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RESEARCH ARTICLE

Novel Wind Power Station Site Selection Framework Based on Multipolar Fuzzy Schweizer-Sklar Aggregation Operators

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ABSTRACT Nowadays, wind power stations play a significant role in eco-friendly energy production by efficiently harnessing wind energy to produce electricity. A crucial factor in constructing a wind power station is the site selection process, which identifies ideal locations for wind turbines to optimize energy generation, minimize costs, and reduce environmental impact. This complex decision-making involves multipolar attributes, including technical and environmental categories. An m-polar fuzzy ($m\mathrm{P}^F$) set model is an effective tool for addressing such uncertain problems involving multi-dimensional parameters. The main goal of this study is to integrate Schweizer-Sklar operations with $m\mathrm{P}^F$ information to determine the aggregated results in a more generalized environment. We develop some novel $m\mathrm{P}^F$-geometric and $m\mathrm{P}^F$-averaging aggregation operators ($\mathrm{A}_g\mathrm{Os}$), including the $m\mathrm{P}^F$ Schweizer-Sklar weighted averaging ($m\mathrm{P}^F$SSWA), $m\mathrm{P}^F$ Schweizer-Sklar ordered weighted averaging ($m\mathrm{P}^F$SSOWA), $m\mathrm{P}^F$ Schweizer-Sklar hybrid averaging ($m\mathrm{P}^F$SSHA), $m\mathrm{P}^F$ Schweizer-Sklar weighted geometric ($m\mathrm{P}^F$SSWG), $m\mathrm{P}^F$ Schweizer-Sklar ordered weighted geometric ($m\mathrm{P}^F$SSOWG), and $m\mathrm{P}^F$ Schweizer-Sklar hybrid geometric ($m\mathrm{P}^F$SSHG) operators. We support these $\mathrm{A}_g\mathrm{Os}$ by presenting numerical examples and some fundamental properties, like monotonicity, boundedness, idempotency, and commutativity. Further, we propose an algorithm for both $m\mathrm{P}^F$SSWA and $m\mathrm{P}^F$SSWG operators to minimize uncertainty in various MCDM problems. Next, we investigate a case study of Sindh province in Pakistan (i.e., choosing the best site for a wind power station) by implementing the suggested algorithm. Finally, we compare the developed $m\mathrm{P}^F$ Schweizer-Sklar $\mathrm{A}_g\mathrm{Os}$ with the preexisting $m\mathrm{P}^F$-Yagar, $m\mathrm{P}^F$-Dombi, $m\mathrm{P}^F$-Aczel-Alsina $\mathrm{A}_g\mathrm{Os}$, $m\mathrm{P}^F$-AHP (Analytical Hierarchy Process), $m\mathrm{P}^F$-TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution), and $m\mathrm{P}^F$-ELECTRE-I (ELimination and Choice Expressing REality)-I methods.

INDEX TERMS m-polar fuzzy sets, Schweizer-Sklar t-norm, aggregation operators, multi-criteria decision-making, wind power station.

I. INTRODUCTION

A key component of decision science is multi-criteria decision-making (MCDM), whose main goal is to find the best option from a set of correlated options. Since every MCDM method involves the evaluation of provided alternatives by a decision-maker (DM) or a group of DMs with a single criterion or several criteria. The DMs' evaluations are delivered as either verbal or unambiguous values. However, these days uncertain information commonly appears for

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MCDM in a variety of contexts like medical diagnosis, clustering, classification, renewable energy sources, and even more.

A. INTRODUCTION OF FUZZY LOGIC

Uncertainties play a significant role in a MCDM procedure, so it can be challenging for DMs to achieve accurate outcomes without addressing inaccurate, ambiguous, or uncertain data. So far multiple frameworks have been introduced to deal with different kinds of uncertain situations. Initially, Zadeh [1] invented the notion of fuzzy sets (FSs) to characterize an object's membership in a set using the membership grade from closed unit interval. Later, the traditional concept of Boolean logic was extended to fuzzy logic (or many valued logic) by Zadeh [2] in 1973, which is established from the principles of FS theory, where truth values are described by certainty degrees as compared to absolute values (that is, false (0) or true (1)). However, fuzzy logic permits a continuity of taking truth values from the closed unit interval, which represents the membership degrees in a FS, as compared to the classical Boolean logic, which deals with entirely true or entirely false truth values.

B. LITERATURE REVIEW

Since both the concepts, fuzzy logic and FSs, only deal with membership degrees, thus, due to the importance and existence of non-membership grades in several practical situations, there was a need to tackle such scenarios. Therefore, concerning the non-membership grades, several extensions of FSs were presented such as the intuitionistic fuzzy set (I_nFS) [3], Pythagorean fuzzy (PF) set [4]. These frameworks described both an object's belonging and its non-belonging grades. Later, Peng and Yang [5] came up with an approach to rank PF numbers (PFNs) in order to make a decision based on more than one factor. In recent times, Ejegwa [6] talked about how to use better composition relationships for PF sets in medical science. Besides, depending on the concept of interval numbers, another extension of FS model called interval-valued FS was proposed by Gorzalczany [7] for dealing with uncertainty and imprecision. A new addition to I_nFS theory called interval-valued I_nFS [8] is being used a lot to deal with complicated uncertainties in many MCDM situations [9], [10], [11].

Nowadays, aggregation operators (A_gOs) are playing an essential role in the procedure of addressing MCDM problems with the primary purpose of aggregating a set of inputs into a single output. To date several A_gOs have been investigated in the context of fuzzy environment to solve different MCDM issues. For example, Xu [12] investigated some A_gOs using I_nFSs, including hybrid weighted and weighted ordered averaging operators. Garg et al. [13] demonstrated Schweizer-Sklar prioritized A_gOs for I_nF numbers along with decision-making applications (see also, Garg et al. [14]). Liu et al. [15] presented Schweizer-Sklar power A_gOs based on complex I_nFSs and discussed their application in decision-making. In order to tackle certain MCDM scenarios, Gao [16] presented a few PF Hamacher prioritized A_gOs. Khan et al. [17] established PF prioritized A_gOs for MCDM (see also, Liang et al. [18]). Furthermore, certain MCDM methods based on the PF Einstein hybrid geometric operator were presented by Rahman and Ali [19]. Merigo and Gil-Lafuente [20] explored fuzzy-induced extended A_gOs and employing them to solve MCDM challenges. Liu and Liu [21] developed certain q-rung Bonferroni mean operators for group decision-making. Kara et al. [22] developed an integrated neutrosophic Schweizer-Sklar operations-based model for evaluating economic activities in organized industrial zones. Jana and Pal [23] studied some mP^F set-based A_gOs and explored their application in tackling a MCDM situation. Ma et al. [24] presented a novel decision-making method based on q-rung orthopair fuzzy set-based Schweizer-Sklar A_gOs with applications in agriculture sectors. For more detail on Schweizer-Sklar operations-based A_gOs, the readers are suggested to [25].

