Received: 10 July 2018 | Revised: 17 February 2019 | Accepted: 1 July 2019

DOI: 10.1002/mana.201800300

ORIGINAL PAPER

MATHEMATISCHE NACHRICHTEN

Generalized boundary triples, I. Some classes of isometric and unitary boundary pairs and realization problems for subclasses of Nevanlinna functions

Volodymyr Derkach[1,2] | **Seppo Hassi**[3] | **Mark Malamud**[4]

[1]Department of Mathematics, Vasyl Stus Donetsk National University, Vinnitsya, 21021, Ukraine

[2]Department of Mathematics, National Pedagogical Dragomanov University, Pirogova 9, Kyiv, 01001, Ukraine

[3]Department of Mathematics and Statistics, University of Vaasa, P.O. Box 700, Vaasa, 65101, Finland

[4]Peoples' Friendship University of Russia, 6 Miklukho-Maklaya St, Moscow, 117198, Russian Federation

Correspondence
Seppo Hassi, Department of Mathematics and Statistics, University of Vaasa, P.O. Box 700, 65101 Vaasa, Finland.
Email: seppo.hassi@uwasa.fi

Funding information
Volkswagen Foundation; Vilho, Yrjö and Kalle Väisälä Foundation of the Finnish Academy of Science and Letters

Abstract

With a closed symmetric operator A in a Hilbert space $\mathfrak{H}$ a triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of a Hilbert space $\mathcal{H}$ and two abstract trace operators Γ_0 and Γ_1 from A^* to $\mathcal{H}$ is called a generalized boundary triple for A^* if an abstract analogue of the second Green's formula holds. Various classes of generalized boundary triples are introduced and corresponding Weyl functions $M(\cdot)$ are investigated. The most important ones for applications are specific classes of boundary triples for which Green's second identity admits a certain maximality property which guarantees that the corresponding Weyl functions are Nevanlinna functions on $\mathcal{H}$, i.e. $M(\cdot) \in \mathcal{R}(\mathcal{H})$, or at least they belong to the class $\widetilde{\mathcal{R}}(\mathcal{H})$ of Nevanlinna families on $\mathcal{H}$. The boundary condition $\Gamma_0 f = 0$ determines a reference operator $A_0 (= \ker \Gamma_0)$. The case where A_0 is selfadjoint implies a relatively simple analysis, as the joint domain of the trace mappings Γ_0 and Γ_1 admits a von Neumann type decomposition via A_0 and the defect subspaces of A. The case where A_0 is only essentially selfadjoint is more involved, but appears to be of great importance, for instance, in applications to boundary value problems e.g. in PDE setting or when modeling differential operators with point interactions. Various classes of generalized boundary triples will be characterized in purely analytic terms via the Weyl function $M(\cdot)$ and close interconnections between different classes of boundary triples and the corresponding transformed/renormalized Weyl functions are investigated. These characterizations involve solving direct and inverse problems for specific classes of operator functions $M(\cdot)$. Most involved ones concern operator functions $M(\cdot) \in \mathcal{R}(\mathcal{H})$ for which

$$\tau_{M(\lambda)}(f, g) = (2i \operatorname{Im} \lambda)^{-1}[(M(\lambda)f, g) - (f, M(\lambda)g)], \quad f, g \in \operatorname{dom} M(\lambda),$$

defines a closable nonnegative form on $\mathcal{H}$. It turns out that closability of $\tau_{M(\lambda)}(f,g)$ does not depend on $\lambda \in \mathbb{C}_\pm$ and, moreover, that the closure then is a form domain invariant holomorphic function on $\mathbb{C}_\pm$ while $\tau_{M(\lambda)}(f,g)$ itself need not be domain invariant. In this study we also derive several additional new results, for instance, Kreĭn-type resolvent formulas are extended to the most general setting of unitary and isometric boundary triples appearing in the present work.

In part II of the present work all the main results are shown to have applications in the study of ordinary and partial differential operators.

KEYWORDS

boundary triple, boundary value problem, Green's identities, resolvent, selfadjoint extension, symmetric operator, trace operator, Weyl family, Weyl function

MSC (2010)

Primary: 47A10, 47B25, Secondary: 47A20, 47A48, 47A56, 47B32

1 | KEY CONCEPTS AND AN OUTLINE OF THE MAIN RESULTS

1.1 | Ordinary boundary triples and Weyl functions

Let $\mathfrak{H}$ be a (complex) Hilbert space, let A be a not necessarily densely defined closed symmetric operator in $\mathfrak{H}$. The adjoint A^* of the operator A is a linear relation, i.e., a subspace of vectors $\widehat{g} = \binom{g}{g'} \in \mathfrak{H}^2$ such that

$$(Af, g) - (f, g') = 0 \qquad \text{for all} \qquad f \in \operatorname{dom} A,$$

see [4, 16]. In what follows the operator A will be identified with its graph, so that the set $\mathcal{C}(\mathcal{H})$ of closed linear operators will be considered as a subset of $\widetilde{\mathcal{C}}(\mathcal{H})$ of closed linear relations in $\mathcal{H}$. Then A is symmetric precisely when $A \subseteq A^*$. The defect subspaces $\mathfrak{N}_\lambda$ and the deficiency indices of A are defined by the equalities $\mathfrak{N}_\lambda := \ker (A^* - \lambda)$, $\lambda \in \mathbb{C}_\pm := \{\lambda \in \mathbb{C} : \pm \operatorname{Im} \lambda > 0\}$, and $n_\pm(A) := \dim \mathfrak{N}_{\pm i}$.

The classical J. von Neumann approach to the extension theory of symmetric operators in Hilbert spaces [61] is based on two fundamental formulas which allow to get a description of all selfadjoint extensions of a symmetric operator by means of isometric operators from $\mathfrak{N}_i$ onto $\mathfrak{N}_{-i}$ (see in this connection the monographs [1, 3, 22]). Another approach to the extension theory that substantially relied on a concept of abstract Green formula was originated by J.W. Calkin [21]. It turned out to be more convenient in the study of boundary value problems for ordinary and especially for partial differential equations (ODE and PDE) (see [19, 20, 32, 33, 36–38, 46, 63, 67]). Some further discussion on Calkin's paper is given below.

Definition 1.1. A collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of a Hilbert space $\mathcal{H}$ and two linear mappings Γ_0 and Γ_1 from A^* to $\mathcal{H}$, is said to be an *ordinary boundary triple* for A^* if:

1.1.1 The following abstract Green's identity holds

$$(f', g) - (f, g') = \left(\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g}\right)_{\mathcal{H}} - \left(\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g}\right)_{\mathcal{H}} \qquad \text{for all} \quad \widehat{f} = \begin{pmatrix} f \\ f' \end{pmatrix}, \widehat{g} = \begin{pmatrix} g \\ g' \end{pmatrix} \in A^*; \tag{1.1}$$

1.1.2 The mapping $\Gamma := \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : A^* \to \mathcal{H}^2$ is surjective.

Note that in the ODE setting formula (1.1) turns into the classical Lagrange identity being a key tool in study of boundary value problems. The advantage of this approach becomes obvious in applications to boundary value problems for elliptic equations where the formula (1.1) becomes a second Green's identity. However, in this case the assumptions of Definition 1.1 are violated and this circumstance was overcome in the classical papers by M. Višik [67] and G. Grubb [38] (see also [39]). Namely, relying on the Lions–Magenes trace theory ([39, 56]) they regularized the classical Dirichlet and Neumann trace mappings to get a proper version of Definition 1.1.

The operator Γ in Definition 1.1 is called the *reduction operator* (in the terminology of [21]). Definition 1.1 immediately yields a parametrization of the set of all selfadjoint extensions $\widetilde{A}$ of A by means of abstract boundary conditions via

$$\widetilde{A} = A_\Theta := \{\widehat{f} \in A^* : \Gamma\widehat{f} \in \Theta\},$$

where A_Θ ranges over the set of all selfadjoint extensions of A when Θ ranges over the set of all selfadjoint relations in $\mathcal{H}$ (subspaces in $\mathcal{H} \times \mathcal{H}$, see [4]). This correspondence is bijective and in this case $\Theta := \Gamma(\widetilde{A})$. The following two selfadjoint extensions of A are of particular interest:

$$A_0 := \ker \Gamma_0 = A_{\Theta_\infty} \quad \text{and} \quad A_1 := \ker \Gamma_1 = A_{\Theta_1};$$

here $\Theta_\infty = \{0\} \times \mathcal{H}$ and $\Theta_1 = \mathbb{O}$. These extensions are *disjoint*, i.e. $A_0 \cap A_1 = A$, and *transversal*, i.e. they are disjoint and $A_0 \widehat{+} A_1 = A^*$. Here the symbol $\widehat{+}$ means the componentwise sum of two linear relations, see (2.1).

In what follows A_0 is considered as a reference extension of A. Let $\rho(A_0)$ be the resolvent set of A_0, and let

$$\widehat{\mathfrak{N}}_\lambda := \left\{ \widehat{f}_\lambda = \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix} : f_\lambda \in \mathfrak{N}_\lambda \right\}, \quad \lambda \in \rho(A_0).$$

The main analytical tool in the description of spectral properties of selfadjoint extensions of A is the abstract Weyl function, introduced and investigated in [30–32].

Definition 1.2 ([30–32])**.** The abstract *Weyl function* and the *γ-field* of A, corresponding to an ordinary boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ are defined by

$$M(\lambda)\Gamma_0 \widehat{f}_\lambda = \Gamma_1 \widehat{f}_\lambda, \quad \gamma(\lambda)\Gamma_0 \widehat{f}_\lambda = f_\lambda, \quad \widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda, \quad \lambda \in \rho(A_0).$$

Notice that when the symmetric operator A is densely defined its adjoint is a single-valued operator and Definitions 1.1 and 1.2 can be used in a simpler form by treating Γ_0 and Γ_1 as operators from $\operatorname{dom} A^*$ to $\mathcal{H}$, see [32, 37, 46]. In what follows this convention will be tacitly used in most of our examples.

Example 1.3. Let A be a minimal symmetric operator in $L^2(\mathbb{R}_+)$ associated with the Sturm–Liouville differential expression

$$\mathcal{L} := -\frac{d^2}{dx^2} + q(x), \qquad q = \overline{q} \in L^1_{loc}([0, \infty)).$$

Assume the limit-point case at infinity, i.e. assume that $n_\pm(A) = 1$. Let $c(\cdot, \lambda)$ and $s(\cdot, \lambda)$ be cosine and sine type solutions of the equation $\mathcal{L}f = \lambda f$ subject to the initial conditions

$$c(0, \lambda) = 1, \quad c'(0, \lambda) = 0; \qquad s(0, \lambda) = 0, \quad s'(0, \lambda) = 1.$$

The defect subspace $\mathfrak{N}_\lambda$ is spanned by the Weyl solution $\psi(\cdot, \lambda)$ of the equation $\mathcal{L}f = \lambda f$ which is given by

$$\psi(x, \lambda) = c(x, \lambda) + m(\lambda)s(x, \lambda) \in L^2(\mathbb{R}_+).$$

The function $m(\cdot)$ is called the Titchmarsh–Weyl coefficient of $\mathcal{L}$. In this case a boundary triple $\Pi = \{\mathbb{C}, \Gamma_0, \Gamma_1\}$ can be defined as $\Gamma_0 f = f(0), \Gamma_1 f = f'(0)$. The corresponding Weyl function $M(\lambda)$ coincides with the classical Titchmarsh–Weyl coefficient, $M(\lambda) = m(\lambda)$.

In this connection let us mention that the role of the Weyl function $M(\lambda)$ in the extension theory of symmetric operators is similar to that of the classical Titchmarsh–Weyl coefficient $m(\lambda)$ in the spectral theory of Sturm–Liouville operators. For instance, it is known (see [32, 52]) that if A is *simple*, i.e. A does not admit orthogonal decompositions with a selfadjoint summand, then the Weyl function $M(\lambda)$ determines the boundary triple Π, in particular, the pair $\{A, A_0\}$, uniquely up to unitary equivalence. Besides, when A is simple, the spectrum of A_Θ coincides with the singularities of the operator function $(\Theta - M(z))^{-1}$; see [32].

As was shown in [32, 33] and [58] the Weyl function $M(\cdot)$ and the γ-field $\gamma(\cdot)$ both are well defined and holomorphic on the resolvent set $\rho(A_0)$ of the operator A_0. Moreover, the γ-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ satisfy the identities

$$\gamma(\lambda) = \left[I + (\lambda-\mu)\left(A_0-\lambda\right)^{-1}\right]\gamma(\mu), \quad \lambda,\mu\in\rho\left(A_0\right), \tag{1.2}$$

$$M(\lambda) - M(\mu)^* = (\lambda-\bar{\mu})\gamma(\mu)^*\gamma(\lambda), \qquad \lambda,\mu\in\rho\left(A_0\right). \tag{1.3}$$

This means that $M(\cdot)$ is a Q-function of the operator A in the sense of Kreĭn and Langer [51].

Denote by $\mathcal{B}(\mathcal{H})$ the set of bounded linear operators in $\mathcal{H}$ and by $\mathcal{R}[\mathcal{H}]$ the class of Nevanlinna functions, i.e., operator valued functions $F(\lambda)$ with values in $\mathcal{B}(\mathcal{H})$, which are holomorphic on $\mathbb{C}\setminus\mathbb{R}$ and satisfy the conditions

$$F(\lambda) = F\left(\bar{\lambda}\right)^* \quad \text{and} \quad \operatorname{Im} F(\lambda)\geq 0 \quad \text{for all} \quad \lambda\in\mathbb{C}_+, \tag{1.4}$$

see [44]. It follows from (1.2) and (1.3) that M belongs to the Nevanlinna class $\mathcal{R}[\mathcal{H}]$. Furthermore, since $\gamma(\lambda)$ isomorphically maps $\mathcal{H}$ onto $\mathfrak{N}_\lambda$, the relation (1.3) ensures that the imaginary part $\operatorname{Im} M(z)$ of $M(z)$ is positively definite, i.e. $M(\cdot)$ belongs to the subclass $\mathcal{R}^u[\mathcal{H}]$ of *uniformly strict* Nevanlinna functions:

$$\mathcal{R}^u[\mathcal{H}] := \{F(\cdot)\in\mathcal{R}[\mathcal{H}] : 0\in\rho(\operatorname{Im} F(i))\}.$$

The converse is also true.

Theorem 1.4 ([33, 52])**.** *The set of Weyl functions corresponding to ordinary boundary triples coincides with the class $\mathcal{R}^u[\mathcal{H}]$ of uniformly strict Nevanlinna functions.*

1.2 | B-generalized and AB-generalized boundary triples

In BVP's for Sturm–Liouville operators with an operator potential, for partial differential operators [26], and in point interaction theory it seems natural to consider more general boundary triples by weakening the surjectivity assumption 1.1.2 in Definition 1.1. The following notion was introduced in [33] with the name *generalized boundary-value space*, see also [25], where the term *generalized boundary triplet* was used.

Definition 1.5. ([25, 33]) Let A be a closed symmetric operator in a Hilbert space $\mathfrak{H}$ with equal deficiency indices and let A_* be a linear relation in $\mathfrak{H}$ such that $A\subset A_*\subset\overline{A_*}=A^*$. Then the collection $\Pi=\left\{\mathcal{H},\Gamma_0,\Gamma_1\right\}$, where $\mathcal{H}$ is a Hilbert space and $\Gamma=\left\{\Gamma_0,\Gamma_1\right\}$ is a single-valued linear mapping from A_* into $\mathcal{H}^2$, is said to be a *B-generalized boundary triple* for A^*, if:

1.5.1 the abstract Green's identity (1.1) holds for all $\widehat{f}=\begin{pmatrix} f\\ f'\end{pmatrix}, \widehat{g}=\begin{pmatrix} g\\ g'\end{pmatrix}\in A_*$;

1.5.2 $A_0 := \ker\Gamma_0$ is a selfadjoint relation in $\mathfrak{H}$;

1.5.3 $\operatorname{ran}\Gamma_0=\mathcal{H}$.

The Weyl function $M(\lambda)$ and the γ-field corresponding to a B-generalized boundary triple are defined by

$$M(\lambda)\Gamma_0\widehat{f}_\lambda=\Gamma_1\widehat{f}_\lambda, \quad \gamma(\lambda)\Gamma_0\widehat{f}_\lambda=f_\lambda, \quad \widehat{f}_\lambda\in\widehat{\mathfrak{N}}_\lambda(A_*) := \widehat{\mathfrak{N}}_\lambda\cap A_*, \quad \lambda\in\rho\left(A_0\right). \tag{1.5}$$

For every $\lambda\in\rho(A)$ the Weyl function $M(\lambda)$ takes values in $\mathcal{B}(\mathcal{H})$ and this justifies the present usage of the term B-generalized boundary triple, where "B" stands for a *Weyl function whose values are "bounded" operators*.

Example 1.6. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the Laplace operator $-\Delta$ in $L^2(\Omega)$. Let γ_D and γ_N be the Dirichlet and Neumann trace mappings. Moreover, let A_* be the pre-maximal operator defined as the restriction of the maximal Laplace operator $A_{\max}$ to the domain

$$\operatorname{dom} A_* = H^{3/2}_\Delta(\Omega) := H^{3/2}(\Omega)\cap\operatorname{dom} A_{\max} = \left\{f\in H^{3/2}(\Omega) : \Delta f\in L^2(\Omega)\right\}.$$

It is well known (see e.g. [39, 56]) that the mappings $\gamma_D : H^{3/2}_\Delta(\Omega)\to H^1(\partial\Omega)$ and $\gamma_N : H^{3/2}_\Delta(\Omega)\to H^0(\partial\Omega)=L^2(\partial\Omega)$ are well defined and surjective.

Using the key mapping properties of γ_D and γ_N one can extend the classical Green's formula to the domain dom A_*. Notice that the condition $\gamma_N f = 0$, $f \in \mathrm{dom}\, A_*$, determines the Neumann realization Δ_N of the Laplace operator. Since Δ_N is selfadjoint and $\gamma_N(\mathrm{dom}\, A_*) = H^0(\partial\Omega)$, the triple $\Pi = \{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ with

$$\Gamma_0 = \gamma_N \upharpoonright \mathrm{dom}\, A_* \quad \text{and} \quad \Gamma_1 = \gamma_D \upharpoonright \mathrm{dom}\, A_*$$

is a B-generalized boundary triple for A^* with dom Γ = dom A_*. Besides, the corresponding Weyl function $M(\cdot)$ coincides with the inverse of the classical Dirichlet-to-Neumann map $\Lambda(\cdot)$, i.e. $M(\cdot) = \Lambda(\cdot)^{-1}$; see Part II of the present work for further details.

As was shown in [27] for every *B-generalized boundary triple* there exists an ordinary boundary triple $\{\mathcal{H}, \Gamma_0^0, \Gamma_1^0\}$ and operators $G, E = E^* \in \mathcal{L}(\mathcal{H})$ such that ker $G = \{0\}$ and

$$\begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} = \begin{pmatrix} G^{-1} & 0 \\ EG^{-1} & G^* \end{pmatrix} \begin{pmatrix} \Gamma_0^0 \\ \Gamma_1^0 \end{pmatrix}. \tag{1.6}$$

Weyl functions M and M_0 corresponding to the boundary triples $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{H}, \Gamma_0^0, \Gamma_1^0\}$, are connected by

$$M(\lambda) = G^* M_0(\lambda) G + E, \quad \lambda \in \rho(A_0). \tag{1.7}$$

It should be noted that the Weyl function $M(\cdot)$ of a B-generalized boundary triple satisfies the properties (1.2)–(1.4). However, instead of the property $0 \in \rho(\mathrm{Im}\, M(i))$ one has a weaker condition $0 \notin \sigma_p(\mathrm{Im}\, M(i))$. This motivates the following definition. Denote by $\mathcal{R}^s[\mathcal{H}]$ the class of strict Nevanlinna functions

$$\mathcal{R}^s[\mathcal{H}] := \{F(\cdot) \in \mathcal{R}[\mathcal{H}] : 0 \notin \sigma_p(\mathrm{Im}\, F(i))\}.$$

In fact, it was also shown in [33, Chapter 5] that every $M(\cdot) \in \mathcal{R}^s[\mathcal{H}]$ can be realized as the Weyl function of a certain B-generalized boundary triple and hence the following statement holds.

Theorem 1.7 ([33])**.** *The set of Weyl functions corresponding to B-generalized boundary triples coincides with the class $\mathcal{R}^s[\mathcal{H}]$ of strict Nevanlinna functions.*

This realization result as well as the technique of B-generalized boundary triples have recently been applied also e.g. to problems in scattering theory, see [13], in the analysis of discrete and continuous time system theory, and in the boundary control theory; for some recent achievements, see e.g. [5, 6, 8, 9, 40, 53, 54, 59, 66].

In the present paper we introduce the new class of AB-generalized boundary triples which is obtained by a weakening of the surjectivity condition 1.5.3 in Definition 1.5.

Definition 1.8. A collection $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is said to be an *almost B-generalized boundary triple*, or briefly, an *AB-generalized boundary triple* for A^*, if $A_* :=$ dom Γ is dense in A^*, the conditions 1.5.1, 1.5.2 are satisfied and

1.8.1 ran Γ_0 is dense in $\mathcal{H}$.

The Weyl function corresponding to an AB-generalized boundary triple is again defined by (1.5). One of the main results of the paper is Theorem 4.4 which states that every AB-generalized boundary triple can be regularized to produce a B-generalized boundary triple in the spirit of (1.6). Another result — Theorem 4.6 gives a characterization of the set of the Weyl functions M of AB-generalized boundary triples in the form

$$M(\lambda) = E + M_0(\lambda), \quad \text{where} \quad M_0 \in \mathcal{R}[\mathcal{H}]$$

and E is a densely defined symmetric operator in $\mathcal{H}$, such that ker Im $M_0(\lambda) \cap \mathrm{dom}\, E = \{0\}$.

The class of AB-generalized boundary triples contains the class of so-called quasi boundary triples, which has been studied in J. Behrndt and M. Langer [11]. In the definition of a quasi boundary triple Assumption 1.5.3 is replaced by the assumption that

$$\mathrm{ran}\, \Gamma \quad \text{is dense in} \quad \mathcal{H} \times \mathcal{H}.$$

A connection between quasi boundary triples and AB-generalized boundary triples is given in Corollary 4.9. A joint feature in AB-generalized boundary triples and quasi boundary triples is that without additional assumptions on the mapping $\Gamma = \{\Gamma_0, \Gamma_1\}$

these boundary triples are not unitary in the sense of Definition 1.9 presented below. Consequently, their Weyl functions need not be Nevanlinna functions, i.e., the values $M(\lambda)$ need not be maximal dissipative (accumulative) in $\mathbb{C}_+$ ($\mathbb{C}_-$); see definitions in Section 2.1. Special type of isometric boundary triples that will appear in Part II of the present paper are so-called *essentially unitary* boundary triples/pairs. As shown therein (cf. [29, Section 7]) quasi boundary triples studied in [11, 12] for elliptic operators are either special type of unitary boundary triples or they are essentially unitary boundary triples, depending on the choice of the underlying regularity index of the space used as the domain A_* for the boundary triple. For a very recent contribution and some further development on essentially unitary boundary pairs see also [43].

Different applications of quasi boundary triples in boundary value problems including applications to elliptic theory and trace formulas can be found e.g. in [11, 14, 15, 40, 62].

1.3 | Unitary boundary triples

A general class of boundary triples, to be called here unitary boundary triples, was introduced in [25]. This concept was motivated by the realization problem for the most general class of Nevanlinna functions: realize each Nevanlinna function as the Weyl function of an appropriate type generalized boundary triple.

To this end denote by $\mathcal{R}(\mathcal{H})$ the class of all operator valued holomorphic Nevanlinna functions on $\mathbb{C}_+$ (in the resolvent sense) with values in the set of maximal dissipative (not necessarily bounded) linear operators in $\mathcal{H}$. Each $M(\cdot) \in \mathcal{R}(\mathcal{H})$ is extended to $\mathbb{C}_-$ by symmetry with respect to the real line $M(\lambda) = M(\bar{\lambda})^*$; see [25, 51]. Analogous to the subclass $\mathcal{R}^s[\mathcal{H}]$ of Nevanlinna functions $\mathcal{R}[\mathcal{H}]$, the class $\mathcal{R}(\mathcal{H})$ contains a subclass $\mathcal{R}^s(\mathcal{H})$ of strict Nevanlinna functions which satisfy the condition

$$\mathcal{R}^s(\mathcal{H}) := \{F(\cdot) \in \mathcal{R}(\mathcal{H}) : \ (\operatorname{Im} F(i)h, h) = 0 \Rightarrow h = 0, \quad h \in \operatorname{dom} F(i)\}. \tag{1.8}$$

In order to present the definition of a unitary boundary triple, introduce the fundamental symmetries

$$J_{\mathfrak{H}} := \begin{pmatrix} 0 & -iI_{\mathfrak{H}} \\ iI_{\mathfrak{H}} & 0 \end{pmatrix}, \quad J_{\mathcal{H}} := \begin{pmatrix} 0 & -iI_{\mathcal{H}} \\ iI_{\mathcal{H}} & 0 \end{pmatrix}, \tag{1.9}$$

and the associated Kreĭn spaces $(\mathfrak{H}^2, J_{\mathfrak{H}})$ and $(\mathcal{H}^2, J_{\mathcal{H}})$ (see [7, 17]) obtained by endowing the Hilbert spaces $\mathfrak{H}^2$ and $\mathcal{H}^2$ with the following indefinite inner products

$$[\hat{f}, \hat{g}]_{\mathfrak{H}^2} = (J_{\mathfrak{H}}\hat{f}, \hat{g})_{\mathfrak{H}^2}, \quad [\hat{h}, \hat{k}]_{\mathcal{H}^2} = (J_{\mathcal{H}}\hat{h}, \hat{k})_{\mathcal{H}^2}, \quad \hat{f}, \hat{g} \in \mathfrak{H}^2, \quad \hat{h}, \hat{k} \in \mathcal{H}^2. \tag{1.10}$$

This allows to rewrite Green's identity (1.1) in the form

$$[\hat{f}, \hat{g}]_{\mathfrak{H}^2} = [\Gamma\hat{f}, \Gamma\hat{g}]_{\mathcal{H}^2}, \tag{1.11}$$

which means that the mapping Γ is in fact a $(J_{\mathfrak{H}}, J_{\mathcal{H}})$-isometric mapping from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$. If $\Gamma^{[*]}$ denotes the Kreĭn space adjoint of the operator Γ (see Definition (2.4)), then (1.11) can be simply rewritten as $\Gamma^{-1} \subset \Gamma^{[*]}$. The surjectivity of Γ implies that $\Gamma^{-1} = \Gamma^{[*]}$. Following Yu. L. Shmuljan [64] a linear operator $\Gamma : (\mathfrak{H}^2, J_{\mathfrak{H}}) \to (\mathcal{H}^2, J_{\mathfrak{H}})$ will be called $(J_{\mathfrak{H}}, J_{\mathcal{H}})$-unitary, if $\Gamma^{-1} = \Gamma^{[*]}$.

Definition 1.9 ([25]). A collection $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a *unitary (resp. isometric) boundary triple* for A^*, if $\mathcal{H}$ is a Hilbert space and $\Gamma = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix}$ is a linear operator from $\mathfrak{H}^2$ to $\mathcal{H}^2$ such that:

1.9.1 $A_* := \operatorname{dom} \Gamma$ is dense in A^* with respect to the topology on $\mathfrak{H}^2$;

1.9.2 The operator Γ is $(J_{\mathfrak{H}}, J_{\mathcal{H}})$-unitary (resp. $(J_{\mathfrak{H}}, J_{\mathcal{H}})$-isometric).

The Weyl function $M(\lambda)$ and the γ-field corresponding to a unitary boundary triple Π are defined again by the same formula (1.5). The *transposed boundary triple* $\Pi^\top := \{\mathcal{H}, \Gamma_1, -\Gamma_0\}$ associated with a unitary boundary triple Π is also a unitary boundary triple, the corresponding Weyl function takes the form $M^\top(\lambda) = -M(\lambda)^{-1}$.

The main realization theorem in [25] gives a solution to the inverse problem mentioned above.

Theorem 1.10 ([25]). *The class of Weyl functions corresponding to unitary boundary triples coincides with the class $\mathcal{R}^s(\mathcal{H})$ of (in general unbounded) strict Nevanlinna functions.*

In fact, in [25, Theorem 3.9] a stronger result is stated showing that the class $\mathcal{R}^s(\mathcal{H})$ can be replaced by the class $\mathcal{R}(\mathcal{H})$ or even by the class $\widetilde{\mathcal{R}}(\mathcal{H})$ of Nevanlinna pairs when one allows multi-valued linear mappings Γ in Definition 1.9; see Theorem 3.3 in Section 3.2. Theorem 1.10 plays a key role in the construction of generalized resolvents in the framework of coupling method that was originally introduced in [24] and developed in its full generality in [26]. It is worth to mention that in [6] it is shown that a counterpart of the main transform of a unitary boundary triple (with some extra properties) naturally appears in impedance conservative continuous time *input/state/output systems*, and, moreover, that the transfer function of such systems is directly connected with the Weyl function of the unitary boundary triplet. A systematic study of so-called *conservative state/signal systems* has been initiated in [5] and, as shown in [6], conservative state/signal systems have a close connection to general unitary boundary triples in Theorem 1.10; see also Remark 5.7.

Ordinary and B-generalized boundary triples give examples of unitary boundary triples; see [25], and as noted above the conditions defining AB-generalized or quasi boundary triples do not guarantee their unitarity; for a criterion see Corollary 4.7. Some necessary and sufficient conditions which characterize unitary boundary triples and which differ from the purely analytic criterion in Theorem 1.10 can be found in [25, Proposition 3.6], [27, Theorem 7.51], some general criteria of geometric nature have been established in [68, 69], and a further characterization, useful e.g. in applications to elliptic equations, can be found in Part II of the present paper.

In connection with Definition 1.9 we wish to make some comments on a seminal paper [21] by J. W. Calkin, where a concept of the *reduction operator* is introduced and investigated. Although no proper geometric machinery appears in the definition of Calkin's reduction operator this notion in the case of a densely defined operator A essentially coincides with concept of a unitary operator between Kreĭn spaces as in Definition 1.9. An overview on the early work of Calkin and some connections to later developments can be found in the papers in the monograph [40]; for a further discussion see also Section 3.5.

In Theorem 5.8 we extend Kreĭn's resolvent formula to the general setting of unitary boundary triples. Namely, for any proper extension $A_\Theta \in \operatorname{Ext}_S$ satisfying $A_\Theta \subset \operatorname{dom}\Gamma$ the following Kreĭn-type formula holds:

$$\left(A_\Theta - \lambda\right)^{-1} - \left(A_0 - \lambda\right)^{-1} = \gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma\left(\bar{\lambda}\right)^*, \qquad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

It is emphasized that in this formula A_Θ *is not necessarily closed and it is not assumed that* $\lambda \in \rho\left(A_\Theta\right)$, in particular, here the inverses $\left(A_\Theta - \lambda\right)^{-1}$ and $\big(\Theta - M(\lambda)\big)^{-1}$ are understood in the sense of relations.

1.4 | S-generalized boundary triples

Following [25] we consider a special class of unitary boundary triples singled out by the condition that $A_0 := \ker\Gamma_0$ is a selfadjoint extension of A.

Definition 1.11 ([25]). A unitary boundary triple $\Pi = \left\{\mathcal{H}, \Gamma_0, \Gamma_1\right\}$ is said to be an S*-generalized boundary triple* for A^* if the assumption 1.5.2 holds, i.e. $A_0 := \ker\Gamma_0$ is a selfadjoint extension of A.

Next following [27, Theorem 7.39] and [25, Theorem 4.13] we present a complete characterization of the Weyl functions $M(\cdot)$ corresponding to S-generalized boundary triples.

Theorem 1.12. *([25, 27]) Let* $\Pi = \left\{\mathcal{H}, \Gamma_0, \Gamma_1\right\}$ *be a unitary boundary triple for* A^* *and let* $M(\cdot)$ *and* $\gamma(\cdot)$ *be the corresponding Weyl function and* γ*-field, respectively. Then the following statements are equivalent:*

(i) $A_0 = \ker\Gamma_0$ *is selfadjoint, i.e.* Π *is an* S*-generalized boundary triple;*

(ii) $A_* = A_0 \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(A_*)$ *and* $A_* = A_0 \,\widehat{+}\, \widehat{\mathfrak{N}}_\mu(A_*)$ *for some (equivalently for all)* $\lambda \in \mathbb{C}_+$ *and* $\mu \in \mathbb{C}_-$*;*

(iii) $\operatorname{ran}\Gamma_0 = \operatorname{dom} M(\lambda) = \operatorname{dom} M(\mu)$ *for some (equivalently for all)* $\lambda \in \mathbb{C}_+$ *and* $\mu \in \mathbb{C}_-$*;*

(iv) $\gamma(\lambda)$ *and* $\gamma(\mu)$ *are bounded and densely defined in* $\mathcal{H}$ *for some (equivalently for all)* $\lambda \in \mathbb{C}_+$ *and* $\mu \in \mathbb{C}_-$*;*

(v) $\operatorname{dom} M(\lambda) = \operatorname{dom} M\left(\overline{\lambda}\right)$ *and* $\operatorname{Im} M(\lambda)$ *is bounded for some (equivalently for all)* $\lambda \in \mathbb{C}_+$*;*

(vi) the Weyl function $M(\cdot)$ *belongs to* $\mathcal{R}^s(\mathcal{H})$ *and it admits a representation*

$$M(\lambda) = E + M_0(\lambda), \quad M_0(\cdot) \in \mathcal{R}[\mathcal{H}], \quad \lambda \in \mathbb{C}\setminus\mathbb{R}, \tag{1.12}$$

where $E = E^*$ *is a selfadjoint (in general unbounded) operator in* $\mathcal{H}$*.*

In Theorem 5.17 this result is extended to the case of S-generalized boundary pairs $\{\mathcal{H}, \Gamma\}$, where $\Gamma : A_* \to \mathcal{H}\times\mathcal{H}$ is allowed to be multi-valued (see Definitions 3.1 and 5.11).

Notice that, for instance, the implications (i) ⇒ (ii), (iii) are immediate from the following decomposition of $A_* := \operatorname{dom}\Gamma$:

$$A_* = A_0 \hat{+} \widehat{\mathfrak{N}}_\lambda(A_*), \quad \lambda \in \rho\big(A_0\big). \tag{1.13}$$

In accordance with (1.12) the Weyl function corresponding to an S-generalized boundary triple is an operator valued Nevanlinna function with *domain invariance property*: $\operatorname{dom} M(\lambda) = \operatorname{dom} E = \operatorname{ran}\Gamma_0$, $\lambda \in \mathbb{C}_\pm$. It takes values in the set $\mathcal{C}(\mathcal{H})$ of closed (in general unbounded) operators while the values of the imaginary parts $\operatorname{Im} M(\lambda)$ are bounded operators.

As an example we mention that the transposed triple $\Pi^\top = \big\{L^2(\partial\Omega), \Gamma_1, -\Gamma_0\big\}$ from the PDE Example 1.6 is an S-generalized boundary triple. The corresponding Weyl function coincides (up to sign change) with the Dirichlet-to-Neumann map $\Lambda(\cdot)$, i.e. $M(\cdot)^\top = -\Lambda(\cdot)$; further details are given in Part II of the present work.

1.5 | ES-generalized boundary triples and form domain invariance

Next we discuss one of the main new objects appearing in the present work.

Definition 1.13. A *unitary* boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* is said to be an *essentially selfadjoint generalized boundary triple*, in short, *ES-generalized boundary triple* for A^*, if:

1.13.1 $A_0 := \ker\Gamma_0$ is an essentially selfadjoint linear relation in $\mathfrak{H}$.

