



Infinite-dimensional perturbations, maximally nondensely defined symmetric operators, and some matrix representations

S. Hassi^a, H.S.V. de Snoo^{b,*}, F.H. Szafraniec^c

^a Department of Mathematics and Statistics, University of Vaasa, P.O. Box 700, 65101 Vaasa, Finland

^b Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700 AK Groningen, Netherlands

^c Instytut Matematyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, 30 348 Kraków, Poland

To the memory of Israel Gohberg

Abstract

The notion of a maximally nondensely defined symmetric operator or relation is introduced and characterized. The selfadjoint extensions (including the generalized Friedrichs extension) of a class of maximally nondensely defined symmetric operators are described. The description is given by means of the theory of ordinary boundary triplets and exhibits the extensions as infinite-dimensional perturbations of a certain selfadjoint operator extension of the symmetric operator.

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1. Introduction

As an illustration of the topics in this paper consider the following situation. Let S be a bounded, closed, symmetric operator in a Hilbert space \mathfrak{H} . Then \mathfrak{H} has the orthogonal

* Corresponding author. Tel.: +31 503633963.

E-mail addresses: sha@uvasa.fi (S. Hassi), desnoo@math.rug.nl (H.S.V. de Snoo), umszafra@cyf-kr.edu.pl (F.H. Szafraniec).

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decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ with $\mathfrak{H}_1 = \text{dom } S$ and $\dim \mathfrak{H}_2 > 0$, and

$$Sf_1 = S_{11}f_1 \oplus S_{21}f_1, \quad S_{j1} \in \mathbf{B}(\mathfrak{H}_1, \mathfrak{H}_j), \quad j = 1, 2, \quad f_1 \in \mathfrak{H}_1. \tag{1.1}$$

Referring to this formula let the matrix

$$\tilde{A} \stackrel{\text{df}}{=} \begin{pmatrix} S_{11} & S_{21}^* \\ S_{21} & S_{22} \end{pmatrix}, \tag{1.2}$$

corresponding to the decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, be an arbitrary bounded selfadjoint operator extension of S . Of course S will also have unbounded selfadjoint operator extensions and, in fact, S will have selfadjoint extensions, which are relations, i.e., multivalued linear mappings. The following result may serve as a starting point to approach the main topics of the paper; cf. [21, Proposition 5.1] and [12, Proposition 3.5] for an inverted form. Here and elsewhere the language of boundary triplets will be used freely; cf. [11,12].

Proposition 1.1. *Let S be a closed bounded symmetric operator as in (1.1) and (1.2). Then the following statements hold:*

- (i) S has equal defect numbers (d, d) , $d = \dim \mathfrak{H}_2 \leq \infty$;
- (ii) the adjoint S^* of S in \mathfrak{H} is the relation given by

$$S^* = \{ \widehat{f} = (f, \tilde{A}f + h); \quad f \in \mathfrak{H}, \quad h \in \mathfrak{H}_2 \};$$

- (iii) a boundary triplet for S^* is given by $\Pi = \{ \mathfrak{H}_2, \Gamma_0, \Gamma_1 \}$, where

$$\Gamma_0 \widehat{f} = -h, \quad \Gamma_1 \widehat{f} = f_2; \quad f = f_1 \oplus f_2 \in \mathfrak{H}, \quad \widehat{f} = (f, \tilde{A}f + h) \in S^*;$$

- (iv) the corresponding γ -field γ is given by $\gamma(\lambda) = (\tilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H}_2$ and the Weyl function M is given by

$$M(\lambda) = P_2(\tilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H}_2 = \left(S_{22} - \lambda - S_{21}(S_{11} - \lambda)^{-1}S_{21}^* \right)^{-1},$$

where P_2 stands for the orthogonal projection onto \mathfrak{H}_2 ;

- (v) the selfadjoint extensions $A_\Theta = \ker(\Gamma_0 + \Theta\Gamma_1)$ of S in \mathfrak{H} are in one-to-one correspondence with the selfadjoint relations Θ in \mathfrak{H}_2 via

$$A_\Theta = \begin{pmatrix} S_{11} & S_{21}^* \\ S_{21} & S_{22} + \Theta \end{pmatrix},$$

and their resolvents, with $\lambda \in \rho(A_\Theta) \cap \rho(\tilde{A})$, are connected by

$$(A_\Theta - \lambda)^{-1} = (\tilde{A} - \lambda)^{-1} - \gamma(\lambda)(M(\lambda) + \Theta^{-1})^{-1}\gamma(\bar{\lambda})^*.$$

Note that S^* is a relation with the multivalued part $\text{mul } S^* = \mathfrak{H}_2$. The mappings Γ_0 and Γ_1 generate selfadjoint extensions of S as follows:

$$\ker \Gamma_0 = \tilde{A}, \quad \ker \Gamma_1 = S_{11} \widehat{\oplus} (\{0\} \times \text{mul } S^*),$$

and they are transversal: $S^* = \ker \Gamma_0 \widehat{+} \ker \Gamma_1$, where the sum is componentwise. The selfadjoint extension $\ker \Gamma_1$ corresponds to the choice $\Theta = \{0\} \times \mathfrak{H}_2$. If S is semibounded, then $\ker \Gamma_1$ is the usual Friedrichs extension of S . Since $\text{ran } \gamma(\lambda) = \ker(S^* - \lambda)$, an application of the Frobenius formula for the inverse $(\tilde{A} - \lambda)^{-1}$ yields

$$P_2 \ker(S^* - \lambda) = \{ (S_{22} - S_{21}(S_{11} - \lambda I_1)^{-1}S_{21}^* - \lambda I_2)^{-1}h; \quad h \in \mathfrak{H}_2 \},$$

where P_2 stands for the orthogonal projection onto \mathfrak{H}_2 . This formula implies that the defect subspaces of the bounded nondensely defined symmetric operator S admit the following property

$$\ker(S^* - \lambda) \cap \overline{\text{dom } S} = \{0\} \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (1.3)$$

The present paper deals with the general class of symmetric, not necessarily closed, relations S in a Hilbert space \mathfrak{H} satisfying (1.3). Such relations will be called *maximally nondensely defined*; cf. Definition 3.3. The property (1.3) holds precisely when the componentwise sum

$$S_\infty = S \hat{+} (\{0\} \times \text{mul } S^*) \quad (1.4)$$

is an essentially selfadjoint extension of S ; see Proposition 3.10. Thus the closure of S_∞ is a selfadjoint extension of S in \mathfrak{H} . In particular, this means that S has equal defect numbers and clearly, if S is not itself essentially selfadjoint, then by taking closures in (1.4) one concludes that $\text{mul } S^* \neq \text{mul } S^{**}$. Consequently, the defect numbers (d, d) are nonzero as $d = \dim(\text{mul } S^* \ominus \text{mul } S^{**})$; see Lemma 3.2. The extension S_∞ is selfadjoint if and only if

$$\text{dom } S = \overline{\text{dom } S} \cap \text{dom } S^*; \quad (1.5)$$

a condition, which has appeared earlier in [8]. Various characterizations for symmetric relations satisfying (1.4) or (1.5) will be established; cf. Propositions 3.10 and 3.11.

One of the aims of the paper is to present an extension of Proposition 1.1 which will now be sketched; cf. Theorem 5.1 and Corollary 5.2. Let S be a closed symmetric operator, which satisfies (1.5) so that it is maximally nondensely defined, and let \tilde{A} be some selfadjoint operator extension of S , which is transversal to S_∞ (there always exist such extensions \tilde{A} of S). Then the selfadjoint extensions A_θ of S are in one-to-one correspondence with the selfadjoint relations θ in \mathcal{H} via the perturbation formula

$$A_\theta = \tilde{A} + G\theta G^* P_2, \quad (1.6)$$

where G is a bounded and boundedly invertible operator from the Hilbert space \mathcal{H} onto $\text{mul } S^*$; see Theorem 5.3. The formula (1.6) admits essentially the same simplicity as the block formula in Proposition 1.1. In general, when S is not closed and bounded, one cannot rewrite the perturbation formula (1.6) as a block form; see Example 5.6. However, such a block formula is still possible for unbounded symmetric operators, which are partially bounded; see Sections 4 and 5.3. Notice also that in (1.6) the operator extensions of S are parameterized by operators θ in \mathcal{H} , i.e., by bounded or unbounded (range) *perturbations* of the selfadjoint operator \tilde{A} . On the other hand, the purely multivalued selfadjoint relation $\{0\} \times \mathcal{H}$ in \mathcal{H} corresponds to the selfadjoint extension S_∞ in (1.4). In fact, S_∞ is the only selfadjoint extension of S whose domain is contained in $\overline{\text{dom } S}$ and, hence, it coincides with the so-called generalized Friedrich extension of S ; cf. [20]. If S is semibounded then S_∞ coincides with the standard Friedrichs extension; cf. [8]. Perturbation formulas as in (1.6) often occur in concrete applications; see [1,3,9,10,15,13,25].

One may view the underlying symmetric operator S as a domain restriction of any of its selfadjoint operator extensions A_θ in (1.6). A completely formal reasoning leads to analogous results for range restrictions S of a selfadjoint operator \tilde{A} . All the selfadjoint extensions of S can again be described by means of an ordinary boundary triplet which is constructed in Proposition 5.11, which gives rise to an explicit domain perturbation formula for all selfadjoint extensions of S analogous to (1.6). Such results find applications in problems involving ordinary and partial differential equations; see [1,14,17,28]. A typical case of this situation occurs for

symmetric densely defined operators, for which $\lambda = 0$ is a point of regular type ($\text{ran } S$ is closed, S^{-1} is bounded), or if for instance there is a selfadjoint extension of S with discrete spectrum. Roughly speaking all what is needed to derive such results is to pass to the formal inverse S^{-1} , which in turn becomes (the graph of) a maximally nondensely defined symmetric relation, and then describe the inverses A_{Θ}^{-1} of the selfadjoint extensions A_{Θ} of S as the range perturbation of S^{-1} as in (1.6).

The contents of the paper are as follows. In Section 2 some preliminary results concerning range perturbations and ordinary boundary triplets are presented; see [16] for linear relations and [12,17] for ordinary boundary triplets. The class of maximally nondensely defined symmetric relations is investigated in Section 3 and continued further in Section 4 under the additional assumption that S is partially bounded. The extension theory of maximally nondensely defined symmetric relations is presented in Section 5 with a construction of suitable boundary triplets for S^* yielding in particular the (range) perturbation formula as in (1.6). In that section also connections to block matrix formulas are given covering the known special case of bounded symmetric operators. Finally, the translation to the perturbations on the side of domains is shortly described.

2. Preliminaries

2.1. Some facts about linear relations

Let S be a, not necessarily closed, linear relation in a Hilbert space \mathfrak{H} , with inner product $\langle \cdot, \cdot \rangle$, so that $\mathfrak{H} = \overline{\text{dom } S^*} \oplus \text{mul } S^{**}$. The so-called (*orthogonal*) *operator part* of A is defined by

$$S_{\text{op}} \stackrel{\text{df}}{=} \{(f, f') \in S; f' \in \overline{\text{dom } S^*}\}. \tag{2.1}$$

Let Q stand for the orthogonal projection from \mathfrak{H} onto $\overline{\text{dom } S^*}$. Then S admits a so-called *canonical decomposition* into the (operator wise) sum of its regular and singular parts $S = S_{\text{reg}} + S_{\text{sing}}$ which are defined as follows:

$$S_{\text{reg}} \stackrel{\text{df}}{=} QS = \{(f, Qf'); (f, f') \in S\}, \tag{2.2}$$

$$S_{\text{sing}} \stackrel{\text{df}}{=} (I - Q)S = \{(f, (I - Q)f'); (f, f') \in S\}; \tag{2.3}$$

see [23]. By definition $\text{dom } S_{\text{reg}} = \text{dom } S_{\text{sing}} = \text{dom } S$ and, moreover, the regular part S_{reg} is a regular, i.e. closable, operator and the singular part S_{sing} is a singular relation, i.e. $\text{ran } S_{\text{sing}} \subset \text{mul } S_{\text{sing}}^{**}$.

Then (2.1) and (2.2) show that $S_{\text{op}} \subset S_{\text{reg}}$, which implies that as a restriction of a closable operator, the operator part of S is also closable. For further properties of this and some other related decompositions of linear relations the reader is referred to the papers [23,16].

Let S be a, not necessarily closed, *symmetric* linear relation in a Hilbert space \mathfrak{H} , so that $S \subset S^*$, or equivalently, $\langle f', f \rangle \in \mathbb{R}$. The closure S^{**} is also symmetric with the same adjoint S^* , thus $S \subset S^{**} \subset S^*$. For a symmetric relation S one has

$$\text{ran } (S - \lambda) \cap \text{mul } S^* = \text{mul } S, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.4}$$

To see this, it suffices to show that the left-hand side is contained in the right-hand side. Let $h \in \text{ran } (S - \lambda) \cap \text{mul } S^*$, then $h = f' - \lambda f$ for some $(f, f') \in S$; hence $\langle f', f \rangle = \lambda \langle f, f \rangle$, which leads to $f = 0$ and $h = f' \in \text{mul } S$.

The *defect subspaces* of S are defined by $\mathfrak{N}_\lambda(S^*) = (\text{ran}(S - \bar{\lambda}))^\perp = \ker(S^* - \lambda)$. The *defect numbers* of S are defined by $\dim \ker(S^* - \lambda) (\leq \infty)$ and are constant for $\lambda \in \mathbb{C}^+$ and for $\lambda \in \mathbb{C}^-$. The following notation is used:

$$\widehat{\mathfrak{N}}_\lambda(S^*) \stackrel{\text{df}}{=} \{ \widehat{f}_\lambda \stackrel{\text{df}}{=} (f_\lambda, \lambda f_\lambda); f_\lambda \in \mathfrak{N}_\lambda(S^*) \}, \quad \lambda \in \mathbb{C}. \tag{2.5}$$

The adjoint S^* has the following componentwise sum decomposition

$$S^* = S^{**} \widehat{+} \widehat{\mathfrak{N}}_\lambda(S^*) \widehat{+} \widehat{\mathfrak{N}}_{\bar{\lambda}}(S^*), \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{2.6}$$

due to von Neumann. Observe that $\mathfrak{H} = \overline{\text{dom } S^*} \oplus \text{mul } S^{**}$ and decompose the closed symmetric relation S^{**} accordingly:

$$S^{**} = (S^{**})_{\text{op}} \widehat{\oplus} (\{0\} \oplus \text{mul } S^{**}), \tag{2.7}$$

where the orthogonal operator part $(S^{**})_{\text{op}} = Q S^{**} = S_{\text{reg}}^{**}$ is as in (2.1), (2.2); for more details on such decompositions see also [16]. Taking adjoints one obtains

$$S^* = ((S^{**})_{\text{op}})^* \widehat{\oplus} (\{0\} \oplus \text{mul } S^{**}),$$

which leads in particular to

$$\ker(S^* - \lambda) = \ker(((S^{**})_{\text{op}})^* - \lambda), \quad \lambda \in \mathbb{C}. \tag{2.8}$$

$$\text{mul } S^* \ominus \text{mul } S^{**} = \text{mul } ((S^{**})_{\text{op}})^*. \tag{2.9}$$

Thus the deficiency indices of S in \mathfrak{H} are equal to the ones of its operator part $(S^{**})_{\text{op}}$ in the subspace $\mathfrak{H} \ominus \text{mul } S^{**}$.

