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# Lebesgue type decompositions for nonnegative forms ☆

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To the memory of M.G. Kreĭn (1907–1989)

**Abstract**

A nonnegative form $\mathfrak{t}$ on a complex linear space is decomposed with respect to another nonnegative form $\mathfrak{w}$: it has a Lebesgue decomposition into an almost dominated form and a singular form. The part which is almost dominated is the largest form majorized by $\mathfrak{t}$ which is almost dominated by $\mathfrak{w}$. The construction of the Lebesgue decomposition only involves notions from the complex linear space. An important ingredient in the construction is the new concept of the parallel sum of forms. By means of Hilbert space techniques the almost dominated and the singular parts are identified with the regular and a singular parts of the form. This decomposition addresses a problem posed by B. Simon. The Lebesgue decomposition of a pair of finite measures corresponds to the present decomposition of the forms which are induced by the measures. T. Ando's decomposition of a nonnegative bounded linear operator in a Hilbert space with respect to another nonnegative bounded linear operator is a consequence. It is shown that the decomposition of positive definite kernels involving families of forms also belongs to the present context. The Lebesgue decomposition is an example of a Lebesgue type decomposition, i.e., any decomposition into an almost dominated and a singular part. There is a necessary and sufficient condition for a Lebesgue type decomposition to be unique. This condition is inspired by the work of Ando concerning uniqueness questions.


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## 1. Introduction

Let $\mathfrak{t}$ and $\mathfrak{w}$ be a pair of (nonnegative sesquilinear) forms, defined everywhere on a complex linear space $\mathfrak{V}$. The form $\mathfrak{t}$ is *almost dominated* by the form $\mathfrak{w}$ if there exists a nondecreasing sequence of forms $\mathfrak{t}_n$, each dominated by the form $\mathfrak{w}$, i.e., $\mathfrak{t}_n \leqslant c_n\mathfrak{w}$ for some $c_n \geqslant 0$, such that $\mathfrak{t} = \sup_{n\in\mathbb{N}} \mathfrak{t}_n$. The form $\mathfrak{t}$ is *singular* with respect to the form $\mathfrak{w}$ if for each form $\mathfrak{s}$ on $\mathfrak{V}$ the inequalities $\mathfrak{s} \leqslant \mathfrak{t}$ and $\mathfrak{s} \leqslant \mathfrak{w}$ imply that $\mathfrak{s} = 0$. The form $\mathfrak{t}$ can be decomposed by means of forms $\mathfrak{t}_{\mathrm{reg}}$ and $\mathfrak{t}_{\mathrm{sing}}$ on $\mathfrak{V}$ as

$$\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}} + \mathfrak{t}_{\mathrm{sing}},$$

where the form $\mathfrak{t}_{\mathrm{reg}}$ is almost dominated by $\mathfrak{w}$ and the form $\mathfrak{t}_{\mathrm{sing}}$ is singular with respect to $\mathfrak{w}$. The above decomposition of a form in an almost dominated and a singular part is called a Lebesgue type decomposition. In fact, the form $\mathfrak{t}_{\mathrm{reg}}$ is the maximum of all forms majorized by $\mathfrak{t}$, which are almost dominated by $\mathfrak{w}$. In this case one speaks of a Lebesgue decomposition. The existence of this decomposition can be shown by means of the parallel sum of forms, which induces an idempotent sublinear operator between forms. The parallel sum of forms is a new notion: it is shown to be a form itself and it has all the properties usually attributed to the parallel sum of operators. The presentation of the decomposition is given entirely in terms of forms on a complex linear space, but it is inspired by the treatment of the decomposition results by T. Ando [4] and by S.-L. Eriksson and H. Leutwiler [7]. The existence of the decomposition was suggested (under stronger conditions and in a slightly different context) by B. Simon [30, Theorem 2.5].

The almost dominated part in the above decomposition can be characterized in terms of closability of a form. A form $\mathfrak{t}$ on $\mathfrak{V}$ is *closable* with respect to the form $\mathfrak{w}$ if for any sequence $\varphi_n \in \mathfrak{V}$:

$$\mathfrak{t}[\varphi_n - \varphi_m] \to 0, \qquad \mathfrak{w}[\varphi_n] \to 0 \quad \Rightarrow \quad \mathfrak{t}[\varphi_n] \to 0.$$

A Hilbert space construction involving the forms $\mathfrak{t}$ and $\mathfrak{w}$ shows that the part $\mathfrak{t}_{\mathrm{reg}}$ is the largest form majorized by $\mathfrak{t}$ which is closable. The forms $\mathfrak{t}_{\mathrm{reg}}$ and $\mathfrak{t}_{\mathrm{sing}}$ are described in a canonical way, i.e., by means of a metric characterization entirely in terms of the pair of forms $\mathfrak{t}$ and $\mathfrak{w}$, thus answering a question posed by B. Simon [30]. It is similar to the corresponding decomposition from measure theory where a measure is written as the sum of an absolutely continuous part and a singular part, cf. [26]. In fact, a pair of finite measures on a $\sigma$-algebra is shown to induce a pair of forms, such that the Lebesgue decomposition of the forms is induced by the Lebesgue decomposition of the measures. A form $\mathfrak{t}$ which is almost dominated by a form $\mathfrak{w}$ but not dominated by $\mathfrak{w}$ always majorizes a nontrivial singular form (singular with respect to $\mathfrak{w}$). This implies that, in the general context, Lebesgue type decompositions need not be unique.

There is an analog of the Lebesgue decomposition associated with linear operators or relations (multivalued operators). Let $T$ be a linear operator or relation from one Hilbert space $\mathfrak{H}$ to another Hilbert space $\mathfrak{K}$. Then there exists a decomposition of $T$ as an operatorwise sum:

$$T = T_{\text{reg}} + T_{\text{sing}},$$

in a regular part $T_{\text{reg}}$ and a singular part $T_{\text{sing}}$. The closure of $T_{\text{reg}}$ in the Cartesian product $\mathfrak{H} \times \mathfrak{K}$ is the graph of a linear operator, i.e., $T_{\text{reg}}$ is a closable operator, whereas the closure of $T_{\text{sing}}$ is the Cartesian product of two closed linear subspaces in $\mathfrak{H}$ and in $\mathfrak{K}$, respectively. This type of decomposition for linear operators goes back to P.E.T. Jorgensen [14] and S. Ôta [22]. In [12] it is shown that linear relations form the proper context for such results thus giving metric characterizations of $T_{\text{reg}}$ and $T_{\text{sing}}$ by means of their associated forms.

Decompositions of Lebesgue type appear in various guises. For a single form $\mathfrak{t}$, densely defined in a Hilbert space $\mathfrak{H}$, a similar decomposition was obtained by B. Simon, cf. [26,30]. The present decomposition reduces to Simon's decomposition with the choice $\mathfrak{V} = \operatorname{dom} \mathfrak{t}$ and $\mathfrak{w}$ being the Hilbert space inner product on $\mathfrak{V}$. Further work concerning the decomposition of a form can be found in [17]. Another setting in which an analog of the Lebesgue decomposition for measures exists is that of a pair of nonnegative bounded linear operators $A$ and $B$ in a Hilbert space $\mathfrak{H}$. It was shown by T. Ando [4] that there exists a decomposition of $A$ with respect to $B$ into two nonnegative bounded linear operators: the almost dominated part of $A$ and the singular part of $A$, cf. [15,19,20]. The present terminology 'almost dominated' stands for Ando's 'absolutely continuous'. Ando's decomposition was obtained by means of parallel sums of operators. This notion goes back to W.N. Anderson and G.E. Trapp [3] in the matrix case, cf. [1,2]. It was studied in the infinite-dimensional case by E.L. Pekarev and Yu.L. Shmulyan [23,24]. Later S.-L. Eriksson and H. Leutwiler [7] gave a potential-theoretic interpretation of Ando's results. When the pair of forms above is generated by a pair of nonnegative bounded linear operators in a Hilbert space the decomposition of these forms gives rise to Ando's decomposition. An interesting construction of W. Pusz and S.L. Woronowicz [25] shows how to associate a pair of nonnegative operators $A$ and $B$ to a pair of forms $\mathfrak{t}$ and $\mathfrak{w}$. Further decompositions, similar to the Lebesgue decomposition from measure theory, exist. For positive definite kernels involving families of forms as introduced by F.H. Szafraniec [32], there is a decomposition by T. Ando and W. Szymański [5], cf. [33]. H. Kosaki [16] studied the Lebesgue decomposition of states on a von Neumann algebra. A. Gheondea and A.S. Kavruk [9] consider operator-valued completely positive maps on $C^*$-algebras.

The organization of the paper is as follows. Section 2 contains the main ingredients needed in later sections of the paper. A parallel sum of forms is introduced and a decomposition into regular and singular forms is shown by means of parallel sums. In Section 3 Hilbert space methods are introduced, which makes it possible to connect almost dominated forms and closable forms. Section 4 deals with the problem of the uniqueness of Lebesgue type decompositions. It is shown that an almost dominated form which is not dominated always majorizes a singular form; see [30] for examples. In Section 5 it is shown that the Lebesgue decomposition of a pair of finite measures agrees with the decomposition of the forms which are induced by these measures. The decomposition of Ando for a pair of nonnegative bounded linear operators is considered in Section 6. In Section 7 the decomposition of positive definite kernels, cf. [5], is considered.

## 2. The Lebesgue decomposition of nonnegative sesquilinear forms

In this section nonnegative sesquilinear forms on a complex linear space will be studied. In particular the notion of parallel sum of such forms will be developed. The main properties of the parallel sum, familiar in the case of bounded nonnegative operators, will be given in the context of forms and the consequences for the decomposition of one form with respect to another form will be studied.

### *2.1. Some generalities*

Let $\mathfrak{V}$ be a complex linear space. A mapping $\mathfrak{s}(\cdot,\cdot)$ from the Cartesian product $\mathfrak{V} \times \mathfrak{V}$ to $\mathbb{C}$ (or $\mathbb{R}$) is said to be *sesquilinear* if it is linear in the first entry and antilinear in the second entry. The corresponding quadratic form is denoted by $\mathfrak{s}[\cdot]$: $\mathfrak{s}[\varphi] = \mathfrak{s}(\varphi, \varphi)$, $\varphi \in \mathfrak{V}$. This implies that $\mathfrak{s}[\lambda\varphi] = |\lambda|^2 \mathfrak{s}[\varphi]$, $\lambda \in \mathbb{C}$ (or $\lambda \in \mathbb{R}$). The quadratic form $\mathfrak{s}[\cdot]$ satisfies the parallelogram identity

$$\mathfrak{s}[\varphi + \psi] + \mathfrak{s}[\varphi - \psi] = 2\big(\mathfrak{s}[\varphi] + \mathfrak{s}[\psi]\big), \quad \varphi, \psi \in \mathfrak{V}.$$

A sesquilinear form is nonnegative if $\mathfrak{s}[\varphi] \geqslant 0$ for all $\varphi \in \mathfrak{V}$. The square root of a quadratic form corresponding to a nonnegative sesquilinear form is a seminorm; in this case one has the Schwarz inequality

$$\big|\mathfrak{s}(\varphi, \psi)\big| \leqslant \sqrt{\mathfrak{s}[\varphi]}\sqrt{\mathfrak{s}[\psi]}, \quad \varphi, \psi \in \mathfrak{V}.$$

As a simple consequence of the Schwarz inequality, note that

$$\mathfrak{s}[\varphi_0] = 0 \quad \Rightarrow \quad \mathfrak{s}[\varphi + \varphi_0] = \mathfrak{s}[\varphi], \quad \varphi_0, \varphi \in \mathfrak{V}. \tag{2.1}$$

Likewise, if $(\varphi_n)$ is a sequence in $\mathfrak{V}$, then

$$\mathfrak{s}[\varphi_n] \to 0 \quad \Rightarrow \quad \mathfrak{s}[\varphi + \varphi_n] \to \mathfrak{s}[\varphi], \quad \varphi \in \mathfrak{V}. \tag{2.2}$$

If $p : \mathfrak{V} \to [0, \infty)$ is a seminorm on $\mathfrak{V}$, which satisfies the parallelogram law

$$p(\varphi + \psi)^2 + p(\varphi - \psi)^2 = 2\big(p(\varphi)^2 + p(\psi)^2\big), \quad \varphi, \psi \in \mathfrak{V},$$

then, according to a theorem of P. Jordan and J. von Neumann [34, Satz 1.6], it generates a nonnegative sesquilinear form $\mathfrak{s}$:

$$\mathfrak{s}(\varphi, \psi) = \frac{1}{4}\big\{p(\varphi + \psi)^2 - p(\varphi - \psi)^2 + ip(\varphi + i\psi)^2 - ip(\varphi - i\psi)^2\big\}, \quad \varphi, \psi \in \mathfrak{V},$$

such that $p = \mathfrak{s}[\cdot]$. If $\mathfrak{V}$ is a real vector space this formula must be adapted; cf. [34, Satz 1.6]. From now on nonnegative sesquilinear forms will be called *forms*.

The sum $\mathfrak{t} + \mathfrak{s}$ of forms $\mathfrak{t}$ and $\mathfrak{s}$ on $\mathfrak{V}$ is the form defined by

$$(\mathfrak{t} + \mathfrak{s})[f] = \mathfrak{t}[f] + \mathfrak{s}[f], \quad f \in \mathfrak{V}.$$

Two forms $\mathfrak{t}$ and $\mathfrak{s}$ are ordered: $\mathfrak{t} \leqslant \mathfrak{s}$, when

$$\mathfrak{t}(f, f) \leqslant \mathfrak{s}(f, f), \quad f \in \mathfrak{V}.$$

Let $\mathfrak{t}$ and $\mathfrak{t}_n$ be forms on $\mathfrak{V}$. Then the notation $\mathfrak{t} = \sup_{n\in\mathbb{N}} \mathfrak{t}_n$ stands for

$$\mathfrak{t}(f,f) = \sup_{n\in\mathbb{N}} \mathfrak{t}_n(f,f), \quad f \in \mathfrak{V}.$$

A sequence of forms $\mathfrak{t}_n$ on a complex linear space $\mathfrak{V}$ is said to be *bounded from above* by a form $\mathfrak{s}$ on $\mathfrak{V}$ if $\mathfrak{t}_n(f,f) \leqslant \mathfrak{s}(f,f)$, $f \in \mathfrak{V}$; and *bounded from below* by a form $\mathfrak{s}$ on $\mathfrak{V}$ if $\mathfrak{t}_n(f,f) \geqslant \mathfrak{s}(f,f)$, $f \in \mathfrak{V}$. The sequence is said to be *monotonically nondecreasing* or *monotonically nonincreasing* if $m \geqslant n$ implies that $\mathfrak{t}_n(f,f) \leqslant \mathfrak{t}_m(f,f)$ or $\mathfrak{t}_n(f,f) \geqslant \mathfrak{t}_m(f,f)$, $f \in \mathfrak{V}$, respectively. The following result is included for completeness.

**Lemma 2.1.** *Let $\mathfrak{t}_n$ be a sequence of forms on a complex linear space $\mathfrak{V}$ which is monotonically nondecreasing and bounded from above by a form $\mathfrak{s}$, or monotonically nonincreasing, respectively. Then the pointwise limit*

$$\mathfrak{t}(f,f) = \sup_{n\in\mathbb{N}} \mathfrak{t}_n(f,f), \qquad \mathfrak{t}(f,f) = \inf_{n\in\mathbb{N}} \mathfrak{t}_n(f,f), \quad f \in \mathfrak{V},$$

*exists and defines a form, such that $0 \leqslant \mathfrak{t} \leqslant \mathfrak{s}$ or $0 \leqslant \mathfrak{t} \leqslant \mathfrak{t}_1$, respectively.*

**Proof.** The existence of $\mathfrak{t}(f,f)$ for each $f \in \mathfrak{V}$ is straightforward. A limit argument shows that $\mathfrak{t}$ defines a seminorm. In order to show that $\mathfrak{t}$ is a form it suffices to observe that each approximating form $\mathfrak{t}_n$ satisfies the parallelogram identity. The argument is complete by taking the limit as $n \to \infty$. □

*2.2. Parallel sums of forms*

The notion of parallel sum was first introduced for operators in finite-dimensional spaces by W.N. Anderson and G.E. Trapp [3] (see also [1,2]) and further studied by E.L. Pekarev and Yu.L. Shmulyan [23,24]. For the considerations in the present paper it will be useful to introduce the *parallel sum* $\mathfrak{t} : \mathfrak{w}$ of two forms $\mathfrak{t}$ and $\mathfrak{w}$ on a complex linear space $\mathfrak{V}$.

**Proposition 2.2.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on $\mathfrak{V}$. Then the parallel sum $\mathfrak{t} : \mathfrak{w}$, defined by*

$$(\mathfrak{t} : \mathfrak{w})[\varphi] = \inf_{g\in\mathfrak{V}} \{\mathfrak{w}[g+\varphi] + \mathfrak{t}[g]\}, \quad \varphi \in \mathfrak{V}, \tag{2.3}$$

*is a form on $\mathfrak{V}$.*

**Proof.** In the proof the expression $\mathfrak{t} : \mathfrak{w}$ will be denoted by $\mathfrak{z}$. The definition (2.3) shows that $\mathfrak{z}[\varphi] \geqslant 0$ for all $\varphi \in \mathfrak{V}$. In order to prove that $\mathfrak{z}$ in (2.3) is a nonnegative sesquilinear form, it will be shown that $\sqrt{\mathfrak{z}}$ is a seminorm on $\mathfrak{V}$, which satisfies the parallelogram law.

