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Year: 2020

Version: Accepted version

Please cite the original version:

Cao, Y., Wei, W., Mei, S., Shafie-khah, M. & Catalao, J.P.S. (2020). Analyzing and Quantifying the Intrinsic Distributional Robustness of CVaR Reformulation for Chance Constrained Stochastic Programs. *IEEE transactions on power systems,* 1-4. https://doi.org/10.1109/TPWRS.2020.3021285

Analyzing and Quantifying the Intrinsic Distributional Robustness of CVaR Reformulation for Chance Constrained Stochastic Programs

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***Abstract*—Chance constrained program (CCP) is a popular stochastic optimization method in power system planning and operation problems. Conditional Value-at-Risk (CVaR) provides a convex approximation for chance constraints which are non-convex. Although CCP assumes an exact empirical distribution, and the optimum of a stochastic programming model is thought to be sensitive in the designated probability distribution, this letter discloses that CVaR reformulation of chance constraint is intrinsically robust. A pair of indices are proposed to quantify the maximum tolerable perturbation of the probability distribution, and can be computed from a computationally-cheap dichotomy search. An example on the coordinated capacity optimization of energy storage and transmission line for a remote wind farm validates the main claims. The above results demonstrate that stochastic optimization methods are not necessarily vulnerable to distributional uncertainty, and justify the positive effect of the conservatism brought by the CVaR reformulation.**

***Index Terms*—chance constraint, conditional-value-at-risk, distributional robustness, stochastic optimization, uncertainty**

I. Introduction

With the proliferation of renewable energy resources, the uncertainty in power system is growing rapidly. Chance constrained program (CCP) is a popular stochastic optimization method in power system planning and operation related problems. A CCP is generally formulated as

$$\min_{x \in X} \quad f(x) \tag{1a}$$

$$\text{s.t.} \quad \Pr_0\{h(x,\xi) - \lambda \le 0\} \ge 1 - \alpha \tag{1b}$$

where x is the vector of decision variables and X represents physical operating constraints; ξ denotes uncertain parameters; $h(x,\xi)$ is a loss function, and λ is the cap of loss. Chance constraint (1b) requires the probability of event $h(x,\xi) - \lambda \le 0$ evaluated at the empirical probability distribution function (PDF) P_0 of ξ should be at least $1-\alpha$.

Two difficulties exist in CCP model (1). The first one is that the empirical PDF P_0 to validate chance constraint (1b) is generally inexact. Therefore, for the optimum x of model (1), the probability $\Pr\{h(x,\xi) - \lambda \le 0\}$ evaluated at the real PDF in practice can be either greater or smaller than $1-\alpha$. The other is that chance constraint (1b) does not have a closed-form expression and is generally non-convex in decision variable x, making a CCP hard to solve. A chance constraint on x is equivalent to imposing an upper bound on a Value-at-Risk (VaR) index [1], which can be further transformed to a mixed-integer linear program (MILP) based on sampling average approximation [2]. Alternatively, a chance constraint can be approximated by the Conditional Value-at-Risk (CVaR) which is proven to be convex in x [3], and has been used in power system operation problems, such as [4]–[6]. CVaR reformulation is an inner approximation of the original feasible region of x, and thus the optimal solution is conservative, which is sometimes criticized. More precisely, let $x^\star$ be the optimal solution offered by the CVaR reformulation model, then strict inequality $\Pr_0\{h(x^\star,\xi) - \lambda \le 0\} > 1-\alpha$ holds under the empirical PDF P_0. Such conservatism endows CVaR reformulation with intrinsic robustness and resolves the first difficulty mentioned above, because a certain margin is preserved for the probability threshold under PDF perturbation.

Now, a natural question arises: given the optimal solution $x^\star$ of a CVaR reformulation model, as chance constraint (1b) is generally a strict inequality under the empirical PDF P_0, then what is the maximal tolerable perturbation of the PDF such that chance constraint (1b) evaluated at the new PDF still holds true? This letter will answer this question. To this end, we propose a pair of indices $(\bar{\alpha}, d_M)$ to measure the intrinsic robustness brought by the CVaR reformulation, where $\bar{\alpha}$ describes the reliability margin, and d_M describes the maximal perturbation scale of PDFs in the sense of Kullback-Leibler (KL) divergence. An efficient dichotomy algorithm is proposed to search the indices. This work could provide a new perspective and broaden the existing understanding on CVaR reformulation of CCPs.

