Wille Berg

From the inaccurate Black-Scholes model to more efficient delta hedging with smile-adjusted extension

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Author:	Wille Berg
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ABSTRACT:

The Black-Scholes model is the first and still one of the most used models to price and hedge options despite its multiple unrealistic background assumptions, like constant volatility. Since the iterated implied volatility from the Black-Scholes model among index options is documented to be downward sloping, the volatility smile takes the form of a smirk. Due to the volatility smile, the Black-Scholes delta (BS delta) does not minimise the variance of the portfolio in delta hedging. Therefore, researchers have presented models like smile-adjusted delta (SAD) to enhance delta hedging performance by considering the volatility smile.

This thesis provides an in-depth discussion about the background assumptions of the Black-Scholes model and the non-unanimous reasons for the volatility smile. This is followed by research on how the Black-Scholes model can be modified to enhance delta hedging outcomes to consider the presence of volatility smiles. The delta hedging performance of the BS delta and SAD is retested among S&P 500 index options during the COVID-19 pandemic year of 2020, following the methodology by Vähämaa (2004) for robust comparison.

The empirical findings from delta hedging indicate that SAD outperforms the BS delta most distinctly among in-the-money options. When the hedging horizon is longer than a day, SAD outperforms the BS delta regardless of the moneyness and maturity of the option. That outperformance becomes even more notable as the hedging horizon lengthens. Moreover, in stable market conditions, the outperformance of SAD is documented for every moneyness-maturity category. Interestingly, for highly volatile market conditions, the outperformance of SAD is documented only among in-the-money options and out-of-the-money long-term options. Further, the volatility smiles or rather smirks, are documented separately for call and put options. A comparison of volatility smiles reveals that put options are priced with greater implied volatility than call options and that the difference in implied volatility is the smallest among at-the-money options. Moreover, due to the observed negative correlation between volatility and underlying movements, the correct delta is smaller than the BS delta.

KEYWORDS: Black-Scholes, delta hedging, local delta, local volatility, smile-adjusted delta, volatility smile

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TIIVISTELMÄ:

Black–Scholes malli on optioiden hinnoittelun ja riskisuojauksen ensimmäinen ja yhä yksi yleisimmin käytetyistä malleista, vaikka se perustuu useisiin epärealistisiin oletuksiin, kuten muuttumattomaan volatiliteettiin. Koska Black-Scholes mallista iteroidun implisiittisen volatiliteetin on dokumentoitu olevan indeksioptioiden kohdalla laskeva, volatiliteettihymy on muodoltaan virnistävä. Volatiliteettihymyn vuoksi Black-Scholes mallin delta (BS delta) ei minimoi portfolion varianssia deltasuojauksessa. Tämän seurauksena tutkijat ovat esittäneet malleja, kuten hymykorjattu delta, joilla parannetaan deltasuojauksen tehokkuutta ottamalla huomioon volatiliteettihymy.

Tässä tutkielmassa käsitellään perusteellisesti Black-Scholes mallin taustaoletuksia ja volatiliteettihymyn muodostumisen eriäviä syitä. Tämän jälkeen tutkitaan, miten Black-Scholes mallia voidaan muokata ottamaan volatiliteettihymy huomioon optimaalisemman deltasuojautumisen takaamiseksi. BS deltan ja hymykorjatun deltan deltasuojauksen optimaalisuutta testataan S&P 500 indeksioptioilla vuonna 2020, jota varjosti COVID-19 pandemia. Vertailukelpoisuuden varmistamiseksi tutkielma seuraa tutkimuksen Vähämaa (2004) metodologiaa.

Deltasuojaukseen liittyvät empiiriset havainnot osoittavat, että hymykorjatulla deltalla saavutetaan BS deltaan nähden parempi deltasuojaus plusoptioita tarkastellessa. Deltasuojaushorisontin ollessa pidempi kuin yksi päivä, hymykorjattu delta tarjoaa BS deltaa paremman deltasuojauksen riippumatta option perusarvosta ja voimassaoloajasta. Suojaushorisontin pidentyessä ero deltasuojauksen tehokkuudessa muuttuu entistä merkitsevämmäksi. Vakailla markkinoilla hymykorjattu delta voidaan todeta paremmaksi jokaisessa perusarvo-voimassaoloaika-ryhmässä. Mielenkiintoista on, että epävakailla markkinoilla hymykorjatun deltan paremmuus voidaan todeta ainoastaan plusoptioilla ja pitkän maturiteetin miinusoptioilla. Tutkielmassa dokumentoidaan myös virnistävät volatiliteettihymyt erikseen osto- ja myyntioptioille. Hymyjen vertailu paljastaa, että myyntioptiot hinnoitellaan suuremmalla implisiittisellä volatiliteetilla kuin osto-optiot, ja että implisiittisten volatiliteettien ero on pienin tarkastellessa tasaoptioita. Lisäksi voidaan todeta, että volatiliteetin ja kohde-etuuden välisen negatiivisen korrelaation vuoksi oikea delta on pienempi kuin BS delta.

AVAINSANAT: Black-Scholes, deltasuojaus, hymykorjattu delta, paikallinen delta, paikallinen volatiliteetti, volatiliteettihymy

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1 Introduction

The Black-Scholes model is the first and still one of the most used models to price and hedge options despite its multiple unrealistic background assumptions. One of those assumptions is constant volatility, meaning that when all the other factors of the model are known, the iterated implied volatility from the Black-Scholes model should be the same regardless of the option's strike price. Nevertheless, after the Black Monday stock market crash in 1987, the implied volatility of equity options has not been constant or flat but rather downward sloping as the strike price increases. This phenomenon is called volatility smile, or rather smirk/skew, due to its shape. Unfortunately, there is no unambiguous answer as to why there are volatility smiles. The Black-Scholes model's assumptions of log-normal distribution of the underlying price movements, constant volatility and frictionless market have all been proposed as the reasons for the volatility smile in the past literature. Nevertheless, the reasons are not limited to those.

Furthermore, the volatility smile has implications for the risk management method of delta hedging. In delta hedging, the portfolio should be kept delta neutral by reducing the directional risk of the underlying. Delta measures the change in option price relative to the change in the underlying price. Unfortunately, the underlying is correlated with volatility, which indirectly affects the size of delta (Vähämaa, 2004). Because of the correlation, using the Black-Scholes delta in delta hedging does not minimise the variance of the portfolio, leading to the presence of smile risk (Hull & White, 2017).

To consider smile risk, researchers have examined minimum variance delta (MV delta) models to take the smile into account for achieving minimal portfolio variance. The MV delta can be found with many different models. It has been tested for example with using stochastic volatility (see e.g. Bakshi et al.,1997; Bakshi et al.,2000; Kim & Kim, 2004; Lim & Guo, 2000; Nandi, 1998) and deterministic volatility (see e.g. Dumas et al., 1998; Engle & Rosenberg, 2000; Lim & Zhi, 2002; Yung & Zhang, 2003). In practice, deterministic volatility models perform even worse than the Black-Scholes model and stochastic volatility models are demanding to implement in use. Therefore, the MV delta should be researched with a more practical perspective.

Researchers Alexander and Imeraj (2023), Alexander et al. (2012), Attie (2017), Coleman et al. (2000), Crépey (2004), François and Stentoft (2021), Hull and White (2017), and Vähämaa (2004) had a different approach with more practical vega-based local volatility models, smile-adjusted/local delta, where the Black-Scholes model is extended with a smile correction term based on vega and real volatility dynamics. To conclude the findings from previous studies, the real delta should be smaller than the Black-Scholes delta, and the delta hedging with smile-adjusted/local delta should produce smaller hedging errors than with the Black-Scholes delta. The overperformance should be more prominent in highly volatile market conditions than in stable market conditions, and more visible as the hedging horizon lengthens.

1.1 Purpose of the study

Since the research focus of delta hedging performance has recently been on more demanding hedging models, it is interesting to retest the hedging performance of an older but more practical model to hedge with the smile. To further emphasise the practical perspective of this thesis, the concentration is on minor adjustments of delta rather than monitoring multiple Greek letters simultaneously. This is appropriate as the delta is the most crucial hedge parameter (Hull & White, 2017).

Furthermore, considering the smile-adjusted delta, there is a clear research gap to fill. The performance of smile-adjusted delta for index options is not documented using samples after 2016, and particularly not under extended volatile market conditions since the 2008 financial crisis. Particularly, during the recent COVID-19 crisis, increased volatility levels provide a renewed opportunity to investigate the delta hedging performance in highly volatile markets, in which the outperformance of the smile-adjusted delta is documented to be most prominent compared to the Black-Scholes delta. To clarify, during crises, volatilities of option strikes increase, leading to short-term skew to steepen sharply (Derman et al., 2016). Since steeper smile refers to greater negative slope, the smile-adjustment term during crises like COVID-19 is more critical to consider. Thus, significant improvements in delta hedging performance can be expected when extending the basic Black-Scholes delta with the smile-adjustment term.

The purpose of the study is to investigate whether the delta hedging performance of the Black-Scholes delta can be improved by adding a volatility smile sensitivity term to the equation. This could help practitioners to delta hedge more optimally, managing the smile risk with a practical model. The thesis has the following hypotheses built from the past literature:

H_1: The volatility smile should be considered for more efficient delta hedging results.

H_2: Smile-adjusted delta outperforms the Black-Scholes delta in delta hedging.

For robust delta hedging performance comparison, the vega-based local volatility model named smile-adjusted delta from Vähämaa (2004) is retested with a more recent and larger dataset. Particularly, it is retested with S&P 500 index options during the COVID-19 year of 2020, to achieve a strong contribution to the previous literature. Delta hedging performance is analysed in different hedging horizons and market regimes, filling the observed research gap with one of the most followed stock indexes, which has historically had relatively steep volatility smiles (see e.g. Jackwerth,2004; Tompkins, 2001).

1.2 Structure of the study

In this introduction, the purpose of the study and the hypotheses are already defined, and the structure of the thesis will be covered next. The second chapter will cover the basics of options and what affects their pricing. This discussion is advantageous for understanding the Black-Scholes model in depth. Chapter 3 defines the background assumptions, formula, limitations, and inaccuracy of the most famous option pricing model. Chapter 4 covers the iterated implied volatility from the Black-Scholes model and how it can be illustrated with a volatility smile. Then, put-call parity is applied to show that the smiles should be similar for both call and put options. Moreover, in that chapter, the unanimous reasons why the smiles exist are discussed. Chapter 5 combines the concept of volatility smile with delta hedging. The more optimal delta hedging with the smile is examined by taking a closer look at the smile-adjusted delta by reviewing the past literature.

Finally, Chapter 6 covers the methodology and data which are used in the empirical part of this thesis. Next, the quantitative fit and hedging error results of the Black-Scholes delta and smile-adjusted delta are compared in chapter 7. Lastly, the conclusion chapter 8 summarises the findings, highlights the implications for practice, notes the limitations of the study and presents recommendations for future research related to the topic.

2 Options

When evaluating an option pricing model, it is advantageous to first discuss the nature of options and their pricing mechanisms. This main chapter is divided into two parts. The first subchapter introduces what options are, and the second subchapter explains what kind of factors affect the price of options.

2.1 Introduction to options

Hull (2022) defines derivatives as financial instruments whose value depends on, or derives from, underlying variables. The underlying variable can be almost anything, but most often it is the price of a traded asset. In the case of a stock option, the value of the derivative depends on the price of the underlying stock (Hull, 2022).

Derivatives are traded on derivative exchanges and in over-the-counter (OTC) markets (Hull, 2022). According to Hull (2022), there are fewer transactions in the over-the-counter markets than in derivative exchanges, but those transactions are larger. The Chicago Board Options Exchange (CBOE) and EUREX Exchange are examples of well-known options exchanges.

There are two types of options: put and call options. Hull (2022) defines a put option as a right to the option holder to sell the underlying on a certain date for a certain price, whereas a call option holder has a right to buy the underlying on a certain date for a certain price. That contract price is defined as a strike/exercise price, and a certain date is defined as the maturity/expiration date.

Options can be divided into European and American options. This categorisation is not related to the geographical location or the trading exchange. American options can be exercised whenever, but European options can be exercised only on the expiry date (Hull, 2022). The option pricing model, the Black-Scholes model, is developed for simpler European options.

Every put and call option contract has two sides: the option buyer (long position) and the writer (short position) (Hull, 2022). At the beginning of the contract, the option buyer pays the premium, also known as the option price, which the option writer receives. In addition, the writer is obligated to buy or sell in terms of the contract if the option buyer exercises the option. In other words, the buyer of an option has the right, without any obligation, to exercise the option, and this right is obtained through the payment of a premium.

Hull (2022) defines three purposes for options: to speculate in the market to make a profit, find arbitrage, and most importantly, hedge against market risks. Hedgers are, in a sense, paying insurance against adverse price movements in the future and still having benefits if the market movements are favourable (Hull, 2022). Speculators are using options for leverage, which magnifies the financial consequences of an investment; Positive outcomes become great in value and for negative outcomes, the whole initial investment is lost (Hull, 2022). The market is assumed to be non-arbitrageur, but if there is a chance to make risk-free income, some arbitrageurs will take those chances until supply and demand correct the market arbitrage (Hull, 2022).

2.2 Factors affecting the price

The price of the option, also known as the premium, is composed of time value and intrinsic value (Knüpfer, 2024). Intrinsic value is defined as the return which the investor receives if the option is exercised now. For bought call options, the intrinsic value is $MAX(0; S-K)$, and for bought put options it is $MAX(0; K-S)$. In the formulas S denote the current stock price and K denote the strike price. In other words, the premium is the highest revenue the option writer can get, but the loss is unlimited. In the case of the option buyer, the maximum loss is the premium, and the maximum gain is unlimited. This leads to explaining option moneyness with terms out-of-the-money (OTM), at-the-money (ATM), and in-the-money (ITM) (Knüpfer, 2024). The option is profitable when it is ITM, for call options $S > K$ and put options $K > S$. OTM is the opposite of the ITM, and the intrinsic value is 0. When the option is ATM, $S = K$. An option should only be exercised when it is ITM.

The price of an option is determined by various factors, including the price of the underlying asset, strike price, time to maturity, volatility of the underlying asset and risk-free rate (Knüpfer, 2024). Next, the influence of the mentioned factors on the option premium is discussed.

Knüpfer (2024) describes that the underlying asset's price naturally affects the option price. The higher the stock price, the higher the call option price because the option is worth more when exercised. The exercise price has the reverse effect: the greater the exercise price of the call option, the lower the amount received and the option's value. The reasoning for a put option is the opposite.

Higher volatility has a positive influence on the price for both call and put options because it increases the chance that the option is eventually ITM (Knüpfer, 2024). Time to maturity is more problematic when examining European options. According to Hull (2022), the options are often more valuable when the time to maturity increases. Nevertheless, when a large dividend is expected between two different maturities, leading to a decline in the stock price, it can make the longer-term call option worth less than the shorter-term call option (Hull, 2022). Furthermore, future dividends decrease the price of a call option and increase the price of a put option, because dividends tend to decrease the price of a stock (Hull, 2022).