Researchers today recognized the importance of multiple polarities across different kinds of domains, including management, engineering, neural fuzzy advancements and medical sciences because various facets of everyday life are controlled through multipolar attributes, influencing alternative multi-directional characteristics. These facts are the reasons behind the emergence of m-polar information through traditional mathematical techniques like FS theory [1], I_nFS theory [3], and PF set theory [4], within the diverse perspectives of routine existence. In the realm of information technology, multi-polar technologies are able to be used to examine complicated systems of data with varied features likely latency, network range, bandwidth and radio frequency. In neurological science, the interconnection of cells in the human mind collects data from several other nerve cells, resulting in a multi-polar information collection mechanism. In addition, when selecting between a desirable spot and the affordability of a property, consider accessibility to work, facilities, and neighborhood safety. People have to make decisions in each of these cases that take into account a number of poles or factors. No doubt, the theories such as FSs, I_nFSs and PF sets, are very effective mathematical instruments for handling ambiguous and uncertainty, nevertheless, in many cases, these models are ineffective when the datasets under examination are multi-dimensional. In order to address the difficulties of implementing FSs and their further extensions, Chen et al. [26] developed the idea of mP^{F}Ss, which is specially designed to address multi-faceted information from a variety of modern science disciplines. An mP^{F}S-based algorithm has been applied to the selection of non-traditional machining procedures by Jagtap and Karande [27] and mP^F networks have been used to solve challenges related to product manufacturing by Akram et al. [28]. Al-Shamiri et al. [29] integrated TOPSIS and ELECTRE-I approaches with cubic mP^F sets and explored their application to the diagnosis of psychiatric disorders. A number of significant researches have been conducted in the past ten years that concentrate on the aggregation of $m\mathrm{P}^F$ data by utilizing existing methods. For example, Ali et al. [30] presented particular geometric and arithmetic A_gOs to facilitate the aggregation of $m\mathrm{P}^F$ numbers using Yagar's operations. Waseem et al. [31] devised and implemented $m\mathrm{P}^F$ Hamacher A_gOs to address MCDM issues. Khameneh and Kilicman [32] introduced $m\mathrm{P}^F$ soft weighted A_gOs that were successfully employed to solve MCDM challenges. Akram and Adeel [33] established a decision-making method based on 2-tuple linguistic $m\mathrm{P}^F$ Hamacher A_gOs. Akram et al. [34] proposed $m\mathrm{P}^F$ Dombi A_gOs and investigated their potential applications in MCDM.

C. INTRODUCTION OF SCHWEIZER-SKLAR AGGREGATION OPERATORS

The notions of Schweizer-Sklar t-norm (t-N) and t-conorm (t-C_oN) were originated by Schweizer and Sklar [35] in 1960, which involve a more reliable and flexible parameter (λ) to aggregate uncertain and imprecise data. For the parametric values $\lambda = -1, 1$, the Schweizer-Sklar t-N converted to the Hamacher and Lukasiewicz t-Ns. Thus, it is a more generalized family of t-Ns regarding parametric environment as compared to other t-Ns, such as Hamacher, Dombi, Frank, et cetera. Consideration of positive or negative risk-management, and the availability of a flexible parameter makes Schweizer-Sklar t-Ns and t-C_oNs [35], [36] more versatile than other existing operations. As a result, these operations have been widely chosen by various DMs to invent various A_gOs for handling different complex MCDM situations. For instance, Deschrijver and Kerre [37] introduced some new Schweizer-Sklar t-N and t-C_oN based operators for the aggregation of I_nF numbers. Zhang et al. [38] presented a fuzzy logic system using the Schweizer-Sklar operations. Each of these Schweizer-Sklar t-N and t-C_oN-based frameworks is inefficient to tackle $m\mathrm{P}^F$ data. Motivated by these facts, this work is devoted to integrate the Schweizer-Sklar operations with $m\mathrm{P}^F$Ss. To provide a better understanding of the research gaps and advancements in the literature, a summary is presented in Table 1.

The following factors motivate us to develop $m\mathrm{P}^F$ Schweizer-Sklar A_gOs.

1) Site selection problem for large-scale projects in every real-life domain frequently involves complicated data that takes the different formats of uncertainty like multi-agents, multi-attributes and multiple poles. Several extensions of FSs are unable to adequately tackle these complicated, multifaceted decision-making situations. To validate this point, a summary of site selection problems is provided by Table 2.
2) The range of parameter (λ) in the Schweizer-Sklar t-N and t-C_oN is $-\infty$ to $+\infty$ that is greater that other existing operations, including Hamacher, Dombi, Yager, and Aczel-Alsina t-Ns and t-C_oNs.
3) As a comprehensive framework, $m\mathrm{P}^F$ numbers characterize effective performance within the evaluation process concerning ambiguous, imprecise and uncertain multifaceted information. Therefore, $m\mathrm{P}^F$ concept gives an effective strategy to evaluate items under multidimensional data. To understand this useful concept let us suppose that a group of students wishes to plan a tour in summer holidays but they are not clear about the location. This situation can not be explained well by a membership value belongs to [0, 1] because there are different properties of a suitable location for tour which should be evaluated, that is, it must contain lakes and waterfalls, availability of food and other services, weather friendly etc. In other words, say, these are sub-characteristics of location, here each characteristic has a membership value belongs to [0, 1]. So, if we use FS theory to deal information we have to choose a FS with respect to each sub-characteristic which is not a precise way to represent this information. Hence, multipolar FSs as an efficient extension of FSs are more flexible and reliable.
4) The existing mathematical tools, including FSs, I_nFSs, PF sets, et cetera are capable of handling single-dimensional and two-dimensional information, which may result in a loss of information.
5) The $m\mathrm{P}^F$ Schweizer-Sklar A_gOs, like certain other $m\mathrm{P}^F$-A_gOs offer an alternate approach for solving a variety of MCDM issues.

Based on these motives, the proposed work concentrates on developing the $m\mathrm{P}^F$ Schweizer-Sklar A_gOs, and demonstrates their ability to make decisions.

The key elements of this work are summarized in the list below:

1) Some fundamental definitions and features of $m\mathrm{P}^F$ numbers are recalled.
2) Certain innovative Schweizer-Sklar A_gOs for the aggregation of $m\mathrm{P}^F$ numbers are established such as weighted averaging A_gOs ($m\mathrm{P}^F$SSWA, $m\mathrm{P}^F$SSOWA, $m\mathrm{P}^F$SSHWA) and weighted geometric A_gOs ($m\mathrm{P}^F$SS WG, $m\mathrm{P}^F$SSOWG, $m\mathrm{P}^F$SSHWG).
3) A comprehensive investigation of the fundamental characteristics of the proposed A_gOs is performed.
4) An appropriate MCDM mechanism for the aggregation of multi-polar data is provided using $m\mathrm{P}^F$SSWA and $m\mathrm{P}^F$SSWG-A_gOs.
5) The literature has examined several site selection problems using fuzzy set-based hybrid models like Deveci et al. [10], Deveci [11]. That is why, to demonstrate the applicability of the initiated $m\mathrm{P}^F$ Schweizer-Sklar A_gOs in real-life situations, a brief application is offered, i.e., a case study of Sindh province in Pakistan for choosing a suitable location for a wind power station.
6) A comparison is depicted with some preexisting operators, including $m\mathrm{P}^F$ Yager A_gOs [30], $m\mathrm{P}^F$ Aczel-Alsina A_gOs [39], $m\mathrm{P}^F$ Dombi A_gOs [34],

TABLE 1. Literature review summary table.