To characterize the class of ES-generalized boundary triples in terms of the corresponding Weyl functions we associate with each $M(\cdot)$ a family of nonnegative quadratic forms $\mathfrak{t}_{M(\lambda)}$ in $\mathcal{H}$:

$$\mathfrak{t}_{M(\lambda)}[u, v] := \frac{1}{\lambda - \bar{\lambda}}\,[(M(\lambda)u, v) - (u, M(\lambda)v)], \quad u, v \in \operatorname{dom}(M(\lambda)), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{1.14}$$

The forms $\mathfrak{t}_{M(\lambda)}$ are not necessarily closable. However, it is shown that if $\mathfrak{t}_{M(\lambda_0)}$ is closable at one point $\lambda_0 \in \mathbb{C}_+$ ($\lambda_0 \in \mathbb{C}_-$), then $\mathfrak{t}_{M(\lambda)}$ is closable for every $\lambda \in \mathbb{C}_+$ (resp. $\lambda \in \mathbb{C}_-$); for an analytic treatment of this fact see also [28]. In the latter case the domain of the closure $\bar{\mathfrak{t}}_{M(\lambda)}$ does not depend on $\lambda \in \mathbb{C}_+$ ($\lambda \in \mathbb{C}_-$) and therefore the Weyl function $M(\lambda)$ is said to be *form domain invariant* in $\mathbb{C}_+$ (resp. in $\mathbb{C}_-$). In general $\mathfrak{t}_{M(\lambda)}$ need not be closable in both half-planes simultaneously; see Proposition 5.26 and Remark 5.27. On the other hand, if $\mathfrak{t}_{M(\lambda)}$ is closable in both half-planes then the form domain does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$; i.e. form domains coincide also in different half-planes.

In what follows one of the main results established in this connection reads as follows (cf. Theorem 5.24).

Theorem 1.14. *Let $\Pi = \big\{\mathcal{H}, \Gamma_0, \Gamma_1\big\}$ be a unitary boundary triple for A^*. Let also $M(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl function and the γ-field, respectively. Then the following statements are equivalent:*

(i) *Π is an ES-generalized boundary triple for A^*;*
(ii) *$\gamma(i)$ and $\gamma(-i)$ are closable;*
(iii) *$\gamma(\lambda)$ is closable for every $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ and $\operatorname{dom}\overline{\gamma(\lambda)} = \operatorname{dom}\overline{\gamma(\pm i)}$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$;*
(iv) *the forms $\mathfrak{t}_{M(i)}$ and $\mathfrak{t}_{M(-i)}$ are closable;*
(v) *the form $\mathfrak{t}_{M(\lambda)}$ is closable for every $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ and $\operatorname{dom}\bar{\mathfrak{t}}_{M(\lambda)} = \operatorname{dom}\bar{\mathfrak{t}}_{M(\pm i)}$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$;*
(vi) *the Weyl function $M(\cdot)$ belongs to $\mathcal{R}^s(\mathcal{H})$ and is form domain invariant in $\mathbb{C}_+ \cup \mathbb{C}_-$.*

The result relies on Theorem 5.5, which contains some important invariance results that unitary boundary triples are shown to satisfy. If $\big\{\mathcal{H}, \Gamma_0, \Gamma_1\big\}$ is an ES-generalized, but not an S-generalized, boundary triple for A^*, then the equality (1.13) fails to hold and turns out to be an inclusion

$$A_0 \hat{+} \widehat{\mathfrak{N}}_\lambda(A_*) \subsetneq A_* \subset A^* = \overline{A_0} \hat{+} \widehat{\mathfrak{N}}_\lambda(A^*), \quad \lambda \in \rho\big(A_0\big).$$

Indeed, since A_0 is not selfadjoint (while it is essentially selfadjoint), the decomposition $A_* = A_0 \hat{+} \widehat{\mathfrak{N}}_\lambda(A_*)$ doesn't hold; cf. [25, Theorem 4.13]. Then there clearly exists $\widehat{f} \in A_*$ which does not belong to $A_0 \hat{+} \widehat{\mathfrak{N}}_\lambda(A_*)$, so that $\Gamma_0\widehat{f} \neq 0$ as well as $\Gamma_0\widehat{f} \notin \Gamma_0\big(\widehat{\mathfrak{N}}_\lambda(A_*)\big) = \operatorname{dom} M(\lambda)$. In particular, in this case a strict inclusion $\operatorname{dom} M(\lambda) \subsetneq \operatorname{ran}\Gamma_0$ holds and, consequently, the

Weyl function $M(\lambda)$ can loose the domain invariance property. However, the domain of the closure $\overline{\Gamma}_0$ contains the selfadjoint relation $\overline{A_0}$ and admits the decomposition

$$\mathrm{dom}\,\overline{\Gamma}_0 = \overline{A_0} \,\hat{+}\, \left(\mathrm{dom}\left(\overline{\Gamma}_0\right) \cap \widehat{\mathfrak{N}}_\lambda(A^*)\right), \quad \lambda \in \rho\left(\overline{A_0}\right).$$

This implies the equality $\mathrm{dom}\,\overline{\gamma(\lambda)} = \overline{\Gamma}_0\left(\mathrm{dom}\left(\overline{\Gamma}_0\right) \cap \widehat{\mathfrak{N}}_\lambda(A^*)\right) = \mathrm{ran}\,\overline{\Gamma}_0$, which combined with $\mathrm{dom}\,\bar{\mathfrak{t}}_{M(\lambda)} = \mathrm{dom}\,\overline{\gamma(\lambda)}$ yields the form domain invariance property for M:

$$\mathrm{dom}\,\bar{\mathfrak{t}}_{M(\lambda)} = \mathrm{ran}\,\overline{\Gamma}_0.$$

Passing from the case of an S-generalized boundary triple to the case of an ES-generalized boundary triple (which is not S-generalized) means that $A_0 \neq A_0^*$. Then, in particular, conditions (ii) and (iii) in Theorem 1.12 are necessary violated. We split the situation into two different cases:

Assumption 1.15. $M(\lambda)$ isn't domain invariant, i.e. $\mathrm{dom}\,M\left(\lambda_1\right) \neq \mathrm{dom}\,M\left(\lambda_2\right)$ at least for two points $\lambda_1, \lambda_2 \in \mathbb{C}_+$, $\lambda_1 \neq \lambda_2$, while it is form domain invariant, i.e. $\mathrm{dom}\,\bar{\mathfrak{t}}_{M(\lambda)} = \mathrm{dom}\,\bar{\mathfrak{t}}_{M(\pm i)}$, $\lambda \in \mathbb{C}_\pm$.

Assumption 1.16. $\mathrm{dom}\,M(\lambda) = \mathrm{dom}\,M(\pm i)$, $\lambda \in \mathbb{C}_\pm$, while $\mathrm{dom}\,M(\pm i) \subsetneqq \mathrm{ran}\,\Gamma_0$.

Both possibilities appear in the spectral theory. An example of a Nevanlinna function satisfying Assumption 1.15 is presented in Example 5.28. Next we present an example of the Weyl function satisfying Assumption 1.16. Such Nevanlinna functions arise in the theory of Schrödinger operators with local point interactions.

Example 1.17. Let $X = \{x_n\}_1^\infty$ be a strictly increasing sequence of positive numbers such that $\lim_{n\to\infty} x_n = \infty$. Let $x_0 = 0$ and denote $d_n := x_n - x_{n-1} > 0$, $0 \le d_* := \inf_{n\in\mathbb{N}} d_n$, and $d^* := \sup_{n\in\mathbb{N}} d_n \le \infty$.

Let also H_n be a minimal operator associated with the expression $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ in $L_0^2\left[x_{n-1}, x_n\right]$. Then H_n is a symmetric operator with deficiency indices $n_\pm\left(\mathrm{H}_n\right) = 2$ and domain $\mathrm{dom}\left(\mathrm{H}_n\right) = W_0^{2,2}\left[x_{n-1}, x_n\right]$. Consider in $L^2\left(\mathbb{R}_+\right)$ the direct sum of symmetric operators H_n,

$$\mathrm{H} := \mathrm{H}_{\min} = \bigoplus_{n=1}^\infty \mathrm{H}_n, \qquad \mathrm{dom}\left(\mathrm{H}_{\min}\right) = W_0^{2,2}\left(\mathbb{R}_+ \setminus X\right) = \bigoplus_{n=1}^\infty W_0^{2,2}\left[x_{n-1}, x_n\right].$$

It is easily seen that a boundary triple $\Pi_n = \left\{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\right\}$ for H_n^* can be chosen as

$$\Gamma_0^{(n)} f := \begin{pmatrix} f'\left(x_{n-1}+\right) \\ f'\left(x_n-\right) \end{pmatrix}, \qquad \Gamma_1^{(n)} f := \begin{pmatrix} -f\left(x_{n-1}+\right) \\ f\left(x_n-\right) \end{pmatrix}, \qquad f \in W_2^2\left[x_{n-1}, x_n\right].$$

The corresponding Weyl function M_n is given by

$$M_n(\lambda) = \frac{-1}{\sqrt{\lambda}} \begin{pmatrix} \cot\left(\sqrt{\lambda} d_n\right) & -\dfrac{1}{\sin\left(\sqrt{\lambda} d_n\right)} \\ -\dfrac{1}{\sin\left(\sqrt{\lambda} d_n\right)} & \cot\left(\sqrt{\lambda} d_n\right) \end{pmatrix}.$$

Clearly, $\mathrm{H} = \mathrm{H}_{\min}$ is a closed symmetric operator in $L^2(\mathbb{R}_+)$. Next we put

$$\mathcal{H} = l^2(\mathbb{N}) \otimes \mathbb{C}^2, \qquad \Gamma = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} := \bigoplus_{n=1}^\infty \begin{pmatrix} \Gamma_0^{(n)} \\ \Gamma_1^{(n)} \end{pmatrix}$$

and note that in accordance with the definition of the direct sum of linear mappings

$$\mathrm{dom}\,\Gamma := \left\{ f = \oplus_{n=1}^\infty f_n \in \mathrm{dom}\,A^* : \sum_{n\in\mathbb{N}} \left\|\Gamma_j^{(n)} f_n\right\|_{\mathcal{H}_n}^2 < \infty, \ j \in \{0,1\} \right\}.$$

We also put $\overline{\Gamma}_j := \oplus_{n=1}^\infty \Gamma_j^{(n)}$ and note that it is a closure of $\Gamma_j = \overline{\Gamma}_j \upharpoonright \mathrm{dom}\,\Gamma$, $j = 1, 2$. It can be seen that the orthogonal sum $\Pi := \oplus_{n=1}^\infty \Pi_n$ of the boundary triples Π_n determines an ES-generalized boundary triple. Moreover, in the case that $d_* = 0$ the

Weyl function $M(\cdot)$ corresponding to the triple $\Pi = \oplus_{n=1}^{\infty}\Pi_n$ satisfies Assumption 1.16, i.e. it is domain invariant, $\operatorname{dom} M(\lambda) = \operatorname{dom} M(i)$, $\lambda \in \mathbb{C}_{\pm}$, while $\operatorname{dom} M(i) \subsetneq \operatorname{ran}\Gamma_0$. Hence, by Theorem 1.12, $A_0 \neq A_0^*$ and $\Pi = \oplus_{n=1}^{\infty}\Pi_n$ being ES-generalized, is not an S-generalized boundary triple for H^*. In fact, with $d_* = 0$ the Weyl function $M(\cdot)$ as well as its imaginary part $\operatorname{Im} M(\cdot)$ take values in the set of unbounded operators. For the details in this example we refer to Part II of the present work, where also analogous results for *moment and Dirac operators with local point interactions* are established.

Notice that the minimal operator H as well as the corresponding triple Π for H^* in Example 1.17 naturally arise when treating the Hamiltonian $\mathrm{H}_{X,\alpha}$ with δ-interactions in the framework of extension theory. The latter have appeared in various physical problems as exactly solvable models that describe complicated physical phenomena (see e.g. [2, 3, 34, 48, 49] for details).

Theorem 5.32 offers a *renormalization procedure* which produces from a form domain invariant Weyl function a domain invariant Weyl function, whose imaginary part becomes a well-defined bounded operator function on $\mathbb{C} \setminus \mathbb{R}$, i.e., the renormalized boundary triple is AB-generalized. Some related results, showing how B-generalized boundary triples give rise to ES-generalized boundary triples, are established in Part II of the present work, where these results are applied in the analysis of *regularized trace operators* for Laplacians.

Before closing this subsection we wish to mention that other type of examples for ES-generalized boundary triples are the *Kreĭn – von Neumann Laplacian* and the *Zaremba Laplacian* for a mixed boundary value problem treated in Part II of the present work.

1.6 | A short description of the contents

For the convenience of the reader in this Introduction we have restricted the exposition of the main definitions and results to the case of generalized boundary triples, i.e. to boundary triples with a single-valued linear mapping $\Gamma : A_* \to \mathcal{H} \times \mathcal{H}$ which admits a decomposition $\Gamma = \{\Gamma_0, \Gamma_1\}$, where Γ_0 and Γ_1 give rise to a pair of boundary conditions in (the boundary space) $\mathcal{H}$ typically occurring in boundary value problems in ODE and PDE setting. In the paper itself these results are mostly presented in a more general setting of boundary pairs, where Γ is allowed to be multi-valued. This generality unifies the presentation in later Sections and, in fact, often simplifies the description of the particular analytic properties of Weyl functions associated with different classes of generalized boundary triples and boundary pairs.

In Section 2 we recall basic concepts of linear relations (sums of relations, componentwise sums, defect subspaces, etc.) as well as unitary and isometric relations in Kreĭn space. We also introduce the concepts of Nevanlinna functions and families.

In Section 3 we discuss unitary and isometric boundary pairs and triples. We introduce the notions of Weyl functions and families and discuss their properties. A general version of the main realization result, Theorem 3.3, is presented therein, too. It completes and improves Theorem 1.10. Besides certain isometric transforms of boundary triples are discussed.

In Section 4 we investigate AB-generalized boundary pairs and triples. Their main properties can be found in Theorem 4.2 and in various Corollaries appearing in this section. In Theorem 4.4 a connection between B-generalized and AB-generalized boundary triples is established by means of triangular isometric transformations. Connections between AB-generalized boundary triples and quasi boundary triples are also explained. Moreover, a Kreĭn type formula for AB-generalized boundary triples can be found in Theorem 4.12.

In Section 5 we consider two further subclasses of unitary boundary triples and pairs: S-generalized and ES-generalized boundary triples and pairs. For deriving some of the main results in this connection we have established also some new facts on the interaction between $(J_{\mathfrak{H}}, J_{\mathcal{H}})$-unitary relations and unitary colligations appearing e.g. in system theory and in the analysis of Schur functions, see Section 5.1; a background for this connection can be found in [10]. In particular, this connection is applied to extend Theorem 1.12 to the case of S-generalized boundary pairs (see Theorem 5.17). In this case representation (1.12) for the Weyl function remains valid with $M_0 \in \mathcal{R}[\mathcal{H}_0]$ and $\mathcal{H}_0 \subseteq \mathcal{H}$ instead of $M_0 \in \mathcal{R}^s[\mathcal{H}]$. In Theorem 5.24 the class of Weyl functions of ES-generalized boundary pairs is characterized. In Theorem 5.8 it is shown that every unitary boundary triple admits a Kreĭn type resolvent formula. Besides, in Theorem 5.32 a connection between ES-generalized boundary triples and AB-generalized boundary triples is established via an isometric transform introduced in Lemma 3.12 (see formula (3.23)).

2 | PRELIMINARY CONCEPTS

2.1 | Linear relations in Hilbert spaces

A linear relation T from $\mathfrak{H}$ to $\mathfrak{H}'$ is a linear subspace of $\mathfrak{H} \times \mathfrak{H}'$. Systematically a linear operator T will be identified with its graph. It is convenient to write $T : \mathfrak{H} \to \mathfrak{H}'$ and interpret the linear relation T as a multi-valued linear mapping from $\mathfrak{H}$ into

$\mathfrak{H}'$. If $\mathfrak{H}' = \mathfrak{H}$ one speaks of a linear relation T in $\mathfrak{H}$. Many basic definitions and properties associated with linear relations can be found in [4, 16, 22].

The following notions appear throughout this paper. For a linear relation $T : \mathfrak{H} \to \mathfrak{H}'$ the symbols $\operatorname{dom} T$, $\ker T$, $\operatorname{ran} T$, $\operatorname{mul} T$ and $\overline{T}$ stand for the domain, kernel, range, multi-valued part, and closure, respectively. The inverse T^{-1} is a relation from $\mathfrak{H}'$ to $\mathfrak{H}$ defined by $\{ \{f', f\} : \{f, f'\} \in T \}$. The adjoint T^* is the closed linear relation from $\mathfrak{H}'$ to $\mathfrak{H}$ defined by $T^* = \big\{ \{h, k\} \in \mathfrak{H}' \oplus \mathfrak{H} : (k, f)_{\mathfrak{H}} = (h, g)_{\mathfrak{H}'}, \{f, g\} \in T \big\}$. The sum $T_1 + T_2$ and the componentwise sum $T_1 \hat{+} T_2$ of two linear relations T_1 and T_2 are defined by

$$T_1 + T_2 = \left\{ \begin{pmatrix} f \\ g + k \end{pmatrix} : \begin{pmatrix} f \\ g \end{pmatrix} \in T_1, \begin{pmatrix} f \\ k \end{pmatrix} \in T_2 \right\},$$

$$T_1 \hat{+} T_2 = \left\{ \begin{pmatrix} f + h \\ g + k \end{pmatrix} : \begin{pmatrix} f \\ g \end{pmatrix} \in T_1, \begin{pmatrix} h \\ k \end{pmatrix} \in T_2 \right\}. \tag{2.1}$$

If the componentwise sum is orthogonal it will be denoted by $T_1 \oplus T_2$. If T is closed, then the null spaces of $T - \lambda$, $\lambda \in \mathbb{C}$, defined by

$$\mathfrak{N}_\lambda(T) = \ker (T - \lambda), \quad \widehat{\mathfrak{N}}_\lambda(T) = \left\{ \begin{pmatrix} f \\ \lambda f \end{pmatrix} \in T : f \in \mathfrak{N}_\lambda(T) \right\}, \tag{2.2}$$

are also closed. Moreover, $\rho(T)$ ($\hat{\rho}(T)$) stands for the set of regular (regular type) points of T.

Recall that a linear relation T in $\mathfrak{H}$ is called *symmetric, dissipative*, or *accumulative* if $\operatorname{Im} (h', h) = 0, \geq 0$, or ≤ 0, respectively, holds for all $\{h, h'\} \in T$. These properties remain invariant under closures. By polarization it follows that a linear relation T in $\mathfrak{H}$ is symmetric if and only if $T \subset T^*$. A linear relation T in $\mathfrak{H}$ is called *selfadjoint* if $T = T^*$, and it is called *essentially selfadjoint* if $\overline{T} = T^*$. A dissipative (accumulative) linear relation T in $\mathfrak{H}$ is called *m-dissipative* (*m-accumulative*) if it has no proper dissipative (accumulative) extensions.

If the relation T is m-dissipative (m-accumulative), then $\operatorname{mul} T = \operatorname{mul} T^*$ and the orthogonal decomposition $\mathfrak{H} = (\operatorname{mul} T)^\perp \oplus \operatorname{mul} T$ induces an orthogonal decomposition of T as

$$T = \operatorname{gr} T_{\rm op} \oplus (\{0\} \times \mathcal{H}_\infty), \quad \mathcal{H}_\infty = \operatorname{mul} T, \quad \operatorname{gr} T_{\rm op} = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in T : g \in \mathcal{H} \ominus \mathcal{H}_\infty \right\},$$

where $T_\infty := \{0\} \times \mathcal{H}_\infty$ is a purely multi-valued selfadjoint relation in $\mathcal{H}_\infty$ and $T_{\rm op}$ is a densely defined m-dissipative (resp. m-accumulative) operator in $\mathcal{H} \ominus \mathcal{H}_\infty$. In particular, if T is a selfadjoint relation, then there is such a decomposition, where $T_{\rm op}$ is a densely defined selfadjoint operator in $\mathcal{H} \ominus \mathcal{H}_\infty$.

A family of linear relations $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, in a Hilbert space $\mathcal{H}$ is called a *Nevanlinna family* if:

(i) for every $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$ the relation $M(\lambda)$ is m-dissipative (resp. m-accumulative);
(ii) $M(\lambda)^* = M(\bar{\lambda})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
(iii) for some, and hence for all, $\mu \in \mathbb{C}_+(\mathbb{C}_-)$ the operator family $(M(\lambda) + \mu)^{-1} (\in [\mathcal{H}])$ is holomorphic for all $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$.

By the maximality condition, each relation $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is necessarily closed. The *class of all Nevanlinna families* in a Hilbert space is denoted by $\widetilde{\mathcal{R}}(\mathcal{H})$. If the multi-valued part $\operatorname{mul} M(\lambda)$ of $M \in \widetilde{\mathcal{R}}(\mathcal{H})$ is nontrivial, then it is independent of $\lambda \in \mathbb{C} \setminus \mathbb{R}$, so that

$$M(\lambda) = \operatorname{gr} M_{\rm op}(\lambda) \oplus M_\infty \quad \mathcal{H}_\infty = \operatorname{mul} M(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{2.3}$$

where $M_\infty = \{0\} \times \mathcal{H}_\infty$ is a purely multi-valued linear relation in $\mathcal{H}_\infty := \operatorname{mul} M(\lambda)$ and $M_{\rm op}(\cdot) \in \mathcal{R}\big(\mathcal{H} \ominus \mathcal{H}_\infty\big)$, cf. [51, 52, 55]. Identifying operators in $\mathcal{H}$ with their graphs one can consider classes

$$\mathcal{R}^u[\mathcal{H}] \subset \mathcal{R}^s[\mathcal{H}] \subset \mathcal{R}^s(\mathcal{H}) \subset \mathcal{R}(\mathcal{H})$$

introduced in Section 1 as subclasses of $\widetilde{\mathcal{R}}(\mathcal{H})$. In addition, a Nevanlinna family $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, which admits a holomorphic extrapolation to the negative real line $(-\infty, 0)$ (in the resolvent sense as in item (iii) of the above definition) and whose values $M(x)$ are nonnegative (nonpositive) selfadjoint relations for all $x < 0$ is called a *Stieltjes family* (an *inverse Stieltjes family*, respectively).

2.2 | Unitary and isometric relations in Kreĭn spaces

Let $\mathfrak{H}$ and $\mathcal{H}$ be Hilbert spaces and let $(\mathfrak{H}^2, J_{\mathfrak{H}})$ and $(\mathcal{H}^2, J_{\mathcal{H}})$ be Kreĭn spaces with fundamental symmetries $J_{\mathfrak{H}}$, $J_{\mathcal{H}}$ and indefinite inner products $[\cdot,\cdot]_{\mathfrak{H}}$, $[\cdot,\cdot]_{\mathcal{H}}$ defined in (1.9) and (1.10), respectively. If Γ is a linear relation from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$, then the adjoint linear relation $\Gamma^{[*]}$ is defined by

$$\Gamma^{[*]} = \left\{ \begin{pmatrix} \hat{k} \\ \hat{g} \end{pmatrix} \in \begin{pmatrix} \mathcal{H}^2 \\ \mathfrak{H}^2 \end{pmatrix} : [\hat{f}, \hat{g}]_{\mathfrak{H}^2} = [\hat{h}, \hat{k}]_{\mathcal{H}^2} \text{ for all } \begin{pmatrix} \hat{f} \\ \hat{h} \end{pmatrix} \in \Gamma \right\}. \tag{2.4}$$

Definition 2.1. ([64]) A linear relation Γ from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$ is said to be $(J_{\mathfrak{H}}, J_{\mathcal{H}})$-*isometric* if $\Gamma^{-1} \subset \Gamma^{[*]}$ and $(J_{\mathfrak{H}}, J_{\mathcal{H}})$-*unitary*, if $\Gamma^{-1} = \Gamma^{[*]}$.

The following two statements are due to Yu. L. Shmul'jan [64]; see also [25].

Proposition 2.2. *Let Γ be a $(J_{\mathfrak{H}}, J_{\mathcal{H}})$-unitary relation from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$. Then:*

(i) $\operatorname{dom}\Gamma$ *is closed if and only if* $\operatorname{ran}\Gamma$ *is closed;*

(ii) the following equalities hold: $\ker\Gamma = (\operatorname{dom}\Gamma)^{[\perp]}$, $\quad \operatorname{mul}\Gamma = (\operatorname{ran}\Gamma)^{[\perp]}$.

A $(J_{\mathfrak{H}}, J_{\mathcal{H}})$-unitary relation $\Gamma : (\mathfrak{H}^2, J_{\mathfrak{H}}) \to (\mathcal{H}^2, J_{\mathcal{H}})$ may be multi-valued, nondensely defined, and unbounded. It is the graph of an operator if and only if its range is dense in $\mathcal{H}^2$. In this case it need not be densely defined or bounded; and even if it is bounded it need not be densely defined.

3 | UNITARY AND ISOMETRIC BOUNDARY PAIRS AND THEIR WEYL FAMILIES

3.1 | Definitions and basic properties

Let A be a closed symmetric linear relation in the Hilbert space $\mathfrak{H}$. It is not assumed that the defect numbers of A are equal or finite. Following [25, 27] a unitary/isometric boundary pair for A^* is defined as follows.

Definition 3.1. Let A be a closed symmetric linear relation in a Hilbert space $\mathfrak{H}$, let $\mathcal{H}$ be an auxiliary Hilbert space and let Γ be a linear relation from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$. Then $\{\mathcal{H}, \Gamma\}$ is called a *unitary/isometric boundary pair for* A^*, if:

3.1.1 $A_* := \operatorname{dom}\Gamma$ is dense in A^* with respect to the topology on $\mathfrak{H}^2$;

3.1.2 the linear relation Γ is $(J_{\mathfrak{H}}, J_{\mathcal{H}})$-unitary/isometric.

In particular, it follows from this definition that for all vectors $\{\hat{f}, \hat{h}\}$, $\{\hat{g}, \hat{k}\} \in \Gamma$ of the form (1.10) the abstract *Green's identity* (cf. Definition 1.1) holds

$$(f', g)_{\mathfrak{H}} - (f, g')_{\mathfrak{H}} = (h', k)_{\mathcal{H}} - (h, k')_{\mathcal{H}}. \tag{3.1}$$

Let $\{\mathcal{H}, \Gamma\}$ be a unitary boundary pair for A^* and let $A_* = \operatorname{dom}\Gamma$. According to [25, Proposition 2.12] the domain A_* of Γ is a linear relation in $\mathfrak{H}$, such that

$$A \subset A_* \subset A^*, \quad \overline{A_*} = A^*.$$

The eigenspaces $\mathfrak{N}_\lambda(A_*)$ and $\hat{\mathfrak{N}}_\lambda(A_*)$ of A_* are defined as in (2.2),

$$\mathfrak{N}_\lambda(A_*) = \ker(A_* - \lambda), \quad \hat{\mathfrak{N}}_\lambda(A_*) = \left\{ \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix} \in A_* : f_\lambda \in \mathfrak{N}_\lambda(A_*) \right\}.$$

Definition 3.2. The *Weyl family* M of A corresponding to the unitary or isometric boundary pair $\{\mathcal{H},\Gamma\}$ is defined by $M(\lambda):=\Gamma\big(\widehat{\mathfrak{N}}_\lambda(A_*)\big)$, i.e.,

$$M(\lambda) := \left\{ \widehat{h} \in \mathcal{H}^2 : \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma \text{ for some } \widehat{f}_\lambda = \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix} \in \mathfrak{H}^2 \right\} \quad (\lambda \in \mathbb{C}\setminus\mathbb{R}).$$

In the case where M is single-valued it is called the *Weyl function* of A corresponding to $\{\mathcal{H},\Gamma\}$. The γ*-field* of A corresponding to the unitary/isometric boundary pair $\{\mathcal{H},\Gamma\}$ is defined by

$$\gamma(\lambda) := \left\{ \{h, f_\lambda\} \in \mathcal{H}\times\mathfrak{H} : \left\{ \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \text{ for some } h' \in \mathcal{H} \right\},$$

where $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Moreover, $\widehat{\gamma}(\lambda)$ stands for

$$\widehat{\gamma}(\lambda) := \left\{ \{h, \widehat{f}_\lambda\} \in \mathcal{H}\times\mathfrak{H}^2 : \left\{ \widehat{f}_\lambda, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \text{ for some } h' \in \mathcal{H} \right\}. \tag{3.2}$$

With $\gamma(\lambda)$ the relation $\Gamma\upharpoonright\widehat{\mathfrak{N}}_\lambda(A_*)$ can be rewritten as follows

$$\Gamma\upharpoonright\widehat{\mathfrak{N}}_\lambda(A_*) := \left\{ \left\{ \begin{pmatrix} \gamma(\lambda)h \\ \lambda\gamma(\lambda)h \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} : \begin{pmatrix} h \\ h' \end{pmatrix} \in M(\lambda) \right\}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{3.3}$$

Associate with Γ the following two linear relations which are not necessarily closed:

$$\Gamma_0 = \left\{ \{\widehat{f}, h\} : \{\widehat{f},\widehat{h}\} \in \Gamma,\ \widehat{h} = \begin{pmatrix} h \\ h' \end{pmatrix} \right\}, \quad \Gamma_1 = \left\{ \{\widehat{f}, h'\} : \{\widehat{f},\widehat{h}\} \in \Gamma,\ \widehat{h} = \begin{pmatrix} h \\ h' \end{pmatrix} \right\}. \tag{3.4}$$

The γ-field $\gamma(\cdot)$ associated with $\{\mathcal{H},\Gamma\}$ is the first component of the mapping $\widehat{\gamma}(\lambda)$ in (3.2). Observe, that

$$\widehat{\gamma}(\lambda) := \big(\Gamma_0\upharpoonright\widehat{\mathfrak{N}}_\lambda(A_*)\big)^{-1}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

is a linear mapping from $\Gamma_0\big(\widehat{\mathfrak{N}}_\lambda(A_*)\big) = \operatorname{dom} M(\lambda)$ onto $\widehat{\mathfrak{N}}_\lambda(A_*)$: it is single-valued in view of (3.1); cf. (3.7), (3.8). Consequently, the γ-field is a single-valued mapping from $\operatorname{dom} M(\lambda)$ onto $\mathfrak{N}_\lambda(A_*)$ and it satisfies $\gamma(\lambda)\Gamma_0\widehat{f}_\lambda = f_\lambda$ for all $\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(A_*)$.

If Γ is single-valued then these component mappings decompose Γ, $\Gamma = \Gamma_0\times\Gamma_1$, and the triple $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ will be called a *unitary/isometric boundary triple* for A^*. In this case the Weyl function corresponding to the unitary/isometric boundary triple $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ can be also defined via

$$M(\lambda)\Gamma_0\widehat{f}_\lambda = \Gamma_1\widehat{f}_\lambda, \quad \widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(A_*). \tag{3.5}$$

When A admits real regular type points it is useful to extend Definition 3.2 of the Weyl family to the points on the real line by setting $M(x) := \Gamma\big(\widehat{\mathfrak{N}}_x(A_*)\big)$ or, more precisely,

$$M(x) := \left\{ \widehat{h} \in \mathcal{H}^2 : \{\widehat{f}_x, \widehat{h}\} \in \Gamma \ \text{ for some } \widehat{f}_x = \begin{pmatrix} f_x \\ x f_x \end{pmatrix} \in \mathfrak{H}^2,\ x \in \mathbb{R} \right\}.$$

3.2 | Unitary boundary pairs and unitary boundary triples

The following theorem shows that the set of all Weyl families of unitary boundary pairs coincides with $\widetilde{\mathcal{R}}(\mathcal{H})$ (see [25, Theorem 3.9]). Recall that a unitary boundary pair $\{\mathcal{H},\Gamma\}$ for A^* is said to be *minimal*, if

$$\mathfrak{H} = \mathfrak{H}_{\min} := \overline{\operatorname{span}}\,\{\, \mathfrak{N}_\lambda(A_*) : \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \,\}.$$

Theorem 3.3. *Let $\{\mathcal{H},\Gamma\}$ be a unitary boundary pair for A^*. Then the corresponding Weyl family M belongs to the class of Nevanlinna families $\widetilde{\mathcal{R}}(\mathcal{H})$.*

Conversely, if M belongs to the class $\widetilde{\mathcal{R}}(\mathcal{H})$, then there exists a unique (up to a unitary equivalence) minimal unitary boundary pair $\{\mathcal{H},\Gamma\}$ whose Weyl function coincides with M.

Notice that Theorem 1.10 contains a general analytic criterion for an isometric boundary triple to be unitary; the Weyl function should be a Nevanlinna function, cf. Theorem 1.10.

Corollary 3.4. *The class of Weyl functions corresponding to unitary boundary triples coincides with the class $\mathcal{R}^s(\mathcal{H})$ of (in general unbounded) strict Nevanlinna functions.*

Proof. The statement is immediate when combining Theorem 3.3 with Proposition 4.5 from [25]. □

As a consequence of (3.1) and (3.3) the following identity holds (cf. (2.3))

$$(\lambda-\bar{\mu})(\gamma(\lambda)h,\gamma(\mu)k)_{\mathfrak{H}} = \left(M_{\text{op}}(\lambda)h,k\right)_{\mathcal{H}} - \left(h, M_{\text{op}}(\mu)k\right)_{\mathcal{H}}, \tag{3.6}$$

where $h\in \operatorname{dom} M(\lambda)$ and $k \in \operatorname{dom} M(\mu)$, $\lambda,\mu\in\mathbb{C}\setminus\mathbb{R}$.

As was already mentioned in Section 1 every operator valued function M from $\mathcal{R}^u[\mathcal{H}]$ ($\mathcal{R}^s[\mathcal{H}]$) can be realized as a Weyl function of some ordinary boundary triple (B-generalized boundary triple, respectively).

The multi-valued analog for the notion of B-generalized boundary triple was introduced in [25, Section 5.3], a formal definition reads as follows.

Definition 3.5. Let A be a symmetric operator (or relation) in the Hilbert space $\mathfrak{H}$ and let $\mathcal{H}$ be another Hilbert space. Then a linear relation $\Gamma : A^* \to \mathcal{H}\oplus\mathcal{H}$ with dense domain in A^* is said to be a *B-generalized boundary pair* for A^*, if the following three conditions are satisfied:

3.5.1 the abstract Green's identity (3.1) holds;

3.5.2 $\operatorname{ran}\Gamma_0 = \mathcal{H}$;

3.5.3 $A_0 = \ker\Gamma_0$ is selfadjoint,

where Γ_0 stands for the first component of Γ; see (3.4).

As was shown in [25, Proposition 5.9] every Weyl function of a B-generalized boundary pair belongs to the class $\mathcal{R}[\mathcal{H}]$ and, conversely, every operator valued function $M\in\mathcal{R}[\mathcal{H}]$ can be realized as the Weyl function of a B-generalized boundary pair.

3.3 | Isometric boundary pairs and isometric boundary triples

Let Γ be a $\left(J_{\mathfrak{H}}, J_{\mathcal{H}}\right)$-isometric relation from the Kreĭn space $\left(\mathfrak{H}^2, J_{\mathfrak{H}}\right)$ to the Kreĭn space $\left(\mathcal{H}^2, J_{\mathcal{H}}\right)$. In view of (1.9)–(1.11) this just means that the abstract Green's identity (3.1) holds. It follows from (3.1) that

$$\ker\Gamma\subset(\operatorname{dom}\Gamma)^{[\perp]},\quad \operatorname{mul}\Gamma\subset(\operatorname{ran}\Gamma)^{[\perp]},$$

compare Proposition 2.2. Let Γ_0 and Γ_1 be the linear relations determined by (3.4). The kernels $A_0 := \ker\Gamma_0$ and $A_1 := \ker\Gamma_1$ need not be closed, but they are symmetric extensions of $\ker\Gamma$ which are contained in the domain $A_* = \operatorname{dom}\Gamma$ of Γ; cf. [25, Proposition 2.13]. If $A_* = \operatorname{dom}\Gamma$ is dense in A^* then the pair $\{\mathcal{H},\Gamma\}$ is viewed as an isometric boundary pair for A^*; cf. Definition 3.1. In general $A := (A_*)^* = (\operatorname{dom}\Gamma)^{[\perp]}$ is an extension of $\ker\Gamma$ which need not belong to $\operatorname{dom}\Gamma$; for some sufficient conditions for the equality $A=\ker\Gamma$, see [26, Section 2.3] and [27, Section 7.8].

With $\{\widehat{f}_\lambda,\widehat{h}\}, \{\widehat{g}_\mu,\widehat{k}\}\in\Gamma$, $\lambda,\mu\in\mathbb{C}$, Green's identity (3.1) gives, cf. (3.6),

$$(h',k)_{\mathcal{H}} - (h,k')_{\mathcal{H}} = (\lambda-\bar{\mu})\left(f_\lambda, g_\mu\right)_{\mathfrak{H}}. \tag{3.7}$$

In particular, with $\mu=\lambda$ (3.7) implies that $\operatorname{Im}(h',h)_{\mathcal{H}} = \operatorname{Im}\lambda\|f_\lambda\|^2$. Hence, for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$

$$\ker\left(\Gamma\upharpoonright\widehat{\mathfrak{N}}_\lambda(A_*)\right)=\{0\}\quad\text{and}\quad\ker\left(\Gamma_j\upharpoonright\widehat{\mathfrak{N}}_\lambda(A_*)\right)=\{0\}\quad(j=0,1). \tag{3.8}$$

Moreover, with $\mu=\bar{\lambda}$ (3.7) implies that

$$M\left(\bar{\lambda}\right)\subseteq M(\lambda)^*,\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{3.9}$$

Here equality does not hold if Γ is not unitary. However, with the Weyl family the multi-valued part of Γ can be described explicitly; see [27, Lemma 7.57], cf. also [25, Lemma 4.1].

