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Spectral decompositions of selfadjoint relations in Pontryagin spaces and factorizations of generalized Nevanlinna functions

Seppo Hassi & Hendrik Luit Wietsma

Dedicated to V.E. Katsnelson on the occasion of his 75th birthday

Abstract. Selfadjoint relations in Pontryagin spaces do not possess a spectral family completely characterizing them in the way that is known to hold for selfadjoint relations in Hilbert spaces. Here it is shown that a combination of a factorization of generalized Nevanlinna functions with the standard spectral family of selfadjoint relations in Hilbert spaces can function as a spectral family for selfadjoint relations in Pontryagin spaces. By this technique additive decompositions are established for generalized Nevanlinna functions and selfadjoint relations in Pontryagin spaces.

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1. Introduction

It is well known that the class of generalized Nevanlinna functions can be realized by means of selfadjoint relations in Pontryagin spaces (cf. Section 2.2 below). In [16] it has been shown that there is a strong connection between the factorization result for scalar generalized Nevanlinna functions and the invariant subspace properties of selfadjoint relations in Pontryagin spaces. Here that approach is extended to the case of operator-valued generalized Nevanlinna functions whose values are bounded operators on a Hilbert space $\mathcal{H}$; in what follows this class is denoted by $\mathfrak{N}_\kappa(\mathcal{H})$, where $\kappa \in \mathbb{N}$ refers to the number of negative squares of the associated Nevanlinna kernel; see [11, 12]. More precisely, by combining the multiplicative factorization for operator-valued generalized Nevanlinna functions established in [14] with the well-known spectral family results for selfadjoint operators in Hilbert spaces the following additive decomposition is obtained.

Theorem 1.1. *Let $F \in \mathfrak{N}_\kappa(\mathcal{H})$ and let Δ be a measurable subset of $\mathbb{R} \cup \{\infty\}$ or a closed symmetric subset of $\mathbb{C} \setminus \mathbb{R}$. Then F can be written as $F_\Delta + F_R$, where*

(i) $\sigma(F_\Delta) \subseteq \operatorname{clos}\Delta$ *and* $\operatorname{int}\Delta \subseteq \rho(F_R)$*;*

(ii) $F_\Delta \in \mathfrak{N}_{\kappa_\Delta}(\mathcal{H})$*,* $F_R \in \mathfrak{N}_{\kappa_R}(\mathcal{H})$ *and* $\kappa_\Delta + \kappa_R \geq \kappa$*.*

If $\partial\Delta \cap \mathrm{GPNT}\,(F) = (\operatorname{clos}(\Delta) \setminus \operatorname{int}(\Delta)) \cap \mathrm{GPNT}\,(F) = \emptyset$, then the decomposition may be chosen such that F_Δ and F_R do not have a generalized pole in common. In this case, $\kappa_\Delta + \kappa_R = \kappa$.

In Theorem 1.1 $\rho(F)$ denotes the set of holomorphy of $F \in \mathfrak{N}_\kappa(\mathcal{H})$ in $\mathbb{C}\cup\{\infty\}$ and $\sigma(F)$ stands for its complement in $\mathbb{C} \cup \{\infty\}$. For the definition of generalized poles and generalized poles not of positive type (GPNTs), see Section 2.2 below. It should be mentioned that Theorem 1.1 generalizes a result obtained for matrix-valued generalized Nevanlinna functions by K. Daho and H. Langer in [2, Prop. 3.3].

For the proof of Theorem 1.1 spectral families for Pontryagin space selfadjoint relations are replaced by factorizations of generalized Nevanlinna functions in combination with the standard spectral decompositions of selfadjoint Hilbert space operators (or relations); this is the main contribution of this paper. Such an approach is needed because spectral families for Pontryagin space selfadjoint relations do not exist in an appropriate form to establish Theorem 1.1; cf. [13]. This approach can be extended to decompose for instance definitizable functions (and operators) in a Kreĭn space setting. Starting from the essentially multiplicative representation of an definitizable function F in [10, Thm. 3.6] one can for example show that F can be written as the sum of two definitizable functions F_+ and F_-, where F_+ has no points of negative type and F_- has no points of positive type.

The intimate connection between generalized Nevanlinna functions and selfadjoint relations in Pontryagin spaces, see e.g. Section 2.2 below, means that the following analogue of Theorem 1.1 holds for selfadjoint relations in Pontryagin spaces. For the notation ENT (A) in the following theorem, see Section 2.1 below.

Theorem 1.2. *Let A be a selfadjoint relation in a Pontryagin space $\{\Pi, [\cdot,\cdot]\}$ with $\rho(A) \neq \emptyset$ and let Δ be either a measurable subset of $\mathbb{R}\cup\{\infty\}$ or a closed symmetric subset of $\mathbb{C} \setminus \mathbb{R}$. Then there exists a selfadjoint relation A_e in a Pontryagin space $\{\Pi_e, [\cdot,\cdot]_e\}$ with $\operatorname{gr}(A) \subseteq \operatorname{gr}(A_e)$ and a decomposition $\Pi_\Delta[+]\Pi_R$ of Π_e such that*

(i) $\{\Pi_\Delta, [\cdot,\cdot]\}$ *and* $\{\Pi_R, [\cdot,\cdot]\}$ *are Pontryagin spaces;*

(ii) Π_Δ *and* Π_R *are A_e-invariant;*

(iii) $\sigma(A_e \upharpoonright_{\Pi_\Delta}) \subseteq \operatorname{clos}\Delta$ *and* $\operatorname{int}\Delta \subseteq \rho(A_e \upharpoonright_{\Pi_R})$*.*

If $\partial\Delta \cap \mathrm{ENT}\,(A) = \emptyset$, then A_e and Π_e can be taken to be A and Π, respectively, and the decomposition can be taken such that

$$\sigma_p(A \upharpoonright_{\Pi_\Delta}) \cap \sigma_p(A \upharpoonright_{\Pi_R}) = \emptyset.$$

In the particular case that Δ is a closed symmetric subset of $\mathbb{C} \setminus \mathbb{R}$ the decomposition in Theorem 1.2 is directly obtained by means of Riesz projection operators; see e.g. [1, Ch. 2: Thm 2.20 & Cor. 3.12]. However, Theorem 1.2 cannot always be established by means of spectral families of selfadjoint relations in Pontryagin spaces if $\partial\Delta \cap \mathrm{ENT}\,(A) \neq \emptyset$. Indeed the eigenspaces of ENTs can be neutral or even degenerate; in such cases the corresponding eigenvalues are critical points and the spectral family might not be extendable to sets having these points as their endpoints; cf. [13, Comments following Thm. 5.7].

To mention another example of decompositions included in Theorem 1.2 consider $\Delta = (-\infty, a) \cup (b, \infty) \cup \{\infty\}$, where $a, b \in \mathbb{R} \setminus \mathrm{ENT}\,(A)$ and $a < b$. Then Theorem 1.2 says that a selfadjoint relation in a Pontryagin spaces can be decomposed into an unbounded selfadjoint relation in a Pontryagin space and a bounded selfadjoint operator in a Pontryagin space; for selfadjoint operators this last result can be found in [11]; see also the references therein. Note that intervals Δ of the given type naturally arise in connection with rational functions; for instance when considering definitizable operators or the products of (generalized) Nevanlinna functions with rational functions, see e.g. [8].

Finally the contents of the paper are shortly outlined. The first half of Section 2 consists of an introduction to selfadjoint relations (multi-valued operators) in Pontryagin spaces together with a short overview of minimal operator realizations of (operator-valued) generalized Nevanlinna functions. In the latter half of this section we recall some results about how non-minimal realizations can be reduced to minimal ones and also consider the (minimality of the) realization for the sum of generalized Nevanlinna functions. In Section 3 we first establish the connection between a factorization of a generalized Nevanlinna function and the spectral properties of its operator realization. This result is a key tool for using the factorization of generalized Nevanlinna functions as a replacement for a spectral decomposition of selfadjoint relations in Pontryagin spaces. Finally, in the second and third subsections of Section 3 Theorems 1.1 and 1.2 are proven, respectively.

2. Preliminaries

The first two subsections contain introductions to (unbounded) operators or more generally linear relations in Pontryagin spaces and (minimal) operator realizations for generalized Nevanlinna functions, respectively. In the third subsection it is shown how non-minimal realizations may be reduced to minimal ones. Finally, in the fourth subsection the sum of generalized Nevanlinna functions is considered.

2.1. Linear relations in Pontryagin spaces

A linear space Π together with a sesqui-linear form $[\cdot,\cdot]$ defined on it, is a *Pontryagin space* if there exists an orthogonal decomposition $\Pi^+ + \Pi^-$ of Π such that $\{\Pi^+, [\cdot,\cdot\}$ and $\{\Pi^-, -[\cdot,\cdot]\}$ are Hilbert spaces either of which is finite dimensional;

here orthogonal means that $[f^+, f^-] = 0$ for all $f^+ \in \Pi^+$ and $f^- \in \Pi^-$. For our purposes it suffices to consider only Pontryagin spaces for which Π^- is finite-dimensional; its dimension (which is independent of the orthogonal decomposition $\Pi^+ + \Pi^-$) is *the negative index* of Π.

A *(linear) relation* H in $\{\Pi, [\cdot,\cdot]\}$ is a *multi-valued (linear) operator* whose domain is a linear subspace of Π, denoted by $\operatorname{dom} H$, and which linearly maps each element $x \in \operatorname{dom} H$ to a subset $Hx := H(x)$ of Π. (Graphs of) linear relations on Π can be identified with subspaces of $\Pi \times \Pi$; in what follows this identification will tacitly be used. The linear subspace $H(0)$ is called the *multi-valued part* of H and is denoted by $\operatorname{mul} H$.