Typical Reference	Major Findings	Improvements over Existing Findings	Research Gaps
Zadeh [1]	Fuzzy sets or FSs	Address vagueness and uncertainty in a data-set using a belongingness function that assigns values between truth(1) and falsehood(0).	The non-belongingness function, which is also important for alternatives in a data-set, is missing. The FSs are also inefficient to demonstrate multiple features of an attribute.
Atanassov [3]	Intuitionistic fuzzy sets or I_nFSs	Presence of belongingness and non-belongingness functions	Sum of belongingness and non-belongingness values is bounded by 1. Moreover, unable to depict different sub-features of an attribute.
Yager [4]	Pythagorean fuzzy sets or PF sets	Enlarge the membership and nonmembership evaluations space as compared to IFSs	Inadequate to tackle data involving multipolar(m-polar) properties of objects
Chen et al. [26]	m-Polar fuzzy sets or $m\mathrm{P}^{\mathrm{F}}$ sets	Enlarge the concepts of Zadeh's FS theory because $m\mathrm{P}^{\mathrm{F}}$ set has ability to deal with multipolar properties of objects regarding a certain attribute	The $m\mathrm{P}^{\mathrm{F}}$ sets are handicap to deal with different sub-characteristics of an object with respect to multiple attributes
Waseem et al. [31]	$m\mathrm{P}^{\mathrm{F}}$ Hamacher $\mathrm{A_gOs}$	Ability to aggregate $m\mathrm{P}^{\mathrm{F}}$ numbers using Hamacher t-Ns and t-$\mathrm{C_oNs}$	The range of parameter involved in Hamacher's operations is form 0 to positive infinity.
Akram et al. [34]	$m\mathrm{P}^{\mathrm{F}}$ Dombi $\mathrm{A_gOs}$	Efficiency to aggregate $m\mathrm{P}^{\mathrm{F}}$ numbers using Dombi's operations	The range of parameter (p) involved in Dombi t-Ns and t-$\mathrm{C_oNs}$ is given by $0 \leq p \leq +\infty$.
Ali et al. [30]	$m\mathrm{P}^{\mathrm{F}}$ Yager $\mathrm{A_gOs}$	Capable to aggregate $m\mathrm{P}^{\mathrm{F}}$ information using Yager t-Ns and t-$\mathrm{C_oNs}$	The variation in parameter (p) involved in Yager's operations is provided by $0 \leq p \leq +\infty$.
Rahman et al. [39]	$m\mathrm{P}^{\mathrm{F}}$ Aczél–Alsina $\mathrm{A_gOs}$	Competency to aggregate $m\mathrm{P}^{\mathrm{F}}$ numbers using Aczél–Alsina t-Ns and t-$\mathrm{C_oNs}$	The parameter involved in Aczél–Alsina's operations lies between 0 and positive infinity.

$m\mathrm{P}^F$ AHP [40], $m\mathrm{P}^F$ TOPSIS [33], [41] and $m\mathrm{P}^F$ ELECTRE-I [33] methods. Moreover, a detailed sensitivity analysis is depicted by taking parametric values from -1 to -10 for the presented $m\mathrm{P}^F$ Schweizer-Sklar $\mathrm{A}_g\mathrm{Os}$, and applied them on the explored application.

This article is organized in the following order: In Section II, we recall fundamental definitions and features of $m\mathrm{P}^F$ numbers. In Section III, present the $m\mathrm{P}^F$ Schweizer-Sklar weighted averaging ($m\mathrm{P}^F$SSWA) operator, $m\mathrm{P}^F$ Schweizer-Sklar ordered weighted averaging ($m\mathrm{P}^F$SSOWA) operator, $m\mathrm{P}^F$ Schweizer-Sklar hybrid averaging ($m\mathrm{P}^F$SSHA) operator, $m\mathrm{P}^F$ Schweizer-Sklar weighted geometric ($m\mathrm{P}^F$SSWG) operator, $m\mathrm{P}^F$ Schweizer-Sklar ordered weighted geometric operator ($m\mathrm{P}^F$SSOWG) and $m\mathrm{P}^F$ Schweizer-Sklar hybrid geometric ($m\mathrm{P}^F$SSHG) operators. In Section IV, we develop an MCDM method using developed $m\mathrm{P}^F$ Schweizer-Sklar $\mathrm{A}_g\mathrm{Os}$ to solve real-life issues having complex $m\mathrm{P}^F$ information. Next in this section, we explore an MCDM application that involves a case study of Sindh province in Pakistan for selecting a best site for wind power station. In Section V, we compare the developed approach of $m\mathrm{P}^F$ Schweizer Sklar $\mathrm{A}_g\mathrm{Os}$ with $m\mathrm{P}^F$-Yagar [30], $m\mathrm{P}^F$-Aczel-Alsina [39] and $m\mathrm{P}^F$-Dombi [34] $\mathrm{A}_g\mathrm{Os}$. At the end, in Section VI, we conclude our study and provide potential future directions.

II. PRELIMINARIES

This section reviews the definition of $m\mathrm{P}^F$Ss and some operations of $m\mathrm{P}^F$ numbers ($m\mathrm{P}^F$Ns).

Definition 1 [26]: An $m\mathrm{P}^F$ set on a universal set $\mathcal{P}$ is a mapping $\gamma : \mathcal{P} \to [0,1]^{\mathfrak{m}}$. $\gamma(\mathfrak{u}) = (\mathfrak{p}_1 \circ \gamma(\mathfrak{p}), \mathfrak{p}_2 \circ \gamma(\mathfrak{p}), \ldots, \mathfrak{p}_{\mathfrak{m}} \circ \gamma(\mathfrak{p}))$ is known as belongingness degree, where $\mathfrak{p} \in \mathcal{P}$, and $\mathfrak{p}_k \circ \gamma : [0,1]^{\mathfrak{m}} \to [0,1]$ with $k = 1, 2, \ldots, \mathfrak{m}$, is the k^{th} projection mapping.

Let $\gamma = (\mathfrak{p}_1 \circ \gamma, \mathfrak{p}_2 \circ \gamma, \ldots, \mathfrak{p}_{\mathfrak{m}} \circ \gamma)$ be an $m\mathrm{P}^F$N where $\mathfrak{p}_k \circ \gamma \in [0,1]$, for all $(k = 1, 2, \ldots, \mathfrak{m})$ the accuracy and score function of γ are given below:

Definition 2 [31]: The score function of $m\mathrm{P}^F$N $\gamma = (\mathfrak{p}_1 \circ \gamma, \ldots, \mathfrak{p}_{\mathfrak{m}} \circ \gamma)$ is defined as follows:

$$\mathcal{S}(\gamma) = \frac{1}{\mathfrak{m}}\Big(\sum_{k=1}^{\mathfrak{m}}(\mathfrak{p}_k \circ \gamma)\Big), \quad \mathcal{S}(\gamma) \in [0,1].$$

TABLE 2. Summary table of site selection problem reviews.

References	Aggregation operators	Problems descriptions
Ali et al. [30]	mP^F Yager A_gOs	Site selection for an oil refinery.
Rahman et al. [39]	mP^F Aczél–Alsina A_gOs	Sites identification for wind power and desalination plants.
Seikh and Mandal [42]	q-Rung orthopair fuzzy Archimedean A_gOs	Software operating unit site selection.
Gao et al. [43]	Intuitionistic linguistic A_gOs	Site selection for an offshore wind farm.
Debnath and Roy [44]	Power partitioned neutral A_gOs based on T-spherical FSs	Site selection for H_2 refueling station.
Javed et al. [45]	Spherical fuzzy neutrality A_gOs	Site selection for olive trees plantation.
Rehman et al. [46]	Dombi exponential A_gOs based on neutrosophic cubic hesitant FSs	Solid waste disposal site selection.
Attaullah et al. [47]	q-rung orthopair hesitant fuzzy rough Einstein A_gOs	Wind power plant site selection.
Seikh and Mandal [48]	Quasi-rung orthopair FSs	Electric vehicle charging station site selection.
Guleria and Bajaj [49]	(R, S)-Norm Pythagorean Fuzzy information measures based on TOPSIS method	Hydrogen power plant site selection.
Mishra et al. [50]	Single-valued neutrosophic similarity measures	Optimal site selection of electric vehicle charging station
Chinram et al. [51]	Spherical fuzzy Yager A_gOs	Site selection for a wind power station.

