$:

$$J(TT^*h) := TT^*h, \qquad h \in \operatorname{dom} TT^*.$$

Note that by (4.12) we have the identity $\operatorname{dom} T = \{T^*g : \, g \in \operatorname{dom} TT^*\}$. Consequently,

$$\begin{aligned}
\operatorname{dom}(J^{**}J^*)^{1/2} &= \operatorname{dom} J^* = \mathcal{D}_*(TT^*)\\
&= \Big\{h \in \mathcal{H} : \, \sup\Big\{|(TT^*f \,|\, h)| : \, f \in \operatorname{dom} TT^*, \, \|T^*f\|^2 \leq 1\Big\} < +\infty\Big\}\\
&= \operatorname{dom} T^* = \operatorname{dom}(T^{**}T^*)^{1/2}.
\end{aligned}$$

At the same time we have that

$$\begin{aligned}
\|(J^{**}J^*)^{1/2}h\|^2 &= \|J^*h\|^2_{\mathcal{E}}\\
&= \sup\big\{|\langle TT^*f, J^*h\rangle^2| : \, f \in \operatorname{dom} TT^*, \, \|T^*f\|^2 \leq 1\big\}\\
&= \sup\big\{|(T^*f \,|\, T^*h)|^2 : \, f \in \operatorname{dom} TT^*, \, \|T^*f\|^2 \leq 1\big\}\\
&= \|T^*h\|^2,
\end{aligned}$$

for every $h \in \operatorname{dom} T^*$. We have therefore proved that $T^{**}T^* \leq J^{**}J^*$, and since $J^{**}J^*$ is the smallest positive self-adjoint extension of TT^*, we obtain that $T^{**}T^* = J^{**}J^*$. $\square$

5 The Friedrichs extension

A densely defined positive symmetric operator A on a real or complex Hilbert space $\mathcal{K}$ always has a positive self-adjoint extension. Indeed, the generalized Schwarz inequality

$$|(Ag \,|\, h)|^2 \leq (Ag \,|\, g)(Ah \,|\, h), \qquad g, h \in \operatorname{dom} A$$

implies that $\operatorname{dom} A \subseteq \mathcal{D}_*(A)$ and, therefore, A admits a positive self-adjoint extension according to Theorem 4.1. In particular, by the same theorem, the Kreĭn-von Neumann extension A_N of A

exists. In that case it is known that the so-called Friedrichs extension, that is, the largest positive self-adjoint extension, exists as well. Using the procedure described in Theorem 4.1, we prove the existence of the Friedrichs extension. Our method is very similar to that of Prokaj & Sebestyén (1996a), but somewhat simpler.

Theorem 5.1. *Let A be a densely defined positive symmetric operator in the real or complex Hilbert space $\mathcal{K}$. Then there exists the largest positive self-adjoint extension A_F of A.*

Proof. Let us recall the construction of the proof of Theorem 4.1 and consider the energy Hilbert space $\mathcal{E}$ and the inclusion operator $J_A : \mathcal{E} \supseteq \operatorname{ran} A \to \mathcal{K}$ defined by

$$J_A(Ag) := Ag, \qquad g \in \operatorname{dom} A.$$

By (4.5) we have $\operatorname{dom} J_A^* = \mathcal{D}_*(A) \supseteq \operatorname{dom} A$, and therefore we may consider the restriction Q_A of J_A^* to $\operatorname{dom} A$, i.e.,

$$Q_A := J_A^*|_{\operatorname{dom} A}.$$

By (4.6),

$$Q_A g = Ag \in \mathcal{E}, \qquad g \in \operatorname{dom} A.$$

On the other hand, from $Q_A \subset J_A^*$ we get $J_A^{**} \subset Q_A^*$ and $Q_A^{**} \subset J_A^*$, whence it follows that $A_F := Q_A^* Q_A^{**}$ is a positive self-adjoint extension of A. We claim that A_F is the largest among the set of all positive self-adjoint extensions of A. Indeed, let $B \supset A$ be any positive self-adjoint extension of A. Repeating the above process we apparently have $B = Q_B^* Q_B^{**}$. Then

$$\begin{aligned}\operatorname{dom}(Q_A^* Q_A^{**})^{1/2} &= \operatorname{dom} Q_A^{**} = \operatorname{dom} \overline{Q_A}\\ &= \{k \in \mathcal{K} : \exists (k_n)_{n\in\mathbb{N}} \subset \operatorname{dom} A,\, k_n \to k,\, (A(k_n - k_m) \,|\, k_n - k_m) \to 0\},\end{aligned}$$

and, accordingly,

$$\begin{aligned}\operatorname{dom}(Q_B^* Q_B^{**})^{1/2} &= \{k \in \mathcal{K} : \exists (k_n)_{n\in\mathbb{N}} \subset \operatorname{dom} B,\, k_n \to k,\, (B(k_n - k_m) \,|\, k_n - k_m) \to 0\}\\ &\supseteq \{k \in \mathcal{K} : \exists (k_n)_{n\in\mathbb{N}} \subset \operatorname{dom} A,\, k_n \to k,\, (A(k_n - k_m) \,|\, k_n - k_m) \to 0\}\\ &= \operatorname{dom}(Q_A^* Q_A^{**})^{1/2}.\end{aligned}$$