This work is essentially different from those on distributional robust optimization (DRO), e.g., the moment based one [1], the KL-divergence based one [7], the Wasserstein-metric based one [8], and the distributionally robust CCP [9]. The above DRO problems assume a predefined ambiguity set and aim to seek an optimal solution $x^\star$ with statistically robust performance guarantee. However, this work targets at quantifying the conservatism and distributional robustness of a given optimal solution $x^\star$, which is an evaluation process.

This work was supported in part by NSFC 51807101 and in part by the energy storage project from State Grid Corporation of China under Special Grant 522800180003. Corresponding to: W. Wei and S. Mei.

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In this process, the range of PDF perturbation is unknown, and the maximal range is to be determined. For any strictly feasible $x^\star$ other than the optimal one, the proposed method can give two indices reflecting its robustness for the chance constraint. So the result of evaluation is decision-dependent.

II. Quantifying the Distributional Robustness

In risk theory, the quantile for the probability of loss $h(x,\xi)$ no greater $1-\alpha$ is defined as the Value-at-Risk (VaR)

$$\begin{aligned}&(1-\alpha)\text{-VaR}(x,\xi,\mathrm{P}_0)=\\&\min\left\{z\in\mathbb{R}\,\middle|\,\int_{h(x,\xi)\le z}p_0(\xi)\mathrm{d}\xi\ge 1-\alpha\right\}\end{aligned}\tag{2}$$

where $p_0(\xi)$ is the probability density function of P_0. By this definition, constraint

$$(1-\alpha)\text{-VaR}(x,\xi,\mathrm{P}_0)\le\lambda\tag{3}$$

is equivalent to the non-convex chance constraints (1b) for x. So constraint (3) is also non-convex.

The most widely used risk measure for approximating a chance constraint is CVaR, which is defined as the conditional expectation of loss no less than the VaR, i.e.,

$$\begin{aligned}&(1-\alpha)\text{-CVaR}(x,\xi,\mathrm{P}_0)=\\&\frac{1}{\alpha}\int_{h(x,\xi)\ge(1-\alpha)\text{-VaR}(x,\xi,\mathrm{P}_0)}h(x,\xi)p_0(\xi)\mathrm{d}\xi\end{aligned}\tag{4}$$

The loss less than VaR does not contribute to CVaR, hence

$$(1-\alpha)\text{-VaR}(x,\xi,\mathrm{P}_0)<(1-\alpha)\text{-CVaR}(x,\xi,\mathrm{P}_0)\tag{5}$$

The advantage of CVaR is its convexity [3]. In this regard, chance constraint (1b) can be conservatively approximated by

$$(1-\alpha)\text{-CVaR}(x,\xi,\mathrm{P}_0)\le\lambda\tag{6}$$

Conservatism means if (6) holds, then (3) must hold because of (5), and so does (1b), given the equivalence between (1b) and (3). In other words, the feasible set of x defined by (6) is smaller than that induced by (3). CVaR constraint (6) usually comes down to linear equalities in most practical optimization problems [3] after performing sampling average approximation. In the following, we discuss how much distributionally robustness is brought by the CVaR reformulation.

To characterize the distributional robustness, it is essential to define the distance between two PDFs. Suppose $p_0(\xi)/p(\xi)$ is the probability density function of P_0 and P, then the Kullback-Leibler (KL) divergence defined by

$$D_{KL}(P\parallel P_0)=\int_\Omega p(\xi)\log\frac{p(\xi)}{p_0(\xi)}\mathrm{d}\xi\tag{7}$$

is a measure on the distance between P and P_0. For discrete distributions on a set of representative scenarios $\{\pi_n\}_{n=1}^N$, the KL divergence has the form of

$$D_{KL}(P\parallel P_0)=\sum_{n=1}^N\pi_n\log\frac{\pi_n}{\pi_n^0}\tag{8}$$

where π_n/π_n^0 is the probability of ξ_n under distribution P/P_0. With this divergence measure, we can explain the meaning of distributional robustness.

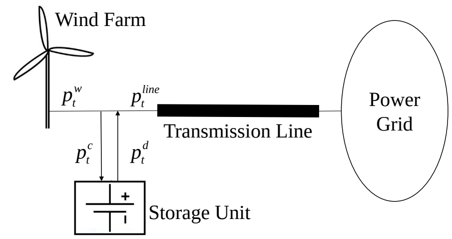

Fig. 1. Diagram of system structure.