Lemma 3.6. *Let $\{\mathcal{H},\Gamma\}$ be an isometric boundary pair with the Weyl family M. Then the following equalities hold for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$:*

(i) $M(\lambda)\cap M(\lambda)^*=\operatorname{mul}\Gamma$;

(ii) $\ker M(\lambda)\times\{0\}=\operatorname{mul}\Gamma\cap(\mathcal{H}\times\{0\})$;

(iii) $\{0\}\times\operatorname{mul}M(\lambda)=\operatorname{mul}\Gamma\cap(\{0\}\times\mathcal{H})$;

(iv) $\ker(M(\lambda)-M(\lambda)^*)=\operatorname{mul}\Gamma_0$;

(v) $\ker\big(M(\lambda)^{-1}-M(\lambda)^{-*}\big)=\operatorname{mul}\Gamma_1$.

If Γ itself is single-valued, then the Weyl family M is an operator valued function, i.e. $\operatorname{mul}M(\lambda)=0$, belonging to the class $\mathcal{R}^s(\mathcal{H})$, see [25, Proposition 4.5]. Moreover, $\ker\operatorname{Im}(M(\lambda))=\{0\}$ and $\ker\operatorname{Im}\big(M(\lambda)^{-1}\big)=\{0\}$, in particular, $\ker M(\lambda)=0$. Recall that when Γ is single-valued $M(\lambda)$ can equivalently be defined by the equality (3.5). Hence, if $h\in\mathcal{H}$ is given and $h\in\Gamma_0\big(\widehat{\mathfrak{N}}_\lambda(A_*)\big)$, then $\gamma(\lambda)h$ solves a boundary eigenvalue problem, i.e., $\gamma(\lambda)h\in\ker(A^*-\lambda)$ and $\Gamma_0\widehat{\gamma}(\lambda)h=h$, while $\Gamma_1\widehat{\gamma}(\lambda)h=M(\lambda)h$. Also for an operator valued function $M(\cdot)$ the identity (3.7) can be rewritten in the form

$$(\lambda-\bar{\mu})(\gamma(\lambda)h,\gamma(\mu)k)_{\mathfrak{H}}=(M(\lambda)h,k)_{\mathcal{H}}-(h,M(\mu)k)_{\mathcal{H}},\tag{3.10}$$

where $h\in\operatorname{dom}M(\lambda)$ and $k\in\operatorname{dom}M(\mu)$, $\lambda,\mu\in\mathbb{C}\setminus\mathbb{R}$. This is an analog of (3.6) for an isometric boundary triple.

Let Γ be an isometric relation and let $A_0=\ker\Gamma_0$. Then A_0 is a symmetric, not necessarily closed, relation and one can write for every $\lambda\in\mathbb{C}\setminus\mathbb{R}$,

$$A_0=\left\{\begin{pmatrix}\big(A_0-\lambda\big)^{-1}h\\ h+\lambda\big(A_0-\lambda\big)^{-1}h\end{pmatrix}:\ h\in\operatorname{ran}\big(A_0-\lambda\big)\right\}.$$

The linear mapping

$$H(\lambda):h\to\left\{\begin{pmatrix}\big(A_0-\lambda\big)^{-1}h\\ h+\lambda\big(A_0-\lambda\big)^{-1}h\end{pmatrix}\right\}\tag{3.11}$$

from $\operatorname{ran}\big(A_0-\lambda\big)$ onto A_0 is clearly bounded with bounded inverse.

Lemma 3.7. *Let $\{\mathcal{H},\Gamma\}$ be an isometric boundary pair and let $A_0=\ker\Gamma_0$. Then the following assertions hold:*

(i) $\Gamma_1H(\lambda)$ *is closable for one (equivalently for all)* $\lambda\in\mathbb{C}\setminus\mathbb{R}$ *if and only if* $\Gamma_1\restriction A_0$ *is closable;*

(ii) $\Gamma_1H(\lambda)$ *is closed for one (equivalently for all)* $\lambda\in\mathbb{C}\setminus\mathbb{R}$ *if and only if* $\Gamma_1\restriction A_0$ *is closed;*

(iii) $\Gamma_1H(\lambda)$ *is a bounded operator for one (equivalently for all)* $\lambda\in\mathbb{C}\setminus\mathbb{R}$ *if and only if* $\Gamma_1\restriction A_0$ *is a bounded operator;*

(iv) $\operatorname{dom}\Gamma_1H(\lambda)$ *is dense in* $\mathfrak{H}$ *for some (equivalently for all)* $\lambda,\bar{\lambda}\in\mathbb{C}\setminus\mathbb{R}$ *if and only if* A_0 *is essentially selfadjoint;*

(v) $\operatorname{dom}\Gamma_1H(\lambda)=\mathfrak{H}$ *for some (equivalently for all)* $\lambda,\bar{\lambda}\in\mathbb{C}\setminus\mathbb{R}$ *if and only if* A_0 *is selfadjoint;*

(vi) $\operatorname{ran}\Gamma_1H(\lambda)=\Gamma_1\big(A_0\big)\ \big[=\operatorname{ran}\big(\Gamma_1\restriction A_0\big)\big]$ *for all* $\lambda\in\mathbb{C}\setminus\mathbb{R}$.

Proof. By definition $A_0=\ker\Gamma_0\subset\operatorname{dom}\Gamma_1$, so that $\operatorname{dom}\Gamma_1H(\lambda)=\operatorname{ran}(A_0-\lambda)$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Since $H(\lambda):\operatorname{ran}\big(A_0-\lambda\big)\to A_0$ is bounded with bounded inverse, all the statements are easily obtained by means of the equality $\Gamma_1\restriction A_0=\big(\Gamma_1H(\lambda)\big)H(\lambda)^{-1}$. □

Similar facts can be stated for the restriction $\Gamma_0\restriction A_1$, where $A_1=\ker\Gamma_1$.

The inclusion (3.13) in the next proposition was stated for a single-valued Γ with dense range in [27, Proposition 7.59]; here a direct proof for this inclusion is given in the general case.

Lemma 3.8. *Let $\{\mathcal{H},\Gamma\}$ be an isometric boundary pair, let $\gamma(\lambda)$ be its γ-field, and let $H(\lambda)$ be as defined in (3.11). Then*

$$\Gamma H(\lambda)\subset\begin{pmatrix}0\\ \gamma\big(\bar{\lambda}\big)^*\end{pmatrix}\hat{+}(\{0\}\times\operatorname{mul}\Gamma),\quad\lambda\in\mathbb{C}\setminus\mathbb{R},\tag{3.12}$$

where the adjoint $\gamma(\bar{\lambda})^*$ *of* $\gamma(\bar{\lambda})$ *is in general a linear relation. In particular,*

$$\Gamma_1 H(\lambda) \subset \gamma(\bar{\lambda})^* \hat{+} (\{0\} \times \operatorname{mul}\Gamma_1), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{3.13}$$

and if, in addition, $\operatorname{mul}\Gamma_1 = \{0\}$, *then*

$$\Gamma_1 H(\lambda) \subset \gamma(\bar{\lambda})^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.14}$$

Furthermore, the following statements hold:

(i) if $\gamma(\bar{\lambda})$ *is densely defined for some* $\bar{\lambda} \in \mathbb{C} \setminus \mathbb{R}$, *then* $\gamma(\bar{\lambda})^*$ *is a closed operator and if, in addition,* $\operatorname{mul}\Gamma_1 = \{0\}$, *then* $\Gamma_1 H(\lambda)$ *is a closable operator;*

(ii) if $A_0 = \ker\Gamma_0$ *is essentially selfadjoint, then* $\gamma(\bar{\lambda})$ *is closable for all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*;*

(iii) if $A_0 = \ker\Gamma_0$ *is selfadjoint, then* $\operatorname{dom}\gamma(\bar{\lambda})^* = \mathfrak{H}$ *and* $\gamma(\bar{\lambda})$ *is a bounded operator for all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. Let $h \in \operatorname{dom}\gamma(\bar{\lambda}) = \operatorname{dom} M(\bar{\lambda})$ and $k_\lambda \in \operatorname{ran}(A_0 - \lambda)$. Then $\{\hat{\gamma}(\bar{\lambda})h, \{h, h'\}\} \in \Gamma$ and, since $H(\lambda)k_\lambda \in A_0 = \ker\Gamma_0$, one has the inclusion $\{H(\lambda)k_\lambda, \{0, k''\}\} \in \Gamma$ for some $k'' \in \mathcal{H}$. On the other hand, $\{k_\lambda, k'\} \in \Gamma_1 H(\lambda)$ means that $\{k_\lambda, \{k, k'\}\} \in \Gamma H(\lambda)$ for some $k \in \mathcal{H}$ which combined with $\{H(\lambda)k_\lambda, \{0, k''\}\} \in \Gamma$ implies that $\{\{0,0\}, \{k, k' - k''\}\} \in \Gamma$.

Now applying Green's identity (3.1) shows that

$$\left(\bar{\lambda}\gamma(\bar{\lambda})h, (A_0 - \lambda)^{-1}k_\lambda\right) - \left(\gamma(\bar{\lambda})h, \left(I + \lambda(A_0 - \lambda)^{-1}\right)k_\lambda\right) = 0 - (h, k'')_{\mathcal{H}}.$$

This identity can be rewritten equivalently in the form

$$\left(\gamma(\bar{\lambda})h, k_\lambda\right) = (h, k'')_{\mathcal{H}}$$

for all $h \in \operatorname{dom}\gamma(\bar{\lambda})$ and $k_\lambda \in \operatorname{ran}(A_0 - \lambda)$. This proves that $\{k_\lambda, k''\} \in \gamma(\bar{\lambda})^*$. Hence, if $\left\{k_\lambda, \begin{pmatrix} k \\ k' \end{pmatrix}\right\} \in \Gamma H(\lambda)$ then

$$\left\{k_\lambda, \begin{pmatrix} k \\ k' \end{pmatrix}\right\} = \left\{k_\lambda, \begin{pmatrix} 0 \\ k'' \end{pmatrix}\right\} + \left\{0, \begin{pmatrix} k \\ k' - k'' \end{pmatrix}\right\} \tag{3.15}$$

with $\{k_\lambda, k''\} \in \gamma(\bar{\lambda})^*$ and $\{k, k' - k''\} \in \operatorname{mul}\Gamma$ from which the formulas (3.12) and (3.13) follow. If $\operatorname{mul}\Gamma_1 = \{0\}$, then $\{k, k' - k''\} \in \operatorname{mul}\Gamma$ implies that $k' = k''$ and therefore the above argument shows that $\{k_\lambda, k'\} \in \gamma(\bar{\lambda})^*$ for all $\{k_\lambda, k'\} \in \Gamma_1 H(\lambda)$; i.e. (3.14) is satisfied.

It remains to prove the statements (i)–(iii).

(i) If $\gamma(\bar{\lambda})$ is densely defined then clearly $\gamma(\bar{\lambda})^*$ is a closed operator and if Γ_1 is single-valued then (3.14) shows that $\Gamma_1 H(\lambda)$ is closable as a restriction of $\gamma(\bar{\lambda})^*$.

(ii) By Lemma 3.7 A_0 is essentially selfadjoint if and only if $\Gamma_1 H(\lambda)$ is densely defined, in which case also $\gamma(\bar{\lambda})^*$ is densely defined, so that $\gamma(\bar{\lambda})$ is closable.

(iii) If A_0 is selfadjoint, then $\operatorname{dom}\Gamma_1 H(\lambda) = \mathfrak{H}$ and, therefore, also $\operatorname{dom}\gamma(\bar{\lambda})^* = \mathfrak{H}$. In addition $\gamma(\bar{\lambda})$ is closable, thus $\operatorname{clos}\gamma(\bar{\lambda})$ and $\gamma(\bar{\lambda})$ are bounded operators.

□

Proposition 3.9. *Let* A *be a closed symmetric relation in the Hilbert space* $\mathfrak{H}$ *and let* $\{\mathcal{H}, \Gamma\}$ *be an isometric boundary pair, let* $M(\cdot)$ *and* $\gamma(\cdot)$ *be the corresponding Weyl function and the* γ*-field and, in addition, assume that* $A_0 = \ker\Gamma_0$ *is selfadjoint. Then:*

(i) $A_* := \operatorname{dom}\Gamma$ *admits the decomposition* $A_* = A_0 \hat{+} \hat{\mathfrak{N}}_\lambda(A_*)$ *and* $\hat{\mathfrak{N}}_\lambda(A_*)$ *is dense in* $\hat{\mathfrak{N}}_\lambda(A^*)$ *for all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*;*

(ii) with a fixed $\lambda \in \mathbb{C} \setminus \mathbb{R}$ *the graph of* Γ *admits the following representation:*

$$\Gamma = \Gamma A_0 \hat{+} \left\{\left\{\begin{pmatrix} \gamma(\lambda)h \\ \lambda\gamma(\lambda)h \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix}\right\} : \begin{pmatrix} h \\ h' \end{pmatrix} \in M(\lambda)\right\};$$

(iii) if $\widetilde{\Gamma} : (\mathfrak{H}^2, J_{\mathfrak{H}}) \to (\mathcal{H}^2, J_{\mathcal{H}})$ is an isometric extension of Γ with the Weyl function $\widetilde{M}$ and the γ-field $\widetilde{\gamma}(\cdot)$ such that $\widetilde{A}_* := \operatorname{dom}\widetilde{\Gamma} \subset A^*$, then with a fixed $\lambda \in \mathbb{C}\setminus\mathbb{R}$ the following equivalence holds:

$$\widetilde{\Gamma} = \Gamma \quad \Leftrightarrow \quad \widetilde{M}(\lambda) = M(\lambda).$$

Proof.

(i) By von Neumann's formula $A^* = A_0 \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(A^*)$. Since $A_* := \operatorname{dom}\Gamma$ is dense in A^* and $A_0 \subset A_*$, it follows that $A_* = A_0 \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(A_*)$ and that $\widehat{\mathfrak{N}}_\lambda(A_*)$ is dense in $\widehat{\mathfrak{N}}_\lambda(A^*)$ for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

(ii) In view of (i) for every $\{\widehat{f}, \widehat{k}\} \in \Gamma$ there exist unique elements $\widehat{f}_0 \in A_0$ and $\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(A_*)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, such that $\widehat{f} = \widehat{f}_0 + \widehat{f}_\lambda$. Moreover, if $\{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma$ then $\widehat{h} = \{h, h'\} \in M(\lambda)$ and one can write (uniquely) $\widehat{f}_\lambda = \widehat{\gamma}(\lambda)h$; see (3.3). The stated representation for Γ is now clear.

(iii) It follows from $\Gamma \subset \widetilde{\Gamma}$ that $A_0 \subset \ker\widetilde{\Gamma}_0$. Since $\ker\widetilde{\Gamma}_0$ is symmetric and A_0 is selfadjoint, the equality $A_0 = \ker\widetilde{\Gamma}_0$ holds. Now recall that two linear relations with $H_1 \subset H_2$ are equal precisely when the equalities $\operatorname{dom} H_1 = \operatorname{dom} H_2$ and $\operatorname{mul} H_1 = \operatorname{mul} H_2$ hold; see [4]. By Lemma 3.6 (i) $\operatorname{mul}\Gamma = M(\lambda) \cap M(\lambda)^*$. Therefore, $\widetilde{M}(\lambda) = M(\lambda)$ implies that $\operatorname{mul}\widetilde{\Gamma} = \operatorname{mul}\Gamma$. Moreover, we have $\operatorname{dom}\widetilde{M}(\lambda) = \operatorname{dom} M(\lambda)$ and, since $\widehat{\widetilde{\gamma}}(\lambda)$ maps $\operatorname{dom}\widetilde{M}(\lambda)$ onto $\widehat{\mathfrak{N}}_\lambda(\widetilde{A}_*)$ and $\widehat{\gamma}(\lambda)$ maps $\operatorname{dom} M(\lambda)$ onto $\widehat{\mathfrak{N}}_\lambda(A_*)$, we conclude from (i) that $\operatorname{dom}\widetilde{\Gamma} = \operatorname{dom}\Gamma$. Therefore, $\widetilde{M}(\lambda) = M(\lambda)$ implies $\widetilde{\Gamma} = \Gamma$. The reverse implication is clear. □

The Weyl function of an isometric or a unitary boundary pair takes values which need not be invertible, and in general can be unbounded, possibly multi-valued, operators. In what follows Weyl functions $M(\lambda)$, whose domain (or form domain) does not dependent on $\lambda \in \mathbb{C}\setminus\mathbb{R}$ are of special interest. Here a characterization for domain invariant Weyl families will be established. We start with the next lemma concerning the domain inclusion $\operatorname{dom} M(\lambda) \subset \operatorname{dom} M(\mu)$.

Lemma 3.10. *Let $\{\mathcal{H}, \Gamma\}$ be an isometric boundary pair with $A_* = \operatorname{dom}\Gamma$, let $M(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl family and the γ-field, and let $A_0 = \ker\Gamma_0$. Then for each fixed $\lambda, \mu \in \mathbb{C}\setminus\mathbb{R}$ with $\lambda \neq \mu$ the inclusion*

$$\operatorname{dom} M(\mu) \subset \operatorname{dom} M(\lambda) \tag{3.16}$$

is equivalent to the inclusion

$$\operatorname{ran}\gamma(\mu) \subset \operatorname{ran}(A_0 - \lambda). \tag{3.17}$$

If one of these conditions is satisfied, then the γ-field $\gamma(\cdot)$ satisfies the identity

$$\gamma(\lambda)h = \left[I + (\lambda-\mu)(A_0-\lambda)^{-1}\right]\gamma(\mu)h, \quad h \in \operatorname{dom}\gamma(\mu). \tag{3.18}$$

Proof. By Definition 3.2 $\operatorname{dom} M(\lambda) = \operatorname{dom}\gamma(\lambda) = \Gamma_0(\widehat{\mathfrak{N}}_\lambda(A_*))$ and, moreover, $\operatorname{ran}\gamma(\lambda) = \mathfrak{N}_\lambda(A_*)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Now assume that (3.16) holds and let $h \in \operatorname{dom} M(\mu) \subset \operatorname{dom} M(\lambda)$. It follows from (3.2) that there exist $h', h'' \in \mathcal{H}$ such that

$$\left\{\begin{pmatrix}\gamma(\lambda)h\\ \lambda\gamma(\lambda)h\end{pmatrix}, \begin{pmatrix}h\\ h'\end{pmatrix}\right\} \in \Gamma\restriction\widehat{\mathfrak{N}}_\lambda(T) \subset \Gamma, \qquad \left\{\begin{pmatrix}\gamma(\mu)h\\ \mu\gamma(\mu)h\end{pmatrix}, \begin{pmatrix}h\\ h''\end{pmatrix}\right\} \in \Gamma\restriction\widehat{\mathfrak{N}}_\mu(T) \subset \Gamma.$$

This implies

$$\left\{\begin{pmatrix}(\gamma(\lambda)-\gamma(\mu))h\\ (\lambda\gamma(\lambda)-\mu\gamma(\mu))h\end{pmatrix}, \begin{pmatrix}0\\ h'-h''\end{pmatrix}\right\} \in \Gamma,$$

and hence

$$\begin{pmatrix}(\gamma(\lambda)-\gamma(\mu))h\\ (\lambda\gamma(\lambda)-\mu\gamma(\mu))h\end{pmatrix} \in A_0 \text{ and } \begin{pmatrix}(\gamma(\lambda)-\gamma(\mu))h\\ (\lambda-\mu)\gamma(\mu)h\end{pmatrix} \in A_0 - \lambda. \tag{3.19}$$

Therefore, $\gamma(\mu)h \in \operatorname{ran}(A_0 - \lambda)$ for every $h \in \operatorname{dom} M(\mu)$ and thus (3.17) follows.

Conversely, assume that (3.17) holds and let $h \in \operatorname{dom} M(\mu) = \operatorname{dom} \gamma(\mu)$. This implies that

$$\left\{ \begin{pmatrix} \gamma(\mu)h \\ \mu\gamma(\mu)h \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \tag{3.20}$$

for some $h' \in \mathcal{H}$. Moreover, since $\gamma(\mu)h \in \operatorname{ran}(A_0 - \lambda)$, there exists an element $k \in \mathfrak{H}$ such that $\{k, \gamma(\mu)h + \lambda k\} \in A_0 = \ker \Gamma_0$. Consequently, there exists $\varphi \in \mathcal{H}$ such that

$$\left\{ \begin{pmatrix} (\lambda - \mu)k \\ (\lambda - \mu)\gamma(\mu)h + \lambda(\lambda - \mu)k \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \right\} \in \Gamma. \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$\left\{ \begin{pmatrix} \gamma(\mu)h + (\lambda - \mu)k \\ \lambda(\gamma(\mu)h + (\lambda - \mu)k) \end{pmatrix}, \begin{pmatrix} h \\ h' + \varphi \end{pmatrix} \right\} \in \Gamma.$$

Therefore, $h \in \Gamma_0(\widehat{\mathfrak{N}}_\lambda(A_*)) = \operatorname{dom} M(\lambda)$. This proves the inclusion (3.16).

Finally, observe that the assumption (3.16) implies (3.19). Since A_0 is symmetric, $(A_0 - \lambda)^{-1}$ is a bounded operator on $\operatorname{ran}(A_0 - \lambda)$ and, thus, (3.19) leads to (3.18). □

The next result characterizes domain invariance of the Weyl family corresponding to an arbitrary isometric boundary pair $\{\mathcal{H}, \Gamma\}$. In the special case of a unitary boundary pair $\{\mathcal{H}, \Gamma\}$ items (i) and (iii) contain [25, Proposition 4.11, Corollary 4.12].

Proposition 3.11. *Let the assumptions and notations be as in Lemma 3.10. Then:*

(i) $\operatorname{dom} M(\lambda)$ *is independent from* $\lambda \in \mathbb{C}_+$ *(resp. from* $\lambda \in \mathbb{C}_-$*) if and only if*

$$\mathfrak{N}_\mu(A_*) \subset \operatorname{ran}(A_0 - \lambda) \quad \textit{for all } \lambda, \mu \in \mathbb{C}_+ \quad (\textit{resp. for all } \lambda, \mu \in \mathbb{C}_-), \quad \lambda \neq \mu,$$

in this case the γ*-field* $\gamma(\cdot)$ *satisfies*

$$\gamma(\lambda) = \left[I + (\lambda - \mu)(A_0 - \lambda)^{-1} \right] \gamma(\mu), \quad \lambda, \mu \in \mathbb{C}_+(\mathbb{C}_-);$$

(ii) if A_0 *is selfadjoint, then* $\operatorname{dom} M(\lambda)$ *does not dependent on* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*;*

(iii) if $\operatorname{dom} M(\lambda)$ *does not dependent on* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*, then* A_0 *is essentially selfadjoint.*

Proof. The assertions (i) and (ii) follow directly from Lemma 3.10.

To see (iii) one can use the same argument that is presented in [25, Corollary 4.12]. □

3.4 | Some transforms of boundary triples

In this subsection a specific transform of isometric boundary triples is treated. In what follows such transforms are used repeatedly and, in fact, they appear also in concrete boundary value problems in ODE and PDE setting. To formulate a general result in the abstract setting consider in the Kreĭn space $(\mathcal{H}^2, J_\mathcal{H})$ the transformation operator V whose action is determined by the triangular operator

$$V = \begin{pmatrix} G^{-1} & 0 \\ EG^{-1} & G^* \end{pmatrix}, \quad E \subset E^*, \ \overline{\operatorname{dom}}\, E = \overline{\operatorname{dom}}\, G = \overline{\operatorname{ran}}\, G = \mathcal{H}, \ \ker G = \{0\}. \tag{3.22}$$

By assumptions on G one has $\ker G^* = \operatorname{mul} G^* = \{0\}$, so that the adjoint G^* is an injective operator in $\mathcal{H}$. To keep a wider generality, G is not assumed to be a closed operator, while in applications that will often be the case. In particular, it is possible that G^* is not densely defined and also its range need not be dense. Since E is a densely defined symmetric operator, it is closable and its closure $\overline{E} \subset E^*$ is also symmetric. With the assumptions on V in (3.22) a direct calculation shows that

$$(J_\mathcal{H} V f, V g)_{\mathcal{H}^2} = (J_\mathcal{H} f, g)_{\mathcal{H}^2}, \quad f, g \in \operatorname{dom} V.$$

Hence, V is an isometric operator in the Kreĭn space $(\mathcal{H}^2, J_\mathcal{H})$. Moreover, V is injective. These observations lead to the following (unbounded) extension of [26, Proposition 3.18].

Lemma 3.12. *Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an isometric boundary triple for A^* such that* $\ker \Gamma = A$, *let $\gamma(\lambda)$ and $M(\lambda)$ be the corresponding γ-field and the Weyl function, and let V be as defined in (3.22). Then V is isometric in the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$ and moreover:*

(i) the transform $\widetilde{\Gamma} = V \circ \Gamma$

$$\begin{pmatrix} \widetilde{\Gamma}_0 \hat{f} \\ \widetilde{\Gamma}_1 \hat{f} \end{pmatrix} = \begin{pmatrix} G^{-1}\Gamma_0 \hat{f} \\ EG^{-1}\Gamma_0 \hat{f} + G^*\Gamma_1 \hat{f} \end{pmatrix}, \quad \hat{f} \in \operatorname{dom} \Gamma, \tag{3.23}$$

defines an isometric boundary triple with domain $\widetilde{A}_* := \operatorname{dom} \widetilde{\Gamma}$ and kernel $\ker \widetilde{\Gamma} = A$;

(ii) the γ-field and the Weyl function of $\widetilde{\Gamma}$ are in general unbounded nondensely defined operators given by

$$\widetilde{\gamma}(\lambda)k = \gamma(\lambda)Gk, \quad \widetilde{M}(\lambda)k = Ek + G^* M(\lambda)Gk, \quad k \in \operatorname{dom} \widetilde{M}(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Proof.

(i) By the assumptions in (3.22) V is an isometric operator in the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$ and since Γ is an isometric operator from $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to $(\mathcal{H}^2, J_{\mathcal{H}})$ the composition operator $V \circ \Gamma$ is also an isometric operator from $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to $(\mathcal{H}^2, J_{\mathcal{H}})$. Since V is injective, one has $\ker \widetilde{\Gamma} = \ker \Gamma = A$. In general V is not everywhere defined, so that $\widetilde{A}_*$ is typically a proper linear subset of $A_* = \operatorname{dom} \Gamma$ which is not necessarily dense in A^*.

(ii) By Definition 3.2 the Weyl function $\widetilde{M}(\lambda)$ of $\widetilde{\Gamma}$ is given by $\widetilde{M}(\lambda) = V \circ M(\lambda)$ or, more explicitly, by

$$\begin{aligned} \widetilde{M}(\lambda) &= \left\{ \begin{pmatrix} G^{-1}h \\ EG^{-1}h + G^* M(\lambda)h \end{pmatrix} : \ h \in \operatorname{dom} EG^{-1} \cap \operatorname{dom} G^* M(\lambda) \right\} \\ &= \left\{ \begin{pmatrix} k \\ Ek + G^* M(\lambda)Gk \end{pmatrix} : \ \begin{matrix} h = Gk \in \operatorname{dom} G^* M(\lambda), \\ k \in \operatorname{dom} G \cap \operatorname{dom} E \end{matrix} \right\} \\ &= E + G^* M(\lambda)G. \end{aligned}$$

Similarly, $\left(G^{-1}\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(\widetilde{A}_*)\right)^{-1} = \left(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(\widetilde{A}_*)\right)^{-1} G$ implies that $\widetilde{\gamma}(\lambda) = \gamma(\lambda)G$ with $\operatorname{dom} \widetilde{\gamma}(\lambda) = \operatorname{dom} \widetilde{M}(\lambda)$.

□

Example 3.13.

(i) If $G = I_{\mathcal{H}}$ then the condition $\widetilde{\Gamma}_1 \hat{f} = 0$ reads as $\Gamma_1 \hat{f} + E\Gamma_0 \hat{f} = 0$. In applications such conditions are called Robin type boundary conditions. This corresponds to the transposed boundary triple $\{\mathcal{H}, \Gamma_1 + E\Gamma_0, -\Gamma_0\}$ which is also isometric and has $-(M(\lambda) + E)^{-1}$ as its Weyl function.

(ii) As indicated G need not be closable. An extreme situation appears when G is a *singular operator*; cf. [47]. By definition this means that $\operatorname{dom} G \subset \ker \overline{G}$ or, equivalently, that $\operatorname{ran} G \subset \operatorname{mul} \overline{G}$. Thus, in this case $\operatorname{dom} G^* = \operatorname{ran} G^* = \{0\}$. If, for instance, Γ is an ordinary boundary triple for A^* then $A_0 = \ker \Gamma_0$ and $A_1 = \ker \Gamma_1$ are selfadjoint. It is easy to check that

$$\widetilde{A}_* = \left\{ \hat{f} \in A^* : \ \Gamma_1 \hat{f} = 0 \right\} = \ker \Gamma_1 = A_1, \quad \ker \widetilde{\Gamma}_0 = A_0 \cap A_1 = A.$$

Moreover, $\operatorname{ran} \widetilde{\Gamma} = E \upharpoonright \operatorname{dom} G$ is a symmetric operator in $\mathcal{H}$ and $\operatorname{dom} \widetilde{M}(\lambda) = \operatorname{dom} \widetilde{\gamma}(\lambda)$ is trivial.

3.5 | Some additional remarks

Despite of the fact that the paper [21] has been quoted by M. G. Kreĭn [50] and a discussion on [21] appears in the monograph [23] the actual results of Calkin on reduction operators remained widely unknown among experts in extension theory. Apparently this was caused by the fact that the paper [21] was ahead of time – it was using the new language of binary linear relations with hidden ideas on geometry of indefinite inner product spaces, concepts which were not well developed at that time. The concept of a bounded reduction operator investigated therein (see [21, Chapter IV]) essentially covers the notion of an ordinary boundary triple in Definition 1.1 as well as the notion of D-boundary triple introduced in [60] for symmetric operators with unequal defect numbers. An overview on the early work of Calkin and more detailed description on its connections to boundary triples and unitary boundary pairs (boundary relations) can be found in the monograph [40]. In fact, [40] contains a collection of articles reflecting various recent activities in different fields of applications with related realization results for Weyl functions, including analysis of differential operators, continuous time state/signal systems and boundary control theory with interconnection analysis of port-Hamiltonian systems involving Dirac and Tellegen structures etc.

4 | *AB*-GENERALIZED BOUNDARY PAIRS AND BOUNDARY TRIPLES

In this section we present a new generalization of the class of B-generalized boundary triples from [33] (cf. Definition 1.5).

Definition 4.1. Let A be a symmetric operator (or relation) in the Hilbert space $\mathfrak{H}$ and let $\mathcal{H}$ be another Hilbert space. Then a linear relation $\Gamma : A^* \to \mathcal{H} \oplus \mathcal{H}$ with domain dense in A^* is said to be an *almost B-generalized boundary pair*, in short, *AB-generalized boundary pair* for A^*, if the following three conditions are satisfied:

4.1.1 the abstract Green's identity (3.1) holds;

4.1.2 $\operatorname{ran}\Gamma_0$ is dense in $\mathcal{H}$;

4.1.3 $A_0 = \ker\Gamma_0$ is selfadjoint.

A *single-valued AB-generalized boundary pair* is also said to be an *almost B-generalized boundary triple*, shortly, an *AB-generalized boundary triple* for A^*.

If Γ is an AB-generalized boundary pair for A^*, then the same is true for its closure. Indeed, since $\overline{\Gamma}$ is an extension of Γ, it is clear that $\operatorname{dom}\overline{\Gamma}$ is dense in A^* and $\operatorname{ran}(\overline{\Gamma})_0$ is dense in $\mathcal{H}$. By Assumption 4.1.1 Γ is isometric (in the Kreĭn space sense), i.e. $\Gamma^{-1} \subset \Gamma^{[*]}$. Thus, clearly $\overline{\Gamma}^{-1} \subset \Gamma^{[*]} = \overline{\Gamma}^{[*]}$. Hence, the closure satisfies Green's identity (3.1) and this implies that the corresponding kernels $\ker(\overline{\Gamma})_0 \supset \ker\Gamma_0 = A_0$ and $\ker(\overline{\Gamma})_1 \supset \ker\Gamma_1 = A_1$ are symmetric. Therefore, $\ker(\overline{\Gamma})_0 = A_0$ must be selfadjoint.

4.1 | Characteristic properties of *AB*-generalized boundary pairs

The next theorem describes the central properties of an AB-generalized boundary pair.

Theorem 4.2. *Let A be a closed symmetric relation in $\mathfrak{H}$, let $\{\mathcal{H},\Gamma\}$ be an AB-generalized boundary pair for A^*, and let Γ_0 and Γ_1 be the corresponding component mappings from* $\operatorname{dom}\Gamma$ *into $\mathcal{H}$. Moreover, let $\gamma(\cdot)$ and $M(\cdot)$ be the corresponding γ-field and the Weyl function, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then:*

(i) $\ker\Gamma = A$;

(ii) $A_* := \operatorname{dom}\Gamma$ *admits the decomposition* $A_* = A_0 \hat{+} \widehat{\mathfrak{N}}_\lambda(A_*)$ *and* $\widehat{\mathfrak{N}}_\lambda(A_*)$ *is dense in* $\widehat{\mathfrak{N}}_\lambda(A^*)$;

(iii) the γ-field $\gamma(\lambda)$ is a densely defined bounded operator from $\operatorname{ran}\Gamma_0$ *onto* $\widehat{\mathfrak{N}}_\lambda(A_*)$*. It is domain invariant and*

$$\operatorname{dom}\gamma(\lambda) = \operatorname{ran}\Gamma_0, \quad \ker\gamma(\lambda) = \operatorname{mul}\Gamma_0;$$

(iv) the adjoint $\gamma(\lambda)^*$ is a bounded everywhere defined operator and, moreover, equalities hold in (3.12), (3.13),

$$\Gamma H(\lambda) = \begin{pmatrix} 0 \\ \gamma(\bar{\lambda})^* \end{pmatrix} \hat{+} (\{0\} \times \operatorname{mul}\Gamma), \quad \Gamma_1 H(\lambda) = \gamma(\bar{\lambda})^* \hat{+} (\{0\} \times \operatorname{mul}\Gamma_1); \tag{4.1}$$

(v) the closure of the γ-field $\overline{\gamma(\lambda)}$ is a bounded operator from $\mathcal{H}$ into $\widehat{\mathfrak{N}}_\lambda(A^*)$ satisfying the identity

$$\overline{\gamma(\lambda)} = \left[I + (\lambda - \mu)(A_0 - \lambda)^{-1}\right]\overline{\gamma(\mu)}, \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R};$$

(vi) the Weyl function $M(\cdot)$ is a densely defined operator function which is domain invariant, $\operatorname{dom}M(\lambda) = \operatorname{ran}\Gamma_0$, $M(\lambda) \subset M(\bar{\lambda})^*$*, and the imaginary part* $\operatorname{Im}M(\lambda) = (M(\lambda) - M(\lambda)^*)/2i$ *is bounded with* $\operatorname{dom}\operatorname{Im}M(\lambda) = \operatorname{ran}\Gamma_0$ *and* $\ker\operatorname{Im}M(\lambda) = \operatorname{mul}\Gamma_0$*. Furthermore, $M(\lambda)$ admits the representation*

$$M(\lambda) = E + M_0(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{4.2}$$

where $E = \operatorname{Re} M(\mu)$ *is a symmetric densely defined operator in* $\mathcal{H}$ *and* $M_0(\cdot)$ *is the restriction of a Nevanlinna function* $\overline{M_0}(\cdot) \in \mathcal{R}[\mathcal{H}]$ *onto the domain* $\operatorname{dom} E$.