Definition 3 [31]: The accuracy function of mP^{F}N $\gamma = (\mathfrak{p}_1 \circ \gamma, \ldots, \mathfrak{p}_m \circ \gamma)$ is provided by

$$\mathcal{H}(\gamma) = \frac{1}{\mathrm{m}}\Big(\sum_{k=1}^{\mathrm{m}}(-1)^{k+1}(\mathfrak{p}_k \circ \gamma - 1)\Big), \quad \mathcal{H}(\gamma) \in [-1, 1].$$

Clearly, the above Definitions 2 and 3 provide an ordered relation criterion for mP^{F}Ns as below:

Definition 4 [31]: For any two mP^{F}Ns $\gamma_1 = (\mathfrak{p}_1 \circ \gamma_1, \ldots, \mathfrak{p}_m \circ \gamma_1)$ and $\gamma_1 = (\mathfrak{p}_1 \circ \gamma_2, \ldots, \mathfrak{p}_m \circ \gamma_2)$, we have

1) $\gamma_1 < \gamma_2$, *if* $\mathcal{S}(\gamma)(\gamma_1) < \mathcal{S}(\gamma)(\gamma_2)$,
2) $\gamma_1 > \gamma_2$, *if* $\mathcal{S}(\gamma)(\gamma_1) > \mathcal{S}(\gamma)(\gamma_2)$,
3) If $\mathcal{S}(\gamma)(\gamma_1) = \mathcal{S}(\gamma)(\gamma_2)$, *then*
   - $\gamma_1 < \gamma_2$ *if* $\mathcal{H}(\gamma_1) < \mathcal{H}(\gamma_2)$,
   - $\gamma_1 > \gamma_2$ *if* $\mathcal{H}(\gamma_1) > \mathcal{H}(\gamma_2)$,
   - $\gamma_1 = \gamma_2$ *if* $\mathcal{H}(\gamma_1) = \mathcal{H}(\gamma_2)$.

Some fundamental operations for mP^{F}Ns are given as [31]:

1) $\gamma_1 \boxplus \gamma_2 = \big(\mathfrak{p}_1 \circ \gamma_1 + \mathfrak{p}_1 \circ \gamma_2 - \mathfrak{p}_1 \circ \gamma_1 . \mathfrak{p}_1 \circ \gamma_2, \ldots, \mathfrak{p}_m \circ \gamma_1 + \mathfrak{p}_m \circ \gamma_2 - \mathfrak{p}_m \circ \gamma_1 . \mathfrak{p}_m \circ \gamma_2\big)$.
2) $\gamma_1 \boxtimes \gamma_2 = \big(\mathfrak{p}_1 \circ \gamma_1 . \mathfrak{p}_1 \circ \gamma_2, \ldots, \mathfrak{p}_m \circ \gamma_1 . \mathfrak{p}_m \circ \gamma_2\big)$.
3) $\phi\gamma = \big(1-(1-\mathfrak{p}_1 \circ \gamma)^{\phi}), \ldots, 1-(1-\mathfrak{p}_m \circ \gamma)^{\phi}\big), \quad \phi \geq 0$.
4) $(\gamma)^{\phi} = ((\mathfrak{p}_1 \circ \gamma)^{\phi}, \ldots, (\mathfrak{p}_m \circ \gamma)^{\phi}), \quad \phi \geq 0$.
5) $\gamma^c = (1 - \mathfrak{p}_1 \circ \gamma, \ldots, 1 - \mathfrak{p}_m \circ \gamma)$.
6) $\gamma_1 \subseteq \gamma_2$, if and only if $\mathfrak{p}_1 \circ \gamma_1 \leq \mathfrak{p}_1 \circ \gamma_2, \ldots, \mathfrak{p}_m \circ \gamma_1 \leq \mathfrak{p}_m \circ \gamma_2$.
7) $\gamma_1 \cup \gamma_2 = \big(\max(\mathfrak{p}_1 \circ \gamma_1, \mathfrak{p}_1 \circ \gamma_2), \ldots, \max(\mathfrak{p}_m \circ \gamma_1, \mathfrak{p}_m \circ \gamma_2)\big)$.
8) $\gamma_1 \cap \gamma_2 = \big(\min(\mathfrak{p}_1 \circ \gamma_1, \mathfrak{p}_1 \circ \gamma_2), \ldots, \min(\mathfrak{p}_1 \circ \gamma_1, \mathfrak{p}_1 \circ \gamma_2)\big)$.

Theorem 1 [31]: For any two arbitrary mP^{F}Ns $\gamma_1 = (\mathfrak{p}_1 \circ \gamma_1, \ldots, \mathfrak{p}_m \circ \gamma_1)$ and $\gamma_2 = (\mathfrak{p}_1 \circ \gamma_2, \ldots, \mathfrak{p}_m \circ \gamma_2)$ with $\phi, \phi_1, \phi_2 > 0$, we have

1) $\gamma_1 \boxplus \gamma_2 = \gamma_2 \boxplus \gamma_1$,
2) $\gamma_1 \boxtimes \gamma_2 = \gamma_2 \boxtimes \gamma_1$,
3) $\phi(\gamma_1 \boxplus \gamma_2) = \phi(\gamma_1) \boxplus \phi(\gamma_2)$,
4) $(\gamma_1 \boxtimes \gamma_2)^{\phi} = (\gamma_2)^{\phi} \boxtimes (\gamma_1)^{\phi}$,
5) $\phi_1(\gamma_1) \boxplus \phi_2(\gamma_1) = (\phi_1 + \phi_2)\gamma_1$,
6) $(\gamma_1)^{\phi_1} \boxtimes (\gamma_2)^{\phi_2} = (\gamma_1)^{\phi_1+\phi_2}$,
7) $((\gamma_1)^{\phi_1})^{\phi_2} = (\gamma_1)^{\phi_1\phi_2}$.

Schweizer-Sklar [35], [36] suggested operations, namely, Schweizer-Sklar product $\otimes$ and Schweizer-Sklar sum $\oplus$, which are respectively t-N and t-C$_o$N, and given as:

$$\mathcal{S}(x, y) = x \otimes y = \big((x)^{\lambda} + (y)^{\lambda} - 1\big)^{1/\lambda}, \tag{1}$$

$$\mathcal{S}^*(x, y) = x \oplus y = 1 - \big((1-x)^{\lambda} + (1-y)^{\lambda} - 1\big)^{1/\lambda}, \tag{2}$$

where $\lambda \in (-\infty, +\infty)$ and $x, y \in [0, 1]$.