Knüpfer (2024) explains why the risk-free rate has a positive effect on the price of call options and a negative effect on the price of put options. The explanation is approached by considering the option as a cash-flow delaying strategy. A call option buyer desires a share but invests in an option instead. In this manner, the call option buyer postpones payment of the share's purchase price until the time of exercise and can invest the purchase money to enhance the interest rate for the duration of the option. The higher the risk-free interest rate, the greater the advantage of postponing the purchase price, so the value of the call option rises in line with the interest rate. In the case of a put option, the delaying cash flow is positive; the buyer of a put option expects to receive payment for the share only at the time of exercise, instead of now. High interest rates now negatively affect the benefit of delay, implying the put option price is lower.

3 The Black-Scholes model

In the year of 1973, two economists, Fischer Black and Myron Scholes, developed an option pricing model, which is widely used to price European options. The Black-Scholes model, also known as the Black-Scholes-Merton model when Robert Merton improved the model with a different approach, has greatly influenced how traders hedge and price derivatives (Hull, 2022). The model was remarkable when making large-scale option trading in derivatives exchanges possible. Therefore, Scholes and Merton were awarded the Nobel Memorial Prize in Economic Sciences in 1997. Unfortunately, Fischer Black passed away before receiving the earned prize, which cannot be received posthumously. In this section, this influential model is discussed through its background assumptions and formula, concluding with its limitations and inaccuracy.

3.1 Background assumptions

The Black-Scholes model was derived using several underlying assumptions. The model originated from the idea that the risk can be fully eliminated by buying and selling the basic instrument and a risk-free asset in a certain ratio. In other words, the price of the option is indirectly influenced by how much the underlying asset's price changes (Janková, 2018). To understand the ideal market conditions the model is based on, Black and Scholes (1973) mentioned seven categories of assumptions for the model:

1. The short-term interest rate and volatility are known and constant over time
2. The underlying pays no dividends
3. No transaction costs, commissions, or taxes
4. Short selling of securities is permitted
5. The market is efficient, so there are no riskless arbitrage opportunities
6. The underlying can be traded continuously

7. The underlying follows a random walk; the price is lognormally distributed, meaning the returns on the underlying are normally distributed

The last assumption should be focused on in more detail. Random walk means that there is an equal probability of an increase and a decrease in the underlying stock price. The normal distribution, also known as the Gaussian distribution, is commonly assumed to describe random variation (Limpert et al., 2001). The assumption of the random walk of the underlying price can be seen as a random variation, as the probability of events is the same. The normal distribution is a symmetrical bell-shaped curve in which the mean (arithmetic average), mode (the most common value) and median (the middle value of the sample) are all the same (Zucchi, 2024). As a result, 68% of the results fall within one standard deviation and 95% within two standard deviations.

As learnt, an option is a right and therefore the value of it is always non-negative. The normal distribution can also get negative values, which an option cannot get. Therefore, the lognormal distribution is used to price the option. The use of only non-negative values makes the lognormal distribution a right-skewed curve (Zucchi, 2024). Zucchi (2024) explains how a lognormal distribution is derived from values which are normally distributed. Zucchi (2024) clarifies this with an example where the future stock price is a computation of various rates of return. When continuously compounding those returns, which are assumed to be normally distributed, the lognormal distribution is obtained.

3.2 Formula

Black and Scholes (1973) used the capital asset pricing model (CAPM) when finding the relationship between the required return of the option and the underlying stock. Whereas Merton (1973) developed the model further by creating a riskless portfolio of underlying stock and its option so that the return of the portfolio should be the risk-free rate in the short term. In the book, Hull (2022) presented mathematical formulas created by Black and Scholes (1973), which are improved by Merton (1973) for pricing European call (1) and European put (2) options.

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{1}$$

and

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \tag{2}$$

where

$$d_1 = \frac{ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \tag{3}$$

$$d_2 = \frac{ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{4}$$

c = price of the European call option

p = price of the European put option

T = time to maturity

S_0 = stock price at time zero

K = strike price

r = risk-free rate

σ = volatility of the stock

$N(x)$ = cumulative standard normal distribution for a variable x

3.3 Limitations and inaccuracy of the model

It is worth noting that all the strict assumptions of the model cannot hold in real financial markets, which leads to examining the inaccuracy and limitations of the model. Especially, the assumptions of the lognormal distribution and the constant implied volatility are crucial for this thesis. Therefore, the assumption of lognormal distribution is discussed separately, first in the assumptions section, and further in the reasons behind the smile section alongside the assumption of constant implied volatility. However, in this section, the real-world holding of the other assumptions will be discussed. Furthermore, the limitations of the model are considered.

As follows, Teneng (2011) challenges the assumption of the random walk of stock prices. The assumption does not hold because stock prices are driven by many factors that cannot be assigned equal probability of how they affect stock price movements. Furthermore, according to the Martingale property of Brownian motion, the price of a stock at time $t + 1$ is independent of the price at time t. Next, according to the Martingale representation theorem, there may not be a single source or factor driving two assets, even if one is a derivative of the other (Teneng, 2011).

As mentioned, the Black-Scholes model was created to price European options. Nevertheless, there are also American options, which need to be more valuable than European options due to greater flexibility since they can be exercised whenever during the lifetime of an option (Teneng, 2011). The model assumes that the underlying should be possible to trade continuously so that the market is not only efficient but also liquid. This is not the case for the underlying or the option. The trading volume or open interest can be used as an indication of how liquid the options market is (Langager, 2022). Langager (2022) writes that options are traded far less actively than stocks.

The model assumes that short selling of securities is permitted. However, certain option exchanges may have guidelines and limitations on short selling, including requiring short sellers to hold a specified amount of capital to protect against potential losses. Considering the market, the market should be efficient, meaning the current prices should fully reflect all the available information (Fama, 1970). To contradict this, history has shown that there are market anomalies and other arbitrage opportunities.

In the real financial world, there can be dividends during the option's lifetime. However, Teneng (2011) writes that the basic Black-Scholes model can be adjusted for dividends when subtracting the discounted value of a future dividend from the price of the stock. The assumption of no transaction costs is not valid because stockbrokers charge rates (Teneng, 2011). Further, investors are paying taxes and commissions for their investments.

In addition to the previously mentioned violating assumptions, the risk-neutral valuing of options is interestingly not a violating assumption alone if the risk-free rate is correct. When valuing an option, it is assumed that investors are risk-neutral, meaning that investors do not increase expected return requirements to compensate for the increased risk of the investment (Hull, 2022). Hence, the expected return on a stock is the risk-free rate, which is also used as a discount rate for option payoffs. That is referred to as a risk-neutral world. Thus, it is reasonable to assume that in real financial markets, investors require more returns if the risk is higher. Counter-intuitively, the assumption of risk-neutrality provides the same option value for both in the risk-neutral world and in the real financial world (Hull, 2022). As an explanation, risk preferences are irrelevant when valuing options in terms of underlying stock prices. When investors become more risk-averse, stock prices decline, but the formulas linking option pricing to stock prices stay the same (Hull, 2022).

Nevertheless, the validity of the risk-free rate used in risk-neutral option pricing is a violating assumption. For the model, the risk-free rate is assumed to remain constant and therefore known with certainty. In the real financial world, by definition, the risk-free rate is not truly risk-free since even the safest investments, like the United States government T-Bills, include some default risk (Hayes, 2024b). Moreover, the risk-free rate fluctuates due to monetary policy, economic conditions, inflation expectations, market demand for government securities, and investor sentiment (Hayes, 2024b). The risk-free rate can also be negative, meaning one may absurdly get paid for lending. Nevertheless, when the risk-free rate varies, the present value of cash flows further in the future fluctuates greatly. Therefore, the assumption of always constant risk-free rate has more consequences for long-term than short-term options.

4 Implied volatility and volatility smile

The volatility is not constant over time. This chapter emphasises how the constant volatility assumption highlights the inaccuracy of the Black-Scholes model through observed volatility smiles and surfaces. According to put-call parity, the observed volatility smile should be similar for both put and call options. Lastly, non-unanimous reasons behind the volatility smile are discussed.

4.1 Introduction to implied volatility and volatility smile

The only factor in the Black-Scholes model that cannot be directly observed is volatility (Hull, 2022). Even though realised volatility can be computed from historical data, the theoretical price of an option depends on the volatility that will be experienced over the option's lifetime (Figlewski, 1997). That is why implied volatility is more coherent for options and is used to predict future volatility. The implied volatility can be solved from the Black-Scholes model with an iterative search procedure (Hull, 2022). According to Hull (2022), traders usually quote on implied volatility rather than the option price because it tends to be less variable than the price of the option.

Before the Black Monday market crash in 1987, the Black-Scholes model appeared to describe the options market quite well, after which the volatility smile occurred (Derman et al, 2016). The volatility smile is the implied volatility of an option with a specific maturity as a function of its strike price (Hull, 2022). When considering equities, the volatility smile is also referred to as the volatility skew/smirk due to its shape. Nevertheless, the smile shown in figure 1 should not exist if the model was accurate. The smile (figure 1) shows that lower strike price options are priced with higher implied volatility. As learnt, the higher the volatility, the higher the price of the option. To conclude, the smile is overpricing deep in-the-money calls and deep out-of-the-money puts, and vice versa for in-the-money puts and out-of-the-money calls. The smile implies that the model cannot efficiently calculate implied volatility (Hayes, 2024a). In addition, Ederington and Guan (2005) showed that implied volatility iterated from out-of-the-money and at-the-money strikes is a less informative and biased prediction of future volatility, but when iterated from in-the-money strike prices, the estimation is better.

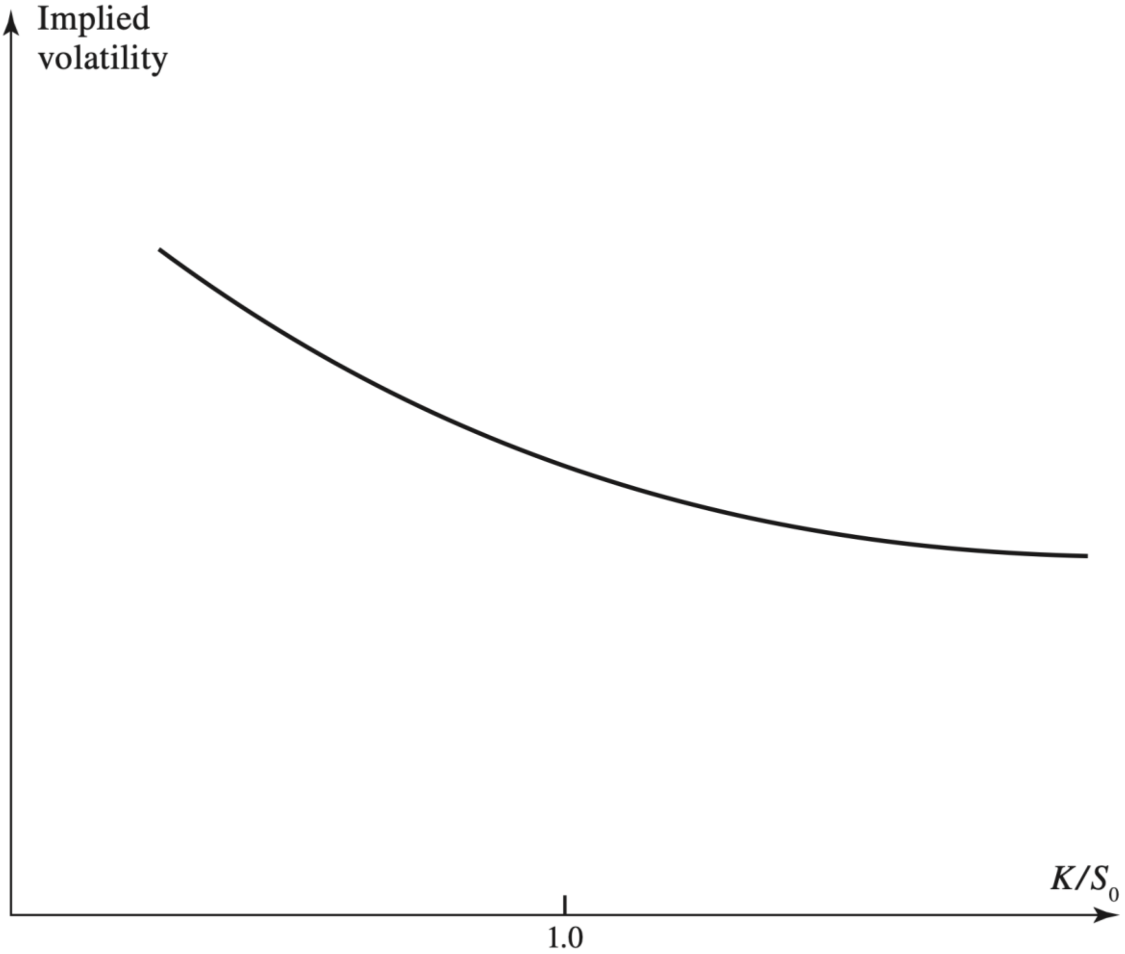


Figure 1 Volatility smile of equity options (Hull, 2022).

The volatility smile phenomenon has been documented all around the world. For example, Tompkins (2001) documented smiles from the United Kingdom, Japan and Germany, but those smiles are not as steep as the S&P 500 index smile in the United States. Moreover, individual option smiles in the United States are not as steep as the index smile there, and this result presumably applies to other markets as well, though it has not been confirmed (Dennis & Mayhew, 2000).

4.2 From put-call parity to the common smile of puts and calls

In theory, the occurred volatility smile is the same for both put and call options. Hull (2022) proves that with the following modified expression from put-call parity that provides a relationship between European put and European call options with common maturity T and strike price K (Hull, 2022). To define the rest of the factors p is the European put and c the European call price. The risk-free rate is r, and S_0 is the stock price today. The proving process begins with the put-call parity formula (5) expressed below.

$$p + S_0 e^{-rT} = c + Ke^{-rT} \tag{5}$$

Because put-call parity holds for the market prices, the formula (6) below must also be true.

$$p_{BS} + S_0 e^{-rT} = c_{BS} + Ke^{-rT} \tag{6}$$

When the assumption of the non-arbitrageurs' markets is made, the formula (7) below is created.

$$p_{mkt} + S_0 e^{-rT} = c_{mkt} + Ke^{-rT} \tag{7}$$

Subtracting formulas (6) and (7), formula (8) is obtained.

$$p_{BS} - p_{mkt} = c_{BS} - c_{mkt} \tag{8}$$

Formula (8) can be solved by following terms (9) & (10).

$$p_{BS} = p_{mkt} \tag{9}$$

$$c_{BS} = c_{mkt} \tag{10}$$

In other words, implied volatility for both puts and calls is the same when the maturity and strike price are common. Therefore, the volatility smile is the same for both puts and calls. In addition, the volatility surface is then the same for both. The volatility surface describes the relationship between implied volatility and strike price for a specific maturity (Hull, 2022). The theory of the common smile of put and call options is convenient, as there is no need to thread the volatility smiles separately. However, since the theory can be inconsistent with reality, it is interesting to document the smiles separately for put and call options to identify potential inconsistencies and learn more about the real market dynamics.

4.3 Volatility term structure and volatility surface

Not only does the strike price influence implied volatility, but also time to maturity. When illustrating the volatility smile, the maturity is fixed, but the strike price is not. In contrast, when illustrating the volatility term structure, the strike price is fixed but the time to maturity is variable (Derman et al., 2016). The term structure shows how the market expects short-term volatility to evolve in the future (Derman et al., 2016).