First it will be shown that $\sqrt{\mathfrak{z}}$ is a seminorm on $\mathfrak{V}$. Note that $\lambda \neq 0$ implies that

$$\mathfrak{z}[\lambda\varphi] = \inf_{g\in\mathfrak{V}} \{\mathfrak{w}[g+\lambda\varphi] + \mathfrak{t}[g]\} = \inf_{g\in\mathfrak{V}} \{\mathfrak{w}[\lambda(g+\varphi)] + \mathfrak{t}[\lambda g]\} = |\lambda|^2 \mathfrak{z}[\varphi], \quad \varphi \in \mathfrak{V},$$

which shows the homogeneity property. Now let $\varphi, \psi \in \mathfrak{V}$. Then for all $g, g' \in \mathfrak{V}$ it follows from (2.3) that

$$\mathfrak{z}[\varphi+\psi]=\inf_{g\in\mathfrak{V}}\bigl\{\mathfrak{w}[g+\varphi+\psi]+\mathfrak{t}[g]\bigr\}\leqslant\mathfrak{w}[g+g'+\varphi+\psi]+\mathfrak{t}[g+g'],$$

and, therefore, $\mathfrak{z}[\varphi+\psi]$ is dominated by

$$\begin{aligned}&\mathfrak{w}[g+\varphi]+2\operatorname{Re}\mathfrak{w}(g+\varphi,g'+\psi)+\mathfrak{w}[g'+\psi]+\mathfrak{t}[g]+2\operatorname{Re}\mathfrak{t}(g,g')+\mathfrak{t}[g']\\&\quad\leqslant\mathfrak{w}[g+\varphi]+2\sqrt{\mathfrak{w}[g+\varphi]}\sqrt{\mathfrak{w}[g'+\psi]}+\mathfrak{w}[g'+\psi]+\mathfrak{t}[g]+2\sqrt{\mathfrak{t}[g]}\sqrt{\mathfrak{t}[g']}+\mathfrak{t}[g']\\&\quad\leqslant\mathfrak{w}[g+\varphi]+\mathfrak{t}[g]+2\sqrt{\mathfrak{w}[g+\varphi]+\mathfrak{t}[g]}\sqrt{\mathfrak{w}[g'+\psi]+\mathfrak{t}[g']}+\mathfrak{w}[g'+\psi]+\mathfrak{t}[g']\\&\quad=\Bigl(\sqrt{\mathfrak{w}[g+\varphi]+\mathfrak{t}[g]}+\sqrt{\mathfrak{w}[g'+\psi]+\mathfrak{t}[g']}\,\Bigr)^2,\end{aligned}$$

for all $g,g'\in\mathfrak{V}$. Hence, by taking the infimum over $g,g'\in\mathfrak{V}$, it follows from (2.3) that

$$\mathfrak{z}[\varphi+\psi]\leqslant\bigl(\sqrt{\mathfrak{z}[\varphi]}+\sqrt{\mathfrak{z}[\psi]}\,\bigr)^2,\quad \varphi,\psi\in\mathfrak{V},$$

which shows the subadditivity of $\sqrt{\mathfrak{z}}$.

For the forms $\mathfrak{t}$ and $\mathfrak{w}$ it is straightforward to check that

$$\begin{aligned}&2\bigl\{\mathfrak{w}[g+\varphi]+\mathfrak{t}[g]+\mathfrak{w}[g'+\psi]+\mathfrak{t}[g']\bigr\}\\&\quad=\mathfrak{w}[g+g'+\varphi+\psi]+\mathfrak{t}[g+g']+\mathfrak{w}[g-g'+\varphi-\psi]+\mathfrak{t}[g-g'],\end{aligned}\tag{2.4}$$

for all $g,g'\in\mathfrak{V}$. The infimum over all $g,g'\in\mathfrak{V}$ in the left-hand side of (2.4) is equal to

$$2\bigl\{\mathfrak{z}[\varphi]+\mathfrak{z}[\psi]\bigr\},$$

as follows from (2.3). Furthermore, the infimum over all $g,g'\in\mathfrak{V}$ in the right-hand side of (3.6) is greater or equal to

$$\begin{aligned}&\inf_{g,g'\in\mathfrak{V}}\bigl\{\mathfrak{w}[g+g'+\varphi+\psi]+\mathfrak{t}[g+g']\bigr\}+\inf_{g,g'\in\mathfrak{V}}\bigl\{\mathfrak{w}[g-g'+\varphi-\psi]+\mathfrak{t}[g-g']\bigr\}\\&\quad=\mathfrak{z}[\varphi+\psi]+\mathfrak{z}[\varphi-\psi],\end{aligned}$$

where the identity follows from (2.3). Hence, the following inequality

$$2\bigl\{\mathfrak{z}[\varphi]+\mathfrak{z}[\psi]\bigr\}\geqslant\mathfrak{z}[\varphi+\psi]+\mathfrak{z}[\varphi-\psi],\quad \varphi,\psi\in\mathfrak{V},\tag{2.5}$$

has been shown.

Formally replacing $g$ and $g'$ in (2.4) by $(g+g')/2$ and $(g-g')/2$ leads to

$$\begin{aligned}&2\biggl\{\mathfrak{w}\biggl[\frac{g+g'}{2}+\varphi\biggr]+\mathfrak{t}\biggl[\frac{g+g'}{2}\biggr]+\mathfrak{w}\biggl[\frac{g-g'}{2}+\psi\biggr]+\mathfrak{t}\biggl[\frac{g-g'}{2}\biggr]\biggr\}\\&\quad=\mathfrak{w}[g+\varphi+\psi]+\mathfrak{t}[g]+\mathfrak{w}[g'+\varphi-\psi]+\mathfrak{t}[g'],\end{aligned}\tag{2.6}$$

for all $g,g'\in\mathfrak{V}$. The infimum over all $g,g'\in\mathfrak{V}$ in the right-hand side of (2.6) is equal to

$$\mathfrak{z}[\varphi+\psi]+\mathfrak{z}[\varphi-\psi],$$

as follows from (2.3). The infimum over all $g, g' \in \mathfrak{V}$ in the left-hand side of (2.6) is greater or equal to

$$\inf_{g,g'\in\mathfrak{V}}\left\{\mathfrak{w}\left[\frac{g+g'}{2}+\varphi\right]+\mathfrak{t}\left[\frac{g+g'}{2}\right]\right\}+\inf_{g,g'\in\mathfrak{V}}\left\{\mathfrak{w}\left[\frac{g-g'}{2}+\psi\right]+\mathfrak{t}\left[\frac{g-g'}{2}\right]\right\}$$
$$=2\big\{\mathfrak{z}[\varphi]+\mathfrak{z}[\psi]\big\},$$

where the identity follows from (2.3). Hence, the following inequality

$$\mathfrak{z}[\varphi+\psi]+\mathfrak{z}[\varphi-\psi]\geqslant 2\big\{\mathfrak{z}[\varphi]+\mathfrak{z}[\psi]\big\},\quad \varphi,\psi\in\mathfrak{V}, \tag{2.7}$$

has been shown.

Combining (2.5) and (2.7) leads to the identity

$$\mathfrak{z}[\varphi+\psi]+\mathfrak{z}[\varphi-\psi]=2\big\{\mathfrak{z}[\varphi]+\mathfrak{z}[\psi]\big\},\quad \varphi,\psi\in\mathfrak{V}.$$

By the theorem of Jordan and von Neumann it follows that $\mathfrak{z}$ is a form on $\mathfrak{V}$. □

### 2.3. *Properties of parallel sums*

The parallel sum of forms satisfies several useful properties which are given in the following lemma; cf. [3,23,24].

**Lemma 2.3.** *Let* $\mathfrak{t}$, $\mathfrak{t}_n$, $\mathfrak{w}$, $\mathfrak{w}_n$, $\mathfrak{s}$, *and* $\mathfrak{u}$ *be forms on* $\mathfrak{V}$, *and let* $\lambda$ *and* $\mu$ *be positive numbers. Then*

(i) $\mathfrak{t}:\mathfrak{w}=\mathfrak{w}:\mathfrak{t}$;
(ii) $(\lambda\mathfrak{t}):(\lambda\mathfrak{w})=\lambda(\mathfrak{w}:\mathfrak{t})$;
(iii) $(\mathfrak{t}:\mathfrak{w}):\mathfrak{s}=\mathfrak{t}:(\mathfrak{w}:\mathfrak{s})$;
(iv) $\mathfrak{t}:\mathfrak{w}\leqslant\mathfrak{t}$;
(v) $\mathfrak{t}\leqslant\mathfrak{s}\Rightarrow\mathfrak{t}:\mathfrak{w}\leqslant\mathfrak{s}:\mathfrak{w}$;
(vi) $(\mathfrak{t}:\mathfrak{w})+(\mathfrak{s}:\mathfrak{u})\leqslant(\mathfrak{t}+\mathfrak{s}):(\mathfrak{w}+\mathfrak{u})$;
(vii) $\mathfrak{t}_n\downarrow\mathfrak{t}$, $\mathfrak{w}_n\downarrow\mathfrak{w}\Rightarrow\mathfrak{t}_n:\mathfrak{w}_n\downarrow\mathfrak{t}:\mathfrak{w}$;
(viii) $(\lambda\mathfrak{t}):(\mu\mathfrak{t})=\frac{\lambda\mu}{\lambda+\mu}\mathfrak{t}$;
(ix) $\mathfrak{s}\leqslant\mathfrak{t}$, $\mathfrak{s}\leqslant\mathfrak{w}\Rightarrow(1/2)\mathfrak{s}\leqslant\mathfrak{t}:\mathfrak{w}$;
(x) $\mathfrak{s}\leqslant\mathfrak{t}$, $\mathfrak{s}\leqslant\mathfrak{w}$, *and* $\mathfrak{t}:\mathfrak{w}\leqslant(1/2)\mathfrak{s}\Rightarrow\mathfrak{t}:\mathfrak{w}=(1/2)\mathfrak{s}$.

**Proof.** (i), (ii), and (v) are immediate consequences of the definition.

(iii) Observe that (2.3) implies for $\varphi\in\mathfrak{V}$ that

$$\big((\mathfrak{t}:\mathfrak{w}):\mathfrak{s}\big)[\varphi]=\inf_{g\in\mathfrak{V}}\big\{\mathfrak{s}[g+\varphi]+(\mathfrak{t}:\mathfrak{w})[g]\big\}$$
$$=\inf_{g\in\mathfrak{V}}\inf_{h\in\mathfrak{V}}\big\{\mathfrak{s}[g+\varphi]+\mathfrak{w}[g+h]+\mathfrak{t}[h]\big\}.$$

Moreover, (2.3) also implies for $\varphi\in\mathfrak{V}$ that

$$\begin{aligned}\big(\mathfrak{t}:(\mathfrak{w}:\mathfrak{s})\big)[\varphi] &= \inf_{h\in\mathfrak{V}}\big\{(\mathfrak{w}:\mathfrak{s})[h+\varphi]+\mathfrak{t}[h]\big\}\\ &= \inf_{h\in\mathfrak{V}}\inf_{g\in\mathfrak{V}}\big\{\mathfrak{s}[g+h+\varphi]+\mathfrak{w}[g]+\mathfrak{t}[h]\big\}\\ &= \inf_{h\in\mathfrak{V}}\inf_{g\in\mathfrak{V}}\big\{\mathfrak{s}[g-h+\varphi]+\mathfrak{w}[g]+\mathfrak{t}[h]\big\}\\ &= \inf_{h\in\mathfrak{V}}\inf_{g'\in\mathfrak{V}}\big\{\mathfrak{s}[g'+\varphi]+\mathfrak{w}[g'+h]+\mathfrak{t}[h]\big\}.\end{aligned}$$

Comparison of the two expressions gives the required result.

(iv) Take $g=-\varphi$ in (2.3) and conclude that $(\mathfrak{t}:\mathfrak{w})[\varphi]\leqslant\mathfrak{t}[\varphi]$.

(vi) Observe that

$$\begin{aligned}&\inf_{g\in\mathfrak{V}}\big(\mathfrak{w}[g+\varphi]+\mathfrak{t}[g]\big)+\inf_{g'\in\mathfrak{V}}\big(\mathfrak{u}[g'+\varphi]+\mathfrak{s}[g']\big)\\ &\quad\leqslant\big(\mathfrak{w}[g+g'+\varphi]+\mathfrak{t}[g+g']\big)+\big(\mathfrak{u}[g+g'+\varphi]+\mathfrak{s}[g+g']\big)\\ &\quad=\big((\mathfrak{w}+\mathfrak{u})[g+g'+\varphi]+(\mathfrak{t}+\mathfrak{s})[g+g']\big).\end{aligned}$$

Since this inequality is true for all $g,g'\in\mathfrak{V}$, the assertion follows.

(vii) The inequality $\lim_{n\to\infty}(\mathfrak{t}_n:\mathfrak{w}_n)\geqslant\mathfrak{t}:\mathfrak{w}$ follows from part (v). On the other hand, for every $\varepsilon>0$ there exists $g_\varepsilon\in\mathfrak{V}$ such that

$$(\mathfrak{t}:\mathfrak{w})[\varphi]>\mathfrak{w}[g_\varepsilon+\varphi]+\mathfrak{t}[g_\varepsilon]-\varepsilon. \tag{2.8}$$

Moreover, for all $n\geqslant n_{g_\varepsilon,\varepsilon}$ one has

$$\mathfrak{w}_n[g_\varepsilon+\varphi]+\mathfrak{t}_n[g_\varepsilon]-\varepsilon<\mathfrak{w}[g_\varepsilon+\varphi]+\mathfrak{t}[g_\varepsilon]. \tag{2.9}$$

The inequalities (2.8) and (2.9) yield

$$\inf_{g\in\mathfrak{V}}\big\{\mathfrak{w}_n[g+\varphi]+\mathfrak{t}_n[g+\varphi]\big\}<(\mathfrak{t}:\mathfrak{w})[\varphi]+2\varepsilon,$$

for all $n\geqslant n_{g_\varepsilon,\varepsilon}$, $\varepsilon>0$. This implies the reverse inequality $\lim_{n\to\infty}(\mathfrak{t}_n:\mathfrak{w}_n)\leqslant\mathfrak{t}:\mathfrak{w}$.

(viii) Completing squares leads to the following identities

$$\begin{aligned}\lambda\mathfrak{t}[g+\varphi]+\mu\mathfrak{t}[g] &= \lambda\mathfrak{t}[\varphi]+2\lambda\,\mathrm{Re}\,\mathfrak{t}(g,\varphi)+(\lambda+\mu)\mathfrak{t}[g]\\ &= \lambda\mathfrak{t}[\varphi]+\mathfrak{t}\left[\sqrt{\lambda+\mu}\,g+\frac{\lambda}{\sqrt{\lambda+\mu}}\varphi\right]-\frac{\lambda^2}{\lambda+\mu}\mathfrak{t}[\varphi]\\ &= \frac{\lambda\mu}{\lambda+\mu}\mathfrak{t}[\varphi]+\mathfrak{t}\left[\sqrt{\lambda+\mu}\,g+\frac{\lambda}{\sqrt{\lambda+\mu}}\varphi\right].\end{aligned}$$

Observe that the infimum over all $g\in\mathfrak{V}$ in the second term of the right-hand side is 0. This shows the assertion.

(ix) An application of (vii) leads to $(1/2)\mathfrak{s}=\mathfrak{s}:\mathfrak{s}$. Hence, if $\mathfrak{s}\leqslant\mathfrak{t}$ and $\mathfrak{s}\leqslant\mathfrak{w}$, then $(1/2)\mathfrak{s}=\mathfrak{s}:\mathfrak{s}\leqslant\mathfrak{s}:\mathfrak{w}\leqslant\mathfrak{t}:\mathfrak{w}$, by a repeated application of (v).

(x) This statement is clear from (ix). □

### 2.4. An idempotent sublinear operator between forms

The parallel sums $\mathfrak{t} : n\mathfrak{w}$ form a monotonically nondecreasing sequence which is bounded above by the form $\mathfrak{t}$ by Lemma 2.3. Therefore

$$\mathbf{D}_{\mathfrak{w}}\,\mathfrak{t} = \sup_{n\in\mathbb{N}}(\mathfrak{t} : n\mathfrak{w})$$

is a well-defined form. Hence $\mathbf{D}_{\mathfrak{w}}$ can be seen as an operator on the set of all forms on $\mathfrak{V}$. Observe that $\mathbf{D}_{c\mathfrak{w}}\,\mathfrak{t} = \mathbf{D}_{\mathfrak{w}}\,\mathfrak{t}$ for each $c > 0$.

**Lemma 2.4.** *Let $\mathfrak{t}$, $\mathfrak{s}$, $\mathfrak{v}$, and $\mathfrak{w}$ be forms on $\mathfrak{V}$ and let $\lambda > 0$. Then*

(i) $(\mathfrak{t} : m\mathfrak{w}) \leqslant \mathbf{D}_{\mathfrak{w}}\,\mathfrak{t} \leqslant \mathfrak{t}$, $m \in \mathbb{N}$;
(ii) $\mathfrak{t} \leqslant \mathfrak{s}$, $\mathfrak{v} \leqslant \mathfrak{w} \Rightarrow \mathbf{D}_{\mathfrak{v}}\,\mathfrak{t} \leqslant \mathbf{D}_{\mathfrak{w}}\,\mathfrak{s}$;
(iii) $\mathbf{D}_{\mathfrak{w}}(\lambda\mathfrak{t}) = \lambda\mathbf{D}_{\mathfrak{w}}\,\mathfrak{t}$;
(iv) $\mathbf{D}_{\mathfrak{w}}\,\mathfrak{t} + \mathbf{D}_{\mathfrak{w}}\,\mathfrak{s} \leqslant \mathbf{D}_{\mathfrak{w}}(\mathfrak{t} + \mathfrak{s})$;
(v) $\mathbf{D}_{\mathfrak{w}}$ *is idempotent, i.e.,* $\mathbf{D}_{\mathfrak{w}}(\mathbf{D}_{\mathfrak{w}}\,\mathfrak{t}) = \mathbf{D}_{\mathfrak{w}}\,\mathfrak{t}$.

**Proof.** (i) By Lemma 2.3 the sequence $\mathfrak{t} : n\mathfrak{w}$ is monotonically nondecreasing and satisfies

$$\mathfrak{t} : m\mathfrak{w} \leqslant \mathfrak{t} : n\mathfrak{w} \leqslant \mathfrak{t}.$$

The result follows by talking the supremum with respect to $n$.

(ii) By Lemma 2.3 the inequalities $\mathfrak{t} \leqslant \mathfrak{s}$ and $\mathfrak{v} \leqslant \mathfrak{w}$ imply that

$$\mathfrak{t} : n\mathfrak{v} \leqslant \mathfrak{s} : n\mathfrak{w}.$$

The result follows by talking the supremum with respect to $n$.

(iii) Observe that by Lemma 2.3

$$\lambda\mathfrak{t} : n\mathfrak{w} = \lambda\mathfrak{t} : \lambda\frac{n}{\lambda}\mathfrak{w} = \lambda\left(\mathfrak{t} : \frac{n}{\lambda}\mathfrak{w}\right).$$

The result follows by talking the supremum in this identity.

