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**Title:** The Valuation of European Option Under Subdiffusive Fractional Brownian Motion of the Short Rate

**Year:** 2020

**Version:** Accepted manuscript

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<https://www.worldscientific.com/worldscinet/ijtaf>. Electronic version of an article published as *International Journal of Theoretical and Applied Finance* 23(4), 1-16.  
<https://doi.org/10.1142/S0219024920500223>

**Please cite the original version:**

Shokrollahi, F. (2020). The Valuation of European Option Under Subdiffusive Fractional Brownian Motion of the Short Rate. *International Journal of Theoretical and Applied Finance* 23(4), 1-16.  
<https://doi.org/10.1142/S0219024920500223>

# THE VALUATION OF EUROPEAN OPTION UNDER SUBDIFFUSIVE FRACTIONAL BROWNIAN MOTION MECHANISM OF THE SHORT RATE

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ABSTRACT. In this paper we propose an extension of the Merton model. We apply the subdiffusive mechanism to analyze European option in a fractional Black-Scholes environment, when the short rate follows the subdiffusive fractional Black-Scholes model. We derive a pricing formula for call and put options and discuss the corresponding fractional Black-Scholes equation. We present some features of our model pricing model for the cases of  $\alpha$  and  $H$ .

## 1. INTRODUCTION

The pioneer study of the option pricing was introduced by Black-Scholes [1] in 1973. In the Black-Scholes (*BS*) model has been assumed that the underlying assets follows a geometric Brownian motion. While, there exist a series of evidence which show the *BS* model unable to cover substantial behavior from financial markets such as: long-range dependence, heavy-tailed and periods of constant values. Hence, they proposed various modifications of the *BS* model to capture these shortcomings.

One of well developed modifications of the *BS* model is the fractional Black-Scholes model which, describes long-range dependence and self-similarity from financial data. In the fractional Black-Scholes (*FBS*) model, the Brownian motion is substituted with the fractional Brownian motion (*FBM*) in the *BS* model. For more details about fractional Black-Scholes model, you can see [16, 14, 2, 13].

Furthermore, analysis of financial data displays that various processes viewed in finance show special terms in which they are constant [8]. The same property is observed in physical system with subdiffusion. The fixed terms of financial processes according to the trapping event in which the subdiffusive examination particle is constant [4]. The mathematical interpretation of subdiffusion is in terms of Fractional Fokker Planck equation (*FFPE*). This equation was introduced from the continuous time random walk strategy with fat tail waiting times [12], later used as a substantial tool to evaluate complex system with slow dynamics. In this paper, we use the *FBS* model in subdiffusive mechanism to better describe

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*Date:* April 15, 2019.

*2010 Mathematics Subject Classification.* 91G20; 91G80; 60G22.

*Key words and phrases.* Merton short rate model; Subdiffusive processes; Fractional Brownian motion; Option pricing.

behaviour from financial markets. We use the same strategy in [11, 15], which the objective time  $t$  is replaced by the inverse  $\alpha$ -stable subordinator  $T_\alpha(t)$  in the *FBS* model. Then, the dynamic of asset price is given by the following subdiffusive *FBS*

$$(1.1) \quad \begin{aligned} dS_\alpha(t) &= \\ dS(T_\alpha(t)) &= \mu_s S(T_\alpha(t)) d(T_\alpha(t)) + \sigma_s S(T_\alpha(t)) dB_1^H(T_\alpha(t)), \end{aligned}$$

where  $\mu_s, \sigma_s$  are constant,  $B_1^H$  is *FBM* with Hurst parameter  $H \in [\frac{1}{2}, 1)$ .  $T_\alpha(t)$  is the inverse  $\alpha$ -stable subordinator with  $\alpha \in (0, 1)$  defined as follows

$$(1.2) \quad T_\alpha(t) = \inf\{\tau > 0 : U_\alpha(\tau) > t\},$$

$T_\alpha(t)$  is assumed to be independent of  $B_1^H$ .  $\{U_\alpha(t)\}_{t \geq 0}$  is a  $\alpha$ -stable Levy process with nonnegative increments and Laplace transform:  $E(e^{-uU_\alpha(t)}) = e^{-tu^\alpha}$  [5, 17, 7]. when  $\alpha \uparrow 1$ , the  $T_\alpha(t)$  degenerates to  $t$ .

On the other hand, all above studies have assumed that the short rate is constant during the life of an option. However, in reality the short rate is evolving randomly over time. Hence, in order to take into account the stochastic short rate, we assume that the short rate  $r(t) = \hat{S}(T_\alpha(t))$  follows:

$$(1.3) \quad \begin{aligned} d\hat{S}_\alpha(t) &= \\ d\hat{S}(T_\alpha(t)) &= \mu_r dT_\alpha(t) + \sigma_r dB_2^H(T_\alpha(t)), \end{aligned}$$

here  $\mu_r, \sigma_r$  are constant,  $B_2^H$  is *FBM* with Hurst parameter  $H \in [\frac{1}{2}, 1)$  and  $T_\alpha(t)$  is assumed to be independent of  $B_2^H$ .