2.2. Range perturbations of linear relations

Recall that for closed subspaces \mathfrak{M} and \mathfrak{N} of a Hilbert space \mathfrak{H} the sum $\mathfrak{M} + \mathfrak{N}$ is closed if and only if $\mathfrak{M}^\perp + \mathfrak{N}^\perp$ is closed; see [24, IV, Theorem 4.8]. The following lemma is a weakening of a known result; cf. [12,16].

Lemma 2.1. *Let \mathfrak{M} be a linear, not necessarily closed, subspace, let \mathfrak{N} be a closed linear subspace of a Hilbert space \mathfrak{H} , and let P be the orthogonal projection from \mathfrak{H} onto \mathfrak{N}^\perp . Then*

- (i) $\mathfrak{M} + \mathfrak{N}$ is closed if and only if $P \mathfrak{M}$ is closed;
- (ii) $\mathfrak{M} + \mathfrak{N} = \mathfrak{H}$ if and only if $P \mathfrak{M} = \mathfrak{N}^\perp$;
- (iii) $\ker(P \upharpoonright \mathfrak{M}) = \mathfrak{M} \cap \mathfrak{N}$.

Proof. Since \mathfrak{N} is closed, the Hilbert space \mathfrak{H} allows the orthogonal decomposition $\mathfrak{H} = \mathfrak{N} \oplus \mathfrak{N}^\perp$. Now it is straightforward to check the following identity:

$$\mathfrak{M} + \mathfrak{N} = P \mathfrak{M} \oplus \mathfrak{N}.$$

This implies immediately the statements in (i)–(ii). Statement (iii) is clear. □

In the sequel also the following closely related fact is needed.

Lemma 2.2. *Let \mathfrak{M} and \mathfrak{N} be linear, not necessarily closed, subspaces of a Hilbert space \mathfrak{H} , and let P be the orthogonal projection from \mathfrak{H} onto the closed subspace \mathfrak{N}^\perp . Then $\mathfrak{M} + \mathfrak{N}$ is dense in \mathfrak{H} if and only if $P \mathfrak{M}$ is dense in \mathfrak{N}^\perp .*

Proof. Observe the following inclusions:

$$\mathfrak{M} + \mathfrak{N} \subset \mathfrak{M} + \text{clos } \mathfrak{N} = P \mathfrak{M} \oplus \text{clos } \mathfrak{N} \subset \text{clos } (\mathfrak{M} + \mathfrak{N}).$$

Hence $\text{clos } (P \mathfrak{M}) \oplus \text{clos } \mathfrak{N} = \text{clos } (\mathfrak{M} + \mathfrak{N})$, which implies the claim. \square

For the calculus of linear relations in a Hilbert space, involving adjoints and componentwise sums, see for instance [11,16].

Lemma 2.3. *Let S and T be closed linear relations from a Hilbert space \mathfrak{H} to a Hilbert space \mathfrak{K} . Then the following statements are equivalent:*

- (i) $S \widehat{+} T$ is a closed relation from \mathfrak{H} to \mathfrak{K} ;
- (ii) $S^* \widehat{+} T^*$ is a closed relation from \mathfrak{K} to \mathfrak{H} .

Proposition 2.4. *Let T be a relation from a Hilbert space \mathfrak{H} to a Hilbert space \mathfrak{K} , let \mathfrak{L} be a closed linear subspace in \mathfrak{K} , and let $P_{\mathfrak{L}}$ be the orthogonal projection from \mathfrak{K} onto \mathfrak{L} . Then the following statements are equivalent:*

- (i) $T^{**} \widehat{+} (\{0\} \times \mathfrak{L})$ is closed in $\mathfrak{H} \times \mathfrak{K}$;
- (ii) $\text{dom } T^* + \mathfrak{L}^{\perp}$ is closed in \mathfrak{K} ;
- (iii) $\text{dom } T^* + \mathfrak{L}^{\perp} = (\text{mul } T^{**} \cap \mathfrak{L})^{\perp}$;
- (iv) $P_{\mathfrak{L}}(\text{dom } T^*)$ is closed in \mathfrak{K} .

Moreover, if $\text{mul } T^{**} + \mathfrak{L}$ is closed in \mathfrak{K} or, equivalently, if $\overline{\text{dom } T^*} + \mathfrak{L}^{\perp}$ is closed in \mathfrak{K} , then each of the statements (i)–(iv) is equivalent to each of the following statements:

- (v) $\overline{\text{dom } T^*} \subset \text{dom } T^* + \mathfrak{L}^{\perp}$;
- (vi) $P_{\mathfrak{L}}(\text{dom } T^*) = P_{\mathfrak{L}}(\overline{\text{dom } T^*})$.

In particular, if T is a closable operator from \mathfrak{H} to \mathfrak{K} , i.e., if $\text{mul } T^{**} = \{0\}$, then the statements (v) and (vi) reduce to:

- (vii) $\text{dom } T^* + \mathfrak{L}^{\perp} = \mathfrak{K}$;
- (viii) $P_{\mathfrak{L}}(\text{dom } T^*) = \mathfrak{L}$,

respectively.

Proof. (i) \Leftrightarrow (ii) The assumption that $T^{**} \widehat{+} (\{0\} \times \mathfrak{L})$ is a closed subspace in $\mathfrak{H} \times \mathfrak{K}$ is equivalent, via Lemma 2.3, to the closedness of the subspace

$$T^* \widehat{+} (\mathfrak{L}^{\perp} \times \mathfrak{H}) = (\text{dom } T^* + \mathfrak{L}^{\perp}) \times \mathfrak{H},$$

which is closed precisely when $\text{dom } T^* + \mathfrak{L}^{\perp}$ is closed.

(ii) \Leftrightarrow (iii) Observe that $\text{mul } T^{**} = (\text{dom } T^*)^{\perp}$, which implies that

$$(\text{dom } T^* + \mathfrak{L}^{\perp})^{\perp} = \text{mul } T^{**} \cap \mathfrak{L},$$

since \mathfrak{L} is closed. Therefore,

$$\text{clos } (\text{dom } T^* + \mathfrak{L}^{\perp}) = (\text{mul } T^{**} \cap \mathfrak{L})^{\perp}, \tag{2.10}$$

which shows that $\text{dom } T^* + \mathfrak{L}^{\perp}$ is closed if and only if (iii) is satisfied.

(ii) \Leftrightarrow (iv) Apply Lemma 2.1 with $\mathfrak{M} = \text{dom } T^*$, $\mathfrak{N} = \mathfrak{L}^{\perp}$, and $P = P_{\mathfrak{L}}$.

Next observe that the subspace $\text{mul } T^{**} + \mathfrak{L}$ is closed if and only if the subspace $(\text{mul } T^{**})^{\perp} + \mathfrak{L}^{\perp} = \overline{\text{dom } T^*} + \mathfrak{L}^{\perp}$ is closed, and that in this case

$$(\text{mul } T^{**} \cap \mathfrak{L})^{\perp} = \overline{\text{dom } T^*} + \mathfrak{L}^{\perp}. \tag{2.11}$$

(iii) \Leftrightarrow (v) When $\text{mul } T^{**} + \mathfrak{L}$ is closed, it follows from (2.11) that the condition $\text{dom } T^* + \mathfrak{L}^\perp = (\text{mul } T^{**} \cap \mathfrak{L})^\perp$ can be rewritten as

$$\text{dom } T^* + \mathfrak{L}^\perp = \overline{\text{dom } T^*} + \mathfrak{L}^\perp,$$

which is equivalent to

$$\overline{\text{dom } T^*} \subset \text{dom } T^* + \mathfrak{L}^\perp.$$

(v) \Rightarrow (vi) It is clear that in general $P_{\mathfrak{L}}(\text{dom } T^*) \subset P_{\mathfrak{L}}(\overline{\text{dom } T^*})$. The reverse inclusion follows directly from (v) as $\overline{\text{dom } T^*} \subset \text{dom } T^* + \mathfrak{L}^\perp$. Hence (vi) holds.

(vi) \Rightarrow (v) Let $f \in \overline{\text{dom } T^*}$. Then there exists $h \in \text{dom } T^*$ such that $f - h \in \mathfrak{L}^\perp$ and $f = h + (f - h) \in \text{dom } T^* \widehat{+} \mathfrak{L}^\perp$. Hence (v) holds.

Finally observe that if T is a closable operator, then $(\text{dom } T^*)^\perp = \text{mul } T^{**} = \{0\}$. Hence, in this case $\text{mul } T^{**} + \mathfrak{L} = \mathfrak{L}$ is closed and furthermore $\overline{\text{dom } T^*} = \mathfrak{R}$. Therefore the statements (v) and (vi) clearly reduce to the statements (vii) and (viii), respectively. \square

Remark 2.5. In [12, Definition 2.1] a linear relation T satisfying the property stated in (i) of Proposition 2.4 has been called \mathfrak{L} -regular; in [12, Proposition 2.5] the equivalence of (i) and (iv) in Proposition 2.4 has been also proved.

2.3. Ordinary boundary triplets

Let S be a closed symmetric relation in a Hilbert space \mathfrak{H} . If the defect numbers are equal, then S has selfadjoint extensions in the Hilbert space \mathfrak{H} . Let A_0 and A_1 be selfadjoint extensions of S ; they are called *disjoint* with respect to S if $A_0 \cap A_1 = S$ and *transversal* with respect to S if $A_0 \widehat{+} A_1 = S^*$; see [12, Definition 1.7]. Some further definitions and facts which can be found in [12] are now given.

Definition 2.6 ([12]). Let S be a symmetric relation in a Hilbert space \mathfrak{H} with equal deficiency indices and let S^* be its adjoint. Then the triplet $\Pi = (\mathcal{H}, \Gamma_0, \Gamma_1)$, where \mathcal{H} is a Hilbert space and $\Gamma \stackrel{\text{df}}{=} (\Gamma_0, \Gamma_1)$ is a linear single-valued surjection of S^* onto $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$, is said to be an *ordinary boundary triplet* for S^* if the abstract Green’s identity

$$\langle f', g \rangle_{\mathfrak{H}} - \langle f, g' \rangle_{\mathfrak{H}} = \langle \Gamma_1 \widehat{f}, \Gamma_0 \widehat{g} \rangle_{\mathcal{H}} - \langle \Gamma_0 \widehat{f}, \Gamma_1 \widehat{g} \rangle_{\mathcal{H}}, \tag{2.12}$$

holds for all $\widehat{f} = (f, f'), \widehat{g} = (g, g') \in S^*$. If S is closed one may think of $(\mathcal{H}, \Gamma_0, \Gamma_1)$ as the ordinary boundary triplet of S itself.

If $(\mathcal{H}, \Gamma_0, \Gamma_1)$ is a boundary triplet for S^* , then $\dim \mathcal{H} = n_{\pm}(S)$. Moreover, $S = \ker \Gamma \subset \ker \Gamma_0 \cap \ker \Gamma_1$ and the relations A_0 and A_1 defined by

$$A_0 \stackrel{\text{df}}{=} \ker \Gamma_0, \quad A_1 \stackrel{\text{df}}{=} \ker \Gamma_1, \tag{2.13}$$

are selfadjoint extensions of S and they are transversal with respect to S . Conversely, for any two selfadjoint extensions A_0 and A_1 of S which are transversal with respect to S , there exists a boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ for S^* such that (2.13) holds. In particular, a boundary triplet is not unique if the defect numbers of S are not equal to zero.

Boundary triplets are particularly convenient for the parameterization and description of the *intermediate* extensions H of S , i.e., the extensions H of S which satisfy $S \subset H \subset S^*$. More

precisely, the mapping

$$\Theta \mapsto A_\Theta \stackrel{\text{df}}{=} \{\widehat{f} \in S^*; \Gamma \widehat{f} \in \Theta\} = \ker(\Gamma_1 - \Theta \Gamma_0) \tag{2.14}$$

establishes a bijective correspondence between the closed relations Θ in \mathcal{H} and the closed intermediate extensions A_Θ of S . Furthermore, it can be shown that

$$A_{\Theta^*} = (A_\Theta)^*. \tag{2.15}$$

In particular, a closed extension A_Θ of S is symmetric or selfadjoint if and only if the relation Θ is symmetric or selfadjoint, respectively. A specific symmetric subspace in \mathcal{H} , which is called in [12, p. 141] a *forbidden manifold*, is defined as follows:

$$\mathcal{F}_\Pi \stackrel{\text{df}}{=} \Gamma(\{0\} \times \text{mul } S^*). \tag{2.16}$$

Note that the symmetric extension of S corresponding to \mathcal{F}_Π in (2.14) is given by

$$S_\infty \stackrel{\text{df}}{=} A_{\mathcal{F}_\Pi} = S \widehat{+} (\{0\} \times \text{mul } S^*);$$

this extension has an important role in later sections. Finally recall that the intermediate extensions in (2.14) have the following properties (see [12, Proposition 1.4]):

$$A_\Theta \cap A_0 = S \text{ (disjoint)} \iff \Theta \text{ operator}, \tag{2.17}$$

and

$$A_\Theta \widehat{+} A_0 = S^* \text{ (transversal)} \iff \Theta \text{ bounded operator}. \tag{2.18}$$

Definition 2.7 ([12]). Let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triplet for S^* with $A_0 = \ker \Gamma_0$. The Γ -field γ is defined by (see (2.5))

$$\gamma(\lambda) = \{(\Gamma_0 \widehat{f}_\lambda, f_\lambda); \widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(S^*)\}, \quad \lambda \in \rho(A_0), \tag{2.19}$$

and the *Weyl function* M is defined by

$$M(\lambda) = \{(\Gamma_0 \widehat{f}_\lambda, \Gamma_1 \widehat{f}_\lambda); \widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(S^*)\}, \quad \lambda \in \rho(A_0). \tag{2.20}$$

Denote by Π_1 the orthogonal projection in $\mathfrak{H} \oplus \mathfrak{H}$ onto the first component. Observe that the restriction $\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^*)$ of the mapping Γ_0 to $\widehat{\mathfrak{N}}_\lambda(S^*)$ is a bijective mapping onto \mathcal{H} . Hence, $\gamma(\lambda)$ is the graph of a bounded linear operator from \mathcal{H} to $\mathfrak{N}_\lambda(S^*)$ and M is a $\mathbf{B}(\mathcal{H})$ -valued function, given by

$$\gamma(\lambda) = \Pi_1(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^*))^{-1}, \quad M(\lambda) = \Gamma_1(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^*))^{-1}, \quad \lambda \in \rho(A_0).$$

For all $\lambda, \mu \in \rho(A_0)$ the Γ -field γ satisfies the identity

$$\gamma(\lambda) = (I + (\lambda - \mu)(A_0 - \lambda)^{-1})\gamma(\mu), \tag{2.21}$$

which, in particular, shows that γ is holomorphic on $\rho(A_0)$. The Weyl function M and the Γ -field γ are related via the identity

$$\frac{M(\lambda) - M(\mu)^*}{\lambda - \bar{\mu}} = \gamma(\mu)^* \gamma(\lambda), \quad \lambda, \mu \in \rho(A_0). \tag{2.22}$$

Since $\gamma(\lambda)$ is injective and maps \mathcal{H} onto $\mathfrak{N}_\lambda(S^*)$, (2.22) shows that $\mathfrak{I}mM(\lambda)$ is boundedly invertible. By means of M the identity (2.14) can be rewritten as

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)(M(\lambda) - \Theta)^{-1}\gamma(\bar{\lambda})^*. \tag{2.23}$$

The class $\mathbf{N}_0(\mathcal{H})$ is introduced as the collection of all Nevanlinna functions M , which satisfy

$$\sup_{y>0} y \langle \mathfrak{I}mM(iy)h, h \rangle < \infty, \tag{2.24}$$

for all $h \in \mathcal{H}$; cf. [20]. In this case there exists a selfadjoint operator $E \in \mathbf{B}(\mathcal{H})$ such that

$$\lim_{\lambda \rightarrow \infty} M(\lambda)h = Eh, \quad h \in \mathcal{H}.$$

It is well-known that for any Nevanlinna function M there exists an operator $B \in \mathbf{B}(\mathcal{H})$ such that

$$\lim_{\lambda \rightarrow \infty} \frac{M(\lambda)h}{\lambda} = Bh, \quad h \in \mathcal{H}.$$

If B is boundedly invertible then

$$s - \lim_{y \uparrow \infty} iyM(iy)^{-1} = B^{-1}. \tag{2.25}$$

A Nevanlinna function is said to be *uniformly strict*, when $0 \in \rho(\mathfrak{I}mM(\lambda))$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$; in this case also $-M(\lambda)^{-1}$ belongs to $\mathbf{N}(\mathcal{H})$ and is uniformly strict. The boundedness of the limit value in (2.25) shows that the function $-M(\lambda)^{-1}$ belongs to the subclass $\mathbf{N}_0(\mathcal{H})$.