Finally, for $k \in \operatorname{dom}(Q_A^* Q_A^{**})^{1/2} \subseteq \operatorname{dom}(Q_B^* Q_B^{**})^{1/2}$, take $(k_n)_{n\in\mathbb{N}} \subset \operatorname{dom} A$ such that

$$k_n \to k \quad \text{and} \quad (A(k_n - k_m) \,|\, k_n - k_m) \to 0,$$

then $Q_A k_n \to Q_A^{**} k$ in $\mathcal{E}$, and hence

$$\|(A_F)^{1/2} k\|^2 = (Q_A^* Q_A^{**})^{1/2} k\|^2 = \|Q_A^{**} k\|_{\mathcal{E}}^2 = \lim_{n\to\infty} \|Q_A k_n\|_{\mathcal{E}}^2 = \lim_{n\to\infty} (Ak_n \,|\, k_n).$$

Moreover, since $B \supset A$,

$$\|B^{1/2} k\|^2 = \|(Q_B^* Q_B^{**})^{1/2} k\|^2 = \lim_{n\to\infty} (Bk_n \,|\, k_n) = \lim_{n\to\infty} (Ak_n \,|\, k_n).$$

As a consequence we see that $A_F \geq B$, as desired. □

Theorem 5.2. *Let $T : \mathcal{K} \to \mathcal{H}$ be a densely defined linear operator satisfying*

$$\operatorname{ran} T \subseteq \operatorname{dom} T^*. \tag{5.1}$$

Then T is closable and the Friedrichs extension of the positive symmetric operator T^*T is equal to T^*T^{**}:

$$(T^*T)_F = T^*T^{**}. \tag{5.2}$$

Proof. Condition (5.1) guarantees, according to Lemma 3.1, that T is closable. Hence, T^{**} exists and T^*T^{**} is a positive self-adjoint extension of T^*T, thanks to von Neumann, see (Schmüdgen, 2012: Proposition 3.18). Our duty is therefore to establish identity (5.2). To this end we need only to prove the domain inclusion

$$\mathrm{dom}\,(T^*T)_F^{1/2} \supseteq \mathrm{dom}\,(T^*T^{**})^{1/2}, \tag{5.3}$$

because we know that $\mathrm{dom}\,(T^*T)_F^{1/2} \subseteq \mathrm{dom}\,(T^*T^{**})^{1/2}$ and that

$$\|(T^*T)_F^{1/2}k\|^2 = \|(T^*T^{**})^{1/2}k\|^2, \qquad k \in \mathrm{dom}\,(T^*T)_F^{1/2},$$

see the proof of Theorem 5.1. First we note that

$$\mathrm{dom}\,(T^*T^{**})^{1/2} = \mathrm{dom}\,T^{**} = \{k \in \mathcal{K} : \exists (k_n)_{n\in\mathbb{N}} \subset \mathrm{dom}\,T,\, k_n \to k,\, Tk_n - Tk_m \to 0\}.$$

Recalling the proof of Theorem 5.1, let us denote by $\mathcal{E}$ the "energy space" associated with T^*T, that is, the completion of $\mathrm{ran}\,T^*T$ endowed with the inner product

$$\langle T^*Tk, T^*Tf\rangle := (Tk\,|\,Tf), \qquad k, f \in \mathrm{dom}\,T^*T.$$

Consider the operator $Q : \mathcal{K} \to \mathcal{E}$ given by $\mathrm{dom}\,Q = \mathrm{dom}\,T^*T = \mathrm{dom}\,T$,

$$Q(T^*Tk) := T^*Tk \in \mathcal{E}, \qquad k \in \mathrm{dom}\,T,$$

then we have $(T^*T)_F = Q^*Q^{**}$, again according to the proof of Theorem 5.1. Consequently, the domain $\mathrm{dom}\,(T^*T)_F^{1/2}$ can be described as follows:

$$\begin{aligned}
\mathrm{dom}\,(T^*T)_F^{1/2} &= \mathrm{dom}\,(Q^*Q^{**})^{1/2} = \mathrm{dom}\,Q^{**} \\
&= \{k \in \mathcal{K} : \exists (k_n)_{n\in\mathbb{N}} \subset \mathrm{dom}\,T,\, k_n \to k,\, \|T^*Tk_n - T^*Tk_m\|_{\mathcal{E}}^2 \to 0\} \\
&= \{k \in \mathcal{K} : \exists (k_n)_{n\in\mathbb{N}} \subset \mathrm{dom}\,T,\, k_n \to k,\, \|Tk_n - Tk_m\|_{\mathcal{K}}^2 \to 0\} \\
&= \mathrm{dom}\,T^{**} = \mathrm{dom}\,(T^*T^{**})^{1/2}.
\end{aligned}$$