(iv) It follows from Lemma 2.3 that

$$\mathfrak{t} : m\mathfrak{w} + \mathfrak{s} : n\mathfrak{w} \leqslant (\mathfrak{t} + \mathfrak{s}) : (m+n)\mathfrak{w}.$$

The result follows by taking successive suprema.

(v) It follows from Lemma 2.3 and items (i) and (ii) that

$$\mathfrak{t} : n\mathfrak{w} = \mathfrak{t} : (2n\mathfrak{w} : 2n\mathfrak{w}) = (\mathfrak{t} : 2n\mathfrak{w}) : 2n\mathfrak{w} \leqslant \mathbf{D}_{\mathfrak{w}}\,\mathfrak{t} : 2n\mathfrak{w} \leqslant \mathbf{D}_{\mathfrak{w}}(\mathbf{D}_{\mathfrak{w}}\,\mathfrak{t}) \leqslant \mathbf{D}_{\mathfrak{w}}\,\mathfrak{t}.$$

The result follows by taking the supremum. □

*2.5. Almost dominated and singular forms*

Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on a complex linear space $\mathfrak{V}$. The form $\mathfrak{t}$ is said to be *dominated* by the form $\mathfrak{w}$ if $\mathfrak{t} \leqslant c\mathfrak{w}$ for some $c > 0$. The form $\mathfrak{t}$ is said to be *almost dominated* by the form $\mathfrak{w}$ if there exists a monotonically nondecreasing sequence of forms $\mathfrak{t}_n$, each dominated by $\mathfrak{w}$, such that $\mathfrak{t} = \sup_{n\in\mathbb{N}} \mathfrak{t}_n$. The notion of almost domination goes back to H. Dye [6]; for the terminology see H. Kosaki [16, p. 10]. It expresses a certain approximation property of $\mathfrak{t}$ with respect to $\mathfrak{w}$. It is not difficult to see that if two forms are almost dominated by $\mathfrak{w}$, then also their sum is almost dominated by $\mathfrak{w}$. The form $\mathfrak{t}$ is said to be *singular* with respect to the form $\mathfrak{w}$ if for each form $\mathfrak{s}$ on $\mathfrak{V}$ the inequalities $\mathfrak{s} \leqslant \mathfrak{t}$ and $\mathfrak{s} \leqslant \mathfrak{w}$ imply that $\mathfrak{s} = 0$. Note that $\mathfrak{t}$ is singular with respect to $\mathfrak{w}$ if and only if $\mathfrak{w}$ is singular with respect to $\mathfrak{t}$. Furthermore, observe that if $\mathfrak{t} \leqslant \mathfrak{s}$ and $\mathfrak{s}$ is singular with respect to $\mathfrak{w}$, then also $\mathfrak{t}$ is singular with respect to $\mathfrak{w}$.

**Lemma 2.5.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on a complex linear space* $\mathfrak{V}$. *If* $\mathfrak{t}$ *is dominated by* $\mathfrak{w}$, *then* $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}$. *In particular,* $\mathbf{D}_{\mathfrak{w}}\mathfrak{w} = \mathfrak{w}$.

**Proof.** Assume that $\mathfrak{t} \leqslant c\mathfrak{w}$ for some $c > 0$. Then, by Lemma 2.3, for each $n \in \mathbb{N}$:

$$\frac{n}{n+c}\mathfrak{t} = \mathfrak{t} : \left(\frac{n}{c}\mathfrak{t}\right) \leqslant \mathfrak{t} : n\mathfrak{w} \leqslant \mathfrak{t}.$$

This leads to $\sup_{n\in\mathbb{N}}(\mathfrak{t} : n\mathfrak{w}) = \mathfrak{t}$. □

The next result gives a characterization for the form $\mathfrak{t}$ to be almost dominated by the form $\mathfrak{w}$.

**Theorem 2.6.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on a complex linear space* $\mathfrak{V}$.

(i) *If* $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}$, *then* $\mathfrak{t}$ *is almost dominated by* $\mathfrak{w}$.
(ii) *If* $\mathfrak{t} = \sup_{n\in\mathbb{N}} \mathfrak{t}_n$ *with each form* $\mathfrak{t}_n$ *dominated by* $\mathfrak{w}$, *then* $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}$. *In particular, if* $\mathfrak{t}$ *is almost dominated by* $\mathfrak{w}$, *then* $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}$.

**Proof.** (i) Assume that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}$. Then, by definition, the sequence $\mathfrak{t}_n = \mathfrak{t} : n\mathfrak{w}$ is a monotonically nondecreasing with $\sup_{n\in\mathbb{N}} \mathfrak{t}_n = \mathfrak{t}$. Furthermore, $\mathfrak{t}_n \leqslant n\mathfrak{w}$ shows that each $\mathfrak{t}_n$ is dominated by $\mathfrak{w}$. Hence, $\mathfrak{t}$ is almost dominated by $\mathfrak{w}$.

(ii) Assume that $\mathfrak{t} = \sup_{n\in\mathbb{N}} \mathfrak{t}_n$ with each form $\mathfrak{t}_n$ dominated by $\mathfrak{w}$, so that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}_n = \mathfrak{t}_n$ by Lemma 2.5. Now Lemma 2.4 shows that

$$\mathfrak{t}_n = \mathbf{D}_{\mathfrak{w}}\mathfrak{t}_n \leqslant \mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leqslant \mathfrak{t}.$$

The result follows by taking the supremum. □

The operator version of the following results is due to Eriksson and Leutwiler; see [7, Lemmas 2.6, 2.7].

**Proposition 2.7.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on a complex linear space* $\mathfrak{V}$. *Then*

$$\mathbf{D}_{\mathfrak{w}}(\mathfrak{t} : \mathfrak{w}) = (\mathbf{D}_{\mathfrak{w}}\mathfrak{t}) : \mathfrak{w} = \mathfrak{t} : \mathfrak{w}. \tag{2.10}$$

*Moreover, for any form* $\mathfrak{s}$,

$$\mathfrak{t} : \mathfrak{w} \leqslant \mathfrak{s} : \mathfrak{w} \quad \Leftrightarrow \quad \mathbf{D}_{\mathfrak{w}} \mathfrak{t} \leqslant \mathbf{D}_{\mathfrak{w}} \mathfrak{s} \ (\leqslant \mathfrak{s}). \tag{2.11}$$

**Proof.** Observe that $\mathfrak{t} : \mathfrak{w}$ is dominated by $\mathfrak{w}$. Hence by Lemma 2.5

$$\mathfrak{t} : \mathfrak{w} = \mathbf{D}_{\mathfrak{w}} (\mathfrak{t} : \mathfrak{w}) = \sup_{n \in \mathbb{N}} \big( (\mathfrak{t} : \mathfrak{w}) : n\mathfrak{w} \big) = \sup_{n \in \mathbb{N}} \big( (\mathfrak{t} : n\mathfrak{w}) : \mathfrak{w} \big) \leqslant (\mathbf{D}_{\mathfrak{w}} \mathfrak{t}) : \mathfrak{w} \leqslant \mathfrak{t} : \mathfrak{w},$$

which shows (2.10); here the last two inequalities are obtained by applying (v) in Lemma 2.3 and (i) in Lemma 2.4.

Clearly, if $\mathbf{D}_{\mathfrak{w}} \mathfrak{t} \leqslant \mathbf{D}_{\mathfrak{w}} \mathfrak{s}$ then by (2.10) it follows that

$$\mathfrak{t} : \mathfrak{w} = \mathbf{D}_{\mathfrak{w}} \mathfrak{t} : \mathfrak{w} \leqslant \mathbf{D}_{\mathfrak{w}} \mathfrak{s} : \mathfrak{w} = \mathfrak{s} : \mathfrak{w}.$$

Conversely, if $\mathfrak{t} : \mathfrak{w} \leqslant \mathfrak{s} : \mathfrak{w}$, then for all $n \in \mathbb{N}$

$$\mathfrak{t} : n\mathfrak{w} \leqslant \mathfrak{s} : n\mathfrak{w}, \tag{2.12}$$

which leads to (2.11). To show the identity (2.12), observe that by Lemma 2.3 it is equivalent to

$$\left(\frac{1}{n}\mathfrak{t}\right) : \mathfrak{w} \leqslant \left(\frac{1}{n}\mathfrak{s}\right) : \mathfrak{w}.$$

To prove the previous inequality, use induction. The statement is clear for $n = 1$. Assume it holds for $n$, then it follows from Lemma 2.4 that

$$\begin{aligned} \left(\frac{1}{n+1}\mathfrak{t}\right) : \mathfrak{w} &= \left(\left(\frac{1}{n}\mathfrak{t}\right) : \mathfrak{w}\right) : \mathfrak{t} \leqslant \left(\left(\frac{1}{n}\mathfrak{s}\right) : \mathfrak{w}\right) : \mathfrak{t} \\ &\leqslant \left(\left(\frac{1}{n}\mathfrak{s}\right) : \mathfrak{w}\right) : \mathfrak{s} = \left(\frac{1}{n+1}\mathfrak{s}\right) : \mathfrak{w}, \end{aligned}$$

which completes the proof. □

**Corollary 2.8.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on a complex linear space* $\mathfrak{V}$. *Then among all forms* $\mathfrak{x}$ *on* $\mathfrak{V}$ *solving the equation* $\mathfrak{x} : \mathfrak{w} = \mathfrak{t} : \mathfrak{w}$, *the form* $\mathbf{D}_{\mathfrak{w}} \mathfrak{t}$ *is the minimal solution.*

**Proof.** By Proposition 2.7 the form $\mathfrak{x} = \mathbf{D}_{\mathfrak{w}} \mathfrak{t}$ is a solution of $\mathfrak{x} : \mathfrak{w} = \mathfrak{t} : \mathfrak{w}$. Furthermore, if the form $\mathfrak{x}$ satisfies $\mathfrak{x} : \mathfrak{w} = \mathfrak{t} : \mathfrak{w}$, then it follows from Proposition 2.7 that $\mathbf{D}_{\mathfrak{w}} \mathfrak{x} = \mathbf{D}_{\mathfrak{w}} \mathfrak{t}$. Since $\mathbf{D}_{\mathfrak{w}} \mathfrak{x} \leqslant \mathfrak{x}$, it follows that $\mathbf{D}_{\mathfrak{w}} \mathfrak{t} \leqslant \mathfrak{x}$. □

**Proposition 2.9.** *Let* $\mathfrak{t}$, $\tilde{\mathfrak{t}}$, $\mathfrak{w}$, *and* $\widetilde{\mathfrak{w}}$ *be forms on a complex linear space* $\mathfrak{V}$.

- (i) *If* $\mathfrak{t}$ *is almost dominated by* $\mathfrak{w}$ *and* $\tilde{\mathfrak{t}}$ *is almost dominated by* $\widetilde{\mathfrak{w}}$, *then the form* $\lambda\mathfrak{t} + \mu\tilde{\mathfrak{t}}$ *with* $\lambda, \mu \geqslant 0$ *is almost dominated by* $\mathfrak{w} + \widetilde{\mathfrak{w}}$.
- (ii) *If* $\mathfrak{t}$ *and* $\tilde{\mathfrak{t}}$ *are almost dominated by* $\mathfrak{w}$, *then also the form* $\lambda\mathfrak{t} + \mu\tilde{\mathfrak{t}}$ *with* $\lambda, \mu \geqslant 0$ *is almost dominated by* $\mathfrak{w}$.

(iii) *If* $\mathfrak{t} = \sup_{n\in\mathbb{N}} \mathfrak{t}_n$ *with each form* $\mathfrak{t}_n$ *almost dominated by* $\mathfrak{w}$, *then the form* $\mathfrak{t}$ *is almost dominated by* $\mathfrak{w}$.

**Proof.** (i) If $\mathfrak{t}$ is almost dominated by $\mathfrak{w}$ and $\tilde{\mathfrak{t}}$ is almost dominated by $\widetilde{\mathfrak{w}}$ then $\mathfrak{t} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ and $\tilde{\mathfrak{t}} = \mathbf{D}_{\widetilde{\mathfrak{w}}}\tilde{\mathfrak{t}}$ by Theorem 2.6. Now Lemma 2.4 shows that

$$\lambda\mathfrak{t} + \mu\tilde{\mathfrak{t}} = \lambda\mathbf{D}_{\mathfrak{w}}\mathfrak{t} + \mu\mathbf{D}_{\widetilde{\mathfrak{w}}}\tilde{\mathfrak{t}} \leqslant \lambda\mathbf{D}_{\mathfrak{w}+\widetilde{\mathfrak{w}}}\mathfrak{t} + \mu\mathbf{D}_{\mathfrak{t}+\widetilde{\mathfrak{w}}}\tilde{\mathfrak{t}} \leqslant \mathbf{D}_{\mathfrak{w}+\widetilde{\mathfrak{w}}}(\lambda\mathfrak{t} + \mu\tilde{\mathfrak{t}}) \leqslant \lambda\mathfrak{t} + \mu\tilde{\mathfrak{t}}.$$

Therefore, $\lambda\mathfrak{t} + \mu\tilde{\mathfrak{t}} = \mathbf{D}_{\mathfrak{w}+\widetilde{\mathfrak{w}}}(\lambda\mathfrak{t} + \mu\tilde{\mathfrak{t}})$, and the statement follows from Theorem 2.6.

(ii) This follows immediately from (i).

(iii) It follows from (2.10) in Proposition 2.7 that

$$\mathfrak{t}_n : \mathfrak{w} \leqslant \mathfrak{t} : \mathfrak{w} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} : \mathfrak{w}.$$

Since $\mathfrak{t}_n$ is almost dominated by $\mathfrak{w}$, Theorem 2.6 and Lemma 2.4 imply that

$$\mathfrak{t}_n = \mathbf{D}_{\mathfrak{w}}\mathfrak{t}_n \leqslant \mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leqslant \mathfrak{t},$$

which, by taking the supremum, leads to $\mathfrak{t} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$. Hence, again by Theorem 2.6 the form $\mathfrak{t}$ is almost dominated by $\mathfrak{w}$. □

**Proposition 2.10.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on a complex linear space* $\mathfrak{V}$. *The following statements are equivalent*:

(i) $\mathfrak{t}$ *is singular with respect to* $\mathfrak{w}$;
(ii) $\mathfrak{t} : \mathfrak{w} = 0$;
(iii) $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = 0$.

**Proof.** (i) ⇒ (ii) Let $\mathfrak{t}$ be singular with respect to $\mathfrak{w}$. Hence the inequalities $\mathfrak{t} : \mathfrak{w} \leqslant \mathfrak{t}$ and $\mathfrak{t} : \mathfrak{w} \leqslant \mathfrak{w}$ imply that $\mathfrak{t} : \mathfrak{w} = 0$.

(ii) ⇒ (iii) Assume that $\mathfrak{t} : \mathfrak{w} = 0$. Since clearly $0 : \mathfrak{w} = 0$, one concludes from Corollary 2.8 that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = 0$.

(iii) ⇒ (i) Assume that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = 0$ and let $\mathfrak{s}$ be a form which satisfies $\mathfrak{s} \leqslant \mathfrak{t}$ and $\mathfrak{s} \leqslant \mathfrak{w}$. Then Lemma 2.3 and Lemma 2.4 show that

$$\frac{1}{2}\mathfrak{s} \leqslant \mathfrak{t} : \mathfrak{w} \leqslant \mathbf{D}_{\mathfrak{w}}\mathfrak{t} = 0.$$

Hence $\mathfrak{s} = 0$ and, therefore, $\mathfrak{t}$ is singular with respect to $\mathfrak{w}$. □

Observe, that using the definition of parallel sum of forms in Proposition 2.2 one can rewrite part (ii) in Proposition 2.10 in the following equivalent form: for every $\varphi \in \mathfrak{V}$ there exists a sequence $\varphi_n \in \mathfrak{V}$, such that

$$\mathfrak{w}[\varphi_n] \to 0, \quad \mathfrak{t}[\varphi_n - \varphi] \to 0.$$

In the case of one densely defined form $\mathfrak{t}$ in a Hilbert space $\mathfrak{H}$, this reduces to the definition of singularity appearing in [17, Definition 5.1] with $\mathfrak{w}[\varphi] = \|\varphi\|^2$.

Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on a complex linear space $\mathfrak{V}$. Assume that $\mathfrak{t}$ is almost dominated by $\mathfrak{w}$ and that $\mathfrak{t}$ is singular with respect to $\mathfrak{w}$. Then by Theorem 2.6 $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}$ and it follows from Proposition 2.10 that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = 0$. Therefore, $\mathfrak{t} = 0$ is the only form which is simultaneously almost dominated by $\mathfrak{w}$ and singular with respect to $\mathfrak{w}$.

### *2.6. The Lebesgue decomposition*

Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on a complex linear space $\mathfrak{V}$. Any decomposition of $\mathfrak{t}$ in an almost dominated form and a singular form is called a *Lebesgue type decomposition*. It will be shown that every form has a particular Lebesgue type decomposition which will be called the *Lebesgue decomposition*; see [7] for a connection with the Parreau decomposition in potential theory.

**Theorem 2.11** *(Lebesgue decomposition). Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on a complex linear space* $\mathfrak{V}$. *Then*

$$\mathfrak{t} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} + (\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}), \tag{2.13}$$

*is a decomposition of the form* $\mathfrak{t}$ *in a form* $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$, *which is almost dominated by* $\mathfrak{w}$, *and a form* $\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$, *which is singular with respect to* $\mathfrak{w}$. *In fact, the form* $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ *is the maximum of all forms majorized by* $\mathfrak{t}$, *which are almost dominated by* $\mathfrak{w}$.

**Proof.** Recall from Lemma 2.4 that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leqslant \mathfrak{t}$. By Lemma 2.4 $\mathbf{D}_{\mathfrak{w}}(\mathbf{D}_{\mathfrak{w}}\mathfrak{t}) = \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ and hence Theorem 2.6 shows that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is almost dominated by $\mathfrak{w}$.

Furthermore, $\mathbf{D}_{\mathfrak{w}}(\mathbf{D}_{\mathfrak{w}}\mathfrak{t}) = \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ and Lemma 2.4 imply that

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} + \mathbf{D}_{\mathfrak{w}}(\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}) = \mathbf{D}_{\mathfrak{w}}(\mathbf{D}_{\mathfrak{w}}\mathfrak{t}) + \mathbf{D}_{\mathfrak{w}}(\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}) \leqslant \mathbf{D}_{\mathfrak{w}}\mathfrak{t}.$$

This leads to $\mathbf{D}_{\mathfrak{w}}(\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}) = 0$, which shows that the form $\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is singular with respect to $\mathfrak{w}$; see Proposition 2.10.