A relation H is closed if (the graph of) H is a closed subspace of $\Pi \times \Pi$. For any relation H in $\{\Pi, [\cdot,\cdot]\}$, its adjoint, denoted as $H^{[*]}$, is defined via its graph:

$$\mathrm{gr} H^{[*]} = \{\{f, f'\} \in \Pi \times \Pi : [f, g'] = [f', g], \ \ \forall \{g, g'\} \in \mathrm{gr} H\}.$$

A relation A in $\{\Pi, [\cdot,\cdot]\}$ is *symmetric* if $A \subseteq A^{[*]}$ and *selfadjoint* if $A = A^{[*]}$. An operator V from (a Pontryagin space) $\{\Pi_1, [\cdot,\cdot]_1\}$ to (a Pontryagin space) $\{\Pi_2, [\cdot,\cdot]_2\}$ is *isometric* if $[f, g]_1 = [Vf, Vg]_2$ for all $f, g \in \operatorname{dom} V$. An isometric operator U from $\{\Pi_1, [\cdot,\cdot]_1\}$ to $\{\Pi_2, [\cdot,\cdot]_2\}$ is a *standard unitary operator* if $\operatorname{dom} U = \Pi_1$ and $\operatorname{ran} U = \Pi_2$.

For a closed relation H in $\{\Pi, [\cdot,\cdot]\}$, the resolvent set, $\rho(H)$, and the spectrum, $\sigma(H)$, are defined as usual:

$$\rho(H) = \{z \in \mathbb{C} : \ \ker(H - z) = \{0\}, \ \operatorname{ran}(H - z) = \Pi\} \quad \text{and} \quad \sigma(H) = \mathbb{C} \setminus \rho(H).$$

Moreover, the point spectrum $\sigma_p(H)$ is defined as the set

$$\sigma_p(H) = \{z \in \mathbb{C} \cup \{\infty\} : \ \exists\, x\, (\neq 0) \in \Pi \text{ s.t. } \{x, zx\} \in \mathrm{gr}(H)\}.$$

These sets have the normal properties, see e.g. [4]. Below we also use the convention that $\infty \in \sigma_p(H)$ if and only if $\operatorname{mul} H \neq \{0\}$ or, equivalently, $0 \in \sigma_p(H^{-1})$, where H^{-1} stands for the inverse (linear relation) of H. Similarly, $\infty \in \rho(H)$ means that $0 \in \rho(H^{-1})$ or, equivalently, that H is a bounded everywhere defined operator, i.e., $H \in \mathbf{B}(\Pi)$.

A subspace $\mathfrak{L}$ of Π is said to be *invariant* under a relation H with $\rho(H) \neq \emptyset$, or H-invariant for short, if

$$(H - z)^{-1}\mathfrak{L} \subseteq \mathfrak{L}, \qquad \forall z \in \rho(H).$$

Here $(H - z)^{-1} \in \mathbf{B}(\Pi)$ is defined via its graph as

$$\mathrm{gr}((H - z)^{-1}) = \{\{f' - zf, f\} \in \Pi \times \Pi : \{f, f'\} \in \mathrm{gr}(H)\}.$$

Recall that the spectrum and resolvent set $\sigma(A)$ and $\rho(A)$ of a selfadjoint relation A in a Pontryagin space are symmetric with respect to the real line:

$$\rho(A) = \overline{\rho(A)}, \quad \sigma(A) = \overline{\sigma(A)} \quad \text{and} \quad \sigma_p(A) = \overline{\sigma_p(A)}. \tag{2.1}$$

Moreover, if $\rho(A) \neq \emptyset$ then $\rho(A)$ contains $\mathbb{C} \setminus \mathbb{R}$ except finitely many points; see [4].

Finally $\alpha \in \mathbb{C} \cup \{\infty\}$ is an *eigenvalue not of positive type*, or ENT for short, of a selfadjoint relation A in a Pontryagin space, if there exists a non-trivial non-positive A-invariant subspace $\mathfrak{L}$ such that $\sigma(A \upharpoonright_{\mathfrak{L}}) = \alpha$. Recall that selfadjoint relations in Pontryagin spaces possess at most finitely many ENTs, see e.g. [9, Thm. 12.1']. The set of all ENTs of a selfadjoint relation A in $\mathbb{C} \cup \{\infty\}$ is denoted by $\mathrm{ENT}\,(A)$.

2.2. Minimal realizations of generalized Nevanlinna functions

The concept of an operator-valued generalized Nevanlinna function has been introduced and studied by M.G. Kreĭn and H. Langer; see [11, 12]. In particular, with some additional analytic assumptions, operator-valued generalized Nevanlinna functions were described as so-called *Q-functions* of symmetric operators in a Pontryagin space. Those additional conditions were removed by allowing selfadjoint relations in model spaces; cf. [3] for the case of matrix functions and [7] for operator-valued functions.

If A is a selfadjoint relation in (a Pontryagin space) $\{\Pi, [\cdot,\cdot]\}$ with a nonempty resolvent set $\rho(A)$, C is a bounded selfadjoint operator in a Hilbert space $\{\mathcal{H}, (\cdot,\cdot)\}$ and Γ is an everywhere defined operator from $\mathcal{H}$ to Π, then F defined by

$$F(z) = C + \overline{z_0}\Gamma^{[*]}\Gamma + (z - \overline{z_0})\Gamma^{[*]}\left(I + (z - z_0)(A - z)^{-1}\right)\Gamma, \qquad z, z_0 \in \rho(A), \tag{2.2}$$

is a generalized Nevanlinna function. Conversely, if F is a generalized Nevanlinna function, then there exist $A = A^{[*]}$ with $\rho(A) \neq \emptyset$, Γ and C as above such that (2.2) holds; in this case $C + \overline{z_0}\Gamma^{[*]}\Gamma = F(z_0)^* = F(\overline{z_0})$.

If (2.2) holds for some generalized Nevanlinna function F, then the pair $\{A, \Gamma\}$ *realizes* F (at z_0). In particular, in the term realization the *realizing space* $\{\Pi, [\cdot,\cdot]\}$ is suppressed; also the selection of the arbitrarily fixed point z_0 is suppressed when it doesn't play a role. With a realizing pair $\{A, \Gamma\}$ (at z_0) we associate a bounded operator-valued function Γ_z, called the *γ-field* associated with $\{A, \Gamma\}$, via

$$\Gamma_z := \left(I + (z - z_0)(A - z)^{-1}\right)\Gamma, \qquad z \in \rho(A). \tag{2.3}$$

Using the γ-field and the resolvent identity, (2.2) can be rewritten into a symmetric form:

$$\frac{F(z) - F(w)^*}{z - \overline{w}} = \Gamma_w^{[*]}\Gamma_z, \qquad z, w \in \rho(A). \tag{2.4}$$

The pair $\{A, \Gamma\}$ is said to realize F (as in (2.2)) *minimally* if

$$\Pi = \text{c.l.s.}\,\{\Gamma_z h : z \in \rho(A), \ \ h \in \mathcal{H}\}.$$

For the existence of a minimal realization for any generalized Nevanlinna function see e.g. [7, Thm. 4.2].

By means of a minimal realization the index of a generalized Nevanlinna function can be characterized: F is a generalized Nevanlinna function with index κ, $F \in \mathfrak{N}_\kappa(\mathcal{H})$, if the negative index of the realizing (Pontryagin) space for any minimal realization is κ. In fact, all minimal realizations are connected by means of (standard) unitary operators.

Proposition 2.1. *(*[7, Thm. 3.2]*) Let* $\{A_i, \Gamma_i\}$ *realize* $F \in \mathfrak{N}_\kappa(\mathcal{H})$ *minimally for* $i = 1, 2$. *Then there exists a standard unitary operator from* $\{\Pi_1, [\cdot,\cdot]_1\}$ *to* $\{\Pi_2, [\cdot,\cdot]_2\}$ *such that* $A_2 = UA_1U^{-1}$ *and* $\Gamma_2 = U\Gamma_1$.

For a generalized Nevanlinna function F the notation $\rho(F)$ and $\sigma(F)$ is used to denote the *domain of holomorphy* of F in $\mathbb{C} \cup \{\infty\}$ and its complement (in $\mathbb{C} \cup \{\infty\}$), respectively. In particular, (2.2) implies that

$$\rho(A) \subseteq \rho(F) \quad \text{and} \quad \sigma(F) \subseteq \sigma(A). \tag{2.5}$$

For minimal realizations the reverse inclusions also hold.

Theorem 2.2. *(*[11, Satz 4.4]*) Let* $F \in \mathfrak{N}_\kappa(\mathcal{H})$ *be minimally realized by* $\{A, \Gamma\}$. *Then* $\rho(A) = \rho(F)$.

Finally, $\alpha \in \mathbb{C} \cup \{\infty\}$ is a *generalized pole* of a generalized Nevanlinna function F if $\alpha \in \sigma_p(A)$ for any minimal realization $\{A, \Gamma\}$ of F. Furthermore, the set of *generalized poles of not of positive type of* F, GPNT$\,(F)$, is defined to be ENT$\,(A)$ (see Section 2.1). Note that Proposition 2.1 guarantees that these concepts are well-defined.

2.3. Reduction of non-minimal realizations

Realizations for a generalized Nevanlinna function need not be minimal. For instance, if the negative index of the realizing Pontryagin space is greater than the negative index of a generalized Nevanlinna function, then the realization is not minimal. Even if the negative index of the realizing space is equal to the negative index of a generalized Nevanlinna function, the realization might still be non-minimal; cf. Section 2.4 below. The following operator-valued analog of [16, Prop. 2.2] shows how non-minimal realizations can be reduced to minimal ones; see also [11] and [7, Section 2].