This proves identity (5.3) and therefore $(T^*T)_F = T^*T^{**}$, as is claimed. □

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THE CHARACTERIZATION OF BROWNIAN MOTION AS AN ISOTROPIC I.I.D.-COMPONENT LÉVY PROCESS

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This work is dedicated to Professor Seppo Hassi for his 60th birthday

1 Introduction

The Brownian motion is arguably the most important stochastic process there is. It has a long history in particle physics dating back to at least the Roman poet and philosopher Lucretius and his scientific poem *De rerum natura* ca. 50 BC; for a recent English translation, see Slavitt (2008). Since that time the Brownian motion has proven to be central in such diverse fields as physics, economics, quantitative finance, and evolutionary biology, just to mention a few.

The process is christened "Brownian motion" to honor the pioneering work of the Scottish botanist Robert Brown in his work in 1827 on pollen movement in water; although it is obvious that Brown was not the first one to observe the Brownian motion. The first mathematical study of the Brownian motion is apparently by the Danish astronomer Thorvald Nicolai Thiele in 1880, see Lauritzen (1981) for a discussion of Thiele's work.

The Brownian motion is also sometimes called the "Wiener process" in honor of Norbert Wiener for his pioneering contributions to the mathematical study of the process. To honor Wiener, we use the symbol W for the Brownian motion.

From a mathematical and modeling point of view, the Brownian motion is extremely convenient. It belongs to the intersection of many mathematical models: it is Gaussian, it is Markovian, it is a Lévy process, it is a martingale, and it is a self-similar process.

The Brownian motion has many characterizations. It is, for example, the scaling limit of random walks, the independent-increment stationary-increment Gaussian process, the $\frac{1}{2}$-self-similar stationary-increment Gaussian process, the Markov process with Laplacian as its generator, the continuous Lévy process, or the continuous local martingale with the (raw) bracket $[W]_t = t$. Of all the characterizations of the Brownian motion, let us just give the shortest one here.

Definition 1.1 (The Brownian motion as a Gaussian process)**.** A d-dimensional centered stochastic process $W = (W^1, \ldots, W^d)$ with $W_0 = 0$ is the *Brownian motion* if it is Gaussian with variance-covariance matrix given by $\mathbb{E}[W_t^i W_s^i] = (t \wedge s)\delta_{ij}$, where $t \wedge s = \min(t, s)$ and

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Definition 1.1 is of course concise and opaque. We will have better definitions later. Nevertheless, let us show that Definition 1.1 is not vacuous, i.e., such a process does indeed exist. We do this by using the Karhunen-Loève type expansion by which the Brownian motion is constructed as the *isonormal Gaussian process* on the separable Hilbert space $L^2([0, T])$.

Theorem 1.2 (Brownian motion series construction)**.** *Let ξ_k^j, $j = 1, \dots, d$, $k \in \mathbb{N}$, be independent and identically distributed standard Gaussian random variables. For every $t \in [0, T]$ set*

$$W_t^j = \sum_{k=1}^{\infty} \int_0^t e_k(s)\,\mathrm{d}s\,\xi_k^j, \qquad j = 1, \dots, d, \tag{1.1}$$

where $(e_k)_{k\geq 0}$ is your favorite orthonormal basis of $L^2([0,T])$. Then the series in (1.1) *converges in L^2 and the process $W = (W^1, \dots, W^d)$ is the Brownian motion on the time interval* $[0, T]$.

To see that (1.1) defines the Brownian motion (in the sense of Definition 1.1), one simply calculates the covariance.

In this paper we provide a new characterization, or a definition if you like, for the d-dimensional Brownian motion for $d \geq 2$ as the isotropic (i.e., rotationally invariant) Lévy process with independent and identically distributed (i.i.d.) components. Our proof is short and simple, but not elementary. Moreover, in proving the rise of Gaussianity we do not use, at least not directly, the central limit theorem.

The rest of the paper is organized as follows. In Section 2 we give some basic results of Lévy processes for the convenience of those readers who are not familiar with Lévy processes, and put our result in context. In particular, we recall the Lévy–Khintchine representation theorem for Lévy processes, and show by using it that the Brownian motion can be characterized as being the continuous Lévy process. In Section 3 we state and prove our new characterization: Theorem 3.1. In Section 4 we discuss our new characterization and its implications. In particular, Remark 4.4 gives an open problem on generalizing our result to the Markovian setting, and its connection to a qualitative characterization of the Laplace operator. Finally, in Remarks 4.5 and 4.6 we discuss the implications of our result for the modeling point of view.

2 Brownian motion and Lévy processes

The following is a common textbook definition of the Brownian motion, see, e.g., the recent book on Brownian motion by Mörters & Peres (2010).