Proof. (i) It is clear from Green's identity that $\ker\Gamma \subset (\operatorname{dom}\Gamma)^* = (A_*)^* = A$; cf. [27, Lemma 7.3]. To prove the reverse inclusion, the property that $\gamma(\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, is densely defined will be used (and this is independently proved in (iii) below). Assumption 4.1.3 implies that $A = (A_*)^* \subset A_0^* = A_0 = \ker\Gamma_0 \subset \operatorname{dom}\Gamma$. On the other hand, if $k_\lambda \in \operatorname{ran}(A-\lambda)$ then by Lemma 3.8 $\{k_\lambda, k''\} \in \gamma(\bar\lambda)^*$ for some k'' and thus

$$(k'', h)_{\mathcal{H}} = (k_\lambda, \gamma(\bar\lambda)h) = 0 \quad \text{for all} \quad h \in \operatorname{dom}\gamma(\bar\lambda).$$

Assumption 4.1.2 combined with $\operatorname{dom}\gamma(\bar\lambda) = \operatorname{ran}\Gamma_0$ (see proof of (iii) below) shows that $\gamma(\bar\lambda)$ is densely defined and, hence, $\gamma(\bar\lambda)^*$ is an operator and $k'' = \gamma(\bar\lambda)^* k_\lambda = 0$. Now apply the formula (3.15) in the proof of Lemma 3.8 to $k_\lambda \in \operatorname{ran}(A-\lambda)$: therein $\{k_\lambda, \{k, k'\}\} \in \Gamma H(\lambda)$ and $k'' = 0$ so that (3.15) reads as

$$\left\{k_\lambda, \begin{pmatrix} k \\ k' \end{pmatrix}\right\} = \left\{k_\lambda, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} + \left\{0, \begin{pmatrix} k \\ k' \end{pmatrix}\right\}, \quad \begin{pmatrix} k \\ k' \end{pmatrix} \in \operatorname{mul}\Gamma.$$

Hence, $H(\lambda)k_\lambda \in \ker\Gamma$ and $A = H(\lambda)(\operatorname{ran}(A-\lambda)) \subset \ker\Gamma$. Therefore, $\ker\Gamma = A$.

(ii) This holds by Proposition 3.9 (i).

(iii) & (iv) The decomposition of A_* in (ii) combined with $A_0 = \ker\Gamma_0$ implies that

$$\Gamma_0(A_*) = \Gamma_0(\widehat{\mathfrak{N}}_\lambda(A_*)) = \operatorname{dom} M(\lambda) = \operatorname{dom}\gamma(\lambda), \quad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

Hence, $\operatorname{dom} M(\lambda) = \operatorname{dom}\gamma(\lambda) = \operatorname{ran}\Gamma_0$ does not depend on $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Now Assumption 4.1.2 shows that $\gamma(\lambda)$ and $M(\lambda)$ are densely defined for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Moreover, according to Lemma 3.8 (iii) $\gamma(\lambda)$ is a bounded operator and the equality $\operatorname{dom}\gamma(\lambda)^* = \mathfrak{H}$ holds for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Since $\gamma(\lambda)$ is densely defined in $\mathcal{H}$, the adjoint $\gamma(\lambda)^*$ is a bounded everywhere defined operator from $\mathfrak{H}$ into $\Gamma_1(A_0)$. Since $M(\bar\lambda) \subset M(\lambda)^*$, see (3.9), the adjoint $M(\lambda)^*$ and the closure of $M(\bar\lambda)$ are also densely defined operators. In view of (3.10) one has

$$(\lambda - \bar\mu)(\gamma(\lambda)h, \gamma(\mu)k)_{\mathfrak{H}} = \left((M(\lambda) - M(\mu)^*)h, k\right)_{\mathcal{H}}, \quad \lambda, \mu \in \mathbb{C}\setminus\mathbb{R}, \tag{4.3}$$

for all $h, k \in \operatorname{dom}\gamma(\lambda) = \operatorname{ran}\Gamma_0$. In particular, $2i\operatorname{Im}\lambda\|\gamma(\lambda)h\|^2_{\mathfrak{H}} = \left((M(\lambda) - M(\lambda)^*)h, h\right)_{\mathcal{H}}$ holds for all $h \in \operatorname{dom}\gamma(\lambda) = \operatorname{dom} M(\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$. By Lemma 3.6 (4.3) implies that

$$\ker\gamma(\lambda) = \ker(M(\lambda) - M(\lambda)^*) = \operatorname{mul}\Gamma_0, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

It remains to prove (4.1). Observe, that $\operatorname{dom}\Gamma_1 H(\lambda) = \operatorname{dom}\gamma(\bar\lambda)^* = \mathfrak{H}$ and clearly the multi-valued parts on both sides of the inclusion in (3.12), (3.13) are equal. Hence, the inclusions (3.12), (3.13) must prevail actually as equalities (by the criterion from [4]).

(v) Since $\operatorname{dom} M(\lambda) = \operatorname{dom}\gamma(\lambda) = \operatorname{ran}\Gamma_0$ does not depend on $\lambda \in \mathbb{C}\setminus\mathbb{R}$, the following equality holds

$$\left(I + (\lambda-\mu)(A_0-\lambda)^{-1}\right)\gamma(\mu)h = \gamma(\lambda)h \quad \text{for all } \lambda, \mu \in \mathbb{C}\setminus\mathbb{R}, \; h \in \operatorname{ran}\Gamma_0 \tag{4.4}$$

by Proposition 3.11. According to (iii) $\gamma(\lambda)$ is bounded and densely defined, so that its closure $\overline{\gamma(\lambda)}$ is bounded and defined everywhere on $\mathcal{H}$. The formula in (iv) is obtained by taking closures in (4.4).

(vi) It suffices to prove the representation (4.2) for $M(\lambda)$, since all the other assertions were already shown above when proving (iii) & (iv). It follows from (4.3) and (4.4) that

$$\begin{aligned}(M(\lambda)h, k) &= (M(\mu)^*h, k) + (\lambda-\bar\mu)\left(\left(I + (\lambda-\mu)(A_0-\lambda)^{-1}\right)\gamma(\mu)h, \gamma(\mu)k\right)\\ &= (\operatorname{Re} M(\mu)h, k) + \left(\left((\lambda - \operatorname{Re}\mu) + (\lambda-\mu)(\lambda-\bar\mu)(A_0-\lambda)^{-1}\right)\gamma(\mu)h, \gamma(\mu)k\right),\end{aligned}$$

$h, k \in \operatorname{dom}\gamma(\lambda) = \operatorname{dom} M(\mu) = \operatorname{ran}\Gamma_0$, $\lambda, \mu \in \mathbb{C}\setminus\mathbb{R}$. Here $2\operatorname{Re} M(\mu) = M(\mu) + M(\mu)^*$ and hence $2(\operatorname{Re} M(\mu))^* \supset M(\mu)^* + \overline{M(\mu)} \supset 2\operatorname{Re} M(\mu)$, so that $E := \operatorname{Re} M(\mu)$ is a symmetric operator with $\operatorname{dom} E = \operatorname{dom} M(\mu) = \operatorname{ran}\Gamma_0$. On the other hand, since $\overline{\gamma(\lambda)}$ and its adjoint $\gamma(\lambda)^*$ are bounded everywhere defined operators, it follows that the closure of

$$M_0(\lambda) := \gamma(\mu)^*\left((\lambda - \operatorname{Re}\mu) + (\lambda-\mu)(\lambda-\bar\mu)(A_0-\lambda)^{-1}\right)\gamma(\mu)$$

defines a holomorphic operator valued Nevanlinna function in the class $\mathcal{R}[\mathcal{H}]$, such that $M(\lambda) = E + M_0(\lambda)$. This completes the proof. □

For an AB-generalized boundary pair it is possible to describe the graph of Γ, $(\operatorname{ran}\Gamma)^{[\perp]}$, and the closure of $\operatorname{ran}\Gamma$ explicitly.

Corollary 4.3. *Let Γ be an AB-generalized boundary pair for A^* and let $\gamma(\cdot)$ and $M(\cdot) = E + M_0(\cdot)$ be the corresponding γ-field and Weyl function as in Theorem 4.2 with $E = \operatorname{Re} M(\mu)$ for some fixed $\mu \in \mathbb{C} \setminus \mathbb{R}$. Then:*

(i) with a fixed $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the graph of Γ admits the following representation:

$$\Gamma = \left\{ \left\{ H(\lambda)k_\lambda, \begin{pmatrix} 0 \\ \gamma(\bar{\lambda})^* k_\lambda \end{pmatrix} \right\} + \left\{ \begin{pmatrix} \gamma(\lambda)h \\ \lambda\gamma(\lambda)h \end{pmatrix}, \begin{pmatrix} h \\ M(\lambda)h \end{pmatrix} \right\} : \begin{matrix} k_\lambda \in \operatorname{ran}\left(A_0 - \lambda\right) \\ h \in \operatorname{dom} M(\lambda) \end{matrix} \right\};$$

(ii) the range of Γ satisfies

$$(\operatorname{ran}\Gamma)^{[\perp]} = E^* \upharpoonright \ker \overline{\gamma(\lambda)} \quad \text{and} \quad \overline{\operatorname{ran}}\,\Gamma = \left(E^* \upharpoonright \ker \overline{\gamma(\lambda)}\right)^*,$$

and here $\ker \overline{\gamma(\lambda)} = \ker\left(\overline{\operatorname{Im} M(\lambda)}\right) = \ker\left(\overline{M_0(\lambda)}\right)$ does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In particular, $\operatorname{ran}\Gamma$ is dense in $\mathcal{H}$ if and only if $\operatorname{dom} E^* \cap \ker \overline{\gamma(\lambda)} = \{0\}$ for some or, equivalently, for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

(iii) Γ is a single-valued mapping if and only if $\operatorname{mul}\Gamma_0 = \{0\}$ or, equivalently, if and only if $\ker \operatorname{Im} M(\lambda)\,(= \ker \gamma(\lambda)) = 0$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof.

(i) Using the representation of $\Gamma H(\lambda)$ in (4.1), the inclusion $\operatorname{mul}\Gamma \subset M(\lambda)$ in Lemma 3.6, and the fact that by Theorem 4.2 $M(\lambda)$ is an operator, one concludes that the representation of Γ given in Proposition 3.9 (ii) can be rewritten in the form as stated in (i).

(ii) The description in (i) shows that

$$\operatorname{ran}\Gamma = \Gamma\left(A_0\right) \hat{+} M(\lambda) = \begin{pmatrix} 0 \\ \operatorname{ran}\gamma\left(\bar{\lambda}\right)^* \end{pmatrix} \hat{+} M(\lambda), \tag{4.5}$$

for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Therefore, $(\operatorname{ran}\Gamma)^{[\perp]} = \left(\{0\} \times \operatorname{ran}\gamma\left(\bar{\lambda}\right)^*\right)^{[\perp]} \cap M(\lambda)^*$. Hence $\hat{k} = \{k, k'\} \in (\operatorname{ran}\Gamma)^{[\perp]}$ if and only if $\hat{k} \in M(\lambda)^*$ and $k \in \left(\operatorname{ran}\gamma\left(\bar{\lambda}\right)^*\right)^\perp = \ker \overline{\gamma\left(\bar{\lambda}\right)}$. Since $\gamma(\lambda)$ and $\operatorname{Im} M(\lambda)$ are bounded and $E = \operatorname{Re} M(\mu)$, one has $\operatorname{Re} M_0(\mu) = 0$ and $\ker \overline{\gamma(\mu)} = \ker\left(\overline{\operatorname{Im} M_0(\mu)}\right) = \ker\left(\overline{M_0(\mu)}\right)$. This kernel does not depend on $\mu \in \mathbb{C} \setminus \mathbb{R}$ due to $\overline{M_0(\cdot)} \in \mathcal{R}[\mathcal{H}]$; cf. Theorem 4.2 (v). This proves that

$$(\operatorname{ran}\Gamma)^{[\perp]} = M(\lambda)^* \upharpoonright \ker \overline{\gamma(\lambda)} = \left(E^* + M_0(\lambda)^*\right) \upharpoonright \ker \overline{\gamma(\lambda)} = E^* \upharpoonright \ker \overline{\gamma(\lambda)}.$$

As to the closure observe that

$$\overline{\operatorname{ran}}\,\Gamma = \left((\operatorname{ran}\Gamma)^{[\perp]}\right)^{[\perp]} = ((\operatorname{ran}\Gamma)^*)^* = (E^* \upharpoonright \ker \overline{\gamma(\lambda)})^*.$$

Thus, $\overline{\operatorname{ran}}\,\Gamma = \mathcal{H} \times \mathcal{H}$ if and only if $E^* \upharpoonright \ker \overline{\gamma(\lambda)} = \{0, 0\}$ or, equivalently, $\operatorname{dom} E^* \cap \ker \overline{\gamma(\lambda)} = \{0\}$, since E^* together with $E\,(\subset E^*)$ is a densely defined operator in $\mathcal{H}$.

(iii) In view of (i) this follows from $\operatorname{mul}\Gamma_0 = \ker \operatorname{Im} M(\lambda) = \ker \gamma(\lambda)$; see Lemma 3.6. □

Corollary 4.3 shows that for an AB-generalized boundary pair the inclusion $\operatorname{mul}\Gamma \subset (\operatorname{ran}\Gamma)^{[\perp]}$ is in general strict. In particular, the range of Γ for a single-valued AB-generalized boundary pair, i.e., an AB-generalized boundary triple, need not be dense in $\mathcal{H} \times \mathcal{H}$. Notice that an AB-generalized boundary pair with the surjectivity condition $\operatorname{ran}\Gamma_0 = \mathcal{H}$ is called a B-generalized boundary pair for A^*; see Definition 3.5. The next result gives a connection between AB-generalized boundary pairs and B-generalized boundary pairs.

Theorem 4.4. *Let $\{\mathcal{H},\Gamma\}$ be a B-generalized boundary pair for A^*, and let $M(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl function and γ-field. Let also E be a symmetric densely defined operator in $\mathcal{H}$ and let $\Gamma = \{\Gamma_0, \Gamma_1\}$ where $\Gamma_i = \pi_i\Gamma$, $i = 0, 1$, be the corresponding components of Γ as in (3.4). Then the transform*

$$\begin{pmatrix}\widetilde{\Gamma}_0\\ \widetilde{\Gamma}_1\end{pmatrix} = \begin{pmatrix} I & 0\\ E & I\end{pmatrix}\begin{pmatrix}\Gamma_0\\ \Gamma_1\end{pmatrix} \tag{4.6}$$

defines an AB-generalized boundary pair for A^*. The corresponding Weyl function $\widetilde{M}(\cdot)$ and $\widetilde{\gamma}(\cdot)$-field are connected by

$$\widetilde{M}(\lambda) = E + M(\lambda), \quad \widetilde{\gamma}(\lambda) = \gamma(\lambda)\upharpoonright \operatorname{dom} E, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

Furthermore, $\widetilde{\Gamma} := \{\widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ in (4.6) is closed if and only if E is a closed symmetric operator in $\mathcal{H}$, in particular, the closure of $\widetilde{\Gamma}$ is given by (4.6) with E replaced by its closure $\overline{E}$.

Conversely, if $\{\mathcal{H},\widetilde{\Gamma}\}$ is an AB-generalized boundary pair for A^* then there exists a B-generalized boundary pair $\{\mathcal{H},\Gamma\}$ for A^* and a densely defined symmetric operator E in $\mathcal{H}$ such that $\widetilde{\Gamma}$ is given by (4.6).

Proof. ($\Rightarrow$) By Lemma 3.12 the block triangular transformation V in (4.6) acting on $\mathcal{H}\times\mathcal{H}$ is an isometric operator. Consequently, $\widetilde{\Gamma} = V\circ\Gamma$ is isometric. It is clear from (4.6) that $A_0 := \ker\Gamma_0 \subset \ker\widetilde{\Gamma}_0$, which by symmetry of $\ker\widetilde{\Gamma}_0$ implies that $\ker\widetilde{\Gamma}_0 = A_0$. Clearly $\operatorname{ran}\widetilde{\Gamma}_0$ is dense in $\mathcal{H}$, since $\operatorname{ran}\Gamma_0 = \mathcal{H}$ and E is densely defined. Thus $\widetilde{\Gamma}$ admits all the properties in Definition 4.1. Since in addition $\ker\widetilde{\Gamma} = \ker\Gamma$, it follows from Theorem 4.2 (i) that $\widetilde{A}_* = \operatorname{dom}\widetilde{\Gamma}$ is dense in A^*. Therefore, $\{\mathcal{H},\widetilde{\Gamma}\}$ is an AB-generalized boundary pair for A^*. The connections between the Weyl functions and γ-fields are clear from the definitions; cf. Lemma 3.12.

To treat the closedness properties of $\widetilde{\Gamma}$ consider the representation of $\widetilde{\Gamma}$ in Corollary 4.3. Let $\lambda\in\mathbb{C}\setminus\mathbb{R}$ be fixed and assume that the sequence $\{\widehat{f}_n, \widehat{k}_n\} \in \widetilde{\Gamma}$ converges to $\{\widehat{f}, \widehat{k}\}$. Then $\widehat{f}_n = H(\lambda)k_{n,\lambda} + \widehat{\gamma}(\lambda)h_n$ with unique $k_{n,\lambda}\in\operatorname{ran}(A_0-\lambda)$ and $h_n \in \operatorname{dom}\widetilde{M}(\lambda) = \operatorname{dom} E$ and, since the angle between the graphs of A_0 and $\widehat{\mathfrak{N}}_\lambda(\widetilde{A}_*)$ is positive, it follows that $k_{n,\lambda}\to k_\lambda \in \operatorname{ran}(A_0-\lambda)$. Moreover, the representation of $\{\widehat{f}_n, \widehat{k}_n\}\in\widetilde{\Gamma}$ in Corollary 4.3 shows that $h_n \to h\in\mathcal{H}$. According to Theorem 4.2 $\gamma(\lambda)$ and $\gamma(\bar\lambda)^*$ are bounded operators and, since $\widetilde{M}(\lambda) = E + M(\lambda)$, where $M(\lambda)$ is bounded (see [26, Proposition 3.16]), it follows from Corollary 4.3 that

$$\left\{H(\lambda)k_{n,\lambda}, \begin{pmatrix}0\\ \gamma(\bar\lambda)^* k_{n,\lambda}\end{pmatrix}\right\} + \left\{\begin{pmatrix}\gamma(\lambda)h_n\\ \lambda\gamma(\lambda)h_n\end{pmatrix}, \begin{pmatrix}h_n\\ Eh_n + M(\lambda)h_n\end{pmatrix}\right\} \in \widetilde{\Gamma}$$

converges to

$$\left\{H(\lambda)k_{\lambda}, \begin{pmatrix}0\\ \gamma(\bar\lambda)^* k_{\lambda}\end{pmatrix}\right\} + \left\{\begin{pmatrix}\gamma(\lambda)h\\ \lambda\gamma(\lambda)h\end{pmatrix}, \begin{pmatrix}h\\ h'' + M(\lambda)h\end{pmatrix}\right\} \in \operatorname{clos}\widetilde{\Gamma}, \tag{4.7}$$

where $\{h, h''\}\in\overline{E}$. It is also clear that the limit element in (4.7) belongs to $\widetilde{\Gamma}$ if and only if $\lim_{n\to\infty}\{h_n, Eh_n\} = \{h, h''\}\in E$. Therefore, $\widetilde{\Gamma}$ is closed if and only if E is closed and, moreover, the closure of $\widetilde{\Gamma}$, which is also an AB-generalized boundary pair for A^* (as stated after Definition 4.1), is given by (4.6) with E replaced by its closure $\overline{E}$.

($\Leftarrow$) Let $\{\mathcal{H},\widetilde{\Gamma}\}$ be an AB-generalized boundary pair. According to Theorem 4.2 the corresponding Weyl function $\widetilde{M}$ is of the form $\widetilde{M} = E + M$, where $\overline{M}\in\mathcal{R}[\mathcal{H}]$ and $E\,(=\operatorname{Re}\widetilde{M}(\mu))$ is a symmetric densely defined operator in $\mathcal{H}$.

To construct $\widetilde{\Gamma}$ directly from an associated B-generalized boundary pair, define

$$\begin{pmatrix}\widehat{\Gamma}_0\\ \widehat{\Gamma}_1\end{pmatrix} := \begin{pmatrix} I & 0\\ -E & I\end{pmatrix}\begin{pmatrix}\widetilde{\Gamma}_0\\ \widetilde{\Gamma}_1\end{pmatrix}. \tag{4.8}$$

Since $\widetilde{M}(\lambda) = \widetilde{\Gamma}(\widehat{\mathfrak{N}}_\lambda(\widetilde{A}_*)) \subset \operatorname{ran}\widetilde{\Gamma}$, where $\widetilde{A}_* = \operatorname{dom}\widetilde{\Gamma}$, and $\operatorname{dom}\widetilde{M}(\lambda) = \operatorname{dom} E$, it follows that the graph of $\widetilde{M}(\lambda)$ belongs to the domain of the block operator $\begin{pmatrix} I & 0\\ -E & I\end{pmatrix}$, i.e., $\widehat{\mathfrak{N}}_\lambda(\widetilde{A}_*)\subset\operatorname{dom}\widehat{\Gamma}$ for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Moreover,

$$\widehat{\Gamma}(\widehat{\mathfrak{N}}_\lambda(\widetilde{A}_*)) = -E + \widetilde{M}(\lambda) = M(\lambda)\upharpoonright\operatorname{dom} E \subset \operatorname{ran}\widehat{\Gamma}$$

for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Since $\overline{M}\in\mathcal{R}[\mathcal{H}]$ this implies that $\operatorname{ran}\widehat{\Gamma}_0$ is dense in $\mathcal{H}$. Clearly, $\ker\widehat{\Gamma}_0 = \ker\widetilde{\Gamma}_0 = A_0$ and since $\widetilde{A}_* = A_0 \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(\widetilde{A}_*)$ one concludes that $\widetilde{A}_* = \operatorname{dom}\widehat{\Gamma} = \operatorname{dom}\widetilde{\Gamma}$ is dense in A^*. Thus, $\widehat{\Gamma}$ is also an AB-generalized boundary pair for A^* and, consequently, also its closure is an AB-generalized boundary pair for A^*, too. Denote the closure of $\widehat{\Gamma}$ by $\Gamma^{(0)}$. Then the corresponding Weyl function $M^{(0)}(\cdot)$ is an extension of M and its closure is equal to $\overline{M}$. Since $\Gamma^{(0)}$ is closed, it must be unitary by [27, Theorem 7.51] (cf. [25, Proposition 3.6]). In particular, $M^{(0)}(\cdot)$ is also closed, i.e., $M^{(0)}(\cdot) = \overline{M} \in \mathcal{R}[\mathcal{H}]$. Thus, $\operatorname{ran}\Gamma_0^{(0)} = \operatorname{dom} M^{(0)}(\cdot) = \mathcal{H}$ and hence $\Gamma^{(0)}$ is a B-generalized boundary pair for A^*; see Definition 3.5. Finally, in view of (4.8) one has

$$\begin{pmatrix}\widetilde{\Gamma}_0\\ \widetilde{\Gamma}_1\end{pmatrix} = \begin{pmatrix} I & 0\\ E & I\end{pmatrix}\begin{pmatrix}\widehat{\Gamma}_0\\ \widehat{\Gamma}_1\end{pmatrix} \subset \begin{pmatrix} I & 0\\ E & I\end{pmatrix}\Gamma^{(0)} =: \widetilde{\Gamma}^{(0)}.$$

Here equality $\widetilde{\Gamma} = \widetilde{\Gamma}^{(0)}$ holds by Proposition 3.9 (iii), since $\widetilde{M}^{(0)}(\cdot) = E + \overline{M}(\cdot) = \widetilde{M}(\cdot)$. □

The proof of Theorem 4.4 contains also the following result.

Corollary 4.5. *If $\{\mathcal{H}, \widetilde{\Gamma}\}$ is an AB-generalized boundary pair for A^* with the Weyl function $\widetilde{M}(\cdot)$ and $E = \operatorname{Re}\widetilde{M}(\mu)$ for some $\mu \in \rho(M)$, then the closure of $\Gamma = \begin{pmatrix} I & 0\\ -E & I\end{pmatrix}\widetilde{\Gamma}$ defines a B-generalized boundary pair for A^* with the bounded Weyl function $M(\cdot) = \operatorname{clos}(\widetilde{M}(\cdot) - E)$.*

Theorems 4.2 and 4.4 imply the following characterization for the Weyl functions corresponding to AB-generalized boundary pairs.

Corollary 4.6. *The class of AB-generalized boundary pairs coincides with the class of isometric boundary pairs whose Weyl function is of the form*

$$M(\lambda) = E + M_0(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{4.9}$$

with E a symmetric densely defined operator in $\mathcal{H}$ and $M_0(\cdot) \in \mathcal{R}[\mathcal{H}]$. In particular, every function M of the form (4.9) is a Weyl function of some AB-generalized boundary pair.

Proof. By Theorem 4.2 the Weyl function M of an AB-generalized boundary pair $\{\mathcal{H}, \widetilde{\Gamma}\}$ is of the form (4.9), where $E \subset E^*$ is densely defined and $M_0(\cdot) \in \mathcal{R}[\mathcal{H}]$.

Conversely, if M is given by (4.9) with $M_0(\cdot) \in \mathcal{R}[\mathcal{H}]$, then by [26, Proposition 3.16] $M_0(\cdot)$ is the Weyl function of a B-generalized boundary pair $\{\mathcal{H}, \Gamma\}$ for A^*. Now according to the first part of Theorem 4.4 the transform $\widetilde{\Gamma}$ of Γ defined in (4.6) is an AB-generalized boundary pair for A^* such that the corresponding Weyl function is equal to (4.9). □

By Definition 3.5 every B-generalized boundary pair is also an AB-generalized boundary pair. Hence, the notions of an AB-generalized boundary triple and AB-generalized boundary pair generalize the earlier notions of “a generalized boundary value space” as introduced in [33, Definition 6.1] and “a boundary relation with the Weyl function in $\mathcal{R}[\mathcal{H}]$” as defined in [25, Proposition 5.9]. It is emphasized that B-generalized boundary pairs are not only isometric: they are also unitary in the Kreĭn space sense, see Definition 3.1. The characteristic properties of the classes of B-generalized boundary triples and pairs can be found in Theorem 1.7, see also [25, Propositions 5.7, 5.9] and [26, Proposition 3.16]. Some further characterizations connected with AB-generalized boundary pairs are given in the next corollary.

Corollary 4.7. *Let $\{\mathcal{H}, \widetilde{\Gamma}\}$ be an AB-generalized boundary pair for A^* as in Theorem 4.4 and let E be a symmetric densely defined operator in $\mathcal{H}$ as in (4.6). Then:*

- *(i) $\{\mathcal{H}, \widetilde{\Gamma}\}$ is a unitary boundary pair (boundary relation) for A^* if and only if the operator E is selfadjoint;*
- *(ii) $\{\mathcal{H}, \widetilde{\Gamma}\}$ has an extension to a unitary boundary pair for A^* if and only if the operator E has equal defect numbers and in this case the formula (4.6) defines a unitary extension of $\widetilde{\Gamma}$ when E is replaced by some selfadjoint extension E_0 of E;*
- *(iii) $\{\mathcal{H}, \widetilde{\Gamma}\}$ is a B-generalized boundary pair for A^* if and only if the operator E is bounded and everywhere defined (hence selfadjoint);*
- *(iv) $\{\mathcal{H}, \widetilde{\Gamma}\}$ is an ordinary boundary triple for A^* if and only if* $\operatorname{ran}\Gamma = \mathcal{H} \oplus \mathcal{H}$, *or equivalently, if and only if* $\operatorname{ran}\Gamma$ *is closed, E is bounded, and* $\ker \operatorname{Im} M(\lambda) = 0$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof.

(i) By Theorem 4.4 $\widetilde{\Gamma}$ is closed if and only if E is closed. Moreover, $E = E^*$ if and only if M is a Nevanlinna function. Now the statement follows from [25, Proposition 3.6] (or [27, Theorem 7.51]).

(ii) This is clear from part (i) and Theorem 4.4.

(iii) This follows from Theorem 4.2 (v) by the equalities $\operatorname{ran}\widetilde{\Gamma}_0 = \operatorname{dom}\widetilde{M} = \operatorname{dom} E\,(=\mathcal{H})$.

(iv) The first equivalence is contained in [25, Proposition 5.3]. To prove the second criterion, we apply Corollary 4.3, in particular, the representation of $\operatorname{ran}\Gamma$ in (4.5):

$$\operatorname{ran}\Gamma = \Gamma\big(A_0\big) \hat{+} M(\lambda) = \big(\{0\}\times \operatorname{ran}\gamma\big(\bar{\lambda}\big)^*\big) \hat{+} M(\lambda). \tag{4.10}$$

Clearly, E is bounded precisely when $M(\lambda)$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, is bounded. In this case the angle between the last two subspaces in (4.10) is positive and then $\operatorname{ran}\Gamma$ is closed if and only if $\operatorname{ran}\gamma\big(\bar{\lambda}\big)^*$ and $M(\lambda)$ both are closed. By Theorem 4.2$\gamma(\lambda)$ is bounded and $\operatorname{dom}\gamma(\lambda) = \operatorname{dom} M(\lambda) = \mathcal{H}$, when $M(\lambda)$ is closed and bounded. Then $\gamma(\lambda)$ is closed and $\big(\operatorname{ran}\gamma(\bar{\lambda})^*\big)^{\perp} = \ker\gamma(\lambda) = \ker \operatorname{Im} M(\lambda)$. Therefore, the conditions $\operatorname{ran}\Gamma$ is closed, E is bounded, and $\ker \operatorname{Im} M(\lambda) = 0$ imply that $\operatorname{ran}\Gamma$ is also dense in $\mathcal{H}\times\mathcal{H}$ and, thus, Γ is surjective. The converse is clear.

□

The class of AB-generalized boundary triples contains the class of so-called *quasi boundary triples*, which has been studied in J. Behrndt and M. Langer [11].

Definition 4.8 ([11]). Let A be a densely defined symmetric operator in $\mathfrak{H}$. A triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is said to be a *quasi boundary triple* for A^*, if $A_* := \operatorname{dom}\Gamma$ is dense in A^* and the following conditions are satisfied:

4.8.1 Green's identity (1.1) holds for all $\hat{f}, \hat{g} \in A_*$;

4.8.2 $A_0 = \ker\Gamma_0$ is a selfadjoint operator in $\mathfrak{H}$;

4.8.3 the range of Γ is dense in $\mathcal{H}\times\mathcal{H}$.

For isometric boundary pairs $\operatorname{mul}\Gamma \subset (\operatorname{ran}\Gamma)^{[\perp]}$ and thus the condition 4.8.3 implies that Γ is single-valued. Since the condition 4.8.3 implies 4.1.2, quasi boundary triples are AB-generalized boundary triples. Corollary 4.3 gives the following characterization for quasi boundary triples.

Corollary 4.9. *An AB-generalized boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* with the Weyl function $M = E + M_0(\cdot)$ represented in the form (4.2) is a quasi boundary triple (with single-valued Γ) for A^* if and only if $\operatorname{ran}\Gamma$ is dense in $\mathcal{H}\oplus\mathcal{H}$, or equivalently,*

$$\operatorname{dom} E^* \cap \ker \overline{\operatorname{Im} M(\lambda)} \left(= \operatorname{dom} E^* \cap \ker \operatorname{Im} \overline{M_0(\lambda)}\right) = \{0\}, \tag{4.11}$$

for some or, equivalently, for every $\lambda\in\mathbb{C}\setminus\mathbb{R}$.

Proof. Item (ii) of Corollary 4.3 shows that $\operatorname{ran}\Gamma$ is dense in $\mathcal{H}$ if and only if $\operatorname{dom}\overline{E^*} \cap \ker\overline{\gamma(\lambda)} = \{0\}$ for some or, equivalently, for every $\lambda\in\mathbb{C}\setminus\mathbb{R}$. This is equivalent to the conditions in (4.11), since $\ker\overline{\gamma(\lambda)} = \ker \operatorname{Im} M(\lambda) = \ker \operatorname{Im}\overline{M_0(\lambda)}$; see Corollary 4.3. □

Remark 4.10. A connection between B-generalized boundary triples and quasi boundary triples can be found in [27, Theorem 7.57], [70, Propositions 5.1, 5.3]. In fact, each of them is special case of Theorem 4.4. Moreover, it should be noted that in the formulation of the converse part in [27, Theorem 7.57] one should use a B-generalized boundary pair $\{\mathcal{H}, \Gamma\}$, instead of a B-generalized boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, since $\ker\gamma(\lambda) = \ker \operatorname{Im} M(\lambda) = 0$ (M is strict) does not imply in general that $\ker\overline{\gamma(\lambda)} = \ker \overline{\operatorname{Im} M(\lambda)} = \ker \operatorname{Im}\overline{M_0(\lambda)} = 0$, i.e. $\overline{M_0} \in \mathcal{R}[\mathcal{H}]$ as in the proof of Theorem 4.4 above: only the factor mapping $\Gamma/\operatorname{mul}\Gamma$ (see [7], [42, Eq. (2.15)]) becomes single-valued (equivalently the corresponding Weyl function is strict, cf. [25, Proposition 4.7]). It should be also noted that a condition which is equivalent to (4.11) appears in [70, Section 5.1]; see also [69]. For some further related facts, see Corollary 5.18 and Remark 5.20 in Section 5.

The next result describes a connection between B-generalized boundary pairs and ordinary boundary triples. In the special case of B-generalized boundary triples the corresponding result is presented in [27, Theorem 7.24].

Theorem 4.11. *Let $\{\mathcal{H},\Gamma\}$ be a B-generalized boundary pair for A^* and let $M(\cdot)$ be the corresponding Weyl function. Then there exists an ordinary boundary triple $\{\mathcal{H}_s,\Gamma_0^0,\Gamma_1^0\}$ with $\mathcal{H}_s = \overline{\operatorname{ran}}\operatorname{Im} M(\lambda)$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, and operators $E = E^*\in\mathcal{B}(\mathcal{H})$ and $G\in\mathcal{B}(\mathcal{H},\mathcal{H}_s)$ with $\ker G = \mathcal{H}\ominus\mathcal{H}_s$ such that (1.6) holds with G^{-1} standing for the inverse of G as a linear relation. If $M_0(\cdot)$ is the Weyl function corresponding to the ordinary boundary triple $\{\mathcal{H}_s,\Gamma_0^0,\Gamma_1^0\}$, then*

$$M(\lambda) = G^* M_0(\lambda) G + E, \quad \lambda\in\rho(A_0).$$

Proof. The proof relies on [27, Theorem 7.24] and [26, Propositions 3.18, 4.1].

Let $E = \operatorname{Re} M(i)$. Then by [26, Propositions 3.18] (cf. Lemma 3.12) the transform

$$\widetilde{\Gamma} = \left\{ \left\{ \widehat{f}, \begin{pmatrix} h \\ -Eh+h' \end{pmatrix} \right\} : \{\widehat{f},\widehat{h}\}\in\Gamma \right\} \tag{4.12}$$

defines a new B-generalized boundary pair for A^* with the Weyl function $M(\cdot)-E$ and the original γ-field $\gamma(\cdot)$ of $\{\mathcal{H},\Gamma\}$.

Let P_s be the orthogonal projection onto $\mathcal{H}_s := \overline{\operatorname{ran}}\operatorname{Im} M(\lambda)$. Then according to [26, Proposition 4.1] the transform $\Gamma^{(s)} : \mathfrak{H}^2 \to (\mathcal{H}_s)^2$ given by

$$\Gamma^{(s)} = \left\{ \left\{ \widehat{f}, \begin{pmatrix} k \\ P_s k' \end{pmatrix} \right\} : \{\widehat{f},\widehat{k}\}\in\widetilde{\Gamma},\ (I-P_s)k = 0 \right\} \tag{4.13}$$

determines a B-generalized boundary pair $\{\mathcal{H}_s,\Gamma^{(s)}\}$ for $(A^{(s)})^*$, where $A^{(s)}$ is defined by

$$A^{(s)} := \ker\Gamma^{(s)}. \tag{4.14}$$

The corresponding Weyl function and γ-field are given by

$$M^{(s)}(\lambda) = P_s(M(\lambda)-E)\restriction\mathcal{H}_s, \quad \gamma^{(s)}(\lambda) = \gamma(\lambda)\restriction\mathcal{H}_s.$$

Recall that $\ker(M(\lambda)-E) = \ker\operatorname{Im} M(\lambda)$ does not depend on $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Consequently, $M(\lambda)-E = M^{(s)}(\lambda)\oplus 0_{\mathcal{H}\ominus\mathcal{H}_s}$. Since $\ker\gamma(\bar\lambda) = \ker(M(\bar\lambda)-E) = \ker P_s$ one has $\operatorname{ran}\gamma(\bar\lambda)^*\subset\mathcal{H}_s$ and it follows from Corollary 4.3 that $\operatorname{ran}\widetilde{\Gamma}_1\subset\mathcal{H}_s$. Therefore, (4.13) implies that $A^{(s)}$ defined in (4.14) coincides with A: $\ker\Gamma^{(s)} = \ker\Gamma = A$. By construction $M^{(s)}(\cdot)\in\mathcal{R}^s[\mathcal{H}_s]$ and hence $\Gamma^{(s)}$ is single-valued; i.e. $\{\mathcal{H}_s,\Gamma_0^{(s)},\Gamma_1^{(s)}\}$ is in fact a B-generalized boundary triple for A^*; cf. [25, Proposition 4.7].