III. M-POLAR FUZZY SCHWEIZER-SKLAR OPERATORS

This section firstly provides Schweizer-Sklar operations for mP^{F}Ns via Schweizer-Sklar-t-N and t-C$_o$N, and then gives $m\mathrm{P}^F$ geometric and arithmetic A_gOs based on these operations. Suppose $\gamma_1 = (\mathfrak{p}_1 \circ \gamma_1, \ldots, \mathfrak{p}_m \circ \gamma_1)$, $\gamma_2 = (\mathfrak{p}_1 \circ \gamma_2, \ldots, \mathfrak{p}_m \circ \gamma_2)$ and $\gamma = (\mathfrak{p}_1 \circ \gamma, \ldots, \mathfrak{p}_m \circ \gamma)$ are three $m\mathrm{P}^F$Ns. Then, some basic Schweizer-Sklar operations of $m\mathrm{P}^F$Ns are provided as follows:

$$\gamma_1 \oplus \gamma_2 = \Big(1 - \big((1-\mathfrak{p}_1\circ\gamma_1)^\lambda + (1-\mathfrak{p}_1\circ\gamma_2)^\lambda - 1\big)^{1/\lambda}, \ldots, 1 - \big((1-\mathfrak{p}_m\circ\gamma_1)^\lambda + (1-\mathfrak{p}_m\circ\gamma_2)^\lambda - 1\big)^{1/\lambda}\Big),$$

$$\gamma_1 \otimes \gamma_2 = \Big(\big((\mathfrak{p}_1\circ\gamma_1)^\lambda + (\mathfrak{p}_1\circ\gamma_2)^\lambda - 1\big)^{1/\lambda}, \ldots, \big((\mathfrak{p}_m\circ\gamma_1)^\lambda + (\mathfrak{p}_m\circ\gamma_2) - 1\big)^{1/\lambda}\Big),$$

$$\phi\gamma = \Big(1 - \big(\phi(1-\mathfrak{p}_1\circ\gamma)^\lambda - (\phi-1)\big)^{1/\lambda}, \ldots, 1 - \big(\phi(1-\mathfrak{p}_m\circ\gamma)^\lambda - (\phi-1)\big)^{1/\lambda}\Big),$$

$$\gamma^\phi = \Big(\big(\phi(\mathfrak{p}_1\circ\gamma)^\lambda - (\phi-1)\big)^{1/\lambda}, \ldots, \big(\phi(\mathfrak{p}_m\circ\gamma)^\lambda - (\phi-1)\big)^{1/\lambda}\Big),$$

where $\lambda \in (-\infty, +\infty)$.

A. M-POLAR FUZZY SCHWEIZER-SKLAR ARITHMETIC AGGREGATION OPERATORS

We are now ready to present $m\mathrm{P}^F$ Schweizer-Sklar arithmetic A_gOs in the following manner:

Definition 5 : Consider a set of $m\mathrm{P}^F$Ns $\gamma_\mathfrak{r} = (\mathfrak{p}_1 \circ \gamma_\mathfrak{r}, \ldots, \mathfrak{p}_m \circ \gamma_\mathfrak{r})$ with a mapping $mP^F SSWA_\nu : \gamma^\mathfrak{n} \to \gamma$ where $(\mathfrak{r} = 1, 2, \ldots, n)$ is called an $m\mathrm{P}^F$SSWA operator, which is provided as:

$$mP^F SSWA_\nu(\gamma_1, \gamma_2, \ldots, \gamma_\mathfrak{n}) = \bigoplus_{\mathfrak{r}=1}^{\mathfrak{n}} (\nu_\mathfrak{r}\gamma_\mathfrak{r}), \tag{3}$$

here $\nu = (\nu_1, \nu_2, \ldots, \nu_\mathfrak{n})$ shows the weights of each $\gamma_\mathfrak{r}$, $\forall\ \mathfrak{r} = 1, 2, 3, \ldots, n$ and $\nu_\mathfrak{r} > 0$ with $\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_\mathfrak{r} = 1$.

The following theorem and example provide a phenomenon to execute Schweizer-Sklar averaging operators on $m\mathrm{P}^F$Ns as given in Equation (3).

Theorem 2 : For a set of $m\mathrm{P}^F$Ns $\gamma_\mathfrak{r} = (\mathfrak{p}_1\circ\gamma_\mathfrak{r}, \ldots, \mathfrak{p}_m\circ\gamma_\mathfrak{r})$ where $\mathfrak{r} = 1, 2, \ldots, n$, the aggregated value of these $m\mathrm{P}^F$Ns obtained by applying an $m\mathrm{P}^F$SSWA operator is given by the following formula:

$$\begin{aligned} & mP^F SSWA_\nu(\gamma_1, \gamma_2, \ldots, \gamma_n) \\ &= \bigoplus_{\mathfrak{r}=1}^{n} (\nu_\mathfrak{r}\gamma_\mathfrak{r}), \\ &= \Big(1 - \big(\sum_{\mathfrak{r}=1}^{n} \nu_\mathfrak{r}(1-\mathfrak{p}_1\circ\gamma_\mathfrak{r})^\lambda - \sum_{\mathfrak{r}=1}^{n}\nu_\mathfrak{r} + 1\big)^{1/\lambda}, \ldots, \\ &\quad 1 - \big(\sum_{\mathfrak{r}=1}^{n} \nu_\mathfrak{r}(1-\mathfrak{p}_m\circ\gamma_\mathfrak{r})^\lambda - \sum_{\mathfrak{r}=1}^{n}\nu_\mathfrak{r} + 1\big)^{1/\lambda}\Big). \end{aligned} \tag{4}$$

Proof : We utilize the induction technique to prove it.

Case 1: For $\mathfrak{n} = 1$, the Equation (4) becomes as:

$$\begin{aligned} & mP^F SSWA_\nu(\gamma_1, \gamma_2, \ldots, \gamma_\mathfrak{n}) \\ &= \nu_1\gamma_1,\ (since\ \nu_1 = 1) \\ &= \Big(1 - \big((1-\mathfrak{p}_1\circ\gamma_1)^\lambda\big)^{1/\lambda}, \ldots, 1 - \big((1-\mathfrak{p}_m\circ\gamma_1)^\lambda\big)^{1/\lambda}\Big), \\ &= (\mathfrak{p}_1\circ\gamma_1, \ldots, \mathfrak{p}_m\circ\gamma_1), \quad (using\ \lambda = 1) \\ &= \gamma_1. \end{aligned}$$

Therefore, when $\mathfrak{n} = 1$, Equation (4) satisfied.

Case 2: Consider the Equation (4) holds for $\mathfrak{n} = \mathfrak{k}$ where $\mathfrak{k}$ is any arbitrarily natural integer, then

$$\begin{aligned} & mP^F SSWA_\nu(\gamma_1, \gamma_2, \ldots, \gamma_\mathfrak{k}) \\ &= \bigoplus_{\mathfrak{r}=1}^{\mathfrak{k}} (\nu_\mathfrak{r}\gamma_\mathfrak{r}), \\ &= \Big(1 - \big(\sum_{\mathfrak{r}=1}^{\mathfrak{k}} \nu_\mathfrak{r}(1-\mathfrak{p}_1\circ\gamma_\mathfrak{r})^\lambda - \sum_{\mathfrak{r}=1}^{\mathfrak{k}}\nu_\mathfrak{r} + 1\big)^{1/\lambda}, \ldots, \\ &\quad 1 - \big(\sum_{\mathfrak{r}=1}^{\mathfrak{k}} \nu_\mathfrak{r}(1-p_m\circ\gamma_\mathfrak{r})^\lambda - \sum_{\mathfrak{r}=1}^{\mathfrak{k}}\nu_\mathfrak{r} + 1\big)^{1/\lambda}\Big). \end{aligned}$$