Often, market traders are dealing with many different strike prices and maturities. In that case, the volatility surface can be convenient to describe how implied volatility varies when the time to maturity and strike price are both flexible to change (Derman et al., 2016). Derman et al. (2016) describe that the volatility surface is like a summary of the options market of the same underlying. They explain that, unlike the yield curve for bonds, option pricing depends on both maturity and strike price, thus requiring a surface (figure 2) rather than a curve for illustration.

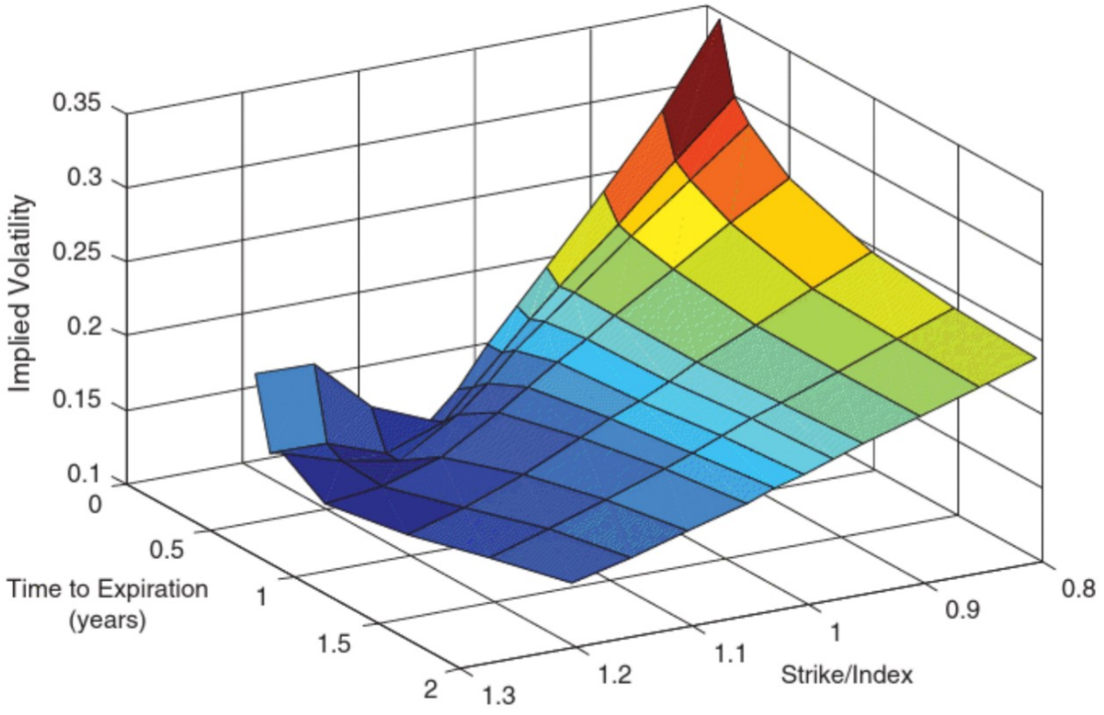


Figure 2 S&P 500 volatility surface on 31.12.2015 (Derman et al., 2015).

It can be convenient to describe the volatility surface with numbers or spreads. Volatility skew is used to determine the change in volatility between two different strike prices (Derman et al., 2016). To avoid ambiguity, this is measured as volatility points, where one volatility point is the same as the percentage point learnt in math. Rather than calculating volatility spreads between different strike prices, market participants usually calculate spreads between options with definite deltas (Derman et al., 2016). This will be focused on more when exploring delta hedging.

4.4 Reasons behind the smile

There is no unanimous answer to why the smile of equity options has that shape (figure 1). Based on past literature, the reasons can be divided into three categories: violated background assumptions of the model, market imperfections, and supply and demand factors.

4.4.1 Violated background assumptions of the model

The inaccuracy of the Black-Scholes model has been discussed earlier, but two main assumptions of the model, which cause the volatility smile, are left out to be discussed in this section: the assumption of constant volatility and the lognormal distribution of option prices. Firstly, Black and Scholes (1973) assumed that the volatility is known and constant over time. In opposition, there is a relationship to the underlying meaning that the volatility tends to increase when the underlying falls (Sinclair, 2010). In other words, there is a negative correlation between volatility changes and stock prices, which leads to a negatively skewed stock return distribution. Sinclair (2010) further explains that the options strike prices below ATM have higher implied volatility, as if they become ATM strikes, the expected volatility is higher. Figlewski (1997) argues that the volatility can remain quite the same in the short term, but in long contracts, it should not be assumed as constant, or there can be considerable pricing errors. The volatility should not only be known and constant but also independent of the strike price and the time to maturity of any option of the stock (Derman et al, 2016). Therefore, Derman et al. (2016) state that if the model is accurate, the plot of the volatilities of options having different strike prices but the same maturity should be a flat line. As learnt, that is not the case.

Secondly, the violated assumption of the lognormal distribution of the Black-Scholes model is considered one of the reasons for the volatility smile. Since asset prices cannot be negative, the Black-Scholes model assumes stock prices to follow a lognormal distribution (Hayes, 2024a). However, the risk-neutral probability of a three (four) standard deviation decrease in the S&P 500 index is 10 (100) times more expected than under an assumption of lognormality (Jackwerth & Rubinstein, 1996). Furthermore, they calculated that such a crash in the S&P 500 two-month future happened on 19.10.1987 was a -27 standard deviation event that should happen with a probability of 10^{-160}. According to them, that kind of event should be virtually impossible, as well as an event two years later. On 13.10.1989, the S&P 500 index fell approximately 6%, which is a -5 standard deviation event that should happen once in 14 756 years since the probability is only 0,00000027 (Jackwerth & Rubinstein, 1996). An extremely rare, unpredictable event with severe consequences, after which a concoction of explanation is made to make it appear less random, is called a black swan (Taleb, 2007).

Figlewski (1997) agrees that the assumption of the lognormal distribution of the underlying stock price movements is not valid when there is more weight in the tails of the actual return distribution than in a lognormal distribution with similar variance. Commonly, this phenomenon is called fat tails (Figlewski, 1997). The fat tails can be seen in figure 3 where the solid line corresponds to the observed implied distribution and the dotted line corresponds to the lognormal distribution with the same mean and standard deviation (Hull, 2022). In practice, the fat tails show that the market is pricing great downward moves to happen more likely than the lognormal distribution indicates (Hayes, 2024a). These movements outside of considered normally expected are referred to as tail risk. Furthermore, the implied distribution expects a higher median and expected value than the lognormal distribution does (figure 3). In other words, the distribution is more concentrated around its expected value, being highly leptokurtic.

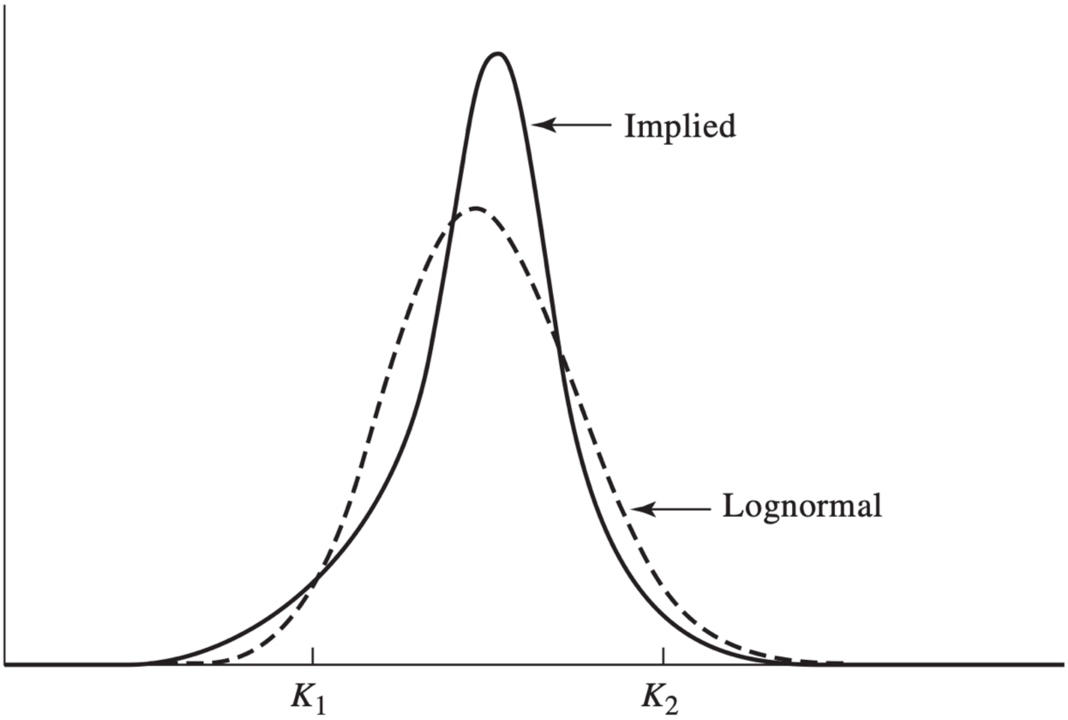


Figure 3 Implied distribution versus normal distribution of equity options (Hull, 2022).

The implied distribution does not make a great fit with the lognormal distribution (figure 3). Several clues of non-normality have been detected from market time series, which encourages to find for a more suitable alternative hypothecated distribution (Ruttiens, 2013). For example, Ruttiens (2013) presented one alternative for the Gaussian distribution to be the Poisson distribution. However, using more complex models with extra parameters can lead to error measurement, implying that the result can be less reliable than relying on the simple but robust hypothesis of normal and lognormal distributions (Ruttiens, 2013).

4.4.2 Market imperfections

Market imperfections are suggested to be one of the reasons causing the volatility smile. Option pricing can be impacted by market frictions such as transaction costs, liquidity restrictions, or short-selling restrictions, which can cause a volatility smile. Market frictions are an argument against the efficient capital markets hypothesis by Fama (1970).

A leverage effect is suggested to be one of the most common explanations of the smile. Hull (2022) explains that leverage rises as a company's equity value decreases. This implies that the stock becomes riskier and more volatile as a result. Contrary, leverage declines as the value of a company's equity rises. The equity's volatility then decreases and becomes less hazardous. According to this reasoning, it is anticipated that a stock's volatility will decline as a function of its price.

Dennis and Mayhew (2000) discovered proof that variables associated with risk, such as the stock's beta with the S&P 500 index and firm size, have an impact on the smile on individual stock options. Furthermore, they discovered that a key factor in understanding the smile of the S&P 500 index options is the put-to-call volume ratio. However, the researchers found conflicting evidence regarding the put-to-call volume ratios of specific equities. The constant variance model for the firm's returns may not be properly stated for firms with more liquidity in the underlying stock since these firms typically have steeper smiles (Dennis & Mayhew, 2000). For example, small market capitalisation options have more liquidity risk. The researchers documented that implied volatility is positively connected to volume and the correlation with the S&P 500 and is adversely associated with leverage and company size. Additionally, the study showed a statistically significant connection between the underlying stock's trading volume and implied volatility.

Rubinstein and Jackwerth (2001) introduced three possible reasons why volatility smiles occur. First, they examined market imperfections. However, they concluded that it was not the case when they focused on longer maturity options in the S&P 500 index options market, which is a rather unfettered, deep, and liquid market. This conclusion was made when they noticed that the volatility smile did not change when daily notional volume, which is sizable, increased sixfold from \$1.5 billion in 1989 to \$8.5 billion in 1995. Furthermore, they found that the S&P 500 index was rather high during 1986-1995, causing option value to be high compared to the bid-ask spread. They expect the true option price to be most often close to the mid-point quote.

Secondly, Rubinstein and Jackwerth (2001) wrote about whether option prices are measured correctly, but the implied probabilities are incorrectly calculated. For instance, a dense set of option prices spanning strike prices is obtained using the incorrect extrapolation or interpolation method. However, in the previous paper, Jackwerth and Rubinstein (1996) demonstrated that if there are enough strike prices, such as 15, the method that is used does not matter markedly because most methods produce roughly the same risk-neutral distribution. Thirdly, Rubinstein and Jackwerth (2001) wrote that the observed option prices can be systematically distorted, so one might profit from the mispricing in the options market.

Ederington and Guan (2002) suggested that the smile is not incompatible with market efficiency. The smile may arise because the volatilities using the Black-Scholes model are calculated incorrectly. If one buys options at the bottom of the smile and sells at the top of the smile in a delta-neutral ratio, the profit should be zero before the transaction costs. In contrast, the researchers found that this strategy makes pre-transaction cost profits. As a result, the true correctly calculated smile is somewhat flatter than the smile the Black-Scholes model presents, but still far from truly flat. This strategy offers significant profit only if the positions are held for a week or longer (Ederington & Guan, 2002). Nevertheless, when the trading period is long, the portfolio does not remain risk-free due to lost delta neutrality. The risk can be reduced by rebalancing the portfolio, but then the transaction costs, like brokerage commissions and bid-ask spreads, will cut the profits (Ederington & Guan, 2002).

4.4.3 Supply and demand factors

Fundamentals in economics, supply and demand factors, can create the volatility smile. The demand for options may change based on market sentiment, news events, or investor expectations. Incorporated with the supply and demand effect, the higher demand for options at specific strike prices increases the implied volatility, which creates a volatility smile (Sinclair, 2010).

Ederington and Guan (2005) suggested that hedging pressure is seen as largely responsible for the smile rather than market imperfections like bid-ask spreads and non-synchronous prices. They explained that the excess demand for low strike put options to hedge the long position of the underlying should increase the price and implied volatility of those strike prices. Combined with the put-call parity, the smile has occurred.

Sinclair (2010) presented static causes and dynamic causes as being responsible for the volatility smile. The hedging pressure is a static cause since it would exist even without the trading of the underlying. In addition to this insuring or hedging pressure, strike prices above the at-the-money have lower implied volatility because people who own stocks may want to sell calls, causing selling pressure. When the realised volatility of the underlying correlates with the movement of the underlying, that is referred to as a dynamic cause (Sinclair, 2010). In contrast to many commodities, considering equities, volatility often increases when the price of the underlying decreases and vice versa (Sinclair, 2010). To conclude, excess demand for out-of-the-money options relative to at-the-money options causes the convexity of the smile. Sinclair (2010) explains that many traders are over-willing to pay a few cents for deep out-of-the-money options, which is a poor bet, but the potential payoff is markedly appealing. Moreover, there are even more traders that will never short these options, causing an asymmetry supply and demand scenario that raises implied volatilities (Sinclair, 2010). Lastly, due to the implied change of a takeover, higher strikes frequently trade at a premium to the at-the-money strikes in the stock market (Sinclair, 2010).

Rubinstein (1994) examined how the Black-Scholes model worked in pricing the S&P 500 index options before and after the crash in 1987. Rubinstein (1994) suggests that “crash-o-phobia” may be one of the reasons behind the occurrence of the volatility smile. To clarify, options are priced under traders' concerns about the likelihood of a crash identical to the one in October 1987. This theory has some empirical support mentioned by Hull (2022). Hull (2022) explains, the declines of the S&P 500 typically coincide with a steepening of the volatility skew. Moreover, the skew tends to become less steep as the S&P 500 rises.