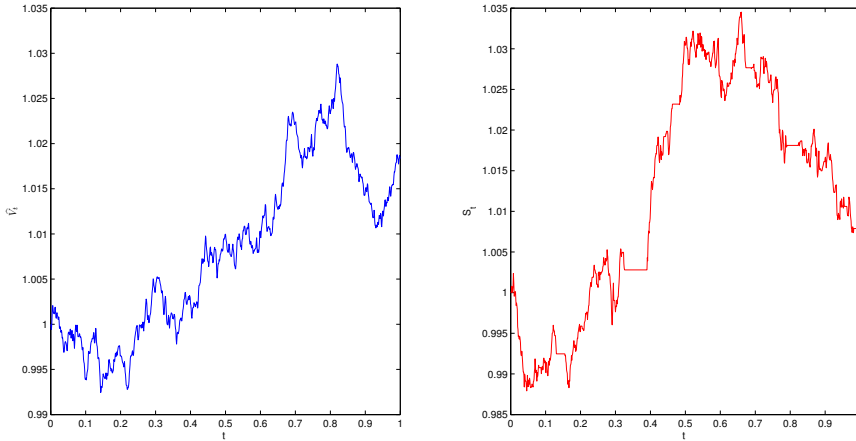


FIGURE 1. discrepancy and relation between the sample paths of the stock price in the *FBS* model (left) and the subdiffusive *FBS* model (right) for  $r = 0.01, \alpha = 0.9, H = 0.8, \sigma = 0.1, S_0 = 1$ .

The first contribution of this paper is to propose a valuation model to price a zero-coupon bond by applying the subdiffusive mechanism of the short rate. The

second contribution is to value an European option when the asset price and short rate are follow subdiffusive *FBS* model. This paper is organized as follows. In the next section, we derive a new model to value a riskless zero-coupon bond paying \$1 at maturity. In Section 3, we obtain the corresponding *FBS* equation by using delta hedging argument and discuss some special cases of this equation. In Section 4, we propose a pricing model for the European call and put options. Some particular features and simulation studies of our sudiffusive model are discussed in Section 5. Section 6 concludes this research.

## 2. PRICING MODEL FOR A ZERO-COUPON BOND

We assume that the short rate  $r(t)$  satisfy Equation (1.3),  $\alpha \in (\frac{1}{2}, 1)$  and  $2\alpha - \alpha H > 1$ , then by using the Taylor series expansion to  $P(r, t, T)$ , we obtain

$$(2.1) \quad \begin{aligned} P(r + \Delta r, t + \Delta t) &= P(r, t, T) + \frac{\partial P}{\partial r} \Delta r + \frac{\partial P}{\partial t} \Delta t \\ &+ \frac{1}{2} \frac{\partial^2 P}{\partial r^2} (\Delta r)^2 + \frac{1}{2} \frac{\partial^2 P}{\partial r \partial t} \Delta r (\Delta t) + \frac{1}{2} \frac{\partial^2 P}{\partial t^2} (\Delta t)^2 + O(\Delta t). \end{aligned}$$

From, Equation (1.3) and [17], we have

$$(2.2) \quad \begin{aligned} \Delta r &= \mu_r (\Delta T_\alpha(t)) + \sigma_r B_1^H(T_\alpha(t)) \\ &= \mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} (\Delta t)^{2H} + \sigma_r \Delta B_1^H(T_\alpha(t)) + O((\Delta t)^{2H}). \end{aligned}$$

$$(2.3) \quad (\Delta r)^2 = \sigma_r^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} (\Delta t)^{2H} + O((\Delta t)^{2H}).$$

$$(2.4) \quad \Delta r (\Delta t) = O((\Delta t)^{2H}).$$

Then from the Lemma 1 in [17], we can get

$$(2.5) \quad \begin{aligned} dP(r, t, T) &= \left[ \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \left( \mu_r \frac{\partial P}{\partial r} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 P}{\partial r^2} \right) 2Ht^{2H-1} + \frac{\partial P}{\partial t} \right] dt \\ &+ \sigma_r \frac{\partial P}{\partial t} dB_1^H(T_\alpha(t)). \end{aligned}$$

Assuming

$$(2.6) \quad \begin{aligned} \mu &= \frac{1}{P} \left[ \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \left( \mu_r \frac{\partial P}{\partial r} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 P}{\partial r^2} \right) 2Ht^{2H-1} + \frac{\partial P}{\partial t} \right], \\ \sigma &= \frac{1}{P} \left( \frac{\partial P}{\partial r} \right), \end{aligned}$$

and letting the local expectations hypothesis holds for the term structure of interest rates (i.e.  $\mu = r$ ), we have

$$(2.7) \quad \begin{aligned} & \frac{\partial P}{\partial t} + 2Ht^{2H-1}\mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial P}{\partial r} \\ & + Ht^{2H-1}\sigma_r^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 P}{\partial r^2} - rP = 0. \end{aligned}$$

Then, zero-coupon bond  $P(r, t, T)$  with boundary condition  $P(r, t, T) = 1$  satisfy the following partial differential equation

$$(2.8) \quad \begin{aligned} & \frac{\partial P}{\partial t} + 2Ht^{2H-1}\mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial P}{\partial r} \\ & + Ht^{2H-1}\sigma_r^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 P}{\partial r^2} - rP = 0. \end{aligned}$$

To solve Equation (2.8) for  $P(r, t, T)$ , let  $\tau = T - t$ ,  $P(r, t, T) = \exp\{f_1(\tau) - rf_2(\tau)\}$ , then we can get

$$(2.9) \quad \frac{\partial P}{\partial t} = P \left( -\frac{\partial f_1(\tau)}{\partial t} + r \frac{\partial f_2(\tau)}{\partial t} \right),$$

$$(2.10) \quad \frac{\partial P}{\partial r} = -P f_2(\tau),$$

$$(2.11) \quad \frac{\partial^2 P}{\partial r^2} = P f_2(\tau)^2.$$

Replacing Equations (2.10) and (2.11) into Equation (2.9) and simplifying Equation (2.8) becomes

$$(2.12) \quad \begin{aligned} & P \left[ Ht^{2H-1}\sigma_r^2 f_2(\tau)^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} - 2Ht^{2H-1}\mu_r f_2(\tau) \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \right. \\ & \left. - \frac{\partial f_1(\tau)}{\partial \tau} + r \left( \frac{\partial f_2(\tau)}{\partial t} - 1 \right) \right] = 0. \end{aligned}$$