Definition 2.1 (The Brownian motion, a textbook definition)**.** A d-dimensional centered stochastic process $W = (W^1, \dots, W^d)$ with $W_0 = 0$ is the *Brownian motion* if:

(i) for all times $0 \leq t_1 \leq \cdots \leq t_n$ the random vectors

$$W_{t_n} - W_{t_{n-1}}, W_{t_{n-1}} - W_{t_{n-2}}, \dots, W_{t_2} - W_{t_1}$$

are independent; i.e., the process W has *independent increments*;

(ii) for every $t \geq 0$ and $h \geq 0$, the distribution of the increment $W_{t+h} - W_t$ does not depend on t; i.e., the process W has *stationary increments*;

(iii) the process $(W_t)_{t\geq 0}$ has almost surely continuous paths;

(iv) for every $t \geq 0$ and $h \geq 0$ the increment $W_{t+h} - W_t$ is multivariate normally distributed with mean zero and variance-covariance matrix $h\,\mathrm{Id}$, where Id is the $d \times d$ identity matrix.

Remark 2.2 (Standard Brownian motion and relaxed definition)**.** Property (iv) of Definition 2.1 states that, in addition to being Gaussian, the process W also has i.i.d.-components W^i, $i = 1, \ldots, d$, and $\mathbb{E}[(W_1^i)^2] = 1$. Sometimes one only insists that the process W is Gaussian. Under the assumptions (i)–(iii) of Definition 2.1 this would mean that $W_{t+h} - W_t$ is a centered Gaussian vector with variance-covariance matrix $h\Sigma$, where Σ is the variance-covariance matrix of W_1. With this more relaxed definition, one usually says that if $\Sigma = \mathrm{Id}$, then W is the *standard Brownian motion*. The connection between the standard Brownian motion and the relaxed definition of the Brownian motion is simple. Indeed, if W is the standard Brownian motion, and we decompose Σ as KK^{T}, then $\mathrm{K}W$ is a Brownian motion in the relaxed sense.

The following is a common textbook definition for Lévy processes, see, e.g, Bertoin (1996) or Sato (2013).

Definition 2.3 (Lévy process, textbook definition)**.** A stochastic process $L = (L^1, \ldots, L^d)$ with $L_0 = 0$ is a *Lévy process* if

(i) it has independent increments;

(ii) it has stationary increments;

(iii) it is stochastically continuous, i.e., for every $t \geq 0$ and $\varepsilon > 0$

$$\lim_{h \to 0} \mathbb{P}\left[|L_{t+h} - L_t| > \varepsilon\right] = 0.$$

Remark 2.4 (Lévy process, càdlàg version)**.** Sometimes one adds the following property to the definition of Lévy processes

(iv) the paths $(L_t)_{t \geq 0}$ are right-continuous with left limits, i.e., they are càdlàg (continue à droite, limite à gauche).

However, it can be shown that under the assumptions (i)–(iii) of Definition 2.3 a Lévy process admits a version with càdlàg paths. Thus, the continuity-type assumption (iv) is not really necessary. Finally, regarding the mild continuity assumption (iii) in Definition 2.3, it should be noted that due to assumption (ii), the stationarity of the increments, it is actually equivalent to assuming that for every $\varepsilon > 0$

$$\lim_{h \to 0} \mathbb{P}\left[|L_h| > \varepsilon\right] = 0.$$

So, the assumption (iii) is mild, indeed.

Thus, Lévy processes are processes with stationary independent increments satisfying a mild continuity assumption. The Brownian motion is a continuous Lévy process that is also Gaussian. Actually, the Gaussianity of the Brownian motion follows from the Lévy property and the continuity. Indeed, property (iv) of Definition 2.1 can be replaced by the following much weaker property

(iv) the process $W = (W^1, \ldots, W^d)$ has i.d.d.-components with $\mathbb{E}[(W_1^i)^2] = 1$.

The fact that Gaussianity follows from continuity is usually not appreciated in the common textbook definitions, such as Definition 2.1 above. That fact follows from the following Lévy–Khintchine representation theorem for Lévy processes. For that we recall the typical notation

$$\langle x, y\rangle = \sum_{j=1}^{d} x_j y_j$$

for the Euclidean inner product on $\mathbb{R}^d$,

$$\|x\| = \sqrt{\langle x, x\rangle}$$

for the Euclidean norm in $\mathbb{R}^d$, and

$$\mathbf{1}_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases}$$

for the indicator of the set A. Finally let $\mathbb{B}_d$ be the closed unit ball in $\mathbb{R}^d$.