Now let $\tilde{\mathfrak{t}}$ be any form which is majorized by $\mathfrak{t}$ and which is almost dominated by $\mathfrak{w}$. Then $\tilde{\mathfrak{t}} \leqslant \mathfrak{t}$, Lemma 2.4, and Theorem 2.6 imply that

$$\tilde{\mathfrak{t}} = \mathbf{D}_{\mathfrak{w}}\tilde{\mathfrak{t}} \leqslant \mathbf{D}_{\mathfrak{w}}\mathfrak{t}.$$

This proves the assertion concerning the maximum. □

In the decomposition (2.13), called the Lebesgue decomposition of the form $\mathfrak{t}$ with respect to the form $\mathfrak{w}$, the almost dominated part $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ has the maximality property as stated in Theorem 2.11. In Section 5 the decomposition in (2.13) is connected to the Lebesgue decomposition of measures. In general, the form $\mathfrak{t}$ can have also other decompositions in an almost dominated form and a singular form with respect to $\mathfrak{w}$, i.e., a Lebesgue type decomposition need not be unique; see Section 4. The presentation leading towards the Lebesgue decomposition in Theorem 2.11 is influenced by the treatment of Ando's results in [7]. The new ingredient here is the concept of parallel addition of forms on $\mathfrak{V}$. The basic idea is that any form $\mathfrak{w}$ induces a sublinear idempotent operator $\mathbf{D}_{\mathfrak{w}}$ on the set of all forms on $\mathfrak{V}$. This operator provides the Lebesgue decomposition of an arbitrary form. The terminology of dominated and almost dominated is used instead of bounded and quasi-bounded as in [7] and instead of strongly continuous and absolutely continuous as in [4].

## 3. Hilbert spaces associated with forms

Let $\mathfrak{t}$ and $\mathfrak{w}$ be nonnegative forms on a complex linear space $\mathfrak{V}$. The decomposition (2.13) of $\mathfrak{t}$ into an almost dominated and a singular part has a geometric interpretation in terms of Hilbert spaces associated with the pair $\mathfrak{t}$ and $\mathfrak{w}$. This construction goes back to B. Simon [29,30], who decomposed a densely defined form in a Hilbert space.

### *3.1. A Hilbert space interpretation of the parallel sum*

Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on a complex linear space $\mathfrak{V}$. Let $\ker\mathfrak{t}$ and $\ker\mathfrak{w}$ be the kernels of the corresponding quadratic forms; then the quotient spaces

$$\mathfrak{V}/\ker\mathfrak{t} \quad \text{and} \quad \mathfrak{V}/\ker\mathfrak{w}$$

with the inner products $\mathfrak{t}(\cdot,\cdot)$ and $\mathfrak{w}(\cdot,\cdot)$ are inner product spaces. Their Hilbert space completions are denoted by $\mathfrak{H}_\mathfrak{t}$ and $\mathfrak{H}_\mathfrak{w}$. The sum $\mathfrak{t}+\mathfrak{w}$ is a form on $\mathfrak{V}$ whose quadratic form has a kernel given by

$$\ker(\mathfrak{t}+\mathfrak{w}) = \ker\mathfrak{t} \cap \ker\mathfrak{w},$$

so that

$$\ker(\mathfrak{t}+\mathfrak{w}) \subset \ker\mathfrak{t}, \qquad \ker(\mathfrak{t}+\mathfrak{w}) \subset \ker\mathfrak{w}.$$

The quotient space

$$\mathfrak{V}/\ker(\mathfrak{t}+\mathfrak{w})$$

with the inner product $(\mathfrak{t}+\mathfrak{w})(\cdot,\cdot)$ is an inner product space. Its Hilbert space completion is denoted by $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$. Define the linear relation $U$ from $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$ to $\mathfrak{H}_\mathfrak{w} \oplus \mathfrak{H}_\mathfrak{t}$ by

$$U = \bigl\{\bigl\{h+\ker(\mathfrak{t}+\mathfrak{w}), (h+\ker\mathfrak{w}) \oplus (h+\ker\mathfrak{t})\bigr\} \in \mathfrak{H}_{\mathfrak{t}+\mathfrak{w}} \times (\mathfrak{H}_\mathfrak{w} \oplus \mathfrak{H}_\mathfrak{t})\colon h \in \mathfrak{V}\bigr\}, \quad (3.1)$$

so that

$$\operatorname{dom} U = \mathfrak{V}/\ker(\mathfrak{t}+\mathfrak{w}), \qquad \operatorname{ran} U = (\mathfrak{V}/\ker\mathfrak{w}) \oplus (\mathfrak{V}/\ker\mathfrak{t}).$$

Clearly, the relation $U$ is isometric and therefore $U$ is an operator. Its closure $U^{**}$ is an isometry from all of $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$ into $\mathfrak{H}_\mathfrak{w} \oplus \mathfrak{H}_\mathfrak{t}$. Let $P$ denote the orthogonal projection from $\mathfrak{H}_\mathfrak{w} \oplus \mathfrak{H}_\mathfrak{t}$ onto $(\operatorname{ran} U^{**})^\perp$.

**Proposition 3.1.** *Let $f, g \in \mathfrak{V}$, then*

$$\bigl\|P(f+\ker\mathfrak{w}, g+\ker\mathfrak{t})\bigr\|^2 = (\mathfrak{t}:\mathfrak{w})[f-g].$$

**Proof.** Let $f, g \in \mathfrak{V}$, $\varphi \in \ker \mathfrak{t}$, and $\psi \in \ker \mathfrak{w}$. Calculate the distance from the element $(f+\psi, g+\varphi) \in \mathfrak{H}_{\mathfrak{w}} \oplus \mathfrak{H}_{\mathfrak{t}}$ to the subspace $\operatorname{ran} U^{**}$ via

$$\begin{aligned}\operatorname{dist}\big((f+\psi, g+\varphi), \operatorname{ran} U\big)^2 &= \inf_{h\in\mathfrak{V}}\big\{\mathfrak{w}[f+h+\ker\mathfrak{w}] + \mathfrak{t}[h+g+\ker\mathfrak{t}]\big\}\\ &= \inf_{h\in\mathfrak{V}}\big\{\mathfrak{w}[f+h] + \mathfrak{t}[h+g]\big\}.\end{aligned}$$

Replace $g+h$ by $g'$, so that

$$\begin{aligned}\operatorname{dist}\big((f+\psi, g+\varphi), \operatorname{ran} U\big)^2 &= \inf_{g'\in\mathfrak{V}}\big\{\mathfrak{w}[f-g+g'] + \mathfrak{t}[g']\big\}\\ &= (\mathfrak{w}:\mathfrak{t})[f-g].\end{aligned}$$

Clearly, the left-hand side is equal to $\|P(f+\ker\mathfrak{w}, g+\ker\mathfrak{t})\|^2$. □

This proposition provides another proof, now involving Hilbert space techniques, that the parallel sum $\mathfrak{t}:\mathfrak{w}$ as defined in (2.3) of Proposition 2.2 is actually a form.

**Corollary 3.2.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on a complex linear space* $\mathfrak{V}$, *and let the isometric operator* $U:\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}} \to \mathfrak{H}_{\mathfrak{w}} \oplus \mathfrak{H}_{\mathfrak{t}}$ *be defined by* (3.1). *Then the following statements are equivalent*:

(i) $\mathfrak{t}$ *is singular with respect to* $\mathfrak{w}$;
(ii) $\operatorname{ran} U^{**} = \mathfrak{H}_{\mathfrak{w}} \oplus \mathfrak{H}_{\mathfrak{t}}$.

**Proof.** By Proposition 2.10 item (i) is equivalent to $\mathfrak{t}:\mathfrak{w}=0$. It follows from Proposition 3.1 that $\mathfrak{t}:\mathfrak{w}=0$ is equivalent to $P=0$, i.e., $\operatorname{ran} U^{**} = \mathfrak{H}_{\mathfrak{w}} \oplus \mathfrak{H}_{\mathfrak{t}}$. □

Define the linear relation $\iota_{\mathfrak{t}}$ from $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$ to $\mathfrak{H}_{\mathfrak{w}}$ by

$$\iota_{\mathfrak{t}} = \big\{\{\varphi+\mathfrak{K}, \varphi+\mathfrak{L}\} \in \mathfrak{H}_{\mathfrak{t}+\mathfrak{w}} \times \mathfrak{H}_{\mathfrak{w}}\colon \varphi \in \mathfrak{V}\big\}, \tag{3.2}$$

where $\mathfrak{K} = \ker(\mathfrak{t}+\mathfrak{w})$, $\mathfrak{L} = \ker\mathfrak{w}$, and denote $\mathfrak{D} = \operatorname{dom}\iota_{\mathfrak{t}}$, $\mathfrak{R} = \operatorname{ran}\iota_{\mathfrak{t}}$. Since $\mathfrak{K} \subset \mathfrak{L}$ the relation $\iota_{\mathfrak{t}}$ is the graph of an operator. In fact, $\iota_{\mathfrak{t}}$ is a densely defined contraction from the Hilbert space $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$ to the Hilbert space $\mathfrak{H}_{\mathfrak{w}}$, and hence its closure $\iota_{\mathfrak{t}}^{**}$ is a contraction with $\operatorname{dom}\iota_{\mathfrak{t}}^{**} = \mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$ and $\operatorname{ran}\iota_{\mathfrak{t}}^{**}$ dense in $\mathfrak{H}_{\mathfrak{w}}$. Therefore, also $\iota_{\mathfrak{t}}^{*}$ is a contraction from $\operatorname{dom}\iota_{\mathfrak{t}}^{*} = \mathfrak{H}_{\mathfrak{w}}$ to $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$, and $\operatorname{ran}\iota_{\mathfrak{t}}^{*}$ is dense in $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}} \ominus \ker\iota_{\mathfrak{t}}^{**}$. Moreover, $\ker\iota_{\mathfrak{t}}^{*} = \{0\}$. Since $\mathfrak{R}$ is dense in $\mathfrak{H}_{\mathfrak{w}}$, the image $\iota_{\mathfrak{t}}^{*}(\mathfrak{R})$ is dense in $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}} \ominus \ker\iota_{\mathfrak{t}}^{**}$, too. From the definition (3.2) one obtains the following description for the kernel of the closure of $\iota_{\mathfrak{t}}$.

**Lemma 3.3.** *The kernel of the closure of the mapping* $\iota_{\mathfrak{t}}$ *defined in* (3.2) *is given by*

$$\ker\iota_{\mathfrak{t}}^{**} = \Big\{\varphi = \lim_{k\to\infty}\{\varphi_k+\mathfrak{K}\} \in \mathfrak{H}_{\mathfrak{t}+\mathfrak{w}} : \mathfrak{t}[\varphi_k-\varphi_\ell] \to 0, \mathfrak{w}[\varphi_k] \to 0,\ k,\ell\to\infty\Big\}.$$

**Proof.** By definition (3.2) the inclusion $\{\varphi, 0\} \in \iota_{\mathfrak{t}}^{**}$ means that $\{\varphi, 0\} \in \mathfrak{H}_{\mathfrak{t}+\mathfrak{w}} \times \mathfrak{H}_{\mathfrak{w}}$ such that

$$[\varphi-\varphi_k]_{\mathfrak{t}+\mathfrak{w}} \to 0, \qquad [\varphi_k]_{\mathfrak{w}} \to 0,$$

for some sequence $\varphi_k \in \mathfrak{V}$, i.e., $\varphi = \lim_{k\to\infty}\{\varphi_k + \mathfrak{K}\} \in \mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$, where

$$\mathfrak{t}[\varphi_k - \varphi_\ell] + \mathfrak{w}[\varphi_k - \varphi_\ell] \to 0, \qquad \mathfrak{w}[\varphi_k] \to 0,$$

or equivalently,

$$\mathfrak{t}[\varphi_k - \varphi_\ell] \to 0, \qquad \mathfrak{w}[\varphi_k] \to 0.$$

This completes the proof. □

*3.2. The regular and singular parts and their metric characterization*

Let $Q_\mathfrak{t}$ be the orthogonal projection from $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$ onto $\ker \iota_\mathfrak{t}^{**}$. Define the sesquilinear form $\mathfrak{t}_{\mathrm{sing}}$ on $\mathfrak{V}$ via

$$\mathfrak{t}_{\mathrm{sing}}[\varphi] = \left\| Q_\mathfrak{t}(\varphi + \mathfrak{K})\right\|^2_{\mathfrak{t}+\mathfrak{w}}, \quad \varphi \in \mathfrak{V}. \tag{3.3}$$

The form $\mathfrak{t}_{\mathrm{sing}}$ is clearly nonnegative. It is called the *singular part* of the form $\mathfrak{t}$ with respect to the form $\mathfrak{w}$. For a single form $\mathfrak{t}$, densely defined in a Hilbert space $\mathfrak{H}$, this definition coincides with the one given by B. Simon [30]. The next theorem gives a characterization for the singular part.

**Theorem 3.4.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be nonnegative forms on a complex linear space* $\mathfrak{V}$. *The form* $\mathfrak{t}_{\mathrm{sing}}$ *on* $\mathfrak{V}$, *defined in* (3.3), *is given by*

$$\mathfrak{t}_{\mathrm{sing}}[\varphi] = \mathfrak{t}[\varphi] + \inf_{h\in\mathfrak{V}}\Big\{\mathfrak{w}[\varphi+h] - \inf_{g\in\mathfrak{V}}\big\{\mathfrak{t}[g] + \mathfrak{w}[g+h]\big\}\Big\}, \quad \varphi \in \mathfrak{V}, \tag{3.4}$$

*so that the form* $\mathfrak{t}_{\mathrm{sing}}$ *is nonnegative and dominated by* $\mathfrak{t}$:

$$0 \leqslant \mathfrak{t}_{\mathrm{sing}} \leqslant \mathfrak{t}. \tag{3.5}$$

**Proof.** Since $\iota_\mathfrak{t}^*(\mathfrak{R})$ is dense in $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}} \ominus \ker \iota_\mathfrak{t}^{**}$, it follows that

$$\begin{aligned}
&\big(Q_\mathfrak{t}(\varphi+\mathfrak{K}), Q_\mathfrak{t}(\varphi+\mathfrak{K})\big)_{\mathfrak{t}+\mathfrak{w}} \\
&\quad = \inf_{h\in\mathfrak{V}}\big\{\big((\varphi+\mathfrak{K}) + \iota_\mathfrak{t}^*(h+\mathfrak{L}), (\varphi+\mathfrak{K}) + \iota_\mathfrak{t}^*(h+\mathfrak{L})\big)_{\mathfrak{t}+\mathfrak{w}}\big\} \\
&\quad = \inf_{h\in\mathfrak{V}}\big\{\mathfrak{t}[\varphi] + \mathfrak{w}[\varphi] + \mathfrak{w}(\varphi,h) + \mathfrak{w}(h,\varphi) + \big(\iota_\mathfrak{t}^*(h+\mathfrak{L}), \iota_\mathfrak{t}^*(h+\mathfrak{L})\big)_{\mathfrak{t}+\mathfrak{w}}\big\} \\
&\quad = \mathfrak{t}[\varphi] + \inf_{h\in\mathfrak{V}}\big\{\mathfrak{w}[\varphi+h] - \mathfrak{w}[h] + \big(\iota_\mathfrak{t}^*(h+\mathfrak{L}), \iota_\mathfrak{t}^*(h+\mathfrak{L})\big)_{\mathfrak{t}+\mathfrak{w}}\big\}.
\end{aligned} \tag{3.6}$$

Furthermore, since $\mathfrak{D}$ is dense in $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$, one obtains

$$\begin{aligned}0 &= \inf_{g\in\mathfrak{V}}\bigl\{\bigl((g+\mathfrak{K})+\iota_{\mathfrak{t}}^*(h+\mathfrak{L}),(g+\mathfrak{K})+\iota_{\mathfrak{t}}^*(h+\mathfrak{L})\bigr)_{\mathfrak{t}+\mathfrak{w}}\bigr\}\\ &= \bigl(\iota_{\mathfrak{t}}^*(h+\mathfrak{L}),\iota_{\mathfrak{t}}^*(h+\mathfrak{L})\bigr)_{\mathfrak{t}+\mathfrak{w}}+\inf_{g\in\mathfrak{V}}\bigl\{\mathfrak{t}[g]+\mathfrak{w}[g]+\mathfrak{w}(g,h)+\mathfrak{w}(h,g)\bigr\}\\ &= -\mathfrak{w}[h]+\bigl(\iota_{\mathfrak{t}}^*(h+\mathfrak{L}),\iota_{\mathfrak{t}}^*(h+\mathfrak{L})\bigr)_{\mathfrak{t}+\mathfrak{w}}+\inf_{g\in\mathfrak{V}}\bigl\{\mathfrak{t}[g]+\mathfrak{w}[g+h]\bigr\}. \end{aligned} \tag{3.7}$$

Combining (3.3), (3.6), and (3.7) leads to

$$\mathfrak{t}_{\mathrm{sing}}[\varphi]=\mathfrak{t}[\varphi]+\inf_{h\in\mathfrak{V}}\Bigl\{\mathfrak{w}[\varphi+h]-\inf_{g\in\mathfrak{V}}\bigl\{\mathfrak{t}[g]+\mathfrak{w}[g+h]\bigr\}\Bigr\}. \tag{3.8}$$

This completes the proof of (3.4).