Changes in interest rates, dividend payments, and maturity can cause extra supply or demand, further causing the smile due to asymmetry. If interest rates increase, it decreases the present value of an option, causing call options to be less valuable and put options to be more valuable. As discussed earlier, the call (put) option value is decreased (increased) if the dividend payment is higher than expected. Moreover, the effect is more significant the closer the dividend payment is. This is explained through the time value. In addition, time to maturity explains the smile. The shorter the time to maturity, the higher the implied volatility. When trading, it is important to know that implied volatility increases when the maturity date is approaching. This term structure can be explained as follows: when there is still a longer time till maturity, the movements in the underlying price have still time to smooth out, unlike in the short term. In other words, the longer the time till maturity, the greater the probability of eventually being ITM.

5 Delta hedging

As previously mentioned, one of the uses of options is risk management. One well-known and relatively simple risk-management method is delta hedging. Thus, in practice, having effective delta hedges is challenging due to the volatility smile. To conclude, this chapter focuses on how volatility smile affects delta hedging and finally introduces one practical solution for better hedging performance, coupled with hedging performance observations from past literature.

5.1 Introduction to delta hedging

The Black-Scholes model developers applied the concept of hedging when the model was derived (Sinclair, 2010). Hedging is a method to minimise risks that an investor does not want to take and simultaneously to provide exposure to the ones the investor wants to take (Sinclair, 2010). Delta hedging an option aims to reduce directional risk (changes in the price of the underlying) while incurring little cost (Sinclair, 2010). Mitigating the delta risk is an options trading strategy that can be used for both single stocks and entire portfolios. This strategy is commonly used by investment companies and institutional investors.

To understand delta hedging, the delta (Δ) needs to be explained. Delta is the most important “Greek letter / Greek” for hedging and pricing options. It is the ratio of a stock option's price change to its underlying stock price change (Hull, 2022). In other words, delta is the number of shares of the stock that should be held for each shorted option to build a riskless portfolio.

As follows (11), the delta can also be seen as the slope of a tangent where the price of the option is compared to the price of the underlying (Sinclair, 2010).

$$\varDelta = \frac{\partial c}{\partial S} \approx \frac{c(S_2)-c(S_1)}{S_2-S_1}, \tag{11}$$

where ∂ refers to the partial derivative.

Delta is not constant, so traders need to rebalance their portfolio by buying or selling the underlying stock frequently if they want to retain the portfolio delta neutral (delta is equal to zero). Thus, if there were no market frictions, the portfolio should always be kept delta neutral, meaning the profit is the function of the difference between the implied volatility and the realised volatility (Sinclair, 2010). When rebalancing regularly, market frictions, such as trading costs, cut profits. In contrast to rebalancing the portfolio regularly, which is known as dynamic hedging, rebalancing only once is known as static hedging, also sometimes referred to as "hedge-and-forget" (Hull, 2022).

5.2 The impact of volatility smiles on delta hedging

It is reasonable to assume that the volatility smile influences not only the pricing of options but also delta hedging, making it more challenging in practice. The iterated implied volatility has raised the volatility smile, which further leads to a non-linear relationship between an option’s delta and the underlying asset’s price. In practice, there is a negative relationship between equity price and its volatility, which affects delta hedging. Therefore, if the BS delta (11) is used for hedging, it does not minimise the variance of the position (Hull & White, 2017). In literature, this is called smile risk.

However, using the BS delta for hedging might not always be inappropriate. Derman (1999) had intuitions of volatility behaviours in three different market conditions, referred to as regimes of volatility. The researcher stated that in stable market conditions, volatility tends to be independent of the underlying index levels, implying $\varDelta_{BS} = \varDelta$.

Meaning, the BS delta is appropriate for delta hedging since it is the same as the delta of the option.

In trending and jumpy markets, the use of the BS delta may be inappropriate according to the following reasoning by Derman (1999). In trending market conditions, there may be a positive relationship between the index level and volatility, implying $\Delta_{BS} > \Delta$. Finally, in jumpy market conditions, that is also referred to as highly volatile market conditions, $\Delta_{BS} < \Delta$ because volatility and index levels tend to move in opposite directions. If volatility is time-varying and correlates with the return of the underlying, like in the previous two cases, the delta must also control the indirect impact of simultaneous change in volatility alongside the direct impact of underlying price change on the option price (Vähämaa, 2004). The importance of this correlation is also empirically emphasised by Bakshi et al. (2000), Coleman et al. (2000) and Lim et al. (2002) when they documented that deltas differentiate from the BS delta.

This inconsistency has led researchers to further examine better methods to calculate the delta so that it considers the inverse movements of volatility and underlying price, discovered as a smile. Hull and White (2017) write that the smile can be adjusted into the BS model by minimum variance delta (MV delta), which takes account of both price changes and the expected change in volatility conditional on the price change. Hull and White (2017) write that several researchers have implemented stochastic volatility models by changing the basic delta to MV delta by applying the assumptions of the models. For example, Bakshi et al. (1997, 2000), Kim and Kim (2004), Lim and Guo (2000) and Nandi (1998) have studied the hedging performance with stochastic volatility models. Those researchers found that it produces an improvement in delta hedging performance, especially for OTM options (Hull & White, 2017).

Even though stochastic volatility models outperform the BS model in terms of pricing, it is often too demanding to implement for hedging purposes and only outperforms when OTM options are hedged (Bakshi et al., 1997). Moreover, the Black-Scholes model has been shown to outperform deterministic volatility models in delta hedging, which are studied for example by Dumas et al. (1998), Engle and Rosenberg (2000), Lim and Zhi (2002), and Yung and Zhang (2003). Further, considering the daily volatility smile correctly in the model calibration has more effect on delta hedging outcomes than the exact choice between models (Aleksander & Kaeck, 2012). Therefore, with a simple smile-effective model, more effective delta hedging should be possible. Besides finding the MV delta through stochastic volatility models or deterministic volatility models, there is also another way to find the MV delta with vega-based local volatility models. Vähämaa (2004) introduced a model-independent extension to the Black-Scholes model, smile-adjusted delta (SAD), to minimise the smile risk while retaining the methodology simple to exercise in practice.

Considering local volatility models, the MV delta can be found when noting that the Black-Scholes delta and vega times the partial derivative of the anticipated implied volatility concerning the asset price produce the delta of the minimum variance (Hull & White, 2017). Implying, to improve the delta, assuming the partial derivative of the expected implied volatility concerning the asset price is necessary (Hull & White, 2017). In addition to Hull and White (2017), researchers Alexander and Imeraj (2023), Alexander et al. (2012), Attie (2017), Coleman et al. (2000), Crépey (2004), François & Stentoft (2021) and Vähämaa (2004) have investigated the MV delta with local volatility models. To narrow down the focus to one methodology, the construction of the smile-adjusted delta by Vähämaa (2004) is covered in the next subchapter. Nevertheless, observations from the other research papers are also discussed when closely related to Vähämaa's (2004) research paper.

5.3 More optimal delta hedging with smile-adjusted delta

Vähämaa (2004) empirically examined delta hedging with the smile. The researcher states that hedging performance with the Black-Scholes model can be improved by simply adjusting the delta. According to the researcher, the delta can be adjusted to account for the inverse movements between volatility and stock prices. Next, the formula of the smile-adjusted delta is formed step by step. As follows (12), Vähämaa (2004) presented the Black-Scholes model delta for a European non-dividend paying call option.

$$\delta_{BS} = \frac{\partial c(S,K,\sigma,r,T)}{\partial S} = N(\frac{ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}), \tag{12}$$

Where other factors are already mentioned, except $c(x)$, which denotes the BS call option pricing formula.

As learnt, the implied volatility from the BS model is smiling, and so does the delta. Therefore, the delta should be adjusted. The delta should not only be affected directly by the change in the price of the underlying but also indirectly by the volatility change, which is correlated with the underlying price change (Vähämaa, 2004). When assuming the volatility to be a deterministic function of S, K, and T, the chain rule gives the formula (13) below (Vähämaa, 2004).

$$\delta = \frac{\partial c}{\partial S} + \frac{\partial c}{\partial \sigma}\frac{\partial \sigma}{\partial S} \tag{13}$$

In the formula (13) term $\frac{\partial c}{\partial \sigma}$ is the vega of the option. Shortly, vega (ν) is a Greek letter which describes the rate of change in the value of an option to the implied volatility. The obtained vega is always positive. Therefore, if there is a negative correlation between stock returns and changes in volatility, the BS delta is greater than it should be (Vähämaa, 2004). The last term $\frac{\partial \sigma}{\partial S}$ from equation (13) is difficult to quantify, but for example, Rubinstein (1994) and Derman (1999) suggested that it can be approximated by the slope of the smile, implying $\frac{\partial \sigma}{\partial S} = \frac{\partial \sigma}{\partial K}$ (Vähämaa, 2004). By substituting that into equation (13), the formula (14) for smile-adjusted delta is obtained.

$$\delta_{SAD} = \frac{\partial c}{\partial S} + \frac{\partial c}{\partial \sigma}\frac{\partial \sigma}{\partial K} = \delta_{BS} + \nu_{BS}\frac{\partial \sigma}{\partial K} \tag{14}$$

Vähämaa (2004) explains that the formed smile-adjusted delta considers the correlation between volatility changes and underlying stock returns by applying the volatility smile paired with the option's vega to the basic BS delta equation. Therefore, when the smile has a form of a smirk (downward slope), the BS delta is adjusted downwards to offset the movement between the underlying and the volatility. Thus, if the smile is flat, the BS delta and smile-adjusted delta are the same.

Vähämaa (2004) empirically tested this smile-adjusted delta for the European-style FTSE 100 index options, which is the most active equity options market in the United Kingdom. The final sample consisted of options with 5 to 120 trading days to expiration and moneyness between 0,90 and 1,10 during the period 2.1.2001 - 29.12.2001. That period consisted of a trending market regime in May and a jumpy market regime in September, caused by the World Trade Center Attacks (Crépey, 2004).

Vähämaa (2004) documented the smile-adjusted delta’s constant smaller hedging errors and outperformance against the Black-Scholes delta in hedging performance. The outperformance was most striking for short-term OTM and ATM options and more visible as the horizon of hedging lengthens. Considering results from different market regimes, the smile-adjusted delta had a smaller standard deviation than the BS delta in May, but that difference was even more significant in September. This implies the outperformance of smile-adjusted delta in trending market conditions and even more significant outperformance in jumpy market conditions.

To conclude Vähämaa’s (2004) findings, the investigation of the smile adjustments has significant practical relevance because the sample moneyness of OTM and ATM options are the most traded options in the index option market, and smile-adjusted delta is easy to implement for practice (Vähämaa, 2004). The paper also reported the importance of the correlation between stock returns and volatility changes for risk management

(Vähämaa, 2004). Because index option volatility smiles generally slope downwards and vega is always positive for both calls and puts, smile-adjusted deltas are consistently smaller than the corresponding BS deltas (Vähämaa, 2004). Vähämaa (2004) explains that the gap between the deltas is greatest among ATM options because the vega is also largest for ATM options. Implying, relative to the Black-Scholes delta, traders should under-hedge equity call options and over-hedge equity put options (Hull & White, 2017).

In addition to Vähämaa (2004), there are also other research papers about smile-adjusted deltas that are methodologically comparable. Crépey (2004) presents similar results from DAX and the same FTSE 100 index as Vähämaa (2004). The sample period was 1999-2000 for the FTSE 100 index and 2001 for DAX. Crépey's (2004) smile-adjusted delta, which is referred to as the local delta, should be preferred over the BS delta in negatively skewed markets when the physical underlying process and the risk-neutral process are also negatively skewed. According to the research, the more physical negative skewness of jumpy markets compared to stable and trending markets might be the explanation why Vähämaa (2004) concluded a more significant outperformance of smile-adjusted delta compared to the BS delta in jumpy markets.

Coleman et al. (2000) compared the performance of the local delta to the BS delta. They observed that the BS delta is typically larger than the local delta in both the S&P 500 index options and futures options markets. Moreover, they found that the average hedging errors were always smaller using the local delta. Like Vähämaa (2004), Coleman et al. (2000) found absolute hedging errors to be smaller as the hedge horizon is sufficiently long. Moreover, Alexander et al. (2012) did the same comparison with a 16-year sample from FTSE 100 index options. They discovered that the errors are almost half the size of the BS delta errors and just slightly less when hedging with very short-term options. Furthermore, they documented that regardless of rebalancing horizons, the effectiveness is greatest in volatile markets.

To compare, results from Vähämaa (2004), Coleman et al. (2000), Alexander et al. (2012) and Crépey (2004) all support Derman’s (1999) intuition that the delta of the option is lower than the BS delta in jumpy markets. Crépey’s (2004) results also support Derman’s (1999) intuition of smile-adjusted deltas’ superior outperformance only during extremely volatile periods. However, when Vähämaa (2004) and Coleman et al. (2000) found outperformance also during trending markets, the intuition is not clear for trending market regimes due to mixed results. To conclude, smile-adjusted/local delta outperforms clearly on the jumpy market regime but not so certainly in trending markets.

Nevertheless, there is more recent research focusing on the trending market conditions. Attie (2017) examined the delta hedging performance of three different smile-adjusted deltas during the sample period of 1.9.2013-31.8.2014 in the S&P 500. During the research period, the S&P 500 had an upward trend, so the trending market performance of the smile-adjusted delta is retested. According to Attie (2017), the mean error is over 30% higher for the Black-Scholes delta compared to the three different smile-adjusted models that provide quite similar hedging performance. To conclude, Derman’s (1999) intuition that smile-adjusted delta should outperform the BS delta only during volatile market conditions has even more contradicting observations.

More recently, the delta hedging performance with smile-adjusted delta has also been examined with a significantly different asset class, cryptocurrencies. Alexander and Imeraj (2023) researched the performance of different kinds of smile-adjusted deltas among bitcoin options during 1.1.2020-1.1.2022. They found that with smile adjustments, it is possible to gain over 40% efficiency gains compared to the Black-Scholes delta among OTM options. They also found that the smile of bitcoin options can be relatively symmetric since the bitcoin price does not follow similar price movements as the S&P 500 and other equity indexes. Equity indexes tend to have smaller upward changes in trending markets and then suddenly crash, but bitcoin price changes can be large in both directions regularly.

Poor hedging performance for smile-adjusted delta has also been documented. François and Stentoft (2021) had a large-scale sample that consisted of S&P 500 index options from 4.1.1996 to 29.4.2016. Interestingly, they documented the outperformance of the Black-Scholes delta against all the considered approaches, including the smile-adjusted delta and more sophisticated models.

6 Data and methodology

In this chapter, the data used in this research is discussed and presented with descriptive statistics, and the inverse correlation of the index and implied volatility is revealed. Like Vähämaa (2004), the index option data is divided into 12 categories in which the hedging performance is analysed. In addition to explaining the methodology of the construction of delta-hedged portfolios and the calculation of hedging errors, the volatility smiles are also separately documented for put and call options.