Theorem 2.5 (Levy–Khintchine representation theorem)**.** *A stochastic process $L = (L^1, \ldots, L^d)$ is a Lévy process if and only if its characteristic function is of the form*

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle\theta, L_t\rangle}\right] = \mathrm{e}^{-t\Psi(\theta)},$$

where the characteristic exponent *is of the form*

$$\Psi(\theta) = \mathrm{i}\langle m, \theta\rangle + \frac{1}{2}\langle\theta, \Sigma\theta\rangle + \int_{\mathbb{R}^d}\left[1 - \mathrm{e}^{\mathrm{i}\langle\theta, x\rangle} + \mathrm{i}\langle\theta, x\rangle\mathbf{1}_{\mathbb{B}^d}(x)\right]\nu(\mathrm{d}x).$$

Here $m \in \mathbb{R}^d$ is the drift parameter, the symmetric non-negative definite matrix $\Sigma \in \mathbb{R}^{d\times d}$ is the diffusion parameter, and ν, the so-called Lévy measure*, is a measure on $\mathbb{R}^d$ satisfying*

$$\nu(\{0\}) = 0 \quad \textit{and} \quad \int_{\mathbb{R}^d}\left[\|x\|^2 \wedge 1\right]\nu(\mathrm{d}x) < \infty.$$

The triplet (m, Σ, ν) is called the Lévy triplet *of the process L.*

Now, to see that the Gaussianity of the Brownian motion follows from the Lévy–Khintchine representation, just note that

- 1^{st} for continuous Lévy processes one must have $\nu \equiv 0$;
- 2^{nd} then, for centered Lévy processes one must have $m \equiv 0$;
- 3^{rd} and finally, for i.i.d.-component Lévy processes one must have $\Sigma = \sigma\,\mathrm{Id}$, and since one has $\mathbb{E}[(W_1^i)^2] = 1$, it follows that $\sigma = 1$.

Thus $\Psi(\theta) = \frac{1}{2}\|\theta\|^2$, and the Gaussianity follows from this.

Remark 2.6. If we did not assume independence (and identical distribution) of the components in the reasoning above, we would still obtain from the continuity that

$$\Psi(\theta) = \frac{1}{2}\langle\theta, \Sigma\theta\rangle,$$

which would still imply Gaussianity. Thus the (relaxed sense) Brownian motion is characterized as being the continuous Lévy process.

3 A new characterization

Theorem 3.1. *Let $d \geq 2$. A Lévy process $W = (W^1, \ldots, W^d)$ is (a multiple of) the standard Brownian motion if and only if it is centered and isotropic with independent and identically distributed components.*

Proof. The only if part is clear. For the if part, let (m, Σ, ν) be the Lévy triplet of W. Since W has independent components, ν is concentrated on the coordinate axes. Since W is isotropic, ν is also isotropic. Consequently, $\nu \equiv 0$. Since W is centered, $m \equiv 0$. Finally, since W has i.i.d.-components, $\Sigma = \sigma\,\mathrm{Id}$. Thus $(m, \Sigma, \nu) = (0, \sigma\,\mathrm{Id}, 0)$, proving the claim. $\square$

4 Discussion

Remark 4.1 (The importance of having $d \geq 2$)**.** In Theorem 3.1 it is important that $d \geq 2$. Indeed, for $d = 1$ symmetric processes are isotropic and there are many discontinuous symmetric Lévy processes for $d = 1$. Indeed, one can construct such Lévy processes by using the Lévy triplet $(0, 0, \nu)$, where ν is any symmetric measure on $\mathbb{R}$ satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} \left[x^2 \wedge 1\right] \nu(\mathrm{d}x) < \infty.$$

Remark 4.2 (Gaussianity without the central limit theorem)**.** As stated in the introduction, we did not (directly) invoke the central limit theorem in proving the rise of Gaussianity in our new characterization of the Brownian motion in Theorem 3.1. Instead, we used the Lévy–Khintchine representation of Theorem 2.5. Now, the classical way of proving the Lévy–Khintchine representation theorem does involve the central limit theorem and the so-called infinitely divisible distributions that are closely related to central limit-type theorems. There are, however, ways of proving the Lévy–Khintchine representation without resorting to the central limit theorem. For example, Jacod & Shiryaev (2003: Chapter 4) contains a nice derivation of the Lévy–Khinchine formula that only uses stochastic analysis and "compensator calculus".

Remark 4.3 (The history of the rise of Gaussianity)**.** It seems that the rise of Gaussianity through independence of the components and the rotational invariance has a long history dating back at least to the works of Herschel and Maxwell in 1850's, see Jaynes (2003: Section 7.2). The key ingredients in the Herschel–Maxwell argument are (looking at a fixed time point and assuming continuous distribution) that the components are i.i.d. and that the process is isotropic. Hereby the first property implies that the distribution takes the form

$$p(x)\,\mathrm{d}x = \prod_{j=1}^{d} f(x_j)\,\mathrm{d}x_j \tag{4.1}$$

in Cartesian coordinates, and the second property implies that the distribution takes the form

$$p(x)\,\mathrm{d}x = g(r) r\,\mathrm{d}r\mathrm{d}\theta \tag{4.2}$$

in polar (or hyper-spherical) coordinates.

Equating (4.1) and (4.2) one is given the functional equation

$$\prod_{j=1}^{d} f(x_j) = g(\|x\|). \tag{4.3}$$

The solution of the functional equation (4.3) is of the form

$$f(x) = c_1 \mathrm{e}^{-c_2\|x\|^2},$$

which is the Gaussian density.