One can now apply [27, Theorem 7.24] with $R = \operatorname{Re} M^{(s)}(i) = 0$ and $K = (\operatorname{Im} M^{(s)}(i))^{1/2}$ to conclude existence of an ordinary boundary triple $\{\mathcal{H}_s,\Gamma_0^0,\Gamma_1^0\}$ with the Weyl function $M_0(\cdot)$ such that

$$\Gamma_0^{(s)} = K^{-1}\Gamma_0^0, \quad \Gamma_1^{(s)} = K\Gamma_1^0, \quad \text{and} \quad M^{(s)}(\lambda) = KM_0(\lambda)K, \quad \lambda\in\mathbb{C}\setminus\mathbb{R}.$$

In particular, $M(i) = E + iK^2P_s$ and $M(\lambda) = E + P_sKM_0(\lambda)KP_s$. The statement follows by taking $G = KP_s$. Indeed, since $\operatorname{ran}\widetilde{\Gamma}_1\subset\mathcal{H}_s$ and $\operatorname{mul}\widetilde{\Gamma}_0 = \ker\operatorname{Im} M(\lambda) = \ker P_s$ (see Lemma 3.6) (4.13) shows that $\operatorname{dom}\Gamma^{(s)} = \operatorname{dom}\widetilde{\Gamma}$ and

$$\widetilde{\Gamma} = \Gamma^{(s)} \oplus \left\{ \left\{ \widehat{0}, \begin{pmatrix} k \\ 0 \end{pmatrix} \right\} : P_s k = 0 \right\} = \left\{ \left\{ \widehat{k}, \begin{pmatrix} P_s^{-1}\Gamma_0^{(s)}\widehat{k} \\ P_s\Gamma_1^{(s)}\widehat{k} \end{pmatrix} \right\} : \widehat{k}\in\operatorname{dom}\Gamma^{(s)} \right\}.$$

Finally, using $G^{-1} = P_s^{-1}K^{-1} = K^{-1}\oplus(\{0\}\times\ker P_s)$ and (4.12) yields the formulas (1.6) and (1.7). □

The notion of an AB-generalized boundary pair introduced in Definition 4.1 appears to be useful in characterizing the class of Nevanlinna functions with unbounded values (and multi-valued Nevanlinna families) whose imaginary parts generate closable forms $\tau_{M(\lambda)} = [(M(\lambda)\cdot,\cdot)-(\cdot,M(\lambda)\cdot)]/2i$ via (3.6) and whose closures are domain invariant. All such functions, after renormalization by a bounded operator $G\in[\mathcal{H}]$, turn out to be Weyl functions of AB-generalized boundary triples, i.e., for a suitable choice of G, G^*MG is a function of the form (4.2): see Theorem 5.32 and Corollary 5.34 in Section 5.

4.2 | A Kreĭn type formula for AB-generalized boundary triples

In this section a Kreĭn type (resolvent) formula for AB-generalized boundary triples will be presented. We refer to [27, Proposition 7.27] where a special case of B-generalized boundary triples was treated, and [11, 12] for a special case of quasi boundary triples. The form of the formula as given in Theorem 4.12 below is new even in the standard case of ordinary boundary triples.

If $A_0 = \ker \Gamma_0$ is selfadjoint, then it follows from the first von Neumann's formula that for each $\lambda \in \rho(A_0)$ the domain of Γ can be decomposed as follows:

$$\operatorname{dom}\Gamma = A_0 \,\widehat{+}\, \left(\operatorname{dom}\Gamma \cap \widehat{\mathfrak{N}}_\lambda(A^*)\right).$$

Now let Γ be single-valued and let Γ be decomposed as $\Gamma = \{\Gamma_0, \Gamma_1\}$. Let $\widetilde{A}$ be an extension of A which belongs to the domain of Γ and let Θ be a linear relation in $\mathcal{H}$ corresponding to $\widetilde{A}$:

$$\Theta = \Gamma(\widetilde{A}), \quad \widetilde{A} \subset \operatorname{dom}\Gamma \quad \Leftrightarrow \quad \widetilde{A} = A_\Theta := \Gamma^{-1}(\Theta), \quad \Theta \subset \operatorname{ran}\Gamma. \tag{4.15}$$

Theorem 4.12. *Let A be a closed symmetric relation, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an AB-generalized boundary triple for A^* with $A_0 = \ker \Gamma_0$, and let $M(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl function and γ-field, respectively. Then for any extension $A_\Theta \in \operatorname{Ext}_A$ satisfying $A_\Theta \subset \operatorname{dom}\Gamma$ the following Kreĭn-type formula holds*

$$\left(A_\Theta - \lambda\right)^{-1} = \left(A_0 - \lambda\right)^{-1} + \gamma(\lambda)\left(\Theta - M(\lambda)\right)^{-1}\gamma(\bar{\lambda})^*, \qquad \lambda \in \rho(A_0). \tag{4.16}$$

Here the inverses in the first and last terms are taken in the sense of linear relations.

The proof of this theorem is postponed until Section 5.2, where an analogous resolvent formula is proved for unitary boundary triples. However, some remarks and consequences of Theorem 4.12 are in order already here.

Remark 4.13. We emphasize that in the Kreĭn-type formula (4.16) it is not assumed that $\lambda \in \rho(A_\Theta)$. In particular, $A_\Theta - \lambda$ need not be invertible; A_Θ and Θ need not even be closed. Hence, even when $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple for A^* the formula (4.16) uses only the assumption $\lambda \in \rho(A_0)$ instead of the standard assumption that $\lambda \in \rho(A_0) \cap \rho(A_\Theta)$.

The following statement is an immediate consequence of Theorem 4.12.

Corollary 4.14. *Let the assumptions be as in Theorem 4.12 and let $\lambda \in \rho(A_0)$. Then:*

(i) $\ker(A_\Theta - \lambda) = \gamma(\lambda)\ker(\Theta - M(\lambda))$;

(ii) if $(\Theta - M(\lambda))^{-1}$ is a bounded operator, then the same is true for $(A_\Theta - \lambda)^{-1}$;

(iii) if $0 \in \rho(\Theta - M(\lambda))$ then $\lambda \in \rho(A_\Theta)$.

5 | SOME CLASSES OF UNITARY BOUNDARY TRIPLES AND WEYL FUNCTIONS

5.1 | Unitary boundary pairs and unitary colligations

Some formulas from Section 3 can be essentially improved when using the interrelations between unitary relations and unitary colligations, see [10]. Let $\{\mathcal{H}, \Gamma\}$ be a unitary boundary pair. As was shown in [25, Proposition 2.10] the linear relation, the so-called *main transform* of Γ,

$$\widetilde{\mathcal{A}} := \left\{ \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' \\ -h' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \right\} \tag{5.1}$$

is selfadjoint in $\mathfrak{H} \oplus \mathcal{H}$. Denote its Cayley transform $I - 2i(\widetilde{\mathcal{A}} + i)^{-1}$ by $\omega(\Gamma)$:

$$\omega(\Gamma) := \left\{ \left\{ \begin{pmatrix} f + if' \\ h - ih' \end{pmatrix}, \begin{pmatrix} -f + if' \\ -h - ih' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \right\} \tag{5.2}$$

Then $\omega(\Gamma)$ is the graph of a unitary operator $\mathcal{U} : \begin{pmatrix} \mathfrak{H} \\ \mathcal{H} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathcal{H} \end{pmatrix}$. The mapping $\omega : \Gamma \mapsto \mathcal{U}$ establishes a one-to-one correspondence between the set of unitary boundary pairs and the set of unitary operators in $\mathfrak{H} \oplus \mathcal{H}$.

The inverse transform $\Gamma = \omega^{-1}(\mathcal{U})$ takes the form

$$\Gamma = \left\{ \left\{ \begin{pmatrix} g' - g \\ i(g' + g) \end{pmatrix}, \begin{pmatrix} u' - u \\ -i(u' + u) \end{pmatrix} \right\} : \left\{ \begin{pmatrix} g \\ u \end{pmatrix}, \begin{pmatrix} g' \\ u' \end{pmatrix} \right\} \in \operatorname{gr} \mathcal{U} \right\}. \tag{5.3}$$

As was shown in [29] $U = \omega(\Gamma)$ is the Potapov–Ginzburg transform of Γ in the sense of [7]. Let us consider the unitary operator $\mathcal{U}$ and the pair of Hilbert spaces $\mathfrak{H}$ and $\mathcal{H}$ as a *unitary colligation* (see [18]) written in the block form

$$\mathcal{U} = \begin{pmatrix} T & F \\ G & H \end{pmatrix} \in \mathcal{B}(\mathfrak{H} \oplus \mathcal{H}), \tag{5.4}$$

where $T \in \mathcal{B}(\mathfrak{H})$, $F \in \mathcal{B}(\mathcal{H}, \mathfrak{H})$, $G \in \mathcal{B}(\mathfrak{H}, \mathcal{H})$, and $H \in \mathcal{B}(\mathcal{H})$. Then the representation (5.3) for Γ takes the form

$$\Gamma = \left\{ \left\{ \begin{pmatrix} (T-I)g + Fu \\ i(T+I)g + iFu \end{pmatrix}, \begin{pmatrix} Gg + (H-I)u \\ -iGg - i(H+I)u \end{pmatrix} \right\} : g \in \mathfrak{H}, u \in \mathcal{H} \right\}. \tag{5.5}$$

Since $\mathcal{U} = (\mathcal{U}^*)^{-1}$, then

$$\mathcal{U} = \left\{ \left\{ \begin{pmatrix} T^*g' + G^*u' \\ F^*g' + H^*u' \end{pmatrix}, \begin{pmatrix} g' \\ u' \end{pmatrix} \right\} : g' \in \mathfrak{H}, u' \in \mathcal{H} \right\}$$

and hence Γ admits a dual representation

$$\Gamma = \left\{ \left\{ \begin{pmatrix} (I-T^*)g' - G^*u' \\ i(I+T^*)g' + iG^*u' \end{pmatrix}, \begin{pmatrix} -F^*g' + (I-H^*)u' \\ -iF^*g' - i(I+H^*)u' \end{pmatrix} \right\} : \begin{matrix} g' \in \mathfrak{H}, \\ u' \in \mathcal{H} \end{matrix} \right\}. \tag{5.6}$$

Observe, that for each element $\{\hat{f}, \hat{h}\} \in \Gamma$ the vectors $g, g' \in \mathfrak{H}$ and $h, h' \in \mathcal{H}$ are in fact uniquely determined in (5.5) and (5.6). Let us collect some formulas concerning Γ and $\mathcal{U}$ which are immediate from (5.5) and (5.6) (see also [10]).

Proposition 5.1. *Let $\{\mathcal{H}, \Gamma\}$ be a unitary boundary pair for A^* with Γ given by (5.5), and let $A_* = \operatorname{dom}\Gamma$, $A_0 = \ker \Gamma_0$. Then:*

$$A_* = \operatorname{ran}\begin{pmatrix} T-I & F \\ i(T+I) & iF \end{pmatrix} = \operatorname{ran}\begin{pmatrix} (I-T)^* & -G^* \\ i(I+T)^* & iG^* \end{pmatrix},$$

$$\operatorname{mul} A_* = (I-T)^{-1}\operatorname{ran} F = (I-T^*)^{-1}\operatorname{ran} G^*;$$

$$\operatorname{mul} A = \ker(I-T) = \ker(I-T^*);$$

$$\begin{aligned} A_0 &= \left\{ \begin{pmatrix} (T-I)g + Fu \\ i(T+I)g + iFu \end{pmatrix} : Gg + (H-I)u = 0,\ g \in \mathfrak{H},\ u \in \mathcal{H} \right\} \\ &= \left\{ \begin{pmatrix} (I-T^*)g' - G^*u' \\ i(I+T^*)g' + iG^*u' \end{pmatrix} : F^*g' + (H^*-I)u' = 0,\ g' \in \mathfrak{H},\ u' \in \mathcal{H} \right\} \end{aligned} \tag{5.7}$$

$$\operatorname{ran}\Gamma_0 = \operatorname{ran}(I-H) + \operatorname{ran} G = \operatorname{ran}(I-H^*) + \operatorname{ran} F^*; \tag{5.8}$$

$$\operatorname{mul}\Gamma = \left\{ \begin{pmatrix} (H-I)u \\ -i(H+I)u \end{pmatrix} : u \in \ker F \right\} = \left\{ \begin{pmatrix} (I-H^*)u' \\ -i(I+H^*)u' \end{pmatrix} : u' \in \ker G^* \right\},$$

in particular,

$$\operatorname{mul}\Gamma = \{0\} \iff \ker F = \{0\} \iff \ker G^* = \{0\}.$$

The characteristic function (or transfer function) of the unitary colligation $\mathcal{U}$ (see [18])

$$\theta(\zeta) = H + \zeta G(I - \zeta T)^{-1} F \quad (\zeta \in \mathbb{D})$$

is holomorphic in $\mathbb{D}$ and takes values in the set of contractive operators in $\mathcal{H}$.

Proposition 5.2. *Let $\{\mathcal{H}, \Gamma\}$ be a unitary boundary pair for A^* with Γ given by (5.5), let $\lambda \in \mathbb{C}_+$ and let $\zeta = \frac{\lambda - i}{\lambda + i}$. Then the γ-field admits the representations*

$$\gamma(\lambda) = \left\{ \left\{ (\theta(\zeta) - I)u, (1-\zeta)(I - \zeta T)^{-1} F u \right\} : u \in \mathcal{H} \right\}, \tag{5.9}$$

$$\gamma(\bar\lambda) = \left\{ \left\{ (\theta(\zeta)^* - I)u, (1-\bar\zeta)(I - \bar\zeta T^*)^{-1} G^* u \right\} : u \in \mathcal{H} \right\}, \tag{5.10}$$

and its kernel does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$:

$$\ker \gamma(\lambda) = \operatorname{mul} \Gamma_0 = \ker \gamma\left(\bar\lambda\right). \tag{5.11}$$

In particular,

$$\gamma(i) = \{\{(H - I)u, Fu\} : u \in \mathcal{H}\}, \quad \gamma(-i) = \{\{(H^* - I)u, G^* u\} : u \in \mathcal{H}\}. \tag{5.12}$$

The Weyl function M corresponding to the boundary pair $\{\mathcal{H}, \Gamma\}$ and the characteristic function θ are connected by

$$M(\lambda) = i(I + \theta(\zeta))(I - \theta(\zeta))^{-1}, \quad M(\bar\lambda) = -i(I + \theta(\zeta)^*)(I - \theta(\zeta)^*)^{-1} \tag{5.13}$$

If Γ is single-valued then $\operatorname{dom} \gamma(\lambda)$ *and* $\operatorname{dom} \gamma(\bar\lambda)$ *are dense in $\mathcal{H}$.*

Proof. Since $\zeta = \frac{\lambda - i}{\lambda + i} \in \mathbb{D}$ the operator $(I - \zeta T)$ has a bounded inverse. Using the substitution $g = f + \zeta(I - \zeta T)^{-1} F u$ one can rewrite the expression (5.5) for Γ in the form

$$\left\{ \left\{ \begin{pmatrix} (T - I)f + (1-\zeta)(I - \zeta T)^{-1} F u \\ i(T + I)f + i(1+\zeta)(I - \zeta T)^{-1} F u \end{pmatrix}, \begin{pmatrix} Gf + (\theta(\zeta) - I)u \\ -iGf - i(\theta(\zeta) + I)u \end{pmatrix} \right\} : \begin{matrix} f \in \mathfrak{H} \\ u \in \mathcal{H} \end{matrix} \right\}. \tag{5.14}$$

Since $\lambda = i\frac{1+\zeta}{1-\zeta}$ the choice $f = 0$ in (5.14) leads to

$$\Gamma \upharpoonright \widehat{\mathfrak{N}}_\lambda = \left\{ \left\{ \begin{pmatrix} (1-\zeta)(I - \zeta T)^{-1} F u \\ \lambda(1-\zeta)(I - \zeta T)^{-1} F u \end{pmatrix}, \begin{pmatrix} (\theta(\zeta) - I)u \\ -i(\theta(\zeta) + I)u \end{pmatrix} \right\} : u \in \mathcal{H} \right\}$$

and hence (5.9) and the first equalities in (5.12) and (5.13) follow.

Similarly, the substitution $g' = f' + \bar\zeta\left(I - \bar\zeta T^*\right)^{-1} G^* u'$ in (5.5) shows that the linear relation Γ coincides with the set of vectors

$$\left\{ \begin{pmatrix} (I - T^*)f' + \left(\bar\zeta - 1\right)\left(I - \bar\zeta T^*\right)^{-1} G^* u' \\ i(I + T^*)f' + i\left(\bar\zeta + 1\right)\left(I - \bar\zeta T^*\right)^{-1} G^* u' \end{pmatrix}, \begin{pmatrix} -F^* f' + (I - \theta(\zeta)^*)u' \\ -iF^* f' - i(I + \theta(\zeta)^*)u' \end{pmatrix} \right\}, \tag{5.15}$$

where $f' \in \mathfrak{H}$, $u' \in \mathcal{H}$. Hence with $f' = 0$ one obtains from (5.15)

$$\Gamma \upharpoonright \widehat{\mathfrak{N}}_{\bar\lambda} = \left\{ \left\{ \begin{pmatrix} \left(\bar\zeta - 1\right)\left(I - \bar\zeta T^*\right)^{-1} G^* u' \\ \bar\lambda\left(\bar\zeta - 1\right)\left(I - \bar\zeta T^*\right)^{-1} G^* u' \end{pmatrix}, \begin{pmatrix} (I - \theta(\zeta)^*)u' \\ i(I + \theta(\zeta)^*)u' \end{pmatrix} \right\} : u' \in \mathcal{H} \right\}. \tag{5.16}$$

Now the formula (5.10) and the second equalities in (5.12) and (5.13) are implied by (5.16).

The equalities in (5.11) hold by the definition of the γ-field and, in fact, are also clear from (5.9), (5.10), and the description of $\operatorname{mul} \Gamma$ in Proposition 5.1.

Finally, if $\operatorname{mul} \Gamma = \{0\}$, then using the fact that $\gamma(\pm i)$ is single-valued, one concludes that

$$\ker(H - I) \subset \ker F, \quad \ker(H^* - I) \subset \ker G^*, \tag{5.17}$$

and hence Proposition 5.1 shows that $\ker(I - H) = \ker(I - H^*) = \{0\}$. Therefore $\operatorname{dom} \gamma(-i) = \operatorname{ran}(I - H^*)$ and $\operatorname{dom} \gamma(i) = \operatorname{ran}(I - H)$ are dense in $\mathcal{H}$. Equivalently, $\operatorname{dom} \gamma(\lambda)$ is dense in $\mathcal{H}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. ☐

Proposition 5.3. *Let $\{\mathcal{H}, \Gamma\}$ be a unitary boundary pair for A^*. Then the closure of $\widehat\gamma(\lambda)$ is given by*

$$\overline{\widehat\gamma(\lambda)} = \left(\overline{\Gamma}_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A^*)\right)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{5.18}$$

In particular, $\ker\overline{\widehat{\gamma}(\lambda)} = \operatorname{mul}\overline{\Gamma}_0$, $\operatorname{mul}\overline{\widehat{\gamma}(\lambda)} = \big(\ker\overline{\Gamma}_0\big) \cap \widehat{\mathfrak{N}}_\lambda(A^*)$, *and*

$$\operatorname{ran}\overline{\widehat{\gamma}(\lambda)} = \big(\operatorname{dom}\overline{\Gamma}_0\big) \cap \widehat{\mathfrak{N}}_\lambda(A^*), \quad \operatorname{dom}\overline{\widehat{\gamma}(\lambda)} = \overline{\Gamma}_0\big(\big(\operatorname{dom}\overline{\Gamma}_0\big) \cap \widehat{\mathfrak{N}}_\lambda(A^*)\big).$$

Proof. By definition $\widehat{\gamma}(\lambda) = \big(\Gamma_0 \restriction \widehat{\mathfrak{N}}_\lambda(A_*)\big)^{-1} = \big(\Gamma_0 \cap \big(\widehat{\mathfrak{N}}_\lambda(A_*) \times \mathcal{H}\big)\big)^{-1}$, which implies that

$$\overline{\widehat{\gamma}(\lambda)}^{-1} \subset \overline{\Gamma}_0 \restriction \widehat{\mathfrak{N}}_\lambda(A^*), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

To prove the reverse inclusion, assume that $\{\widehat{f}_\lambda, h\} \in \overline{\Gamma}_0 \cap \big(\widehat{\mathfrak{N}}_\lambda(A^*) \times \mathcal{H}\big)$. With $\lambda \in \mathbb{C}_+$ it follows from (5.14) that there are sequences $f_n \in \mathfrak{H}$ and $u_n \in \mathcal{H}$, such that

$$\left\{ \begin{pmatrix} (T-I)f_n + (1-\zeta)(1-\zeta T)^{-1}Fu_n \\ i(T+I)f_n + i(1+\zeta)(1-\zeta T)^{-1}Fu_n \end{pmatrix}, Gf_n + (\theta(\zeta) - I)u_n \right\} \xrightarrow{\mathfrak{H}^2} \left\{ \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix}, h \right\}.$$

This implies that $(I - \zeta T)f_n \xrightarrow{\mathfrak{H}} 0$ and hence $f_n \xrightarrow{\mathfrak{H}} 0$, since $\lambda \in \mathbb{C}_+$ or, equivalently, $\zeta \in \mathbb{D}$. Thus

$$\left\{ \begin{pmatrix} (1-\zeta)(1-\zeta T)^{-1}Fu_n \\ \lambda(1-\zeta)(1-\zeta T)^{-1}Fu_n \end{pmatrix}, (\theta(\zeta) - I)u_n \right\} \xrightarrow{\mathfrak{H}^2} \left\{ \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix}, h \right\},$$

which by (5.9) in Proposition 5.2 means that $\{\widehat{f}_\lambda, h\} \in \overline{\widehat{\gamma}(\lambda)}^{-1}$.

Similarly, with $\bar{\lambda} \in \mathbb{C}_-$ it follows from (5.15) that for every $\{\widehat{f}_{\bar{\lambda}}, h\} \in \overline{\Gamma}_0 \cap \big(\widehat{\mathfrak{N}}_{\bar{\lambda}}(A^*) \times \mathcal{H}\big)$ there exists a sequence $u_n' \in \mathcal{H}$ such that

$$\left\{ \begin{pmatrix} (\bar{\zeta}-1)(1-\bar{\zeta}T)^{-1}G^*u_n' \\ \bar{\lambda}(\bar{\zeta}-1)(1-\bar{\zeta}T)^{-1}G^*u_n' \end{pmatrix}, (I - \theta(\zeta)^*)u_n' \right\} \xrightarrow{\mathfrak{H}^2} \left\{ \begin{pmatrix} f_{\bar{\lambda}} \\ \bar{\lambda} f_{\bar{\lambda}} \end{pmatrix}, h \right\},$$

which by (5.10) in Proposition 5.2 means that $\{\widehat{f}_{\bar{\lambda}}, h\} \in \overline{\widehat{\gamma}(\bar{\lambda})}^{-1}$. This completes the proof of (5.18) and the remaining statements follow easily from this identity. □

Corollary 5.4. *Let* $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a unitary boundary triple for* A^* *and let* $M(\cdot)$ *be the corresponding Weyl function. Then the mapping* Γ_0 *is closable if and only if for some, equivalently for every,* $\lambda \in \mathbb{C} \setminus \mathbb{R}$ *the Weyl function satisfies the following condition:*

$$h_n \xrightarrow{\mathcal{H}} h \quad \textit{and} \quad \operatorname{Im}\big(M(\lambda)h_n, h_n\big) \to 0 \quad (n \to \infty) \quad \Rightarrow \quad h = 0. \tag{5.19}$$

Proof. By Lemma 3.6 $M(\cdot)$ is an operator valued function with $\ker(M(\lambda) - M(\lambda)^*) = \{0\}$. In this case (3.10) implies that

$$\big(\lambda - \bar{\lambda}\big)\|\gamma(\lambda)h\|_{\mathfrak{H}}^2 = 2i\,\operatorname{Im}\,(M(\lambda)h, h)_{\mathcal{H}},$$

$h \in \operatorname{dom} M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. From this formula it is clear that the condition (5.19) is equivalent to $\ker\overline{\gamma(\lambda)} = \{0\}$. Therefore, the result follows from Proposition 5.3. □

Clearly, the condition (5.19) is stronger than the condition (1.8) appearing in the definition of strict Nevanlinna functions. If $M(\cdot) \in \mathcal{R}[\mathcal{H}]$ then the condition (5.19) simplifies to $\ker \operatorname{Im} M(\lambda) = \{0\}$, i.e., for bounded Nevanlinna functions the conditions (5.19) and (1.8) are equivalent. Hence, if $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a B-generalized boundary triple then Γ_0 is closable by Theorem 1.7. However, when $M(\cdot)$ is an unbounded Nevanlinna function, the condition in Corollary 5.4 need not be satisfied. Example 5.19 shows that already for S-generalized boundary triples $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ the mapping Γ_0 need not be closable.

The next result contains an essential improvement of Lemma 3.8 for unitary boundary pairs.

Theorem 5.5. *Let* $\{\mathcal{H}, \Gamma\}$ *be a unitary boundary pair for* A^**, let its Potapov–Ginzburg transform* $\mathcal{U} = \omega(\Gamma)$ *be given by (5.4), let* $H(\lambda)$ *be defined by*

$$H(\lambda) : h \to \left\{ \begin{pmatrix} (A_0 - \lambda)^{-1}h \\ h + \lambda(A_0 - \lambda)^{-1}h \end{pmatrix} \right\},$$

see (3.11), and denote $\zeta = \frac{\lambda - i}{\lambda + i}$ *for* $\lambda \in \mathbb{C}_+$. *Then:*

(i) *with* $\lambda \in \mathbb{C}_+$ *the adjoint of the* γ*-field is given by the formulas*

$$\gamma(\bar{\lambda})^* = \{\{g, v\} : (\theta(\zeta) - I)v + (\zeta - 1)G(I - \zeta T)^{-1}g = 0,\ g \in \mathfrak{H},\ v \in \mathcal{H}\}, \tag{5.20}$$

$$\gamma(\lambda)^* = \{\{g', v'\} : (\theta(\zeta)^* - I)v' + (\bar{\zeta} - 1)F^*(I - \bar{\zeta}T^*)^{-1}g' = 0,\ g' \in \mathfrak{H},\ v' \in \mathcal{H}\}; \tag{5.21}$$

(ii) *for all* $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ *one has the equality*

$$\Gamma H(\lambda) = \begin{pmatrix} 0 \\ \gamma(\bar{\lambda})^* \end{pmatrix} \hat{+} (\{0\} \times \operatorname{mul} \Gamma),$$

and, in particular,

$$\Gamma_1 H(\lambda) = \gamma(\bar{\lambda})^* \ \hat{+}\ (\{0\} \times \operatorname{mul} \Gamma_1),$$

and, furthermore, here $\operatorname{ran}(\Gamma_1 H(\lambda)) = \operatorname{ran}(\Gamma_1 H(\bar{\lambda}))$ *does not depend on* $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$;

(iii) *the range of* $\gamma(\lambda)^*$ *does not depend on* $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$:

$$\operatorname{ran} \gamma(\bar{\lambda})^* = (I - H)^{-1}(\operatorname{ran} G) = (I - H^*)^{-1}(\operatorname{ran} F^*) = \operatorname{ran} \gamma(\lambda)^*; \tag{5.22}$$

(iv) *the multi-valued part of* $\gamma(\lambda)^*$ *does not depend on* $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$:

$$\operatorname{mul} \gamma(\bar{\lambda})^* = \ker(H - I) = \ker(H^* - I) = \operatorname{mul} \gamma(\lambda)^*.$$

In particular, if $\operatorname{mul} \Gamma_1 = \{0\}$ *then the equality* $\operatorname{ran}(\Gamma_1 H(\lambda)) = \operatorname{ran} \gamma(\bar{\lambda})^*$ *holds for all* $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$. *In this case also* $\operatorname{mul} \gamma(\lambda)^* = \{0\}$ *or, equivalently, the* γ*-field* $\gamma(\lambda)$ *is a densely defined operator for all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof.

(i) Let $\lambda \in \mathbb{C}_+$ and assume that $\{g, v\} \in \gamma(\bar{\lambda})^*$ for some $g \in \mathfrak{H}$ and $v \in \mathcal{H}$. Then by (5.10) this means that

$$((1 - \bar{\zeta})(I - \bar{\zeta}T^*)^{-1}G^*u, g) = ((\theta(\zeta)^* - I)u, v)_{\mathcal{H}}$$

for all $u \in \mathcal{H}$ or, equivalently, $(\theta(\zeta) - I)v + (\zeta - 1)G(I - \zeta T)^{-1}g = 0$. This proves the identity (5.20) in (ii) with $\lambda \in \mathbb{C}_+$. Similarly, using (5.9) it is seen that $\{g', v'\} \in \gamma(\lambda)^*(\lambda \in \mathbb{C}_+)$ is equivalent to

$$(\theta(\zeta)^* - I)v + (\bar{\zeta} - 1)F^*(I - \bar{\zeta}T^*)^{-1}g' = 0,$$

which proves (5.21).

(ii) By Lemma 3.8 (cf. also [27, Lemma 7.38]) the inclusions "⊂" hold in (ii). For the reverse inclusions "⊃" in (ii) it suffices to prove the inclusion $\gamma(\bar{\lambda})^* \subset \Gamma_1 H(\lambda)$. For this purpose we first derive a formula for the mapping $H(\lambda)$ defined in (3.11) analogous to what appears in Proposition 5.2. It follows from (5.14) and (5.15) (with $\lambda \in \mathbb{C}_+$) that

$$A_0 - \lambda = \left\{ \begin{pmatrix} (T - I)f + (1 - \zeta)(I - \zeta T)^{-1}Fu \\ \frac{2i}{1 - \zeta}(I - \zeta T)f \end{pmatrix} : \begin{matrix} f \in \mathfrak{H},\ u \in \mathcal{H} \\ Gf + (\theta(\zeta) - I)u = 0 \end{matrix} \right\}, \tag{5.23}$$

$$A_0 - \bar{\lambda} = \left\{ \begin{pmatrix} (I - T^*)f' + (\bar{\zeta} - 1)(I - \bar{\zeta}T^*)^{-1}G^*u' \\ \frac{-2i}{1 - \bar{\zeta}}(I - \bar{\zeta}T^*)f' \end{pmatrix} : \begin{matrix} f' \in \mathfrak{H},\ u' \in \mathcal{H} \\ -F^*g' + (I - \theta(\zeta)^*)u' \end{matrix} \right\}. \tag{5.24}$$

In particular, using (5.23) and the substitution $g = 2i(1 - \zeta T)(1 - \zeta)^{-1}f$ one obtains

$$h := (A_0 - \lambda)^{-1}g = \frac{1 - \zeta}{2i}(T - I)(I - \zeta T)^{-1}g + (1 - \zeta)(I - \zeta T)^{-1}Fu,$$

where $u \in \mathcal{H}$ satisfies the equality

$$\frac{1-\zeta}{2i} G(I-\zeta T)^{-1} g + (\theta(\zeta) - I)u = 0,$$

or, equivalently,

$$(\theta(\zeta) - I)(-2iu) + (\zeta - 1)G(I-\zeta T)^{-1} g = 0. \tag{5.25}$$

Now denoting $\hat{h} = \begin{pmatrix} h \\ h' \end{pmatrix} = H(\lambda)g \in A_0$ one concludes from (5.14) that

$$\left\{\hat{h}, -2iu\right\} = \left\{\hat{h}, -iGf - i(\theta(\zeta)+I)u\right\} \in \Gamma_1,$$

so that $\{g, v\} \in \Gamma_1 H(\lambda)$, where $v = -2iu$ and g satisfies (5.25) or, equivalently, $f = (1-\zeta)(2i(I-\zeta T))^{-1}g$ satisfies $Gf + (\theta(\zeta) - I)u = 0$. It remains to compare (5.20) with (5.25) to conclude that the elements $\{g, v\} \in \gamma(\bar{\lambda})^*$ in fact belong to $\Gamma_1 H(\lambda)$, i.e., the inclusion $\gamma(\bar{\lambda})^* \subset \Gamma_1 H(\lambda)$ holds $(\lambda \in \mathbb{C}_+)$.
Similarly, one proves the inclusion $\gamma(\lambda)^* \subset \Gamma_1 H(\bar{\lambda})$ $(\lambda \in \mathbb{C}_+)$ by means of (5.21), (5.15) and (5.24).
Finally, the equality $\operatorname{ran}(\Gamma_1 H(\lambda)) = \operatorname{ran}(\Gamma_1 H(\bar{\lambda}))$ and the independence from $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ holds by item (vi) of Lemma 3.7.

(iii) Let $u \in (H-I)^{-1}(\operatorname{ran} G \cap \operatorname{ran}(H-I))$ and $\lambda \in \mathbb{C}_+$. Then $(H-I)u \in \operatorname{ran} G$ and hence

$$(\theta(\zeta) - I)u = (H-I)u + \zeta G(I-\zeta T)^{-1} Fu \in \operatorname{ran} G. \tag{5.26}$$

In view of (5.20) this proves that $u \in \operatorname{ran}\gamma(\bar{\lambda})^*$. Conversely, if $u \in \operatorname{ran}\gamma(\bar{\lambda})^*$ then in view of (5.20) and (5.26) $(H-I)u \in \operatorname{ran} G$, which proves the first equality in (5.22). Similarly, for $\lambda \in \mathbb{C}_-$ the last equality in (5.22) is implied by (5.21).
Finally, the equality $(H-I)^{-1}\operatorname{ran} G = (H^*-I)^{-1}\operatorname{ran} F^*$ is implied by the identities $\mathcal{U}^*\mathcal{U} = \mathcal{U}\mathcal{U}^* = I_{\mathfrak{H}\oplus\mathcal{H}}$. To see this assume that $u \in (H-I)^{-1}\operatorname{ran} G$, i.e., that $(H-I)u = Gg$ for some $g \in \mathfrak{H}$. Then $H^*(H-I)u = H^*Gg = -F^*Tg$ and using $H^*H = I - F^*F$ one obtains $(I-H^*)u = F^*(Fu - Tg)$. Hence $(H-I)^{-1}\operatorname{ran} G \subset (H^*-I)^{-1}\operatorname{ran} F^*$ and the reverse inclusion is proved similarly.

(iv) Part (i) shows that the adjoint of the γ-field at $\lambda = \pm i$ is given by

$$\gamma(i)^* = \left\{\{f, f'\} : (H^*-I)f' = F^*f\right\}, \quad \gamma(-i)^* = \left\{\{f, f'\} : (H-I)f' = Gf\right\},$$

see also (5.12). On the other hand, $\operatorname{mul}\gamma(\lambda)^* = (\operatorname{dom}\gamma(\lambda))^\perp = (\operatorname{dom} M(\lambda))^\perp = \operatorname{mul} M(\lambda)$ does not depend on $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$. The identities in (iv) are now clear from the formulas in (i).

The last statement is obtained from (ii). □

Corollary 5.6. *(Cf. [27].) Assume that Γ_1 in Theorem 5.5 is a single-valued operator. Then the operator $\Gamma_1 H(\lambda)$ is bounded or, equivalently, $\Gamma_1 \upharpoonright A_0$ is bounded if and only if A_0 is closed.*

Proof. By Lemma 3.7 the operator $\Gamma_1 H(\lambda)$ is bounded if and only if the restriction $\Gamma_1 \upharpoonright A_0$ is bounded. Since $\Gamma_1 H(\lambda) = \gamma(\bar{\lambda})^*$ by Theorem 5.5, this mapping is closed and it follows from the closed graph theorem that $\Gamma_1 \upharpoonright A_0$ is bounded if and only if its domain $\operatorname{dom}(\Gamma_1 \upharpoonright A_0) = A_0$ is closed. □

Remark 5.7. The unitary colligation $\{\mathcal{U}, \mathfrak{H}, \mathcal{H}, \mathcal{H}\}$ from (5.4) is an operator formalization of a discrete time input/state/output system $\begin{pmatrix} x(t+1) \\ y(t) \end{pmatrix} = \mathcal{U}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}$ $(t \in \mathbb{N})$ with the input $u(t) \in \mathcal{H}$ and the output $y(t) \in \mathcal{H}$. The transfer function of this discrete time input/state/output system coincides with the characteristic function $\theta(z)$ of the unitary colligation $(\mathcal{U}, \mathfrak{H}, \mathcal{H}, \mathcal{H})$, see [5, 57].