6.1 Data

Recently, the research focus has been on more complex volatility models than the vega-based local volatility models, smile-adjusted/local delta. Thus, it can be noted that the topic has only a few more recent research papers, and none of those have a sample period after 2016 for index options. Naturally, there is then no published research on the topic during the recent COVID-19 crisis among index options. Further, since risk management becomes more crucial in such volatile market periods, researching delta hedging performance should be appealing again, as explained in the purpose of the study section. In this thesis, my contribution is to apply Vähämaa's (2004) methodology and fill the earlier-mentioned research gap with one of the most followed stock indexes, with S&P 500 index options during the COVID-19 year of 2020.

That said, this thesis researches the hedging performance of the smile-adjusted delta compared to the Black-Scholes delta in the United States during the first full COVID-19 year of 2020. The chosen S&P 500 index is a float-adjusted market capitalisation-weighted stock index that tracks the performance of the leading 500 companies in the United States (S&P Global, n.d.). The index committee considers both quantitative and qualitative aspects when including companies. The assembled index covers approximately 80% of available market capitalisation in the United States, consisting of large-capitalisation companies in diverse sectors. For this thesis, the daily closing prices of the S&P 500 index for 2020 are gathered from Yahoo Finance.

Since the prices of S&P 500 index options (ticker: SPX) are derived from the S&P 500 index, SPX can be used in delta hedging the stock index. SPX is a cash-settled European-style index option, so the risk of early settlement is eliminated. The settlement date is the third Friday of the month. Trading with SPX ends the day before expiration, and the settlement time is AM (opening price). Unfortunately, this exposes the hedger to an additional last trading day overnight volatility, which will be further discussed in the limitations of the study section. To continue, the SPX settlement prices are gathered from DataStream.

In addition to the daily closing prices of the index and the option settlement prices, the risk-free rate needs to be proxied. Like Vähämaa (2004), the risk-free rate is proxied by a 3-month interest rate related to the sample country. Thus, daily data of the U.S. 3-month treasury bill rate gathered from DataStream is used as the risk-free rate.

The descriptive statistics of the S&P 500 index and the 1-month ATM implied volatility series are presented in table 1. In the year of 2020, the S&P 500 index ranged between 2 237,40 and 3 756,07 with a mean continuously compounded daily return of nearly zero, and 0,0219 as the standard deviation of daily returns. To convert this standard deviation or daily volatility to annualised volatility, it is multiplied by the square root of 252, the trading days of the year. Calculated annualised volatility is then 34,77%, which is much higher than the mean ATM implied volatility of 28,28%. This indicates that ATM options are relatively inexpensive and that markets expect a decrease in volatility. This is reasonable since the largest market shock was in March, and the months after that were bullish and less volatile. Nevertheless, converting daily volatility to annualised volatility as mentioned above unrealistically assumes that volatility is constant, and returns are independent and normally distributed. To clarify, for the sample, volatility is time varying (figure 4) and strike varying (figure 5).

Furthermore, the assumption of normal distribution is also contradicted in the sample. Table 1 indicates that the S&P 500 index has a relatively symmetric distribution in levels, but in logarithmic first differences, the distribution is skewed to the left. In comparison, ATM implied volatility is skewed to the right in both calculations. Logarithmic first differences of the S&P 500 show that the distribution is highly leptokurtic. To conclude, the distribution assumptions do not hold with the sample data.

Table 1 Descriptive statistics.

Panel A: Level

	S&P 500 Index	*ATM Implied Volatility*
Mean	3 217,86	28,28%
Median	3 276,02	25,82%
Minimum	2 237,40	10,20%
Maximum	3 756,07	88,20%
Standard Deviation	319,23	13,10%
Skewness	-0,70	1,94%
Kurtosis	0,19	4,83%

Panel B: Logarithmic first differences

	S&P 500 Index	*ATM Implied Volatility*
Mean	0,0006	0,0020
Median	0,0024	-0,0073
Minimum	-0,1277	-0,3063
Maximum	0,0897	0,4020
Standard Deviation	0,0219	0,1033
Skewness	-0,8662	0,9343
Kurtosis	8,6631	2,4826

Panel C: Correlation

	S&P 500 Index	*ATM Implied Volatility*
S&P 500	1,00	-0,79
ATM Volatility	-0,79	1,00

Figure 4 shows the level of the S&P 500 index and the level of ATM implied volatility. ATM implied volatility is calculated by using the three shortest maturity option series, always calculated by the two shortest maturity options of those which still have at least 5 days to maturity. Figure 4 shows a strong inverse correlation between the index level and volatility, which supports why the approximation $\frac{\partial\sigma}{\partial S} = \frac{\partial\sigma}{\partial K}$ can be made. Further, this is confirmed by the negative correlation between index returns and volatility changes shown in panel C of table 1.

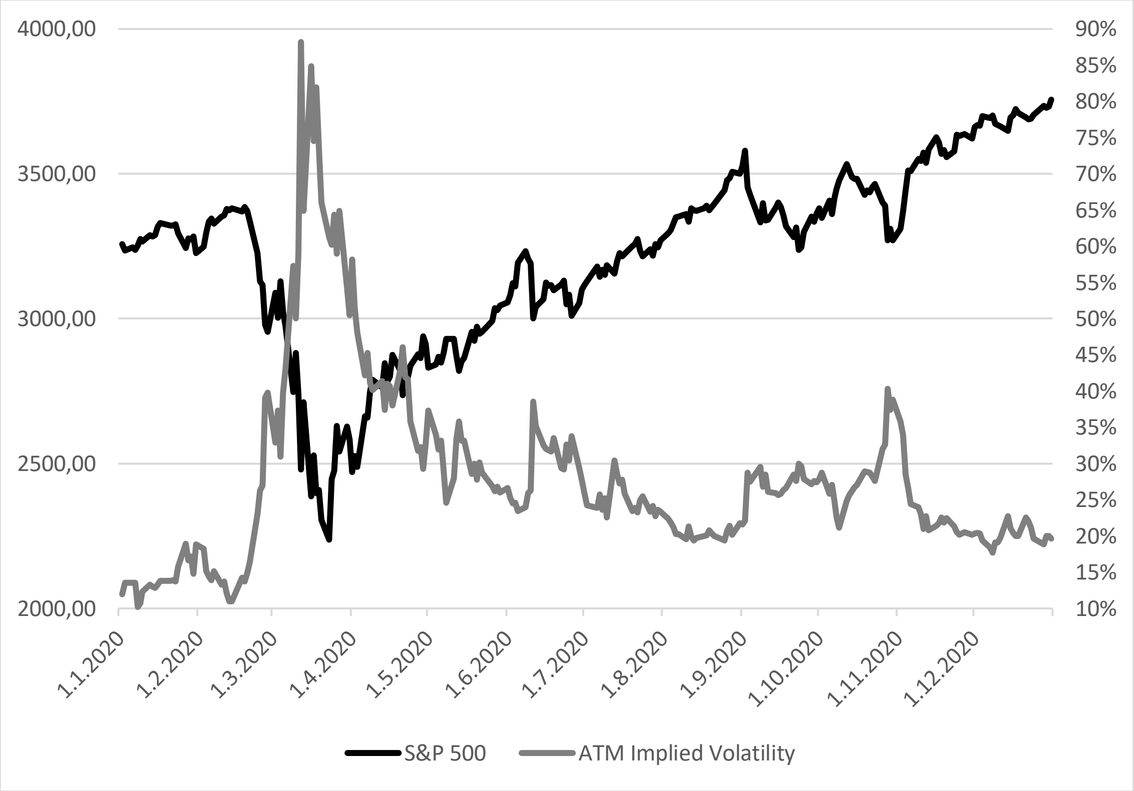


Figure 4 S&P 500 index and ATM implied volatility.

To make the SPX sample more specific, the same sample criteria as Vähämaa (2004) are applied. That said, only contracts with a maturity of 5-120 trading days and moneyness 0,90-1,10 are included, which leads to the final sample of the most actively traded option contracts. Similarly, as Vähämaa (2004), the contracts will be divided into categories based on maturity and moneyness. An option is considered out-of-the-money if its moneyness is less than 0,97, at-the-money if it is between 0,97 and 1,03, and in-the-money if it is more than 1,03. An option is considered short-term if its maturity is less than 40 days to expiration and long-term otherwise. As a result, there are 12 categories for the empirical analysis.

In table 2, summary statistics of each 12 categories are reported for the final sample containing 79 417 call and 125 907 put option settlement prices. In addition to both option types, both maturity classes are well represented, including 83 336 short-term and 121 988 long-term settlement prices. By moneyness classes, OTM options are slightly more represented (74 493) than ATM (64 025) and ITM options (66 806). To conclude, the sample size in total is large, including 205 324 settlement prices to contrast 35 180 that Vähämaa (2004) had. This should improve the reliability of the results. Nevertheless, one more important observation from table 2 is that the average option price increases with moneyness and maturity, as Vähämaa (2004) also expected and observed.

Table 2 S&P 500 index option sample.

Moneyness	Time to Maturity		
	Short	Long	Subtotal
OTM average settlement price	28,61	83,21	
OTM average implied volatility	31,41%	29,35%	
OTM total number of observations	30 214	44 279	74 493
ATM average settlement price	87,24	157,88	
ATM average implied volatility	27,78 %	27,58%	
ATM total number of observations	26 104	37 921	64 025
ITM average settlement price	227,28	281,34	
ITM average implied volatility	28,30%	27,34%	
ITM total number of observations	27 018	39 788	66 806
Subtotal	83 336	121 988	205 324

Before taking the focus into the methodology of constructing the delta-hedged portfolios and the calculation of hedging errors, the volatility smiles are documented first. In figure 5, the volatility smiles of call and put options are documented for the entire research period of 2020. The vertical axis presents the implied volatility, whereas the horizontal axis presents the moneyness of the option. The volatility smiles are obtained by calculating the averages of implied volatilities for each moneyness rounded to two decimals. In figure 5, there are no observations for 0,90 moneyness call options, which is observable as a lacking datapoint of the chart.

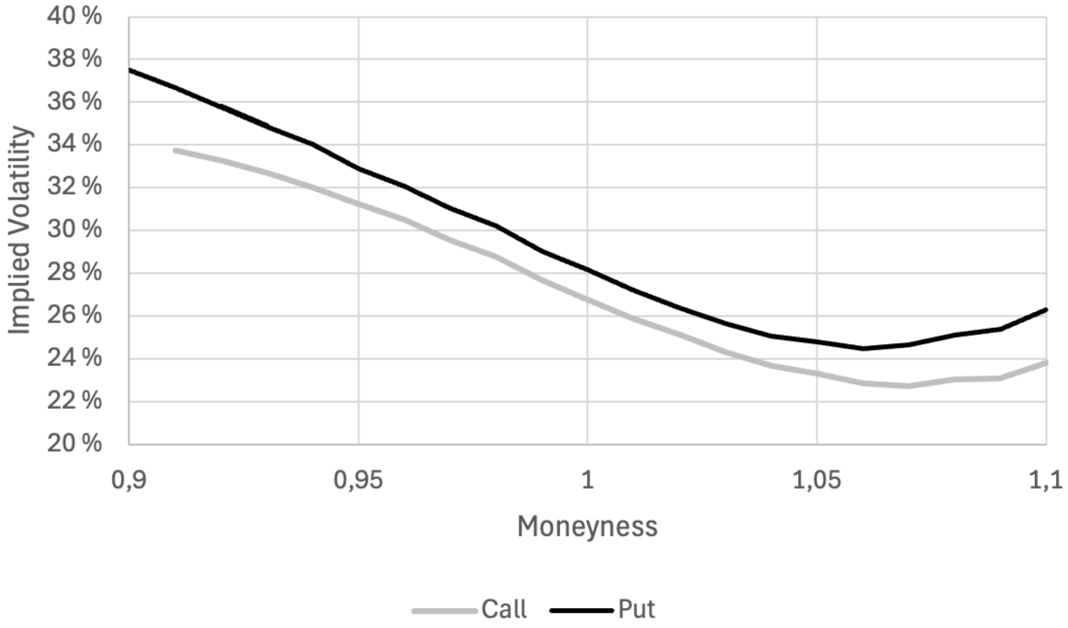


Figure 5 Average implied volatilities.

The main observation from figure 5 is that both call and put volatility smiles are asymmetric; OTM implied volatility is greater than ITM volatility. Implied volatility decreases as moneyness decreases until the moneyness is 1,07, and after that, it increases. Furthermore, the implied volatility of put options is always higher than the relative implied volatility of call options, which means that the options are skewed to the downside. Kenton (2023) explains that downside skews are typical for times when the market expects downward price movements. Kenton (2023) continues that with those expectations, investors buy put options, known as protective puts, to protect their investments. The greater demand for put options increases their implied volatilities and thus premiums of put options compared to call options.

The greater implied volatility or overpriced put options is also previously documented in the S&P 500 (see e.g. Bakshi & Kapadia, 2003; Bondarenko, 2014; Coval & Shumway, 2001; Gârleanu et al., 2009; Hull & White, 2017). Nevertheless, the findings are not in line with the reasoning from put-call parity that the volatility smile should be the same for both put and call options. The slopes of the smiles (figure 5) are rather similar compared to the one which Jackwerth (2004) documented nearly 20 years ago for the S&P 500. To mention, chapter 4.4 “Reasons behind the smile” covered the possible reasons for the observed shape of the smiles more in depth.

6.2 Methodology

The methodology of this thesis relies on the paper “delta hedging with the smile” written by Vähämaa (2004). The researcher states that hedging performance with the Black-Scholes model can be improved by simply adjusting the delta. According to the research, the delta can be adjusted to account for the inverse movements between volatility and stock prices. The formulas for the Black-Scholes delta and smile-adjusted delta were already explained in chapter 5.3 “More optimal delta hedging with smile-adjusted delta”.

As Vähämaa (2004), the quantitative fit of the previously explained BS delta and SAD are examined by applying first-order Taylor series expansion regressions (15) and (16), respectively.

$$\varDelta c = \beta_0 + \beta_1 \left[\frac{\delta c}{\delta S} \varDelta S\right] + \varepsilon \tag{15}$$

$$\varDelta c = \beta_0 + \beta_1 \left[\frac{\delta c}{\delta S} \varDelta S\right] + \beta_2 \left[\frac{\delta c}{\delta \sigma} \frac{\delta \sigma}{\delta K} \varDelta S\right] + \varepsilon \tag{16}$$

In equations (15) and (16), c denotes the option price, S is the price of the underlying, K is the strike price of the option, and $\varDelta$ denotes a change. If the model works well to estimate delta, β_0 should be slightly negative, β_1 should be one, and the regression coefficient for determination R^2should also be one. Simply, the regression which have greater R^2 values is the more accurate estimate of the correct delta. Since there is one more independent term in equation (16) compared to (15), the Wald chi-square statistic, also known as the Wald test, is applied to measure whether the new independent term is significant/insignificant for the regression.