3. Maximally nondensely defined symmetric relations

3.1. Orthogonal projections of nondensely defined symmetric relations

Let S be a symmetric relation in a Hilbert space \mathfrak{H} and assume it is nondensely defined so that $\text{mul } S^*$ is nontrivial. The Hilbert space \mathfrak{H} admits the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ with $\mathfrak{H}_1 = \overline{\text{dom } S}$ and $\mathfrak{H}_2 = \text{mul } S^*$. Note that $\mathfrak{H}_1 \subset \overline{\text{dom } S^*}$. Let P denote the orthogonal projection from \mathfrak{H} onto $\text{mul } S^*$. Note that $\text{mul } S^{**} \subset \text{mul } S^*$ so that $(\text{mul } S^*)^\perp \subset (\text{mul } S^{**})^\perp$. Define the linear relations $S_1 = (I - P)S$ and $S_2 = PS$ in \mathfrak{H} by

$$S_1 = \{(f, (I - P)f'); (f, f') \in S\}, \quad S_2 = \{(f, Pf'); (f, f') \in S\}. \tag{3.1}$$

It is clear that $\text{dom } S_1 = \text{dom } S \subset \mathfrak{H}_1$, $\text{ran } S_1 \subset \overline{\text{dom } S} = \mathfrak{H}_1$, and that S_1 is a symmetric operator in \mathfrak{H} . Moreover,

$$S_1^* = S^*(I - P), \quad S_1^{**} = (S^*(I - P))^* \supset (I - P)S^{**}.$$

Note that $\text{mul } S^* = \ker(I - P) \subset \text{dom } S_1^*$. Likewise it is clear that $\text{dom } S_2 = \text{dom } S \subset \mathfrak{H}_1$, $\text{ran } S_2 \subset \text{mul } S^* = \mathfrak{H}_2$, and that S_2 is a relation in \mathfrak{H} with $\text{mul } S_2 = \text{mul } S$. Moreover,

$$S_2^* = S^*P, \quad S_2^{**} = (S^*P)^* \supset PS^{**}.$$

Note that $\overline{\text{dom } S} = \ker P \subset \text{dom } S_2^*$, which implies that

$$\overline{\text{dom } S_2^*} = \overline{\text{dom } S^*} = \mathfrak{H} \ominus \text{mul } S^{**} = \overline{\text{dom } S} \oplus (\text{mul } S^* \ominus \text{mul } S^{**}), \tag{3.2}$$

in particular, $\text{mul } S_2^{**} = \text{mul } S^{**}$.

In some sense S_1 and S_2 resemble the regular and singular parts of the relation S . However it will be useful to consider them in spaces related to the decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. The relation S_1 is a linear subspace of the product $\mathfrak{H} \times \mathfrak{H}$. However, the inclusions $\text{dom } S_1 \subset \mathfrak{H}_1$, $\text{ran } S_1 \subset \mathfrak{H}_1$ show that S_1 may also be considered as a linear subspace of the product $\mathfrak{H}_1 \times \mathfrak{H}_1$; therefore define

$$S_{11} = S_1 \cap (\mathfrak{H}_1 \times \mathfrak{H}_1), \tag{3.3}$$

in other words S_{11} as a graph is the same as S_1 but considered in the product space $\mathfrak{H}_1 \times \mathfrak{H}_1$, rather than in the product space $\mathfrak{H} \times \mathfrak{H}$. Clearly S_{11} is a symmetric operator in \mathfrak{H}_1 with $\text{dom } S_{11} = \text{dom } S$; hence S_{11} is densely defined and closable in \mathfrak{H}_1 . Observe that

$$(S_{11})^* = S^* \cap (\mathfrak{H}_1 \times \mathfrak{H}_1), \tag{3.4}$$

so that $S_{11}^* \subset S^*$. The relation S_2 is a linear subspace of the product $\mathfrak{H} \times \mathfrak{H}$. However, the inclusions $\text{dom } S_2 \subset \mathfrak{H}_1$, $\text{ran } S_2 \subset \mathfrak{H}_2$ show that S_2 may also be considered as a linear subspace of the product $\mathfrak{H}_1 \times \mathfrak{H}_2$; therefore define

$$S_{21} = S_2 \cap (\mathfrak{H}_1 \times \mathfrak{H}_2), \tag{3.5}$$

in other words S_{21} as a graph is the same as S_2 but considered in the product space $\mathfrak{H}_1 \times \mathfrak{H}_2$, rather than in the product space $\mathfrak{H} \times \mathfrak{H}$. Clearly $\text{dom } S_{21} = \text{dom } S$, so that the relation S_{11} is densely defined in \mathfrak{H}_1 . Observe that

$$(S_{21})^* = S^* \cap (\mathfrak{H}_2 \times \mathfrak{H}_1), \tag{3.6}$$

so that $S_{21}^* \subset S^*$. The adjoints in (3.4) and (3.6) of S_{11} and S_{21} have been taken with respect to the spaces $\mathfrak{H}_1 \times \mathfrak{H}_1$ and $\mathfrak{H}_1 \times \mathfrak{H}_2$, respectively. This convention will be used in the rest of the paper; in fact, the notations S_{ij}^* and S_{ij}^{**} , $i, j = 1, 2$, will stand for $(S_{ij})^*$ and $(S_{ij})^{**}$, respectively. The next lemma can be derived from [22]; see also [30,27]. Observe that if the multi-valued relation S is not symmetric, then the following matrix representation fails to hold in general.

Lemma 3.1. *Let S be a symmetric relation in the Hilbert space \mathfrak{H} . Then S admits the following block representation as a multi-valued column operator from \mathfrak{H}_1 to $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ with entries S_{11} and S_{21} defined in (3.3), (3.5):*

$$S = \begin{pmatrix} S_{11} \\ S_{21} \end{pmatrix}. \tag{3.7}$$

Moreover, the adjoint S^* is a, not necessarily densely defined, operator from $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ to \mathfrak{H}_1 such that

$$S^* \supset \begin{pmatrix} S_{11}^* & S_{21}^* \end{pmatrix}, \quad S^{**} \subset \begin{pmatrix} S_{11}^* & S_{21}^* \end{pmatrix}^* = \begin{pmatrix} S_{11}^{**} \\ S_{21}^{**} \end{pmatrix}. \tag{3.8}$$

Conversely, if S_{11} is a symmetric operator in \mathfrak{H}_1 and S_{21} is a linear relation from \mathfrak{H}_1 to \mathfrak{H}_2 such that $\text{dom } S_{11} \cap \text{dom } S_{21}$ is dense in \mathfrak{H}_1 , then S defined by (3.7) is a symmetric relation in \mathfrak{H} with a dense domain in \mathfrak{H}_1 .

Proof. Clearly the inclusion “ \subset ” in (3.7) holds for an arbitrary (not necessarily symmetric) relation S . Since S is symmetric, $\text{mul } S \subset \text{mul } S^* = \mathfrak{H} \ominus \text{dom } S = \mathfrak{H}_2$ and hence $\text{mul } S = \text{mul } S_{21}$; this implies that the inclusion “ \supset ” in (3.7) is also satisfied. The formulas in (3.8) follow from (3.7) using the general result on block relations in [22, Proposition 2.1].

The converse statement can be checked in a straightforward manner. \square

In particular, **Lemma 3.1** implies that S is closable if and only if S_{21} is closable and that $\text{mul } S_{21}^{**} = \text{mul } S^{**}$ or, equivalently,

$$\overline{\text{dom}} S_{21}^* = \text{mul } S^* \ominus \text{mul } S^{**} = \overline{\text{dom}} S^* \ominus \mathfrak{H}_1. \tag{3.9}$$

If the adjoint of S is calculated in \mathfrak{H} then it is given by $S^* \widehat{\uparrow} (\{0\} \times \text{mul } S^*)$, where S^* is the adjoint as in **Lemma 3.1**.

The defect subspaces of the densely defined symmetric operator S_{11} in \mathfrak{H}_1 are said to be *the semidefect subspaces* of the original symmetric relation S in \mathfrak{H} : they are denoted by

$$\mathfrak{N}_\lambda(S_{11}^*) = \ker(S_{11}^* - \lambda) = \mathfrak{H}_1 \ominus \overline{\text{ran}}(S_{11} - \bar{\lambda}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.10}$$

The dimensions of the semidefect subspaces of S , i.e. the defect numbers of S_{11} in \mathfrak{H}_1 , are called *the semidefect numbers of S* ; see [2,31, Section 1.5]. To formulate the next lemma denote by P_λ the orthogonal projection from \mathfrak{H} onto $\ker(S^* - \lambda)$, so that $\ker P_\lambda = \overline{\text{ran}}(S - \bar{\lambda})$.

Lemma 3.2. *Let S be a symmetric relation in a Hilbert space \mathfrak{H} . Then*

$$\mathfrak{N}_\lambda(S_{11}^*) = \mathfrak{N}_\lambda(S^*) \cap \overline{\text{dom}} S, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{3.11}$$

and moreover,

$$P_\lambda(\mathfrak{N}_\lambda(S_{11}^*) \oplus \text{mul } S^*) = \mathfrak{N}_\lambda(S^*), \tag{3.12}$$

$$\ker(P_\lambda \upharpoonright (\mathfrak{N}_\lambda(S_{11}^*) \oplus \text{mul } S^*)) = \text{mul } S^{**}. \tag{3.13}$$

In particular,

$$\dim \mathfrak{N}_\lambda(S^*) = \dim \mathfrak{N}_\lambda(S_{11}^*) + \dim(\text{mul } S^* \ominus \text{mul } S^{**}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Proof. The identity (3.11) follows directly from (3.4). It follows from (3.10) that

$$\overline{\text{ran}}(S_{11} - \bar{\lambda}) \oplus \mathfrak{N}_\lambda(S_{11}^*) \oplus \text{mul } S^* = \mathfrak{H}. \tag{3.14}$$

Observe that $S_1 = (I - P)S$ leads to

$$\text{ran}(S_{11} - \bar{\lambda}) \subset \text{ran}(S - \bar{\lambda}) + \text{mul } S^*. \tag{3.15}$$

Hence a combination of (3.14) and (3.15) gives

$$\overline{\text{ran}}(S - \bar{\lambda}) + (\mathfrak{N}_\lambda(S_{11}^*) \oplus \text{mul } S^*) = \mathfrak{H},$$

which leads to (3.12). Finally apply (2.4) with S^{**} to obtain (3.13). \square

It follows from **Lemma 3.2** that for a nondensely defined symmetric relation S there are two extreme cases. The one extreme case occurs when $\mathfrak{N}_\lambda(S_{11}^*) = \{0\}$ or, equivalently, when S_{11} is essentially selfadjoint in \mathfrak{H}_1 . Since this case has an important role in the present paper, the following definition is introduced.

Definition 3.3. A symmetric, not necessarily closed, relation S is said to be maximally nondensely defined if the semidefect numbers of S are equal to $(0, 0)$, or equivalently, if the following equality holds:

$$\mathfrak{N}_\lambda(S^*) \cap \overline{\text{dom}} S = \{0\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

The other extreme case occurs when $\mathfrak{N}_\lambda(S_{11}^*) = \mathfrak{N}_\lambda(S^*)$ or, equivalently, when $\mathfrak{N}_\lambda(S^*) \subset \overline{\text{dom}} S$. For the sake of completeness this last case will be described in the rest of the present subsection.

Lemma 3.4. *Let S be a symmetric relation in a Hilbert space \mathfrak{H} . Then the following statements are equivalent:*

- (i) $\mathfrak{N}_\lambda(S^*) \subset \overline{\text{dom}} S$ for some (equivalently for every) $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
- (ii) $\text{mul } S^{**} = \text{mul } S^*$;
- (iii) $\text{mul } S_{21}^{**} = \text{mul } S^*$ or, equivalently, S_{21} is singular.

If the relation S is a closable operator, then (ii) is equivalent to

- (iv) $\text{dom } S$ is dense in \mathfrak{H} .

If the relation S is closed, then each of (i), (ii), or (iii) is equivalent to

- (v) $S = S_{\text{op}} \hat{+} (\{0\} \oplus \text{mul } S^*)$.

Proof. (i) \Leftrightarrow (ii) In view of (3.11) one has

$$\mathfrak{N}_\lambda(S^*) \subset \overline{\text{dom}} S \Leftrightarrow \mathfrak{N}_\lambda(S_{11}^*) = \mathfrak{N}_\lambda(S^*), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

In this case it follows from

$$\mathfrak{H} = \overline{\text{ran}} (S_{11} - \bar{\lambda}) \oplus \mathfrak{N}_\lambda(S_{11}^*) \oplus \text{mul } S^* = \overline{\text{ran}} (S - \bar{\lambda}) \oplus \mathfrak{N}_\lambda(S^*),$$

that

$$\overline{\text{ran}} (S_{11} - \bar{\lambda}) \oplus \text{mul } S^* = \overline{\text{ran}} (S - \bar{\lambda}),$$

and hence

$$\text{mul } S^* \subset \overline{\text{ran}} (S - \bar{\lambda}) \subset \overline{\text{ran}} (S^{**} - \bar{\lambda}) = \text{ran} (S^{**} - \bar{\lambda}),$$

which combined with (2.4) shows that $\text{mul } S^* \subset \text{mul } S^{**}$, i.e., $\text{mul } S^* = \text{mul } S^{**}$.