Proposition 2.3. *Let* $\{A, \Gamma\}$ *realize* $F \in \mathfrak{N}_\kappa(\mathcal{H})$ *and let* κ_m *denote the negative index of the realizing Pontryagin space* $\{\Pi, [\cdot,\cdot]\}$. *Moreover, with*

$$\mathfrak{M} := \text{span}\,\{\left(I + (z - z_0)(A - z)^{-1}\right)\Gamma h : z \in \rho(A), \ \ h \in \mathcal{H}\},$$

define $\mathfrak{L}$*,* Π_s *and* Π_r *as*

$$\mathfrak{L} = (\operatorname{clos}\mathfrak{M}) \cap \mathfrak{M}^{[\perp]}, \quad \Pi_s = (\operatorname{clos}\mathfrak{M})/\mathfrak{L} \quad \text{and} \quad \Pi_r = \mathfrak{M}^{[\perp]}/\mathfrak{L}.$$

Then the following statements hold:

(i) $\mathfrak{L}$ *is an* A*-invariant neutral subspace of* $\{\Pi, [\cdot,\cdot]\}$ *with* $\kappa_{\mathfrak{L}} := \dim\mathfrak{L} \leq \kappa_m$*;*

(ii) A_s *and* A_r*, defined via*

$$\operatorname{gr}A_s = \{\{f + [\mathfrak{L}], f' + [\mathfrak{L}]\} : \{f, f'\} \in \operatorname{gr}A \cap (\Pi_s \times \Pi_s)\};$$
$$\operatorname{gr}A_r = \{\{f + [\mathfrak{L}], f' + [\mathfrak{L}]\} : \{f, f'\} \in \operatorname{gr}A \cap (\Pi_r \times \Pi_r)\},$$

are selfadjoint relations in the Pontryagin spaces $\{\Pi_s, [\cdot,\cdot]\}$ *and* $\{\Pi_r, [\cdot,\cdot]\}$ *with negative index* κ *and* $\kappa_m - \kappa - \kappa_{\mathfrak{L}}$*, respectively;*

(iii) $\{A_s, \Gamma + [\mathfrak{L}]\}$ *realizes* f *minimally;*

(iv) $\mathfrak{M}^{[\perp]}$ *is the largest* A*-invariant subspace contained in* $\ker\Gamma^{[*]}$.

Proof. (i) Let $\mathfrak{M}$ be as in the statement, then $(A-\xi)^{-1}\mathfrak{M} \subseteq \mathfrak{M}$ for every $\xi \in \rho(A)$ by the resolvent identity. From the preceding inclusion it follows by elementary arguments that $\left((A-\xi)^{-1}\right)^{[*]}\mathfrak{M}^{[\perp]} \subseteq \mathfrak{M}^{[\perp]}$ or, equivalently, using the selfadjointness of A that $(A-\overline{\xi})^{-1}\mathfrak{M}^{[\perp]} \subseteq \mathfrak{M}^{[\perp]}$. Another application of the same argument yields that $(A-\xi)^{-1}\operatorname{clos}\mathfrak{M} \subseteq \operatorname{clos}\mathfrak{M}$. Since $\rho(A)$ is symmetric with respect to the real line for selfadjoint relations, see (2.1), $\mathfrak{M}$, $\operatorname{clos}\mathfrak{M}$ and $\mathfrak{M}^{[\perp]}$ are A-invariant and, hence, $\mathfrak{L}$ is A-invariant, too.

(ii) Since $\mathfrak{L}$ is neutral in a Pontryagin space, it is a finite-dimensional (closed) subspace. Therefore $\{\mathfrak{L}^{[\perp]}/\mathfrak{L}, [\cdot,\cdot]\}$ is a Pontryagin space with negative index $\kappa_m - \kappa_{\mathfrak{L}}$, see [1, Ch. 1: Cor. 9.14]. A calculation, using the A-invariance and neutrality of $\mathfrak{L}$, shows that $A_{\mathfrak{L}}$, defined via

$$\operatorname{gr}(A_{\mathfrak{L}}) = \Big\{\{f + [\mathfrak{L}], f' + [\mathfrak{L}]\} \in \mathfrak{L}^{[\perp]}/\mathfrak{L} \times \mathfrak{L}^{[\perp]}/\mathfrak{L} : \{f, f'\} \in \operatorname{gr}A \cap (\mathfrak{L}^{[\perp]} \times \mathfrak{L}^{[\perp]})\Big\}$$

is a symmetric linear relation in the introduced quotient space. To establish that A is selfadjoint, it suffices by [4, Thm. 4.6] to show that

$$\rho(A) \subseteq \rho(A_{\mathfrak{L}}). \tag{2.6}$$

Let $z \in \rho(A)$ be arbitrary. Since $\mathfrak{L}$ is A-invariant (see (i)), $\mathfrak{L}^{[\perp]}$ is also A-invariant and $\mathfrak{L}^{[\perp]} \subseteq \operatorname{ran}(A - z)$, because $z \in \rho(A)$ by assumption. Thus for every $g \in \mathfrak{L}^{[\perp]}$ there exists $\{f, f'\} \in A$, such that $g = f' - zf$. Now the A-invariance of $\mathfrak{L}^{[\perp]}$ implies that $f = (A-z)^{-1}g \in \mathfrak{L}^{[\perp]}$ and thus also $f' \in \mathfrak{L}^{[\perp]}$. Therefore,

$$\mathfrak{L}^{[\perp]} \subseteq \{\, f' - zf : \{f, f'\} \in \operatorname{gr}A \cap (\mathfrak{L}^{[\perp]} \times \mathfrak{L}^{[\perp]})\}.$$

Consequently, $\operatorname{ran}(A_{\mathfrak{L}} - z) = \mathfrak{L}^{[\perp]}/\mathfrak{L}$ and this implies that $z \in \rho(A_{\mathfrak{L}})$. Since $z \in \rho(A)$ was arbitrary, the above argument shows that (2.6) holds.

Now $\Gamma_{\mathfrak{L}}$, defined via $\Gamma_{\mathfrak{L}}h := \Gamma h + [\mathfrak{L}]$ for $h \in \mathcal{H}$, is an everywhere defined mapping from $\mathcal{H}$ to $\mathfrak{L}^{[\perp]}/\mathfrak{L}$. Using $\Gamma_{\mathfrak{L}}$ and $A_{\mathfrak{L}}$ define the subspace $\mathfrak{M}_{\mathfrak{L}}$ of $\mathfrak{L}^{[\perp]}/\mathfrak{L}$

as

$$\mathfrak{M}_{\mathfrak{L}} := \operatorname{span}\{\left(I + (z - z_0)(A_{\mathfrak{L}} - z)^{-1}\right)\Gamma_{\mathfrak{L}} h : z \in \rho(A_{\mathfrak{L}}),\ \ h \in \mathcal{H}\}. \tag{2.7}$$

By means of $\mathfrak{M}_{\mathfrak{L}}$ introduce in $\{\mathfrak{L}^{[\perp]}/\mathfrak{L}, [\cdot,\cdot]\}$ the subspaces $\Pi_s := \operatorname{clos}\mathfrak{M}_{\mathfrak{L}}$ and $\Pi_r := \Pi_s^{[\perp]_{\mathfrak{L}}}$; here $[\perp]_{\mathfrak{L}}$ denotes the orthogonal complement in $\{\mathfrak{L}^{[\perp]}/\mathfrak{L}, [\cdot,\cdot]\}$. Then clearly $\Pi_s = \operatorname{clos}(\mathfrak{M})/\mathfrak{L}$ and $\Pi_r = \mathfrak{M}^{[\perp]}/\mathfrak{L}$. Since $\mathfrak{L} = \operatorname{clos}(\mathfrak{M}) \cap \mathfrak{M}^{[\perp]}$, Π_s and Π_r are non-degenerate. Therefore $\{\Pi_s, [\cdot,\cdot]\}$ and $\{\Pi_r, [\cdot,\cdot]\}$ are Pontryagin spaces, see [1, Ch. 1: Thm. 7.16 & Thm. 9.9]. The same arguments used in (i) yield

$$(A_{\mathfrak{L}} - \xi)^{-1}\Pi_s \subseteq \Pi_s \quad \text{and} \quad (A_{\mathfrak{L}} - \xi)^{-1}\Pi_r \subseteq \Pi_r, \quad \xi \in \rho(A_{\mathfrak{L}}) \supseteq \rho(A). \tag{2.8}$$

Let A_s and A_r be as in (ii) with Π_s and Π_r as defined following (2.7), then A_s and A_r, being restrictions of the selfadjoint relation $A_{\mathfrak{L}}$, are symmetric. Moreover, (2.8) together with the decomposition $\mathfrak{L}^{[\perp]}/\mathfrak{L} = \Pi_s[\dot{+}]\Pi_r$ implies that $\rho(A_s)\cap\mathbb{C}_+$, $\rho(A_s)\cap\mathbb{C}_-$, $\rho(A_r)\cap\mathbb{C}_+$ and $\rho(A_r)\cap\mathbb{C}_-$ are all non-empty. Therefore A_s and A_r are selfadjoint relations; again cf. [4]. The last assertion on the negative indices of the Pontryagin spaces is a consequence of the result in (iii) combined with the fact that the negative index of the Pontryagin space $\{\mathfrak{L}^{[\perp]}/\mathfrak{L}, [\cdot,\cdot]\}$ is $\kappa_m - \dim\mathfrak{L} = \kappa_m - \kappa_{\mathfrak{L}}$.

(iii) Let Γ_z be the γ-field associated with the realization $\{A, \Gamma\}$ as in (2.3). Then for every $\omega_g, \omega_h \in \mathfrak{L}$ we have by definition of $\mathfrak{L}$ that

$$[\Gamma_z h + \omega_h, \Gamma_w g + \omega_g] = [\Gamma_z h, \Gamma_w g] = g^* \frac{F(z) - F(w)^*}{z - \overline{w}} h, \quad g, h \in \mathcal{H}.$$

Hence, $\{A_s, \Gamma_{\mathfrak{L}}\}$ realizes F, see (2.4). Moreover, this realization is minimal by construction, see the proof of (ii). Therefore the negative index of $\{\Pi_s, [\cdot,\cdot]\}$ is κ by Proposition 2.1 and the discussion preceding it.