From Equation (2.12), we obtain

$$(2.13) \quad \begin{aligned} \frac{\partial f_1(\tau)}{\partial \tau} &= Ht^{2H-1} \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} (\sigma_r^2 f_2(\tau)^2 - 2\mu_r f_2(\tau)), \\ \frac{\partial f_2(\tau)}{\partial \tau} &= 1. \end{aligned}$$

Then,

$$(2.14) \quad \begin{aligned} f_1(\tau) &= \frac{H\sigma_r^2}{(\Gamma(\alpha))^{2H}} \int_0^\tau (T-s)^{(\alpha-1)2H+2H-1} s^2 ds \\ &\quad - \frac{2H\mu_r}{(\Gamma(\alpha))^{2H}} \int_0^\tau (T-s)^{(\alpha-1)2H+2H-1} s ds, \end{aligned}$$

$$(2.15) \quad f_2(\tau) = \tau.$$

Therefore, we derive a pricing model for a riskless zero-coupon bond.

$$(2.16) \quad P(r, t, T) = e^{-r\tau + f_1(\tau)}.$$

**Corollary 2.1.** *When  $\alpha \uparrow 1$ , Equations (1.3) and (1.1) reduce to the FBM, we obtain*

$$(2.17) \quad f_1(\tau) = H\sigma_r^2 \int_0^\tau (T-s)^{2H-1} s^2 ds - 2H\mu_r \int_0^\tau (T-s)^{2H-1} s ds,$$

specially, if  $t = 0$

$$(2.18) \quad f_1(\tau) = \sigma_r^2 \frac{T^{2H+2}}{(2H+1)(2H+2)} - \mu_r \frac{T^{2H+1}}{2H+1},$$

then

$$(2.19) \quad P(r, t, T) = \exp \left\{ -rT + \sigma_r^2 \frac{T^{2H+2}}{(2H+1)(2H+2)} - \mu_r \frac{T^{2H+1}}{2H+1} \right\}.$$

**Corollary 2.2.** *If  $H = \frac{1}{2}$ , from Equation (2.14), we obtain*

$$(2.20) \quad f_1(\tau) = \frac{1}{2} \frac{\sigma_r^2}{\Gamma(\alpha)} \int_0^\tau (T-s)^{\alpha-1} s^2 ds - \frac{\mu_r}{\Gamma(\alpha)} \int_0^\tau (T-s)^{\alpha-1} s ds,$$

then the result is consistent with the result in [6].

Further, if  $\alpha \uparrow 1$  and  $H = \frac{1}{2}$ , Equations (1.3) and (1.1) reduce to the geometric Brownian motion, then we have

$$(2.21) \quad f_1(\tau) = \frac{1}{6} \sigma_r^2 \tau^3 - \frac{1}{2} \mu_r \tau^2,$$

then

$$(2.22) \quad P(r, t, T) = e^{-r\tau + \frac{1}{6} \sigma_r^2 \tau^3 - \frac{1}{2} \mu_r \tau^2}.$$

which is consistent with the result in [9, 3].

### 3. FRACTIONAL BLACK-SCHOLES EQUATION

This section provides corresponding *FBS* equation for European options when the short rate and stock price satisfy Equations (1.3) and (1.1), respectively, here  $B_1^H$  and  $B_2^H$  are two dependent *FBM* with Hurst parameter  $H \in [\frac{1}{2}, 1)$  and correlation coefficient  $\rho$ .

Let  $C = C(S, r, t)$  be the price of a European call option at time  $t$  with a strike price  $K$  that matures at time  $T$ . Then we have.

**Theorem 3.1.** *Assume that the short rate  $r(t)$  and stock price  $S(t)$  satisfy Equations (1.3) and (1.1), respectively. Then,  $C(S, r, t)$  is the solution the following equation:*

$$(3.1) \quad \begin{aligned} & \frac{\partial C}{\partial t} + \tilde{\sigma}_s^2(t) S^2 \frac{\partial^2 C}{\partial S^2} + \tilde{\sigma}_r^2(t) \frac{\partial^2 C}{\partial r^2} + 2\rho \tilde{\sigma}_r(t) \tilde{\sigma}_s(t) \frac{\partial^2 C}{\partial S \partial r} \\ & + 2Ht^{2H-1} \mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial C}{\partial r} + rS \frac{\partial C}{\partial S} - rC = 0, \end{aligned}$$

where

$$(3.2) \quad \tilde{\sigma}_s^2(t) = Ht^{2H-1}\sigma_s^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H},$$

$$(3.3) \quad \tilde{\sigma}_r^2(t) = Ht^{2H-1}\sigma_r^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H}.$$

$\sigma_s, \sigma_r, \mu_s, \mu_r$ , are constant,  $H \in [\frac{1}{2}, 1)$  and  $\alpha \in (\frac{1}{2}, 1)$  and  $2\alpha - \alpha H > 1$ .