Einstein (1905) also used similar arguments in his investigation of the Brownian motion in connection to the existence of atoms and molecules. Of course, neither the Herschel–Maxwell nor the Einstein derivation can be used in our setting of Theorem 3.1, since the distribution of a Lévy process at any fixed time is not *a priori* continuous.

Remark 4.4 (Open problem: from Lévy to Markov)**.** Lévy processes in general, and Brownian motion in particular, are Markovian processes that admit generators. Indeed, recall that the generator, if it exists, of a Markov process $X = (X^1, \ldots, X^d)$ is the linear operator given as

$$Af(x) = \lim_{t\to 0} \frac{1}{t}\left[\mathbb{E}^x\left[f(X_t)\right] - f(x)\right].$$

Here $\mathbb{E}^x$ means that the process X is started from x, or, in the case of Lévy processes, we are considering the translated process $X + x$. The generator of a Lévy process with triplet (m, Σ, ν) is

$$\begin{aligned} Af(x) = {} & \langle m, \nabla f(x)\rangle + \frac{1}{2}\langle \nabla, \Sigma\nabla f(x)\rangle \\ & + \int_{\mathbb{R}^d} \left[f(x+y) - f(x) - \langle y, \nabla f(x)\rangle \mathbf{1}_{\mathbb{B}^d}(y)\right] \nu(\mathrm{d}y). \end{aligned} \tag{4.4}$$

Thus the generator of the Brownian motion is the Laplacian up to a factor $1/2$. This provides also an alternative proof for Theorem 3.1. Indeed, one only has to show that (a multiple) of the Laplacian is the only linear operator of the form (4.4) with $d \geq 2$ that satisfies

(i) $A_x = A_y$ for any rotation $y = \mathrm{R}x$ (isotropy);

(ii) $A_x = \sum_{j=1}^d A_{x_j}$ (i.i.d.-components).

It would be interesting to know in which class of operators the properties (i)–(ii) above characterize the (multiple of the) Laplacian. For example, could it be possible to extend Theorem 3.1 from Lévy processes to (a larger class of) Markov processes?

Remark 4.5 (Fractional Laplacian)**.** Recently there has been much interest in models involving the fractional Laplacian $-(-\Delta)^{\alpha/2}$, which is a non-local pseudo-differential operator given by the Cauchy principal value integral

$$-(-\Delta)^{\alpha/2} f(x) = \frac{2^\alpha \Gamma(\alpha/2 + 1/2)}{\pi^{1/2}\Gamma(-\alpha/2)} \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{\|x-y\|^{1+\alpha}}\,\mathrm{d}y.$$

From the probabilistic point of view the fractional Laplacian can be understood as the generator of an isotropic α-stable Lévy process. With $\alpha = 2$ the fractional Laplacian is just the Laplacian (actually, with our probabilistic parametrization it is $\frac{1}{2}\Delta$). Now, α-stable processes X are natural models in the sense that they are Lévy processes satisfying the scaling property

$$(cX_{c^{-\alpha}t})_{t\geq 0} \stackrel{d}{=} (X_t)_{t\geq 0},$$

where $\stackrel{d}{=}$ means equality in the sense of finite-dimensional distributions. Thus, assuming α-stability of some given random Lévy d-variate time-series X, we are faced with two natural (and maybe contradicting) assumptions:

(α_1) The time-series X is rotationally invariant.

(α_2) The time-series X has i.i.d.-components.

If $\alpha = 2$, then the time-series is generated by a Brownian motion, and the assumptions (α_1) and (α_2) are the same. If, however, $\alpha \neq 2$ (and then necessarily $\alpha \in (0, 2)$), the assumptions (α_1) and (α_2) are mutually exclusive. Assumption (α_1) corresponds to the Lévy process with generator $-(-\Delta)_x^{\alpha/2}$ that is rotationally invariant, while assumption (α_2) corresponds to the Lévy process with generator

$$A_x^\alpha = \sum_{j=1}^{d} -(-\Delta_{x_j}^{\alpha/2})$$

that acts coordinatewise.

Remark 4.6 (Modeling implications)**.** Remark 4.5 above illustrates the message of our new characterization, Theorem 3.1, for the modeling point of view. In the context of d-variate Lévy processes for $d \geq 2$ the two natural assumptions

- rotational invariance;
- independent components,

are mutually exclusive unless your model is the Brownian motion.

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THE ROLE OF MATHEMATICS AND STATISTICS IN THE UNIVERSITY OF VAASA; THE FIRST FIVE DECADES

Ilkka Virtanen

In dedication to Professor Seppo Hassi on the occasion of his 60th birthday

The tragedy of the world is
that those who are imaginative
have but slight experience,
and those who are experienced
have feeble imaginations.
Fools act on imagination
without experience.
Pedants act on experience
without imagination.
The task of the university is
to weld together imagination and experience.