By definition the form $\mathfrak{t}_{\mathrm{sing}}$ is nonnegative. As to the second inequality in (3.5), observe that by (3.4)

$$\mathfrak{t}_{\mathrm{sing}}[\varphi]\leqslant\mathfrak{t}[\varphi]+\mathfrak{w}[\varphi+h]-\inf_{g\in\mathfrak{V}}\bigl\{\mathfrak{t}[g]+\mathfrak{w}[g+h]\bigr\},\quad \varphi\in\mathfrak{V}, \tag{3.9}$$

holds for all $h\in\mathfrak{V}$. Take in (3.9) $h=-\varphi$ and $\mathfrak{t}_{\mathrm{sing}}[\varphi]\leqslant\mathfrak{t}[\varphi]$, $\varphi\in\mathfrak{V}$, follows. □

The nonnegative forms $\mathfrak{t}$ and $\mathfrak{t}_{\mathrm{sing}}$ give rise to a new form $\mathfrak{t}_{\mathrm{reg}}$ as follows:

$$\mathfrak{t}_{\mathrm{reg}}[\varphi]=\mathfrak{t}[\varphi]-\mathfrak{t}_{\mathrm{sing}}[\varphi],\quad \varphi\in\mathfrak{V}. \tag{3.10}$$

The form $\mathfrak{t}_{\mathrm{reg}}$ is called the *regular part* of the form $\mathfrak{t}$ with respect to the form $\mathfrak{w}$. The regular part admits a characterization similar to the one of the singular part.

**Theorem 3.5.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be nonnegative forms on a complex linear space* $\mathfrak{V}$. *The sesquilinear form* $\mathfrak{t}_{\mathrm{reg}}$ *on* $\mathfrak{V}$, *defined in* (3.10), *is given by*

$$\mathfrak{t}_{\mathrm{reg}}[\varphi]=\sup_{h\in\mathfrak{V}}\bigl\{(\mathfrak{t}:\mathfrak{w})[h]-\mathfrak{w}[\varphi+h]\bigr\},\quad \varphi\in\mathfrak{V}, \tag{3.11}$$

*so that* $\mathfrak{t}_{\mathrm{reg}}$ *is nonnegative and majorized by* $\mathfrak{t}$:

$$0\leqslant\mathfrak{t}_{\mathrm{reg}}\leqslant\mathfrak{t}. \tag{3.12}$$

**Proof.** The identities (3.10) and (3.4) give the following representation

$$\mathfrak{t}_{\mathrm{reg}}[\varphi]=\sup_{h\in\mathfrak{V}}\Bigl\{\inf_{g\in\mathfrak{V}}\bigl\{\mathfrak{w}[g+h]+\mathfrak{t}[g]\bigr\}-\mathfrak{w}[\varphi+h]\Bigr\},\quad \varphi\in\mathfrak{V}. \tag{3.13}$$

With the definition of $\mathfrak{t}:\mathfrak{w}$ in (2.3) the identity (3.13) leads to (3.11).

The inequalities $0\leqslant\mathfrak{t}_{\mathrm{reg}}\leqslant\mathfrak{t}$ are clear from (3.10) and the inequalities (3.5) in Theorem 3.4. □

Observe that the definition (3.3) of the form $\mathfrak{t}_{\mathrm{sing}}$, and consequently the definition (3.10) of the form $\mathfrak{t}_{\mathrm{reg}}$, was given via a construction involving the Hilbert spaces $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$ and $\mathfrak{H}_{\mathfrak{t}}$. However, Theorems 3.4 and 3.5 imply that the resulting forms $\mathfrak{t}_{\text{sing}}$ and $\mathfrak{t}_{\text{reg}}$ in (3.4) and (3.11), respectively, only depend on the forms $\mathfrak{t}$ and $\mathfrak{w}$.

The expression in the right-hand side of (3.11) is a form on $\mathfrak{V}$: it can be viewed as the *parallel difference* $(\mathfrak{t} : \mathfrak{w}) \div \mathfrak{w}$ of the forms $\mathfrak{t} : \mathfrak{w}$ and $\mathfrak{w}$; cf. [23,24] for the case of bounded operators. A different approach to the parallel sum and parallel difference of a pair of forms is presented in [11].

### 3.3. Closable forms

Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on a complex linear space $\mathfrak{V}$. The form $\mathfrak{t}$ is said to be closable with respect to $\mathfrak{w}$ if and only if for any sequence $\varphi_n \in \mathfrak{V}$:

$$\mathfrak{t}[\varphi_n - \varphi_m] \to 0, \qquad \mathfrak{w}[\varphi_n] \to 0 \quad \Rightarrow \quad \mathfrak{t}[\varphi_n] \to 0. \tag{3.14}$$

The next theorem characterizes closability via the regular part.

**Theorem 3.6.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on a complex linear space* $\mathfrak{V}$. *Then* $\mathfrak{t}_{\text{reg}}$ *is closable with respect to* $\mathfrak{w}$. *Moreover, the following statements are equivalent*:

(i) $\mathfrak{t}$ *is closable with respect to* $\mathfrak{w}$;
(ii) $\mathfrak{t} = \mathfrak{t}_{\text{reg}}$.

**Proof.** Recall that the form $\mathfrak{t}_{\text{reg}}$ has the representation (3.11). Let $\varphi_n \in \mathfrak{V}$ be a sequence which satisfies the conditions in (3.14) with $\mathfrak{t} = \mathfrak{t}_{\text{reg}}$, and let $\varepsilon > 0$. According to the first condition in (3.14) there is an $N \in \mathbb{N}$ such that for all $n, m \geqslant N$

$$\mathfrak{t}_{\text{reg}}[\varphi_n - \varphi_m] < \varepsilon.$$

Hence, due to (3.11), it follows for all $h \in \mathfrak{V}$ and $n, m \geqslant N$ that

$$(\mathfrak{t} : \mathfrak{w})[h] - \mathfrak{w}[\varphi_n - \varphi_m + h] < \varepsilon.$$

Letting $m \to \infty$ it follows from the second condition in (3.14) that for all $h \in \mathfrak{V}$ and $n \geqslant N$:

$$(\mathfrak{t} : \mathfrak{w})[h] - \mathfrak{w}[\varphi_n + h] \leqslant \varepsilon,$$

cf. (2.2). Therefore, according to (3.11), for all $n \geqslant N$ one obtains

$$\mathfrak{t}_{\text{reg}}[\varphi_n] = \sup_{h \in \mathfrak{V}} \bigl\{ (\mathfrak{t} : \mathfrak{w})[h] - \mathfrak{w}[\varphi_n + h] \bigr\} \leqslant \varepsilon.$$

This means that $\mathfrak{t}_{\text{reg}}[\varphi_n] \to 0$. Hence the form $\mathfrak{t}_{\text{reg}}$ is closable with respect to the form $\mathfrak{w}$.

Now the equivalence of the statements (i) and (ii) will be shown.

(i) $\Rightarrow$ (ii) Assume that $\mathfrak{t}$ is closable with respect to $\mathfrak{w}$. It suffices to show that $\mathfrak{t}_{\text{sing}} = 0$. Recall that $Q_{\mathfrak{t}}$ in (3.3) is the orthogonal projection from $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$ onto $\ker \iota_{\mathfrak{t}}^{**}$, so that it is enough to prove that $\ker \iota_{\mathfrak{t}}^{**} = \{0\}$. This equality follows immediately by combining the representation of $\ker \iota_{\mathfrak{t}}^{**}$ in Lemma 3.3 with the definition of closability in (3.14).

(ii) ⇒ (i) Assume that $\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}}$. Then the form $\mathfrak{t}$ is closable, since the form $\mathfrak{t}_{\mathrm{reg}}$ is closable by what has been shown in the first part of the proof. □

By Theorem 3.6 the regular part $\mathfrak{t}_{\mathrm{reg}}$ is closable. In fact, the regular part $\mathfrak{t}_{\mathrm{reg}}$ is the largest closable form on $\mathfrak{V}$ majorized by $\mathfrak{t}$. This is a consequence of Theorem 3.6 and the following monotonicity property of the regular part.

**Proposition 3.7.** *Assume that* $\tilde{\mathfrak{t}}$ *and* $\mathfrak{t}$ *are nonnegative forms on a complex linear space* $\mathfrak{V}$ *and let* $\tilde{\mathfrak{t}}_{\mathrm{reg}}$ *and* $\mathfrak{t}_{\mathrm{reg}}$ *be their regular parts with respect to the form* $\mathfrak{w}$. *Then*

$$\tilde{\mathfrak{t}} \leqslant \mathfrak{t} \quad \Rightarrow \quad \tilde{\mathfrak{t}}_{\mathrm{reg}} \leqslant \mathfrak{t}_{\mathrm{reg}}. \tag{3.15}$$

*In particular, the regular part* $\mathfrak{t}_{\mathrm{reg}}$ *is the largest closable form on* $\mathfrak{V}$ *majorized by* $\mathfrak{t}$:

$$\tilde{\mathfrak{t}}_{\mathrm{reg}} = \tilde{\mathfrak{t}} \leqslant \mathfrak{t} \quad \Rightarrow \quad \tilde{\mathfrak{t}} \leqslant \mathfrak{t}_{\mathrm{reg}}.$$

**Proof.** The monotonicity property (3.15) is an immediate consequence of the representation (3.11) in Theorem 3.5 and Lemma 2.3(v).

By Theorem 3.5 one has $0 \leqslant \mathfrak{t}_{\mathrm{reg}} \leqslant \mathfrak{t}$ and according to Theorem 3.6 the form $\mathfrak{t}_{\mathrm{reg}}$ is closable with respect to $\mathfrak{w}$. Moreover, if $\tilde{\mathfrak{t}}$ is closable with respect to $\mathfrak{w}$, then $\tilde{\mathfrak{t}} = \tilde{\mathfrak{t}}_{\mathrm{reg}}$ by Theorem 3.6. Now the second statement follows from the monotonicity property (3.15). □

The next theorem gives the important connection between the notions of closability and almost domination of forms.

**Theorem 3.8.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be nonnegative forms on a complex linear space* $\mathfrak{V}$. *Then the following statements are equivalent*:

(i) $\mathfrak{t}$ *is almost dominated by* $\mathfrak{w}$;
(ii) $\mathfrak{t}$ *is closable with respect to* $\mathfrak{w}$.

**Proof.** (i) ⇒ (ii) Let $\mathfrak{t}$ be almost dominated by $\mathfrak{w}$. Then by definition there exists a nondecreasing sequence of nonnegative forms $\mathfrak{t}_n$ on $\mathfrak{V}$ and a sequence of numbers $\alpha_n > 0$, such that for all $\varphi \in \mathfrak{V}$

$$\mathfrak{t}_n[\varphi] \leqslant \alpha_n \mathfrak{w}[\varphi], \qquad \mathfrak{t}_n[\varphi] \nearrow \mathfrak{t}[\varphi].$$

The assumption $\mathfrak{t}_n \leqslant \alpha_n \mathfrak{w}$ implies that the form $\mathfrak{t}_n$ is closable with respect to $\mathfrak{w}$; cf. (3.14). Hence $\mathfrak{t}_n = (\mathfrak{t}_n)_{\mathrm{reg}}$ by Theorem 3.6. The assumption $\mathfrak{t}_n \leqslant \mathfrak{t}$ and Proposition 3.7 imply that $\mathfrak{t}_n = (\mathfrak{t}_n)_{\mathrm{reg}} \leqslant \mathfrak{t}_{\mathrm{reg}}$, where $\mathfrak{t}_{\mathrm{reg}}$ is the regular part of $\mathfrak{t}$. Therefore, $\mathfrak{t}[\varphi] = \lim_{n\to\infty} \mathfrak{t}_n[\varphi] \leqslant \mathfrak{t}_{\mathrm{reg}}[\varphi]$ and this forces the equality $\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}}$ (see Theorem 3.5), so that $\mathfrak{t}$ is closable by Theorem 3.6.

(ii) ⇒ (i) Let $\mathfrak{t}$ be closable with respect to $\mathfrak{w}$, so that $\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}}$ by Theorem 3.6. According to Proposition 2.7 $(\mathbf{D}_{\mathfrak{w}}\mathfrak{t}) : \mathfrak{w} = \mathfrak{t} : \mathfrak{w}$ and hence it follows from (3.11) in Theorem 3.5 that $(\mathbf{D}_{\mathfrak{w}}\mathfrak{t})_{\mathrm{reg}} = \mathfrak{t}_{\mathrm{reg}}$. Therefore, by applying (i) in Lemma 2.4 and (3.12) in Theorem 3.5 one gets

$$\mathfrak{t} \geqslant \mathbf{D}_{\mathfrak{w}}\mathfrak{t} \geqslant (\mathbf{D}_{\mathfrak{w}}\mathfrak{t})_{\mathrm{reg}} = \mathfrak{t}_{\mathrm{reg}} = \mathfrak{t}.$$

Thus, $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}$ and hence $\mathfrak{t}$ is almost dominated by $\mathfrak{w}$ by Theorem 2.6. □

### 3.4. The Lebesgue decomposition via associated Hilbert spaces

Let $\mathfrak{t}$ and $\mathfrak{w}$ be nonnegative forms on a complex linear space $\mathfrak{V}$. The introduction of the forms $\mathfrak{t}_{\mathrm{reg}}$ and $\mathfrak{t}_{\mathrm{sing}}$ in (3.3) and (3.10) leads to the decomposition $\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}} + \mathfrak{t}_{\mathrm{sing}}$. It will now be shown that this decomposition coincides with the Lebesgue decomposition in Theorem 2.11.

**Theorem 3.9.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be nonnegative forms on a complex linear space* $\mathfrak{V}$. *Then*

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}}, \qquad \mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}_{\mathrm{sing}},$$

*so that* $\mathfrak{t}_{\mathrm{reg}}$ *is almost dominated by* $\mathfrak{w}$ *and* $\mathfrak{t}_{\mathrm{sing}}$ *is singular with respect to* $\mathfrak{w}$.

**Proof.** Observe that $\mathfrak{t}_{\mathrm{reg}} \leqslant \mathfrak{t}$, which implies by (ii) of Lemma 2.4 that

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t}_{\mathrm{reg}} \leqslant \mathbf{D}_{\mathfrak{w}}\mathfrak{t}.$$

Now $\mathfrak{t}_{\mathrm{reg}}$ is closable (Theorem 3.6) and, hence, almost dominated by $\mathfrak{w}$ according to Theorem 3.8. Therefore, $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}_{\mathrm{reg}} = \mathfrak{t}_{\mathrm{reg}}$ by Theorem 2.6. Furthermore, by (i) of Lemma 2.4, $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leqslant \mathfrak{t}$. Now $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is almost dominated by $\mathfrak{w}$ (cf. Theorem 2.11), so that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is closable by Theorem 3.8. From Proposition 3.7 one concludes that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leqslant \mathfrak{t}_{\mathrm{reg}}$. Combining these results gives

$$\mathfrak{t}_{\mathrm{reg}} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t}_{\mathrm{reg}} \leqslant \mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leqslant \mathfrak{t}_{\mathrm{reg}},$$

which leads to the required identity. □

The following result is now immediate.

**Corollary 3.10.** *The regular part of* $\mathfrak{t}_{\mathrm{reg}}$ *is* $\mathfrak{t}_{\mathrm{reg}}$ *itself, or, equivalently,*

$$(\mathfrak{t}_{\mathrm{reg}})_{\mathrm{sing}} = 0, \tag{3.16}$$

*and the singular part of* $\mathfrak{t}_{\mathrm{sing}}$ *is* $\mathfrak{t}_{\mathrm{sing}}$ *itself, or, equivalently,*

$$(\mathfrak{t}_{\mathrm{sing}})_{\mathrm{reg}} = 0. \tag{3.17}$$

In Theorem 3.6 it has been shown that $\mathfrak{t}$ is closable with respect to $\mathfrak{w}$ if and only if $\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}}$. For the sake of completeness also the following result will be stated.

**Corollary 3.11.** *The following statements are equivalent*:

(i) $\mathfrak{t} = \mathfrak{t}_{\mathrm{sing}}$;
(ii) $\mathfrak{t}$ *is singular with respect to* $\mathfrak{w}$.

**Proof.** By Proposition 2.10 $\mathfrak{t}$ is singular with respect to $\mathfrak{w}$ if and only if $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = 0$. Hence, the statement is obtained from Theorem 3.9. □

It was shown in Proposition 2.10 that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = 0$ is equivalent to $\mathfrak{t} : \mathfrak{w} = 0$. Singular forms have been characterized by Koshmanenko and Ôta [17,18]. The condition $\mathfrak{t} : \mathfrak{w} = 0$ is equivalent to their characterization. It is interesting to note that there is also a weaker (pointwise) version of this result.

**Proposition 3.12.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be nonnegative forms on a complex linear space* $\mathfrak{V}$. *Let* $\varphi \in \mathfrak{V}$, *then*

$$(\mathfrak{t} : \mathfrak{w})[\varphi] = 0 \quad \Leftrightarrow \quad \mathfrak{t}_{\mathrm{reg}}[\varphi] = 0.$$

**Proof.** ($\Leftarrow$) Observe, that (see Proposition 2.7, Lemma 2.4, and Theorem 3.9)

$$\mathfrak{t} : \mathfrak{w} = \mathbf{D}_{\mathfrak{w}}(\mathfrak{t} : \mathfrak{w}) \leqslant \mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}}. \tag{3.18}$$

Hence, if $\mathfrak{t}_{\mathrm{reg}}[\varphi] = 0$, then it follows from (3.18) that $(\mathfrak{t} : \mathfrak{w})[\varphi] = 0$.