**Proof:** We consider a portfolio with  $D_{1t}$  units of stock and  $D_{2t}$  units of zero-coupon bond  $P(r, t, T)$  and one unit of  $C = C(r, t, T)$ . Then, the value of the portfolio at current time  $t$  is

$$(3.4) \quad \Pi_t = C - D_{1t}S_t - D_{2t}P_t.$$

Then, from [6] we have

$$(3.5) \quad \begin{aligned} d\Pi_t &= C_t - D_{1t}dS_t - D_{2t}dP_t \\ &= \left[ \frac{\partial C}{\partial t} dt + Ht^{2H-1}\sigma_s^2 S_t^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 C}{\partial S^2} + Ht^{2H-1}\sigma_r^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 C}{\partial r^2} \right. \\ &\quad \left. + 2Ht^{2H-1}\rho\sigma_r\sigma_s S \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 C}{\partial S \partial r} \right] dt + \left[ \frac{\partial C}{\partial t} - D_{1t} \right] dS_t \\ &\quad + \left[ \frac{\partial C}{\partial r} - D_{2t} \frac{\partial P}{\partial r} \right] dr + D_{2t} \left[ \frac{\partial P}{\partial t} + Ht^{2H-1}\sigma_r^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 P}{\partial r^2} \right] dt. \end{aligned}$$

By setting  $D_{1t} = \frac{\partial C}{\partial S}$ ,  $D_{2t} = \frac{\partial C}{\partial P}$ , to eliminate the stochastic noise, then

$$(3.6) \quad \begin{aligned} d\Pi_t &= \\ &= \left[ \frac{\partial C}{\partial t} + Ht^{2H-1} \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \left( \sigma_s^2 S^2 \frac{\partial^2 C}{\partial S^2} + \sigma_r^2 \frac{\partial^2 C}{\partial r^2} + 2\rho\sigma_r\sigma_s S \frac{\partial^2 C}{\partial S \partial r} \right) \right] dt \\ &- \frac{\partial C}{\partial r} \left[ rP - 2Ht^{2H-1}\mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial P}{\partial r} \right] dt. \end{aligned}$$

The return of an amount  $\Pi_t$  invested in bank account is equal to  $r(t)\Pi_t dt$  at time  $dt$ ,  $\mathbb{E}(d\Pi_t) = r(t)\Pi_t dt = r(t)(C - D_{1t}S_t - D_{2t}P_t)$ , hence from Equation (3.6) we have

$$(3.7) \quad \begin{aligned} &\frac{\partial C}{\partial t} + Ht^{2H-1} \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \left( \sigma_s^2 S^2 \frac{\partial^2 C}{\partial S^2} + \sigma_r^2 \frac{\partial^2 C}{\partial r^2} + 2\rho\sigma_r\sigma_s S \frac{\partial^2 C}{\partial S \partial r} \right) \\ &+ 2Ht^{2H-1}\mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial C}{\partial r} + rS \frac{\partial C}{\partial S} - rC = 0. \end{aligned}$$

Let

$$(3.8) \quad \tilde{\sigma}_s^2(t) = Ht^{2H-1}\sigma_s^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H},$$

$$(3.9) \quad \tilde{\sigma}_r^2(t) = Ht^{2H-1}\sigma_r^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H}.$$

Then

$$(3.10) \quad \begin{aligned} & \frac{\partial C}{\partial t} + \tilde{\sigma}_s^2(t) S_t^2 \frac{\partial^2 C}{\partial S_t^2} + \tilde{\sigma}_r^2(t) \frac{\partial^2 C}{\partial r^2} + 2\rho \tilde{\sigma}_r(t) \tilde{\sigma}_s(t) \frac{\partial^2 C}{\partial S \partial r} \\ & + 2Ht^{2H-1} \mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial C}{\partial r} + rS \frac{\partial C}{\partial S} - rC = 0, \end{aligned}$$

proof is completed.

From Theorem (3.1), we can get the following corollaries

**Corollary 3.1.** *If  $\rho = 0$  and  $r(t)$  be a constant, then the European call option  $C = C(S, r, T)$  satisfies*

$$(3.11) \quad \frac{\partial C}{\partial t} + Ht^{2H-1} \sigma_s^2 S_t^2 \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial^2 C}{\partial S_t^2} + rS \frac{\partial C}{\partial S} - rC = 0,$$

which is a fractional BS equation considered in [10].

**Corollary 3.2.** *When  $\alpha \uparrow 1$ , we obtain*

$$(3.12) \quad \begin{aligned} & \frac{\partial C}{\partial t} + Ht^{2H-1} \sigma_s^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} + Ht^{2H-1} \sigma_r^2 \frac{\partial^2 C}{\partial r^2} + 2Ht^{2H-1} \rho \sigma_r \sigma_s \frac{\partial^2 C}{\partial S \partial r} \\ & + 2Ht^{2H-1} \mu_r \frac{\partial C}{\partial r} + rS \frac{\partial C}{\partial S} - rC = 0, \end{aligned}$$

Further, if  $\rho = 0$ ,  $H = \frac{1}{2}$ , and  $r(t)$  be a constant, from Equation (3.12) we have the celebrated BS equation

$$(3.13) \quad \frac{\partial C}{\partial t} + \frac{1}{2} \sigma_s^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} + rS \frac{\partial C}{\partial S} - rC = 0,$$

#### 4. PRICING FORMULA UNDER SUBDIFFUSIVE FRACTIONALBLACK-SCHOLES MODEL

In this section, we propose an explicit formula for European call option when its value satisfies the partial differential equation (3.1) with boundary condition  $C(S, r, T) = (S_T - K)^+$ . Then, we can get