(iv) In (i) it has been established that $\mathfrak{M}^{[\perp]}$ is A-invariant. The inclusion $\mathfrak{M}^{[\perp]} \subseteq \ker\Gamma^{[*]}$ follows directly from the fact that $\operatorname{ran}\Gamma \subseteq \mathfrak{M}$. Therefore to prove the assertion it suffices to show that all A-invariant subspaces $\mathfrak{N}$ contained in $\ker\Gamma^{[*]}$ are orthogonal to $\mathfrak{M}$. Let $\mathfrak{N}$ be any such subspace. Then for all $h \in \mathcal{H}$ and $z \in \rho(A)$

$$[(I + (z - z_0)(A - z)^{-1})\Gamma h, \mathfrak{N}] = (h, \Gamma^{[*]}(I + (\overline{z} - \overline{z_0})(A - \overline{z})^{-1})\mathfrak{N}) = 0.$$

This shows that $\mathfrak{N} \subseteq \mathfrak{M}^{[\perp]}$. □

Corollary 2.4. *Let $F \in \mathfrak{N}_\kappa(\mathcal{H})$ be realized by $\{A, \Gamma\}$ and let κ_m denote the negative index of the realizing Pontryagin space $\{\Pi, [\cdot,\cdot]\}$. Then*

$$\kappa_m - \kappa = \max_{\mathfrak{N}}\{\dim\mathfrak{N} : \mathfrak{N} \text{ is } A\text{-invariant}, \ \ \mathfrak{N} \subseteq \ker\Gamma^{[*]}\};$$

here the maximum is over all nonpositive subspaces $\mathfrak{N}$ of $\{\Pi, [\cdot,\cdot]\}$.

Proof. Using the notation as in Proposition 2.3, Proposition 2.3 (ii) shows that $\kappa_m - \kappa = \dim\mathfrak{L} + \kappa_r$; here κ_r is defined to be the negative index of the Pontryagin space $\{\Pi_r, [\cdot,\cdot]\}$. Since the negative index of the subspace $\mathfrak{M}^{[\perp]}$ of $\{\Pi, [\cdot,\cdot]\}$ is equal to $\dim \mathfrak{L} + \kappa_r$ and $\{\mathfrak{N} : \mathfrak{N} \text{ is } A\text{-invariant}, \ \mathfrak{N} \subseteq \ker \Gamma^{[*]}\} \subseteq \mathfrak{M}^{[\perp]}$ by Proposition 2.3 (iv), the statement is proven if the existence of a nonpositive A-invariant subspace of dimension $\dim \mathfrak{L} + \kappa_r$ contained in $\ker \Gamma^{[*]}$ is established.

Since A_r, the restriction of A to $\{\Pi_r, [\cdot,\cdot]\}$ (see Proposition 2.3 (ii)), is a selfadjoint relation in the Pontryagin space $\{\Pi_r, [\cdot,\cdot]\}$, the invariant subspace theorem states that there exists a κ_r-dimensional nonpositive subspace $\mathfrak{L}_r$ of $\{\Pi_r, [\cdot,\cdot]\}$ which is A_r-invariant, see e.g. [9, Thm. 12.1']. Therefore $\mathfrak{N} := \mathfrak{L}_r + \mathfrak{L}$ is a $(\dim \mathfrak{L} + \kappa_r)$-dimensional nonpositive A-invariant subspace contained in $\mathfrak{M}^{[\perp]}$. □

2.4. The sum of generalized Nevanlinna functions

A particular situation where non-minimal realizations may be encountered is when the sum of generalized Nevanlinna functions is considered; cf. [6]. Let $F_i \in \mathfrak{N}_{\kappa_i}(\mathcal{H})$ be (minimally) realized by $\{A_i, \Gamma_i\}$, for $i = 1, 2$. Then the sum $F_1 + F_2$ is realized by $\{A_1 \widehat{\oplus} A_2, \operatorname{col}(\Gamma_1, \Gamma_2)\}$, where

$$\begin{aligned} \operatorname{gr}(A_1 \widehat{\oplus} A_2) &= \{\{\{f_1, f_2\}, \{f_1', f_2'\}\} : \{f_i, f_i'\} \in \operatorname{gr}(A_i)\}; \\ \operatorname{col}(\Gamma_1, \Gamma_2)h &= \begin{pmatrix} \Gamma_1 h \\ \Gamma_2 h \end{pmatrix}. \end{aligned} \tag{2.9}$$

Here the realizing space is $\{\Pi_{\text{sum}}, [\cdot,\cdot]_{\text{sum}}\}$ where $\Pi_{\text{sum}} = \Pi_1 \times \Pi_2$ and

$$[\{f_1, f_2\}, \{g_1, g_2\}]_{\text{sum}} = [f_1, g_1]_1 + [f_2, g_2]_2, \quad \{f_1, f_2\}, \{g_1, g_2\} \in \Pi_1 \times \Pi_2. \tag{2.10}$$

To see this note that

$$\begin{aligned} &(\operatorname{col}(\Gamma_1, \Gamma_2))^{[*]}(I + (z - z_0)(A_1 \widehat{\oplus} A_2 - z)^{-1})\operatorname{col}(\Gamma_1, \Gamma_2) \\ =&\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}^{[*]} \begin{pmatrix} I + (z - z_0)(A_1 - z)^{-1} & 0 \\ 0 & I + (z - z_0)(A_2 - z)^{-1} \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} \\ =&\Gamma_1^{[*]}(I + (z - z_0)(A_1 - z)^{-1})\Gamma_1 + \Gamma_2^{[*]}(I + (z - z_0)(A_2 - z)^{-1})\Gamma_2 \\ =&\frac{F_1(z) - F_1(\overline{z_0})}{z - \overline{z_0}} + \frac{F_2(z) - F_2(\overline{z_0})}{z - \overline{z_0}} = \frac{F_1(z) + F_2(z) - (F_1(\overline{z_0}) + F_2(\overline{z_0}))}{z - \overline{z_0}}, \end{aligned}$$

where in the third step (2.2) was used. In view of (2.2) this calculation shows that $\{A_1 \widehat{\oplus} A_2, \operatorname{col}(\Gamma_1, \Gamma_2)\}$ realizes $F_1 + F_2$; cf. [6, Prop. 4.1]. In particular, $F_1 + F_2 \in \mathfrak{N}_{\kappa_{\text{sum}}}(\mathcal{H})$ where $\kappa_{\text{sum}} \leq \kappa_1 + \kappa_2$; cf. Proposition 2.3.

Notice, conversely, that if $\{A, \Gamma\}$ realizes the function $F \in \mathfrak{N}_\kappa(\mathcal{H})$ and there exists a decomposing (regular) subspace Π_1 of Π, i.e. $\Pi = \Pi_1 [\dot{+}] \Pi_2$ with $\Pi_2 = \Pi_1^{[\perp]}$, which also reduces A, $A = A_1 \widehat{\oplus} A_2$, then $\{A_1, P_1\Gamma\}$ and $\{A_2, P_2\Gamma\}$, where P_j with $j = 1, 2$ is the Π-orthogonal projection onto Π_j, produce realizations for generalized Nevanlinna functions F_1 and F_2 such that $F = F_1 + F_2$.

Proposition 2.5 below contains sufficient conditions for the index of $F_1 + F_2$ to be the sum of the indices of F_1 and F_2; see [2, Prop. 3.2] for a similar statement for matrix-valued generalized Nevanlinna functions.

Proposition 2.5. *Let $F_1 \in \mathfrak{N}_{\kappa_1}(\mathcal{H})$, $F_2 \in \mathfrak{N}_{\kappa_2}(\mathcal{H})$ and assume that F_1 and F_2 do not have a generalized pole in common. Then $F_1 + F_2 \in \mathfrak{N}_{\kappa_1+\kappa_2}(\mathcal{H})$.*

Proof. Let $\{A_i, \Gamma_i\}$ be a minimal realization for F_i where the realizing space is $\{\Pi_i, [\cdot,\cdot]_i\}$, for $i = 1, 2$. Then, as the discussion preceding this statement demonstrated, $F_1 + F_2$ is realized by $\{A, \Gamma\} := \{A_1 \widehat{\oplus} A_2, \operatorname{col}(\Gamma_1, \Gamma_2)\}$ where the realizing space is the Pontryagin space $\{\Pi, [\cdot,\cdot]\} := \{\Pi_{\mathrm{sum}}, [\cdot,\cdot]_{\mathrm{sum}}\}$ whose negative index is $\kappa_1 + \kappa_2$, see (2.10). Hence $F_1 + F_2$ is a generalized Nevanlinna function. In order to establish that its index is $\kappa_1 + \kappa_2$, the non-minimal part of its realization $\{A, \Gamma\}$ should be investigated; cf. Proposition 3.1. But first note that if P_1 and P_2 are the orthogonal projections onto Π_1 and Π_2 in Π_{sum}, then

$$(A_1 \widehat{\oplus} A_2 - z)^{-1} P_i = (A_i - z)^{-1} P_i = P_i (A_1 \widehat{\oplus} A_2 - z)^{-1}, \qquad i = 1, 2, \tag{2.11}$$

see (2.9). Denote by $\mathfrak{M}^{[\perp]}$ the non-minimal part of the realization $\{A, \Gamma\}$ as in Proposition 2.3. If $\mathfrak{L} := \operatorname{clos}(\mathfrak{M}) \cap \mathfrak{M}^{[\perp]} \neq \{0\}$, then $\mathfrak{L}$, being finite-dimensional and A-invariant (see Proposition 2.3 (i)), contains an eigenvector x for $A = A_1 \widehat{\oplus} A_2$ such that $x \in \ker \Gamma^{[*]}$. But, then (2.11) implies that $P_1 x$ and $P_2 x$ are eigenvectors for A_1 and A_2, respectively. Since $\sigma_p(A_1) \cap \sigma_p(A_2) = \emptyset$ by assumption, this implies that either of the two vectors is zero; say $P_2 x = 0$. Thus $P_1 x$ is an eigenvector for A_1, $P_2 x = 0$ and $x \in \ker \Gamma^{[*]}$. The last two conditions together yield that $P_1 x \in \ker \Gamma_1^{[*]}$; cf. (2.9). But then the realization $\{A_1, \Gamma_1\}$ for F_1 is not minimal by Proposition 2.3; in contradiction to the assumption. I.e., $\mathfrak{L} = \{0\}$.