($\Rightarrow$) Recall from Lemma 2.3 that $\mathfrak{w} \geqslant \mathfrak{t} : \mathfrak{w}$. Hence, if $(\mathfrak{t} : \mathfrak{w})[\varphi] = 0$, then

$$\mathfrak{w}[\varphi + h] \geqslant (\mathfrak{t} : \mathfrak{w})[\varphi + h] = (\mathfrak{t} : \mathfrak{w})[h], \quad h \in \mathfrak{V},$$

where the implication in (2.1) has been used. Then (3.11) shows that $\mathfrak{t}_{\mathrm{reg}}[\varphi] \leqslant 0$, i.e., $\mathfrak{t}_{\mathrm{reg}}[\varphi] = 0$. □

The orthogonal projection $Q_{\mathfrak{t}}$ from $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$ onto $\ker \iota_{\mathfrak{t}}^{**}$ defines the singular part $\mathfrak{t}_{\mathrm{sing}}$ of $\mathfrak{t}$ by (3.3). The definition of $\mathfrak{t}_{\mathrm{reg}}$ in (3.10) leads to the representation

$$(\mathfrak{w} + \mathfrak{t}_{\mathrm{reg}})[\varphi] = \left\| (I - Q_{\mathfrak{t}})(\varphi + \mathfrak{K}) \right\|_{\mathfrak{t}+\mathfrak{w}}^2, \quad \varphi \in \mathfrak{V}/\mathfrak{K}. \tag{3.19}$$

**Proposition 3.13.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on a complex linear space* $\mathfrak{V}$. *Then the form* $\mathfrak{x} = \mathfrak{t}_{\mathrm{reg}}$ *is a solution of the equation*

$$(\mathfrak{t} - \mathfrak{x}) : (\mathfrak{w} + \mathfrak{x}) = 0. \tag{3.20}$$

**Proof.** With $g, \varphi \in \mathfrak{V}$ the representations (3.3) and (3.19) lead to

$$\begin{aligned} \mathfrak{w}[g - \varphi] + \mathfrak{t}_{\mathrm{reg}}[g - \varphi] + \mathfrak{t}_{\mathrm{sing}}[g] &= \left\| (I - Q_{\mathfrak{t}})(g - \varphi) \right\|_{\mathfrak{t}+\mathfrak{w}}^2 + \| Q_{\mathfrak{t}} g \|_{\mathfrak{t}+\mathfrak{w}}^2 \\ &= \left\| g - (I - Q_{\mathfrak{t}})\varphi \right\|_{\mathfrak{t}+\mathfrak{w}}^2. \end{aligned} \tag{3.21}$$

Hence it follows that

$$\inf_{g \in \mathfrak{V}} \left\{ \mathfrak{w}[g - \varphi] + \mathfrak{t}_{\mathrm{reg}}[g - \varphi] + \mathfrak{t}_{\mathrm{sing}}[g] \right\} = \inf_{g \in \mathfrak{D}} \left\| g - (I - Q_{\mathfrak{t}})\varphi \right\|_{\mathfrak{t}+\mathfrak{w}}^2 = 0,$$

which shows $\mathfrak{t}_{\mathrm{sing}} : (\mathfrak{w} + \mathfrak{t}_{\mathrm{reg}}) = 0$ or, equivalently (3.20). □

Some further similar identities are given in the next corollary.

**Corollary 3.14.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on a complex linear space* $\mathfrak{V}$. *Then*

$$\mathfrak{t}_{\mathrm{sing}} : \mathfrak{w} = 0, \qquad \mathfrak{t}_{\mathrm{sing}} : \mathfrak{t}_{\mathrm{reg}} = 0, \qquad \mathfrak{t}_{\mathrm{reg}} : \mathfrak{w} = \mathfrak{t} : \mathfrak{w}. \tag{3.22}$$

**Proof.** The inequalities

$$\mathfrak{t}_{\mathrm{sing}} : \mathfrak{w} \leqslant \mathfrak{t}_{\mathrm{sing}} : (\mathfrak{w} + \mathfrak{t}_{\mathrm{reg}}), \qquad \mathfrak{t}_{\mathrm{sing}} : \mathfrak{t}_{\mathrm{reg}} \leqslant \mathfrak{t}_{\mathrm{sing}} : (\mathfrak{w} + \mathfrak{t}_{\mathrm{reg}}),$$

imply the first two identities in (3.22). Moreover, from Proposition 2.7 and Theorem 3.9 one obtains $\mathfrak{t} : \mathfrak{w} = (\mathbf{D}_{\mathfrak{w}}\mathfrak{t}) : \mathfrak{w} = \mathfrak{t}_{\mathrm{reg}} : \mathfrak{w}$. □

Observe that the first identity in this corollary is also clear from Proposition 2.10 and Corollary 3.11.

## 4. The uniqueness of Lebesgue type decompositions

Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on a complex linear space $\mathfrak{V}$. Recall that a decomposition of $\mathfrak{t}$ into an almost dominated form and a singular form with respect to $\mathfrak{w}$ is called a Lebesgue type decomposition. The Lebesgue decomposition in Theorem 2.11 of $\mathfrak{t}$ into the (maximal) almost dominated form $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ and the corresponding singular form (with respect to $\mathfrak{w}$) is unique. These forms have been also identified in Theorem 3.9 with the regular and singular parts of $\mathfrak{t}$ (with respect to $\mathfrak{w}$). In general, a Lebesgue type decomposition into an almost dominated and a singular form is not unique. In this section necessary and sufficient conditions for uniqueness will be presented.

### *4.1. A decomposition of densely defined closable operators*

The following simple observation is useful; see for earlier versions [21] and [31].

**Lemma 4.1.** *Let* $T$ *be a linear relation from a Hilbert space* $\mathfrak{H}$ *to a Hilbert space* $\mathfrak{K}$. *Then the following statements are equivalent*:

(i) $T^{**}$ *is* (*the graph of*) *a bounded linear operator*;
(ii) $\operatorname{ran} T^{**} \subset \operatorname{dom} T^*$;
(iii) $\operatorname{dom} T^* = \mathfrak{K}$.

**Proof.** (i) ⇔ (iii) $T^{**}$ is (the graph of) a bounded operator if and only if $\operatorname{dom} T^{**}$ is closed and $\operatorname{mul} T^{**} = \{0\}$. Since $\operatorname{dom} T^{**}$ is closed precisely when $\operatorname{dom} T^*$ is closed (see e.g. [28]) and $\operatorname{mul} T^{**} = \{0\}$ means that $\operatorname{dom} T^*$ is dense, one concludes that then, equivalently, $\operatorname{dom} T^* = \mathfrak{K}$.

(ii) ⇔ (iii) Observe that for any relation $T$ the adjoint $T^*$ is given by $T^* = JT^{\perp}$, where $J\{f, f'\} = \{f', -f\}$. Hence $\mathfrak{H} \times \mathfrak{K} = \overline{T} \oplus T^{\perp} = T^{**} \oplus JT^*$, leading to

$$\mathfrak{H} = \operatorname{dom} T^{**} + \operatorname{ran} T^*, \qquad \mathfrak{K} = \operatorname{ran} T^{**} + \operatorname{dom} T^*, \tag{4.1}$$

from which the statement easily follows. □

For completeness, it is shown that bounded invertibility of $T^{**}$ can be characterized in a similar way.

**Lemma 4.2.** *Let $T$ be a linear relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$. Then the following statements are equivalent*:

(i) *$T^{**}$ has a bounded inverse*;
(ii) $\operatorname{dom} T^{**} \subset \operatorname{ran} T^*$;
(iii) $\operatorname{ran} T^* = \mathfrak{H}$.

**Proof.** (i) $\Leftrightarrow$ (iii) The inverse of $T^{**}$ is (the graph of) a bounded operator if and only if $\operatorname{ran} T^{**}$ is closed and $\ker T^{**} = \{0\}$. Then, equivalently, $\operatorname{ran} T^*$ is closed and dense, i.e., $\operatorname{ran} T^* = \mathfrak{H}$.
(ii) $\Leftrightarrow$ (iii) Again, this is an immediate consequence of (4.1). □

Let $T$ be a closable operator, so that $T^{**}$ is an operator. Clearly, $T$ is bounded if and only if $T^{**}$ is bounded. Hence, according to Lemma 4.1, $T$ is not bounded only in case $\operatorname{dom} T^* \neq \mathfrak{K}$.

**Proposition 4.3.** *Let $T$ be a densely defined closable operator from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$. Let $v \in \mathfrak{K}$ and let $P_v$ be the orthogonal projection from $\mathfrak{K}$ onto the linear space spanned by $v$. Then the operator $T$ admits the* (*orthogonal*) *decomposition*

$$T = A + B, \tag{4.2}$$

*with the densely defined operators $A$ and $B$ defined by*

$$A = (I - P_v)T, \qquad B = P_v T. \tag{4.3}$$

*Here $A$ is closable and, furthermore,*

(i) *if $v \in \operatorname{dom} T^*$, then $B^{**} \in \mathbf{B}(\mathfrak{H}, \mathfrak{K})$*;
(ii) *if $v \in \mathfrak{H} \setminus \operatorname{dom} T^*$, then $B$ is a singular operator, i.e.,* $\operatorname{ran} B \subset \operatorname{mul} B^{**}$.

*In fact, in case* (i) *one has $B^{**}h = (h, T^*v)_{\mathfrak{H}} v$, $h \in \mathfrak{H}$, and in case* (ii) $B^{**} = \mathfrak{H} \times \operatorname{span}\{v\}$.

**Proof.** Clearly the decomposition $T = A + B$ is valid and orthogonal with respect to the ranges. Since $T$ is densely defined and closable, the adjoint $T^*$ is a closed densely defined operator. Let $v \in \mathfrak{K}$ be normalized by $\|v\|_{\mathfrak{K}} = 1$. Then it follows from the definition (4.3) that the adjoint of $A$ is given by

$$A^* = T^*(I - P_v) = \bigl\{\{f, g\} \in \mathfrak{K} \times \mathfrak{H}\colon\ f - (f, v)_{\mathfrak{K}} v \in \operatorname{dom} T^*,\ g = T^*\bigl(f - (f, v)_{\mathfrak{K}} v\bigr)\bigr\}.$$

In particular,

$$\operatorname{dom} A^* = \operatorname{span}\{v\} \oplus \bigl(\operatorname{dom} T^* \cap \operatorname{span}\{v\}^\perp\bigr).$$

Since $\dim \operatorname{span}\{v\} = 1 < \infty$, $\operatorname{dom} T^* \cap \operatorname{span}\{v\}^\perp$ is dense in $\operatorname{span}\{v\}^\perp$ (see [10, Lemma 2.1] or [27]) and therefore $\operatorname{dom} A^*$ is dense in $\mathfrak{K}$. Hence, $\operatorname{mul} A^{**} = (\operatorname{dom} A^*)^\perp = \{0\}$ and $A^{**}$ is the graph of an operator, i.e., the operator $A$ is closable.

On the other hand, it follows from the definition (4.3) that

$$B^* = T^* P_v = \big\{\{f, g\} \in \mathfrak{K} \times \mathfrak{H}\colon \big\{(f, v)_{\mathfrak{K}} v, g\big\} \in T^*\big\}. \tag{4.4}$$

(i) If $v \in \operatorname{dom} T^*$, then (4.4) shows that $\operatorname{dom} B^* = \mathfrak{K}$. Lemma 4.1 implies that $B$ is a densely defined bounded operator and $B^{**} \in \mathbf{B}(\mathfrak{H}, \mathfrak{K})$. Moreover, it is clear that in this case the closure of $B$ is given by $B^{**}h = (h, T^* v)_{\mathfrak{H}} v$, $h \in \mathfrak{H}$.

(ii) If $v \in \mathfrak{K} \setminus \operatorname{dom} T^*$ then (4.4) shows that $\{f, g\} \in B^*$ if and only if $(f, v)_{\mathfrak{K}} = 0$ and $g = 0$. Hence the adjoint of $B$ is given by

$$B^* = \big\{\{f, 0\}\colon (f, v)_{\mathfrak{K}} = 0\big\} = \operatorname{span}\{v\}^{\perp} \times \{0\}. \tag{4.5}$$

Thus, $\operatorname{mul} B^{**} = (\operatorname{dom} B^*)^{\perp} = \operatorname{span}\{v\} \supset \operatorname{ran} B$, i.e., $B$ is a singular operator. In this case the formula for the closure of $B$ is obtained by taking adjoints in (4.5). □

For singular operators and relations, and for decomposition of general linear relations via such objects the reader is referred to [12]; see also references therein.

### *4.2. A decomposition of almost dominated forms*

The statement of the following result is inspired by the proof of the uniqueness result of Ando [4, Theorem 6], cf. [13]. The present proof builds on the decomposition of a densely defined closable unbounded operator as presented in Proposition 4.3; in this sense the present proof is simpler than Ando's proof since there is no need for von Neumann's result as in [8, Theorem 3.6].

**Theorem 4.4.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on a complex linear space* $\mathfrak{V}$ *and let* $\mathfrak{t}$ *be almost dominated by* $\mathfrak{w}$. *Then the following statements are equivalent*:

(i) $\mathfrak{t}$ *is not dominated by* $\mathfrak{w}$;
(ii) $\mathfrak{t}$ *has a decomposition* $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$ *where the form* $\mathfrak{t}_1$ *is almost dominated by* $\mathfrak{w}$ *and the nontrivial form* $\mathfrak{t}_2$ *is singular with respect to* $\mathfrak{w}$.

**Proof.** Recall that $\mathfrak{t}$ is almost dominated by $\mathfrak{w}$ if and only if $\mathfrak{t}$ is closable with respect to $\mathfrak{w}$, cf. Theorem 3.8. Define the relation $T$ from $\mathfrak{H}_{\mathfrak{w}}$ to $\mathfrak{H}_{\mathfrak{t}}$ by

$$T = \big\{\{\varphi + \ker \mathfrak{w}, \varphi + \ker \mathfrak{t}\} \in \mathfrak{H}_{\mathfrak{w}} \times \mathfrak{H}_{\mathfrak{t}}\colon \varphi \in \mathfrak{V}\big\}, \tag{4.6}$$

cf. (3.2). Clearly, $T$ is densely defined and, moreover, the form $\mathfrak{t}$ is closable with respect to $\mathfrak{w}$ if and only if the relation $T$ is the graph of a closable operator (i.e., the closure of $T$ in $\mathfrak{H}_{\mathfrak{w}} \times \mathfrak{H}_{\mathfrak{t}}$ is the graph of an operator); cf. (3.14). Furthermore, $\mathfrak{t}$ is dominated by $\mathfrak{w}$ if and only if $T$ is a bounded operator.

(i) $\Rightarrow$ (ii) If $\mathfrak{t}$ is not dominated by $\mathfrak{w}$, then the densely defined closable operator $T$ in (4.6) is not bounded. Hence $\operatorname{dom} T^* \neq \mathfrak{H}_{\mathfrak{t}}$ (see Lemma 4.1) and one can select a unit vector $v \in \mathfrak{H}_{\mathfrak{t}} \setminus \operatorname{dom} T^*$. Let $P_v$ be the orthogonal projection from $\mathfrak{H}_{\mathfrak{t}}$ onto the one-dimensional space spanned by $v$. Then the decomposition of the operator $T$ in Proposition 4.3,

$$T = (I - P_v)T + P_v T, \tag{4.7}$$

leads to the decomposition of the form $\mathfrak{t}$:

$$\mathfrak{t}[\varphi] = \mathfrak{t}_1[\varphi] + \mathfrak{t}_2[\varphi], \quad \varphi \in \mathfrak{V},$$

where the form $\mathfrak{t}_1$ is defined by

$$\mathfrak{t}_1[\varphi] = \big\|(I - P_v)T(\varphi + \ker \mathfrak{w})\big\|_{\mathfrak{t}}^2 = \big\|(I - P_v)(\varphi + \ker \mathfrak{t})\big\|_{\mathfrak{t}}^2, \quad \varphi \in \mathfrak{V}, \tag{4.8}$$

and the form $\mathfrak{t}_2$ is defined by

$$\mathfrak{t}_2[\varphi] = \big\|P_v T(\varphi + \ker \mathfrak{w})\big\|_{\mathfrak{t}}^2 = \big\|\big(T(\varphi + \ker \mathfrak{w}), v\big)_{\mathfrak{t}} v\big\|_{\mathfrak{t}}^2, \quad \varphi \in \mathfrak{V}. \tag{4.9}$$

It follows from Proposition 4.3 that the form $\mathfrak{t}_1$ is closable with respect to $\mathfrak{w}$ and hence almost dominated by $\mathfrak{w}$. The condition $v \in \mathfrak{H}_{\mathfrak{t}} \setminus \operatorname{dom} T^*$ implies that the form $\mathfrak{t}_2$ is nontrivial. Furthermore, by Proposition 4.3 the operator $P_v T$ is singular and $(P_v T)^{**} = \mathfrak{H}_{\mathfrak{w}} \times \operatorname{span}\{v\}$. In particular, $\operatorname{dom} P_v T \subset \ker(P_v T)^{**} = \mathfrak{H}_{\mathfrak{w}}$ and therefore, for every $\varphi \in \mathfrak{V}$

$$\inf_{g + \ker \mathfrak{w} \in \mathfrak{H}_{\mathfrak{w}}} \big\{\big\|(g - \varphi) + \ker \mathfrak{w}\big\|_{\mathfrak{w}}^2 + \big\|P_v T(\varphi + \ker \mathfrak{w})\big\|_{\mathfrak{t}}^2\big\} = 0,$$

or equivalently,

$$\inf_{g \in \mathfrak{V}} \big\{\mathfrak{w}[g - \varphi] + \mathfrak{t}_2[g]\big\} = 0.$$

Hence from (2.3) in Proposition 2.2 it follows that $\mathfrak{t}_2 : \mathfrak{w} = 0$. Now Proposition 2.10 shows that $\mathfrak{t}_2$ is singular with respect to $\mathfrak{w}$.

(ii) $\Rightarrow$ (i) If $\mathfrak{t}$ is dominated by $\mathfrak{w}$, then also $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are dominated by $\mathfrak{w}$. In particular, $\mathfrak{t}_2$ is almost dominated by $\mathfrak{w}$, which leads to $\mathfrak{t}_2 = 0$, a contradiction. □

Hence, if $\mathfrak{t}$ is almost dominated by $\mathfrak{w}$, but not dominated by $\mathfrak{w}$, then $\mathfrak{t}$ has a decomposition $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$ where the form $\mathfrak{t}_1$ is almost dominated by $\mathfrak{w}$ and the nontrivial form $\mathfrak{t}_2$ is singular with respect to $\mathfrak{w}$. In particular, $\mathfrak{t}$ is an almost dominated form (i.e., a closable form) which majorizes a nontrivial singular form.

In the decomposition in (ii) of Theorem 4.4 a seemingly weaker statement is that $\mathfrak{t}_2$ is not almost dominated by $\mathfrak{w}$. However, by the Lebesgue decomposition, the form $\mathfrak{t}_2$ is decomposed as $\mathfrak{t}_2 = \mathfrak{t}_3 + \mathfrak{t}_4$ with $\mathfrak{t}_3$ almost dominated by $\mathfrak{w}$ and $\mathfrak{t}_4$ singular with respect to $\mathfrak{w}$. Now recall that the sum of forms $\mathfrak{t}_1 + \mathfrak{t}_3$ is a form which is almost dominated by $\mathfrak{w}$; see Proposition 2.9.

The operator version of the following result is due to Eriksson and Leutwiler [7, Proposition 3.3].