**Theorem 4.1.** *Let  $r(t)$  satisfies Equation (1.3) and  $S(t)$  satisfies Equation (1.1), then the price of European call and put options with strike price  $K$  and maturity  $T$  are given by*

$$(4.1) \quad C(S, r, t) = S\phi(d_1) - KP(r, t, T)\phi(d_2),$$

$$(4.2) \quad P(S, r, t) = KP(r, t, T)\phi(-d_2) - \phi(-d_1).$$

where

$$(4.3) \quad d_1 = \frac{\ln \frac{S}{K} - \ln P(r, t, T) + \frac{H}{(\Gamma(\alpha))^{2H}} \int_t^T \hat{\sigma}^2(s) s^{(\alpha-1)2H+2H-1} ds}{\sqrt{\frac{2H}{(\Gamma(\alpha))^{2H}} \int_t^T \hat{\sigma}^2(s) s^{(\alpha-1)2H+2H-1} ds}},$$

$$(4.4) \quad d_2 = d_1 - \sqrt{\frac{2H}{(\Gamma(\alpha))^{2H}} \int_t^T \hat{\sigma}^2(s) s^{(\alpha-1)2H+2H-1} ds},$$

$$(4.5) \quad \hat{\sigma}^2(t) = \sigma_s^2 + 2\rho \sigma_r \sigma_s (T - t) + \sigma_r^2 (T - t)^2.$$



$P(r, t, T)$  is given by Equation (2.16) and  $\phi(\cdot)$  is the cumulative normal distribution function.

**Proof:**

Consider the partial differential equation (3.1) of the European call option with boundary condition  $C(S, r, T) = (S_T - K)^+$

$$(4.6) \quad \begin{aligned} & \frac{\partial C}{\partial t} + \tilde{\sigma}_s^2(t) S_t^2 \frac{\partial^2 C}{\partial S_t^2} + \tilde{\sigma}_r^2(t) \frac{\partial^2 C}{\partial r^2} + 2\rho \tilde{\sigma}_r(t) \tilde{\sigma}_s(t) \frac{\partial^2 C}{\partial S \partial r} \\ & + 2Ht^{2H-1} \mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial C}{\partial r} + rS \frac{\partial C}{\partial S} - rC = 0. \end{aligned}$$

Denote

$$(4.7) \quad z = \frac{S}{P(r, t, T)}, \quad \Theta(z, t) = \frac{C(S, r, t)}{P(r, t, T)},$$

therefore by computing, we get

$$(4.8) \quad \begin{aligned} \frac{\partial C}{\partial t} &= \Theta \frac{\partial P}{\partial t} + P \frac{\partial \Theta}{\partial t} - z \frac{\partial \Theta}{\partial z} \frac{\partial P}{\partial t}, \\ \frac{\partial C}{\partial r} &= \Theta \frac{\partial P}{\partial r} - z \frac{\partial \Theta}{\partial z} \frac{\partial P}{\partial r}, \\ \frac{\partial C}{\partial S} &= \frac{\partial \Theta}{\partial z}, \\ \frac{\partial^2 C}{\partial r^2} &= \Theta \frac{\partial^2 P}{\partial r^2} - z \frac{\partial \Theta}{\partial z} \frac{\partial^2 P}{\partial r^2} + \frac{z^2}{P} \frac{\partial^2 \Theta}{\partial z^2} \left( \frac{\partial P}{\partial r} \right)^2, \\ \frac{\partial^2 C}{\partial r \partial S} &= -\frac{z}{P} \frac{\partial^2 \Theta}{\partial z^2} \frac{\partial P}{\partial r}, \\ \frac{\partial^2 C}{\partial S^2} &= \frac{1}{P} \frac{\partial^2 \Theta}{\partial z^2}. \end{aligned}$$

Inserting Equation (4.8) into Equation (4.6)

$$(4.9) \quad \begin{aligned} & \frac{\partial \Theta}{\partial t} + \frac{\partial^2 \Theta}{\partial z^2} \left[ \tilde{\sigma}_s^2(t) \frac{S^2}{P^2} + 2\rho z^2 \tilde{\sigma}_r(t) \tilde{\sigma}_s(t) \frac{1}{P} \frac{\partial P}{\partial r} + \tilde{\sigma}_r^2(t) z^2 \left( \frac{1}{P} \frac{\partial P}{\partial r} \right)^2 \right] \\ & - \frac{z}{P} \left[ \frac{\partial P}{\partial t} + \tilde{\sigma}_r^2(t) \frac{\partial^2 P}{\partial r^2} + 2Ht^{2H-1} \mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial P}{\partial r} - r \frac{S}{z} \right] \\ & + \frac{\Theta}{P} \left[ \frac{\partial P}{\partial t} + \tilde{\sigma}_r^2(t) \frac{\partial^2 P}{\partial r^2} + 2Ht^{2H-1} \mu_r \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right)^{2H} \frac{\partial P}{\partial r} - rP \right] = 0. \end{aligned}$$

From Equation (2.8), we can obtain

$$(4.10) \quad \frac{\partial \Theta}{\partial t} + \tilde{\sigma}^2(t) z^2 \frac{\partial^2 \Theta}{\partial z^2} = 0,$$

with boundary condition  $\Theta(z, T) = (z - K)^+$ ,

where

$$(4.11) \quad \bar{\sigma}^2(t) = \tilde{\sigma}_s^2(t) + 2\rho\tilde{\sigma}_r(t)\tilde{\sigma}_s(t)(T-t) + \tilde{\sigma}_r(t)^2(T-t)^2.$$