Therefore $\mathfrak{M}^{[\perp]}$ is A-invariant and $\{\mathfrak{M}^{[\perp]}, [\cdot,\cdot]\}$ is a Pontryagin space, see Proposition 2.3 (ii). The exact same argument as used in the preceding paragraph shows that $\sigma_p(A \restriction_{\mathfrak{M}^{[\perp]}}) = \emptyset$. Hence Pontryagin's invariant subspace theorem (applied to the selfadjoint relation $A \restriction_{\mathfrak{M}^{[\perp]}}$ in $\{\mathfrak{M}^{[\perp]}, [\cdot,\cdot]\}$) implies that $\{\mathfrak{M}^{[\perp]}, [\cdot,\cdot]\}$ is a Hilbert space, see e.g. [9, Thm. 12.1']. Consequently, the statement holds by Proposition 2.3 (ii); cf. Corollary 2.4. □

Extending upon Proposition 2.5, the following result shows when a minimal realization for $F_1 + F_2$ can be obtained when starting from minimal realizations for F_1 and F_2.

Proposition 2.6. *Let $\{A_i, \Gamma_i\}$ minimally realize the generalized Nevanlinna function $F_i \in \mathfrak{N}_{\kappa_i}(\mathcal{H})$, for $i = 1, 2$, and assume that*

$$\sigma_p(A_1) \cap \sigma_p(A_2) = \emptyset \quad \text{and} \quad \sigma(A_1) \cap \sigma(A_2) = \{\gamma_1, \ldots, \gamma_n\} \subseteq \mathbb{R} \cup \{\infty\}.$$

Then $F_1 + F_2 \in \mathfrak{N}_{\kappa_1+\kappa_2}(\mathcal{H})$ is minimally realized by $\{A_1 \widehat{\oplus} A_2, \operatorname{col}(\Gamma_1, \Gamma_2)\}$.

Proof. As the above discussion demonstrated, $F_1 + F_2$ is realized by $\{A, \Gamma\} := \{A_1 \widehat{\oplus} A_2, \operatorname{col}(\Gamma_1, \Gamma_2)\}$ where the realizing space is the Pontryagin space $\{\Pi, [\cdot,\cdot]\} := \{\Pi_{\mathrm{sum}}, [\cdot,\cdot]_{\mathrm{sum}}\}$ whose negative index is $\kappa_1 + \kappa_2$, see (2.10). To prove the minimality of the realization for $F_1 + F_2$ let $\mathfrak{M}$ be as in Proposition 2.3.

Since the index of $F_1 + F_2$ is equal to the negative index of $\{\Pi, [\cdot,\cdot]\}$ by Proposition 2.5, Proposition 2.3 yields that $\{\mathfrak{M}^{[\perp]}, [\cdot,\cdot]\}$ is a Hilbert space and that A_r, defined via $\mathrm{gr}(A_r) = \mathrm{gr}(A) \cap (\mathfrak{M}^{[\perp]} \times \mathfrak{M}^{[\perp]})$, is a selfadjoint relation in $\{\mathfrak{M}^{[\perp]}, [\cdot,\cdot]\}$. In particular, $\sigma(A_r) \subseteq \mathbb{R} \cup \{\infty\}$. We claim that

$$\sigma(A_r) \subseteq \sigma(A_1) \quad \text{and} \quad \sigma(A_r) \subseteq \sigma(A_2). \tag{2.12}$$

If the first inclusion does not hold, then, since $\sigma(A_1) \cap (\mathbb{R} \cup \{\infty\})$ and $\sigma(A_r)$ are closed subsets of $\mathbb{R} \cup \{\infty\}$, there exists a closed interval $\Delta = [a, b]$ of $\mathbb{R}$ such that

$$\Delta \cap \sigma(A_r) \neq \emptyset \quad \text{and} \quad \Delta \subseteq \rho(A_1). \tag{2.13}$$

Let E_t be the spectral family of A_r and let P_i be the orthogonal projections onto Π_i in Π, for $i = 1, 2$. Then the assumption $\Delta \cap \sigma(A_r) \neq \emptyset$ implies that

$$\mathfrak{L} := (E_b - E_a)\mathfrak{M}^{[\perp]} \neq \{0\}.$$

Consider $\mathfrak{L}_1 := P_1\mathfrak{L} \subseteq \Pi_1$. Then, on the one hand,

$$\sigma(A\restriction_{\mathfrak{L}_1}) \subseteq \sigma(A\restriction_{\mathfrak{L}}) \subseteq \Delta \subseteq \rho(A_1).$$

On the other hand, the A_1-invariance of $\mathfrak{L}_1$ implies that $\sigma(A\restriction_{\mathfrak{L}_1}) \subseteq \sigma(A_1)$. The preceding two results together imply that $\mathfrak{L}_1 = \{0\}$; cf. (2.13). In other words, $\mathfrak{L} \subseteq \{0\} \times \Pi_2$. But then $\mathfrak{L} \subseteq \ker \Gamma_2^{[*]}$, because $\mathfrak{L} \subseteq \mathfrak{M}^{[\perp]} \subseteq \ker \Gamma^{[*]}$. Consequently, the realization $\{A_2, \Gamma_2\}$ is not minimal. This contradiction shows that the first inclusion in (2.12) holds. By symmetry the second inclusion also holds.

Combining the inclusions from (2.12) together with the assumption about $\sigma(A_1) \cap \sigma(A_1)$ yields that $\sigma(A_r)$ consists at most of isolated points. I.e., all the spectrum of A_r is point spectrum. Let x be an eigenvector for A_r. Then P_1x and P_2x are eigenvectors for A_1 and A_2, respectively. Since $\sigma_p(A_1) \cap \sigma_p(A_2) = \emptyset$, either P_1x or P_2x should be equal to zero. Assume the latter. Since $x \in \mathfrak{M}^{[\perp]} \subseteq \ker \Gamma^{[*]}$ (see Proposition 2.3), it follows that $x = P_1x \subseteq \ker \Gamma_1^{[*]}$; but this is in contradiction to the assumed minimality of the realization $\{A_1, \Gamma_1\}$ of F_1. □

3. Decompositions of generalized Nevanlinna functions

For $\alpha, \beta \in \mathbb{C} \cup \{\infty\}$, with $\alpha \neq \beta$, and for non-orthogonal vectors η and ξ in a Hilbert space $\mathcal{H}$ define the operator-valued rational function R as:

$$R(z; \alpha, \beta, \eta, \xi) = I - P + \frac{z-\alpha}{z-\beta}P, \qquad P = \frac{\xi\eta^*}{\eta^*\xi}, \quad \eta^*\xi \neq 0; \tag{3.1}$$

here $R(z; \infty, \beta, \eta, \xi)$ and $R(z; \alpha, \infty, \eta, \xi)$ should be interpreted to be $I - P + (z - \beta)^{-1}P$ and $I - P + (z - \alpha)P$, respectively. Note that

$$(R(z; \alpha, \beta, \eta, \xi))^{\#} = R(z; \overline{\alpha}, \overline{\beta}, \xi, \eta) \quad \text{and} \quad (R(z; \alpha, \beta, \eta, \xi))^{-1} = (R(z; \beta, \alpha, \eta, \xi));$$

here for any operator-valued function $Q(z)$, $Q^{\#}(z)$ is defined to be $Q(\overline{z})^*$.

With this notation, (realizations for) products of the form $R^{\#}FR$, where $R(z) = R(z;\alpha,\beta,\eta,\xi)$, are investigated in the first subsection. In the second subsection these considerations are combined with a factorization from [14] to decompose generalized Nevanlinna functions with respect to their analytic behavior as stated in Theorem 1.1. These results are in turn used to prove Theorem 1.2 in the third and final subsection.

3.1. Multiplication with an order one term

Here an explicit realization for $R^{\#}FR$, where R is as in (3.1), is generated from any given realization for $F \in \mathfrak{N}_\kappa(\mathcal{H})$. This realization expresses can be seen as a modification and extension of [16, Thm. 1.3] from scalar-valued to operator-valued functions. Note that the explicit resolvent formula in Proposition 3.1 reflects how the invariant subspaces of the realizing relation for $R^{\#}FR$ are connected to the invariant subspaces of the realizing relation for the original function F.