To investigate the hedging performance of SAD and the BS delta, Vähämaa (2004) constructed a self-financed delta-hedged portfolio $\prod$ of one shorted option, δ times the underlying, and B units of the risk-free bond. Without considering transaction cost, $\prod$ should be 0 when time $t = 0$ (17).

$$\prod_t = \delta_t S_t + B_t - c_{\ t} \tag{17}$$

As Vähämaa (2004), the hedging performance of SAD and the BS delta is compared in 1-day, 5-day and 10-day hedging horizons. The portfolio will be rebalanced daily at hedge-revision time t by using the closing prices, the hedge parameter is recomputed, and the new bond position for the effective hedge is (18).

$$B_t = e^{rt} B_{t-1} + S_t(\delta_{t-1} - \delta_t). \tag{18}$$

Now the delta hedging error ε from $t - 1$ to t is calculated as follows (19).

$$\varepsilon_t = \delta_{t-1} S_t - c_t + e^{rt} B_{t-1} \tag{19}$$

Lastly, in equation (20), the total hedging error during the focused hedging horizon τ is calculated, which is the same as the value of the portfolio at the end of the hedging horizon Π_T.

$$\varepsilon_\tau \ = \ \sum_{t=1}^{T} \varepsilon_t \ = \Pi_T \tag{20}$$

As Vähämaa (2004), to measure the delta hedging effectiveness of SAD to the BS delta, two commonly used error statistics are applied: mean absolute hedging error (MAHE)

(21) and root mean square hedging error (RMSHE) (22), both measured in dollars. To make the methodology more robust without violating distribution assumptions, bootstrapping is used to measure the statistical significance of the difference between SAD and the BS delta.

$$MAHE = \frac{1}{n}\sum_{i=1}^{n}\left|\varepsilon_{t_i}\right| \tag{21}$$

$$RMSHE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\varepsilon_{t_i}^{2}} \tag{22}$$

7 Results

In this chapter, the significant differences between the Black-Scholes delta and smile-adjusted delta are documented, and the quantitative fit of both deltas is tested. Next, the results from the delta hedging performance of both deltas measured by mean absolute hedging error (MAHE) and root mean square hedging error (RMSHE) are reported in 1-day, 5-day and 10-day hedging horizons. Moreover, 1-day hedging performance is also reported in stable and highly volatile market conditions.

7.1 Observed differences between the deltas

Table 3 compares the deltas calculated with the Black-Scholes (BS) model and smile-adjusted delta (SAD) method for both call and put options. Since the observed volatility smile of equity options is usually downwards sloping, as documented in figure 5, and vega is positive for both call and put options, the SAD is always smaller than the corresponding BS delta. The data is organised not only by option type but also by moneyness. For the entire sample, the average delta for both call and put options differs by 0,09. Looking closer, the differences are slightly greater for ATM options being 0,10. This is rational since the vega is the largest among ATM options (Vähämaa, 2004). Overall, the results show even greater differences between the deltas than what Vähämaa (2004) documented.

Table 3 Comparison of the Black-Scholes and smile-adjusted deltas.

Moneyness	Call Options			Put Options		
	BS	SAD	Difference	BS	SAD	Difference
Full Sample	0,48	0,40	0,09	-0,48	-0,57	0,09
OTM	0,21	0,14	0,07	-0,25	-0,33	0,08
ATM	0,51	0,41	0,10	-0,48	-0,58	0,10
ITM	0,75	0,66	0,08	-0,76	-0,83	0,08

Figure 6 presents the stability and co-movement of the Black-Scholes (BS) and smile-adjusted delta (SAD) during the research period. The deltas are presented for the call option with the second-shortest maturity and the strike price closest to the value of the underlying index on the observation day.

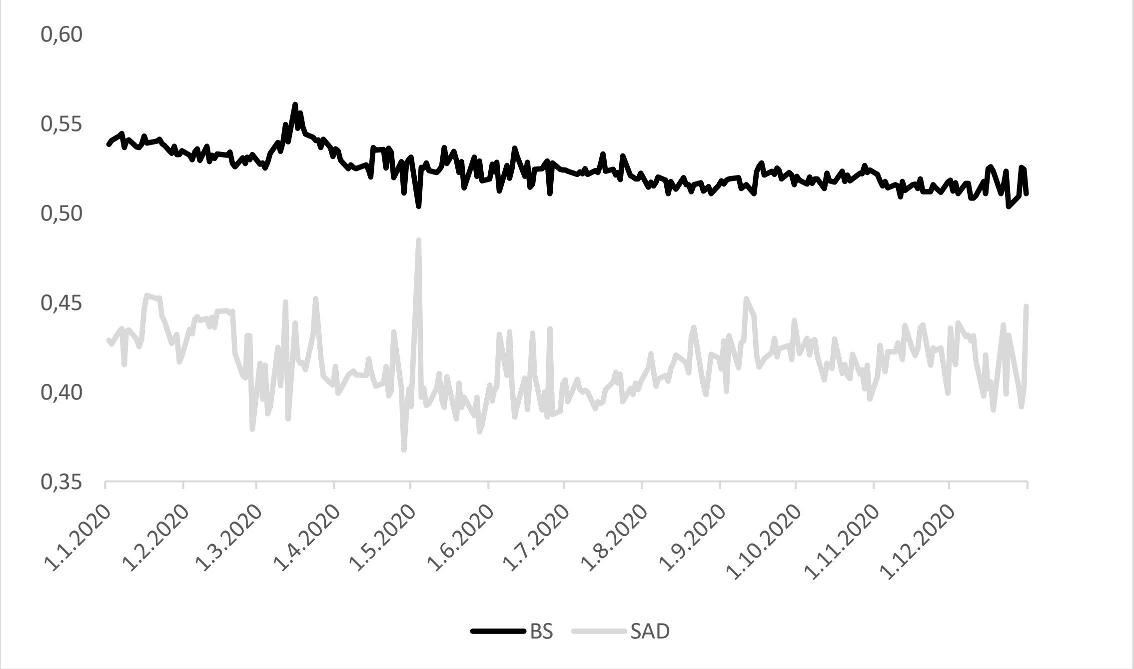
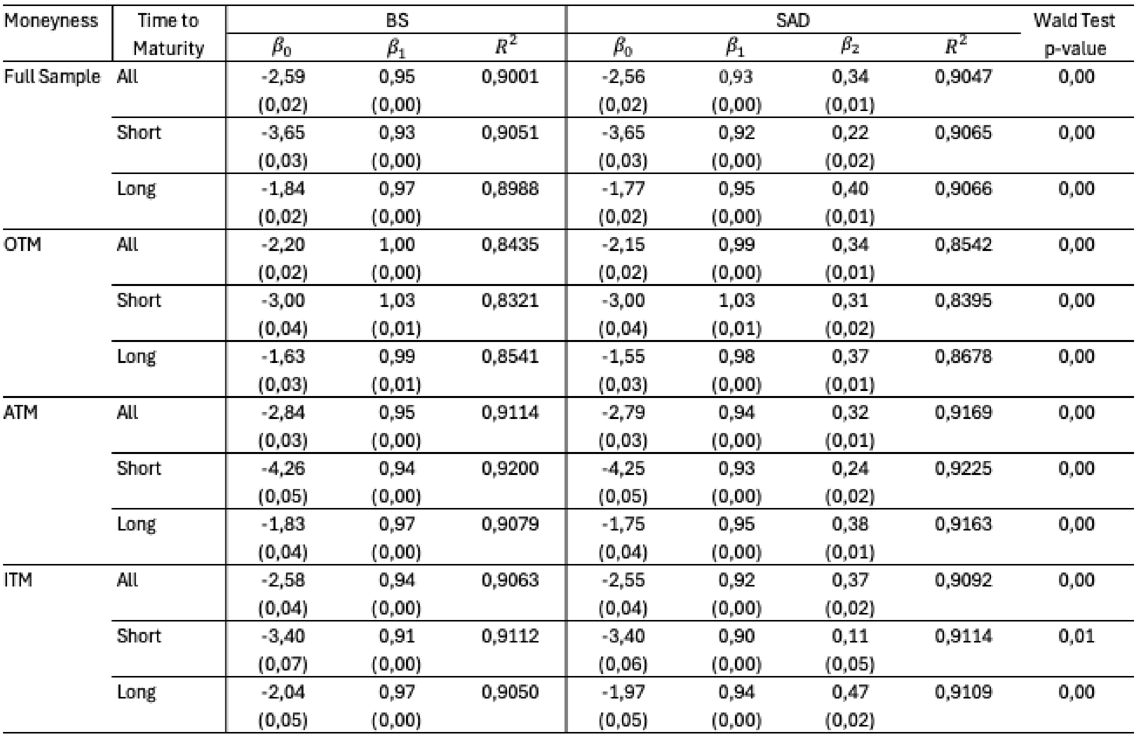


Figure 6 Comparison of the Black-Scholes and smile-adjusted deltas.

Figure 6 shows that the deltas move in the same direction. The daily movement of the BS delta appears relatively stable, but the daily movement of SAD is greater than that. This is reasonable to observe since SAD considers the volatility sensitivity of the option, which the BS delta does not. The BS delta takes one positive spike in March, after which it has a downward trend. In contrast, SAD had the largest negative spike in late April, after which it had the largest positive spike in early May, resulting in the smallest difference between the BS delta and SAD during the research period.

Similarly, as Vähämaa (2004), table 4 concludes the regression results of equations (15) and (16) in the previously explained 12 categories. As before, BS denotes the Black-Scholes delta, and SAD denotes smile-adjusted delta. R^2 denotes the adjusted $R^2$of the regressions. White's heteroskedasticity-consistent standard errors are present in parentheses below each coefficient estimate. To test the significance of the new β_2 term on the basic BS equation, the Wald chi-square test results are presented with the null hypothesis $\beta_2 = 0$.

Table 4 Quantitative fit of the Black-Scholes and smile-adjusted deltas.



Moneyness	Time to Maturity	BS			SAD				Wald Test p-value
		β_0	β_1	R^2	β_0	β_1	β_2	R^2	
Full Sample	All	-2,59 (0,02)	0,95 (0,00)	0,9001	-2,56 (0,02)	0,93 (0,00)	0,34 (0,01)	0,9047	0,00
	Short	-3,65 (0,03)	0,93 (0,00)	0,9051	-3,65 (0,03)	0,92 (0,00)	0,22 (0,02)	0,9065	0,00
	Long	-1,84 (0,02)	0,97 (0,00)	0,8988	-1,77 (0,02)	0,95 (0,00)	0,40 (0,01)	0,9066	0,00
OTM	All	-2,20 (0,02)	1,00 (0,00)	0,8435	-2,15 (0,02)	0,99 (0,00)	0,34 (0,01)	0,8542	0,00
	Short	-3,00 (0,04)	1,03 (0,01)	0,8321	-3,00 (0,04)	1,03 (0,01)	0,31 (0,02)	0,8395	0,00
	Long	-1,63 (0,03)	0,99 (0,01)	0,8541	-1,55 (0,03)	0,98 (0,00)	0,37 (0,01)	0,8678	0,00
ATM	All	-2,84 (0,03)	0,95 (0,00)	0,9114	-2,79 (0,03)	0,94 (0,00)	0,32 (0,01)	0,9169	0,00
	Short	-4,26 (0,05)	0,94 (0,00)	0,9200	-4,25 (0,05)	0,93 (0,00)	0,24 (0,02)	0,9225	0,00
	Long	-1,83 (0,04)	0,97 (0,00)	0,9079	-1,75 (0,04)	0,95 (0,00)	0,38 (0,01)	0,9163	0,00
ITM	All	-2,58 (0,04)	0,94 (0,00)	0,9063	-2,55 (0,04)	0,92 (0,00)	0,37 (0,02)	0,9092	0,00
	Short	-3,40 (0,07)	0,91 (0,00)	0,9112	-3,40 (0,06)	0,90 (0,00)	0,11 (0,05)	0,9114	0,01
	Long	-2,04 (0,05)	0,97 (0,00)	0,9050	-1,97 (0,05)	0,94 (0,00)	0,47 (0,02)	0,9109	0,00

The results from table 4 reveal that the smile-adjusted delta (SAD) consistently provides higher R^2 values compared to the Black-Scholes delta (BS) across every moneyness and maturity category. For the full sample, R^2 under the BS delta is 0,9001, while R^2 for SAD improves to 0,9047. This means that the BS delta can explain 90,01% and SAD 90,47%

of the observed option price changes for the full sample. Overall, both deltas have relatively high R^2 values ranging from 0,8321 to 0,9200 for the BS delta and from 0,8395 to 0,9225 for SAD. The coefficient smile-adjustment term β_2 is highly statistically significant across every moneyness and maturity category. Moreover, the Wald chi-square test demonstrates that there is a statistically significant difference between the models, meaning that SAD is a significant improvement over the BS delta.

The differences between the two deltas are most pronounced for OTM and ATM options, where the SAD exhibits a significantly better fit. For ITM options, the improvement is relatively smaller but still significant. Both models provide better quantitative fits for ATM and ITM options compared to OTM options. Differences in quantitative fit are greater for long-term options compared to short-term ones.

The previously mentioned main result that SAD outperforms the BS delta in quantitative fit is in line with what Vähämaa (2004) observed. Nevertheless, both models' quantitative fit has been better when pricing FTSE 100 index options during 2001 compared to pricing S&P 500 index options in 2020. The difference in the quantitative fit for the full sample is 6,50%-points for the BS delta and 6,70%-points for SAD. One possible reason for this may be found in the steepness of smiles, since the volatility smiles in the United Kingdom have been historically flatter than in the S&P 500, as Tompkins (2001) documented. Nevertheless, compared to Vähämaa (2004), who documented the best quantitative fit to be among ITM options, the quantitative fit of both models in this research is highest for ATM options, slightly lower for ITM options and notably worse for OTM options.

7.2 Comparison of the hedging performances in different hedging horizons

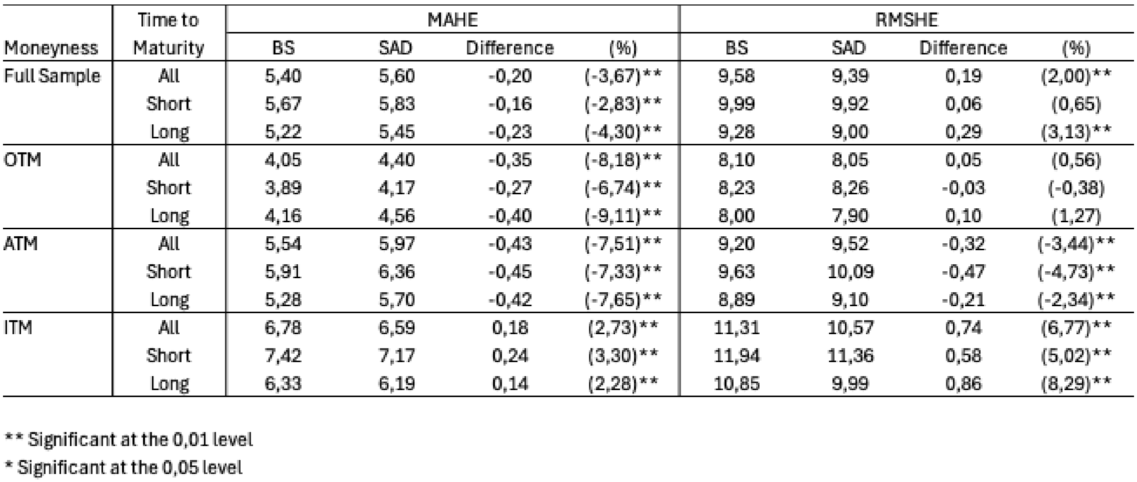
After proving the better quantitative fit of smile-adjusted delta (SAD) compared to the Black-Scholes delta (BS), the hedging performance of both deltas is explored similarly to Vähämaa (2004). In contrast to Vähämaa’s (2004) results, regardless of moneyness or maturity, the mean hedging error is found to be slightly lower under SAD than under the BS delta. Instead, the result that the standard deviation of hedging errors is constantly lower when using the SAD is in line with Vähämaa’s (2004) calculations. Therefore, SAD not only produces more stable and predictable hedging outcomes, as Vähämaa (2004) observed, but the hedging errors of SAD are also closer to zero.