The solution of partial differential Equation (4.10) with boundary condition  $\Theta(z, T) = (z - K)^+$ , is given by

$$(4.12) \quad \Theta(z, t) = z\phi(\hat{d}_1) - K\phi(\hat{d}_2),$$

here

$$(4.13) \quad \hat{d}_1 = \frac{\ln \frac{z}{K} + \int_t^T \bar{\sigma}^2(s) ds}{\sqrt{2 \int_t^T \hat{\sigma}^2(s) ds}},$$

$$(4.14) \quad \hat{d}_2 = \hat{d}_1 - \sqrt{2 \int_t^T \bar{\sigma}^2(s) ds}.$$

Thus, from Equations (4.7) and (4.12)-(4.14) we obtain

$$(4.15) \quad C(S, r, t) = S\phi(d_1) - KP(r, t, T)\phi(d_2),$$

where

$$(4.16) \quad d_1 = \frac{\ln \frac{S}{K} - \ln P(r, t, T) + \frac{H}{(\Gamma(\alpha))^{2H}} \int_t^T \hat{\sigma}^2(s) s^{(\alpha-1)2H+2H-1} ds}{\sqrt{\frac{2H}{(\Gamma(\alpha))^{2H}} \int_t^T \hat{\sigma}^2(s) s^{(\alpha-1)2H+2H-1} ds}},$$

$$(4.17) \quad d_2 = d_1 - \sqrt{\frac{2H}{(\Gamma(\alpha))^{2H}} \int_t^T \hat{\sigma}^2(s) s^{(\alpha-1)2H+2H-1} ds}.$$

Letting  $\alpha \uparrow 1$ , from Theorem 4.1, we obtain

**Corollary 4.1.** *The price of European call and put options with strike price  $K$  and maturity  $T$  are given by*

$$(4.18) \quad C(S, r, T) = S\phi(d_1) - KP(r, t, T)\phi(d_2),$$

$$(4.19) \quad P(S, r, T) = KP(r, t, T)\phi(-d_2) - S\phi(-d_1).$$

where

$$(4.20) \quad d_1 = \frac{\ln \frac{S}{K} - \ln P(r, t, T) + H \int_t^T \hat{\sigma}^2(s) s^{2H-1} ds}{\sqrt{2H \int_t^T \hat{\sigma}^2(s) s^{2H-1} ds}},$$

$$(4.21) \quad d_2 = d_1 - \sqrt{2H \int_t^T \hat{\sigma}^2(s) s^{2H-1} ds},$$

$$(4.22) \quad \hat{\sigma}^2(t) = \sigma_s^2 + 2\rho\sigma_r\sigma_s(T-t) + \sigma_r^2(T-t)^2,$$

$$(4.23) \quad P(r, t, T) = \exp \left\{ -r\tau + H\sigma_r^2 \int_0^\tau (T-s)^{2H-1} s^2 ds - 2H\mu_r \int_0^\tau (T-s)^{2H-1} s ds \right\}, \tau = T-t.$$

More specifically, if  $H = \frac{1}{2}$ , we have

$$(4.24) \quad d_1 = \frac{\ln \frac{S}{K} - \ln P(r, t, T) + \frac{1}{2}\varphi(t, T)}{\sqrt{\varphi(t, T)}},$$

$$(4.25) \quad d_2 = d_1 - \sqrt{\varphi(t, T)},$$

$$(4.26) \quad \varphi(t, T) = \sigma_s^2(T-t) + \rho\sigma_r\sigma_s(T-t)^2 + \frac{1}{3}\sigma_r^2(T-t)^3,$$

$$(4.27) \quad P(r, t, T) = \exp \left\{ -r(T-t) - \frac{1}{2}\mu_r(T-t)^2 + \frac{1}{6}\sigma_r^2(T-t)^3 \right\}.$$

which is consistent with result in [3].

Letting  $H = \frac{1}{2}$ , from Theorem 4.1, we can get

**Corollary 4.2.** *The price of European call and put options with strike price  $K$  and maturity  $T$  are given by*

$$(4.28) \quad C(S, r, T) = S\phi(d_1) - KP(r, t, T)\phi(d_2),$$

$$(4.29) \quad P(S, r, T) = KP(r, t, T)\phi(-d_2) - \phi(-d_1).$$

where

$$(4.30) \quad d_1 = \frac{\ln \frac{S}{K} - \ln P(r, t, T) + \frac{1}{2\Gamma(\alpha)} \int_t^T \hat{\sigma}^2(s) s^{\alpha-1} ds}{\sqrt{\frac{1}{\Gamma(\alpha)} \int_t^T \hat{\sigma}^2(s) s^{\alpha-1} ds}},$$

$$(4.31) \quad d_2 = d_1 - \sqrt{\frac{1}{\Gamma(\alpha)} \int_t^T \hat{\sigma}^2(s) s^{\alpha-1} ds},$$

$$(4.32) \quad \hat{\sigma}^2(t) = \sigma_s^2 + 2\rho\sigma_r\sigma_s(T-t) + \sigma_r^2(T-t)^2,$$

$$(4.33) \quad P(r, t, T) = \exp \left\{ -r\tau + \frac{\sigma_r^2}{2\Gamma(\alpha)} \int_0^\tau (T-s)^{\alpha-1} s^2 ds - \frac{\mu_r}{\Gamma(\alpha)} \int_0^\tau (T-s)^{\alpha-1} s ds \right\}.$$