Consider $x^\star$ satisfying constraint (6). Evaluate the probability of event $h(x^\star,\xi)-\lambda\le 0$ at P_0, the result is:

$$\mathrm{Pr}_0\{h(x^\star,\xi)-\lambda\le 0\}=1-\alpha^\star\tag{9}$$

where $\alpha^\star$ must be smaller than α, because of the conservatism introduced by (5). Denoted by $\bar\alpha=\alpha-\alpha^\star$ the margin of reliability. Furthermore, if the actual distribution varies in

$$W=\{P|D_{KL}(P\parallel P_0)\le d_M\}\tag{10}$$

Then, $\Pr\{h(x^\star,\xi)-\lambda\le 0\}$ evaluated at distribution P may fluctuate around $1-\alpha^\star$ but is still no less than $1-\alpha$ due to the margin $\bar\alpha$. Let d_M be the maximum value such that

$$\inf_{P\in W}\Pr\{h(x^\star,\xi)-\lambda\le 0\}=1-\alpha\tag{11}$$

The pair $(\bar\alpha,d_M)$ quantifies the conservatism and distributional robustness brought by the CVaR reformulation.

To compute the proposed indices $(\bar\alpha,d_M)$, we have to find the relationship among α, $\alpha^\star$ and d_M, which can be derived from Theorem 1 and Proposition 4 in [7]. In brief, if (11) holds with fixed α and d_M, then the probability of event $h(x^\star,\xi)-\lambda\le 0$ evaluated at P_0 is $1-\alpha^\star$, where

$$\alpha^\star=\max\left\{0,1-\inf_{z\in(0,1)}\left\{\frac{e^{-d_M}z^{1-\alpha}-1}{z-1}\right\}\right\}\tag{12}$$

where the univariate function $\psi(z)=(e^{-d_M}z^{1-\alpha}-1)/(z-1)$ is convex in z over the open interval $(0,1)$ [7], so the minimum can be easily found using one-dimensional search. In the algorithmic implementation, we use the renowned golden section search algorithm which entails only function value comparison and is very efficient.

Equation (12) predicts $\alpha^\star$ from the known values of α and d_M, while in the proposed method, both of α and d_M are unknown. Based on relation (12), a dichotomy algorithm is proposed to compute the indices $\bar\alpha$ and d_M. The flowchart is given in Algorithm 1. The robustness quantification is performed by steps 3∼6; the solution $x^\star$ is offered by the CVaR reformulation model in step 1. Because no complex optimization problem is solved in the main loop of steps 3∼6, Algorithm 1 is actually very fast.

III. Coordinated Capacity Planning Problem

Consider a remote wind farm shown in Fig. 1, and its capacity is known. The power grid company plans to build a local energy storage unit and a transmission line connecting the wind farm with the power grid. The target is to minimize

Algorithm 1

1: Replace chance constraint (1b) with CVaR constraint (6), solve the following problem, the optimal solution is $x^\star$;

$$\min_{x\in X}\{f(x)|(1-\alpha)\text{-CVaR}(x,\xi,\mathrm{P}_0)\le\lambda\}$$

2: Evaluate probability $\mathrm{Pr}_0\{h(x^\star,\xi)-\lambda\le 0\}$ at P_0, and the result is $1-\alpha^\star$; then $\bar{\alpha}=\alpha-\alpha^\star$;

3: Choose a convergence tolerance $\varepsilon>0$; Initialize the upper and lower bounds $d_M^{\min}$, $d_M^{\max}$;

4: Set $d_M=(d_M^{\max}+d_M^{\min})/2$ and solve

$$\beta=\max\left\{0,1-\inf_{z\in(0,1)}\left\{\frac{e^{-d_M}z^{1-\alpha}-1}{z-1}\right\}\right\}$$

5: If $\beta>\alpha$, update $d_M^{\min}=d_M$; else update $d_M^{\max}=d_M$.