In contrast to Vähämaa’s (2004) results, both deltas show positively skewed hedging errors instead of negative ones. The positively skewed hedging errors, coupled with highly leptokurtic distributions of both deltas, indicate the presence of fat tails and a tendency for positive errors. Notably, the difference in skewness and kurtosis between the deltas is minimal.

Because RMSHE squares the errors before averaging rather than considering them equally as MAHE, it provides greater weight to larger errors, making it a preferable calculation to consider when the objective is to reduce extreme financial losses. Hence, with observed highly non-normal error distributions, the RMSHE is a better estimate for hedging performance. Therefore, like Vähämaa (2004), RMSHE will receive more importance when discussing the hedging error results.

Finally, the hedging error results of the Black-Scholes delta (BS) and smile-adjusted delta (SAD) are shown in tables 5, 6 and 7 for hedging horizon analysis and later in tables 9 and 10 for market regimes analysis. To clarify the tables, the documented error statistics for each maturity-moneyness category include the mean absolute hedging error (MAHE) and root mean squared hedging error (RMSHE), both measured in dollars. In each table, the third and seventh columns provide the mean differences of delta hedging errors between the two models, while the fourth and eighth columns show the mean percentage differences in parentheses. Bootstrapping is applied to determine statistically significant differences in hedging errors.

Table 5 Hedging errors for the 1-day hedging horizon.



	Time to	MAHE				RMSHE			
Moneyness	Maturity	BS	SAD	Difference	(%)	BS	SAD	Difference	(%)
Full Sample	All	5,40	5,60	-0,20	(-3,67)**	9,58	9,39	0,19	(2,00)**
	Short	5,67	5,83	-0,16	(-2,83)**	9,99	9,92	0,06	(0,65)
	Long	5,22	5,45	-0,23	(-4,30)**	9,28	9,00	0,29	(3,13)**
OTM	All	4,05	4,40	-0,35	(-8,18)**	8,10	8,05	0,05	(0,56)
	Short	3,89	4,17	-0,27	(-6,74)**	8,23	8,26	-0,03	(-0,38)
	Long	4,16	4,56	-0,40	(-9,11)**	8,00	7,90	0,10	(1,27)
ATM	All	5,54	5,97	-0,43	(-7,51)**	9,20	9,52	-0,32	(-3,44)**
	Short	5,91	6,36	-0,45	(-7,33)**	9,63	10,09	-0,47	(-4,73)**
	Long	5,28	5,70	-0,42	(-7,65)**	8,89	9,10	-0,21	(-2,34)**
ITM	All	6,78	6,59	0,18	(2,73)**	11,31	10,57	0,74	(6,77)**
	Short	7,42	7,17	0,24	(3,30)**	11,94	11,36	0,58	(5,02)**
	Long	6,33	6,19	0,14	(2,28)**	10,85	9,99	0,86	(8,29)**

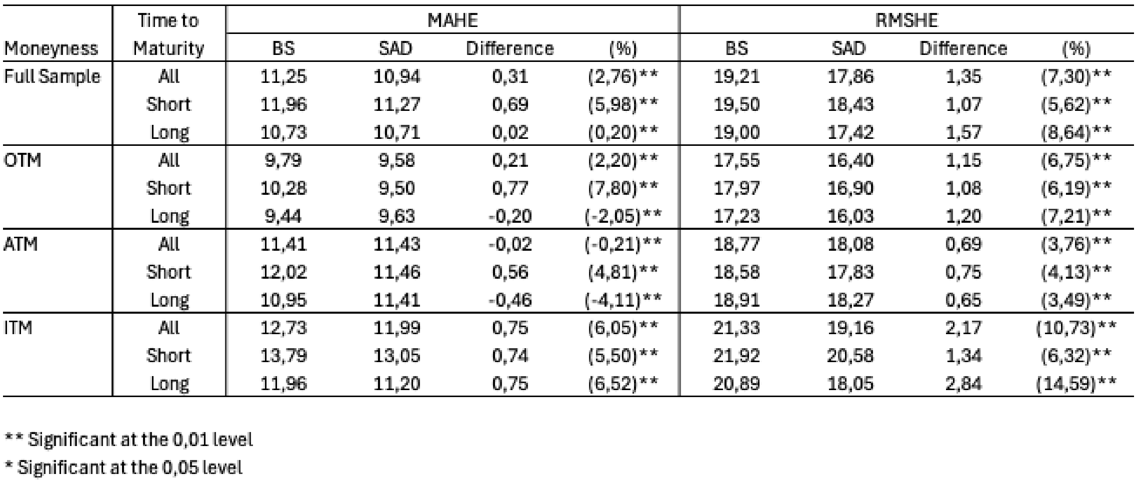
** Significant at the 0,01 level
* Significant at the 0,05 level

Table 5 presents the hedging errors for the 1-day hedging horizon. Vähämaa (2004) documented higher hedging errors for ATM options due greatest vega and gamma of those options. In contrast, table 5 as well as tables 6 and 7 show that the MAHE and RMSHE increase by moneyness. As a result, the hedging errors are the greatest for ITM options. What is aligned with Vähämaa (2004) is that the delta hedging errors are smaller for long-term options due to the decrease of gamma as the option maturity increases.

When analysing hedging performances from table 5 in terms of MAHE, all sample categories provide statistically significant results at the 0,01 level. MAHE results indicate delta hedging outperformance for the BS delta among OTM and ATM options. When analysing ITM options, SAD outperforms the BS delta in delta hedging.

When considering RMSHE results from table 5, the outperformance of SAD compared to the BS delta among ITM options is confirmed with statistical significance at the 0,01 level. RMSHE results indicate that the BS delta does not outperform SAD among OTM options, but the results for the BS delta outperformance among ATM options remain statistically significant at the 0,01 level.

Table 6 Hedging errors for the 5-day hedging horizon.



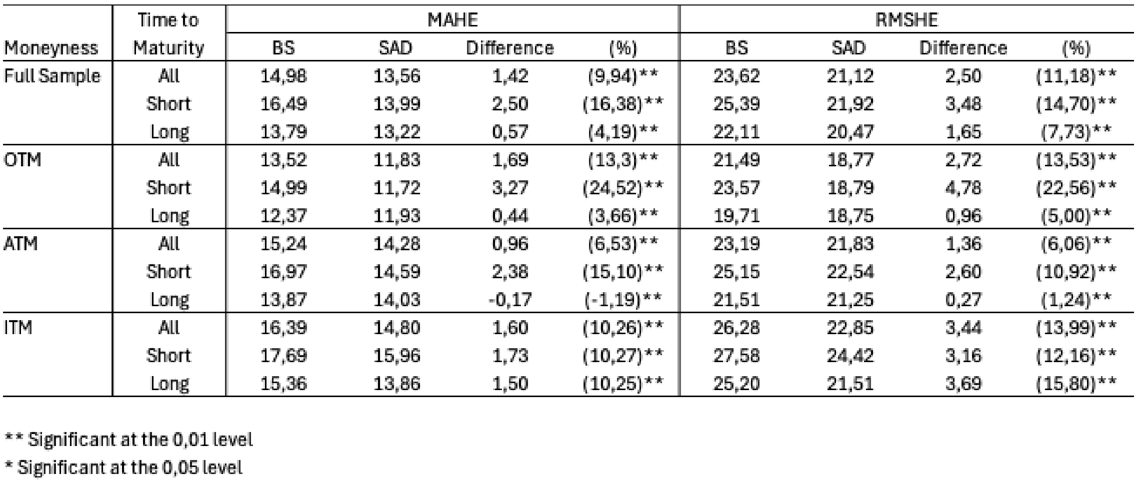
	Time to	MAHE				RMSHE			
Moneyness	Maturity	BS	SAD	Difference	(%)	BS	SAD	Difference	(%)
Full Sample	All	11,25	10,94	0,31	(2,76)**	19,21	17,86	1,35	(7,30)**
	Short	11,96	11,27	0,69	(5,98)**	19,50	18,43	1,07	(5,62)**
	Long	10,73	10,71	0,02	(0,20)**	19,00	17,42	1,57	(8,64)**
OTM	All	9,79	9,58	0,21	(2,20)**	17,55	16,40	1,15	(6,75)**
	Short	10,28	9,50	0,77	(7,80)**	17,97	16,90	1,08	(6,19)**
	Long	9,44	9,63	-0,20	(-2,05)**	17,23	16,03	1,20	(7,21)**
ATM	All	11,41	11,43	-0,02	(-0,21)**	18,77	18,08	0,69	(3,76)**
	Short	12,02	11,46	0,56	(4,81)**	18,58	17,83	0,75	(4,13)**
	Long	10,95	11,41	-0,46	(-4,11)**	18,91	18,27	0,65	(3,49)**
ITM	All	12,73	11,99	0,75	(6,05)**	21,33	19,16	2,17	(10,73)**
	Short	13,79	13,05	0,74	(5,50)**	21,92	20,58	1,34	(6,32)**
	Long	11,96	11,20	0,75	(6,52)**	20,89	18,05	2,84	(14,59)**

** Significant at the 0,01 level
* Significant at the 0,05 level

Table 6 presents the hedging errors for the 5-day hedging horizon. Regardless of the sample category, or calculating with MAHE or RMSHE, the results are statistically significant at the 0,01 level. MAHE results indicate that the increase in the hedging horizon leads to the overperformance of SAD when delta hedging OTM and ATM short-term options, but the BS delta outperformance remains among OTM and ATM long-term options. For delta hedging ITM options, SAD outperforms the BS delta as in table 5. When considering table 6 RMSHE results, the delta hedging outperformance of SAD is present in every category.

Table 7 presents the hedging errors for the 10-day hedging horizon. Except for the MAHE result among ATM long-term options, the results highlight the outperformance of SAD in delta hedging at 0,01 level in every category. Vähämaa (2004) concluded that the delta hedging outperformance of SAD is most distinct among short-term OTM and ATM options. Unfortunately, from tables 5, 6 and 7, simple patterns cannot be detected. 1-day and 5-day hedging outperformance of SAD is most distinct among ITM long-term options, but as the hedging horizon lengthens to 10 days, the outperformance of SAD is also highlighted for short-term options regardless of moneyness class. Nevertheless, the outperformance of SAD is more prominent as the hedging horizon lengthens, as Vähämaa (2004) also noted.

Table 7 Hedging errors for the 10-day hedging horizon.



Moneyness	Time to Maturity	MAHE				RMSHE			
		BS	SAD	Difference	(%)	BS	SAD	Difference	(%)
Full Sample	All	14,98	13,56	1,42	(9,94)**	23,62	21,12	2,50	(11,18)**
	Short	16,49	13,99	2,50	(16,38)**	25,39	21,92	3,48	(14,70)**
	Long	13,79	13,22	0,57	(4,19)**	22,11	20,47	1,65	(7,73)**
OTM	All	13,52	11,83	1,69	(13,3)**	21,49	18,77	2,72	(13,53)**
	Short	14,99	11,72	3,27	(24,52)**	23,57	18,79	4,78	(22,56)**
	Long	12,37	11,93	0,44	(3,66)**	19,71	18,75	0,96	(5,00)**
ATM	All	15,24	14,28	0,96	(6,53)**	23,19	21,83	1,36	(6,06)**
	Short	16,97	14,59	2,38	(15,10)**	25,15	22,54	2,60	(10,92)**
	Long	13,87	14,03	-0,17	(-1,19)**	21,51	21,25	0,27	(1,24)**
ITM	All	16,39	14,80	1,60	(10,26)**	26,28	22,85	3,44	(13,99)**
	Short	17,69	15,96	1,73	(10,27)**	27,58	24,42	3,16	(12,16)**
	Long	15,36	13,86	1,50	(10,25)**	25,20	21,51	3,69	(15,80)**

** Significant at the 0,01 level
* Significant at the 0,05 level

7.3 Comparison of the hedging performance in stable and highly volatile market conditions

When choosing months for the analysis of hedging performance in stable and highly volatile market conditions, the average ATM implied volatility is considered. In addition to the monthly review of the average ATM implied volatilities, table 8 shows the monthly correlations between the S&P 500 index besides the average ATM implied volatility. Again, to test the statistical significance, bootstrapping is applied.

From table 8 it can be observed that for 9 out of 12 months, the correlations were statistically significantly negative. For the rest 3 out of 12 months, the correlations were negative but statistically insignificant. Thus, the inverse correlation is highly present in the research period. Noteworthy, there were no months with a positive correlation. When observing the average ATM implied volatilities, the first two months of the year were stable with the average ATM implied volatility less than 20,00%. This was followed by three higher volatility months, March, April and May, with the average ATM implied volatility over 30,00%. For the remaining 7 out of 12 months, the months after the spring, the average ATM implied volatility decreased to fluctuate between 20,00% and 30,00%.

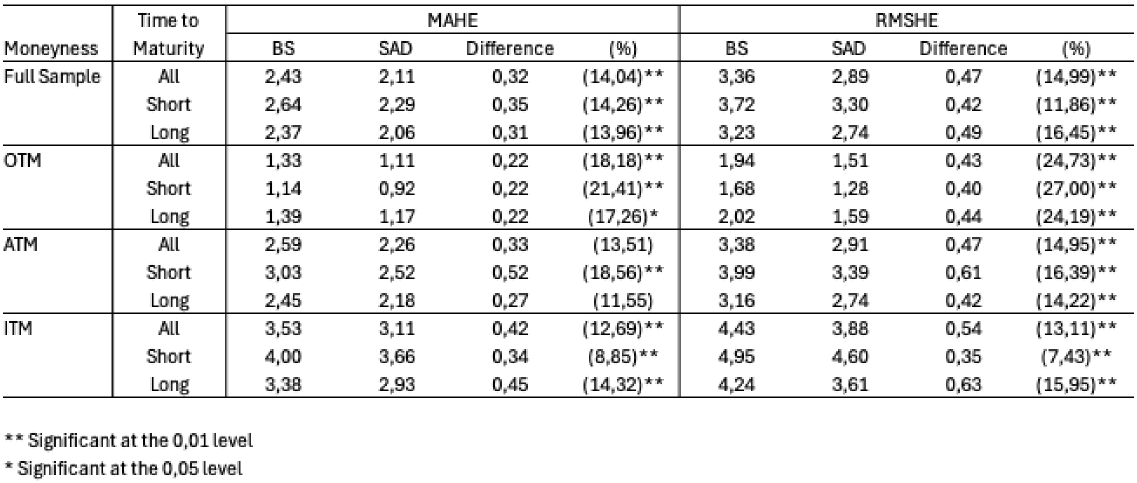
Table 8 Monthly average ATM implied volatility and its correlation with the S&P 500 index.