Proposition 3.1. *Let $F \in \mathfrak{N}_\kappa(\mathcal{H})$ be realized by $\{A,\Gamma\}$ at $z_0 \in \rho(A)\setminus\{\beta,\overline{\beta}\}$, where $\alpha,\beta \in \mathbb{C}\cup\{\infty\}$ satisfy $\alpha \neq \overline{\beta}$, and let $\xi,\eta \in \mathcal{H}$ satisfy $\eta^*\xi \neq 0$. Then $F_R := R^{\#}FR$, where $R(z) = R(z;\overline{\alpha},\beta,\eta,\xi)$ as in* (3.1)*, is realized by $\{A_R,\Gamma_R\}$ which are defined for $z \in \rho(A)\setminus\{\beta,\overline{\beta}\}$ via*

$$(A_R - z)^{-1} = \begin{pmatrix} \frac{1}{\overline{\beta}-z} & \frac{\xi^*\Gamma_{\overline{z}}^{[*]}}{\overline{\beta}-z} & \frac{\xi^*F(z)\xi}{(\overline{\beta}-z)(\beta-z)} \\ 0 & (A-z)^{-1} & \frac{\Gamma_z\xi}{\beta-z} \\ 0 & 0 & \frac{1}{\beta-z} \end{pmatrix}, \quad \Gamma_R = \begin{pmatrix} \frac{\xi^*F(z_0)}{\overline{\beta}-z_0}R(z_0) \\ \Gamma R(z_0) \\ \frac{\overline{\alpha}-\beta}{\beta-z_0}\frac{\eta^*}{\eta^*\xi} \end{pmatrix}. \quad (3.2)$$

Here the realizing space $\{\Pi_2,[\cdot,\cdot]_2\}$ of $\{A_R,\Gamma_R\}$ is defined as

$$[g,h]_2 := [g_c,h_c] + g_r\overline{h_l} + g_l\overline{h_r}, \quad g = \{g_l,g_c,g_r\}, h = \{h_l,h_c,h_r\} \in \Pi_2 := \mathbb{C}\times\Pi\times\mathbb{C},$$

where $\{\Pi,[\cdot,\cdot]\}$ is the realizing space of $\{A,\Gamma\}$.

Recall that Γ_z in Proposition 3.1 is the γ-field associated with the realization $\{A,\Gamma\}$ for F, see (2.3). Furthermore, if $\alpha = \infty$, then Γ_R should be interpreted to be

$$\Gamma_R = \begin{pmatrix} \frac{\xi^*F(z_0)}{\overline{\beta}-z_0}R(z_0) & \Gamma R(z_0) & -\frac{1}{\beta-z_0}\frac{\eta^*}{\eta^*\xi} \end{pmatrix}^T,$$

and if $\beta = \infty$, then $\{A_R,\Gamma_R\}$ should be interpreted to be

$$(A_R - z)^{-1} = \begin{pmatrix} 0 & \xi^*\Gamma_{\overline{z}}^{[*]} & \xi^*F(z)\xi \\ 0 & (A-z)^{-1} & \Gamma_z\xi \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_R = \begin{pmatrix} \xi^*F(z_0)R(z_0) \\ \Gamma R(z_0) \\ \frac{\eta^*}{\eta^*\xi} \end{pmatrix}.$$

Proof. Here only the case $\alpha,\beta \in \mathbb{C}$ is treated; the cases $\alpha = \infty$ or $\beta = \infty$ follow by analogous arguments.

First the selfadjointness of A_R is established. Therefore let $H(z) := (A_R - z)^{-1}$. Then the formula in (3.2) shows that $H(z)$ is an everywhere defined operator for $z \in \rho(A)\setminus\{\beta,\overline{\beta}\}$. In particular, since $\rho(A) \neq \emptyset$, $\rho(A)$ contains all of $\mathbb{C}\setminus\mathbb{R}$ except finitely many points, for all those points $H(z)$ is an everywhere defined bounded operator. Moreover, a direct calculation shows that $H(z)^{[*]} = H(\overline{z})$. Next we establish that H satisfies the resolvent identity. Therefore note that a calculation shows that $H(z)H(w)$ is equal to

$$\begin{pmatrix} \frac{1}{(\overline{\beta}-z)(\overline{\beta}-w)} & \frac{\xi^*}{\overline{\beta}-z}\left(\frac{\Gamma_{\overline{w}}^{[*]}}{\overline{\beta}-w} + \Gamma_{\overline{z}}^{[*]}(A-w)^{-1}\right) & \xi^* \frac{\frac{F(w)}{\overline{\beta}-w} + \Gamma_{\overline{z}}^{[*]}\Gamma_w + \frac{F(z)}{\beta-z}}{(\overline{\beta}-z)(\beta-w)}\xi \\ 0 & (A-z)^{-1}(A-w)^{-1} & \left((A-z)^{-1}\Gamma_w + \frac{\Gamma_z}{\beta-z}\right)\frac{\xi}{\beta-w} \\ 0 & 0 & \frac{1}{(\beta-z)(\beta-w)} \end{pmatrix}.$$

Using (2.3) and the resolvent identity for A we have that

$$\begin{aligned} \Gamma_{\overline{z}}^{[*]}(A-w)^{-1} &= \Gamma^{[*]}\left(I + (z-\overline{z_0})(A-z)^{-1}\right)(A-w)^{-1} \\ &= \Gamma^{[*]}\left((A-w)^{-1} + \frac{z-\overline{z_0}}{z-w}\left((A-z)^{-1} - (A-w)^{-1}\right)\right) \\ &= \frac{\Gamma^{[*]}}{z-w}\left((z-\overline{z_0})(A-z)^{-1} - (w-\overline{z_0})(A-w)^{-1}\right) = \frac{\Gamma_{\overline{z}}^{[*]} - \Gamma_{\overline{w}}^{[*]}}{z-w}. \end{aligned}$$

Moreover, using (2.4) we have that

$$\begin{aligned} \frac{F(w)}{\overline{\beta}-w} + \Gamma_{\overline{z}}^{[*]}\Gamma_w + \frac{F(z)}{\beta-z} &= F(z)\left(\frac{1}{\beta-z} + \frac{1}{z-w}\right) + F(w)\left(\frac{1}{\overline{\beta}-w} - \frac{1}{z-w}\right) \\ &= \frac{1}{z-w}\left(\frac{\beta-w}{\beta-z}F(z) - \frac{\overline{\beta}-z}{\overline{\beta}-w}F(w)\right). \end{aligned}$$

Combining the three preceding expressions and using the resolvent identity for A yields that $H(z)H(w) = \frac{H(z)-H(w)}{z-w}$. Consequently, A_R is a selfadjoint relation in $\{\Pi_2, [\cdot,\cdot]_2\}$, see [4, Prop. 3.4 and Cor. on p. 162].

As the second step towards proving that $\{A_R, \Gamma_R\}$ realizes F_R, the γ-field associated with $\{A_R, \Gamma_R\}$ is determined. Using

$$\frac{z_0-\overline{\alpha}}{z_0-\beta} + \frac{z-z_0}{\beta-z}\frac{\overline{\alpha}-\beta}{\beta-z_0} = \frac{\overline{\alpha}-\beta}{\beta-z} + 1 = \frac{z-\overline{\alpha}}{z-\beta} \tag{3.3}$$

and the identity $(z-z_0)\Gamma_{\overline{z}}^{[*]}\Gamma = F(z) - F(z_0)$, see (2.4), a straight-forward calculation shows that

$$(\Gamma_R)_z := (I + (z-z_0)(A_R-z)^{-1})\Gamma_R = \begin{pmatrix} \frac{\xi^* F(z)}{\overline{\beta}-z}R(z) & \Gamma_z R(z) & \frac{\overline{\alpha}-\beta}{\beta-z}\frac{\eta^*}{\eta^*\xi} \end{pmatrix}^{\top}.$$

Combining this last result with (3.3) and the identity $(z-\overline{z_0})\Gamma^{[*]}\Gamma_z = F(z) - F(\overline{z_0})$ from (2.4) leads to

$$\begin{aligned} (z-\overline{z_0})\Gamma_R^{[*]}(\Gamma_R)_z &= \frac{z-\overline{z_0}}{\overline{\beta}-z}\frac{\alpha-\overline{\beta}}{\overline{\beta}-\overline{z_0}}\frac{\eta\xi^*}{\xi^*\eta}F(z)R(z) + R(z_0)^*F(\overline{z_0})\frac{z-\overline{z_0}}{\beta-\overline{z_0}}\frac{\overline{\alpha}-\beta}{\beta-z}\frac{\xi\eta^*}{\eta^*\xi} \\ &\quad + \left[(I-P^*) + \frac{\overline{z_0}-\alpha}{\overline{z_0}-\overline{\beta}}P^*\right](F(z) - F(\overline{z_0}))[(I-P) + \frac{z-\overline{\alpha}}{z-\beta}P] \\ &= R^{\#}(z)F(z)R(z) - R^{\#}(\overline{z_0})F(\overline{z_0})R(\overline{z_0}) = F_R(z) - F_R(\overline{z_0}). \end{aligned}$$

This shows that $\{A_R, \Gamma_R\}$ realizes F_R, see (2.4). □

3.2. Decomposing generalized Nevanlinna functions

Recall that $\rho(F)$ and $\sigma(F)$ denote the set of holomorphy of a generalized Nevanlinna function F in $\mathbb{C}\cup\{\infty\}$ and its complement, respectively. When F is minimally realized by $\{A, \Gamma\}$, then $\rho(F)$ and $\sigma(F)$ coincide with $\rho(A)$ and $\sigma(A)$, respectively, see Theorem 2.2.

Proof of Theorem 1.1. Let $z_0 \in \rho(F) \cap (\mathbb{C} \setminus \mathbb{R})(\neq \emptyset)$, then there exists an everywhere defined selfadjoint operator C in $\mathcal{H}$ such that $\operatorname{ran}(F(z_0) + C) = \mathcal{H}$, i.e., that $F + C$ is boundedly invertible at z_0. Since the statement clearly holds for F if it holds for $F + C$, we may w.l.o.g. assume that F is boundedly invertible at a point $z_0 \in \rho(F) \cap (\mathbb{C} \setminus \mathbb{R})$, cf. [14, Prop. 2.1]; such operator-valued generalized Nevanlinna functions are called *regular*.