6: If $|d_M^{\max}-d_M^{\min}|<\varepsilon$, terminate and report d_M as the final solution; else go to step 4.

the total investment cost while ensuring that wind power curtailment rate does not exceed a threshold. Here, we first give the definition of wind power curtailment rate $h(x,\xi)$, which is the optimal value of the following problem

$$h(x,\xi)=\min\ \frac{\sum_{t=1}^{T}(\xi_t-p_t^w)}{\sum_{t=1}^{T}\xi_t} \tag{13a}$$

$$\text{s.t.}\ \ p_t^{line}=p_t^w+p_t^d-p_t^c \tag{13b}$$

$$0\le p_t^w\le\xi_t, 0\le p_t^{line}\le x_l \tag{13c}$$

$$W_{t+1}^E=W_t^E\mu^E+(p_t^c\eta_c^E-p_t^d/\eta_d^E)\Delta t \tag{13d}$$

$$0\le\left[p_t^c,p_t^d,W_t^E\right]\le[R_cx_s,R_dx_s,x_s] \tag{13e}$$

where T is the number of periods; x = [x_s(MWh), x_l(MW)] are the capacity of energy storage unit and transmission line; ξ_t is the maximal wind power output depending on wind speed, which is uncertain. Decision variables include the dispatched wind power p_t^w, the transported power p_t^{line}, the charging/discharging power p_t^c/p_t^d, and the energy W_t^E in the storage unit. (13b) prescribes power balance; (13c) limits the dispatched wind power and transmitted power; (13d) represents the charging dynamics of storage unit; (13e) imposes bounds on storage charging power, discharging power, and storage level; $1-\mu^E$ is the self-discharge rate, η_c^E/η_d^E is charging/discharging efficiency, and R_c and R_d are constants.

In the coordinated planning problem, we aim to maintain the wind power curtailment rate $h(x,\xi)$ within a threshold λ, which can be interpreted as an event $h(x,\xi)-\lambda\le 0$. The probability of this event depends on the distribution of ξ, yielding the following CCP

$$\min\ \ C_sx_s+C_lx_l \tag{14a}$$

$$\text{s.t.}\ \ x_s\ge 0,\ x_l\ge 0 \tag{14b}$$

$$\mathrm{Pr}_0\{h(x,\xi)-\lambda\le 0\}\ge 1-\alpha \tag{14c}$$

where C_s(\$/MWh)/C_l(\$/MW) is the unit capacity cost of the energy storage unit and transmission line. Chance constraint (14c) requires that under the empirical distribution P_0, the probability of event $h(x,\xi)-\lambda\le 0$ is at least $1-\alpha$. According to the discussions in Section II, the chance constraint (14c) is approximated by the following CVaR constraint

$$(1-\alpha)\text{-CVaR}(x,\xi,\mathrm{P}_0)\le\lambda \tag{15}$$

Suppose we have N typical scenarios $\boldsymbol{\xi}^1$, $\boldsymbol{\xi}^2$, $\cdots$, $\boldsymbol{\xi}^N$ with probabilities π^1, π^2, $\cdots$, π^N, respectively. Decision variables and constraints in (13b)-(13e) are duplicated for each scenario with a script n. Afterwards, the CVaR constraint (15) comes down to the following linear forms [3]

$$\begin{aligned}&\frac{\sum_{t=1}^{T}(\xi_t^n-p_t^{w,n})}{\sum_{t=1}^{T}\xi_t^n}-\lambda+\gamma\le\sigma^n,\ \forall n\\ &\sum_{n=1}^{N}\pi^n\sigma^n\le\gamma\alpha,\ \sigma^n\ge 0,\ \forall n\end{aligned} \tag{16}$$

where $\sigma^n(n=1,2,...,N)$ and γ are auxiliary variables.

Replacing (14c) with (16), which is known as the sampling average approximation of CVaR constraint (15), the coordinated planning model (14) finally comes down to

$$\min\ \ C_sx_s+C_lx_l \tag{17a}$$

$$\text{s.t.}\ \ x_s\ge 0,\ x_l\ge 0 \tag{17b}$$

$$p_t^{line,n}=p_t^{w,n}+p_t^{d,n}-p_t^{c,n} \tag{17c}$$

$$0\le p_t^{w,n}\le\xi_t^n, 0\le p_t^{line,n}\le x_l \tag{17d}$$

$$W_{t+1}^{E,n}=W_t^{E,n}\mu^E+\left(p_t^{c,n}\eta_c^E-\frac{p_t^{d,n}}{\eta_d^E}\right)\Delta t \tag{17e}$$

$$0\le\left[p_t^{c,n},p_t^{d,n},W_t^{E,n}\right]\le[R_cx_s,R_dx_s,x_s] \tag{17f}$$

$$\frac{\sum_{t=1}^{T}(\xi_t^n-p_t^{w,n})}{\sum_{t=1}^{T}\xi_t^n}-\lambda+\gamma\le\sigma^n \tag{17g}$$

$$\sum_{n=1}^{N}\pi^n\sigma^n\le\gamma\alpha,\ \sigma^n\ge 0 \tag{17h}$$

Problem (17) is a linear program, which can be efficiently solved by off-the-shelf solvers.