**Corollary 4.5.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on a complex linear space $\mathfrak{V}$ and let $\mathfrak{t}$ be almost dominated by $\mathfrak{w}$. Then the following statements are equivalent*:

(i) $\mathfrak{t}$ *is dominated by* $\mathfrak{w}$;
(ii) *if a form* $\mathfrak{s}$ *is bounded by* $\mathfrak{t}$, *then* $\mathfrak{s}$ *is almost dominated by* $\mathfrak{w}$;

(iii) *if a form* $\mathfrak{s}$ *is bounded by* $\mathfrak{t}$ *and singular with respect to* $\mathfrak{w}$*, then* $\mathfrak{s}=0$.

**Proof.** (i) ⇒ (ii) If $\mathfrak{s}$ is bounded by $\mathfrak{t}$, then $\mathfrak{s}$ is dominated by $\mathfrak{w}$ (since $\mathfrak{t}$ is dominated by $\mathfrak{w}$) and, in particular, $\mathfrak{s}$ is almost dominated by $\mathfrak{w}$.

(ii) ⇒ (iii) If $\mathfrak{s}$ is bounded by $\mathfrak{t}$ and singular with respect to $\mathfrak{w}$, then $\mathfrak{s}$ is almost dominated by $\mathfrak{w}$ and singular with respect to $\mathfrak{w}$. Theorem 2.6 and Proposition 2.10 show that $\mathfrak{s}=D_{\mathfrak{w}}\mathfrak{s}=0$.

(iii) ⇒ (i) This follows immediately from Theorem 4.4. □

Hence, if $\mathfrak{t}$ is almost dominated by $\mathfrak{w}$, but not dominated by $\mathfrak{w}$, and the form $\tilde{\mathfrak{t}}$ is dominated (i.e. bounded) by $\mathfrak{t}$, then $\tilde{\mathfrak{t}}$ need not be almost dominated by $\mathfrak{w}$.

### *4.3. The uniqueness of Lebesgue type decompositions*

The following result is the analog of Ando's uniqueness result in [4, Theorem 6] now stated in the context of a pair of forms. Its proof parallels the proof of Ando.

**Theorem 4.6.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on a complex linear space* $\mathfrak{V}$*. A Lebesgue type decomposition of* $\mathfrak{t}$ *is unique if and only if* $\mathfrak{t}_{\mathrm{reg}}$ *is dominated by* $\mathfrak{w}$*. In this case the* (*unique*) *Lebesgue type decomposition coincides with the Lebesgue decomposition* (2.13) *in Theorem* 2.11.

**Proof.** Assume that $\mathfrak{t}_{\mathrm{reg}}=\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is dominated by $\mathfrak{w}$. Let the form $\mathfrak{t}$ have the decomposition $\mathfrak{t}=\mathfrak{t}_1+\mathfrak{t}_2$, where $\mathfrak{t}_1$ is almost dominated by $\mathfrak{w}$ and $\mathfrak{t}_2$ is singular with respect to $\mathfrak{w}$. Then

$$\mathfrak{t}_1+\mathfrak{t}_2=\mathbf{D}_{\mathfrak{w}}\mathfrak{t}+(\mathfrak{t}-\mathbf{D}_{\mathfrak{w}}\mathfrak{t}),$$

which, by means of Lemma 2.4, leads to

$$\mathfrak{t}_2=\mathbf{D}_{\mathfrak{w}}\mathfrak{t}-\mathfrak{t}_1+(\mathfrak{t}-\mathbf{D}_{\mathfrak{w}}\mathfrak{t})\geqslant\mathbf{D}_{\mathfrak{w}}\mathfrak{t}-\mathfrak{t}_1\geqslant 0,$$

where the last inequality follows from Theorem 2.11. Since $\mathfrak{t}_2$ is singular with respect to $\mathfrak{w}$, also the form $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}-\mathfrak{t}_1$ is singular with respect to $\mathfrak{w}$. Moreover,

$$0\leqslant\mathbf{D}_{\mathfrak{w}}\mathfrak{t}-\mathfrak{t}_1\leqslant\mathbf{D}_{\mathfrak{w}}\mathfrak{t},$$

and hence it follows from Corollary 4.5 that $\mathfrak{t}_1=\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$.

Now assume that $\mathfrak{t}_{\mathrm{reg}}=\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is not dominated by $\mathfrak{w}$. Then by Theorem 4.4 there is a decomposition

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t}=\mathfrak{t}_1+\mathfrak{t}_2,$$

where the form $\mathfrak{t}_1$ is almost dominated by $\mathfrak{w}$ and the nontrivial form $\mathfrak{t}_2$ is singular with respect to $\mathfrak{w}$. This shows that $\mathfrak{t}=\mathbf{D}_{\mathfrak{w}}\mathfrak{t}+(\mathfrak{t}-\mathbf{D}_{\mathfrak{w}}\mathfrak{t})$ can be written as

$$\mathfrak{t}=\left[\mathfrak{t}_1+\mathbf{D}_{\mathfrak{w}}(\mathfrak{t}_2+\mathfrak{t}-\mathbf{D}_{\mathfrak{w}}\mathfrak{t})\right]+\left[\mathfrak{t}_2+\mathfrak{t}-\mathbf{D}_{\mathfrak{w}}\mathfrak{t}-\mathbf{D}_{\mathfrak{w}}(\mathfrak{t}_2+\mathfrak{t}-\mathbf{D}_{\mathfrak{w}}\mathfrak{t})\right]. \tag{4.10}$$

Clearly the form $\mathfrak{t}_1+\mathbf{D}_{\mathfrak{w}}(\mathfrak{t}_2+\mathfrak{t}-\mathbf{D}_{\mathfrak{w}}\mathfrak{t})$ is almost dominated by $\mathfrak{w}$, being the sum of two such forms. Furthermore, it is also clear that the form

$$\mathfrak{t}_2 + \mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}(\mathfrak{t}_2 + \mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t})$$

is singular with respect to $\mathfrak{w}$; see Theorem 2.11. It remains to show that the decomposition in (4.10) is different from the decomposition $\mathfrak{t} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} + (\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t})$. To see this assume that these decompositions are equal, so that

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}_1 + \mathbf{D}_{\mathfrak{w}}(\mathfrak{t}_2 + \mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}),$$

which leads to

$$\mathbf{D}_{\mathfrak{w}}(\mathfrak{t}_2 + \mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}) = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} - \mathfrak{t}_1 = \mathfrak{t}_2.$$

This identity implies that $\mathfrak{t}_2$ is almost dominated by $\mathfrak{w}$, which gives a contradiction. Hence the decomposition in (4.10) differs from the Lebesgue decomposition in (2.13). □

According to Theorem 2.11 for each pair of forms $\mathfrak{t}$ and $\mathfrak{w}$ there is a Lebesgue type decomposition as the sum of an almost dominated form and a singular form. In Theorem 3.9 the almost dominated part and the singular part in the Lebesgue decomposition of $\mathfrak{t}$ are identified with the regular and the singular part of $\mathfrak{t}$, respectively. Recall from Theorem 2.11 that the almost dominated part $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is characterized as the maximum of all forms majorized by $\mathfrak{t}$, which are almost dominated by $\mathfrak{w}$. According to Theorem 4.6 the Lebesgue decomposition is the only Lebesgue type decomposition precisely when $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is dominated by $\mathfrak{w}$.

## 5. The Lebesgue decomposition of measures and their forms

The Lebesgue decomposition of one form with respect to another form introduced in the previous sections is inspired by the Lebesgue decomposition of one measure with respect to another measure; cf. [26]. As each measure induces a form the Lebesgue decomposition for measures can be compared with the Lebesgue decomposition of forms. Let $X$ be a set, let $\mathcal{M}$ be a $\sigma$-algebra on it, and let $\lambda$ and $\mu$ be finite positive measures on the $\sigma$-algebra $\mathcal{M}$. Let the complex linear space $\mathfrak{V}$ be defined as the linear span of the characteristic functions of measurable sets of $X$:

$$\mathfrak{V} = \operatorname{span}\{\chi_M\colon\ M \in \mathcal{M}\}.$$

Define the forms $\mathfrak{t}$ and $\mathfrak{w}$ on $\mathfrak{V}$ by means of the measures $\lambda$ and $\mu$:

$$\mathfrak{t}[\varphi] = \int_X |\varphi|^2\,d\lambda, \qquad \mathfrak{w}[\varphi] = \int_X |\varphi|^2\,d\mu, \quad \varphi \in \mathfrak{V}. \tag{5.1}$$

This section contains a connection between the Lebesgue decomposition of forms with the Lebesgue decomposition of finite measures.

### 5.1. Absolute continuity and closability

The positive measure $\lambda$ is said to be *absolutely continuous* with respect to the positive measure $\mu$, if

$$E \in \mathcal{M}, \quad \mu(E) = 0 \quad \Rightarrow \quad \lambda(E) = 0.$$

**Lemma 5.1** *(F. Riesz). Let $\lambda$ be absolutely continuous with respect to $\mu$. Then $\mathfrak{t}$ is closable with respect to $\mathfrak{w}$.*

**Proof.** Assume that $(\varphi_n)$ is a sequence in $\mathfrak{V}$ for which

$$\mathfrak{t}[\varphi_n - \varphi_m] = \int_X |\varphi_n - \varphi_m|^2 \, d\lambda_r \to 0, \qquad \mathfrak{w}[\varphi_n] = \int_X |\varphi_n|^2 \, d\mu \to 0,$$

cf. (3.14). Then by a standard result on $L^2$-convergence there is a subsequence $(\varphi_{n_k})$ of $(\varphi_n)$ such that

$$\varphi_{n_k} \to 0 \quad \text{a.e. } [\mu],$$

which, since $\lambda$ is absolutely continuous with respect to $\mu$, implies that

$$\varphi_{n_k} \to 0 \quad \text{a.e. } [\lambda].$$

Moreover, there exists an element $\varphi \in L^2(X, \lambda)$ such that

$$\int_X |\varphi_n - \varphi|^2 \, d\lambda \to 0 \quad \text{and, in particular,} \quad \int_X |\varphi_{n_k} - \varphi|^2 \, d\lambda \to 0.$$

Hence, there exists a subsequence $(\varphi_{n_{k_l}})$ of $(\varphi_{n_k})$, such that

$$\varphi_{n_{k_l}} \to \varphi \quad \text{a.e. } [\lambda].$$

This implies that $\varphi = 0$, a.e. $[\lambda]$, i.e.,

$$\mathfrak{t}[\varphi_n] = \int_X |\varphi_n|^2 \, d\lambda \to 0.$$

Hence the form $\mathfrak{t}$ is closable with respect to $\mathfrak{w}$. □

*5.2. The Lebesgue decomposition of measures*

Von Neumann's proof of the Radon–Nikodym theorem involved Hilbert spaces techniques, i.e., Riesz lemma. This approach also leads to the Lebesgue decomposition, cf. [26].

**Theorem 5.2.** *Let $\lambda$ and $\mu$ be finite positive measures on a $\sigma$-algebra $\mathcal{M}$ on a set $X$. Then there exists a set $S \in \mathcal{M}$ with $\mu(S) = 0$ and a nonnegative function $f \in L^1(X, d\mu)$, such that*

$$\lambda(M) = \int_M f\, d\mu$$

*for all $M \subset X \setminus S$.*

**Proof.** Here is a sketch of the proof in [26]. Define the measure $\alpha$ by $\alpha = \lambda + \mu$. By the Riesz lemma there exists a function $F$ such that

$$\lambda(M) = \int_M F\, d\alpha, \tag{5.2}$$

for all $M \subset \mathcal{M}$. It follows from $\alpha = \lambda + \mu$ and (5.2) that $0 \leqslant F \leqslant 1$ a.e. with respect to $\lambda$, $\mu$, and $\alpha$. Now define the sets $A_n$ and $A$ by

$$A_n = \left\{ x \in X\colon\ 1 - \frac{1}{n} \leqslant F(x) < 1 - \frac{1}{n+1} \right\}, \qquad A = \bigcup_{n=1}^{\infty} A_n.$$

Since $F(x) = 1$ on $S = X \setminus A$, it follows from (5.2) that $\mu(S) = 0$. Define the nonnegative functions $g_n$ by

$$g_n(x) = \frac{F(x)}{1 - F(x)}, \quad x \in A_n,\ g_n(x) = 0,\ x \notin A_n.$$

It follows from $\alpha = \lambda + \mu$ and (5.2) that

$$\int_X g_n \chi_M\, d\mu = \lambda(A_n \cap M), \quad M \in \mathcal{M}.$$

In particular, with $M = X$ this shows that $\sum_{n=0}^{\infty} \int_X g_n\, d\mu = \lambda(A) < \infty$, so that $f = \sum_{n=0}^{\infty} g_n$ belongs to $L^1(X, d\mu)$ by the monotone convergence theorem. Finally, if $M \subset A = X \setminus S$, then

$$\int_M f\, d\mu = \sum_{n=0}^{\infty} \int_X g_n \chi_M\, d\mu = \sum_{n=0}^{\infty} \lambda(A_n \cap M) = \lambda(M).$$

This completes the proof. $\square$

Define the measures $\lambda_r$ and $\lambda_s$ on $\mathcal{M}$ by

$$\lambda_r(M) = \lambda(M \setminus S), \qquad \lambda_s(M) = \lambda(M \cap S), \quad M \in \mathcal{M}.$$

For the following corollary, see [26, I.4].

**Corollary 5.3** *(Lebesgue decomposition). The measure* $\lambda$ *has a decomposition*

$$\lambda = \lambda_r + \lambda_s, \tag{5.3}$$

*where* $\lambda_r$ *is a positive measure which is absolutely continuous with respect to* $\mu$:

$$E \in \mathcal{M}, \quad \mu(E) = 0 \quad \Rightarrow \quad \lambda_r(E) = 0,$$

*and* $\lambda_s$ *is a positive measure which is singular with respect to* $\mu$:

$$\lambda_s(X \setminus S) = 0, \qquad \mu(S) = 0.$$

The Lebesgue decomposition is unique with respect to the above mentioned properties, cf. [26, Theorem I.20].

**Corollary 5.4** *(Radon–Nikodym). There exists a function* $f \in L^1(X, d\mu)$*, such that*

$$\lambda_r(M) = \int_M f\, d\mu, \quad M \in \mathcal{M}.$$

The Radon–Nikodym derivative $f$ is unique, cf. [26, Theorem I.19].

*5.3. The connection with forms*

It will now be shown that the Lebesgue decomposition of the form $\mathfrak{t}$ with respect to the form $\mathfrak{w}$ corresponds to the Lebesgue decomposition of the positive measure $\lambda$ with respect to the positive measure $\mu$; cf. Corollary 5.3.

**Theorem 5.5.** *Let the forms* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be defined by* (5.1). *Let* $\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}} + \mathfrak{t}_{\mathrm{sing}}$ *be the Lebesgue decomposition of* $\mathfrak{t}$ *with respect to* $\mathfrak{w}$ *as in Theorems* 2.11, 3.9*, and let* $\lambda$ *be decomposed as in* (5.3)*. Then*

$$\mathfrak{t}_{\mathrm{reg}}[\varphi] = \int_X |\varphi|^2\, d\lambda_r \quad \textit{and} \quad \mathfrak{t}_{\mathrm{sing}}[\varphi] = \int_X |\varphi|^2\, d\lambda_s, \quad \varphi \in \mathfrak{V}. \tag{5.4}$$

**Proof.** According to Theorem 3.5 one has

$$\mathfrak{t}_{\mathrm{reg}}[\varphi] = \sup_{h \in \mathfrak{V}} \inf_{g \in \mathfrak{V}} \bigl\{ \mathfrak{w}[g+h] + \mathfrak{t}[g] - \mathfrak{w}[\varphi + h] \bigr\}, \quad \varphi \in \mathfrak{V},$$

cf. (3.13). In particular, with $M \in \mathcal{M}$, this gives

$$\begin{aligned}
\mathfrak{t}_{\mathrm{reg}}[\chi_M] &= \sup_{h\in\mathfrak{V}} \inf_{g\in\mathfrak{V}} \left\{ \int\limits_X |g+h|^2\, d\mu + \int\limits_X |g|^2\, d\lambda - \int\limits_X |\chi_M + h|^2\, d\mu \right\} \\
&= \sup_{h\in\mathfrak{V}} \inf_{g\in\mathfrak{V}} \left\{ \int\limits_{X\setminus S} |g+h|^2\, d\mu + \int\limits_X |g|^2\, d\lambda - \int\limits_{X\setminus S} |\chi_M + h|^2\, d\mu \right\} \\
&\leqslant \sup_{h\in\mathfrak{V}} \left\{ \int\limits_{X\setminus S} |\chi_{M\setminus S}+h|^2\, d\mu + \int\limits_X |\chi_{M\setminus S}|^2\, d\lambda - \int\limits_{X\setminus S} |\chi_M + h|^2\, d\mu \right\} \\
&= \int\limits_X |\chi_{M\setminus S}|^2\, d\lambda,
\end{aligned}$$

where the inequality is obtained by the choice $g = \chi_{M\setminus S} \in \mathfrak{V}$. Note that also the identity $\mu(S) = 0$ has been used. Hence the above inequality leads to

$$\mathfrak{t}_{\mathrm{reg}}[\chi_M] \leqslant \lambda(M \setminus S) = \lambda_r(M) = \int\limits_X |\chi_M|^2\, d\lambda_r, \quad M \subset \mathcal{M}.$$

Therefore it follows that

$$\mathfrak{t}_{\mathrm{reg}}[\varphi] \leqslant \tilde{\mathfrak{t}}[\varphi], \quad \varphi \in \mathfrak{V}, \tag{5.5}$$

where the form $\tilde{\mathfrak{t}}$ on $\mathfrak{V}$ is defined by

$$\tilde{\mathfrak{t}}[\varphi] = \int\limits_X |\varphi|^2\, d\lambda_r, \quad \varphi \in \mathfrak{V}.$$

The definition of the form $\tilde{\mathfrak{t}}$ implies that it is dominated by the form $\mathfrak{t}$:

$$\tilde{\mathfrak{t}}[\varphi] \leqslant \mathfrak{t}[\varphi], \quad \varphi \in \mathfrak{V}.$$

Since $\lambda_r$ is absolutely continuous with respect to $\mu$, the form $\tilde{\mathfrak{t}}$ is closable with respect to the form $\mathfrak{w}$; cf. Lemma 5.1. Hence, by Proposition 3.7

$$\tilde{\mathfrak{t}}[\varphi] = \int\limits_X |\varphi|^2\, d\lambda_r \leqslant \mathfrak{t}_{\mathrm{reg}}[\varphi], \quad \varphi \in \mathfrak{V}. \tag{5.6}$$

The inequalities (5.5) and (5.6) lead to (5.4). □

The Lebesgue decomposition of the form $\mathfrak{t}$ with respect to the form $\mathfrak{w}$ in Theorem 2.11 corresponds precisely to the Lebesgue decomposition of the measure $\lambda$ with respect to the measure $\mu$. However, here the analogy stops. According to Theorem 4.4 an almost dominated form, which is not dominated, majorizes a nontrivial singular form; cf. B. Simon's remark preceding Theorem 2.5 in [30].