Now for $\mathfrak{n} = \mathfrak{k} + 1$,

$$\begin{aligned} & mP^F SSWA_\nu(\gamma_1, \gamma_2, \ldots, \gamma_\mathfrak{k}, \gamma_{\mathfrak{k}+1}) \\ &= \bigoplus_{\mathfrak{r}=1}^{\mathfrak{k}} (\nu_\mathfrak{r}\gamma_\mathfrak{r}) \bigoplus (\nu_{\mathfrak{k}+1}\gamma_{\mathfrak{k}+1}), \\ &= \Big(1 - \big(\sum_{\mathfrak{r}=1}^{\mathfrak{k}} \nu_\mathfrak{r}(1-\mathfrak{p}_1\circ\gamma_\mathfrak{r})^\lambda - \sum_{\mathfrak{r}=1}^{\mathfrak{k}}\nu_\mathfrak{r} + 1\big)^{1/\lambda}, \ldots, \\ &\quad 1 - \big(\sum_{\mathfrak{r}=1}^{\mathfrak{k}} \nu_j(1-\mathfrak{p}_m\circ\gamma_\mathfrak{r})^\lambda - \sum_{\mathfrak{r}=1}^{\mathfrak{k}}\nu_\mathfrak{r} + 1\big)^{1/\lambda}\Big) \\ &\quad \bigoplus \Big(1 - \big(\nu_{\mathfrak{k}+1}(1-\mathfrak{p}_1\circ\gamma_{\mathfrak{k}+1})^\lambda - \nu_{\mathfrak{k}+1} + 1\big)^{1/\lambda}, \ldots, \\ &\quad 1 - \big(\nu_{\mathfrak{k}+1}(1-\mathfrak{p}_m\circ\gamma_{\mathfrak{k}+1})^\lambda - \nu_{\mathfrak{k}+1} + 1\big)^{1/\lambda}\Big), \\ &= \Big(1 - \big(\sum_{\mathfrak{r}=1}^{\mathfrak{k}+1} \nu_\mathfrak{p}(1-\mathfrak{r}_1\circ\gamma_\mathfrak{r})^\lambda - \sum_{\mathfrak{r}=1}^{\mathfrak{k}+1}\nu_\mathfrak{r} + 1\big)^{1/\lambda}, \ldots, \\ &\quad 1 - \big(\sum_{\mathfrak{r}=1}^{\mathfrak{k}+1} \nu_\mathfrak{j}(1-\mathfrak{p}_m\circ\gamma_\mathfrak{r})^\lambda - \sum_{\mathfrak{r}=1}^{\mathfrak{k}+1}\nu_\mathfrak{r} + 1\big)^{1/\lambda}\Big). \end{aligned}$$

Thus, Equation (4) is verified for $\mathfrak{n} = \mathfrak{k} + 1$. This implies that Equation (4) holds for all natural numbers $\mathfrak{n}$.

Example 1: Consider $\gamma_1 = (0.41, 0.70, 0.32, 0.26)$, $\gamma_2 = (0.91, 0.63, 0.14, 0.52)$, $\gamma_3 = (0.39, 0.23, 0.45, 0.73)$ and $\gamma_4 = (0.47, 0.69, 0.81, 0.17)$ are $4\mathrm{P}^F$Ns and $\nu = (0.21, 0.32, 0.10, 0.37)$ contains weights associated to these $4\mathrm{P}^F$Ns. Then, for $\lambda = -3$, we obtain

$$\begin{aligned} & mP^F SSWA_\nu(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \\ &= \bigoplus_{\mathfrak{r}=1}^{4} (\nu_\mathfrak{r}\gamma_\mathfrak{r}), \end{aligned}$$

$$= \Big(1 - (\sum_{\mathfrak{r}=1}^{4} \nu_{\mathfrak{r}}(1 - \mathfrak{p}_1 \circ \gamma_{\mathfrak{r}})^{\lambda} - \sum_{\mathfrak{r}=1}^{4} \nu_{\mathfrak{r}} + 1)^{1/\lambda}, \ldots,$$
$$1 - (\sum_{\mathfrak{r}=1}^{4} \nu_{\mathfrak{r}}(1 - \mathfrak{p}_4 \circ \gamma_{\mathfrak{r}})^{\lambda} - \sum_{\mathfrak{r}=1}^{4} \nu_{\mathfrak{r}} + 1)^{1/\lambda}\Big),$$
$$= \Big(1 - \big(0.21(1-0.41)^{-3} + 0.32(1-0.91)^{-3} + 0.10(1-0.39)^{-3} + 0.37(1-0.47)^{-3}\big)^{-1/3},$$
$$1 - \big(0.21(1-0.70)^{-3} + 0.32(1-0.63)^{-3} + 0.10(1-0.23)^{-3} + 0.37(1-0.69)^{-3}\big)^{-1/3},$$
$$1 - \big(0.21(1-0.32)^{-3} + 0.32(1-0.14)^{-3} + 0.10(1-0.45)^{-3} + 0.37(1-0.81)^{-3}\big)^{-1/3},$$
$$1 - \big(0.21(1-0.26)^{-3} + 0.32(1-0.52)^{-3} + 0.10(1-0.73)^{-3} + 0.37(1-0.17)^{-3}\big)^{-1/3}\Big),$$
$$= (0.8688, 0.6656, 0.7382, 0.5217).$$

We are now ready to investigate some basic properties of mFSSWA operators.

Theorem 3 (Idempotent law): Suppose a collection of '$\mathfrak{n}$' mP^{F}Ns $\gamma_{\mathfrak{r}} = (\mathfrak{p}_1 \circ \gamma_{\mathfrak{r}}, \ldots, \mathfrak{p}_{\mathfrak{m}} \circ \gamma_{\mathfrak{r}})$, which are equal such that $\gamma_{\mathfrak{r}} = \gamma$, then

$$mP^F SSWA_{\nu}(\gamma_1, \gamma_2, \ldots, \gamma_{\mathfrak{n}}) = \gamma.$$

Proof: Since $\gamma_{\mathfrak{r}} = (\mathfrak{p}_1 \circ \gamma_{\mathfrak{r}}, \ldots, \mathfrak{p}_{\mathfrak{m}} \circ \gamma_{\mathfrak{r}})$, here $\mathfrak{r} = 1, 2, \ldots, \mathfrak{n}$, then using Equation (4),

$$mP^F SSWA_{\nu}(\gamma_1, \gamma_2, \ldots, \gamma_{\mathfrak{n}})$$
$$= \bigoplus_{\mathfrak{r}=1}^{\mathfrak{n}} (\nu_{\mathfrak{r}} \gamma_{\mathfrak{r}}),$$
$$= \Big(1 - (\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_{\mathfrak{r}}(1 - \mathfrak{p}_1 \circ \gamma_{\mathfrak{r}})^{\lambda} - \sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_{\mathfrak{r}} + 1)^{1/\lambda}, \ldots,$$
$$1 - (\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_{\mathfrak{r}}(1 - \mathfrak{p}_{\mathfrak{m}} \circ \gamma_{\mathfrak{r}})^{\lambda} - \sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_{\mathfrak{r}} + 1)^{1/\lambda}\Big),$$
$$= \Big(1 - \big((1 - \mathfrak{p}_1 \circ \gamma)^{\lambda}\big)^{1/\lambda}, \ldots, 1 - \big((1 - \mathfrak{p}_{\mathfrak{m}} \circ \gamma)^{\lambda}\big)^{1/\lambda}\Big),$$
$$= (\mathfrak{p}_1 \circ \gamma, \mathfrak{p}_2 \circ \gamma, \ldots, \mathfrak{p}_{\mathfrak{m}} \circ \gamma), \ for\ \lambda = 1,$$
$$= \gamma.$$

Therefore, $mP^F SSWA_{\nu}(\gamma_1, \gamma_2, \ldots, \gamma_{\mathfrak{n}}) = \gamma$ satisfied if $\gamma_{\mathfrak{r}} = \gamma, \forall\, \mathfrak{r} = 1, \ldots, \mathfrak{n}$.

The proofs of the following two properties are similar to the proof of the above property, so we omit them.