IV. Simulations Results

Consider the scene in Fig. 1 with a 100MW wind farm. Parameters of energy storage unit and transmission line are shown in I. Based on the real weather data from wind farms in Qinghai province, China, the predicted wind power outputs in four typical days are extracted from spring, summer, autumn and winter to represent a whole year. Assuming the forecast errors obey Gaussian distribution with zero mean, and the standard deviation is 0.2 multiplying the forecast values. Next, 5000 scenarios are generated via Monte Carlo method to construct the empirical PDF P_0. The wind power output data is available at [10]. In the numerical experiments, the cap of curtailment rate $\lambda=0.05$ and the risk level $\alpha=10\%$. All optimization models are established in MATLAB 2018b and solved by CPLEX 12.8 with YALMIP interface.

The optimal solution of model (17) is $C_s=33.37$ MWh and $C_l=58.08$ MW, which is denoted by $x^\star$. The curtailment rate requirement $h(x^\star,\xi)-\lambda\le 0$ holds with a probability $1-\alpha^\star=96.01\%$ under the empirical distribution P_0, so the reliability margin is $\bar{\alpha}=6.01\%$. By executing Algorithm 1, the value of

TABLE I
PARAMETER SETTINGS

C_s (\$/MWh)	C_l (\$/MW)	μ^E	η_c	η_d	R_c	R_d
2×10^5	1.2×10^6	0.99	0.95	0.95	0.25	0.25

TABLE II
RESULTS WITH DIFFERENT RISK LEVEL α

$1-\alpha$	80%	85%	90%	95%
$1-\alpha^\star$	92.55%	93.90%	96.01%	98.20%
$\bar{\alpha}$	12.55%	8.90%	6.01%	3.20%
d_M	0.0809	0.0503	0.0335	0.0196
Test probabilities for CVaR	80.20%	85.24%	90.27%	95.21%
Test probabilities for VaR	61.18%	72.78%	81.92%	90.83%

d_M is 0.0335, indicating that for all distributions in set (10) with $d_M = 0.0335$, the probability of event $h(x^\star, \xi) - \lambda \le 0$ is no less than $1 - \alpha = 90\%$. In other words, the optimal solution $x^\star$ is robust against the perturbations of actual PDFs in practice. The time for calculating the pair of indices $(\bar{\alpha}, d_M)$ is 2.04s, demonstrating the efficiency of Algorithm 1.

To validate our conclusion, we generate data sets consisting of 10000 scenarios whose distribution is very different from the empirical distribution P_0, while maintaining $D_{KL}(P||P_0) = 0.0335$. Then we test the probability Pr$\{h(x^\star, \xi) - \lambda \le 0\}$ under this unfavorable distribution P; the result is 90.27%, which is slightly higher than the anticipated value $1 - \alpha = 90\%$, validating that the proposed index d_M is accurate. For comparison, the VaR reformulation of CCP model (14), which is an MILP [2], is solved; the optimal solution x' gives $C_s = 31.96$ MWh and $C_l = 56.21$ MW. Both of them are smaller than those in $x^\star$. Then we test the probability Pr$\{h(x', \xi) - \lambda \le 0\}$ under the aforementioned unfavorable distribution P; the result is 81.92%, which is much lower than the anticipated value $1-\alpha = 90\%$. This fact means that the optimal solution of VaR model is not robust and unable to protect system security with a predefined reliability level if the distribution used in the model is inexact.