Similarly, as was shown in [6, Theorem 5.35] any unitary boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with the extra properties $\operatorname{ran}\Gamma_0 = \mathfrak{H}$ and $\operatorname{mul}\widetilde{\mathcal{A}} = \{0\}$, where $\widetilde{\mathcal{A}}$ (a skew-adjoint operator) is the analog of the main transform of Γ (see (5.1)), corresponds to some impedance conservative continuous time input/state/output system

$$\Sigma = \{\widetilde{\mathcal{A}}, \mathfrak{H}, \mathcal{H}, \mathcal{H}\} : \quad \begin{pmatrix} \dot{x}(t) \\ y(t) \end{pmatrix} = \widetilde{\mathcal{A}} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \quad (t \in \mathbb{R}_+).$$

Realization problems for Schur functions via transfer functions of scattering conservative (and passive) continuous time input/state/output systems were studied in [8, 9] and were motivated by the earlier works [65, 66]. On the other hand, connections between general unitary boundary pairs and the notion of conservative state/signal system nodes, whose systematic study was initiated in [5] (see also e.g. [53, 54]), have been established in [6, Theorem 5.34]. Moreover, the connection between conservative state/signal system nodes and so-called Dirac structures can be found in [6, Proposition 5.38], while the connection between Dirac structures and unitary boundary pairs is made explicit in [41].

5.2 | A Kreĭn type formula for unitary boundary triples

In this section Kreĭn's resolvent formula is extended to the setting of general unitary boundary triples. It is analogous to the formula established in Section 4.2. Recall from [25] that for a unitary boundary triple the kernel $A_0 = \ker\Gamma_0$ need not be selfadjoint, it is in general only a symmetric extension of A which can even coincide with A; see e.g. [25, Example 6.6]. For simplicity the next result is formulated for nonreal points $\lambda \in \mathbb{C} \setminus \mathbb{R}$; these points are regular type points for A_0.

As in Section 4.2, let $\widetilde{A}$ be an extension of A which belongs to the domain of Γ and let Θ be a linear relation in $\mathcal{H}$ corresponding to $\widetilde{A}$ via (4.15).

Theorem 5.8. *Let A be a closed symmetric relation, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a unitary boundary triple for A^* with $A_0 = \ker\Gamma_0$, and let $M(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl function and γ-field, respectively. Then for any linear relation $\Theta(\subset \operatorname{ran}\Gamma)$ in $\mathcal{H}$ and the extension $A_\Theta \in \operatorname{Ext}_A$ given by (4.15) the following equality holds*

$$(A_\Theta - \lambda)^{-1} - (A_0 - \lambda)^{-1} = \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{5.27}$$

where the inverses in the first and last term are taken in the sense of linear relations.

Proof. We first prove the inclusion "$\subset$" in (5.27). Since A_0 is symmetric, $(A_0 - \lambda)^{-1}$ is a bounded, in general nondensely defined, operator for every fixed $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Now assume that $\{g, g''\} \in (A_\Theta - \lambda)^{-1} - (A_0 - \lambda)^{-1}$. Then $g \in \operatorname{dom}(A_\Theta - \lambda)^{-1} \cap \operatorname{dom}(A_0 - \lambda)^{-1}$ and $\{g, g'\} \in (A_\Theta - \lambda)^{-1}$ for some $g' \in \mathfrak{H}$, so that $g'' = g' - (A_0 - \lambda)^{-1}g$. Hence $\widehat{g}_\Theta := \{g', g + \lambda g'\} \in A_\Theta \subset \operatorname{dom}\Gamma$,

$$\widehat{g}_0 := \left\{ (A_0 - \lambda)^{-1}g, \left(I + \lambda(A_0 - \lambda)^{-1}\right)g \right\} \in A_0 \subset \operatorname{dom}\Gamma, \tag{5.28}$$

and

$$\widehat{g}_\Theta - \widehat{g}_0 = \left\{ g' - (A_0 - \lambda)^{-1}g, \lambda\left(g' - (A_0 - \lambda)^{-1}g\right) \right\},$$

so that $\widehat{g}_\Theta - \widehat{g}_0 \in \widehat{\mathfrak{N}}_\lambda(A_*)$. Recall that $\widehat{\gamma}(\lambda)$ maps $\operatorname{dom}\widehat{\gamma}(\lambda)$ onto $\widehat{\mathfrak{N}}_\lambda(A_*) \subset \operatorname{dom}\Gamma$ and hence there exists $\varphi \in \operatorname{dom}\widehat{\gamma}(\lambda) = \operatorname{dom} M(\lambda)$ such that

$$\widehat{g}_\Theta - \widehat{g}_0 = \widehat{\gamma}(\lambda)\varphi, \quad \Gamma\widehat{\gamma}(\lambda)\varphi = \{\varphi, M(\lambda)\varphi\}, \tag{5.29}$$

see (3.2), (3.3); notice that $M(\lambda)$ is an operator, since $\operatorname{mul}\Gamma = \{0\}$. Clearly $\Gamma_0\widehat{g}_0 = 0$ and according to Theorem 5.5 one has $\Gamma_1\widehat{g}_0 = \Gamma_1 H(\lambda)g = \gamma(\bar{\lambda})^* g$, where $H(\lambda)$ is defined by (3.11). Observe, that here $\gamma(\bar{\lambda})^*$ is an operator since $H(\lambda)$ and Γ_1 are operators. Now it follows from (5.29) that

$$\{0, \gamma(\bar{\lambda})^* g\} + \{\varphi, M(\lambda)\varphi\} = \Gamma\widehat{g}_0 + \Gamma\widehat{\gamma}(\lambda)\varphi = \Gamma\widehat{g}_\Theta \in \Theta, \tag{5.30}$$

see (4.15). Consequently,

$$\{\varphi, \gamma(\bar{\lambda})^* g + M(\lambda)\varphi\} \in \Theta \text{ and } \{\varphi, \gamma(\bar{\lambda})^* g\} \in \Theta - M(\lambda)$$

or, equivalently, $\{g, \varphi\} \in (\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^*$ and hence (5.29) shows that

$$\{g, g''\} = \{g, \gamma(\lambda)\varphi\} \in \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^*,$$

which proves the first inclusion in (5.27).

To prove the reverse inclusion "⊃" in (5.27) assume that $\{g, g''\} \in \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^*$. Since $\operatorname{dom}(\Theta - M(\lambda)) \subset \operatorname{dom} M(\lambda) = \operatorname{dom}\gamma(\lambda)$ the assumption on $\{g, g''\}$ means that for some $\varphi \in \mathcal{H}$ one has $\{\gamma(\bar{\lambda})^* g, \varphi\} \in (\Theta - M(\lambda))^{-1}$ and

$$\{g, g''\} = \{g, \gamma(\lambda)\varphi\} \in \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^*.$$

It follows from $\{\varphi, \gamma(\bar{\lambda})^* g + M(\lambda)\varphi\} \in \Theta$ and (4.15) that $\Gamma\hat{g}_\Theta = \{\varphi, \gamma(\bar{\lambda})^* g + M(\lambda)\varphi\}$ for some $\hat{g}_\Theta \in A_\Theta$. By Theorem 5.5 $\Gamma_1 H(\lambda) = \gamma(\bar{\lambda})^*$, which shows that $g \in \operatorname{ran}(A_0 - \lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$; see (3.11). Now associate with g the element $\hat{g}_0$ as in (5.28). Since $\Gamma\hat{g}_0 = \{0, \gamma(\bar{\lambda})^* g\}$ and $\Gamma\hat{\gamma}(\lambda)\varphi = \{\varphi, M(\lambda)\varphi\}$ we conclude that (5.30) is satisfied. Therefore, $\hat{g}_0 + \hat{\gamma}(\lambda)g - \hat{g}_\Theta \in \ker\Gamma = A$ and thus $\hat{g}_0 + \hat{\gamma}(\lambda)\varphi \in A_\Theta$ or, equivalently,

$$\{g, (A_0 - \lambda)^{-1} g + \gamma(\lambda)\varphi\} \in (A_\Theta - \lambda)^{-1}.$$

Hence,

$$\{g, g''\} = \{g, \gamma(\lambda)\varphi\} \in (A_\Theta - \lambda)^{-1} - (A_0 - \lambda)^{-1}.$$

This proves the reverse inclusion in (5.27) and completes the proof. □

It is useful to make some further comments on the formula (5.27).

Remark 5.9.

(i) Again notice the generality of the formula (5.27); in particular, as in Theorem 4.12 λ need not belong to $\rho(A_\Theta)$.

(ii) A careful look at the above proof shows that the key elements which in addition to the general properties of γ-fields and Weyl functions of isometric boundary triples are used in the proof are the following two requirements: (1) the equality $\Gamma_1 H(\lambda) = \gamma(\bar{\lambda})^*$, so that $\Gamma_1\hat{g}_0 = \gamma(\bar{\lambda})^* g$ when g and $\hat{g}_0$ are connected by (5.28); (2) $\gamma(\lambda)$ (hence also $M(\lambda)$) is a densely defined operator or, equivalently, $\gamma(\bar{\lambda})^*$ is a closed operator. Hence, the formula (5.27) in Theorem 4.12 remains valid for isometric boundary triples which satisfy these two additional properties.

(iii) In the formula (5.27) the operator $(A_0 - \lambda)^{-1}$ cannot be shifted to the right hand side without loosing the stated equality. Indeed, in that case only the following inclusion remains valid:

$$(A_\Theta - \lambda)^{-1} \supset (A_0 - \lambda)^{-1} - \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^*.$$

Namely, by the equality $\Gamma_1 H(\lambda) = \gamma(\bar{\lambda})^*$ one has $\operatorname{ran}(A_0 - \lambda) = \operatorname{dom}\gamma(\bar{\lambda})^*$ and thus the term $(A_0 - \lambda)^{-1}$ can be shifted to the right side of (5.27) without changing the domain on the right side. However, in this case the range of the right side belongs to the span $\operatorname{dom} A_0 + \mathfrak{N}_\lambda(A_*)$ and for general unitary boundary triples this would restrict the choice of A_Θ; recall that for a unitary boundary triple A_0 need not be even essentially selfadjoint, one can even have $A_0 = A$.

By considering the multi-valued parts we obtain the following statement for the point spectrum of A_Θ from Theorem 5.8.

Corollary 5.10. *With the assumptions in Theorem 5.8 one has $\lambda \in \sigma_p(A_\Theta)$ if and only if $0 \in \sigma_p(\Theta - M(\lambda))$, in which case*

$$\ker(A_\Theta - \lambda) = \gamma(\lambda)\ker(\Theta - M(\lambda)), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

We are now ready to prove also Theorem 4.12 from Section 4.2.

Proof of Theorem 4.12. By assumption $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an AB-generalized boundary triple for A^*. Hence, $\operatorname{mul}\Gamma = 0$ and according to Theorem 4.2 (iv) this implies that $\Gamma_1 H(\lambda) = \gamma(\bar{\lambda})^*$ and, moreover, $\gamma(\bar{\lambda})^*$ is a bounded everywhere defined operator. Thus, from part (ii) in Remark 5.9 one concludes that the formula (5.27) holds. Furthermore, for an AB-generalized boundary triple A_0 is selfadjoint. Thus $\operatorname{dom}(A_0 - \lambda)^{-1} = \mathfrak{H}$ $(\lambda \in \mathbb{C} \setminus \mathbb{R})$ and the formula (5.27) is equivalent to the formula (4.16) in Theorem 4.12. □

5.3 | S-generalized boundary triples

Here we extend Definition 1.11 to the case of boundary pairs.

Definition 5.11. A unitary boundary pair $\{\mathcal{H},\Gamma\}$ is said to be an S-*generalized boundary pair*, if A_0 is a selfadjoint linear relation in $\mathfrak{H}$.

In the following proposition some special boundary triples/pairs are characterized in terms of their Potapov–Ginzburg transform.

Proposition 5.12. *Let $\{\mathcal{H},\Gamma\}$ be a unitary boundary pair, let $\mathcal{U}=\omega(\Gamma)$ be its Potapov–Ginzburg transform given by (5.2) and (5.4), and let $A_*=\operatorname{dom}\Gamma$, $A_0=\ker\Gamma_0$. Then:*

(i) $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ *is an ordinary boundary triple if and only if*

$$\operatorname{ran} G=\mathcal{H}\quad\Longleftrightarrow\quad\operatorname{ran} F^*=\mathcal{H};$$

(ii) $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ *is a B-generalized boundary triple if and only if*

$$\begin{cases}\ker F=\{0\},\\ \operatorname{ran}(I-H)=\mathcal{H}\end{cases}\Longleftrightarrow\quad\begin{cases}\ker G^*=\{0\},\\ \operatorname{ran}(I-H^*)=\mathcal{H}\end{cases};$$

(iii) $\{\mathcal{H},\Gamma\}$ *is a B-generalized boundary pair if and only if*

$$\Gamma_0|\widehat{\mathfrak{N}}_i=\mathcal{H}\quad\Longleftrightarrow\quad\operatorname{ran}(I-H)=\mathcal{H}\quad\Longleftrightarrow\quad\operatorname{ran}(I-H^*)=\mathcal{H};$$

(iv) Γ_0 *is surjective if and only if*

$$\operatorname{ran}(I-H)+\operatorname{ran} G=\mathcal{H}\quad\Longleftrightarrow\quad\operatorname{ran}(I-H^*)+\operatorname{ran} F^*=\mathcal{H};$$

(v) $\{\mathcal{H},\Gamma\}$ *is an S-generalized boundary pair if and only if*

$$\operatorname{ran} G\subset\operatorname{ran}(I-H)\quad\text{and}\quad\operatorname{ran} F^*\subset\operatorname{ran}(I-H^*).$$

Proof. The statements (i)–(iii) can be found in [10, Proposition 5.9, Corollaries 5.11 and 5.12].

(iv) This is implied by (5.8).

(v) This statement follows from the equalities

$$A_0-i=\left\{\begin{pmatrix}(T-I)g+Fu\\ 2ig\end{pmatrix}:\ Gg+(H-I)u=0,\ g\in\mathfrak{H},\ u\in\mathcal{H}\right\},$$

$$A_0+i=\left\{\begin{pmatrix}(I-T^*)g'-G^*u'\\ 2ig'\end{pmatrix}:\ F^*g'+(H^*-I)u'=0,\ g'\in\mathfrak{H},\ u'\in\mathcal{H}\right\}$$

which, in turn, are implied by (5.7). □

Remark 5.13. An example of a unitary boundary triple $\{\mathcal{H},\Gamma\}$, such that A_0 is selfadjoint and Γ_0 is not surjective is presented in [25, Example 6.6]. Observe also that A_0 is a maximal symmetric operator if at least one of the conditions $\operatorname{ran} G\subset\operatorname{ran}(I-H)$ or $\operatorname{ran} F^*\subset\operatorname{ran}(I-H^*)$ is satisfied.

The statement (v) in Proposition 5.12 is closely related to the early work of Calkin on existence of maximal symmetric extensions $\widetilde{A}$ contained in the domain of a reduction operator for A^* (meaning here $\operatorname{dom}\Gamma$); cf. [21, Theorems 4.8, 4.11, 4.12]. His results are described in modern terms in [42, Theorems 2.26, 2.27] by means of an angular representation for A_0.

The following lemma shows that the conditions (iv) and (v) in Proposition 5.12 are not unrelated.

Lemma 5.14. *Let $\mathcal{U}$ be a unitary colligation of the form (5.4). Then the following conditions are equivalent:*

(i) $\operatorname{ran}(I-H)+\operatorname{ran} G=\mathcal{H}$*;*

(ii) $\operatorname{ran}(I-H^*)+\operatorname{ran} F^*=\mathcal{H}$*;*

(iii) $\operatorname{ran}(I - H) = \mathcal{H}$;

(iv) $\operatorname{ran}(I - H^*) = \mathcal{H}$.

Proof. The equivalence of (i) and (ii) is implied by (5.8).

Since $\operatorname{ran}(I - H) \subseteq \operatorname{ran}(I - H) + \operatorname{ran} G$ and $\operatorname{ran}(I - H^*) \subseteq \operatorname{ran}(I - H^*) + \operatorname{ran} F^*$ it remains to prove the implications (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv).

Assume that $\operatorname{ran}(I - H) + \operatorname{ran} G = \mathcal{H}$. Then [35, Theorem 2.2] and the identity $HH^* + GG^* = I$ yield

$$\begin{aligned}\operatorname{ran}(I - H) + \operatorname{ran} G &= \operatorname{ran}\big(((I - H)(I - H^*))^{1/2}\big) + \operatorname{ran}\big((GG^*)^{1/2}\big)\\ &= \operatorname{ran}\big(((I - H)(I - H^*) + GG^*)^{1/2}\big)\\ &= \operatorname{ran}\big((I - 2\operatorname{Re} H + HH^* + GG^*)^{1/2}\big) = \operatorname{ran}\big((I - \operatorname{Re} H)^{1/2}\big).\end{aligned}$$

This implies the equality $\operatorname{ran}(I - \operatorname{Re} H) = \mathcal{H}$ and hence $-I \le \operatorname{Re} H \le qI$ for some $q < 1$. Therefore, the numerical range of H is contained in the half-plane $\operatorname{Re} z \le q$ and hence $1 \in \rho(H)$. This proves (iii). The implication (ii) $\Rightarrow$ (iv) is proved similarly. □

Corollary 5.15. *If $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a unitary boundary triple with $\operatorname{ran}\Gamma_0 = \mathcal{H}$, then $A_0 = A_0^*$ and Π is necessarily a B-generalized boundary triple.*

Remark 5.16. If $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple, then Γ and, consequently, Γ_0 and Γ_1 are surjective. Hence, $A_0 = A_0^*$ and $A_1 = A_1^*$. This conclusion can be made directly also from Proposition 5.12. Indeed, the assumption $\operatorname{ran} G = \mathcal{H}$ implies $0 \in \rho(GG^*)$. In view of the identity $GG^* = I - HH^*$ this implies $1 \in \rho(HH^*)$ and hence $1 \in \rho(H)$. By Proposition 5.12 (v) this condition yields $A_0 = A_0^*$.

We are now ready to prove Theorem 1.12 in a more general setting, where $\{\mathcal{H}, \Gamma\}$ is an arbitrary unitary boundary pair. It gives a complete characterization of the Weyl functions $M(\cdot)$ of S-generalized boundary pairs. In its present general form it completes and extends [25, Theorem 4.13] and [27, Theorem 7.39].

Theorem 5.17. *Let $\Pi = \{\mathcal{H}, \Gamma\}$ be a unitary boundary pair and let $M(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl family and the γ-field. Then the following statements are equivalent:*

(i) A_0 *is selfadjoint, i.e.* Π *is an* S*-generalized boundary pair;*

(ii) $A_* = A_0 \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda$ *and* $A_* = A_0 \,\widehat{+}\, \widehat{\mathfrak{N}}_\mu$ *for some (equivalently for all)* $\lambda \in \mathbb{C}_+$ *and* $\mu \in \mathbb{C}_-$*;*

(iii) $\operatorname{ran}\Gamma_0 = \operatorname{dom} M(\lambda) = \operatorname{dom} M(\mu)$ *for some (equivalently for all)* $\lambda \in \mathbb{C}_+$ *and* $\mu \in \mathbb{C}_-$*;*

(iv) $\gamma(\lambda)$ *and* $\gamma(\mu)$ *are bounded for some (equivalently for all)* $\lambda \in \mathbb{C}_+$ *and* $\mu \in \mathbb{C}_-$*;*

(v) $\operatorname{dom} M(\lambda) = \operatorname{dom} M\big(\overline{\lambda}\big)$ *and* $\operatorname{Im} M_{\rm op}(\lambda)$ *is bounded for some (equivalently for all)* $\lambda \in \mathbb{C}_+$*;*

(vi) The Weyl family $M(\lambda)$*,* $\lambda \in \mathbb{C}\setminus\mathbb{R}$*, admits the representation*

$$M(\lambda) = E + M_0(\lambda), \tag{5.31}$$

where $E = E^*$ *is a selfadjoint relation in* $\mathcal{H}$ *and* $M_0 \in \mathcal{R}[\mathcal{H}_0]$*, with* $\mathcal{H}_0 = \overline{\operatorname{dom}} E$.

Proof. (i) $\Leftrightarrow$ (ii) This equivalence and the independence from $\lambda \in \mathbb{C}_+$ and $\mu \in \mathbb{C}_-$ is proved in [25, Theorem 4.13].

(i) $\Leftrightarrow$ (iii) This can also be obtained from [25, Theorem 4.13], but we present here a different proof. Indeed, it follows from (5.14) that for all $\lambda \in \mathbb{C}_+$ and $\zeta = \frac{\lambda - i}{\lambda + i}$

$$\operatorname{ran}\Gamma_0 = \operatorname{ran} G + \operatorname{ran}(\theta(\zeta) - I). \tag{5.32}$$

If $A_0 = A_0^*$ then by Proposition 5.12 $\operatorname{ran} G \subset \operatorname{ran}(I - H)$ and (5.23) (see the proof of Theorem 5.5) yields $\operatorname{ran} G \subset \operatorname{ran}(\theta(\zeta) - I)$. By (5.32), (5.9), and $\operatorname{dom}\gamma(\lambda) = \operatorname{dom} M(\lambda)$ one obtains

$$\operatorname{ran}\Gamma_0 = \operatorname{ran}(\theta(\zeta) - I) = \operatorname{dom} M(\lambda) \quad \text{for all} \quad \lambda \in \mathbb{C}_+.$$

Similarly, it follows from (5.32) and (5.24) in the proof of Theorem 5.5 that

$$\operatorname{ran}\Gamma_0 = \operatorname{ran} F^* + \operatorname{ran}(\theta(\zeta)^* - I) = \operatorname{dom} M\big(\bar{\lambda}\big) \quad \text{for all} \quad \lambda \in \mathbb{C}_+.$$

Conversely, if for some $\lambda \in \mathbb{C}_+$ one has $\operatorname{ran}\Gamma_0 = \operatorname{dom} M(\lambda) = \operatorname{dom}\gamma(\lambda)$, then (5.8) implies, in particular, that $\operatorname{ran} G \subset \operatorname{ran}(\theta(\zeta) - I)$. Hence, it follows from (5.23) that $\operatorname{ran}(A_0 - \lambda) = \mathfrak{H}$. Similarly the identities $\operatorname{ran}\Gamma_0 = \operatorname{dom} M(\bar{\lambda}) = \operatorname{dom}\gamma(\bar{\lambda})$ imply that $\operatorname{ran}(A_0 - \bar{\lambda}) = \mathfrak{H}$ and, thus, $A_0 = A_0^*$.

(i) ⇒ (iv) This implication was proved in Theorem 4.2 (iv), (v).

(iv) ⇒ (i) If some $\gamma(\lambda) : \overline{\operatorname{dom}}\,\gamma(\lambda) \to \mathfrak{H}$ is bounded then $\operatorname{dom}\gamma(\lambda)^* = \mathfrak{H}$. Then by Theorem 5.5

$$\operatorname{ran}(A_0 - \bar{\lambda}) = \operatorname{dom}\Gamma_1 H(\bar{\lambda}) = \operatorname{dom}\gamma(\lambda)^* = \mathfrak{H}.$$

Similarly if $\gamma(\mu)$ is bounded then $\operatorname{ran}(A_0 - \bar{\mu}) = \mathfrak{H}$. Thus, A_0 is a selfadjoint relation in $\mathfrak{H}$.

(iv) ⇒ (v), (vi) Consider the decomposition (2.3) $M(\lambda) = \operatorname{gr} M_{\mathrm{op}}(\lambda) \oplus M_\infty$ of the Weyl family $M(\lambda)$ with the operator part $M_{\mathrm{op}} \in \mathcal{R}(\mathcal{H}_0)$, where $\mathcal{H}_0 = \overline{\operatorname{dom}}\, M(\lambda)$. As was already shown, now $A_0 = A_0^*$ and $\operatorname{dom} M_{\mathrm{op}}(\lambda) = \operatorname{ran}\Gamma_0$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$. It follows from the equality $M_{\mathrm{op}}(\lambda)^* = M_{\mathrm{op}}(\bar{\lambda})$ that the operator $E_0 = \operatorname{Re} M_{\mathrm{op}}(\lambda_0)$ $(\lambda_0 \in \mathbb{C}_+)$ is selfadjoint with the domain $\operatorname{dom} E_0 = \operatorname{ran}\Gamma_0$. Moreover, since the operator $\gamma(\lambda)$ is bounded for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$ it follows from the equality (3.6) that the operator

$$\operatorname{Im} M_{\mathrm{op}}(\lambda_0) = \operatorname{Im}\lambda_0 \gamma(\lambda_0)^* \gamma(\lambda_0)$$

is also bounded in $\mathcal{H}_0$ and hence the operator $M_{\mathrm{op}}(\lambda) - E_0$ is bounded in $\mathcal{H}_0$ at λ_0. Therefore, its closure, denoted now by $M_0(\lambda)$, is bounded in $\mathcal{H}_0$ at λ_0 and then also for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$; see e.g. [26, Proposition 4.18], [28, Theorem 3.9]. Finally, by setting $E = E_0 \oplus M_\infty$ one arrives at (5.31).

Finally, the implication (vi) ⇒ (v) is clear and (v) ⇒ (iv) (for $\mu = \bar{\lambda}$) follows easily from (3.6). □

Theorem 5.17 implies Theorem 1.12. In the case that Γ is single-valued $M(\lambda)$ is an operator valued Nevanlinna function with $\ker \operatorname{Im} M(\lambda) = \ker(M(\lambda) - M(\lambda)^*) = \{0\}$, i.e., $M(\cdot) \in \mathcal{R}^s(\mathcal{H})$; see (1.8) and Lemma 3.6.

Corollary 5.18. *Let $\{\mathcal{H}, \Gamma\}$ be an S-generalized boundary pair with the Weyl family $M(\cdot) = E + M_0(\cdot)$ as in Theorem 5.17. Then $\operatorname{ran}\Gamma$ is dense in $\mathcal{H}\times\mathcal{H}$, i.e., Γ defines an S-generalized boundary triple if and only if $E\,(= \operatorname{Re} M(\mu))$ is a selfadjoint operator and*

$$\operatorname{dom} E \cap \ker \overline{\gamma(\lambda)} = E \cap \ker \operatorname{Im} M_0(\lambda) = \{0\}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{5.33}$$

Proof. This follows from Lemma 3.6 and Corollary 4.3. □

Corollary 5.18 can be used to give an example of an S-generalized boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ such that the mapping Γ_0 is not closable; cf. Corollary 5.4.

Example 5.19. Let $M_0(\cdot) \in \mathcal{R}[\mathcal{H}]$ be a Nevanlinna function such that $\ker \operatorname{Im} M_0(\lambda)$ is nontrivial and let E be an unbounded selfadjoint operator in $\mathcal{H}$ with $\operatorname{dom} E \cap \ker \operatorname{Im} M_0(\lambda) = \{0\}$. Then the function

$$M(\lambda) = E + M_0(\cdot), \quad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

is a domain invariant Nevanlinna function. It follows from Corollary 5.18 and Theorems 1.10, 1.12 that $M(\cdot)$ can be realized as the Weyl function of some S-generalized boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$. However, $\operatorname{Im}(M(\lambda)h, h) = \operatorname{Im}(M_0(\lambda)h, h)$ with $h \in \operatorname{dom} M(\lambda)$ does not satisfy the condition (5.19) in Corollary 5.4, since the kernel $\ker \overline{\operatorname{Im} M(\lambda)} = \ker \operatorname{Im} M_0(\lambda)$ is nontrivial by construction.

Remark 5.20. Observe that in Theorems 1.12 and 5.17 the function $M_0(\cdot)$ can be considered as the closure of $M(\cdot) - E$. In Theorem 5.17 $M(\cdot)$ is an operator valued function if and only if E is an operator. By Corollary 5.18 even in this case Γ can still be multi-valued if the kernel $\ker M_0(\lambda)$ or $\ker \operatorname{Im} M_0(\lambda) = \ker \overline{\gamma(\lambda)}$ is nontrivial and the condition (5.33) is violated. In fact, any Nevanlinna function with bounded values in $\mathcal{H}$ and $\ker \operatorname{Im} M_0(\lambda) \neq \{0\}$ combined with an unbounded selfadjoint operator E in $\mathcal{H}$ satisfying the condition (5.33) is associated with an S-generalized boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with the Weyl function $M = E + M_0(\cdot)$. If such a function M is regularized by subtracting the unbounded constant operator E, the function $M_0(\cdot) = M(\cdot) - E$ corresponds to an AB-generalized boundary triple $\{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ whose range $\operatorname{ran}\widetilde{\Gamma}$ is not dense in $\mathcal{H}^2$. In particular, $\widetilde{\Gamma}$ whose Weyl function is the regularized function $M(\cdot) - E$ is not a quasi boundary triple. The closure $M_0(\cdot)$ of $M(\cdot) - E$ is the Weyl function of the closure of $\widetilde{\Gamma}$ which in this case is always a (multi-valued) B-generalized boundary pair. An example of an S-generalized boundary triple with $\ker \operatorname{Im} M_0(\lambda) \neq \{0\}$ satisfying the property (5.33) appears in [14, Proposition 2.17].

5.4 | ES-generalized boundary triples and form domain invariance

Recall, see Definition 1.13, that a unitary boundary triple $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^* is called *ES-generalized*, if the extension A_0 is essentially selfadjoint in $\mathfrak{H}$.

As the main result of this section it will be shown that the class of Weyl functions of ES-generalized boundary triples coincides with the class of form domain invariant Nevanlinna functions.

Definition 5.21. A Nevanlinna function $M \in \mathcal{R}(\mathcal{H})$ is said to be *form domain invariant* in $\mathbb{C}_+(\mathbb{C}_-)$, if the quadratic form $\mathfrak{t}_{M(\lambda)}$ in $\mathcal{H}$ generated by the imaginary part of $M(\lambda)$ via

$$\mathfrak{t}_{M(\lambda)}[u,v] := \frac{1}{\lambda-\bar{\lambda}}\,[(M(\lambda)u,v)-(u,M(\lambda)v)],$$

is closable for all $\lambda\in\mathbb{C}_+(\mathbb{C}_-)$ and the closure of the form $\mathfrak{t}_{M(\lambda)}$ has a constant domain. A Nevanlinna family $M\in\widetilde{\mathcal{R}}(\mathcal{H})$ is said to be form domain invariant in $\mathbb{C}_+(\mathbb{C}_-)$, if its operator part $M_{\rm op}(\cdot)$ in the decomposition (2.3) is form domain invariant in $\mathbb{C}_+(\mathbb{C}_-)$.

The following two lemmas are preparatory for the main result.

Lemma 5.22. *Let $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a unitary boundary triple. Then the following statements are equivalent:*

(i) $\operatorname{ran}(A_0-\lambda)$ *is dense in $\mathfrak{H}$ for some or, equivalently, for every $\lambda\in\mathbb{C}_+(\mathbb{C}_-)$;*

(ii) $\gamma(\bar{\lambda})$ *admits a single-valued closure $\overline{\gamma(\bar{\lambda})}$ for some or, equivalently, for every $\lambda\in\mathbb{C}_+(\mathbb{C}_-)$;*

(iii) the form $\mathfrak{t}_{M(\bar{\lambda})}$ is closable for some or, equivalently, for every $\lambda\in\mathbb{C}_+(\mathbb{C}_-)$.

Proof. (i) $\Leftrightarrow$ (ii) By Theorem 5.5 for every $\lambda\in\mathbb{C}_+(\mathbb{C}_-)$

$$\operatorname{dom}\gamma(\bar{\lambda})^* = \operatorname{dom}(\Gamma_1 H(\lambda)) = \operatorname{ran}(A_0-\lambda).$$

Therefore, $\gamma(\bar{\lambda})$ admits a single-valued closure for $\lambda\in\mathbb{C}_+(\mathbb{C}_-)$ if and only if $\operatorname{ran}(A_0-\lambda)$ is dense in $\mathfrak{H}$.

(ii) $\Leftrightarrow$ (iii) The equality (3.6) gives the following representation for $\mathfrak{t}_{M(\lambda)}$:

$$\mathfrak{t}_{M(\lambda)}[u,v] = (\gamma(\lambda)u,\gamma(\lambda)v)_{\mathfrak{H}}.$$

It is well-known (see e.g. [45, Chapter VI]) that the form $(\gamma(\lambda)u,\gamma(\mu)v)_{\mathfrak{H}}$ is closable precisely when the operator $\gamma(\lambda)$ is closable. □

Lemma 5.23. *Let $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be an ES-generalized boundary triple. Then:*

(i) $\ker\overline{\Gamma}_0=\overline{A_0}$ *is selfadjoint and the domain of $\overline{\Gamma}_0$ admits the decomposition*

$$\operatorname{dom}\overline{\Gamma}_0=\overline{A_0}\dotplus\left(\operatorname{dom}\overline{\Gamma}_0\cap\widehat{\mathfrak{N}}_\lambda(A^*)\right)=\overline{A_0}\dotplus\operatorname{ran}\overline{\widehat{\gamma}(\lambda)},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}; \tag{5.34}$$

(ii) $\gamma(\lambda)$ *admits a single-valued closure $\overline{\gamma(\lambda)}$ for every $\lambda\in\mathbb{C}\setminus\mathbb{R}$;*

(iii) the closure of the γ-field satisfies

$$\operatorname{ran}\overline{\Gamma}_0=\operatorname{dom}\overline{\gamma(\lambda)}=\operatorname{dom}\overline{\gamma(\mu)},\quad \lambda,\mu\in\mathbb{C}\setminus\mathbb{R}; \tag{5.35}$$

(iv) $\overline{\gamma(\lambda)}$ *and* $\overline{\gamma(\mu)}$ *are connected by*

$$\overline{\gamma(\lambda)}=\left[I+(\lambda-\mu)\left(\overline{A_0}-\lambda\right)^{-1}\right]\overline{\gamma(\mu)},\quad \lambda,\mu\in\mathbb{C}\setminus\mathbb{R}. \tag{5.36}$$

Proof.

(i) Since the closed linear relation $\overline{\Gamma}_0$ has a closed kernel, one has $\overline{A_0}\subset\ker\overline{\Gamma}_0$. Since $\overline{A_0}$ is selfadjoint, the first von Neumann's formula shows that $A^*=\overline{A_0}\dotplus\widehat{\mathfrak{N}}_\lambda(A^*)$ for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Consequently,

$$\overline{A_0}\subset\operatorname{dom}\overline{\Gamma}_0\subset\overline{A_0}\dotplus\widehat{\mathfrak{N}}_\lambda(A^*),\quad \lambda\in\mathbb{C}\setminus\mathbb{R},$$

and this implies the first equality in (5.34). The second equality in (5.34) holds by Proposition 5.3. Finally, according to Proposition 5.3 $\ker\overline{\Gamma}_0 \cap \widehat{\mathfrak{N}}_\lambda(A^*) = \operatorname{mul}\overline{\widehat{\gamma}(\lambda)} = \{0\}$, since $\gamma(\lambda)$ or, equivalently, $\widehat{\gamma}(\lambda)$ is closable by Lemma 5.22. Since $\overline{A_0} \subset \ker\overline{\Gamma}_0$, the identity $\ker\overline{\Gamma}_0 \cap \widehat{\mathfrak{N}}_\lambda(A^*) = \{0\}$ combined with the first equality in (5.34) implies the equality $\overline{A_0} = \ker\overline{\Gamma}_0$.

(ii) The statement (ii) is implied by Lemma 5.22.