Theorem 4 (Bounded law): Suppose $\gamma_{\mathfrak{r}} = (\mathfrak{p}_1 \circ \gamma_{\mathfrak{r}}, \mathfrak{p}_2 \circ \gamma_{\mathfrak{r}}, \ldots, \mathfrak{p}_{\mathfrak{m}} \circ \gamma_{\mathfrak{r}})$ is a family of '$\mathfrak{n}$' mP^{F}Ns with $\gamma^- = \bigcap_{\mathfrak{r}=1}^{n}(\gamma_{\mathfrak{r}})$, and $\gamma^+ = \bigcup_{\mathfrak{r}=1}^{n}(\gamma_{\mathfrak{r}})$, then

$$\gamma^- \leq mP^F SSWA_{\nu}(\gamma_1, \gamma_2, \ldots, \gamma_{\mathfrak{n}}) \leq \gamma^+. \tag{5}$$

Theorem 5 (Monotonic law): For two families of mP^{F}Ns $\gamma_{\mathfrak{r}}$ and $\gamma'_{\mathfrak{r}}$ with $\mathfrak{r} = 1, \ldots, \mathfrak{n}$, if $\gamma_{\mathfrak{r}} \leq \gamma'_{\mathfrak{r}}$, then

$$mP^F SSWA_{\nu}(\gamma_1, \gamma_2, \ldots, \gamma_{\mathfrak{n}}) \leq mP^F SSWA_{\nu}(\gamma'_1, \gamma'_2, \ldots, \gamma'_{\mathfrak{n}}). \tag{6}$$

Now we present the main concept of mP^{F}SSOWA operators.

Definition 6: For a family of mP^{F}Ns $\gamma_{\mathfrak{r}} = (\mathfrak{p}_1 \circ \gamma_{\mathfrak{r}}, \ldots, \mathfrak{p}_{\mathfrak{m}} \circ \gamma_{\mathfrak{r}})$ with $\mathfrak{r} = 1, \ldots, \mathfrak{n}$, an mP^{F}SSOWA operator is a mapping such that mP^{F}SSOWA:$\gamma^{\mathfrak{n}} \to \gamma$, which is given by

$$mP^F SSOWA_{\Omega}(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{\mathfrak{n}}) = \bigoplus_{\mathfrak{r}=1}^{\mathfrak{n}} (\nu_{\mathfrak{r}} \gamma_{\sigma(\mathfrak{r})}) \tag{7}$$

where $\nu_{\mathfrak{r}} \in (0, 1]$, $\Omega = (\nu_1, \nu_2, \ldots, \nu_{\mathfrak{n}})$ represents the weight of each $\gamma_{\mathfrak{r}}$ with $\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_{\mathfrak{r}} = 1$, and $\sigma(\mathfrak{r})$, $(\mathfrak{r} = 1, 2, \ldots, \mathfrak{n})$ indicates permutations for which $\gamma_{\sigma(\mathfrak{r}-1)} \geq \gamma_{\sigma(\mathfrak{r})}$.

Theorem 6: For a family of mP^{F}Ns $\gamma_{\mathfrak{r}} = (\mathfrak{p}_1 \circ \gamma_{\mathfrak{r}}, \ldots, \mathfrak{p}_{\mathfrak{m}} \circ \gamma_{\mathfrak{r}})$ with $\mathfrak{r} = 1, 2, \ldots, \mathfrak{n}$, the aggregated value of these mP^{F}Ns obtained by applying an mP^{F}SSOWA operators is defined as follows:

$$mP^F SSOWA_{\nu}(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{\mathfrak{n}})$$
$$= \bigoplus_{\mathfrak{r}=1}^{n} (\nu_{\mathfrak{r}} \gamma_{\sigma(\mathfrak{r})})$$
$$= \Big(1 - (\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_{\mathfrak{r}}(1 - \mathfrak{p}_1 \circ \gamma_{\sigma(\mathfrak{r})})^{\lambda} - \sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_{\mathfrak{r}} + 1)^{1/\lambda}, \ldots,$$
$$1 - (\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_{\mathfrak{r}}(1 - \mathfrak{p}_{\mathfrak{m}} \circ \gamma_{\sigma(\mathfrak{r})})^{\lambda} - \sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_{\mathfrak{r}} + 1)^{1/\lambda}\Big). \tag{8}$$

Example 2: Let $\gamma_1 = (0.37, 0.77, 0.42, 0.21)$, $\gamma_2 = (0.62, 0.81, 0.54, 0.18)$, $\gamma_3 = (0.59, 0.30, 0.71, 0.49)$ and $\gamma_4 = (0.28, 0.67, 0.12, 0.75)$ be 4P^{F}Ns with weights $\nu = (0.31, 0.24, 0.33, 0.12)$. Then, for $\lambda = -3$, we compute the score values as:

$$\mathcal{S}(\gamma_1) = \frac{0.37 + 0.77 + 0.42 + 0.21}{4} = 0.4425,$$
$$\mathcal{S}(\gamma_2) = \frac{0.62 + 0.81 + 0.54 + 0.18}{4} = 0.5375,$$
$$\mathcal{S}(\gamma_3) = \frac{0.59 + 0.30 + 0.71 + 0.49}{4} = 0.5225,$$
$$\mathcal{S}(\gamma_4) = \frac{0.28 + 0.67 + 0.12 + 0.75}{4} = 0.4550.$$

Since $\mathcal{S}(\gamma_2) > \mathcal{S}(\gamma_3) > \mathcal{S}(\gamma_4) > \mathcal{S}(\gamma_1)$, this implies

$$\gamma_{\sigma(1)} = \gamma_2 = (0.62, 0.81, 0.54, 0.18),$$
$$\gamma_{\sigma(2)} = \gamma_3 = (0.59, 0.30, 0.71, 0.49),$$
$$\gamma_{\sigma(3)} = \gamma_4 = (0.28, 0.67, 0.12, 0.75),$$
$$\gamma_{\sigma(4)} = \gamma_1 = (0.37, 0.77, 0.42, 0.21).$$

Now using Equation (8), we get

$$mP^F SSOWA_{\nu}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$$
$$= \bigoplus_{\mathfrak{r}=1}^{4} (\nu_{\mathfrak{r}} \gamma_{\sigma(\mathfrak{r})}),$$
$$= \Big(1 - (\sum_{\mathfrak{r}=1}^{4} \nu_{\mathfrak{r}}(1 - \mathfrak{p}_1 \circ \gamma_{\sigma(\mathfrak{r})})^{\lambda} - \sum_{\mathfrak{r}=1}^{4} \nu_{\mathfrak{r}} + 1)^{1/\lambda}, \ldots,$$

$$1-(\sum_{\mathfrak{r}=1}^{4}\nu_{\mathfrak{r}}(1-\mathfrak{p}_4\circ\gamma_{\sigma(\mathfrak{r})})^{\lambda}-\sum_{\mathfrak{r}=1}^{4}\nu_{\mathfrak{r}}+1)^{1/\lambda}\Big),$$
$$=\Big(1-\big(0.31(1-0.62)^{-3}+0.24(1-0.59)^{-3}$$
$$+0.33(1-0.28)^{-3}+0.12(1-0.37)^{-3}\big)^{-1/3},$$
$$+1-\big(0.31(1-0.81)^{-3}+0.24(1-0.30)^{-3}$$
$$0.33(1-0.67)^{-3}+0.12(1-0.77)^{-3}\big)^{-1/3},$$
$$1-\big(0.31(1-0.54)^{-3}+0.24(1-0.71)^{-3}$$
$$+0.33(1-0.12)^{-3}+0.12(1-0.42)^{-3}\big)^{-1/3},$$
$$1-\big(0.31(1-0.18)^{-3}+0.24(1-0.49)^{-3}$$
$$+0.33(1-0.75)^{-3}+0.12(1-0.21)^{-3}\big)^{-1/3}\Big),$$
$$=(0.5433, 0.7512, 0.5863, 0.6520).$$

Remark 1: Observe that the mP^{F}SSOWA operators verify the idempotency, boundedness, and monotonicity features that are respectively described in Theorems 3, 4, and 5 for mP^{F}SSWA operators.