Conversely, observe that (3.11) and (2.4) imply that

$$\mathfrak{N}_\lambda(S_{11}^*) = P_\lambda(\mathfrak{N}_\lambda(S_{11}^*)) = P_\lambda(\mathfrak{N}_\lambda(S_{11}^*) \oplus \text{mul } S^{**}).$$

Hence, if $\text{mul } S^* = \text{mul } S^{**}$, then (3.12) implies that

$$\mathfrak{N}_\lambda(S_{11}^*) = P_\lambda(\mathfrak{N}_\lambda(S_{11}^*) \oplus \text{mul } S^*) = \mathfrak{N}_\lambda(S^*),$$

which with (3.11) shows that $\mathfrak{N}_\lambda(S^*) \subset \overline{\text{dom}} S$.

(ii) \Leftrightarrow (iii) Since $\text{mul } S_{21}^{**} = \text{mul } S^{**}$, this means that (ii) is equivalent to $\text{mul } S_{21}^{**} = \text{mul } S^*$.

If S is a closable operator, then $\text{mul } S^{**} = \{0\}$, and condition (ii) means that $\text{mul } S^* = \{0\}$, which is equivalent with (iv). If S is closed, then (ii) \Leftrightarrow (v) follows from (2.7). \square

Example 3.5. The condition $\mathfrak{N}_\lambda(S^*) \subset \overline{\text{dom}} S$ may be satisfied without S being densely defined. Let S_{21} be a singular operator in a Hilbert space of the form $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ with $\mathfrak{H}_1 = \overline{\text{dom}} S_1$ and $\mathfrak{H}_2 = \overline{\text{ran}} S_2 \neq \{0\}$; such an operator is easily constructed from any singular operator in a, necessarily infinite-dimensional, Hilbert space \mathfrak{H} ; cf. [16]. Let S_{11} be a bounded selfadjoint operator acting in \mathfrak{H}_1 . Then S defined by (3.7) is a symmetric operator in \mathfrak{H} with $\text{dom } S = \text{dom } S_{21}$ and $\text{mul } S^{**} = \text{mul } S_{21}^{**} = \mathfrak{H}_2 = (\text{dom } S)^\perp$. Hence by Lemma 3.4 $\mathfrak{N}_\lambda(S^*) \subset \overline{\text{dom}} S$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, while the operator S is not densely defined in \mathfrak{H} .

Remark 3.6. Example 3.5 shows that the condition $\mathfrak{N}_\lambda(S^*) \subset \overline{\text{dom } S}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, does not imply that S is densely defined in \mathfrak{H} when S is not closed.

Notice also that in Lemma 3.4(i) it suffices to consider the inclusion $\mathfrak{N}_\lambda(S^*) \subset \overline{\text{dom } S}$ for one point $\lambda \in \mathbb{C} \setminus \mathbb{R}$. If one knows in addition that $\mathfrak{N}_\lambda(S^*) \subset \overline{\text{dom } S}$ and $\mathfrak{N}_{\bar{\lambda}}(S^*) \subset \overline{\text{dom } S}$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then from the first von Neumann’s formula one concludes that $\text{dom } S^* \subset \overline{\text{dom } S}$. Since S is closable if and only if S^* is densely defined, it follows that S , together with S^* , is densely defined; hence using two points one obtains a more direct proof for item (iv) in Lemma 3.4 via von Neumann’s formula.

3.2. Some characterizations of symmetric relations

Let S be a symmetric relation in a Hilbert space \mathfrak{H} and assume it is nondensely defined so that $\text{mul } S^*$ is nontrivial. Then the relation S_∞ defined by the forbidden manifold (see (2.16))

$$S_\infty = S \widehat{+} (\{0\} \times \text{mul } S^*), \tag{3.16}$$

is a symmetric extension of S :

$$S \subset S_\infty \subset (S_\infty)^* \subset S^*. \tag{3.17}$$

Observe that the operatorwise sum $S_1 + S_2$ is a symmetric extension of S , which is contained in S_∞ . It follows from $\text{mul } S \subset \text{mul } S^*$ that $\text{mul } S_\infty = \text{mul } S^*$. Clearly, $S \subset \text{clos } S = S^{**}$ and, since $(\text{dom } S^{**})^\perp = \text{mul } S^*$, it is easy to see that the following inclusions hold:

$$S_\infty \subset (S^{**})_\infty \subset \text{clos } (S_\infty) = \text{clos } (S^{**})_\infty. \tag{3.18}$$

Observe that

$$\text{ran } (S_\infty - \lambda) = \text{ran } (S - \lambda) + \text{mul } S^*, \quad \lambda \in \mathbb{C}. \tag{3.19}$$

Due to (2.4) the sum in (3.19) is direct for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ if and only if S is an operator. The operator S_{11} in (3.3) and the relation S_∞ in (3.16) are related by

$$S_\infty = S_{11} \widehat{\oplus} (\{0\} \times \text{mul } S^*), \tag{3.20}$$

where $\{0\} \times \text{mul } S^*$ is a selfadjoint relation in the Hilbert space $\text{mul } S^*$. The next lemma describes S_∞ , its adjoint, and its closure; cf. [16].

Lemma 3.7. *Let S be a symmetric relation in a Hilbert space \mathfrak{H} . Then*

$$S_\infty = \{(f, f') \in S^*; f \in \text{dom } S\}, \tag{3.21}$$

$$(S_\infty)^* = \{(f, f') \in S^*; f \in \overline{\text{dom } S}\}, \tag{3.22}$$

and

$$(S_\infty)^{**} = \{(f, f') \in S^*; f \in \text{dom } S_{11}^{**}\}. \tag{3.23}$$

Proof. The formulas (3.21) and (3.22) can be found in [16,8]. To prove (3.23), observe that the orthogonal sum decomposition in (3.20) implies that

$$(S_\infty)^{**} = S_{11}^{**} \widehat{\oplus} (\{0\} \times \text{mul } S^*). \tag{3.24}$$

In particular, $S_{11}^{**} \subset (S_\infty)^{**} \subset S^*$. Now it is easy to check that the formula (3.24) is equivalent to the formula (3.23). \square

Note that $(S_\infty)^* = ((S^{**})_\infty)^*$; see (3.18). Since S is symmetric, Lemma 3.7 implies that $\overline{\text{dom } (S_\infty)^*} = \overline{\text{dom } S}$ and, hence, $\text{mul clos } (S_\infty) = \text{mul } S^* = \text{mul } S_\infty$.

Lemma 3.8. *Let S be a symmetric relation in a Hilbert space \mathfrak{H} . Then the following statements are equivalent:*

- (i) S_∞ is closed;
- (ii) S_{11} in (3.1) is a closed operator in $\overline{\text{dom } S}$;
- (iii) $\text{ran } (S - \lambda) + \text{mul } S^*$ is closed for some (and hence for all) λ in $\mathbb{C} \setminus \mathbb{R}$;
- (iv) $\text{dom } S = \text{dom } S^{**}$ and $\overline{\text{dom } S^*} = \text{dom } S^* + \overline{\text{dom } S}$.

If the symmetric relation S is closed, then each of the statements (i)–(iv) is also equivalent to the following statement:

- (v) $P_\lambda \text{mul } S^*$ is a closed subspace of $\ker(S^* - \lambda)$ for some (and hence for all) $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. (i) \Leftrightarrow (ii) This equivalence follows from (3.20).

(i) \Leftrightarrow (iii) The symmetric relation S_∞ is closed if and only if $\text{ran } (S_\infty - \lambda)$ is closed for some (and hence for all) $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then recall the identity in (3.19).

(i) \Rightarrow (iv) Since S_∞ is closed, one has $S_\infty = (S^{**})_\infty$; see (3.18). This shows that $\text{dom } S = \text{dom } S^{**}$. Next apply Proposition 2.4 with $T = S^{**}$ and $\mathfrak{L} = \text{mul } S^*$. Since S is symmetric, it follows that $\text{mul } S^{**} \subset \text{mul } S^*$, so that $\text{mul } S^{**} + \mathfrak{L} = \mathfrak{L}$ is closed. By Proposition 2.4 the relation $(S^{**})_\infty$ in (3.16) is closed if and only if

$$\overline{\text{dom } S^*} \subset \text{dom } S^* + (\text{mul } S^*)^\perp = \text{dom } S^* + \overline{\text{dom } S}.$$

Since the right-hand side is clearly contained in $\overline{\text{dom } S^*}$, this shows that (iv) holds.

(iv) \Rightarrow (i) If $\overline{\text{dom } S^*} = \text{dom } S^* + \overline{\text{dom } S}$ then Proposition 2.4 shows that $(S^{**})_\infty$ is closed. On the other hand, $\text{mul } S \subset \text{mul } S^{**} \subset \text{mul } S^*$ and hence the equality $\text{dom } S = \text{dom } S^{**}$ implies the equality $S_\infty = (S^{**})_\infty$ and, thus, (i) is satisfied.

Now assume in addition that S is closed. Then equivalently $\text{ran } (S - \lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is closed.

(iii) \Leftrightarrow (v) Apply Lemma 2.1 to the closed subspaces $\mathfrak{M} = \text{mul } S^*$ and $\mathfrak{N} = \text{ran } (S - \bar{\lambda})$. Note that $\mathfrak{N}^\perp = \ker(S^* - \lambda) = \text{ran } P_\lambda$. Hence, by Lemma 2.1(i)

$$\text{ran } (S - \bar{\lambda}) + \text{mul } S^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

is closed if and only if $P_\lambda \text{mul } S^*$ is closed. \square

Remark 3.9. In the case that S is a closed operator the equivalence of (ii) and (v) in Lemma 3.8 is also proved, for instance, in [2, Theorem 1.5.4] and in fact it goes back to M.A. Krasnoselskiĭ [26]. The equivalence of (i) and (ii) has been also proved e.g. in [12, Proposition 2.5]. Some further equivalent conditions, based on the rigged Hilbert space generated by S^* can be found in [2, Theorem 2.4.1]. In [29] (see also [2,12]) a symmetric operator S is said to be *regular*, if the subspaces $P_\lambda \text{mul } S^*$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, are closed; in the opposite case S is said to be *singular*. In the present paper the terms regular and singular refer to a more general terminology which appears in connection with general linear operators and relations in Hilbert spaces, cf. Section 2; for further details see [16] and the references therein. Finally, observe that in [2, Section 2.4] the term *O-operator* stands for a closed symmetric operator whose semidefect numbers are equal to $(0, 0)$, instead of the present expression “ S is maximally nondensely defined” that appears in Definition 3.3.

The next result gives several criteria for S to be maximally nondensely defined in \mathfrak{H} .

Proposition 3.10. *Let S be a symmetric relation in a Hilbert space \mathfrak{H} . Then the following statements are equivalent:*

- (i) S is maximally nondensely defined in \mathfrak{H} ;
- (ii) S_{11} in (3.1) is an essentially selfadjoint operator in $\overline{\text{dom}} S$, i.e., the semidefect indices of S are equal to $(0, 0)$;
- (iii) S_∞ is essentially selfadjoint;
- (iv) $\text{ran}(S - \lambda) + \text{mul } S^*$ is dense in \mathfrak{H} for some (and hence for all) λ in \mathbb{C}^+ and for some (and hence for all) $\lambda \in \mathbb{C}^-$;
- (v) $\text{dom } S_{11}^{**} = \overline{\text{dom}} S \cap \text{dom } S^*$;
- (vi) $P_\lambda \text{mul } S^*$ is dense in $\ker(S^* - \lambda)$ for some (and hence for all) $\lambda \in \mathbb{C}^+$ and for some (and hence for all) $\lambda \in \mathbb{C}_-$.

Proof. (i) \Leftrightarrow (ii) This follows from Lemma 3.2.

(ii) \Leftrightarrow (iii) This equivalence follows directly from (3.24).

(iii) \Leftrightarrow (iv) This equivalence follows from (3.19).

(ii) \Leftrightarrow (v) By Lemma 3.7 the equality $(S_\infty)^{**} = (S_\infty)^*$ can be rewritten as $\text{dom } S_{11}^{**} = \overline{\text{dom}} S \cap \text{dom } S^*$, since $\text{dom } S_{11}^{**} \subset \text{dom } S^*$; cf. (3.24).

(iv) \Leftrightarrow (vi) Consider the subspaces $\mathfrak{M} = \text{mul } S^*$ and $\mathfrak{N} = \text{ran}(S - \bar{\lambda})$ with $\lambda \in \mathbb{C} \setminus \mathbb{R}$. By Lemma 2.2 the sum $\overline{\text{ran}}(S - \bar{\lambda}) + \text{mul } S^*$ is dense in \mathfrak{H} if and only if $P_\lambda \text{mul } S^*$ is dense in $\mathfrak{N}^\perp = \ker(S^* - \lambda)$. \square

By combining Lemma 3.8 and Proposition 3.10, one obtains the following characterizations for S_∞ to be selfadjoint. The equivalence of the statements (i) and (iv) goes back to [8].

Proposition 3.11. *Let S be a symmetric relation in a Hilbert space \mathfrak{H} . Then the following statements are equivalent:*

- (i) S_∞ is selfadjoint;
- (ii) S_{11} in (3.1) is a selfadjoint operator in $\overline{\text{dom}} S$;
- (iii) $\text{ran}(S - \lambda) + \text{mul } S^* = \mathfrak{H}$ for some (and hence for all) λ in \mathbb{C}^+ and for some (and hence for all) $\lambda \in \mathbb{C}^-$;
- (iv) $\text{dom } S = \overline{\text{dom}} S \cap \text{dom } S^*$;
- (v) S is maximally nondensely defined in \mathfrak{H} and in addition $\text{dom } S = \text{dom } S^{**}$ and $\overline{\text{dom}} S^* = \text{dom } S^* + \overline{\text{dom}} S$.

If the symmetric relation S is closed, then each of the statements (i)–(iv) is also equivalent to the following statement:

- (vi) $P_\lambda \text{mul } S^* = \ker(S^* - \lambda)$ for some (and hence for all) $\lambda \in \mathbb{C}^+$ and for some (and hence for all) $\lambda \in \mathbb{C}_-$.

Proof. The equivalence of (i), (ii), (iii), (v), and (vi) is obtained from Lemma 3.8 and Proposition 3.10. The equivalence (i) \Leftrightarrow (iv) follows from Lemma 3.7. \square

3.3. Characterizations via intermediate extensions

The following theorem gives necessary and sufficient conditions for S_∞ to be selfadjoint by means of a symmetric extension T of S which is transversal to S_∞ .