(iii) Since $\overline{A_0}$ is selfadjoint, the defect subspaces of A are connected by

$$\mathfrak{N}_\lambda(A^*) = \left[I + (\lambda-\mu)\left(\overline{A_0}-\lambda\right)^{-1}\right]\mathfrak{N}_\mu(A^*), \qquad \lambda,\mu \in \mathbb{C}\setminus\mathbb{R}.$$

Hence, if $f_\lambda = \left[I + (\lambda-\mu)\left(\overline{A_0}-\lambda\right)^{-1}\right] f_\mu$, then $\widehat{f}_\mu = \{f_\mu, \mu f_\mu\} \in \widehat{\mathfrak{N}}_\mu(A^*)$ precisely when

$$\widehat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} = \widehat{f}_\mu + (\lambda-\mu)\overline{H(\lambda)} f_\mu \in \widehat{\mathfrak{N}}_\lambda(A^*), \tag{5.37}$$

where $\overline{H(\lambda)} f_\mu = \left\{\left(\overline{A_0}-\lambda\right)^{-1} f_\mu, \left(I + \lambda\left(\overline{A_0}-\lambda\right)^{-1}\right) f_\mu\right\} \in \overline{A_0}$. Since $\overline{A_0} \subset \operatorname{dom}\overline{\Gamma}_0$, it follows from (5.37) that $\widehat{f}_\mu \in \operatorname{dom}\left(\overline{\Gamma}_0\right) \cap \widehat{\mathfrak{N}}_\mu(A^*)$ if and only if $\widehat{f}_\lambda \in \operatorname{dom}\left(\overline{\Gamma}_0\right) \cap \widehat{\mathfrak{N}}_\lambda(A^*)$ and

$$\{\widehat{f}_\mu, h\} \in \overline{\Gamma}_0 \cap \left(\widehat{\mathfrak{N}}_\mu(A^*) \oplus \mathcal{H}\right) \;\Leftrightarrow\; \{\widehat{f}_\lambda, h\} \in \overline{\Gamma}_0 \cap \left(\widehat{\mathfrak{N}}_\lambda(A^*) \oplus \mathcal{H}\right)$$

for some $h \in \mathcal{H}$. Now, using (i) and Proposition 5.3 one gets

$$\operatorname{dom}\overline{\widehat{\gamma}(\lambda)} = \overline{\Gamma}_0\left(\operatorname{dom}\overline{\Gamma}_0 \cap \widehat{\mathfrak{N}}_\lambda(A^*)\right) = \operatorname{ran}\overline{\Gamma}_0 = \overline{\Gamma}_0\left(\operatorname{dom}\overline{\Gamma}_0 \cap \widehat{\mathfrak{N}}_\mu(A^*)\right) = \operatorname{dom}\overline{\widehat{\gamma}(\mu)}$$

Clearly $\operatorname{dom}\overline{\widehat{\gamma}(\lambda)} = \operatorname{dom}\overline{\gamma(\lambda)}$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and hence (iii) is proved.

(iv) The proof of (iii) shows that $\{h, \widehat{f}_\mu\} \in \overline{\widehat{\gamma}(\mu)}$ if and only if $\{h, \widehat{f}_\lambda\} \in \overline{\widehat{\gamma}(\lambda)}$. Consequently, $\{h, f_\mu\} \in \overline{\gamma(\mu)}$ if and only if $\{h, f_\lambda\} = \left\{h, \left[I + (\lambda-\mu)\left(\overline{A_0}-\lambda\right)^{-1}\right] f_\mu\right\} \in \overline{\gamma(\lambda)}$ and, since $\overline{\gamma(\mu)}$ and $\overline{\gamma(\lambda)}$ are operators, this means that (5.36) is satisfied. □

Theorem 5.24. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a unitary boundary triple for A^* and let M and $\gamma(\cdot)$ be the corresponding Weyl function and the γ-field. Then the following statements are equivalent:*

(i) $\operatorname{ran}\left(A_0 - \bar\lambda\right)$ is dense in $\mathfrak{H}$ for some or, equivalently, for every $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$;

(ii) $\gamma(\lambda)$ admits a single-valued closure $\overline{\gamma(\lambda)}$ for one $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$ with a domain dense in $\mathcal{H}$;

(iii) $\gamma(\lambda)$ admits a single-valued closure $\overline{\gamma(\lambda)}$ for every $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$ which is domain invariant with a constant domain dense in $\mathcal{H}$;

(iv) the form $\mathfrak{t}_{M(\lambda)}$ is closable for one $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$;

(v) the Weyl function M belongs to $\mathcal{R}^s(\mathcal{H})$ and is form domain invariant in $\mathbb{C}_+(\mathbb{C}_-)$.

In particular, if the statements (i)–(v) are satisfied both in $\mathbb{C}_+$ and $\mathbb{C}_-$ then Π is an ES-generalized boundary triple and the Weyl function M is form domain invariant with

$$\operatorname{dom}\overline{\mathfrak{t}_{M(\lambda)}} = \operatorname{dom}\overline{\gamma(\lambda)} = \operatorname{ran}\overline{\Gamma}_0, \qquad \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{5.38}$$

Proof. The equivalence (i) ⇔ (ii) is obtained from Lemma 5.22. The fact that the domain of $\overline{\gamma(\lambda)}$ is dense in $\mathcal{H}$ follows from Proposition 5.2.

The equivalences (i) ⇔ (iv), (v) and (ii) ⇔ (iii) follow from Lemmas 5.22 and 5.23.

In particular, Lemma 5.22 shows that the form $\mathfrak{t}_{M(\lambda)}$ is closable for some (and then for every) $\lambda \in \mathbb{C}_+$ and for some (and then for every) $\mu \in \mathbb{C}_-$ if and only if A_0 is essentially selfadjoint. In this case the closure of the form $\mathfrak{t}_{M(\lambda)}$ is given by

$$\overline{\mathfrak{t}_{M(\lambda)}}[u,v] = \left(\overline{\gamma(\lambda)}u, \overline{\gamma(\lambda)}v\right)_{\mathfrak{H}}, \tag{5.39}$$

in particular, $\operatorname{dom}\overline{\mathfrak{t}_{M(\lambda)}} = \operatorname{dom}\overline{\gamma(\lambda)}$. According to Lemma 5.23 this domain does not depend on $\lambda \in \mathbb{C}\setminus\mathbb{R}$ when A_0 is essentially selfadjoint. The last equality in (5.38) is obtained from (5.35). □

Remark 5.25. Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ES-generalized boundary triple, and assume that $(\alpha, \beta) \subset \rho(\overline{A_0})$. Then:

(i) for every $\mu \in (\alpha, \beta)$ $\gamma(\mu)$ admits a single-valued closure $\overline{\gamma(\mu)}$ such that (5.35) and (5.36) hold for all $\lambda,\ \mu \in (\mathbb{C} \setminus \mathbb{R}) \cup (\alpha, \beta)$;
(ii) for every $\mu \in (\alpha, \beta)$ and $u, v \in \mathcal{H}$ there exists a limit

$$\overline{\mathfrak{t}_{M(\mu)}}[u, v] = \lim_{\nu \downarrow 0} \overline{\mathfrak{t}_{M(\mu+i\nu)}}[u, v] = \left(\overline{\gamma(\mu)}u, \overline{\gamma(\mu)}v\right)_{\mathcal{H}}.$$

The proof of the first statement is precisely the same as the proof of Lemma 5.23. The statement (ii) is implied by the equality (5.39), and the continuity of $\overline{\gamma(\mu)}u$ with respect to $\mu \in (\alpha, \beta)$; see (5.36).

The assumption that $\gamma(\lambda)$ admits a single-valued closure for some $\lambda \in \mathbb{C}_-$ does not imply that $\gamma(\mu)$ admits a single-valued closure for some $\mu \in \mathbb{C}_+$. In particular, for a maximal symmetric relation A the following extreme situation holds.

Proposition 5.26. *Let $\Pi = \{\mathcal{H}, \Gamma\}$ be a unitary boundary pair for A^*, let $\gamma(\cdot)$ be the corresponding γ-field, and assume that A is maximal symmetric with $n_-(A) = 0$ and $0 < n_+(A) \le \infty$. Then $\gamma(\lambda)$ is a bounded operator (in fact a zero operator) for every $\lambda \in \mathbb{C}_-$, while $\gamma(\lambda)$ is a singular operator with* mul $\overline{\gamma(\lambda)} = \mathfrak{N}_\lambda$ *for every $\lambda \in \mathbb{C}_+$.*

Proof. First recall that for every closed symmetric relation A there is a unitary boundary pair for A^*; see [25, Proposition 3.7]. By definition $\gamma(\lambda)$ is a single-valued operator and it is known that $\overline{\text{ran}}\,\gamma(\lambda) = \mathfrak{N}_\lambda$ for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$; see [25, Lemma 2.14]. Hence the statement in the lower half-plane $\mathbb{C}_-$ is clear. In particular, one has $\ker \overline{\gamma(\lambda)} = \overline{\text{dom}}\,\gamma(\lambda)$ and consequently $\text{ran}\,\gamma(\lambda)^* = \text{mul}\,\gamma(\lambda)^*$, $\lambda \in \mathbb{C}_-$. On the other hand, by Theorem 5.5 $\text{ran}\,\gamma(\lambda)^*$ and $\text{mul}\,\gamma(\lambda)^*$ do not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Consequently, the equality $\text{ran}\,\gamma(\lambda)^* = \text{mul}\,\gamma(\lambda)^*$ holds also for every $\lambda \in \mathbb{C}_+$. Then equivalently $\overline{\text{dom}}\,\gamma(\lambda) = \ker \overline{\gamma(\lambda)}$, which shows that $\gamma(\lambda)$ is a singular operator with $\text{mul}\,\overline{\gamma(\lambda)} = \overline{\text{ran}}\,\gamma(\lambda) = \mathfrak{N}_\lambda$ for every $\lambda \in \mathbb{C}_+$. □

Observe that in Proposition 5.26 the corresponding Weyl function M is actually domain invariant in each half-plane $\mathbb{C}_+$ and $\mathbb{C}_-$, while it is neither domain nor form domain invariant in $\mathbb{C} \setminus \mathbb{R}$; see Proposition 3.11 (i). For an explicit example demonstrating Proposition 5.26 we refer to [25, Example 6.7], where A is the minimal differential operator generated in $\mathfrak{H} = L_2(0, \infty)$ by the differential expression iD.

Remark 5.27.

(a) If A_+ is a maximal symmetric operator in $\mathfrak{H}$ with $n_-(A_+) = 0$ and $0 < n_+(A_+) \le \infty$ and A_- is a maximal symmetric operator in $\mathfrak{H}$ with $n_+(A_-) = 0$ and $0 < n_-(A_-) \le \infty$, then $A = A_+ \oplus A_-$ is a symmetric operator in $\mathfrak{H} \oplus \mathfrak{H}$ with defect numbers $\{n_+(A_+), n_-(A_-)\}$. Moreover, if $\Pi_\pm = \{\mathcal{H}_\pm, \Gamma_\pm\}$ is a unitary boundary pair for $A_\pm^*$ then clearly the orthogonal sum $\Pi_+ \oplus \Pi_- = \{\mathcal{H}_+ \oplus \mathcal{H}_-, \Gamma_+ \oplus \Gamma_-\}$ is a unitary boundary pair for A^*. Moreover, the corresponding γ-field is $\gamma(\lambda) = \gamma_+(\lambda) \oplus \gamma_-(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Now Proposition 5.26 shows that $\gamma(\lambda)$ is not closable for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Hence, there exists symmetric operators A with arbitrary deficiency indices $n_\pm(A)$ and a unitary boundary pair for A^* such that the corresponding γ-field $\gamma(\lambda)$ is not closable for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$. This also holds in the case of equal deficiency indices $0 < n_-(A_-) = n_+(A_+) \le \infty$. However, in this case the boundary pair Π for A^* is not minimal in general.
(b) Let A_+ be a maximal symmetric operator in $\mathfrak{H}$ with $n_-(A_+) = 0$ and $n_+(A_+) = 1$ and let A_- be a symmetric operator in $\mathfrak{H}$ with equal deficiency indices $n_+(A_-) = n_-(A_-) = \infty$. Then $A = A_+ \oplus A_-$ is a symmetric operator in $\mathfrak{H} \oplus \mathfrak{H}$ with equal deficiency indices $n_+(A) = n_-(A) = \infty$. Moreover, if $\Pi_- = \{\mathcal{H}_-, \Gamma_-\}$ is an ordinary boundary triple for A_-^* then the corresponding γ-field $\gamma_-(\lambda)$ is a bounded operator in $\mathcal{B}(\mathcal{H}_-, \mathfrak{N}_\lambda(A_-^*))$. Considering the boundary pair $\Pi_+ \oplus \Pi_-$ for A^* one concludes from Proposition 5.26 that the corresponding γ-field $\gamma(\lambda) = \gamma_+(\lambda) \oplus \gamma_-(\lambda)$ is a bounded operator for every $\lambda \in \mathbb{C}_-$, while $\gamma(\lambda)$ is not closable for any $\lambda \in \mathbb{C}_+$.

In the next example we present a unitary boundary triple whose Weyl function is form domain invariant but not domain invariant.

Example 5.28. Let $\varphi(\cdot)$ be a scalar Nevanlinna function and $\mathcal{H} = L^2(0, \infty)$. Define an operator valued function $G_\varphi(\cdot)$

$$G_\varphi(\lambda)u = -i\frac{d^2u}{dx^2}, \quad \text{dom}\big(G_\varphi(\lambda)\big) = \left\{u \in W_2^2(\mathbb{R}_+)\ :\ u'(0) + i\varphi(\lambda)u(0) = 0\right\}, \quad \lambda \in \mathbb{C}_+.$$

Clearly, $G_\varphi(\lambda)$ is densely defined, $\rho(G_\varphi(\lambda)) \neq \emptyset$ for each $\lambda \in \mathbb{C}_+$ and the family $G_\varphi(\cdot)$ is holomorphic in $\mathbb{C}_+$ in the resolvent sense. Now consider the form generated by the imaginary part of $G_\varphi(\lambda)$. Integrating by parts one obtains

$$\mathfrak{t}_{\varphi(\lambda)}[u] := \operatorname{Im}(G_\varphi(\lambda)u, u) = \int_{\mathbb{R}_+} |u'(x)|^2\, dx + \operatorname{Im}\varphi(\lambda)|u(0)|^2,$$

where $u \in \operatorname{dom}\mathfrak{t}_{\varphi(\lambda)} = \operatorname{dom}(G_\varphi(\lambda))$. Hence the form $\mathfrak{t}_{\varphi(\lambda)}$ is nonnegative and $G_\varphi(\lambda)$ is m-dissipative for each $\lambda \in \mathbb{C}_+$. Moreover, $G_\varphi(\cdot) \in R^s(\mathcal{H})$ since $\ker \mathfrak{t}_{\varphi(\lambda)} = \{0\}$. Therefore, by Theorem 1.10, there exists a certain unitary boundary triple such that the corresponding Weyl function coincides with $G_\varphi(\cdot)$.

Notice that the form $\mathfrak{t}_{G_\varphi(\lambda)}(\lambda \in \mathbb{C}_+)$ associated with $G_\varphi(\lambda)$ in (1.14) coincides with $\mathfrak{t}_{\varphi(\lambda)}$ up to an inessential renormalization by $\operatorname{Im}\lambda$. Clearly, the form $\mathfrak{t}_{\varphi(\lambda)}$ is closable with the closure given by

$$\bar{\mathfrak{t}}_{\varphi(\lambda)}[u] = \int_{\mathbb{R}_+} |u'(x)|^2\, dx + \operatorname{Im}\varphi(\lambda)|u(0)|^2, \quad \operatorname{dom}\bar{\mathfrak{t}}_{\varphi(\lambda)} = W_2^1(\mathbb{R}_+).$$

Thus, the form domain $\operatorname{dom}(\bar{\mathfrak{t}}_{\varphi(\lambda)}) = W_2^1(\mathbb{R}_+)$ does not depend on $\lambda \in \mathbb{C}_+$ while the domain $\operatorname{dom} G_\varphi(\lambda)$ does, i.e. G_φ satisfies Assumption 1.15. The operator associated with the form $\bar{\mathfrak{t}}_{\varphi(\lambda)}$ is given by

$$G_{\varphi,I}(\lambda)u = -\frac{d^2u}{dx^2}, \quad \operatorname{dom}(G_{\varphi,I}(\lambda)) = \{u \in W_2^2(\mathbb{R}_+) : u'(0) = (\operatorname{Im}\varphi(\lambda))u(0)\}.$$

The operator $G_{\varphi,I}$ can be treated as the imaginary part of the unbounded operator G_φ.

A simple example of a unitary boundary triple whose Weyl function is form domain invariant and γ-field is unbounded can be obtained as follows (see also [25, Example 6.5]).

Example 5.29. Let H be a nonnegative selfadjoint operator in the Hilbert space $\mathfrak{H}$ with $\ker H = \{0\}$. Let

$$A_* = \operatorname{ran} H^{1/2} \times \operatorname{dom} H^{1/2}, \text{ so that } A := (A_*)^* = \{0, 0\} \text{ and } (A)^* = \mathfrak{H}^2,$$

and define

$$\Gamma_0\hat{f} = H^{-1/2}f, \quad \Gamma_1\hat{f} = H^{1/2}f'; \quad \hat{f} = \{f, f'\},\ f \in \operatorname{ran} H^{1/2},\ f' \in \operatorname{dom} H^{1/2}.$$

Then $\{\mathfrak{H}, \Gamma_0, \Gamma_1\}$ is a unitary boundary triple for $A^* = \overline{A_*}$. Indeed, Green's identity (1.1) is satisfied, and $\operatorname{ran}\Gamma$ is dense in $\mathfrak{H}^2$. Moreover, it is straightforward to check that Γ is closed, since $H^{1/2}$ is selfadjoint and, in particular, closed. Observe, that $\hat{f}_\lambda := \{f_\lambda, \lambda f_\lambda\} \in A_*$ if and only if $f_\lambda = H^{1/2}k$ and $\lambda f_\lambda = H^{-1/2}g$, with $k \in \operatorname{dom} H^{1/2}$ and $g \in \operatorname{ran} H^{1/2}$, are connected by $H^{-1/2}g = \lambda H^{1/2}k$. Then $k \in \operatorname{dom} H$ and

$$\Gamma_0\hat{f}_\lambda = k, \quad \Gamma_1\hat{f}_\lambda = \lambda H k.$$

These formulas imply that $\gamma(\lambda) = H^{1/2}$ and $M(\lambda) = \lambda H$, $\lambda \in \mathbb{C}$. In particular, the Weyl function is a Nevanlinna function. According to [25, Proposition 3.6] this implies that Γ is in fact $J_{\mathfrak{H}}$-unitary.

Note that $M(\lambda)$ and its inverse are domain invariant, but in general unbounded Nevanlinna functions with unbounded imaginary parts. Clearly, $A_0 = \ker\Gamma_0 = \{0\} \times \operatorname{dom} H^{1/2}$ and $A_1 = \ker\Gamma_1 = \operatorname{ran} H^{1/2} \times \{0\}$ are essentially selfadjoint and A_0 is selfadjoint (A_1 selfadjoint) if and only if $M(\lambda) = \lambda H$ ($-M(\lambda)^{-1} = -\lambda^{-1}H^{-1}$, respectively) is a Nevanlinna function with bounded values (cf. Theorem 1.12).

In this example the Weyl function is also domain invariant. In fact, domain invariance of a Nevanlinna function M implies its form domain invariance.

Proposition 5.30. *Let M be a Nevanlinna function in the Hilbert space $\mathcal{H}$. If the equality* $\operatorname{dom} M(\lambda) = \operatorname{dom} M(\bar{\lambda})$ *holds for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then M is form domain invariant.*

In particular, if M is domain invariant, then it is also form domain invariant.

Proof. If $\operatorname{dom} M(\lambda) = \operatorname{dom} M(\bar{\lambda})$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then one can write

$$\mathfrak{t}_{M(\lambda)}[u, v] = \left(\frac{M(\lambda) - M(\lambda)^*}{\lambda - \bar{\lambda}}u, v\right)_{\mathcal{H}} = (\gamma(\lambda)u, \gamma(\lambda)v)_{\mathfrak{H}}, \quad u, v \in \operatorname{dom} M(\lambda).$$

Hence, the operator $N(\lambda) := \frac{M(\lambda)-M(\lambda)^*}{\lambda-\bar{\lambda}}$ is nonnegative and densely defined in $\mathcal{H} \ominus \operatorname{mul} M(\lambda)$. Therefore, the form $\mathfrak{t}_{M(\lambda)}$ is closable for $\lambda \in \mathbb{C} \setminus \mathbb{R}$; see [45]. By applying the same reasoning to $\bar{\lambda}$ it is seen that also the form $\mathfrak{t}_{M(\bar{\lambda})}$ is closable. Now by applying Lemma 5.22 it is seen that A_0 is essentially selfadjoint and hence by Theorem 5.24 M is form domain invariant. □

The converse statement does not hold. In fact, in [28] an example of a form domain invariant Nevanlinna function is constructed, such that the domains of $M(\lambda)$ and $M(\mu)$ have a zero intersection:

$$\operatorname{dom} M(\lambda) \cap \operatorname{dom} M(\mu) = \{0\} \text{ for all } \lambda, \mu \in \mathbb{C}_+, \quad \lambda \neq \mu.$$

Remark 5.31. A unitary boundary pair $\{\mathcal{H}, \Gamma\}$ for A^* is said to be ES-generalized if $\overline{A_0} = A_0^*$. ES-generalized boundary pairs can be characterized by the following equivalent conditions:

(i) for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\gamma(\lambda)$ admits a single-valued closure $\overline{\gamma(\lambda)}$ with a constant domain;

(ii) the Weyl family $M \in \mathcal{R}(\mathcal{H})$ is form domain invariant, i.e. its operator part $M_{\mathrm{op}}(\cdot)$ in the decomposition (2.3) is form domain invariant.

Notice, that in the case when (i)–(ii) are in force and $\operatorname{mul}\Gamma$ is nontrivial it may happen that the domain of the form $\mathfrak{t}_{M_{\mathrm{op}}(\lambda)}[u, v] = \left(\overline{\gamma(\lambda)}u, \overline{\gamma(\lambda)}v\right)_{\mathfrak{H}}$ is not dense in $\mathcal{H}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$; for an example involving differential operators; see [29, Example 5.40].

5.5 | Renormalizations of form domain invariant Nevanlinna functions

The next theorem shows that form domain invariant Nevanlinna functions M in $\mathcal{H}$ can be renormalized with a bounded operator G such that the renormalized function G^*MG becomes domain invariant.

Theorem 5.32. *Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a unitary boundary triple for A^* with the γ-field $\gamma(\cdot)$ and the Weyl function M, and assume that $A_0 = \ker \Gamma_0$ is essentially selfadjoint. Then:*

(1) There exists a bounded operator G in $\mathcal{H}$ with $\operatorname{ran} G = \operatorname{dom} \overline{\gamma(\lambda)}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, *and* $\ker G = \{0\}$, *such that*

$$\begin{pmatrix} \widetilde{\Gamma}_0 \\ \widetilde{\Gamma}_1 \end{pmatrix} = \operatorname{clos} \begin{pmatrix} G^{-1} & 0 \\ 0 & G^* \end{pmatrix} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix}$$

defines an AB-generalized boundary pair $\{\mathcal{H}, \widetilde{\Gamma}\}$ for A^*.

(2) The corresponding Weyl function $\widetilde{M}$ is domain invariant and it is given by

$$\widetilde{M}(\lambda) = E + M_0(\lambda),$$

where E is a closed densely defined symmetric operator in $\mathcal{H}$ and $M_0(\cdot)$ is the restriction of a Nevanlinna function $\overline{M_0(\cdot)} \in \mathcal{R}[\mathcal{H}]$ onto the domain $\operatorname{dom} E$.

(3) Furthermore, $\overline{G^*M(\lambda)G}$ is also a Weyl function of a closed AB-generalized boundary pair and it satisfies

$$\overline{G^*M(\lambda)G} = E_0 + M_0(\lambda) \subset \widetilde{M}(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where $E_0 \subset E$ is a closed densely defined symmetric restriction of E.

Proof. The proof is divided into five steps.

1. *Construction of a bounded operator G with the properties*

$$\ker G = \{0\}, \quad \operatorname{ran} G = \operatorname{dom} \overline{\gamma(\mu)} \quad \text{and} \quad \overline{\operatorname{dom} \gamma(\mu) G} = \mathcal{H}, \quad \text{for some} \quad \mu \in \mathbb{C} \setminus \mathbb{R}.$$

Since A_0 is essentially selfadjoint, $\gamma(\lambda)$ is closable and the dense subspace $\mathcal{H}_0 = \operatorname{dom} \overline{\gamma(\lambda)}$ of $\mathcal{H}$ does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$; see Theorem 5.24. Since $\mathcal{H}_0$ is an operator range there exists a bounded selfadjoint operator $G = G^*$ with $\operatorname{ran} G = \mathcal{H}_0$ and $\ker G = \{0\}$; for instance, one can fix $\mu \in \mathbb{C} \setminus \mathbb{R}$ and then take $G = \left(\gamma(\mu)^* \overline{\gamma(\mu)} + I\right)^{-1/2}$. Namely,

$\operatorname{dom}\gamma(\mu) = \operatorname{dom} M(\mu)$ is dense in $\mathcal{H}$, since $\operatorname{mul}\Gamma = \{0\}$ by assumption, and hence $\gamma(\mu)^*\overline{\gamma(\mu)}$ is a selfadjoint operator satisfying $\operatorname{dom}\overline{\gamma(\mu)} = \operatorname{dom}\left(\gamma(\mu)^*\overline{\gamma(\mu)}\right)^{1/2} = \operatorname{dom}\left(\gamma(\mu)^*\overline{\gamma(\mu)} + I\right)^{1/2}$. With this choice of G the domain of $\gamma(\mu)G$ is dense in $\mathcal{H}$ since $\operatorname{dom}\gamma(\mu)$ is a core for the form $\mathfrak{t}_{M(\mu)}$ and due to the equality $\operatorname{dom}\left(\mathfrak{t}_{M(\mu)} + I\right) = \operatorname{ran} G$ one concludes that $\operatorname{dom}\gamma(\mu)$ is also a core for the operator $G^{-1} = \left(\gamma(\mu)^*\overline{\gamma(\mu)} + I\right)^{1/2}$.

2. *Construction of an isometric boundary triple* $\{\mathcal{H}, \Gamma_0^G, \Gamma_1^G\}$ *such that the corresponding* γ*-field* $\gamma^G(\lambda)$ *is a bounded densely defined operator*.
Introduce the transform $\{\mathcal{H}, \Gamma_0^G, \Gamma_1^G\}$ of the boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ by setting
$$\begin{pmatrix}\Gamma_0^G\\ \Gamma_1^G\end{pmatrix} = \begin{pmatrix}G^{-1} & 0\\ 0 & G^*\end{pmatrix}\begin{pmatrix}\Gamma_0\\ \Gamma_1\end{pmatrix}, \tag{5.40}$$
where G has the properties stated above. The block operator is isometric (in the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$) and hence Γ^G is isometric as a composition of isometric mappings; i.e. Γ^G satisfies Green's identity (3.1) (*Assumption 3.1.2*). Since $\left(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A_*)\right) = \widehat{\gamma}(\lambda)^{-1}$ one has
$$\Gamma_0\widehat{\mathfrak{N}}_\lambda(A_*) = \operatorname{dom}\gamma(\lambda) \subset \operatorname{dom}\overline{\gamma(\lambda)} = \operatorname{ran} G,$$
which implies that $\operatorname{ran}\widehat{\gamma}(\lambda) = \widehat{\mathfrak{N}}_\lambda(A_*) = \widehat{\mathfrak{N}}_\lambda(A^*) \cap \operatorname{dom}\Gamma \subset \operatorname{dom}\Gamma^G$ (here $A_* = \operatorname{dom}\Gamma$), and hence $\widehat{\mathfrak{N}}_\lambda\left({A_*}^G\right) = \widehat{\mathfrak{N}}_\lambda(A_*)$. Moreover, it is clear that $\ker\Gamma_0^G = \ker\Gamma_0 = A_0$ is essentially selfadjoint. Since the closure of $A_0 \hat{+} \widehat{\mathfrak{N}}_\lambda(A_*)$ is $\overline{A_0} \hat{+} \widehat{\mathfrak{N}}_\lambda(A^*) = A^*$ one gets $\overline{\operatorname{dom}}\,\Gamma^G = A^*$ (*Assumption 3.1.1*). The corresponding γ-field is given by
$$\gamma^G(\lambda) = \left(\Gamma_0^G \upharpoonright \widehat{\mathfrak{N}}_\lambda\left({A_*}^G\right)\right)^{-1} = \gamma(\lambda)G, \quad \lambda \in \mathbb{C}\setminus\mathbb{R};$$
see Lemma 3.12. Since $\gamma(\lambda)$ is closable and $\gamma(\lambda)G \subset \overline{\gamma(\lambda)}G$, it follows from $\operatorname{ran} G = \operatorname{dom}\overline{\gamma(\lambda)}$ that the closed operator $\overline{\gamma(\lambda)}G$ is everywhere defined and, hence, bounded by the closed graph theorem. Thus also $\gamma(\lambda)G$ is a bounded operator with bounded closure $\overline{\gamma(\lambda)G} \subset \overline{\gamma(\lambda)}G$.
Next recall the operator $H(\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, from (3.11); see also Lemma 3.7. Since A_0 is essentially selfadjoint, $\operatorname{dom}\left(\Gamma_1 H(\lambda)\right) = \operatorname{dom} H(\lambda) = \operatorname{ran}\left(A_0 - \lambda\right)$ is dense in $\mathfrak{H}$. Since $\ker\Gamma_0^G = A_0 \subset \operatorname{dom}\Gamma_1^G$ and $\operatorname{mul}\Gamma_1^G = \{0\}$, it follows from Theorem 5.5 that
$$\Gamma_1^G H(\lambda) = G^*\Gamma_1 H(\lambda) = G^*\gamma\left(\bar{\lambda}\right)^* \subset \left(\gamma\left(\bar{\lambda}\right)G\right)^*, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{5.41}$$
By the construction of G the domain of $\gamma(\mu)G$ is dense in $\mathcal{H}$ for some $\mu \in \mathbb{C}\setminus\mathbb{R}$. Therefore, (5.41) implies that $\Gamma_1^G H(\bar{\mu})$ is a bounded densely defined operator for some $\mu \in \mathbb{C}\setminus\mathbb{R}$ and, since A_0 is essentially selfadjoint, Lemma 3.7 shows that $\Gamma_1^G H(\lambda)$ is bounded and densely defined for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

3. *Verification of (1):* Now consider the closure $\widetilde{\Gamma}$ of Γ^G in (5.40). It is shown below that $\ker\widetilde{\Gamma}_0 = \overline{A_0}$, which means that $\ker\widetilde{\Gamma}_0$ is selfadjoint (*Assumption 1.13.1*), since A_0 is essentially selfadjoint by assumption. By construction Γ^G is defined via the transform $\Gamma^G = \left\{G^{-1}\Gamma_0, G^*\Gamma_1\right\}$ of $\left\{\Gamma_0, \Gamma_1\right\}$. It follows from Lemma 3.8 (see also Theorem 5.5) that the graph of Γ^G contains all elements of the form
$$\{\widehat{f}, \widehat{k}\} = \left\{H(\lambda)k_\lambda, \begin{pmatrix}0\\ G^*\gamma(\bar{\lambda})^*k_\lambda\end{pmatrix}\right\} + \left\{\begin{pmatrix}\gamma(\lambda)Gh\\ \lambda\gamma(\lambda)Gh\end{pmatrix}, \begin{pmatrix}h\\ G^*M(\lambda)Gh\end{pmatrix}\right\}, \tag{5.42}$$
where $k_\lambda \in \operatorname{ran}(A_0 - \lambda)$ and $h \in \operatorname{dom} G^*M(\lambda)G = \operatorname{dom}\gamma(\lambda)G$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Let $\widehat{h} = \{h, h'\} \in \overline{A_0}$ and let $k \in \operatorname{ran}\left(\overline{A_0} - \lambda\right)$ be such that $\widehat{h} = \overline{H(\lambda)}k$, where $\overline{H(\lambda)}$ corresponds to the graph of $\overline{A_0}$; see (3.11). Moreover, let $k_n \in \operatorname{ran}\left(A_0 - \lambda\right)$ be a sequence such that $k_n \to k$ as $n \to \infty$. Then $H(\lambda)k_n \to \widehat{h} \in \overline{A_0}$, since $H(\lambda)$ is bounded. Moreover, by boundedness of $\Gamma_1^G H(\lambda) = G^*\gamma\left(\bar{\lambda}\right)^*$
$$\widehat{h}_n = \Gamma^G H(\lambda)k_n = \left\{0, G^*\gamma(\bar{\lambda})^*k_n\right\} \to \{0, g\}, \quad g \in \mathcal{H}.$$
Since $\widetilde{\Gamma}$ is closed, it follows that $\left\{\widehat{h}; \{0, g\}\right\} \in \widetilde{\Gamma}$ which shows that $\widehat{h} \in \ker\widetilde{\Gamma}_0$. Hence, $\overline{A_0} \subset \ker\widetilde{\Gamma}_0$ and since $\ker\widetilde{\Gamma}_0$ is symmetric this implies that $\ker\widetilde{\Gamma}_0 = \overline{A_0} = A_0^*$.

Since $\overline{\operatorname{dom}\Gamma^G} = A^*$, the closure $\widetilde{\Gamma}$ has dense domain in A^* (*Assumption 1.9.1*). Clearly, $\operatorname{dom} G^*M(\mu)G = \operatorname{dom}\gamma(\mu)G \subset \operatorname{ran}\Gamma_0^G$ and hence the ranges of Γ_0^G and $\widetilde{\Gamma}_0$ are dense in $\mathcal{H}$ (*Assumption 1.8.1*). Furthermore, $\widetilde{\Gamma}$ as the closure of Γ^G is also isometric, i.e., Green's formula (3.1) holds for $\widetilde{\Gamma}$ (*Assumption 1.5.1*). According to Definition 4.1 this means that $\widetilde{\Gamma}$ is an AB-generalized boundary pair for A^*.

4. *Verification of (2):* The form of the Weyl function $\widetilde{M}(\lambda) = E + M_0(\lambda)$ of $\widetilde{\Gamma}$ is obtained from Theorem 4.2. Furthermore, by Theorem 4.4 $\widetilde{\Gamma}$ is closed if and only if E is closed or, equivalently, $\widetilde{M}(\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, is closed.
5. *Verification of (3):* Since $\Gamma_1^G H(\lambda) = G^*\gamma\left(\bar{\lambda}\right)^*$ and $\gamma_G(\lambda) := \gamma(\lambda)G$ are bounded and densely defined for each $\lambda \in \mathbb{C}\setminus\mathbb{R}$, if follows from (5.42) that $\widehat{\Gamma}^G$ defined as

$$\left\{\left\{\overline{H(\lambda)}k_\lambda, \begin{pmatrix} 0 \\ \overline{G^*\gamma\left(\bar{\lambda}\right)^* k_\lambda}\end{pmatrix}\right\} + \left\{\begin{pmatrix}\overline{\gamma_G(\lambda)}h \\ \lambda\overline{\gamma_G(\lambda)}h\end{pmatrix}, \begin{pmatrix} h \\ \overline{M_G(\lambda)}h\end{pmatrix}\right\} : \begin{matrix} k_\lambda \in \operatorname{ran}\left(\overline{A}_0 - \lambda\right) \\ h \in \operatorname{dom}\overline{M_G(\lambda)}\end{matrix}\right\}$$

satisfies $\widehat{\Gamma}^G \subset \widetilde{\Gamma}$; here $\gamma_G(\lambda)$ and $M_G(\lambda) := G^*M(\lambda)G$ are the γ-field and the Weyl function of Γ^G. Notice that $\overline{A_0} \subset \operatorname{dom}\widehat{\Gamma}^G$ and, as shown above, $\operatorname{ran}\widehat{\Gamma}_0^G \supset \operatorname{dom}\gamma(\mu)G$ is dense in $\mathcal{H}$ (*Assumption 1.5.2-1.5.3*). Due to $\widehat{\Gamma}^G \subset \widetilde{\Gamma}$ also Green's identity (3.1) is satisfied (*Assumption 1.5.1*). Therefore, $\widehat{\Gamma}^G$ is also an AB-generalized boundary pair whose Weyl function is clearly $\overline{M_G(\lambda)}$, which is closed. Now by Theorem 4.4 the AB-generalized boundary pair $\widehat{\Gamma}^G$ is also closed and, since $\widehat{\Gamma}^G \subset \widetilde{\Gamma}$, one has

$$\overline{G^*M(\lambda)G} \subset \widetilde{M}(\lambda) = E + M_0(\lambda), \quad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

Now $\overline{G^*M(\lambda)G}$ as a closed restriction of $E + M_0(\lambda)$ is of the form $\overline{G^*M(\lambda)G} = E_0 + M_0(\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, where E_0 is a closed densely defined restriction of E; cf. Theorem 4.2. This proves the last statement.