Specially, If  $\rho = 0$ , from Equations (4.28)-(4.33), we have

$$(4.34) \quad d_1 = \frac{\ln \frac{S}{K} - \ln P(r, t, T) + \frac{1}{2\Gamma(\alpha)} \int_t^T \hat{\sigma}^2(s) s^{\alpha-1} ds}{\sqrt{\frac{1}{\Gamma(\alpha)} \int_t^T \hat{\sigma}^2(s) s^{\alpha-1} ds}},$$

$$(4.35) \quad d_2 = d_1 - \sqrt{\frac{1}{\Gamma(\alpha)} \int_t^T \hat{\sigma}^2(s) s^{\alpha-1} ds},$$

$$(4.36) \quad \hat{\sigma}^2(t) = \sigma_s^2 + \sigma_r^2(T-t)^2,$$

$$(4.37) \quad P(r, t, T) = \exp \left\{ -r\tau + \frac{1}{2\Gamma(\alpha)} \int_0^\tau (T-s)^{\alpha-1} s^2 ds - \frac{\mu_r}{\Gamma(\alpha)} \int_0^\tau (T-s)^{\alpha-1} s ds \right\}.$$

which is similar with results mentioned in [6].

## 5. SIMULATION STUDIES

Let us first discuss about the implied volatility of the subdiffusive *FBS* model, then we will show some simulation findings.

**Corollary 5.1.** *If  $t = 0$ , the value of European call option  $\bar{C}(K, T)$  and put option  $\bar{P}(K, T)$  can be written as*

$$(5.1) \quad \bar{C}(K, T) = S_0 \phi(\bar{d}_1) - K P_0 \phi(\bar{d}_2),$$

$$(5.2) \quad \bar{P}(K, T) = K P_0 \phi(-\bar{d}_2) - S_0 \phi(-\bar{d}_1).$$

where

$$(5.3) \quad P_0 = \exp \left\{ -r_0 T + \frac{2HT^{(\alpha-1)2H+2H+1}}{(\Gamma(\alpha))^{2H}((\alpha-1)2H+2H)((\alpha-1)2H+2H+1)} \right. \\ \left. \times \left( \frac{\sigma_r^2 T}{(\alpha-1)2H+2H+2} - \mu_r \right) \right\}$$

$$(5.4) \quad \bar{d}_1 = \frac{\ln \frac{S_0}{K} + \bar{r}T + \frac{1}{2}\bar{\sigma}^2 T}{\bar{\sigma}\sqrt{T}},$$

$$(5.5) \quad \bar{d}_2 = \bar{d}_1 - \bar{\sigma}\sqrt{T},$$

$$(5.6) \quad \bar{r} = r_0 + \frac{2HT^{(\alpha-1)2H+2H}}{(\Gamma(\alpha))^{2H}((\alpha-1)2H+2H)((\alpha-1)2H+2H+1)} \\ \times \left( \mu_r - \frac{\sigma_r^2 T}{(\alpha-1)2H+2H+2} \right),$$

$$(5.7) \quad \bar{\sigma}^2 = \frac{2HT^{(\alpha-1)2H+2H-1}}{(\Gamma(\alpha))^{2H}((\alpha-1)2H+2H)} \left( \sigma_s^2 + \frac{\rho\sigma_r\sigma_s T}{(\alpha-1)2H+2H+1} \right) \\ + \frac{\sigma_r^2 T^2}{((\alpha-1)2H+2H+1)((\alpha-1)2H+2H+2)}.$$

and  $\phi(\cdot)$  is the cumulative normal distribution function.

Table 1 indicates the theoretical prices from our *FBS* and subdiffusive *FBS* models and Merton and subdiffusive *BS* models, where  $S_0$  shows the stock price,  $P_M$  presents the prices evaluated by the Merton model,  $P_{SBS}$  denotes the price simulated by the subdiffusive *BS* model,  $P_{FBS}$  and  $P_{SFBS}$  show the price obtained by the *FBS* and subdiffusive *FBS* models, respectively.

TABLE 1. Results by different pricing models. Here,  $\alpha = 0.9, H = 0.6, K = 3, \sigma_r = 0.3, \sigma_s = 0.4, \rho = 0.4, \mu_r = 0.5, r_0 = 0.3, T = 0.3, t = 0$ .

S	$T = 0.2$				$T = 1$			
	$P_M$	$P_{SBS}$	$P_{FBS}$	$P_{SFBS}$	$P_M$	$P_{SBS}$	$P_{FBS}$	$P_{SFBS}$
2	0.0174	0.0334	0.0012	0.0036	1.8826	1.9129	1.7986	1.8347
2.25	0.0638	0.0979	0.0122	0.0236	2.1326	2.1629	2.0486	2.0847
2.5	0.1598	0.2126	0.0587	0.0859	2.3826	2.4129	2.2986	2.3347
2.75	0.3094	0.3754	0.1687	0.2094	2.6326	2.6629	2.5486	2.5847
3	0.5023	0.5752	0.3440	0.3900	2.8826	2.1929	2.7986	2.8347
3.25	0.7235	0.7988	0.5630	0.6086	3.1326	3.1629	3.0486	3.0847
3.5	0.9604	1.0360	0.8026	0.8466	3.3826	3.4129	3.2986	3.3347
3.75	1.2094	1.2801	1.0498	1.0926	3.6326	3.6629	3.5486	3.5847
4	1.4527	1.5275	1.2991	1.3414	3.8826	3.9129	3.7986	3.8347

By comparing columns  $P_M, P_{SBS}, P_{FBS}$  and  $P_{SFBS}$  in Table 1, we conclude the call option prices obtained by four pricing models are close to each other in the both in-the-money and out-of-the-money cases with low and high maturities. Meanwhile, we can see that the prices given by the our  $FBS$  and subdiffusive  $FBS$  models are smaller than the prices given by the Merton and subdiffusive Merton models [3, 6].