## 6. The Lebesgue decomposition of nonnegative bounded linear operators

Let $\mathfrak{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and let $A$ and $B$ be bounded nonnegative operators in $\boldsymbol{B}(\mathfrak{H})$. The operator $A$ is said to be *dominated* by $B$ if $(Af, f) \leqslant c(Bf, f)$, $f \in \mathfrak{H}$. The operator $A$ is said to be *almost dominated* by $B$ if there exists a monotonically nondecreasing sequence of nonnegative bounded linear operators $A_n \in \boldsymbol{B}(\mathfrak{H})$, each dominated by $B$, such that $(Af, f) = \sup(A_n f, f)$, $f \in \mathfrak{H}$. In [4] the terminology *strongly continuous* and *absolutely continuous* has been used for dominated and almost dominated, respectively. The operator $A$ is said to be *singular* with respect to $B$ if for any $D \in \boldsymbol{B}(\mathfrak{H})$, the inequalities $0 \leqslant D \leqslant A$ and $0 \leqslant D \leqslant B$ imply that $D = 0$. These definitions agree with the earlier given definitions when $\mathfrak{V} = \mathfrak{H}$ and

$$\mathfrak{t}(f, g) = (Af, g), \qquad \mathfrak{t}_n(f, g) = (A_n f, g), \qquad \mathfrak{w}(f, g) = (Bf, g), \quad f, g \in \mathfrak{H}. \tag{6.1}$$

The parallel sum $\mathfrak{t} : \mathfrak{w}$ now takes the form

$$(\mathfrak{t} : \mathfrak{w})[\varphi] = \inf_{g \in \mathfrak{H}} \bigl\{ \bigl(B(g + \varphi), g + \varphi\bigr) + (Ag, g) \bigr\}, \quad \varphi \in \mathfrak{H}.$$

Observe that

$$\begin{aligned} \inf_{g \in \mathfrak{H}} \bigl\{ \bigl(B(g + \varphi), g + \varphi\bigr) + (Ag, g) \bigr\} &\leqslant \inf_{g \in \mathfrak{H}} \bigl\{ \|B\| \|g + \varphi\|^2 + \|A\| \|g\|^2 \bigr\} \\ &= \frac{\|A\| \|B\|}{\|A\| + \|B\|} \|\varphi\|^2 \end{aligned}$$

(see Lemma 2.3) so that $\mathfrak{t} : \mathfrak{w}$ is a bounded nonnegative form. The corresponding operator is the *parallel sum* $A : B$ of the operators $A$ and $B$:

$$\bigl((A : B)f, f\bigr) = \inf\bigl\{(Ag, g) + (Bh, h)\colon\ g, h \in \mathfrak{H},\ g + h = f\bigr\}, \quad f \in \mathfrak{H}, \tag{6.2}$$

cf. [3], [23, (2.2), (2.3)]. The sequence $A : nB$ is a monotonically nondecreasing sequence and

$$A : nB \leqslant A.$$

Hence, the form $\mathbf{D}_{\mathfrak{w}} \mathfrak{t} = \sup_{n \in \mathbb{N}} (\mathfrak{t} : n\mathfrak{w})$ is bounded. The corresponding bounded nonnegative operator will be denoted by $\mathbf{D}_B A$, i.e.,

$$\mathbf{D}_B A = \sup_{n \in \mathbb{N}} (A : nB).$$

The following decomposition result of T. Ando [4, Theorem 2] for bounded nonnegative operators in now clear.

**Theorem 6.1.** *Let $A$ and $B$ be bounded nonnegative operators in a Hilbert space $\mathfrak{H}$. Then $A$ has a Lebesgue decomposition with respect to $B$*:

$$A = \mathbf{D}_B A + (A - \mathbf{D}_B A). \tag{6.3}$$

*The bounded nonnegative operator $\mathbf{D}_B A$ is almost dominated by $B$, and the bounded nonnegative operator $A - \mathbf{D}_B A$ is singular with respect to $B$. In fact, $\mathbf{D}_B A$ is the maximum of all operators majorized by $A$, which are almost dominated by $B$.*

The operator $\mathbf{D}_B A$ has been denoted by $[B]A$ in [4]. It follows from (3.11) that

$$(\mathbf{D}_B A\varphi, \varphi) = \sup_{h \in \mathfrak{H}} \bigl\{ \bigl((A : B)h, h\bigr) - \bigl(B(\varphi + h), \varphi + h\bigr) \bigr\}, \quad \varphi \in \mathfrak{H}.$$

The right-hand side can be seen to be the short of the operator $A$ to the operator range $\operatorname{ran} B^{\frac{1}{2}}$, cf. [23]. It is the parallel difference $(A : B) \div B$ of $A : B$ and $B$ in the sense of Pekarev.

The following result is Ando's uniqueness result for the Lebesgue (type) decomposition of $A$ into a sum of two bounded nonnegative operators, one of which is almost dominated by $B$, the other one singular with respect to $B$; see [4, Theorem 6].

**Theorem 6.2.** *Let $A$ and $B$ be nonnegative operators in $\boldsymbol{B}(\mathfrak{H})$. Then $A$ admits a unique Lebesgue type decomposition with respect to $B$ if and only if $A_{\rm reg} = \mathbf{D}_B A$ is dominated by $B$.*

The results in Theorem 6.1 and Theorem 6.2 for operators $A$ and $B$ in $\boldsymbol{B}(\mathfrak{H})$ have been obtained here via the forms $\mathfrak{t}$ and $\mathfrak{w}$ associated with them in (6.1). There is an alternative approach where the forms $\mathfrak{t}$ and $\mathfrak{w}$ are given and operators $A$ and $B$ are defined by means of the forms.

Let $\mathfrak{t}$ and $\mathfrak{w}$ be nonnegative forms on a complex linear space $\mathfrak{V}$. Let $\iota_{\mathfrak{t}}$ be the contraction from $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$ to $\mathfrak{H}_{\mathfrak{w}}$ defined by (3.2): $\operatorname{dom} \iota_{\mathfrak{t}} = \mathfrak{V}/\mathfrak{K}$ and $\iota_{\mathfrak{t}}(\varphi + \mathfrak{K}) = \varphi + \mathfrak{L}$. Its closure $\iota_{\mathfrak{t}}^{**}$ is a contraction defined from all of $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$ to $\mathfrak{H}_{\mathfrak{w}}$. Then

$$\bigl(\iota_{\mathfrak{t}}^{*}\iota_{\mathfrak{t}}^{**}(\varphi + \mathfrak{K}), \varphi + \mathfrak{K}\bigr)_{\mathfrak{t}+\mathfrak{w}} = \bigl\| \iota_{\mathfrak{t}}^{**}(\varphi + \mathfrak{K}) \bigr\|_{\mathfrak{w}}^{2} = \bigl\| (\varphi + \mathfrak{L}) \bigr\|_{\mathfrak{w}}^{2} = \mathfrak{w}[\varphi], \quad \varphi \in \mathfrak{V}. \tag{6.4}$$

Define the operators $A$ and $B$ by

$$B = \iota_{\mathfrak{t}}^{*}\iota_{\mathfrak{t}}^{**}, \qquad A = I - B, \tag{6.5}$$

so that $A$ and $B$ are bounded linear operators in $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$ with $0 \leqslant A \leqslant I$ and $0 \leqslant B \leqslant I$. Observe that $A$ and $B$ commute. These observations yield an operator representation for the given forms $\mathfrak{t}$ and $\mathfrak{w}$.

**Lemma 6.3.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be nonnegative forms on a complex linear space $\mathfrak{V}$. Then the nonnegative contractions $A$ and $B$ in* (6.5) *acting in the Hilbert space $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$ with $A + B = I$, satisfy*

$$\mathfrak{t}[\varphi] = \bigl(A(\varphi + \mathfrak{K}), \varphi + \mathfrak{K}\bigr)_{\mathfrak{t}+\mathfrak{w}}, \quad \varphi \in \mathfrak{V}/\mathfrak{K}, \tag{6.6}$$

*and*

$$\mathfrak{w}[\varphi] = \bigl(B(\varphi + \mathfrak{K}), \varphi + \mathfrak{K}\bigr)_{\mathfrak{t}+\mathfrak{w}}, \quad \varphi \in \mathfrak{V}/\mathfrak{K}. \tag{6.7}$$

**Proof.** The identity (6.7) follows from (6.4), and the identity (6.6) follows from (6.5). □

This result offers an approach to consider forms via their representation in terms of bounded nonnegative operators. The construction of the operators $A$ and $B$ in the Hilbert space $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$ and the representations (6.6) and (6.7) for $\mathfrak{t}$ and $\mathfrak{w}$ go back to Pusz and Woronowicz [25, p. 160].

## 7. The Lebesgue decomposition of positive definite kernels

Let $\mathfrak{E}$ be a complex Banach space with dual space $\mathfrak{E}^*$. The dual pairing of $x \in \mathfrak{E}$ and $x^* \in \mathfrak{E}^*$ is denoted by $\langle x, x^* \rangle$. The Banach space of all bounded linear operators from $\mathfrak{E}$ to $\mathfrak{E}^*$ is denoted by $\boldsymbol{B}(\mathfrak{E}, \mathfrak{E}^*)$. Let $S$ be a set and denote by $\mathfrak{D} = \mathfrak{F}(S, \mathfrak{E})$ the vector space of functions on $S$ with values in $\mathfrak{E}$ with finite support. The function $\mathsf{K}$ defined on the Cartesian product $S \times S$ with values in $\boldsymbol{B}(\mathfrak{E}, \mathfrak{E}^*)$ is said to be a *positive definite kernel* if

$$\sum_{s,t \in S} \big\langle f(t), \mathsf{K}(s,t) f(s) \big\rangle \geqslant 0, \quad f \in \mathfrak{D}.$$

Associated with the kernel $\mathsf{K}$ is the form $\mathfrak{w}_{\mathsf{K}}$ defined on $\mathfrak{D} \times \mathfrak{D}$ by

$$\mathfrak{w}_{\mathsf{K}}(f, g) = \sum_{s,t \in S} \big\langle f(t), \mathsf{K}(s,t) g(s) \big\rangle, \quad f, g \in \mathfrak{D}. \tag{7.1}$$

Two positive definite kernels $\mathsf{L}$ and $\mathsf{K}$, defined on $S \times S$, are said to satisfy the inequality $\mathsf{L} \prec \mathsf{K}$ if the associated forms satisfy $\mathfrak{w}_{\mathsf{L}} \leqslant \mathfrak{w}_{\mathsf{K}}$. The interpretation of a family of forms in the present context goes back to [32].

**Lemma 7.1.** *Let $\mathsf{K}$ be a positive definite kernel with associated form $\mathfrak{w}_{\mathsf{K}}$ and let $\mathfrak{w}$ be a form on $\mathfrak{D}$. Then the following statements are equivalent*:

(i) $\mathfrak{w} \leqslant \mathfrak{w}_{\mathsf{K}}$;
(ii) $\mathfrak{w} = \mathfrak{w}_{\mathsf{L}}$ *for a unique kernel* $\mathsf{L} \prec \mathsf{K}$.

**Proof.** (ii) ⇒ (i) This implication follows directly from the definitions.

(i) ⇒ (ii) Let $\mathfrak{w}$ be a form which is majorized by $\mathfrak{w}_{\mathsf{K}}$. Note that each $s \in S$ and $x \in \mathfrak{E}$ define a function $h_s \in \mathfrak{D} = \mathfrak{F}(S, \mathfrak{E})$ by

$$h_s(u) = \delta_s(u) x, \quad u \in S,$$

where $\delta$ indicates the Kronecker symbol. Hence, for each pair $s, t \in S$ the linear mapping $\mathsf{L}(s,t) : \mathfrak{E} \to \mathfrak{E}^*$ in

$$\big\langle x, \mathsf{L}(s,t) y \big\rangle = \mathfrak{w}(\delta_t x, \delta_s y), \quad x, y \in \mathfrak{E},$$

is well defined. Now let $f \in \mathfrak{D} = \mathfrak{F}(S, \mathfrak{E})$, so that it has the representation

$$f = \sum_q \delta_q f(q).$$

Then, by the definition of the kernel $\mathsf{L}(s,t)$, it follows that for all $f \in \mathfrak{D} = \mathfrak{F}(S, \mathfrak{E})$

$$\sum_{s,t} \langle f(t), \mathsf{L}(s,t) f(s) \rangle = \sum_{s,t,q,r} \delta_q(t) \overline{\delta_r(s)} \mathfrak{w}\big(\delta_t f(q), \delta_s f(r)\big)$$
$$= \mathfrak{w}\bigg( \sum_{t,q} \delta_q(t) \delta_t f(q), \sum_{s,r} \delta_r(s) \delta_s f(r) \bigg) \geqslant 0.$$

Hence, the kernel $\mathsf{L}(s,t)$ is positive definite.

It remains to show that $\mathsf{L}(s,t)$ belongs to $\boldsymbol{B}(\mathfrak{E}, \mathfrak{E}^*)$. Observe that the inequality $\mathfrak{w} \leqslant \mathfrak{w}_{\mathsf{K}}$ leads to

$$\begin{aligned}
\langle x, \mathsf{L}(s,t)x \rangle &= \mathfrak{w}(\delta_t x, \delta_s x) \\
&\leqslant \mathfrak{w}_{\mathsf{K}}(\delta_t x, \delta_s x) \\
&= \sum_{q,r} \langle \delta_t(q) x, \mathsf{K}(s,t) \delta_s(r) x \rangle \\
&= \langle x, \mathsf{K}(s,t) x \rangle \\
&\leqslant \|\mathsf{K}(s,t)\| \|x\|^2, \quad x, y \in \mathfrak{E}.
\end{aligned}$$

Since the kernel $\mathsf{L}$ is positive definite, the inequality of Cauchy–Schwarz and the above inequality lead to

$$\big|\langle x, \mathsf{L}(s,t) y \rangle\big|^2 \leqslant \langle x, \mathsf{L}(s,t) x \rangle \langle y, \mathsf{L}(s,t) y \rangle \leqslant \|\mathsf{K}(s,t)\|^2 \|x\|^2 \|y\|^2, \quad x, y \in \mathfrak{E}.$$

Therefore, $\mathsf{L}(s,t)$ belongs to $\boldsymbol{B}(\mathfrak{E}, \mathfrak{E}^*)$. By definition it follows that $\mathfrak{w} = \mathfrak{w}_{\mathsf{L}}$. Recall that $\mathfrak{w}_{\mathsf{L}} \leqslant \mathfrak{w}_{\mathsf{K}}$ gives $\mathsf{L} \prec \mathsf{K}$ by definition. □

Let $\mathsf{L}$ and $\mathsf{K}$ be positive definite kernels. The kernel $\mathsf{L}$ is said to be *absolutely continuous* with respect to the kernel $\mathsf{K}$ if the associated form $\mathfrak{w}_{\mathsf{L}}$ is closable with respect to the associated form $\mathfrak{w}_{\mathsf{K}}$. The kernel $\mathsf{L}$ is said to be *singular* with respect to the kernel $\mathsf{K}$ if the associated form $\mathfrak{w}_{\mathsf{L}}$ is singular with respect to the associated form $\mathfrak{w}_{\mathsf{K}}$. The following result goes back to [5,33].

**Theorem 7.2.** *Let* $\mathsf{L}$ *and* $\mathsf{K}$ *be positive definite kernels. Then there exist positive definite kernels* $\mathsf{L}_a$ *and* $\mathsf{L}_s$*, such that*

(i) $\mathsf{L} = \mathsf{L}_a + \mathsf{L}_s$;
(ii) $\mathsf{L}_a$ *is absolutely continuous with respect to* $\mathsf{K}$;
(iii) $\mathsf{L}_a$ *is the maximum of all kernels* $\mathsf{M}$ *which are absolutely continuous with respect to* $\mathsf{K}$ *and which satisfy* $\mathsf{M} \prec \mathsf{L}$;
(iv) $\mathsf{L}_s$ *is singular with respect to* $\mathsf{K}$.

**Proof.** Let $\mathfrak{w}_{\mathsf{L}}$ and $\mathfrak{w}_{\mathsf{K}}$ be the forms corresponding to the positive definite kernels $\mathsf{L}$ and $\mathsf{K}$, cf. (7.1). The form $\mathfrak{w}_{\mathsf{L}}$ has a Lebesgue decomposition with respect to the form $\mathfrak{w}_{\mathsf{K}}$:

$$\mathfrak{w}_{\mathsf{L}} = \mathfrak{w}_a + \mathfrak{w}_s, \tag{7.2}$$

where the form $\mathfrak{w}_a$ is almost dominated by $\mathfrak{w}_{\mathsf{K}}$ and the form $\mathfrak{w}_s$ is singular with respect to $\mathfrak{w}_{\mathsf{K}}$. Due to the inequalities

$$\mathfrak{w}_a \leqslant \mathfrak{w}_{\mathsf{L}}, \qquad \mathfrak{w}_s \leqslant \mathfrak{w}_{\mathsf{L}},$$

and Lemma 7.1, there exist unique positive definite kernels $\mathsf{L}_a$ and $\mathsf{L}_s$, respectively, such that

$$\mathfrak{w}_a = \mathfrak{w}_{\mathsf{L}_a}, \qquad \mathfrak{w}_s = \mathfrak{w}_{\mathsf{L}_s}.$$

It follows from the decomposition (7.2) that

$$\mathfrak{w}_{\mathsf{L}} = \mathfrak{w}_{\mathsf{L}_a} + \mathfrak{w}_{\mathsf{L}_s} \quad \text{or, equivalently,} \quad \mathsf{L} = \mathsf{L}_a + \mathsf{L}_s.$$

The statements (i), (ii), (iii), and (iv) concerning the kernels now follow from the corresponding properties of the associated forms. □

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