Month	Correlation	Average ATM Implied Volatility
January	-0,19	14,07%
February	-0,99**	18,54%
March	-0,83**	59,06%
April	-0,91**	40,93%
May	-0,87**	30,22%
June	-0,71**	29,47%
July	-0,38*	24,97%
August	-0,19	20,48%
September	-0,80**	27,01%
October	-0,81**	28,73%
November	-0,97**	23,68%
December	-0,30	20,13%

** Significant at the 0,01 level
* Significant at the 0,05 level

The month with the lowest average ATM implied volatility is chosen for the analysis of hedging performance in stable market conditions. During January, the S&P 500 index was relatively stable, the index level fluctuated from 3 225,52 to 3 329,62. The average ATM implied volatility was the lowest of the year, only 14,07% (table 8). Moreover, table 8 shows that the correlation between the S&P 500 and ATM implied volatility was only -0,19 and statistically insignificant. Hence, table 9 presents the hedging errors for the 1-day hedging horizon during stable market conditions of January.

Table 9 Hedging errors in stable market conditions.



	Time to	MAHE				RMSHE			
Moneyness	Maturity	BS	SAD	Difference	(%)	BS	SAD	Difference	(%)
Full Sample	All	2,43	2,11	0,32	(14,04)**	3,36	2,89	0,47	(14,99)**
	Short	2,64	2,29	0,35	(14,26)**	3,72	3,30	0,42	(11,86)**
	Long	2,37	2,06	0,31	(13,96)**	3,23	2,74	0,49	(16,45)**
OTM	All	1,33	1,11	0,22	(18,18)**	1,94	1,51	0,43	(24,73)**
	Short	1,14	0,92	0,22	(21,41)**	1,68	1,28	0,40	(27,00)**
	Long	1,39	1,17	0,22	(17,26)*	2,02	1,59	0,44	(24,19)**
ATM	All	2,59	2,26	0,33	(13,51)	3,38	2,91	0,47	(14,95)**
	Short	3,03	2,52	0,52	(18,56)**	3,99	3,39	0,61	(16,39)**
	Long	2,45	2,18	0,27	(11,55)	3,16	2,74	0,42	(14,22)**
ITM	All	3,53	3,11	0,42	(12,69)**	4,43	3,88	0,54	(13,11)**
	Short	4,00	3,66	0,34	(8,85)**	4,95	4,60	0,35	(7,43)**
	Long	3,38	2,93	0,45	(14,32)**	4,24	3,61	0,63	(15,95)**

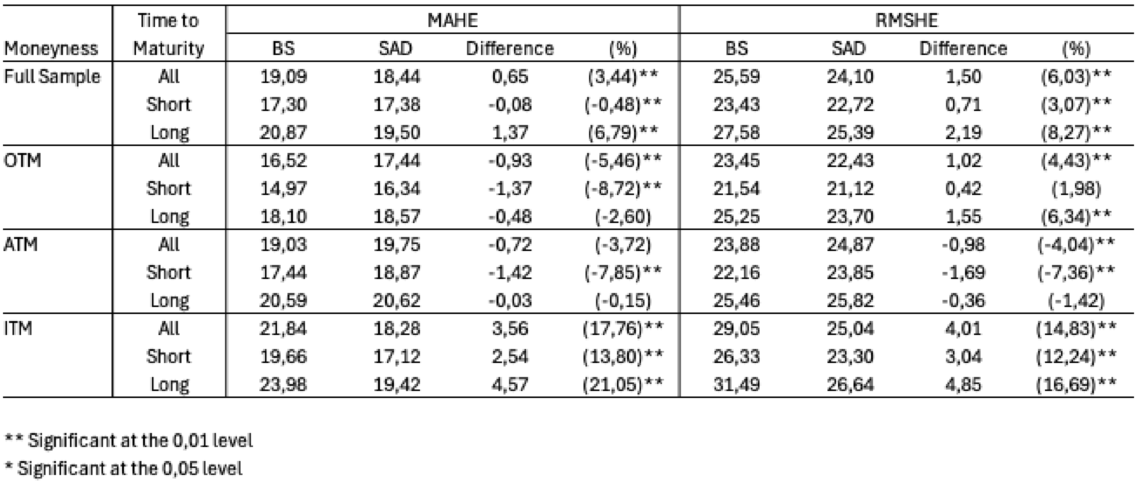
** Significant at the 0,01 level
* Significant at the 0,05 level

When considering the RMSHE results from table 9, SAD provides consistently smaller hedging errors compared to the BS delta. This can be due to possibly relatively steep volatility smile. Further, MAHE results for SAD are smaller for every category, although not all of those are statistically significant. Interestingly, as moneyness increases, the hedging errors increase and the overperformance of SAD decreases. According to Derman's (1999) intuition, in stable market conditions, volatility tends to be independent of the underlying index levels, which is confirmed in table 8. Therefore, according to the research, the Black-Scholes delta should be the same as the real market delta. Hedging errors in table 9 are smaller than those for the full year in table 5. Generally, in stable market conditions, both deltas should provide relatively small hedging errors as expected relating to results from Vähämaa (2004) and Coleman et al. (2000). Surprisingly, even if the hedging errors are small, the differences between SAD and the BS delta hedging errors are notably large (table 9). This indicates that even though the hedging errors are small, with a higher value portfolio, the approximately 15% difference in delta hedging performance should be considered in risk minimisation.

Next, the month with the highest average ATM implied volatility is chosen for the analysis of hedging performance in highly volatile market conditions. During March, the S&P 500 index value dropped to its lowest value of the year, as shown in figure 4. From table 8, it can be observed that in March the average ATM implied volatility was the highest for the research period, being 59,06%. Moreover, from table 8, it can be noted that the correlation between the S&P 500 and ATM implied volatility was -0,83 and statistically significant at the 0,01 level.

The increased volatility in March can be linked to the increased fear of spreading disease, COVID-19. On March 11, the World Health Organization (WHO) declared the COVID-19 outbreak a global pandemic (World Health Organization, 2020). Only a few published articles have yet been peer-reviewed covering the topic. Nevertheless, for example, Yilmazkuday (2021) explains that the historical decomposition of the S&P 500 index indicates that the negative impact of COVID-19 cases in the United States was most pronounced in March 2020. Finally, table 10 presents the hedging errors for the 1-day hedging horizon during the highly volatile market conditions in March.

Table 10 Hedging errors in highly volatile market conditions.



	Time to	MAHE				RMSHE			
Moneyness	Maturity	BS	SAD	Difference	(%)	BS	SAD	Difference	(%)
Full Sample	All	19,09	18,44	0,65	(3,44)**	25,59	24,10	1,50	(6,03)**
	Short	17,30	17,38	-0,08	(-0,48)**	23,43	22,72	0,71	(3,07)**
	Long	20,87	19,50	1,37	(6,79)**	27,58	25,39	2,19	(8,27)**
OTM	All	16,52	17,44	-0,93	(-5,46)**	23,45	22,43	1,02	(4,43)**
	Short	14,97	16,34	-1,37	(-8,72)**	21,54	21,12	0,42	(1,98)
	Long	18,10	18,57	-0,48	(-2,60)	25,25	23,70	1,55	(6,34)**
ATM	All	19,03	19,75	-0,72	(-3,72)	23,88	24,87	-0,98	(-4,04)**
	Short	17,44	18,87	-1,42	(-7,85)**	22,16	23,85	-1,69	(-7,36)**
	Long	20,59	20,62	-0,03	(-0,15)	25,46	25,82	-0,36	(-1,42)
ITM	All	21,84	18,28	3,56	(17,76)**	29,05	25,04	4,01	(14,83)**
	Short	19,66	17,12	2,54	(13,80)**	26,33	23,30	3,04	(12,24)**
	Long	23,98	19,42	4,57	(21,05)**	31,49	26,64	4,85	(16,69)**

** Significant at the 0,01 level
* Significant at the 0,05 level

When analysing the hedging error results from table 10, the delta hedging outperformance of SAD cannot be documented for all the categories, like Vähämaa (2004) observed. Furthermore, since both deltas produce relatively large hedging errors, delta hedging is generally less effective in highly volatile market conditions compared to the full period 1-day delta hedging (table 5) or delta hedging in stable market conditions (table 9). The results indicate that SAD outperforms the BS delta in delta hedging measured by both MAHE and RMSHE among ITM options at the 0,01 level. Moreover, among

OTM long-term options, results from MAHE are insignificant, but the results from RMSHE show better hedging performance for SAD at the 0,01 level for that category. SAD performs worse in the case of ATM options, both MAHE and RMSHE indicate better delta hedging performance for the BS delta among short-term ATM options at the 0,01 level.

8 Conclusions

In this conclusions chapter, the significant empirical findings of this thesis are concluded and compared with the research hypotheses. Implications for practice are highlighted, but the study’s limitations are also acknowledged. Lastly, some recommendations for future research are provided.

8.1 Summary of findings

The purpose of the study was to investigate whether the delta hedging performance of the Black-Scholes delta can be improved by adding a volatility smile sensitivity term to the equation. To provide more background on the topic, this thesis can be observed to consist of three parts. Firstly, to discuss the reasons behind the volatility smile. Secondly, to ascertain how the smile affects delta hedging and how delta hedging could be optimised to take the smile into account. Thirdly, the empirical comparison of delta hedging performances of the Black-Scholes delta and smile-adjusted delta, applying the methodology from Vähämaa (2004).

The Black-Scholes model is inaccurate due to the background assumptions and limitations of the model, which do not replicate the real financial world perfectly. The most violated assumptions of the model are the assumption of constant volatility and lognormal distribution of the price of the underlying. The assumption of constant volatility is not valid for the sample, since the volatility smile is documented to be downward sloping. Further, the documented volatility smiles for put and call options separately are not truly identical. Put options are priced with greater implied volatility than call options, and the difference in implied volatility is the smallest among ATM options. Even though the Black-Scholes model does not appear to be the only reason for the volatility smile, the model’s impact on it is remarkably present since barely any of the model’s assumptions hold in the real financial markets.

To fully understand the market dynamics for optimal delta hedging, the reason for the volatility smile would be important to find. Unfortunately, the unambiguous answer to why there are volatility smiles is still undiscovered. Nevertheless, the suggested reasons can be divided into three categories. Firstly, there are violated background assumptions of constant volatility and lognormal distribution of the model, which cause the smile. Secondly, market imperfections affect the steepness of the smile. To mention a few of those, there are leverage, transaction costs and liquidity risk. Thirdly, there are supply and demand factors including betting with inexpensive deep OTM options, hedging pressure, crash-o-phobia, and changes in interest rates, dividend payments and maturity which cause smiling.

Therefore, this thesis is motivated by the fact that there are volatility smiles with non-unambiguous reasons, and the smiles can be assumed to negatively affect delta hedging outcomes. Finally, it is time to review the constructed thesis hypotheses, which were the following:

H_1: The volatility smile should be considered for more efficient delta hedging results.

H_2: Smile-adjusted delta outperforms the Black-Scholes delta in delta hedging.

The first hypothesis (H_1) is true. The volatility smile has its impact on delta hedging, where the aim is to minimise variance by rebalancing the delta of the portfolio to zero. When there is a negative correlation between equity price and volatility, as in this research sample, the delta calculated from the Black-Scholes model is too large to minimise the portfolio variance. Therefore, researchers have investigated more optimal methods to delta hedge with the presence of a volatility smile. One of the methods is the smile-adjusted delta, a model which is built from the Black-Scholes model, with an additional term to capture the sensitivity of the option to the slope of the volatility smile. In this thesis, the smile-adjusted delta is observed to be on average 0,09 smaller than the BS delta.

Considering the second hypothesis (H_2), the performance of the smile-adjusted delta was retested with data from the S&P 500 index. Both models' quantitative fit is documented to be approximately 90%, but SAD explains daily option price movements slightly better than the BS delta, regardless of the moneyness and maturity of the option. Indicating, it is reliable to finally compare the delta hedging performance of the two deltas.

The overall conclusion is that SAD outperforms the BS delta in delta hedging, meaning that the second hypothesis (H_2) is also true. To contradict Vähämaa's (2004) results, the empirical finding from delta hedging indicates that SAD outperforms the BS delta most distinctly among ITM options. When the hedging horizon is longer than a day, SAD outperforms the BS delta regardless of the moneyness and maturity of the option. That outperformance becomes even more notable as the hedging horizon lengthens. This outperformance increase is even more distinct than what Vähämaa (2004) documented. This may be due to the relatively steep volatility smile making the smile-adjustment term, or specifically considering the slope of the smile, more crucial.

Interestingly, in highly volatile market conditions, the outperformance of SAD is documented only among ITM options and OTM long-term options, with the BS delta outperforming among ATM short-term options. In stable market conditions, both deltas provide relatively small hedging errors. Nevertheless, since the outperformance of SAD is documented for every moneyness-maturity category, SAD outperforms the BS delta also in stable market conditions. To conclude the volatility regimes analysis, SAD do not outperform delta hedging distinctly better in highly volatile markets compared to stable markets.

8.2 Implications for practice

The volatility smile is an important phenomenon for traders to make more informed investment decisions and manage risk effectively. For example, the shape of the smile can give insight into market sentiment. The steeper smile may indicate that investors are more concerned about downside risk than upside risk. Further, the smile might provide opportunities if investors could take advantage of the non-constant implied volatility when trading volatility.

This thesis focused on the volatility smile from the perspective of delta hedging. The thesis highlights how a minor adjustment to the Black-Scholes delta to consider the volatility smile can lead to better delta hedging performance. This information is useful for risk management, especially because the required smile adjustments are easy to implement in practice. The implication is that risk managers can limit the smile risk and therefore delta hedge more optimally regardless of the market regime. To conclude, the thesis findings have significant practical relevance, especially since the research sample is from one of the most followed stock indexes.

8.3 Limitations of the study

One key limitation of the study comes from the proxy used for building the smile-adjusted delta. The unknown relationship between the volatility and the underlying price is approximated with the slope of the volatility smile, so $\frac{\partial\sigma}{\partial S} = \frac{\partial\sigma}{\partial K}$. Since $S = K$ is only true for ATM options with moneyness exactly one, the change in volatility may affect differently for OTM and ITM options.

The used S&P 500 index options (ticker: SPX) for delta hedging do not capture all the characteristics of the underlying S&P 500 index. For example, corporate events like mergers and acquisitions, spin-offs or dividends may not immediately reflect the pricing of SPX. Moreover, trading with SPX ends a day before expiration with the settlement time of AM (opening price), so the delta hedger is exposed to additional last trading day overnight volatility. Although these trading gaps exist every time the stock market is not open, the last night trading gap is the nearest to the settlement date and cannot be ignored just by early settlement, since it is not possible for European options.

8.4 Recommendations for future research

This thesis leaves much to research in the future. The volatility smiles in different continents, asset classes and time series would be interesting to research, as well as the differences between index option smiles compared to single option smiles or differences in smiles between dual-listed companies. For example, the last two mentioned could catch the effect of trading volume differences on hedging errors. Other option pricing models could be examined and compared in performance on pricing options and hedging, as well as investigating the consequences of the smile on other Greek letters' performance. Moreover, since the thesis left out transaction costs, it would be relevant to research the most effective hedge revision horizon. Particularly, the recent tariff announcements by United States President Donald Trump provide an interesting period to conduct research from various angles.

Lastly, an intriguing question is whether the crash in 1987 had not happened, would the volatility smile have occurred with its challenging impacts on delta hedging and pricing options? In addition, is there a chance to permanently eliminate the smile? If not, may there be an even more efficient formula for delta hedging with such practicality?

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