Theorem 3.12. *Let S be a closed symmetric relation in a Hilbert space \mathfrak{H} . Then the following statements are equivalent:*

- (i) S_∞ is selfadjoint;
- (ii) there exists a closed symmetric extension T of S such that

$$S^* = T \widehat{\mp} S_\infty; \tag{3.25}$$

- (iii) there exists a closed symmetric extension T of S such that

$$S^* = T \widehat{\mp} (\{0\} \times \text{mul } S^*); \tag{3.26}$$

- (iv) there exists a closed symmetric extension T of S such that

$$\text{dom } S^* \subset \text{dom } T; \tag{3.27}$$

- (v) there exists a closed symmetric extension T of S such that

$$\ker(S^* - \lambda) \subset \text{dom } T, \quad \ker(S^* - \bar{\lambda}) \subset \text{dom } T, \tag{3.28}$$

for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

If a closed symmetric extension T of S satisfies one of (3.25), (3.26), (3.27), or (3.28), then it automatically satisfies the other three of them. Any such closed symmetric relation T is automatically selfadjoint with $\text{mul } T = \text{mul } S$; and the selfadjoint extensions T and S_∞ are transversal with respect to S : $S^* = T \widehat{\mp} S_\infty$.

Proof. (i) \Rightarrow (ii) In fact there exists a selfadjoint extension T of S , which is transversal to S_∞ ; cf. [12].

(ii) \Leftrightarrow (iii) This equivalence is obvious.

(iii) \Rightarrow (iv) The identity (3.26) implies that $\text{dom } S^* = \text{dom } T$.

(iv) \Rightarrow (iii) Observe that $T \widehat{\mp} (\{0\} \times \text{mul } S^*) \subset S^*$ for any symmetric extension T of S . For the converse inclusion, let $\{f, f'\} \in S^*$. Then $f \in \text{dom } S^* = \text{dom } T$, and there exists an element $h \in \mathfrak{H}$ such that $\{f, h\} \in T \subset S^*$. Hence $\{0, f' - h\} = \{f, f'\} - \{f, h\} \in S^*$, which shows that $\{f, f'\} \in T \widehat{\mp} (\{0\} \times \text{mul } S^*)$. Therefore $S^* \subset T \widehat{\mp} (\{0\} \times \text{mul } S^*)$ and (iii) follows.

(iii) \Rightarrow (i) Since S is closed the identity (3.26) implies the identity

$$S = T^* \cap (\overline{\text{dom } S} \oplus \mathfrak{H}). \tag{3.29}$$

Now let $\{f, f'\} \in (S_\infty)^*$, so that by Lemma 3.7 $\{f, f'\} \in S^*$ and $f \in \overline{\text{dom } S}$. By assumption $f \in \text{dom } T \subset \text{dom } T^*$ and it follows from (3.29) that $f \in \text{dom } S$. Hence $\{f, f'\} \in S_\infty$ by Lemma 3.7. Therefore S_∞ is selfadjoint.

(iv) \Leftrightarrow (v) If (3.27) holds, then certainly (3.28) holds for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, since $\ker(S^* - \lambda) \subset \text{dom } S^*$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Conversely, assume that (3.28) holds for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Recall von Neumann’s decomposition (2.6) of S^* , which shows that (with S being closed)

$$\text{dom } S^* \subset \text{dom } S + \ker(S^* - \lambda) + \ker(S^* - \bar{\lambda}).$$

Now $\text{dom } S \subset \text{dom } T$ since T is an extension of S . Hence (3.27) follows.

The proof of the equivalence of (ii), (iii), (iv), (v), has already shown that if a closed symmetric extension T of S satisfies one of the four conditions (3.25), (3.26), (3.27), or (3.28), it automatically satisfies the other three conditions too.

Finally, let T be a closed symmetric extension of S which satisfies (3.25), (3.26), (3.27), or (3.28). In fact, assume that T satisfies (3.27). Then $\text{dom } T = \text{dom } T^* = \text{dom } S^*$, and it follows that $\text{mul } T = \text{mul } T^*$, since T is closed. If $\{f, f'\} \in T^*$, then $f \in \text{dom } T^* = \text{dom } T$

and there exists an element h such that $\{f, h\} \in T \subset T^*$. Hence $\{0, f' - h\} \in T^*$ or $f' - h \in \text{mul } T^* = \text{mul } T$, which implies that $\{f, f'\} = \{f, h\} + \{0, f' - h\} \in T$. Therefore T is selfadjoint and $\text{mul } T = (\text{dom } T^*)^\perp = (\text{dom } S^*)^\perp = \text{mul } S$. \square

Corollary 3.13. *Let S be a closed symmetric relation in a Hilbert space \mathfrak{H} . Assume that S_∞ is selfadjoint and that T is a selfadjoint extension of S such that T and S_∞ are transversal with respect to S . Then*

$$\mathfrak{N}_\lambda(S^*) = \{(T - \lambda)^{-1}\varphi; \varphi \in \text{mul } S^*\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.30}$$

Proof. Let $(f, \lambda f) \in S^*$. Then it follows from (3.26) that

$$(f, \lambda f) = (h, h') + (0, \varphi), \quad (h, h') \in T, \quad \varphi \in \text{mul } S^*.$$

Therefore, one obtains $(h, h') = (f, \lambda f - \varphi) \in T$ and $(f, -\varphi) \in T - \lambda$, which leads to $f = -(T - \lambda)^{-1}\varphi$. This shows $\mathfrak{N}_\lambda(S^*) \subset (T - \lambda)^{-1} \text{mul } S^*$.

Conversely, if $f = -(T - \lambda)^{-1}\varphi$ with $\varphi \in \text{mul } S^*$, then

$$(f, -\varphi + \lambda f) = (-(T - \lambda)^{-1}\varphi, -\varphi - \lambda(T - \lambda)^{-1}\varphi) \in T,$$

so that by (3.26)

$$(f, \lambda f) = (f, -\varphi + \lambda f) + (0, \varphi) \in T \widehat{+} (\{0\} \times \text{mul } S^*) = S^*.$$

This shows $\{(T - \lambda)^{-1}\varphi; \varphi \in \text{mul } S^*\} \subset \mathfrak{N}_\lambda(S^*)$. \square

The next theorem contains a further characterization for S_∞ to be selfadjoint which involves only properties of the domain of some symmetric extension T of S ; it can be also seen as a weakening of the criterion (vi) in Proposition 3.11.

Theorem 3.14. *Let S be a closed symmetric relation in a Hilbert space \mathfrak{H} . Then S_∞ is selfadjoint if and only if there is a closed symmetric extension T of S satisfying the following properties:*

- (i) $\overline{\text{dom } T} = \overline{\text{dom } S^*}$;
- (ii) $\text{dom } S = \overline{\text{dom } S} \cap \text{dom } T^*$;
- (iii) $P(\text{dom } T^*) = P(\overline{\text{dom } T^*})$.

In this case T is selfadjoint and transversal to $S_\infty : T \widehat{+} S_\infty = S^$. Moreover, if S_∞ is selfadjoint then every selfadjoint extension T of S such that $T \widehat{+} S_\infty = S^*$ admits the properties (i)–(iii).*

In particular, if S is a closed symmetric operator, then S_∞ is selfadjoint if and only if there exists a closed symmetric operator extension T of S such that

- (iv) $\overline{\text{dom } T} = \mathfrak{H}$;
- (v) $\text{dom } S = \overline{\text{dom } S} \cap \text{dom } T^*$;
- (vi) $P(\text{dom } T^*) = \text{mul } S^*$.

Proof. (\Rightarrow) Assume that S_∞ is selfadjoint; then there exists a selfadjoint extension T of S such that $S^* = T \widehat{+} S_\infty$; cf. Theorem 3.12. Moreover, by Theorem 3.12 $\text{mul } T = \text{mul } S$, so that $\overline{\text{dom } T} = \overline{\text{dom } S^*}$ and (i) follows. Taking adjoints in (3.26) leads to $S = T \cap (\overline{\text{dom } S} \times \mathfrak{H})$, which gives (ii). Since the right-hand side of (3.26) is closed and $\text{mul } T + \text{mul } S^* = \text{mul } S^*$ is closed, it follows from Proposition 2.4 that $P(\text{dom } T) = P(\overline{\text{dom } T})$, which is (iii).

(\Leftarrow) Let T be a closed symmetric extension of S , so that $S \subset T \subset T^* \subset S^*$ and assume that (i), (ii), and (iii) are satisfied. Introduce the linear relation

$$H = T \widehat{+} (\{0\} \times \text{mul } S^*).$$

Clearly $S \subset H \subset S^*$ and $\text{mul } T + \text{mul } S^* = \text{mul } S^*$. By condition (iii) and Proposition 2.4 it follows that H is closed. Furthermore, observe that

$$H^* = T^* \cap \overline{(\text{dom } S \times \mathfrak{H})} = S \widehat{\uparrow} (\{0\} \times \text{mul } T^*),$$

where the last identity follows from the assumption (ii). Now by the assumption (i) $\text{mul } T^* = \text{mul } S$ and, therefore, $H^* = S$ or, equivalently, $S^* = H$. Hence, the assumptions (i)–(iii) imply that $S^* = H = T \widehat{\uparrow} (\{0\} \times \text{mul } S^*)$, which by part (iii) of Theorem 3.12 means that S_∞ is selfadjoint and, furthermore, by the same theorem T is necessarily selfadjoint and satisfies $S^* = T \widehat{\uparrow} S_\infty$.

As to the last statement observe that, if S is a closed operator, then S^* is densely defined and, therefore, the statements (i)–(iii) reduce to the statements (iv)–(vi), respectively. \square

Corollary 3.15. *Let S be a closed symmetric relation in a Hilbert space \mathfrak{H} . Then S_∞ is selfadjoint if and only if there is a selfadjoint extension T of S satisfying the following properties:*

- (i) $\text{dom } S = \overline{\text{dom } S} \cap \text{dom } T$;
- (ii) $P(\text{dom } T) = \text{mul } S^* \ominus \text{mul } S^{**}$.

Proof. (\Rightarrow) By Theorem 3.14 there exists a selfadjoint extension T of S with the properties (i)–(iii) stated therein. In particular, (i) and (iii) in Theorem 3.14 combined with (3.9) show that $P(\text{dom } T) = P(\overline{\text{dom } S^*}) = \text{mul } S^* \ominus \text{mul } S^{**}$.

(\Leftarrow) Let T be a selfadjoint extension of S with the properties (i) and (ii). Here (ii) means that $P(\text{dom } T) = P(\overline{\text{dom } S^*})$ and hence $P(\text{dom } T) = P(\overline{\text{dom } T})$ holds. Since $\ker P = \overline{\text{dom } S} \subset \overline{\text{dom } T} \subset \overline{\text{dom } S^*}$, the equality $P(\overline{\text{dom } T}) = P(\overline{\text{dom } S^*})$ implies that $\overline{\text{dom } T} = \overline{\text{dom } S^*}$. To complete the proof it remains to apply Theorem 3.14. \square

4. Partially bounded symmetric relations

Let S be a nondensely defined symmetric relation in a Hilbert space \mathfrak{H} . Then \mathfrak{H} admits the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2 = \overline{\text{dom } S} \oplus \text{mul } S^*$. Recall from Section 3.1 the definitions of S_{11} as a densely defined operator in \mathfrak{H}_1 and of S_{21} as a densely defined relation from \mathfrak{H}_1 to \mathfrak{H}_2 , which satisfy $S_{11}^* \subset S^*$ and $S_{12}^* \subset S^*$. Observe that the corresponding inclusions fail to hold for the linear relations $S_1 = (I - P)S$ and $S_2 = PS$ in \mathfrak{H} , since $\text{mul } S^* \subset \text{dom } S_1^* = \text{dom } S^*(I - P)$ and $\overline{\text{dom } S} \subset \text{dom } S_2^* = \text{dom } S^*P$, but in general $\text{mul } S^* \not\subset \text{dom } S_1^*$ and $\overline{\text{dom } S} \not\subset \text{dom } S_2^*$. This motivates the following definition.

Definition 4.1. A symmetric, not necessarily closed, relation S in a Hilbert space \mathfrak{H} is said to be inner or outer bounded if the closure of S_{11} or S_{21} , respectively, has a closed domain. Moreover, S is said to be partially bounded if it is inner or outer bounded.

Partially bounded symmetric relations can be characterized as follows.

Proposition 4.2. *Let S be a symmetric relation in a Hilbert space \mathfrak{H} . Then the following statements are equivalent:*

- (i) S is inner bounded, i.e., $\text{dom } S_{11}^{**}$ is closed;
- (ii) S_{11} is a bounded symmetric operator in $\overline{\text{dom } S}$;
- (iii) $\text{dom } S \subset \text{dom } S^*$;
- (iv) S_∞ , as well as $(S^{**})_\infty$, is essentially selfadjoint and their closure $(S_\infty)^{**} = (S_\infty)^*$ has a closed domain.

Moreover the following statements are equivalent:

- (v) S is outer bounded, i.e., $\text{dom } S_{21}^{**}$ is closed;
- (vi) $\text{mul } S^* \ominus \text{mul } S^{**} \subset \text{dom } S^*$.

Proof. (i) \Leftrightarrow (ii) Since S_{11} as a densely defined symmetric operator is closable and S_{11} is bounded if and only if S_{11}^{**} is bounded, this equivalence follows from the closed graph theorem.

(ii) \Rightarrow (iii) If S_{11} is bounded, then S_{11}^{**} is bounded with a closed domain. Observe that $S_{11} \subset S_{11}^* \subset S^*$, see (3.8), which implies that $S_{11}^{**} \subset S^*$. Therefore, $\overline{\text{dom } S} = \overline{\text{dom } S_{11}} = \text{dom } S_{11}^{**} \subset \text{dom } S^*$.

(iii) \Rightarrow (iv) By Lemma 3.1 one has

$$(S^{**})_{11} \subset S_{11}^{**}$$

and hence the assumption $\overline{\text{dom } S} \subset \text{dom } S^*$ implies that S_{11}^{**} is bounded. Hence, $(S^{**})_{11}$ is also bounded and consequently

$$\text{dom } (S^{**})_{11}^{**} = \text{dom } S_{11}^{**} = \overline{\text{dom } S} = \overline{\text{dom } S} \cap \text{dom } S^*,$$

which according to Proposition 3.10 means that S_∞ and $(S^{**})_\infty$ are essentially selfadjoint. On the other hand, (3.24) implies that $\text{dom } (S_\infty)^{**} = \text{dom } S_{11}^{**} = \overline{\text{dom } S}$ is closed.

(iv) \Rightarrow (i) This follows from the equality $\text{dom } (S_\infty)^{**} = \text{dom } S_{11}^{**}$; see (3.24).

(v) \Rightarrow (vi) If $\text{dom } S_{21}^{**}$ is closed then, equivalently, $\text{dom } S_{21}^*$ is closed. Hence, (3.8) and (3.9) imply that $\text{dom } S_{21}^* = \text{mul } S^* \ominus \text{mul } S^{**} \subset \text{dom } S^*$.