□

Theorem 5.32 remains valid for all form domain invariant Nevanlinna functions $M(\cdot) \in \mathcal{R}(\mathcal{H})$ that need not be strict. The only essential difference appearing in the proof of Theorem 5.32 in this case is that $\ker\gamma(\lambda) = \operatorname{mul}\Gamma_0$ (see (5.11)) is nontrivial, and then also, $\ker\gamma_G(\lambda) = \ker\gamma(\lambda)G$ is nontrivial. Notice that even if $\ker\gamma(\lambda) = \{0\}$ (i.e. $M(\cdot) \in \mathcal{R}^s(\mathcal{H})$) then the γ-field $\widetilde{\gamma}(\lambda)$ as well as its closure $\overline{\widetilde{\gamma}(\lambda)} = \overline{\gamma_G(\lambda)}$ can have a nontrivial kernel. This explains why the constructed boundary pair $\widetilde{\Gamma}$ can in general be multi-valued even if the original boundary triple $\Gamma = \{\Gamma_0, \Gamma_1\}$ is single-valued.

Theorem 5.32 combined with the next lemma yields an explicit representation for the class of form domain invariant Nevanlinna functions as well as form domain invariant Nevanlinna families.

Lemma 5.33. *Let G be a bounded operator in the Hilbert space $\mathcal{H}$ with* $\ker G = \ker G^* = \{0\}$, *let H be a closed symmetric densely defined operator on $\mathcal{H}$ and let $M_0(\cdot) \in \mathcal{R}[\mathcal{H}]$. Then the function*

$$M(\lambda) = G^{-*}\big(H + M_0(\lambda)\big)G^{-1}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

is form domain invariant if and only if for some, equivalently for every, $\lambda \in \mathbb{C}\setminus\mathbb{R}$

$$\mathfrak{D}_\lambda := \left\{h \in \mathcal{H} : \ \left(\operatorname{Im} M_0(\lambda)\right)^{\frac{1}{2}}h \in \operatorname{ran} G^*\right\} \quad \textit{is dense in} \quad \mathcal{H}.$$

Proof. To calculate the form $\mathfrak{t}_{M(\lambda)}$ let $\lambda \in \mathbb{C}\setminus\mathbb{R}$ be fixed and let $u, v \in \operatorname{dom} M(\lambda)$. Then $u, v \in \operatorname{dom} G^{-1}$ and hence

$$\begin{aligned}
\mathfrak{t}_{M(\lambda)}[u, v] &= \frac{1}{\lambda - \bar{\lambda}}\left[\left(\big(H + M_0(\lambda)\big)G^{-1}u, G^{-1}v\right) - \left(G^{-1}u, \big(H + M_0(\lambda)\big)G^{-1}v\right)\right] \\
&= \frac{1}{\operatorname{Im}\lambda}\left((\operatorname{Im} M_0)(\lambda)G^{-1}u, G^{-1}v\right) \\
&= \frac{1}{\operatorname{Im}\lambda}\left(\left(\operatorname{Im} M_0(\lambda)\right)^{\frac{1}{2}}G^{-1}u, \left(\operatorname{Im} M_0(\lambda)\right)^{\frac{1}{2}}G^{-1}v\right),
\end{aligned}$$

where symmetry of H has been used. This form is closable precisely when the operator $\left(\operatorname{Im} M_0(\lambda)\right)^{\frac{1}{2}}G^{-1}$ is closable or, equivalently, its adjoint $\left(\left(\operatorname{Im} M_0(\lambda)\right)^{\frac{1}{2}}G^{-1}\right)^* = G^{-*}\left(\operatorname{Im} M_0(\lambda)\right)^{\frac{1}{2}}$ is densely defined. Since $\mathfrak{D}_\lambda = \operatorname{dom} G^{-*}\left(\operatorname{Im} M_0(\lambda)\right)^{\frac{1}{2}}$, the closability of $\mathfrak{t}_{M(\lambda)}$ is equivalent for $\mathfrak{D}_\lambda$ to be dense in $\mathcal{H}$.

To prove that this criterion does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$ consider $M_0(\cdot)$ as the Weyl function of some B-generalized boundary pair $(\mathcal{H}, \Gamma)$. Let $\gamma_0(\cdot)$ be the corresponding γ-field and let $A_0 = \ker \Gamma_0$ be the associated selfadjoint operator. Then the form $\mathfrak{t}_{M(\lambda)}[u, v]$ can be also rewritten in the form

$$\mathfrak{t}_{M(\lambda)}[u, v] = \left(\gamma_0(\lambda) G^{-1} u, \gamma_0(\lambda) G^{-1} v\right)$$

and hence the form $\mathfrak{t}_{M(\lambda)}[u, v]$ is closable if and only if $\gamma_0(\lambda) G^{-1}$ is a closable operator. Now for any $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$ one has

$$\left(I + (\lambda - \mu)\left(A_0 - \lambda\right)^{-1}\right)\gamma_0(\mu) G^{-1} = \gamma_0(\lambda) G^{-1},$$

and since $I + (\lambda - \mu)\left(A_0 - \lambda\right)^{-1}$ bounded with bounded inverse, one concludes that $\gamma_0(\mu) G^{-1}$ is closable exactly when $\gamma_0(\lambda) G^{-1}$ is closable and that the closures are connected by

$$\left(I + (\lambda - \mu)\left(A_0 - \lambda\right)^{-1}\right)\overline{\gamma_0(\mu) G^{-1}} = \overline{\gamma_0(\lambda) G^{-1}}.$$

Therefore, if $\mathfrak{t}_{M(\mu)}[u, v]$ is closable for some $\mu \in \mathbb{C} \setminus \mathbb{R}$ then $\mathfrak{t}_{M(\lambda)}[u, v]$ is closable for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and the form domains of these closures coincide. This completes the proof. □

Proposition 5.34. *Let M be a strict form domain invariant operator valued Nevanlinna function in the Hilbert space $\mathcal{H}$. Then there exist a bounded operator $G \in [\mathcal{H}]$ with* $\ker G = \ker G^* = \{0\}$, *a closed symmetric densely defined operator E in $\mathcal{H}$, and a bounded Nevanlinna function $M_0(\cdot) \in \mathcal{R}[\mathcal{H}]$ with the property*

$$\mathcal{H} = \operatorname{clos} \mathfrak{D}_\lambda := \operatorname{clos}\left\{ h \in \mathcal{H} : \left(\operatorname{Im} M_0(\lambda)\right)^{\frac{1}{2}} h \in \operatorname{ran} G^* \right\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{5.43}$$

such that $M(\cdot)$ admits the representation

$$M(\lambda) = G^{-*}\left(E + M_0(\lambda)\right) G^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{5.44}$$

Conversely, every Nevanlinna function $M(\cdot)$ of the form (5.44) is form domain invariant in $\mathbb{C} \setminus \mathbb{R}$, whenever $E \subset E^*$ and $G \in \mathcal{B}(\mathcal{H})$, $\ker G = \ker G^* = \{0\}$, *and $M_0(\cdot) \in \mathcal{R}[\mathcal{H}]$ satisfy the condition (5.43).*

Proof. Let the Nevanlinna function $M \in \mathcal{R}(\mathcal{H})$ be realized as the Weyl function of some boundary pair $\{\mathcal{H}, \Gamma\}$ (see Theorem 5.24, [25, Theorem 3.9]). Since M is form domain invariant, A_0 is essentially selfadjoint by Theorem 5.24. Since M is an operator valued Nevanlinna function, one can apply Theorem 5.32 (see also the discussion after Theorem 5.32), which shows that the inclusion $G^* M(\lambda) G \subset E + M_0(\lambda)$ holds for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$. This implies that

$$M(\lambda) = G^{-*} G^* M(\lambda) G G^{-1} \subset G^{-*}\left(E + M_0(\lambda)\right) G^{-1}, \tag{5.45}$$

where G is a bounded operator with $\ker G = \ker G^* = \{0\}$ (cf. proof of Theorem 5.32 where $\overline{\operatorname{ran}}\, G = \mathcal{H}$ by construction). Clearly, the function $G^{-*}\left(E + M_0(\lambda)\right) G^{-1}$ is dissipative for $\lambda \in \mathbb{C}_+$ and accumulative for $\lambda \in \mathbb{C}_-$. Since M is Nevanlinna function, it is m-dissipative in $\mathbb{C}_+$ and m-accumulative in $\mathbb{C}_-$. Therefore, the inclusion in (5.45) prevails as an equality. Since $M(\cdot)$ is form domain invariant Lemma 5.33 shows that the condition (5.43) holds for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Conversely, if $M(\cdot)$ is a Nevanlinna function of the form (5.44), where E, G and $M_0(\cdot)$ are as indicated and the condition (5.43) holds for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then by Lemma 5.33 $M(\cdot)$ is form domain invariant and the condition holds for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$. □

Remark 5.35. As to the renormalization in Theorem 5.32 we do not know if the renormalized function $\widetilde{M}(\cdot) = E + M_0(\cdot)$ belongs to the class of Nevanlinna functions.

However, the representation of $M(\cdot)$ in Proposition 5.34 combined with $E \subset E^*$ leads to

$$M(\lambda) = M\left(\bar{\lambda}\right)^* \supset G^{-*}\left(E^* + M_0(\lambda)\right) G^{-1} \supset M(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Hence, $M(\lambda)$ can also be represented with E^* instead of E as follows:

$$M(\lambda) = G^{-*}\left(E^* + M_0(\lambda)\right) G^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

In particular, if $\widetilde{E}$ is any maximal symmetric extension of E then one has also the representation

$$M(\lambda) = G^{-*}\big(\widetilde{E} + M_0(\lambda)\big)G^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Remark 5.36. The result in Proposition 5.34 remains valid also for form domain invariant Nevanlinna families. In this case there exist a bounded operator $G \in [\mathcal{H}]$ with $\ker G = \ker G^* = \operatorname{mul} M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, a closed symmetric densely defined operator E in $\mathcal{H}$, and a Nevanlinna function $M_0(\cdot) \in \mathcal{R}[\mathcal{H}]$ satisfying (5.43), such that

$$M(\lambda) = G^{-*}\big(E + M_0(\lambda)\big)G^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

To see this, decompose $M(\lambda) = \operatorname{gr} M_{\mathrm{op}}(\lambda) + M_\infty$, where $M_\infty = \{0\} \times \operatorname{mul} M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, see (2.3). Now as in the proof of Proposition 5.34 the operator part $M_{\mathrm{op}}(\lambda)$ admits the representation $M_{\mathrm{op}}(\lambda) = G_0^{-*}\big(E + M_0(\lambda)\big)G_0^{-1}$ with some operator $G_0 \in [\mathcal{H}_0]$ in $\mathcal{H}_0 = \mathcal{H} \ominus \operatorname{mul} M(\lambda)$ with $\ker G_0 = \ker G_0^* = \{0\}$. The desired representation of M is obtained by letting G to be the zero continuation of G_0 from $\mathcal{H}_0$ to $\mathcal{H} = \mathcal{H}_0 \oplus \operatorname{mul} M(\lambda)$.

The next example contains a wide class of ES-generalized boundary triples and demonstrates the regularization procedure formulated in Theorem 5.32.

Example 5.37. Let $\Pi^0 = \{\mathcal{H}, \Gamma_0^0, \Gamma_1^0\}$ be an ordinary boundary triple for A^* with $A_0^0 = \ker \Gamma_0^0$, $A_1^0 = \ker \Gamma_1^0$, let $M_0(\cdot)$ and $\gamma_0(\cdot)$ be the corresponding Weyl function and the γ-field, and let $G \in \mathcal{B}(\mathcal{H})$ with $\ker G = \ker G^* = \{0\}$. Then the transform

$$\begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} = \begin{pmatrix} G & 0 \\ 0 & G^{-*} \end{pmatrix} \begin{pmatrix} \Gamma_0^0 \\ \Gamma_1^0 \end{pmatrix}, \tag{5.46}$$

defines an ES-generalized boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^*. Indeed, since $G \in \mathcal{B}(\mathcal{H})$ the transform V in (3.22) is unitary in the Kreĭn space $\{\mathcal{H}^2, J_{\mathcal{H}}\}$ and it follows from [26, Theorem 2.10 (ii)] that the composition $\Gamma = V \circ \Gamma^0$ is unitary. By Lemma 3.12 one has $\ker \Gamma = A$ and, since Γ is unitary, $A_* := \operatorname{dom} \Gamma$ is dense in A^*. Since Π^0 is an ordinary boundary triple, $\mathcal{H} \times \{0\} \subset \operatorname{ran} \Gamma^0$ and hence one concludes from (5.46) that

$$\operatorname{ran} \Gamma_0 = \operatorname{ran} G, \quad A_0 := \ker \Gamma_0 = A_0^0 \cap A_*.$$

Consequently, $\operatorname{ran} \Gamma_0$ is dense in $\mathcal{H}$ and A_0 is essentially selfadjoint. Moreover, $A_1 := \ker \Gamma_1 = \ker \Gamma_1^0 = A_1^0$ and $\operatorname{ran} \Gamma_1 = \operatorname{dom} G^* = \mathcal{H}$: this means that the transposed boundary triple $\{\mathcal{H}, \Gamma_1, -\Gamma_0\}$ is B-generalized. Observe, that A_0 is selfadjoint if and only if $\operatorname{ran} G = \mathcal{H}$ or, equivalently, when Π is an ordinary boundary triple for A^*, too.

Next the form domain of the Weyl function M is calculated. By Lemma 3.12 $M(\cdot) = G^{-*} M_0(\cdot) G^{-1}$ and $\gamma(\cdot) = \gamma_0(\cdot) G^{-1}$. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ be fixed and let $u, v \in \operatorname{dom} M(\lambda)$. Then

$$\begin{aligned}
\mathfrak{t}_{M(\lambda)}[u, v] &= \frac{1}{\lambda - \bar\lambda} \big[\big(G^{-*} M_0(\lambda) G^{-1} u, v\big) - \big(u, G^{-*} M_0(\lambda) G^{-1} v\big)\big] \\
&= \frac{1}{\lambda - \bar\lambda} \big[\big(M_0(\lambda) G^{-1} u, G^{-1} v\big) - \big(M_0(\lambda)^* G^{-1} u, G^{-1} v\big)\big] \\
&= \big(\gamma_0(\lambda) G^{-1} u, \gamma_0(\lambda) G^{-1} v\big).
\end{aligned}$$

Since Π^0 is an ordinary boundary triple, $\gamma_0(\lambda) : \mathcal{H} \to \ker(A^* - \lambda)$ is bounded and surjective, i.e., the inverse of this mapping is also bounded. Hence $\gamma_0(\lambda) G$ is closed, when considered on its natural domain $\operatorname{dom} \gamma_0(\lambda) G^{-1} = \operatorname{ran} G\, (\supset \operatorname{dom} M(\lambda))$. Therefore, the closure of the form $\mathfrak{t}_{M(\lambda)}$ is given by

$$\overline{\mathfrak{t}_{M(\lambda)}}[u, v] = \big(\gamma(\lambda) G^{-1} u, \gamma(\lambda) G^{-1} v\big), \quad u, v \in \operatorname{ran} G.$$

In particular, $M(\lambda)$ is a form domain invariant Nevanlinna function whose form domain is equal to $\operatorname{ran} G$. Since G is bounded, one can use G to produce a regularized function $\widetilde{M}$:

$$\widetilde{M} = G^* M G = G^*\big(G^{-*} M_0(\cdot) G^{-1}\big) G = M_0(\lambda),$$

so that $\widetilde{M}$ coincides with the Nevanlinna function $M_0(\cdot)$ which belongs to the class $\mathcal{R}^u[\mathcal{H}]$.

It is emphasized that when G is not surjective, the form domain invariant function $M(\cdot) = G^{-*}M_0(\cdot)G^{-1}$ need not be domain invariant. In fact, in [28] an example of a form domain invariant Nevanlinna function M was given, such that

$$\operatorname{dom} M(\lambda) \cap \operatorname{dom} M(\mu) = \{0\}, \quad \lambda \neq \mu \quad (\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}),$$

and the corresponding regularized function $\widetilde{M}$ therein still belongs to the class $\mathcal{R}^u[\mathcal{H}]$.

In Example 5.37 the boundary triple Π is ES-generalized while the transposed boundary triple $\Pi^\top := \{\mathcal{H}, \Gamma_1, -\Gamma_0\}$ is B-generalized. Therefore, according to [27, Theorem 7.24] there exist an ordinary boundary triple $\widetilde{\Pi}^0$ and operators $R = R^*$, $K \in \mathcal{B}(\mathcal{H})$, $\ker K = \ker K^* = \{0\}$, such that $\Pi^\top$ is the transform (1.6) of $\widetilde{\Pi}^0$. Recall that one can take e.g. $R = \mathrm{Re}\big(-M(i)^{-1}\big)$, $K = \big(\mathrm{Im}\,\big(-M(i)^{-1}\big)\big)^{1/2}$. In particular, this yields the following connections between the associated Weyl functions:

$$-M^{-1}(\cdot) = K^*\widetilde{M}_0(\cdot)K + R.$$

In particular, with $R = 0$ one obtains $M(\cdot) = K^{-1}\big(-\widetilde{M}_0(\cdot)^{-1}\big)K^{-*}$ and here $-\widetilde{M}_0(\cdot)^{-1} \in \mathcal{R}^u[\mathcal{H}]$.

Together with Example 5.37 this characterizes those ES-generalized boundary triples Π for A^* whose transposed boundary triple $\Pi^\top$ is B-generalized.

Recall that Weyl functions of S-generalized boundary pairs are domain invariant, but converse does not hold (explicit examples can be found in Part II). As shown in the next proposition a domain invariant Nevanlinna function can always be renormalized by means of a fixed bounded operator to a Nevanlinna function belonging to the class $\mathcal{R}[\mathcal{H}]$.

Proposition 5.38. *Let $M(\cdot)$ be a domain invariant operator valued Nevanlinna function in the Hilbert space $\mathcal{H}$. Moreover, let G with $\ker G = \ker G^* = \{0\}$ be a bounded operator in $\mathcal{H}$ such that $\operatorname{ran} G = \operatorname{dom} M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then the renormalized function*

$$M_G(\lambda) = G^*M(\lambda)G, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{5.47}$$

is a Nevanlinna function in the class $\mathcal{R}[\mathcal{H}]$. Moreover, $M_G(\cdot) \in \mathcal{R}^s[\mathcal{H}]$ precisely when $M(\cdot) \in \mathcal{R}^s(\mathcal{H})$.

Proof. By assumptions the equality $\operatorname{dom} G^*M(\lambda)G = \operatorname{dom} M(\lambda)G = \mathcal{H}$ holds for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Consequently, the adjoint $M_G(\lambda)^*$ is a closed operator and in view of

$$M_G(\lambda)^* = (G^*M(\lambda)G)^* \supset G^*M\big(\bar{\lambda}\big)G$$

one has $\operatorname{dom} M_G(\lambda)^* = \mathcal{H}$. Therefore, the equality $M_G(\lambda)^* = G^*M\big(\bar{\lambda}\big)G$ holds for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Now clearly $\mathrm{Im}\, M_G(\lambda) = G^*\mathrm{Im}\, M(\lambda)G$, which implies that $M_G \in \mathcal{R}[\mathcal{H}]$ and also proves the last statement. ☐

The assumption $\operatorname{ran} G = \operatorname{dom} M(\lambda)$ in Proposition 5.38 (or more generally the inclusion $\operatorname{dom} M(\lambda) \subset \operatorname{ran} G$) guarantees that $M(\cdot)$ can be recovered from $M_G(\cdot)$ in (5.47) similarly as was done in Proposition 5.34:

$$M(\lambda) = G^{-*}G^*M(\lambda)GG^{-1} = G^{-*}M_G(\lambda)G^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

5.6 | An example on renormalization

The following example demonstrates renormalization of an unbounded form domain invariant Nevanlinna function. In this example the real part of $M(i)$ is strongly subordinated with respect to its imaginary part. In this case the renormalized function $\widetilde{M}(\cdot)$ is a Nevanlinna function in the class $\mathcal{R}[\mathcal{H}]$.

Example 5.39. Let S be a positively definite closed symmetric operator in $\mathcal{H}$, so that $S \geq \varepsilon I$. Let

$$M(z) = zS^*S + S, \quad \operatorname{dom} M(z) = \operatorname{dom} S^*S, \quad z \in \mathbb{C}.$$

Replacing if necessary S by $S + aI$ we can assume that $\varepsilon > 1$. First notice that

$$\|f\|^2 \leq \varepsilon^{-2}\|Sf\|^2 = \varepsilon^{-2}(S^*Sf, f) \leq \varepsilon^{-2}\|S^*Sf\| \cdot \|f\|, \quad f \in \operatorname{dom} S^*S,$$

i.e. $\|S^*Sf\| \geq \varepsilon^2\|f\|$. It follows that S is strongly subordinated with respect to S^*S, i.e.

$$\|Sf\|^2 = (Sf, Sf) = (S^*Sf, f) \leq \|S^*Sf\| \cdot \|f\| \leq \varepsilon^{-2}\|S^*Sf\|^2, \quad f \in \operatorname{dom} S^*S.$$

Since dom $S^*S \subset$ dom $S \subset$ dom S^*, one easily proves that S^* is also strongly subordinated with respect to S^*S. Now, these inequalities imply that both operators S/z and S^*/z are also strongly subordinated to S^*S for $|z| \geq 1$. Therefore,

$$M(\overline{z})^* = (\overline{z}S^*S + S)^* = zS^*S + S^* = zS^*S + S = M(z).$$

Since $M(\cdot)$ is dissipative in $\mathbb{C}_+$, it follows that $M(z)$ is m-dissipative for $z \in \mathbb{C}_+, |z| \geq 1$, and m-accumulative for $z \in \mathbb{C}_-$, $|z| \geq 1$. In turn, the latter implies that $M(z)$ being holomorphic and dissipative is m-dissipative for each $z \in \mathbb{C}_+$. Summing up we conclude that $M(\cdot)$ is an entire Nevanlinna function with values in $\mathcal{C}(\mathcal{H})$.

Furthermore,

$$\mathfrak{t}_{M(z)}(f,g) = \frac{(M(z)f,g) - (f,M(z)g)}{z-\bar{z}} = (Sf,Sg), \quad f,g \in \text{dom}\, S^*S, \quad z \in \mathbb{C}.$$

The form is closable because so is the operator S. Taking the closure we obtain the closed form $\bar{\mathfrak{t}}_{M(z)}(f,g) = (Sf,Sg)$, $f,g \in \text{dom}\, S,\ z \in \mathbb{C}$, with constant domain. In other words, $M(\cdot)$ is a form domain invariant Nevanlinna function and the (selfadjoint) operator associated with $\bar{\mathfrak{t}}_{M(z)}$ in accordance with the second representation theorem is $(S^*S)^{1/2}$.

Now consider the renormalization of $M(\cdot)$ as in Theorem 5.32. The operator $G = (S^*S)^{-\frac{1}{2}}$ is bounded and $\text{ran}\, G = \text{dom}\, \bar{\mathfrak{t}}_{M(z)}$. Moreover, $G^*(S^*S)G = I \upharpoonright \text{dom}\,(S^*S)^{\frac{1}{2}}$ and $G^*SG = G^*U$, where $U : \overline{\text{ran}}\,(S^*S)^{\frac{1}{2}} = \mathcal{H} \to \overline{\text{ran}}\, S$ is the (partial) isometry from the polar decomposition $S = U(S^*S)^{\frac{1}{2}}$. Consequently, $C := G^*SG$ is a bounded selfadjoint operator in $\mathcal{H}$. By Theorem 5.32 one has $\widetilde{M}(z) \supset \text{clos}(G^*M(z)G) = zI + C$. Thus, $\widetilde{M}(z) = zI + C$ is a Nevanlinna function in the class $\mathcal{R}[\mathcal{H}]$.

Some modifications of this example can be found in [29].

ACKNOWLEDGEMENTS

The research was partially supported by a grant from the Vilho, Yrjö and Kalle Väisälä Foundation of the Finnish Academy of Science and Letters. V. D. and M. M. gratefully also acknowledge support by the University of Vaasa. The research of V. D. was also supported by Ministry of Education and Science of Ukraine (projects 0118U003138, 0118U00206), a Volkswagen Foundation grant and a Fulbright grant. The research of M. M. has been prepared with the support of the "RUDN University Program 5-100".

REFERENCES

[1] N. I. Achieser and I. M. Glasmann, *Theorie der linearen Operatoren im Hilbertraum*, 8th edition, Akademie Verlag, Berlin, 1981 (German).
[2] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, *Solvable models in quantum mechanics*, Second Edition, AMS Chelsea Publ., 2005.
[3] S. Albeverio and P. Kurasov, *Singular perturbations of differential operators and Schrödinger type operators*, Cambridge Univ. Press, 2000.
[4] R. Arens, *Operational calculus of linear relations*, Pacific J. Math. **11** (1961), 9–23.
[5] D. Z. Arov and O. J. Staffans, *State/signal linear time-invariant systems theory. I. Discrete time systems*, The State Space Method Generalizations and Applications, Oper. Theory Adv. Appl., vol. 161, Birkhäuser, Basel, 2006, pp. 115–177.
[6] D. Z. Arov, M. Kurula, and O. J. Staffans, *Passive state/signal systems and conservative boundary relations*, London Math. Soc. Lecture Note Ser. **404** (2012), 87–119.
[7] T. Ya. Azizov and I. S. Iokhvidov, *Linear operators in spaces with indefinite metric*, John Wiley and Sons, New York, 1989.
[8] J. A. Ball, M. Kurula, O. J. Staffans, and H. Zwart, *De Branges–Rovnyak realizations of operator valued Schur functions on the complex right half-plane*, Complex Anal. Oper. Theory **9** (2015), 723–792.
[9] J. A. Ball and O. J. Staffans, *Conservative state-space realizations of dissipative system behaviors*, Integral Equations Operator Theory **54** (2006), 151–213.
[10] J. Behrndt, S. Hassi, and H. S. V. de Snoo, *Boundary relations, unitary colligations, and functional models*, Complex Anal. Oper. Theory **3** (2009), 57–98.
[11] J. Behrndt and M. Langer, *Boundary value problems for elliptic partial differential operators on bounded domains*, J. Funct. Anal. **243** (2007), 536–565.
[12] J. Behrndt and M. Langer, *Elliptic operators, Dirichlet-to-Neumann maps and quasi boundary triples*, London Math. Soc. Lecture Note Ser. **404** (2012), 121–160.
[13] J. Behrndt, M. Malamud, and H. Neidhardt, *Scattering matrices and Dirichlet-to-Neumann maps*, J. Funct. Anal. **273** (2017), 1970–2025.
[14] J. Behrndt and T. Micheler, *Elliptic differential operators on Lipschitz domains and abstract boundary value problems*, J. Funct. Anal. **267** (2014), 3657–3709.

[15] J. Behrndt, F. Gesztesy, T. Micheler, and M. Mitrea, *The Kreĭn–von Neumann realization of perturbed laplacians on bounded lipschitz domains*, Oper. Theory Adv. Appl., vol. 255, Birkhäuser/Springer, Cham, 2016, pp. 49–66.
[16] C. Bennewitz, *Symmetric relations on a Hilbert space*, Lect. Notes in Math. **280** (1972), 212–218.
[17] J. Bognar, *Indefinite inner product spaces*, Springer-Verlag, Berlin–Heidelberg–New York, 1974.
[18] M. S. Brodskii, *Unitary operator colligations and their characteristic functions*, Russian Math. Surveys **33** (1978), no. 4, 159–191, Uspekhi Mat. Nauk. 33:4 (202) (1978), 141–168.
[19] V. M. Bruk, *On a class of problems with the spectral parameter in the boundary conditions*, Mat. Sb. **100** (1976), 210–216.
[20] J. Bruning, V. Geyler, and K. Pankrashkin, *Spectra of self-adjoint extensions and applications to solvable Schrodinger operators*, Rev. Math. Phys. **20** (2008), 1–70.
[21] J. W. Calkin, *Abstract symmetric boundary conditions*, Trans. Amer. Math. Soc. **45** (1939), 369–442.
[22] E. A. Coddington, *Extension theory of formally normal and symmetric subspaces*, Mem. Amer. Math. Soc. **134** (1973), 1–80.
[23] N. Danford and J. T. Schwartz, *Linear operators*, Vol. III, Wiley, New York, 1971.
[24] V. A. Derkach, S. Hassi, M. M. Malamud, and H. S. V. de Snoo, *Generalized resolvents of symmetric operators and admissibility*, Methods Funct. Anal. Topology **6** (2000), 24–55.
[25] V. A. Derkach, S. Hassi, M. M. Malamud, and H. S. V. de Snoo, *Boundary relations and Weyl families*, Trans. Amer. Math. Soc. **358** (2006), 5351–5400.
[26] V. A. Derkach, S. Hassi, M. M. Malamud, and H. S. V. de Snoo, *Boundary relations and generalized resolvents of symmetric operators*, Russ. J. Math. Phys. **16** (2009), 17–60.
[27] V. A. Derkach, S. Hassi, M. M. Malamud, and H. S. V. de Snoo, *Boundary triples and Weyl functions. Recent developments*, London Math. Soc. Lecture Note Ser. **404** (2012), 161–220.
[28] V. A. Derkach, S. Hassi, and M. M. Malamud, *Invariance theorems for Nevanlinna families*, arXiv:1503.05606 (2015).
[29] V. A. Derkach, S. Hassi, and M. M. Malamud, *Generalized boundary triples, Weyl functions and inverse problems*, arXiv:1706.07948 (2017), 104 pages.
[30] V. A. Derkach and M. M. Malamud, *Weyl function of Hermitian operator and its connection with characteristic function*, Preprint 85-9 (104) Donetsk Fiz-Techn. Institute AN Ukrain. SSR, 1985 (Russian).
[31] V. A. Derkach and M. M. Malamud, *On Weyl function and Hermitian operators with gaps*, Doklady Akad. Nauk SSSR **293** (1987), no. 5, 1041–1046.
[32] V. A. Derkach and M. M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. **95** (1991), 1–95.
[33] V. A. Derkach and M. M. Malamud, *The extension theory of Hermitian operators and the moment problem*, J. Math. Sci. **73** (1995), 141–242.
[34] P. Exner, *Seize ans après*, Appendix K to "Solvable Models in Quantum Mechanics" by S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden, Sec. Edition, AMS Chelsea Publ., 2004.
[35] P. A. Fillmore and J. P. Williams, *On operator ranges*, Adv. Math. **7** (1971), 254–281.
[36] M. L. Gorbachuk, *Self-adjoint boundary value problems for differential equation of the secoind order with unbounded operator coefficient*, Funct. Anal. Appl. **5** (1971), no. 1, 10–21.
[37] V. I. Gorbachuk and M. L. Gorbachuk, *Boundary problems for differential operator equations*, Naukova Dumka, Kiev, 1984 (Russian).
[38] G. Grubb, *A characterization of the non-local boundary value problems associated with an elliptic operator*, Ann. Sci. Norm. Sup. Pisa Cl. Sci. (3) **22** (1968), no. 3, 425–513.
[39] G. Grubb, *Distributions and operators*, Grad. Texts in Math., vol. 552, Springer, New York, 2009.
[40] S. Hassi, H. S. V. de Snoo, and F. H. Szafraniec, *Operator methods for boundary value problems*, London Math. Soc. Lecture Note Ser. **404**, (2012).
[41] S. Hassi, A. J. van der Schaft, H. S. V. de Snoo, and H. J. Zwart, *Dirac structures and boundary relations*, London Math. Soc. Lecture Note Ser. **404** (2012), 259–274.
[42] S. Hassi and H. L. Wietsma, *On Calkin's abstract symmetric boundary conditions*, London Math. Soc. Lecture Note Ser. **404** (2012), 3–34.
[43] R. Juršėnas, *Weyl families of essentially unitary pairs*, arXiv:1812.07470 (2018).
[44] I. S. Kac and M. G. Kreĭn, *R-functions – analytic functions mapping the upper halfplane into itself.* Supplement to the Russian edition of F. V. Atkinson, *Discrete and continuous boundary problems*, Mir, Moscow, 1968 (Russian); (English transl. in: Amer. Math. Soc. Transl. Ser. 2, **103** (1974), 1–18).
[45] T. Kato, *Perturbation theory for linear operators*, Springer Verlag, Berlin–Heidelberg–New York, 1966.
[46] A. N. Kochubei, *On extentions of symmetric operators and symmetric binary relations*, Mat. Zametki (Berlin) **17** (1975), no. 1, 41–48.
[47] V. Koshmanenko, *Singular quadratic forms in perturbation theory*, Math. Appl. (Berlin), vol. 474, Kluwer Academic Publishers, Dordrecht, 1999.
[48] A. S. Kostenko and M. M. Malamud, *1–D Schrödinger operators with local point interactions on a discrete set*, J. Differential Equations **249** (2010), 253–304.
[49] A. S. Kostenko and M. M. Malamud, *1–D Schrödinger operators with local point interactions: a review*. Spectral Analysis, Integrable Systems, and Ordinary Differential Equations (H. Holden et al., eds.), Proc. Symp. Pure Math., vol. 87, Amer. Math. Soc., Providence, 2013, pp. 235–262.
[50] M. G. Kreĭn, *Theory of self-adjoint extensions of semibounded hermitian operators and applications, II*, Mat. Sb. **21** (1947), no. 3, 365–404.
[51] M. G. Kreĭn and H. Langer, *On defect subspaces and generalized resolvents of Hermitian operator in Pontryagin space*, Funkt. Anal. i Prilozhen. **5** (1971), no. 2, 59–71; ibid. **5** no. 3 (1971), 54–69 (Russian) (English transl. in Funct. Anal. Appl. **5** (1971), 136–146; ibid. **5** (1971), 217–228).

[52] M. G. Kreĭn and H. Langer, *Über die Q-funktion eines π-hermiteschen Operators in Raume* Π_κ, Acta. Sci. Math. (Szeged) **34** (1973), 191–230 (German).
[53] M. Kurula, *On passive and conservative state/signal systems in continuous time*, Integral Equation Operator Theory **67** (2010), 377–424.
[54] M. Kurula and O. Staffans, *Well-posed state/signal systems in continuous time*, Complex Anal. Oper. Theory **4** (2009), 319–390.
[55] H. Langer and B. Textorius, *On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space*, Pacific J. Math. **72** (1977), 135–165.
[56] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, Vol. 1, Springer, Berlin, 1972.
[57] M. S. Livshits and A. A. Yantsevich, *Operator colligations in Hilbert spaces*. Edited and with a foreword by Ronald G. Douglas. Translated from the Russian. V. H. Winston & Sons, Washington, D. C.; A Halsted Press Book, John Wiley & Sons, New York–Toronto, Ont.–London, 1979.
[58] M. M. Malamud, *On the formula of generalized resolvents of a nondensely defined Hermitian operator*, Ukrain. Mat. Zh. **44** (1992), No. 2, 1658–1688.
[59] J. Malinen and O. J. Staffans, *Conservative boundary control systems*, J. Differential Equations **231** (2006), no. 1, 290–312.
[60] V. Mogilevskii, *Boundary triplets and Kreĭn type resolvent formula for symmetric operators with unequal defect numbers*, Methods Funct. Anal. Topology **12** (2006), 258–280.
[61] J. von Neumann, *Über adjungierte Operatoren*, Anal. Math. **33** (1932), no. 2, 294–310 (German).
[62] O. Post, *Boundary pairs associated with quadratic forms*, Math. Nachr. **289** (2016), no. 8–9, 1052–1099.
[63] F. S. Rofe-Beketov, *On selfadjoint extensions of differential operators in a space of vector-functions*, Teor. Funktsiĭ. Funktsional. Anal. i Prilozhen **8** (1969), 3–24.
[64] Yu. L. Shmul'jan, *Theory of linear relations, and spaces with indefinite metric*, Funksional. Anal. i Prilozhen. **10** (1976), no. 1, 67–72 (Russian).
[65] O. J. Staffans, *Well-Posed linear systems*, Cambridge University Press, Cambridge and New York, 2005.
[66] O. J. Staffans and G. Weiss, *A physically motivated class of scattering passive linear systems*, SIAM J. Control Optim. **50** (2012), no. 5, 3083–3112.
[67] M. I. Višik, *On general boundary problems for elliptic differential equations*, Trudy Moskov. Mat. Obshch. **1** (1952), 187–246.
[68] H. L. Wietsma, *Representations of unitary relations between Kreĭn spaces*, Integral Equation Operator Theory **72** (2012), 309–344.
[69] H. L. Wietsma, On unitary relations between Kreĭn spaces, PhD Thesis, Acta Wasaensia 263, University of Vaasa, 2012. available at (http://www.uva.fi/en/research/publications/orders/database/?julkaisu=685)
[70] H. L. Wietsma, *Block representations for classes of isometric operators between Kreĭn spaces*, Oper. Matrices **7** (2013), 651–685.

How to cite this article: Derkach V, Hassi S, Malamud M. Generalized boundary triples, I. Some classes of isometric and unitary boundary pairs and realization problems for subclasses of Nevanlinna functions. *Mathematische Nachrichten*. 2020;293:1278–1327. https://doi.org/10.1002/mana.201800300