The risk level α remarkably influences the optimal solution of the CCP model. In the following tests, we investigate the impact of α on the distributional robustness. First, the value of $1-\alpha^\star$, $\bar{\alpha}$ and d_M is calculated from Algorithm 1 with different values of α. Next, a series of unfavorable distributions with corresponding KL-divergence distances to P_0 is generated. We test Pr$\{h(x^\star, \xi) - \lambda \le 0\}$/Pr$\{h(x^{'}, \xi) - \lambda \le 0\}$, termed as the test probability for CVaR/VaR. Results are summarized in Table II. It is observed that $\bar{\alpha}$ and d_M decrease with the increase of $1 - \alpha$, implying that when the targeted reliability level $1 - \alpha$ grows higher, the CVaR reformulation becomes less conservative, and the distributional robustness is weaker. The test probability for CVaR is always slightly higher than $1 - \alpha$, demonstrating the effectiveness of the proposed characterization on distributional robustness. In contrast, the test probability for VaR is always lower than $1 - \alpha$, because the VaR reformation does not warrant distributional robustness.

Finally, we investigate the impact of curtailment rate cap λ on the distributional robustness indices. Similar to the above analysis, we calculate the values of $1 - \alpha^\star$, $\bar{\alpha}$, d_M and test probabilities with different λ while remaining $1 - \alpha = 90\%$. The results are summarized in Table III. It is observed that the values of $1 - \alpha^\star$, $\bar{\alpha}$, d_M and test probability for CVaR and VaR almost keep unchanged with different values of λ, which means that the distributional robustness of CVaR reformulation are barely affected by the cap of curtailment rate λ.

TABLE III
RESULTS WITH DIFFERENT CURTAILMENT RATE CAP λ

λ	0.02	0.05	0.07	0.1
$1-\alpha^\star$	95.99%	96.01%	95.87%	95.95%
$\bar{\alpha}$	5.99%	6.01%	5.87%	5.95%
d_M	0.0334	0.0335	0.0315	0.0327
Test probabilities for CVaR	90.26%	90.27%	90.24%	90.26%
Test probabilities for VaR	81.94%	81.92%	82.20%	82.03%

V. CONCLUSION

Although chance constrained stochastic programming only accounts for an empirical distribution, this letter reveals that the renowned CVaR reformulation of chance constraint naturally endows the solution with distributional robustness, which can be quantified by a reliability margin and a maximum KL-divergence distance. Numerical results corroborate the proposed indices, and demonstrate that with the growth of reliability level required by the chance constraint, the distributional robustness brought by CVaR reformulation gets weaker, and the induced conservativeness decreases at the same time.

REFERENCES

[1] S. Zymler, D. Kuhn, and B. Rustem, "Distributionally robust joint chance constraints with second-order moment information," *Math. Program.*, vol. 137, no. 1-2, pp. 167–198, Nov. 2013.

[2] R. Xie, W. Wei, M. E. Khodayar, J. Wang, and S. Mei, "Planning fully renewable powered charging stations on highways: A data-driven robust optimization approach," *IEEE Trans. Transp. Electrific.*, vol. 4, no. 3, pp. 817–830, Jun. 2018.

[3] R. T. Rockafellar and S. Uryasev, "Optimization of conditional value-at-risk," *J. Risk*, vol. 2, no. 3, pp. 21–41, Sep. 2000.

[4] Y. Cao, W. Wei, J. Wang, S. Mei, M. Shafie-khah, and J. P. Catalão, "Capacity planning of energy hub in multi-carrier energy networks: a data-driven robust stochastic programming approach," *IEEE Trans. Sustain. Energy*, vol. 11, no. 1, pp. 3–14, Jan. 2020.

[5] E. DallAnese, K. Baker, and T. Summers, "Chance-constrained ac optimal power flow for distribution systems with renewables," *IEEE Trans. Power Syst.*, vol. 32, no. 5, pp. 3427–3438, Sep. 2017.

[6] C. Peng, Y. Hou, N. Yu, and W. Wang, "Risk-limiting unit commitment in smart grid with intelligent periphery," *IEEE Trans. Power Syst.*, vol. 32, no. 6, pp. 4696–4707, Nov. 2017.

[7] R. Jiang and Y. Guan, "Data-driven chance constrained stochastic program," *Math. Program.*, vol. 158, no. 1-2, pp. 291–327, Jan. 2016.

[8] P. M. Esfahani and D. Kuhn, "Data-driven distributionally robust optimization using the wasserstein metric: Performance guarantees and tractable reformulations," *Math. Program.*, vol. 171, no. 1-2, pp. 115–166, Sep. 2018.

[9] W. Xie, "On distributionally robust chance constrained programs with Wasserstein distance," *Math. Program. in press*, 2019.

[10] https://sites.google.com/site/caoyangthu13/data/data-cvar.