Let $\{\alpha_1, \ldots, \alpha_\kappa\}$ and $\{\beta_1, \ldots, \beta_\kappa\}$ be the sets of all GPNTs of F and $-F^{-1}$ in $\mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\}$, respectively; here each GPNT occurs in accordance with its multiplicity. Since F is assumed to be regular, [14, Thm. 5.2 and Cor. 5.3] yield the existence of $\eta_1, \xi_1, \tilde{\eta}_1, \tilde{\xi}_1 \in \mathcal{H}$ satisfying $\eta_1^*\xi_1 \neq 0$ and $\tilde{\eta}_1^*\tilde{\xi}_1 \neq 0$ such that $F_1 := R_1^{\#} F R \in \mathfrak{N}_{\kappa-1}(\mathcal{H})$, where

$$R_1(z) = R(z; \beta_1, \overline{\gamma}, \tilde{\eta}_1, \tilde{\xi}_1) R(z; \gamma, \overline{\alpha_1}, \eta_1, \xi_1);$$

here γ is an arbitrary element of $\mathbb{C} \setminus (\mathbb{R} \cup \operatorname{GPNT}(F) \cup \operatorname{GPNT}(-F^{-1}))$. Moreover, the cited statements yield that $\{\alpha_2, \ldots, \alpha_\kappa\}$ and $\{\beta_2, \ldots, \beta_\kappa\}$ are the sets of all GPNTs of F_1 and $-F_1^{-1}$ in $\mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\}$, respectively. Since F_1 is evidently regular, inductively applying this argument yields that F can be factorized as $R^{\#} F_0 R$, where $F_0 \in \mathfrak{N}_0(\mathcal{H})$ and

$$R(z) = \prod_{j=1}^{\kappa} R(z; \beta_i, \overline{\gamma}, \tilde{\eta}_i, \tilde{\xi}_i) R(z; \gamma, \overline{\alpha_i}, \eta_i, \xi_i); \tag{3.4}$$

here γ is any element of $\mathbb{C} \setminus (\mathbb{R} \cup \operatorname{GPNT}(F) \cup \operatorname{GPNT}(-F^{-1}))$ and $\eta_i, \xi_i, \tilde{\eta}_i, \tilde{\xi}_i \in \mathcal{H}$ satisfy $\eta_i^*\xi_i \neq 0 \neq \tilde{\eta}_i^*\tilde{\xi}_i$ for $i = 1, \ldots, \kappa$. For later usage introduce the set $\mathcal{P}_0$ as

$$\mathcal{P}_0 := \{\gamma, \overline{\gamma}\} \cup \operatorname{GPNT}(F) = \{\gamma, \overline{\gamma}\} \cup \{\alpha_1, \ldots, \alpha_\kappa, \overline{\alpha_1}, \ldots, \overline{\alpha_\kappa}\}. \tag{3.5}$$

Let $\{A_0, \Gamma_0\}$ realize F_0 minimally, then the corresponding realizing space is a Hilbert space $\{\mathfrak{H}, (\cdot, \cdot)\}$, see e.g. [15]. Using the spectral family of A_0, $\mathfrak{H}$ can be decomposed as $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ such that, with A_i defined via $\operatorname{gr}(A_i) = \operatorname{gr}(A) \cap (\mathfrak{H}_i \times \mathfrak{H}_i)$,

(a) $\{\mathfrak{H}_i, (\cdot, \cdot)\}$ is a Hilbert space and $\mathfrak{H}_i$ is A-invariant for $i = 1, 2$;
(b) $\sigma(A_1) \subseteq \operatorname{clos}\Delta$ and $\operatorname{int}\Delta \subseteq \rho(A_2)$;
(c) $\sigma_p(A_1) \cap \sigma_p(A_2) = \emptyset$.

For instance, if $\Delta = (a, b) \subseteq \mathbb{R}$, then the desired decomposition with the properties (a)–(c) can be obtained by taking $\mathfrak{H}_1$ to be $E_b - E_a$, where $\{E_x\}_{x\in\mathbb{R}}$ is the spectral family associated with the Hilbert space selfadjoint relation A_0.

Since $\{A_0, \Gamma_0\}$ is a minimal realization for F_0, the decomposition with the properties (a)–(c) induces an additive representation $F_0 = F_1 + F_2$, where F_j is an ordinary Nevanlinna function realized by $\{A_j, P_j\Gamma_0\}$; here P_j is the Π-orthogonal projection onto $\mathfrak{H}_j$ for $j = 1, 2$, see the discussion following (2.10). Notice that the realizations $\{A_1, P_1\Gamma_0\}$ and $\{A_2, P_2\Gamma_0\}$ are automatically minimal, because the realization $\{A_0, \Gamma_0\}$ is assumed to be minimal. Inserting this additive representation $F_0 = F_1 + F_2$ into the factorization $F = R^{\#}F_0R$ produces the following decomposition for F:

$$F(z) = R^{\#}(z)F_0(z)R(z) = R^{\#}(z)F_1R(z) + R^{\#}(z)F_2(z)R(z). \tag{3.6}$$

Next the terms $R^{\#}F_1R$ and $R^{\#}F_2R$ are considered separately. In order to treat them, divide $\mathcal{P}_0$, see (3.5), into the following three sets:

$$\mathcal{P}_\Delta = \mathcal{P}_0 \cap \operatorname{int}\Delta, \quad \mathcal{P}_c = \mathcal{P}_0 \cap \partial\Delta \quad \text{and} \quad \mathcal{P}_r = \mathcal{P}_0 \setminus (\mathcal{P}_\Delta \cup \mathcal{P}_c). \tag{3.7}$$

$R^{\#}F_1R$**:** By Proposition 3.1 there exist an extension $A_{1,R}$ of A_1 in a Pontryagin space $\{\Pi_{1,R}, [\cdot,\cdot]_{1,R}\}$ (with at most 2κ negative squares since, in addition to the poles $\overline{\alpha_i}$, R in (3.4) can have at most κ additional poles located at $\overline{\gamma}$) and a mapping $\Gamma_{1,R}$ such that $\{A_{1,R}, \Gamma_{1,R}\}$ realizes $R^{\#}F_1R$. Furthermore, Proposition 3.1 shows that

$$\sigma(A_{1,R}) \subseteq \sigma(A_1) \cup \mathcal{P}_0 = \sigma(A_1) \cup \mathcal{P}_\Delta \cup \mathcal{P}_c \cup \mathcal{P}_r.$$

By definition, see (b) and (3.7), $\mathcal{P}_r$ consists of (finitely many) isolated points of the spectrum $\sigma(A_{1,R})$. Therefore $\{\Pi_{1,R}, [\cdot,\cdot]\}$ can by means of Riesz projections (contour integrals of the resolvent, see e.g. [1, Ch. 2: Thm. 2.20]) be decomposed as $\Pi^1_{1,R}[+]\Pi^2_{1,R}$, such that, with $A_{1,R,i}$ defined by $\operatorname{gr}(A_{1,R,i}) = \operatorname{gr}(A_{1,R}) \cap (\Pi^i_{1,R} \times \Pi^i_{1,R})$, the following statements hold:

- (a_1) $\{\Pi^i_{1,R}, [\cdot,\cdot]_{1,R}\}$ is a Pontryagin space and $\Pi^i_{1,R}$ is $A_{1,R}$-invariant for $i = 1, 2$;
- (b_1) $\sigma(A_{1,R,1}) \subseteq \sigma(A_1) \cup \mathcal{P}_\Delta \cup \mathcal{P}_c \subseteq \operatorname{clos}\Delta$;
- (c_1) $\sigma(A_{1,R,2}) \subseteq \mathcal{P}_r$ and, hence, $\operatorname{int}\Delta \subseteq \rho(A_{1,R,2})$.

$R^{\#}F_2R$**:** By Proposition 3.1 there exist an extension $A_{2,R}$ of A_2 in a Pontryagin space $\{\Pi_{2,R}, [\cdot,\cdot]_{2,R}\}$ (again with at most 2κ negative squares) and a mapping $\Gamma_{2,R}$ such that $\{A_{2,R}, \Gamma_{2,R}\}$ realizes $R^{\#}F_2R$. Again Proposition 3.1 shows that

$$\sigma(A_{2,R}) \subseteq \sigma(A_2) \cup \mathcal{P}_0 = \sigma(A_2) \cup \mathcal{P}_\Delta \cup \mathcal{P}_c \cup \mathcal{P}_r.$$

Hence, by construction (see (b)) there exist an open neighborhood $\mathcal{O}$ (in $\mathbb{C}$) containing $\mathcal{P}_\Delta$ such that $\mathcal{O} \setminus \mathcal{P}_\Delta \subseteq \rho(A_{2,R})$. Thus $\{\Pi_{2,R}, [\cdot,\cdot]\}$ can by means of Riesz projections (see [1, Ch. 2: Thm. 2.20]) be decomposed as $\Pi^1_{2,R}[+]\Pi^2_{2,R}$, where, with $A_{2,R,i}$ defined via $\operatorname{gr}(A_{2,R,i}) = \operatorname{gr}(A_{2,R}) \cap (\Pi^i_{2,R} \times \Pi^i_{2,R})$, the following statements hold:

- (a_2) $\{\Pi^i_{2,R}, [\cdot,\cdot]_{2,R}\}$ is a Pontryagin space and $\Pi^i_{2,R}$ is $A_{2,R}$-invariant for $i = 1, 2$;
- (b_2) $\sigma(A_{2,R,1}) \subseteq \mathcal{P}_\Delta \subseteq \Delta$;
- (c_2) $\sigma(A_{2,R,2}) \subseteq \sigma(A_2) \cup \mathcal{P}_c \cup \mathcal{P}_r$ and, hence, $\operatorname{int}\Delta \subseteq \rho(A_{2,R,2})$.