Alfred North Whitehead
A Cambridge mathematician

1 From a business school towards a university with a clearly defined academic profile

1.1 The beginning

The University of Vaasa is one of the four “new universities” founded in Finland in 1966. Practical activities of the new units started two to five years after the foundation decision was made by the state authorities. Three of the new universities were founded in Eastern Finland whereas the University of Vaasa came to be placed on the west coast of Finland. All these new universities had different academic profiles. Vaasa got a private business school which, however, from the beginning – the first students started their studies in 1968 – got the major part of its financing from the state. After ten years the business school became a state university. Thus the University of Vaasa started as a business school, but it was clear from the beginning that the final target was to develop it into a multidisciplinary university. The University of Lappeenranta was a technical university, the University of Kuopio had disciplines in medicine, biosciences, and social sciences and, finally, the University of Joensuu was based on an earlier teacher training college having also natural sciences, humanities, and social sciences in its discipline palette. The Eastern Finland universities were state universities from the beginning.

The post-war baby boom generation was coming to the university in the 1960s and 1970s, and competition for resources between universities was great. So it turned out that the development of the business school towards the multidisciplinary target would be a hard and long-lasting exercise. Step by step the goal was, however, reached, at least to a decent extent.

Today the University of Vaasa is – on the Finnish scale – a small or medium-sized business-oriented and multidisciplinary science university. The strategic focus areas of the university are management and change, energy and sustainable development, as well as financial and economical decision-making. The university has about 5000 students and about 300 of them are international degree students. The total number of employees is about 510 of whom about 100 are international.

The university has been organized into schools. There are four schools for research and teaching: the School of Management, the School of Accounting and Finance, the School of Marketing and Communication, and the School of Technology and Innovations. The unit of mathematical sciences belongs to the School of Technology and Innovations, although it is responsible for all the teaching of mathematics and mathematics-based quantitative methods in the whole university. The teachers and researchers of this unit are in close cooperation with employees of the other schools.

1.2 Strategy and values of the university today

In its strategy for 2030 the university defines itself to be an internationally competitive, productive and specialized research university with a strong focus on impactful basic scientific research. The core competence of the university consists of high-level expertise in business, technology, management, and communications.

The university is focused on responsible business. Its fundamental purpose is to cultivate new knowledge and nurture civilization as a core value of our society. This is why the focus is on global challenges and opportunities. They provide the university with the core source of motivation for its education and research. The university uses its work as a means to advance positive and sustainable development for individuals, communities, and the world at large.

Based on the strategy for 2030, the university has defined its vision, mission and values as follows.

- *Vision:* The University of Vaasa is regarded internationally as a successful and impactful research university.

- *Mission:* The university carries out impactful research and educates experts that address the needs of society today, and in the future. The university advances competitiveness, innovation and sustainable development in business, technology, and society.

- *Values:* The values of the university are Courage, Community, and Responsibility.

2 The role of mathematics and statistics in the university's palette of sciences

2.1 No own degree programme in master level – a recognized role, however

The role of mathematics and statistics in the University of Vaasa is not typical for universities. According to the strategy applied in the university today and during the past decades, these disciplines have never been offered as the main subjects of bachelor and master level degrees. This has not meant, however, that the disciplines were considered as less important for studies and research than the disciplines with their own degree programmes. On the contrary, the methodological disciplines have for example always been represented by professor chairs. It is noteworthy that even the first chair established in the university – a business school in the beginning – and also having got its first office-holder was the combined chair in economic mathematics and statistics.

The holder of the chair in economic mathematics and statistics was also the first rector of the business school. He started his work already one year earlier than the school was opened and received its first students. That is, he had a central role in planning the first year's curriculum for the school. According to the experiential opinion of the author of this article, the result was that the quantitative sciences got a stronger role in the curriculum for the business students than was the case in the other Finnish business schools at that time.

Nowadays the situation has changed. Other business schools have also became aware that modern economic and business education as well as research needs a good comprehension of the quantitative methods as well as the skill to use them. A large number of students graduated as doctors in business economics in the University of Vaasa who, having been recruited as professors and other academic employees to other business schools, have had a marked effect in this process.

2.2 Mathematical sciences – a close and united academic unit

At the beginning the teaching staff consisted of one professor and one lecturer, both posts being combined posts for business mathematics and statistics. As the number of students increased the unit obtained another professorship and an additional lectureship. This made it possible to focus the duties of all the posts only on business mathematics or statistics, respectively. An assistantship was also an important addition to the staff.

In the beginning of the 1990s the university started to enlarge its branch of activities towards technology. This could progress step by step only. First, technical elements were included in the curriculum of two master programs in economics and business administration and an unofficial label "industrial economist" was given to the graduates. The main subject in these programmes was either industrial management or information technology. In the next step the university started to educate civil engineers in co-operation with the Helsinki Technical University. The University of Vaasa recruited the students who carried out half of their studies in Vaasa and graduated from Helsinki Technical University. In the beginning of the 2000s the University of Vaasa was allowed to educate civil engineers completely as its own activity.