(vi) \Rightarrow (v) The adjoint of $S_2 = PS$ in \mathfrak{H} satisfies $\overline{\text{dom } S} \subset \text{dom } S_2^*$ and hence it follows from $\text{mul } S^* \ominus \text{mul } S^{**} \subset \text{dom } S^*$ and the formula (3.2) that

$$\overline{\text{dom } S_2^*} = \overline{\text{dom } S} \oplus (\text{mul } S^* \ominus \text{mul } S^{**}) \subset \text{dom } S^* P = \text{dom } S_2^*.$$

Hence, $\text{dom } S_2^*$ and, therefore, also $\text{dom } S_2^{**}$ is closed or, equivalently, $\text{dom } S_{21}^{**}$ is closed, i.e., S is outer bounded. \square

Some further characterizations for S to be inner bounded may be obtained by combining Proposition 4.2 with Proposition 3.10.

The next proposition gives a characterization for S to be inner bounded and $(S^{**})_\infty$ to be selfadjoint.

Proposition 4.3. *Let S be a symmetric relation in a Hilbert space \mathfrak{H} . Then the following statements are equivalent:*

- (i) S is inner bounded and $(S^{**})_\infty$ is selfadjoint;
- (ii) $(S^{**})_\infty$ is selfadjoint with a closed domain and $\text{mul } S^* \ominus \text{mul } S^{**} \subset \text{dom } S^*$;
- (iii) $\text{dom } S^{**}$ is closed or, equivalently, $\text{dom } S^*$ is closed;
- (iv) S is inner and outer bounded.

In particular, a closable symmetric relation S is inner and outer bounded if and only if S is a bounded symmetric operator.

Proof. (i) \Rightarrow (ii) If S is inner bounded, then by Proposition 4.2 $\overline{\text{dom } S} \subset \text{dom } S^*$ and hence $\overline{\text{dom } S} \cap \text{dom } S^* = \overline{\text{dom } S}$. Since $(S^{**})_\infty$ is selfadjoint, it follows from Proposition 3.11 that

$$\text{dom } (S^{**})_\infty = \text{dom } S^{**} = \overline{\text{dom } S^{**}} \cap \text{dom } S^* = \overline{\text{dom } S}.$$

Hence $\text{dom } S^{**}$ and, thus, also $\text{dom } S^*$ is closed. The inclusion $\text{mul } S^* \ominus \text{mul } S^{**} \subset \text{dom } S^*$ is now obtained from (3.2).

(ii) \Rightarrow (iii) This implication is clear from $\text{dom } (S^{**})_\infty = \text{dom } S^{**}$.

(iii) \Rightarrow (iv) If $\text{dom } S^{**}$ is closed then it is clear from Lemma 3.1 that $\text{dom } S_{21}^{**} = \text{dom } S_{11}^{**} = \text{dom } S^{**} = \overline{\text{dom } S}$, so that S is inner and outer bounded.

(iv) \Rightarrow (i) If S is inner bounded, then by Proposition 4.2 $\overline{\text{dom } S} \subset \text{dom } S^*$ and $\overline{\text{dom } S} \cap \text{dom } S^* = \overline{\text{dom } S}$. Since S is outer bounded, Proposition 4.2 shows that $\text{mul } S^* \ominus \text{mul } S^{**} \subset \overline{\text{dom } S^*}$. Now (3.2) shows that $\overline{\text{dom } S^*} \subset \text{dom } S^*$, i.e. $\text{dom } S^*$ is closed, and thus $\text{dom } S^{**} = \overline{\text{dom } S} = \overline{\text{dom } S} \cap \text{dom } S^*$. Hence $(S^{**})_\infty$ is selfadjoint by Proposition 4.2. \square

As to the last statement in Proposition 4.3 observe, that if S is inner and outer bounded, but not closable, then $\text{mul } S^{**} = \text{mul } S_2^{**}$ is non-trivial.

The situation that S is outer bounded and, in addition, maximally nondensely defined can be characterized as follows.

Proposition 4.4. *Let S be a symmetric relation in a Hilbert space \mathfrak{H} . Then the following statements are equivalent:*

- (i) S is outer bounded and $(S^{**})_\infty$ is selfadjoint;
- (ii) $\text{dom } S^* = \text{dom } S^{**} \oplus (\text{mul } S^* \ominus \text{mul } S^{**})$.

Moreover, the following statements are equivalent:

- (iii) S is outer bounded and S_∞ is selfadjoint;
- (iv) $\text{dom } S^* = \text{dom } S \oplus (\text{mul } S^* \ominus \text{mul } S^{**})$.

Proof. (i) \Rightarrow (ii) By Proposition 4.2 $\text{mul } S^* \ominus \text{mul } S^{**} \subset \text{dom } S^*$, and hence

$$\text{dom } S^{**} \oplus (\text{mul } S^* \ominus \text{mul } S^{**}) \subset \text{dom } S^*.$$

As to the reverse inclusion let $f \in \text{dom } S^*$ and use (3.2) to decompose f as $f = g + h$ with

$$g \in \overline{\text{dom } S}, \quad h \in \text{mul } S^* \ominus \text{mul } S^{**}.$$

This implies that $g = f - h \in \text{dom } S^*$. Since by assumption $g \in \overline{\text{dom } S}$, it follows from the identity $\text{dom } S^{**} = \overline{\text{dom } S} \cap \text{dom } S^*$ in Proposition 3.11 that $g \in \text{dom } S^{**}$.

(ii) \Rightarrow (i) If $\text{dom } S^* = \text{dom } S^{**} \oplus (\text{mul } S^* \ominus \text{mul } S^{**})$ then it follows from Proposition 4.2 that S is outer bounded. Moreover, from the form of $\text{dom } S^*$ it is clear that $\text{dom } S^* \cap \overline{\text{dom } S} = \text{dom } S^{**}$. Hence, by Proposition 3.11 $(S^{**})_\infty$ is selfadjoint.

For the proof of the equivalence of (iii) and (iv) it suffices to replace $\text{dom } S^{**}$ by $\text{dom } S$ in the above arguments. \square

This section is finished with some decomposition result for partially bounded symmetric relations. Recall from (3.2) that the closure of $\text{dom } S^*$ has the following orthogonal decomposition:

$$\overline{\text{dom } S^*} = \overline{\text{dom } S} \oplus (\text{mul } S^* \ominus \text{mul } S^{**}). \tag{4.1}$$

If S is partially bounded then by Proposition 4.2 either the closed subspace $\overline{\text{dom } S}$ or the closed subspace $\text{mul } S^* \ominus \text{mul } S^{**}$ belongs to $\text{dom } S^*$. This means that with a partially bounded S the decomposition of $\overline{\text{dom } S^*}$ in (4.1) induces also a decomposition for $\text{dom } S^*$ itself (as in this case one of the components in (4.1) is a closed subspace of $\text{dom } S^*$). This yields the following block formula for the adjoint relation S^* itself; cf. [22, Proposition 2.1 (iii)] and [27, Proposition 4.5] for a special case.

Proposition 4.5. *Let S be a symmetric relation in a Hilbert space \mathfrak{H} and let S be decomposed with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2 = \overline{\text{dom } S} \oplus \text{mul } S^*$ as in Lemma 3.1. If S is partially bounded, then the inclusions in (3.8) hold as equalities:*

$$S^* = \begin{pmatrix} S_{11}^* & S_{21}^* \end{pmatrix}, \quad S^{**} = \begin{pmatrix} S_{11}^{**} \\ S_{21}^{**} \end{pmatrix}, \tag{4.2}$$

where $S : \mathfrak{H}_1 \rightarrow \mathfrak{H}$ and the adjoint S^* of S is considered as a relation from \mathfrak{H} to \mathfrak{H}_1 . If S and its adjoint S^* are considered as relations in \mathfrak{H} , then the formula for S^* takes the following equivalent form:

$$S^* = \left\{ \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} S_{11}^* f_1 + S_{21}^* f_2 \\ \varphi \end{pmatrix} \right); f_1 \in \text{dom } S_{11}^*, f_2 \in \text{dom } S_{21}^*, \varphi \in \text{mul } S^* \right\}. \tag{4.3}$$

Proof. In view of Lemma 3.1 to prove the stated formula for S^* , it suffices to show the inclusion “ \subset ”. Consider S^* as a linear relation from $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ to $\mathfrak{H}_1 = \overline{\text{dom } S}$ and let $h = h_1 \oplus h_2 \in \mathfrak{H}_1 \oplus \mathfrak{H}_2$, let $k \in \mathfrak{H}_1$, and assume that $(h, k) \in S^*$. This means that for all $(f, g) \in S$ with $g = g_1 \oplus g_2 = S_{11} f \oplus g_2$ and $(f, g_2) \in S_{21}$ one has

$$\langle S_{11} f, h_1 \rangle + \langle g_2, h_2 \rangle = \langle f, k \rangle. \tag{4.4}$$

If S is inner bounded, then it follows from Proposition 4.2 that $\text{dom } S_{11}^*$ is closed, and hence $h_1 \in \text{dom } S_{11}^* = \mathfrak{H}_1$. In this case (4.4) can be rewritten as

$$\langle g_2, h_2 \rangle = \langle f, k - S_{11}^* h_1 \rangle, \quad (f, g_2) \in S_{21}, \tag{4.5}$$

which means that $(h_2, k') \stackrel{\text{df}}{=} (h_2, k - S_{11}^* h_1) \in S_{21}^*$, i.e., $(h, k) = (h_1, S_{11}^* h_1) + (h_2, k') \in (S_{11}^* \ S_{21}^*)$, which proves the inclusion $S^* \subset (S_{11}^* \ S_{21}^*)$ in (4.2).

If S is outer bounded, then it follows from Proposition 4.2 that $\text{dom } S_{21}^*$ is closed, i.e., $\text{dom } S_{21}^* = \text{mul } S^* \ominus \text{mul } S^{**}$, and hence (3.2) implies that $h_2 \in \text{dom } S^* \cap \text{mul } S^* = \text{dom } S_{21}^*$, i.e., $\{h_2, k'\} \in S_{21}^*$ for some $k' \in \mathfrak{H}_2 = \text{mul } S^*$. Therefore, $(g_2, h_2) = (f, k')$ which implies that (4.4) can now be rewritten as

$$(S_{11} f, h_1) = (f, k - k'), \quad f \in \text{dom } S_{11} = \text{dom } S. \tag{4.6}$$

This means that $(h_1, k - k') \in S_{11}^*$, i.e., $S_{11}^* h_1 = k - k'$ and hence $(h, k) = (h, S_{11}^* h_1 + k') \in (S_{11}^* \ S_{21}^*)$, which again proves the inclusion $S^* \subset (S_{11}^* \ S_{21}^*)$ in (4.2).

The formula for the closure S^{**} follows from the equality $S^* = (S_{11}^* \ S_{21}^*)$ by taking adjoints on both sides; see [22, Proposition 2.1 (ii)].

If S and S^* are considered as relations in \mathfrak{H} then S is nondensely defined and, thus, $\text{mul } S^* = (\text{dom } S)^\perp = \mathfrak{H}_2$ is nontrivial. This together with the formula for S^* in (4.2) implies the formula in (4.3). \square

The next example shows that if S is not partially bounded then the adjoint S^* (in particular the domain $\text{dom } S^*$) of a (even maximally) nondensely defined symmetric operator S is not decomposable as in Proposition 4.5.

Example 4.6. Consider an unbounded 2×1 block (matrix) operator S in a Hilbert space $\mathfrak{H} \times \mathfrak{H}$, of the form

$$S = \begin{pmatrix} A \\ A \end{pmatrix},$$

where A is an unbounded selfadjoint operator in \mathfrak{H} . Then S is a symmetric operator in $\mathfrak{H} \times \mathfrak{H}$ and since $S_{11} = A$ is selfadjoint, S is maximally nondensely defined with $\text{mul } S^* = \{0\} \times \mathfrak{H}$; cf. **Proposition 3.10**. Now consider S as a densely defined operator from \mathfrak{H} to $\mathfrak{H} \times \mathfrak{H}$. Since S is closed and $S_{11} = S_{21} = A = A^*$, **Lemma 3.1** shows that

$$S^* \supset (A^* \quad A^*) = (A \quad A), \quad S^{**} \subset (A^* \quad A^*)^* = \begin{pmatrix} A^{**} \\ A^{**} \end{pmatrix} = S.$$

In particular, S is closed and $S^* = \text{clos } (A \quad A)$. However, here the row operator $\text{clos } (A \quad A)$ is not closed, and S^* does not admit a representation with a block formula as in (4.2). In fact, a straightforward calculation using the definition of the adjoint shows that S^* , considered as an operator from $\mathfrak{H} \times \mathfrak{H}$ to \mathfrak{H} , is given by

$$S^* = \left\{ \left(\begin{pmatrix} f \\ h - f \end{pmatrix}, Ah \right); h \in \text{dom } A, f \in \mathfrak{H} \right\}. \tag{4.7}$$

Clearly, the vectors $(f, -f) \in \mathfrak{H} \times \mathfrak{H}$, $f \in \mathfrak{H}$, belong to $\ker S^*$; however $(f, -f) \in \text{dom } (A \quad A)$ if and only if $f \in \text{dom } A$. Therefore, the equality $S^* = (A \quad A)$ holds if and only if A is bounded.

Example 4.6 shows that even in the case that S_∞ is selfadjoint, S^* or its domain $\text{dom } S^*$ need not be decomposable as in **Proposition 4.5**. However, in the case that S_∞ is selfadjoint one can select a transversal selfadjoint extension \tilde{A} , in which case $S^* = \tilde{A} \hat{+} S_\infty$ and construct a boundary triplet for S in an explicit manner. Such a boundary triplet will be constructed in the next section by extending some earlier formulas for boundary triplets known in the bounded case and in the case of finite defect numbers.

5. Boundary triplets for a class of maximally nondensely defined symmetric operators

5.1. Extensions for a class of maximally nondensely defined operators

Let S be a maximally nondensely defined operator in a Hilbert space \mathfrak{H} with S_∞ selfadjoint. Then there exists a selfadjoint operator extension \tilde{A} of S , which is transversal to S_∞ , in which case

$$S^* = \tilde{A} \hat{+} (\{0\} \times \text{mul } S^*), \tag{5.1}$$

and the eigenspace $\mathfrak{N}_\lambda(S^*)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is parameterized as follows:

$$\mathfrak{N}_\lambda(S^*) = \{ (\tilde{A} - \lambda)^{-1} \varphi; \varphi \in \text{mul } S^* \}; \tag{5.2}$$

see **Theorem 3.12** and **Corollary 3.13**. Hence, $(f_\lambda, \lambda f_\lambda) \in \widehat{\mathfrak{N}}_\lambda(S^*)$ with $\lambda \in \mathbb{C} \setminus \mathbb{R}$ has the following unique decomposition as in (5.1):

$$(f_\lambda, \lambda f_\lambda) = ((\tilde{A} - \lambda)^{-1} \varphi, \varphi + \lambda(\tilde{A} - \lambda)^{-1} \varphi) + (0, -\varphi), \quad \varphi \in \text{mul } S^*. \tag{5.3}$$

Let \mathcal{H} be a Hilbert space and let G be a bounded and boundedly invertible operator from \mathcal{H} onto $\text{mul } S^*$, so that

$$G \in \mathbf{B}(\mathcal{H}, \text{mul } S^*), \quad G^{-1} \in \mathbf{B}(\text{mul } S^*, \mathcal{H}). \tag{5.4}$$

Recall that the orthogonal projection onto $\text{mul } S^*$ is denoted by P .