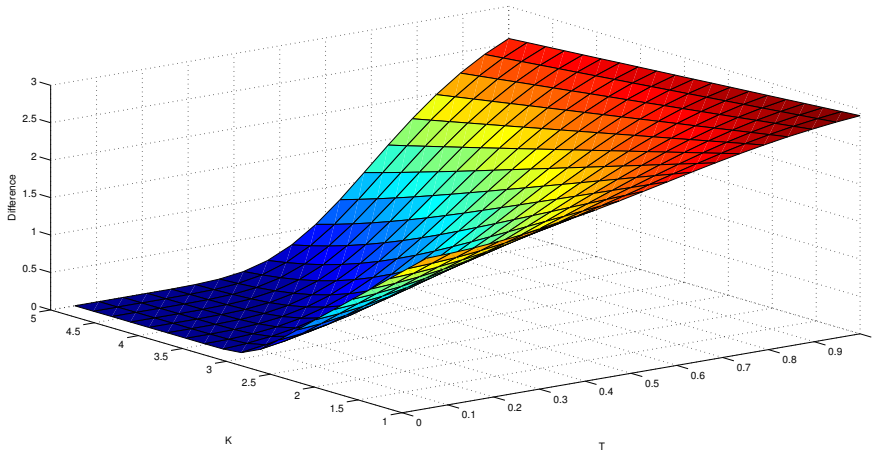


FIGURE 2. The European call option under subdiffusive  $FBS$ . Where  $r_0 = 0.1, \alpha = 0.9, H = 0.8, \sigma_r = 0.3, \sigma_s = 0.4, S_0 = 3, \mu_r = 0.2, \rho = 0.2$ .

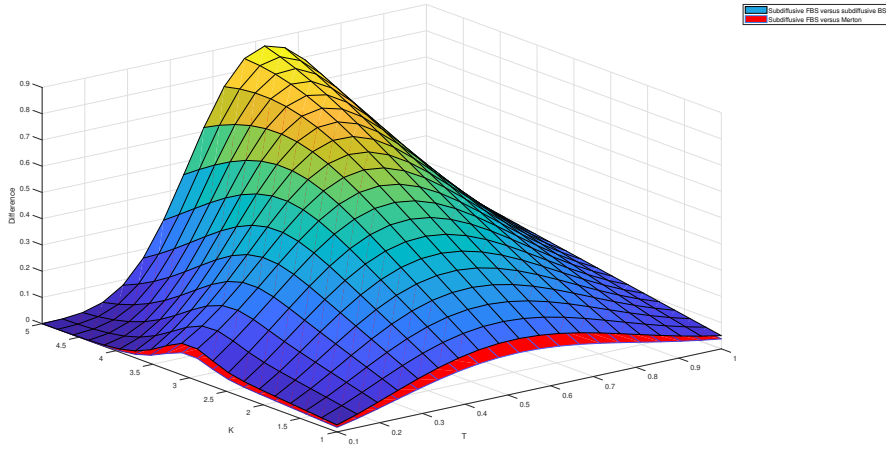


FIGURE 3. The difference between the price of the European call option under subdiffusive *FBS*, subdiffusive Merton and Merton models. Where  $r_0 = 0.1, \alpha = 0.9, H = 0.8, \sigma_r = 0.3, \sigma_s = 0.4, S_0 = 3, \mu_r = 0.2, t = 0, \rho = 0.3$ .

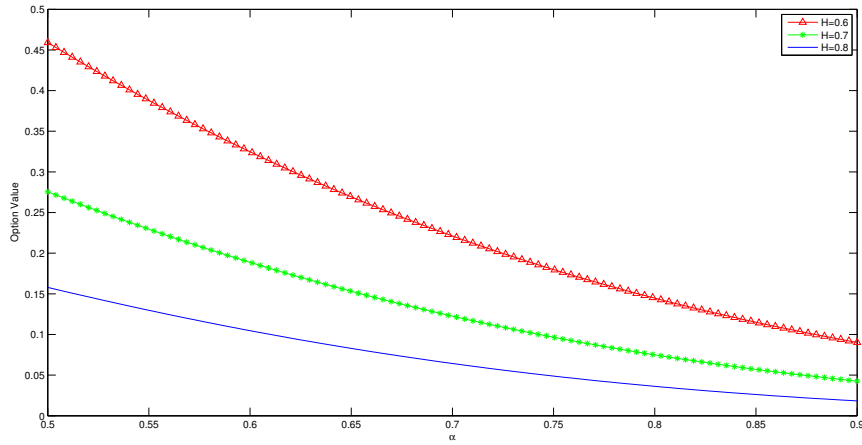


FIGURE 4. The European call option under subdiffusive *FBS*. Where  $r_0 = 0.3, \sigma_r = 0.1, \sigma_s = 0.3, S_0 = 4, \mu_r = 0.2, \rho = 0.2, t = 0, T = 0.2$ .

From Equations (5.1)-(5.7), it is easy to see that  $\bar{\sigma}$  and  $\bar{r}$  is the implied volatility and implied short rate connected to the *FBS* model, respectively (See Fig 2, 3 and 4 ).

## 6. CONCLUSION

Most of prior pricing models have assumed the constant short rate during the life of an option. However, in real life the short rate is evolving randomly through time. For this purpose, we apply the subdiffusive mechanism to get better characteristic

property of stock markets. We propose a pricing model for a zero-coupon bond when the short rate is governed by the subdiffusive fractional Black-Scholes model. Then, we exert these results to develop analytical valuation formulas for European option and corresponding fractional Black-Scholes equation.

We allow to referees to evaluate our manuscript.

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