Now we reconsider F and decompose it as claimed in Theorem 1.1. Therefore let $F_{i,j}$ be the function realized by $\{A_{i,R,j}, \Gamma_{i,R,j}\}$ for $i, j = 1, 2$. By means of these functions define F_Δ and F_R as

$$F_\Delta := F_{1,1} + F_{2,1} \quad \text{and} \quad F_R := F_{1,2} + F_{2,2}.$$

We claim that these functions satisfy all the criteria in Theorem 1.1. Indeed, by construction the functions F_Δ and F_R are (possibly non-minimally) realized by $\{A_\Delta, \Gamma_\Delta\} := \{A_{1,R,1} \oplus A_{2,R,1}, \operatorname{col}(\Gamma_{1,R,1}, \Gamma_{2,R,1})\}$ and $\{A_R, \Gamma_R\} := \{A_{1,R,2} \oplus A_{2,R,2}, \operatorname{col}(\Gamma_{1,R,2}, \Gamma_{2,R,2})\}$, see Section 2.4. Therefore (b)-(c), (b_1)-(c_1) and (b_2)-(c_2) show that

$$\sigma(A_\Delta) \subseteq \operatorname{clos}\Delta, \quad \operatorname{int}(\Delta) \subseteq \rho(A_R) \quad \text{and} \quad \sigma_p(A_\Delta) \cap \sigma_p(A_r) \subseteq \mathcal{P}_c.$$

Since $\mathcal{P}_c$ is by definition equal to $(\{\gamma, \overline{\gamma}\} \cup \operatorname{GPNT}(F)) \cap \partial\Delta$, cf. (3.5) and (3.7), (2.5) and Proposition 2.3 show that all the assertions in Theorem 1.1 hold except the assertions about the sum of the indices κ_Δ and κ_R of F_Δ and F_R. The fact that $\kappa_\Delta + \kappa_R \geq \kappa$ is indicated in the discussion preceding (2.9). The final assertion in Theorem 1.1 that $\kappa_\Delta + \kappa_R = \kappa$ if $\operatorname{GPNT}(F) \cap (\operatorname{clos}(\Delta) \setminus \Delta) = \emptyset$ is now a consequence of Proposition 2.5. □

Inductively applying the preceding statement to the case when Δ is an interval of $\mathbb{R} \cup \{\infty\}$ containing precisely one GPNT in its interior yields Corollary 3.2 below. Note in connection with Corollary 3.2 that since non-real poles of a generalized Nevanlinna function are isolated, we can always write a generalized Nevanlinna function as the sum of a generalized Nevanlinna function holomorphic in $\mathbb{C} \setminus \mathbb{R}$ with rational functions each having a pole only at a non-real point and its conjugate.

Corollary 3.2. *Let $F \in \mathfrak{N}_\kappa(\mathcal{H})$ and let $\operatorname{GPNT}(F) = \{\alpha_1, \ldots, \alpha_n, \overline{\alpha_1}, \ldots, \overline{\alpha_n}\}$ where $\alpha_1, \ldots, \alpha_n$ are distinct elements of $\mathbb{C} \cup \{\infty\}$. Then $F = \sum_{i=1}^n F_i$, where*

(i) *$F_i \in \mathfrak{N}_{\kappa_i}(\mathcal{H})$, for $i = 1, \ldots, n$, and $\sum_{i=1}^n \kappa_i = \kappa$;*
(ii) *$\operatorname{GPNT}(F_i) = \{\alpha_i, \overline{\alpha_i}\}$, for $i = 1, \ldots, n$;*
(iii) *$\sigma(F_i) \cap \sigma(F_j)$ contains at most two points and any point contained in the intersection is not both a generalized pole for F_i and F_j, for $1 \leq i \neq j \leq n$.*

3.3. Decomposing selfadjoint relations in Pontryagin spaces

In order to prove Theorem 1.2 the result from the preceding section is lifted to the setting of selfadjoint relations by associating to (the resolvent) of selfadjoint relations an (operator-valued) generalized Nevanlinna function.

Proof of Theorem 1.2. Let J be any canonical symmetry for the Pontryagin space $\{\Pi, [\cdot,\cdot]\}$ appearing in Theorem 1.2. Then $\{\mathcal{H}, (\cdot,\cdot)\} := \{\Pi, [J\cdot,\cdot]\}$ defines a Hilbert space, see e.g. [1, Ch. 1, § 3]. In addition to the given selfadjoint relation A in the Pontryagin space Π introduce the operator $\Gamma : \mathcal{H} (= \Pi) \to \Pi$ as the identity mapping. Then the pair $\{A, \Gamma\}$ provides a minimal realization for the following generalized Nevanlinna function:

$$F(z) = \overline{z_0}J + (z - \overline{z_0})J\left(I + (z - z_0)(A - z)^{-1}\right), \quad z_0, z \in \rho(A); \tag{3.8}$$

cf. (2.2). Let $F_\Delta + F_R$ be the additive decomposition of F provided by Theorem 1.1 with respect to Δ as in Theorem 1.2. In particular,

$$\sigma(F_\Delta) \subseteq \operatorname{clos}\Delta \quad \text{and} \quad \operatorname{int}\Delta \subseteq \rho(F_R). \tag{3.9}$$

If $\{A_\Delta, \Gamma_\Delta\}$ and $\{A_R, \Gamma_R\}$ are arbitrary minimal realizations for F_Δ and F_R, respectively, then $\{A_\Delta \widehat{\oplus} A_R, \operatorname{col}(\Gamma_\Delta, \Gamma_R)\}$ is a realization for F. Moreover, by Theorem 2.2 $\sigma(A_\Delta) = \sigma(F_\Delta)$ and $\rho(A_R) = \rho(F_R)$. In view of (3.9), the first statement in Theorem 1.2 now holds by Proposition 2.3 and 2.1.

Finally, if $\partial\Delta \cap \mathrm{ENT}\,(A) = \emptyset$, then by definition $\partial\Delta \cap \mathrm{GPNT}\,(F) = \emptyset$. Thus the additive decomposition $F_\Delta + F_R$ of F with respect to Δ provided by Theorem 1.1 has the following properties:

(a) $\sigma(F_\Delta) \subseteq \operatorname{clos}\Delta$ and $\operatorname{int}\Delta \subseteq \rho(F_R)$;
(b) no point of $\operatorname{clos}(\Delta) \setminus \Delta$ is both a generalized pole of F_Δ and F_R.

Let $\{A_\Delta, \Gamma_\Delta\}$ and $\{A_R, \Gamma_R\}$ be arbitrary minimal realizations for the function F_Δ and F_R, respectively. By Theorem 2.2 and the definition of generalized poles (see Section 2.2) the preceding two properties imply that

(a') $\sigma(A_\Delta) \subseteq \operatorname{clos}\Delta$ and $\operatorname{int}\Delta \subseteq \rho(A_R)$;
(b') $\sigma_p(A_\Delta) \cap \sigma_p(A_R) = \emptyset$.

Thus Proposition 2.6 implies that $\{A_\Delta \widehat{\oplus} A_R, \operatorname{col}(\Gamma_\Delta, \Gamma_R)\}$, see (2.9), is a minimal realization for F in (3.8). Therefore the statement has been proven, because all minimal realizations for the same generalized Nevanlinna function are unitarily equivalent by Proposition 2.1. □

The assumption $\rho(A) \neq \emptyset$ in Theorem 1.2 is needed, because there exist selfadjoint relations A (even in finite-dimensional) Pontryagin spaces for which $\sigma_p(A) = \mathbb{C} \cup \{\infty\}$; see [4, p. 155-156].

Applying Theorem 1.2 inductively leads to the following decomposition results for selfadjoint relations. Note that from Corollary 3.3 the *canonical form* of selfadjoint operators in finite-dimensional Pontryagin spaces, see [5, Thm. 5.1.1.], can be derived.

Corollary 3.3. *Let A be a selfadjoint relation in a Pontryagin space $\{\Pi, [\cdot,\cdot]\}$ with $\sigma(A) \cap (\mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\}) = \{\alpha_1, \ldots, \alpha_n\}$. Then there exists a decomposition $\Pi_1[+]\ldots[+]\Pi_n$ of Π such that*

(i) *$\{\Pi_i, [\cdot,\cdot]\}$ is a Pontryagin space for $i = 1, \ldots, n$;*
(ii) *Π_i is A-invariant for $i = 1, \ldots, n$;*
(iii) *$\sigma(A\restriction_{\Pi_i}) = \{\alpha_i, \overline{\alpha_i}\}$ for $i = 1, \ldots, n$.*

Corollary 3.4. *Let A be a selfadjoint relation in a Pontryagin space $\{\Pi, [\cdot,\cdot]\}$ with $\rho(A) \neq \emptyset$ and let $\mathrm{ENT}\,(A) = \{\alpha_1, \ldots, \alpha_n, \overline{\alpha_1}, \ldots, \overline{\alpha_n}\}$. Then there exists a decomposition $\Pi_1[+]\ldots[+]\Pi_n$ of Π such that*

(i) *$\{\Pi_i, [\cdot,\cdot]\}$ is a Pontryagin space for $i = 1, \ldots, n$;*

(ii) Π_i *is* A*-invariant for* $i = 1, \dots, n$*;*
(iii) $\{\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots \alpha_n\} \in \rho(A \upharpoonright_{\Pi_i})$ *for* $i = 1, \dots, n$*;*
(iv) $\sigma_p(A \upharpoonright_{\Pi_i}) \cap \sigma_p(A \upharpoonright_{\Pi_j}) = \emptyset$ *and* $\sigma(A \upharpoonright_{\Pi_i}) \cap \sigma(A \upharpoonright_{\Pi_j})$ *contains at most finitely many points, for* $1 \leq i \neq j \leq n$*.*

Observe that condition (iii) in Corollary 3.4 implies that α_i and $\overline{\alpha_i}$ are the only ENTs of A restricted to Π_i for $i = 1, \dots, n$.

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