Theorem 5.1. *Let S be a closed maximally nondensely defined symmetric operator in a Hilbert space \mathfrak{H} with S_∞ selfadjoint. Let \tilde{A} be a selfadjoint operator extension of S , which is transversal*

to S_∞ , so that

$$S^* = \{ \widehat{f} = (f, \widetilde{A}f + \varphi); f \in \text{dom } \widetilde{A}, \varphi \in \text{mul } S^* \}. \tag{5.5}$$

Then $\Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \}$, where

$$\Gamma_0 \widehat{f} = G^* P f, \quad \Gamma_1 \widehat{f} = G^{-1} \varphi, \quad \widehat{f} \in S^*, \tag{5.6}$$

is an ordinary boundary triplet for S^* with

$$\ker \Gamma_0 = S_\infty \quad \text{and} \quad \ker \Gamma_1 = \widetilde{A}. \tag{5.7}$$

The corresponding Γ -field γ and Weyl function M are given by

$$\begin{aligned} \gamma(\lambda) &= (\widetilde{A} - \lambda)^{-1} [G^* P (\widetilde{A} - \lambda)^{-1} \upharpoonright \text{mul } S^*]^{-1}, \\ M(\lambda) &= -G^{-1} [P (\widetilde{A} - \lambda)^{-1} \upharpoonright \text{mul } S^*]^{-1} G^{-*}. \end{aligned} \tag{5.8}$$

Moreover, $G^* G$ is boundedly invertible and

$$\lim_{\lambda \rightarrow \infty} \frac{M(\lambda)}{\lambda} = (G^* G)^{-1}. \tag{5.9}$$

Proof. Note that $\text{dom } \Gamma = S^*$ and that (5.5) is equivalent to (5.1); see Theorem 3.12. Choose typical elements $\widehat{f} = (f, \widetilde{A}f + \varphi), \widehat{g} = (g, \widetilde{A}g + \psi)$ in S^* as in (5.5). Then (5.6) implies

$$\begin{aligned} \langle \Gamma_1 \widehat{f}, \Gamma_0 \widehat{g} \rangle_{\mathcal{H}} - \langle \Gamma_0 \widehat{f}, \Gamma_1 \widehat{g} \rangle_{\mathcal{H}} &= \langle G^{-1} \varphi, G^* P g \rangle_{\mathcal{H}} - \langle G^* P f, G^{-1} \psi \rangle_{\mathcal{H}} \\ &= \langle \varphi, g \rangle - \langle f, \psi \rangle = \langle \widetilde{A}f + \varphi, g \rangle - \langle f, \widetilde{A}g + \psi \rangle, \end{aligned}$$

since $f, g \in \text{dom } \widetilde{A}$. Therefore (2.12) is satisfied. Since the operator \widetilde{A} is transversal to S_∞ , it follows from Theorem 3.14 that $P(\text{dom } \widetilde{A}) = \text{mul } S^*$. Consequently,

$$\Gamma(S^*) = G^* P(\text{dom } \widetilde{A}) \times G^{-1}(\text{mul } S^*) = \mathcal{H} \times \mathcal{H},$$

so that Γ is surjective. Hence (5.6) defines an ordinary boundary triplet for S^* .

To verify the first equality in (5.7), let $\widehat{f} = (f, \widetilde{A}f + \varphi) \in \ker \Gamma_0$ with $f \in \text{dom } \widetilde{A}$ and $\varphi \in \text{mul } S^*$. By definition $Pf = 0$ or, equivalently, $f \in \text{dom } S$. Since $\text{dom } A = \text{dom } S^*$ one has $f \in \text{dom } S \cap \text{dom } S^*$, so that $f \in \text{dom } S$ by Proposition 3.11. This implies $\widehat{f} \in S_\infty$. Therefore $\ker \Gamma_0 \subset S_\infty$, and since both sides are selfadjoint, equality follows: $\ker \Gamma_0 = S_\infty$. In order to verify the second equality in (5.7), let $\widehat{f} = (f, \widetilde{A}f + \varphi) \in \ker \Gamma_1$ with $f \in \text{dom } A$ and $\varphi \in \text{mul } S^*$. By definition $\varphi = 0$, which shows $\widehat{f} \in \widetilde{A}$. Therefore $\ker \Gamma_1 \subset \widetilde{A}$, and since both sides are selfadjoint, equality follows: $\ker \Gamma_1 = \widetilde{A}$.

Let $\widehat{f}_\lambda \in S^*$ then there exists $\varphi \in \text{mul } S^*$ such that (5.3) holds. It follows from (5.3), (5.5), and (5.6) that

$$\Gamma_0 \widehat{f}_\lambda = G^* P f_\lambda = G^* P (\widetilde{A} - \lambda)^{-1} \varphi, \quad \Gamma_1 \widehat{f}_\lambda = -G^{-1} \varphi.$$

Therefore, according to (2.19) and (2.20) one obtains

$$\gamma(\lambda) = \{ (G^* P (\widetilde{A} - \lambda)^{-1} \varphi, (\widetilde{A} - \lambda)^{-1} \varphi); \varphi \in \text{mul } S^* \},$$

$$M(\lambda) = \{ (G^* P (\widetilde{A} - \lambda)^{-1} \varphi, -G^{-1} \varphi); \varphi \in \text{mul } S^* \},$$

which lead to (5.8). Finally (5.9) can be seen from the observation that

$$\lim_{\lambda \rightarrow \infty} \lambda G^* P (\widetilde{A} - \lambda)^{-1} \varphi = -G^* \varphi, \quad \varphi \in \text{mul } S^*. \quad \square \tag{5.10}$$

A boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ for S^* with Weyl function M and Γ -field γ gives rise to a transposed boundary triplet $(\mathcal{H}, -\Gamma_1, \Gamma_0)$ for S^* with Weyl function $-M^{-1}$ and $(-\Gamma_1, \Gamma_0)$ -field $-\gamma M^{-1}$.

Corollary 5.2. *Let G be as in (5.4) and assume that the conditions of Theorem 5.1 are satisfied. Then, in terms of the decomposition (5.5), $\Pi' = (\mathcal{H}, \Gamma'_0, \Gamma'_1)$ with*

$$\Gamma'_0 \widehat{f} = -G^{-1}\varphi, \quad \Gamma'_1 \widehat{f} = G^*Pf, \quad \widehat{f} \in S^*, \tag{5.11}$$

is an ordinary boundary triplet for S^* with $\ker \Gamma'_0 = \widetilde{A}$ and $\ker \Gamma'_1 = S_\infty$. The corresponding Γ' -field γ' and Weyl function M' are given by

$$\gamma'(\lambda) = -(\widetilde{A} - \lambda)^{-1}G, \quad M'(\lambda) = G^*P(\widetilde{A} - \lambda)^{-1}G. \tag{5.12}$$

Moreover, $M' \in \mathbf{N}_0(\mathcal{H})$ and, in fact,

$$\lim_{\lambda \rightarrow \infty} \lambda M'(\lambda) = -G^*G. \tag{5.13}$$

5.2. Extensions as perturbations

In the situation of Theorem 5.1 the intermediate extensions of S can be seen as perturbations of the selfadjoint extension \widetilde{A} . Let P be the orthogonal projection onto $\text{mul } S^*$.

Theorem 5.3. *Let S be a closed maximally nondensely defined symmetric operator in a Hilbert space \mathfrak{H} with S_∞ selfadjoint, and assume that \widetilde{A} is a selfadjoint operator extension of S , which is transversal to S_∞ . Then the closed intermediate extensions A_Θ of S are in one-to-one correspondence with the closed relations Θ in \mathcal{H} , via*

$$A_\Theta = \widetilde{A} + G\Theta G^*P. \tag{5.14}$$

Moreover, the following statements are equivalent:

- (i) A_Θ is an operator perturbation of \widetilde{A} ;
- (ii) Θ is an operator in \mathcal{H} or $(A_\Theta \cap S_\infty = S)$,

and the following statements are equivalent:

- (iii) A_Θ is a bounded operator perturbation of \widetilde{A} ;
- (iv) Θ is a bounded operator in \mathcal{H} or $(A_\Theta \uparrow S_\infty = S^*)$.

Proof. In terms of the ordinary boundary triplet Γ in (5.6) all (closed) intermediate extensions A_Θ of S are in one-to-one correspondence with the (closed) linear relations Θ in \mathcal{H} via (2.14). For $\widehat{f} = (f, \widetilde{A}f + \varphi) \in S^*$ with $f \in \text{dom } \widetilde{A}$, $\varphi \in \text{mul } S^*$, it follows from (5.6) that

$$\Gamma \widehat{f} = (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}) = (G^*Pf, G^{-1}\varphi) \in \Theta \iff (f, \varphi) \in G\Theta G^*P.$$

Hence, $\widehat{f} = (f, \widetilde{A}f + \varphi) \in S^*$ belongs to A_Θ if and only if $(f, \varphi) \in G\Theta G^*P$, or, equivalently, that $\widehat{f} = (f, \widetilde{A}f + \varphi) \in \widetilde{A} + G\Theta G^*P$. This proves (5.14). It is clear from (5.14) that

$$\text{mul } A_\Theta = \text{mul } G\Theta G^*P = G(\text{mul } \Theta). \tag{5.15}$$

Hence, A_Θ is an operator extension of S if and only if Θ is an operator in \mathcal{H} . Moreover, since $G^* \in \mathbf{B}(\text{mul } S^*, \mathcal{H})$, the product $G\Theta G^*P$ is bounded if and only if Θ is a bounded operator in \mathcal{H} . \square

Corollary 5.4. *Let the intermediate extension A_θ be given by (5.14). Then*

$$(A_\theta)^* = \tilde{A} + G\theta^*G^*P. \tag{5.16}$$

Proof. It has been shown that A_θ in (2.14) is written as (5.14). However, the adjoint $(A_\theta)^*$ of A_θ in (2.14) satisfies (2.15). Therefore, $(A_\theta)^*$ is a perturbation of \tilde{A} as in (5.14) based on the parameter θ^* . \square

Proposition 5.5. *Let the assumptions be as in Theorem 5.3. Then*

$$\text{dom } A_\theta = \{f \in \text{dom } \tilde{A}; G^*Pf \in \text{dom } \theta\}. \tag{5.17}$$

Moreover, if θ_1 and θ_2 are closed relations in \mathcal{H} . Then

$$\text{dom } A_{\theta_1} \subset \text{dom } A_{\theta_2} \Leftrightarrow \text{dom } \theta_1 \subset \text{dom } \theta_2. \tag{5.18}$$

Proof. The identity (5.17) follows from (5.14). Now consider the equivalence in (5.18).

(\Rightarrow) Let $g \in \text{dom } \theta_1$. Since \tilde{A} and S_∞ are transversal, it follows from Theorem 3.14 (iv) that there exists $f \in \text{dom } \tilde{A}$ such that $G^*Pf = g$. Hence, $f \in \text{dom } A_{\theta_1} \subset \text{dom } A_{\theta_2}$ and now (5.17) shows that $g \in \text{dom } \theta_2$.

(\Leftarrow) This follows directly from (5.17). \square

Finally, recall from [16] that a relation T is said to be *domain tight* if $\text{dom } T = \text{dom } T^*$ and *formally domain tight* if $\text{dom } T \subset \text{dom } T^*$. For the extensions in Theorem 5.3 one can show that A_θ is (formally) domain tight if and only if θ is (formally) domain tight.

To show the applicability of the results established in this section they are applied to Example 4.6, where no block formulas for S^* or for the selfadjoint extensions of S are available.

Example 5.6. Consider the unbounded 2×1 block operator S in Example 4.6. Define

$$\tilde{A} = \left\{ \left(\begin{pmatrix} f \\ h - f \end{pmatrix}, \begin{pmatrix} Ah \\ Ah \end{pmatrix} \right); h \in \text{dom } A, f \in \mathfrak{H} \right\}.$$

Then \tilde{A} is clearly a symmetric operator extension of S and it follows from the expression (4.7) that $\text{dom } \tilde{A} = \text{dom } S^*$. By Theorem 3.12 this means that \tilde{A} is a selfadjoint extension of S which is transversal to the selfadjoint extension S_∞ of S , i.e., $\tilde{A} \hat{+} S_\infty = S^*$.

Now one can apply Theorem 5.1 or Corollary 5.2 to construct an ordinary boundary triplet for S^* . For this purpose take $\mathcal{H} = \mathfrak{H}_2 := \text{mul } S^* = \{0\} \times \mathfrak{H}$, let $P_2 = P$ be the orthogonal projection onto \mathfrak{H}_2 , and let G be the identity mapping on \mathfrak{H}_2 , so that $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ takes the form

$$\begin{pmatrix} \Gamma_0 \hat{f} \\ \Gamma_1 \hat{f} \end{pmatrix} = \begin{pmatrix} h - f \\ \varphi \end{pmatrix}, \quad \hat{f} = \left(\begin{pmatrix} f \\ h - f \end{pmatrix}, \begin{pmatrix} Ah \\ Ah + \varphi \end{pmatrix} \right) \in S^*, \quad h \in \text{dom } A, f, \varphi \in \mathfrak{H}.$$

An application of Theorem 5.3 shows that all closed intermediate extensions of S have an expression as (domain) perturbations of \tilde{A} of the form $A_\theta = \tilde{A} + \theta P_2$, where θ is a closed relation in \mathfrak{H}_2 . In particular, all closed intermediate operator extensions of S are parameterized by closed operators θ in \mathfrak{H}_2 via the formula

$$\begin{aligned} A_\theta &= \tilde{A} + \theta P_2 \\ &= \left\{ \left(\begin{pmatrix} f \\ h - f \end{pmatrix}, \begin{pmatrix} Ah \\ Ah + \theta(h - f) \end{pmatrix} \right); h \in \text{dom } A, h - f \in \text{dom } \theta \right\}. \end{aligned}$$

Note that if A is, in addition, assumed to be bounded, then S^* admits the decomposition stated in [Proposition 4.5](#) and the previous formula for A_Θ can be expressed in a block operator matrix form as follows:

$$A_\Theta = \tilde{A} + \Theta P_2 = \begin{pmatrix} A & A \\ A & A + \Theta \end{pmatrix}.$$

Finally, notice that as in [Corollary 5.2](#) one obtains a transposed boundary triplet II' for S^* , whose Γ -field and Weyl function take the following simple forms:

$$\gamma'(\lambda) = (\tilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H}_2, \quad M'(\lambda) = P_2(\tilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H}_2.$$

Only in the case that A is bounded it is possible to rewrite these formulas for the Γ -field and Weyl function, as well as the corresponding functions in (5.8), by means of the blocks of \tilde{A} or proper Schur complements.