It was clear that the enlargement of the branch of activities into technology presumed additional resources also for the unit of mathematical sciences. The total number of students increased and the amount, type, and level of mathematics needed in technical studies is different from what is needed in economics, business administration, and in the social sciences. The requirements were resolved by providing the unit with a professorship and a lectureship in mathematics.

Today the permanent staff in the unit (department) consists of three professors, three senior lecturers with doctor degrees, one university teacher, and a varying number (3–5) of doctoral students and post-doctoral researchers. Although administratively the unit is a part of the School of Technology and Innovations, the main sphere of responsibility of business mathematics and statistics has been supporting the School of Accounting and Finance and other business oriented schools by teaching quantitative methods both for education and research purposes.

The unit of mathematical sciences is small. Therefore it is important that the mathematically oriented disciplines form also administratively an integrated own unit. Together the unit's academic subjects are stronger, they can flexibly use common teaching resources, and they create and maintain a high-level and international academic atmosphere in research. This administrative solution guarantees that the department can form a close and united academic society. One example of the manifestation of this coherence is that the unit takes care also of maintaining close contacts with its staff in retirement.

Holders of the professor offices in the department have been mainly recruited from other universities. Statistics is an exception. The last two professor appointments in statistics have been candidates who have done their doctoral studies and qualified for professorship when working at the University of Vaasa. All the senior lecturers of the unit have received their doctor degrees while working at the university. As a result, the staff members represents a high quality group of experts in their own fields, but, at the same time, they possess understanding of and positive attitude towards the needs of quantitative methods appearing among the students and researchers of other disciplines in the university.

2.3 The unit's supporting role in undergraduate education, its own intensive postgraduate education, and high-quality international research

As has been mentioned earlier, the unit of mathematical sciences doesn't have and has never had any own master-level education programme in the curriculum of the university. In undergraduate studies the role of mathematical sciences is to contribute as a strong and high-standard supporting discipline offering basic university level knowledge and relevant advanced tools in mathematical and statistical modelling for the students of the other education programmes of the university, especially for students in business and technology. The demand of skills in the use of modern quantitative methods and models is outstandingly high in the postgraduate level of the studies. Besides offering methodological courses the professors of the unit participate as tutors in other subject's doctoral seminars.

Postgraduate studies in mathematics and statistics have been possible in the university from the beginning. Their role has become more and more important during the years. The recruiting of students is challenging due to the non-existing own master level education. The achievements are, however, good and the department is for example a pioneer in the university in recruiting international students for doctoral studies. Today, a majority of the unit's postgraduate students is of foreign origin.

As the mathematical sciences are represented on the professor level in the university, scientific research is, of course, strongly on the department's agenda. The research groups in which the department's researchers participate represent the top quality in the university. The researchers have created and maintain one of the university's research programmes with the theme of mathematical modelling. The projects in the programme are both discipline oriented, especially in mathematics, and more application oriented in statistics and business mathematics. The research groups work actively in co-operation with other researchers inside the university and with researchers in other universities, both in Finland and abroad. Visiting foreign scholars are commonly seen working in the department and the staff members work frequently abroad.

The professors and other staff members participate actively in the general management of the University. Two professors in economic mathematics have served as rectors of the University, several professors as faculty deans and as heads of multidisciplinary departments. Professors and other staff members are members of several managerial working groups.

3 Discussion

The chosen policy for the mathematical sciences in the University of Vaasa – i.e., to operate as an academic education and research unit without any own undergraduate education programme – is challenging. But the work done during the five first decades has shown that the chosen option has been successful. The main measures for reaching success are:

- Careful attention has been paid when recruiting new members to the staff. Besides competence, the new members must also have an understanding of and a positive attitude towards the other disciplines in the university with which they are expected to co-operate both in education and research.
- Active co-operation with other disciplines inside the university has been a necessity for being able to give relevant and up-to-date methodological support to both students and researchers of other disciplines.
- To reach and maintain a high level competence in the area of everyone's own expertise an active communication and co-operation with the representatives of one's own discipline in other universities in Finland and abroad have been strongly on the agenda.
- The absence of own undergraduate education has not meant absence of postgraduate education. On the contrary, active doctoral programmes have guaranteed continuity in research and have helped in recruiting to the unit new employees who possess a relevant orientation towards the operating principle of the unit. Of course, external recruiting has also been important for guaranteeing high levels of competence.
- Special attention has been paid for activating international co-operation in research and postgraduate education.

A small university like the University of Vaasa must carefully focus its activities on areas about which it has successful experience from the past, which are still relevant today and are also or are expected to become crucial in the future, and which form a coherent entity. In education and research, also the needs of the region and the whole society as well as the region's possibilities to offer co-operation and support must have a strong role in the agenda. The strategy and values of the university meet these requirements well.

Similarly, the unit of mathematical sciences as a small unit inside the university must have a clear and specific high-standard academic culture in its operations. Conclusions from the scrutiny above show that the unit has been successful in creating this culture.

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