Remark 5.7. In [2, Theorem 2.5.8] a formula to express all regular selfadjoint operator extensions of a regular symmetric O -operator S (meaning that S_{11} is selfadjoint as in [Theorem 5.3](#)) has been established by means of Hilbert space techniques; see [Remark 3.9](#) for this terminology. The parameterization in [2] is given via $(+)$ -selfadjoint and $(+)$ -bounded operators on a $(+)$ -closed subspace \mathfrak{N} satisfying $\mathfrak{H} = \overline{\text{dom } S} \dot{+} \mathfrak{N}$ (cf. [2, Proposition 2.4.2]). The perturbation formula (5.14) is essentially simpler than the formula given in [2, Theorem 2.5.8]; in addition (5.14) parameterizes all intermediate (in particular all selfadjoint) extensions of S in \mathfrak{H} . Note that the formula (5.14) relies on an explicit construction of an ordinary boundary triplet for S^* which was established in [Theorem 5.1](#).

5.3. Matrix decompositions

Let S be a closed symmetric operator in \mathfrak{H} , let P be the orthogonal projection from \mathfrak{H} onto $\text{mul } S^* = (\overline{\text{dom } S})^\perp$, and write S in a block form with entries S_{11} and S_{21} as in [Lemma 3.1](#). Assume in addition that the operator S is partially bounded. Then by [Proposition 4.5](#) also the adjoint S^* of S admits a block representation as in (4.2), (4.3). This makes it possible to specialize the earlier results in this section in this special case and derive proper block formulas for all intermediate extensions A_Θ of S .

If S is a partially bounded closed symmetric operator, then by [Proposition 4.5](#)

$$S^* = \left\{ \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} S_{11}^* f_1 + S_{21}^* f_2 \\ \varphi \end{pmatrix} \right); f_j \in \text{dom } S_{1j}^*, \varphi \in \text{mul } S^*, j = 1, 2 \right\}. \tag{5.19}$$

To apply the results in this section it is now assumed that S_{11} is selfadjoint. It follows from [Proposition 4.3](#) that if S is inner bounded, then it is necessarily also outer bounded. In other words, a partially bounded closed symmetric operator S , such that S_{11} is selfadjoint, is always outer bounded. According to [Proposition 4.4](#) this implies that $\text{dom } S^* = \text{dom } S \oplus \text{mul } S^*$, since here $\text{mul } S^{**} = \text{mul } S = \{0\}$. One is now ready to specialize the main results in this section for a partially bounded closed symmetric operator S .

Parallel to (5.19) the selfadjoint extension S_∞ can be written as:

$$S_\infty = \left\{ \left(\begin{pmatrix} f_1 \\ 0 \end{pmatrix}, \begin{pmatrix} S_{11} f_1 \\ \varphi \end{pmatrix} \right); f_1 \in \text{dom } S_{11}, \varphi \in \text{mul } S^* \right\}. \tag{5.20}$$

Let $S_{22} \in \mathbf{B}(\text{mul } S^*)$ be selfadjoint. Then the extension \tilde{A} defined by

$$\tilde{A} = \left\{ \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} S_{11}f_1 + S_{21}^*f_2 \\ S_{21}f_1 + S_{22}f_2 \end{pmatrix} \right); f_1 \in \text{dom } S_{11}, f_2 \in \text{mul } S^* \right\} \tag{5.21}$$

is selfadjoint and it is an operator. Note that $(S_{21})^{**}$ restricted to $\text{dom } S_{11}$ is S_{21} . It is obvious from (5.20) and (5.21) that S_∞ and \tilde{A} are transversal with respect to S . The identity (5.5) involving S^* and \tilde{A} can be written in explicit form:

$$S^* = \left\{ \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} S_{11}f_1 + S_{21}^*f_2 \\ S_{21}f_1 + S_{22}f_2 + \varphi \end{pmatrix} \right); f_1 \in \text{dom } S_{11}, f_2, \varphi \in \text{mul } S^* \right\};$$

cf. (5.19). Thus the ordinary boundary triplet for S^* in (5.6) has the specific form:

$$\Gamma_0 \hat{f} = f_2, \quad \Gamma_1 \hat{f} = \varphi, \quad \hat{f} \in S^*, \tag{5.22}$$

with the identification of \mathcal{H} with $\text{mul } S^*$. Therefore the intermediate extensions A_θ of S are in one-to-one correspondence with the relations θ in $\text{mul } S^*$ via

$$A_\theta = \left\{ \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} S_{11}f_1 + S_{21}^*f_2 \\ S_{21}f_1 + S_{22}f_2 + \varphi \end{pmatrix} \right); f_1 \in \text{dom } S_{11}, \{f_2, \varphi\} \in \theta \right\}.$$

This last result is sometimes written in a formal way as

$$A_\theta = \begin{pmatrix} S_{11} & S_{21}^* \\ S_{21} & S_{22} + \theta \end{pmatrix}. \tag{5.23}$$

Since \tilde{A} and S_∞ are transversal with respect to S , an application of Corollary 3.13 shows that $f_\lambda \in \mathfrak{N}_\lambda(S^*)$ if and only if there exists $\varphi \in \text{mul } S^*$ such that $f_\lambda = (\tilde{A} - \lambda)^{-1}\varphi$. In terms of the matrix decomposition in (5.21) one obtains

$$P(\tilde{A} - \lambda)^{-1}\varphi = [S_{22} - \lambda - S_{21}(S_{11} - \lambda)^{-1}S_{12}]^{-1}\varphi, \quad \varphi \in \text{mul } S^*.$$

Hence the Weyl function M associated with (5.22) is given by

$$M(\lambda) = \{([S_{22} - \lambda - S_{21}(S_{11} - \lambda)^{-1}S_{12}]^{-1}\varphi, -\varphi); \varphi \in \text{mul } S^*\},$$

cf. (2.20) and (5.8), which leads to the representations

$$M(\lambda) = -S_{22} + \lambda + S_{21}(S_{11} - \lambda)^{-1}S_{12}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

and

$$-M(\lambda)^{-1} = [S_{22} - \lambda - S_{21}(S_{11} - \lambda)^{-1}S_{12}]^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Thus such functions are characteristic functions of colligations in the sense of [4,5]. The results in this section can be seen as extensions of the matrix representations for selfadjoint operator or relation extensions of symmetric contractions (bounded symmetric operators) in [12,21].

5.4. Generalized Friedrichs extensions

Under the conditions of Theorem 5.1 the identity (5.1) can also be written as $S^* = \tilde{A} \upharpoonright Z$, where $Z = \{0\} \times \text{mul } S^*$, so that Z is a closed linear subspace of $\mathfrak{H} \times \mathfrak{H}$. In that case $S = \tilde{A} \cap Z^*$, or in other words

$$S = \{(f, \tilde{A}f) \in \tilde{A}; f \perp \text{mul } S^*\}.$$

Hence the closed symmetric operator S is a domain restriction of the selfadjoint operator \tilde{A} ; this kind of domain restriction was first systematically studied in [7].

Lemma 5.8. *Under the conditions of Theorem 5.1 the selfadjoint extension S_∞ of S is characterized by the formula*

$$S_\infty = \{\widehat{f} \in S^*; f \in \overline{\text{dom } S}\}. \tag{5.24}$$

In particular, S_∞ is the only selfadjoint extension of S whose domain is contained in $\overline{\text{dom } S}$.

Proof. Since S_∞ is selfadjoint, the description of S_∞ in (5.24) is obtained from Lemma 3.7. Now assume that H is a selfadjoint extension of S such that $\text{dom } H \subset \overline{\text{dom } S}$. Then it follows from Proposition 3.11 that $\text{dom } H \cap \overline{\text{dom } S} = \text{dom } S$ and this implies that $H \subset S_\infty$. Since H and S_∞ are selfadjoint, the equality $H = S_\infty$ follows. \square

The extension S_∞ is called the *generalized Friedrichs extension* of S . Observe that if S is semibounded then the Friedrichs extension S_F of S is characterized by

$$S_F = \{\widehat{f} \in S^*; f \in \mathfrak{H}_{+1}\} \tag{5.25}$$

where \mathfrak{H}_{+1} stands for the energy space obtained as a completion of $\text{dom } S$ with respect to the graph norm of S on $\text{dom } S$; see [8]. In particular, S_F is the only selfadjoint extension of S whose domain is contained in \mathfrak{H}_{+1} . Since the topology on \mathfrak{H}_{+1} is in general stronger than the original topology of \mathfrak{H} on $\overline{\text{dom } S}$, one has a strict inclusion $\mathfrak{H}_{+1} \subset \overline{\text{dom } S}$. Hence, in the case that S is maximally nondensely defined, the characterization of S_∞ in Lemma 5.8 is actually stronger than the usual characterization of the Friedrichs extension S_F in (5.25). For the case of nonsemibounded symmetric relations with defect numbers $(1, 1)$, see also [20,15]. The generalized Friedrichs extension can be characterized analytically. In fact, the limiting properties (5.9) and (5.13) are characteristic properties.

Theorem 5.9. *Let S be a (nondensely defined) closed symmetric operator with equal deficiency indices in the Hilbert space \mathfrak{H} . Let $\Pi = (\mathcal{H}, \Gamma_0, \Gamma_1)$ be an ordinary boundary triplet for S^* with corresponding Weyl function M . Assume that*

$$s - \lim_{\lambda \rightarrow \infty} \frac{M(\lambda)}{\lambda} \tag{5.26}$$

is boundedly invertible. Then S_∞ is a selfadjoint extension of S and $\ker \Gamma_0 = S_\infty$.

Proof. Let B denote the limit in (5.26), then according to [12, Proposition 2.6, Corollary 2.6] the forbidden manifold \mathcal{F}_Γ defined in (2.16) is given by

$$\mathcal{F}_\Gamma = \{0\} \times \text{ran } B^{1/2}.$$

Since $B^{-1} \in \mathbf{B}(\mathcal{H})$, clearly $\text{mul } \mathcal{F}_\Gamma = \mathcal{H}$ and since $\{0\} \times \mathcal{H}$ is selfadjoint, \mathcal{F}_Γ as its symmetric extension must coincide with $\{0\} \times \mathcal{H}$, i.e., $S_\infty = \Gamma^{-1}(\mathcal{F}_\Gamma) = \ker \Gamma_0$. \square

Corollary 5.10. *Let S be a (nondensely defined) closed symmetric operator with equal deficiency indices in the Hilbert space \mathfrak{H} . Let $\Pi = (\mathcal{H}, \Gamma_0, \Gamma_1)$ be an ordinary boundary triplet for S^* with corresponding Weyl function M . Assume that $M \in \mathbf{N}_0$. Then S_∞ is a selfadjoint extension of S and $\ker \Gamma_0$ is transversal to S_∞ .*

5.5. A connection with graph perturbations

It should be noted that so far \tilde{A} is a selfadjoint operator and Z is the subspace of a particular form for which $\tilde{A} \hat{+} Z$ is closed. It is possible to consider selfadjoint relations and closed linear subspaces Z (not necessarily of the above form) for which $\tilde{A} \hat{+} Z$ is not closed, in which case more general boundary triplets have to be considered. These more general situations considered from a different point of view, will be treated elsewhere; see also [11].

Finally, it is pointed out that all the results in this paper concerning maximally nondensely defined symmetric relations can be transformed from the domain side to the range side by systematically inverting the graphs of S and S^* . The analog of S_∞ in (3.16) is a symmetric extension of S defined by

$$S_0 = S \hat{+} (\{0\} \times \ker S^*). \tag{5.27}$$

As an example it is mentioned that S is selfadjoint if and only if

$$\text{ran } S = \overline{\text{ran } S} \cap \text{ran } S^*; \tag{5.28}$$

cf. **Proposition 3.10**. If in addition S is nonnegative, then the selfadjoint extension S_0 in (5.27) coincides with the Kreĭn–von Neumann extension of S ; see [8]. If S is not semibounded and S_0 is selfadjoint, then it defines a so-called generalized Kreĭn–von Neumann extension of S ; it admits similar (geometric and analytic) characterizations proved for S_∞ in this paper; in the case of defect numbers $(1, 1)$ such results for a more general class of nonsemibounded symmetric operators (containing the class of all semibounded symmetric operators S and its Kreĭn–von Neumann extension S_N) have been established in [19] along the lines of [20,18] for the generalized Friedrichs extension. As an example it is mentioned that if (5.28) is satisfied, then S_0 is characterized by

$$S_0 = \{\hat{f} \in S^*; f' \in \overline{\text{ran } S}\},$$

and, moreover, S_0 is the generalized Kreĭn–von Neumann extension S_N of S , and it is the only selfadjoint extension of S , whose range is contained in $\overline{\text{ran } S}$. In the scalar case this is a special case of [19, Theorem 8.1]. Using the selfadjoint extension S_0 one can construct a boundary triplet as in **Theorem 5.1** for the adjoint S^* .

Proposition 5.11. *Let S_0 and \tilde{A} be transversal selfadjoint extensions of S , let G be a bounded and boundedly invertible operator from a Hilbert space \mathcal{H} onto $\ker S^*$, and let P be the orthogonal projection from $\tilde{\mathfrak{H}}$ onto $\ker S^*$. Then*

$$S^* = \{\hat{f} = (\tilde{A}^{-1} f' + G\varphi, f'); f' \in \text{ran } \tilde{A}, \varphi \in \mathcal{H}\}$$

and define the operators $\Gamma_0, \Gamma_1 : S^* \rightarrow \mathcal{H}$ by

$$\Gamma_0 \hat{f} = \varphi, \quad \Gamma_1 \hat{f} = G^* P f', \quad \hat{f} \in S^*.$$

Then $(\mathcal{H}, \Gamma_0, \Gamma_1)$ is an ordinary boundary triplet for S^* . The corresponding Γ -field γ and Weyl function M are given by

$$\gamma(\lambda) = (I - \lambda \tilde{A}^{-1})^{-1} G, \quad M(\lambda) = \lambda G^* P (I - \lambda \tilde{A}^{-1})^{-1} G, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

There is a characterization of the Weyl function in **Proposition 5.11** analogous to **Theorem 5.9** and **Corollary 5.10**; cf. [19] for the scalar case.

An analog of Proposition 5.11, involving more general unitary boundary triplets, can be found in [11, Proposition 7.41]; see also [11, Corollary 7.33]. Even in the nonnegative case the result in Proposition 5.11 is useful in various applications (allowing infinite defect numbers, like in the analysis of elliptic PDEs) when studying selfadjoint extensions of S and their spectral properties; see e.g. [14, Chapters 12 and 13], [28, Section 2]. Note that the conditions in Proposition 5.11 are satisfied if, in particular, the symmetric operator S has a bounded inverse (cf. Section 5.3) or S is for instance semibounded with a positive lower bound.

Further results on range, domain, and general graph perturbations can be found in [6,9,10,15] and in [1,13,14,17,24,25].

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