Theorem 7 (Commutative law): For any two families of mP^{F}Ns $\gamma_{\mathfrak{r}}$ and $\gamma'_{\mathfrak{r}}$, where $\mathfrak{r}=1,2,\ldots,\mathfrak{n}$, we get

$$mP^F SSOWA_{\nu}(\gamma_1,\gamma_2,\gamma_3,\ldots,\gamma_{\mathfrak{n}}) = mP^F SSOWA_{\nu}(\gamma'_1,\gamma'_2,\gamma'_3,\ldots,\gamma'_{\mathfrak{n}}), \quad (9)$$

where $\gamma'_{\mathfrak{r}}$ represents arbitrary permutation of $\gamma_{\mathfrak{r}}$. *Proof*: Its proof directly followed by Equation (8).

In the following, we propose a new averaging operator called mP^{F}SSHA operator, which has the characteristics of both mP^{F}SSWA and mP^{F}SSOWA operators.

Definition 7: Considering a set of mP^{F}Ns $\gamma_{\mathfrak{r}}=(\mathfrak{p}_1\circ\gamma_{\mathfrak{r}},\ldots,\mathfrak{p}_{\mathfrak{m}}\circ\gamma_{\mathfrak{r}})$ where $\mathfrak{r}=1,2,3,\ldots,\mathfrak{n}$, an mP^{F}SSHA operator is a function mP^{F}SSHA:$\gamma^{\mathfrak{n}}\rightarrow\gamma$, which is expressed as:

$$mP^F SSHWA_{\nu,\theta}(\gamma_1,\gamma_2,\ldots,\gamma_{\mathfrak{n}}) = \bigoplus_{\mathfrak{r}=1}^{\mathfrak{n}}(\nu_{\mathfrak{r}}\gamma_{\sigma(\mathfrak{r})}), \quad (10)$$

where each $\nu_{\mathfrak{r}}\in(0,1]$ from $\nu=(\nu_1,\nu_2,\ldots,\nu_{\mathfrak{n}})$ that includes the weights of available mP^{F}Ns with $\sum_{\mathfrak{r}=1}^{\mathfrak{n}}\nu_{\mathfrak{r}}=1$, and $\gamma_{\sigma(\mathfrak{r})}$ is the largest mP^{F}N at $\mathfrak{r}^{th}$ place, which is defined as $\gamma_{\sigma(\mathfrak{r})}=(n\theta_{\mathfrak{r}})\gamma_{\mathfrak{r}}$ with $\mathfrak{r}=1,2,\ldots,\mathfrak{n}$. Note that here $\theta=(\theta_1,\theta_2,\ldots,\theta_{\mathfrak{n}})$ having the weights from experts, which satisfy $\theta_{\mathfrak{r}}\in(0,1]$, $\sum_{\mathfrak{r}=1}^{\mathfrak{n}}\theta_{\mathfrak{r}}=1$.

Observe that in the case of $\nu=(\frac{1}{\mathfrak{n}},\frac{1}{\mathfrak{n}},\ldots,\frac{1}{\mathfrak{n}})$, the operator mP^{F}SSHA becomes mP^{F}SSWA operator. The mP^{F}SSHA operator generates mP^{F}SSOWA operator, when $\theta=(\frac{1}{\mathfrak{n}},\frac{1}{\mathfrak{n}},\ldots,\frac{1}{\mathfrak{n}})$. Thus, mP^{F}SSHA operator deals with the degree and ranking of mP^{F}Ns because it is a generalized operator as compared to mP^{F}SSWA and mP^{F}SSOWA operators.

The following theorem can be easily proved using the same arguments as in Theorem 2, therefore, we omit it.

Theorem 8: For any family of mP^{F}Ns $\gamma_{\mathfrak{r}}=(\mathfrak{p}_1\circ\gamma_{\mathfrak{r}},\ldots,\mathfrak{p}_{\mathfrak{m}}\circ\gamma_{\mathfrak{r}})$ with $\mathfrak{r}=1,2,\ldots,\mathfrak{n}$, the accumulated grade of these mP^{F}Ns is computed by applying the proposed mP^{F}SSHA operator as follows:

$$mP^F SSHA_{\nu,\theta}(\gamma_1,\gamma_2,\gamma_3,\ldots,\gamma_{\mathfrak{n}})$$
$$=\bigoplus_{\mathfrak{r}=1}^{\mathfrak{n}}(\nu_{\mathfrak{r}}\gamma_{\sigma(\mathfrak{r})}),$$
$$=\Big(1-(\sum_{\mathfrak{r}=1}^{\mathfrak{n}}\nu_{\mathfrak{r}}(1-\mathfrak{p}_1\circ\gamma_{\sigma(\mathfrak{r})})^{\lambda}-\sum_{\mathfrak{r}=1}^{\mathfrak{n}}\nu_{\mathfrak{r}}+1)^{1/\lambda},\ldots,$$
$$1-(\sum_{\mathfrak{r}=1}^{\mathfrak{n}}\nu_{\mathfrak{r}}(1-\mathfrak{p}_{\mathfrak{m}}\circ\gamma_{\sigma(\mathfrak{r})})^{\lambda}-\sum_{\mathfrak{r}=1}^{\mathfrak{n}}\nu_{\mathfrak{r}}+1)^{1/\lambda}\Big). \quad (11)$$

Example 3: Let $\gamma_1=(0.22, 0.42, 0.37, 0.62)$, $\gamma_2=(0.52, 0.72, 0.68, 0.83)$, $\gamma_3=(0.64, 0.75, 0.93, 0, 32)$ and $\gamma_4=(0.47, 0.79, 0.24, 0.81)$ be 4P^{F}Ns with a weight-vector $\nu=(0.31, 0.24, 0.33, 0.12)$ associated to given 4P^{F}Ns, and a vector $\theta=(0.33, 0.12, 0.24, 0.31)$ from experts to support weights. Then, for $\lambda=-4$, we have

$$\hat{\gamma_1}=\Big(1-\big(n\theta_1(1-\mathfrak{p}_1\circ\gamma_1)^{\lambda}\big)^{1/\lambda},\ldots,$$
$$1-\big(\mathfrak{n}\theta_1(1-\mathfrak{p}_4\circ\gamma_1)^{\lambda}\big)^{1/\lambda}\Big),$$
$$=\Big(1-\big(4(0.33)(1-0.22)^{-4}\big)^{-1/4},$$
$$1-\big(4(0.33)(1-0.42)^{-4}\big)^{-1/4},$$
$$1-\big(4(0.33)(1-0.37)^{-4}\big)^{-1/4},$$
$$1-\big(4(0.33)(1-0.62)^{-4}\big)^{-1/4}\Big),$$
$$=(0.2723, 0.4589, 0.4122, 0.6455).$$

Similarly,

$$\hat{\gamma_2}=\Big(1-\big(4(0.12)(1-0.52)^{-4}\big)^{-1/4},$$
$$1-\big(4(0.12)(1-0.72)^{-4}\big)^{-1/4},$$
$$1-\big(4(0.12)(1-0.68)^{-4}\big)^{-1/4},$$
$$1-\big(4(0.12)(1-0.83)^{-4}\big)^{-1/4}\Big),$$
$$=(0.4233, 0.6636, 0.6156, 0.7958).$$
$$\hat{\gamma_3}=\Big(1-\big(4(0.24)(1-0.64)^{-4}\big)^{-1/4},$$
$$1-\big(4(0.24)(1-0.75)^{-4}\big)^{-1/4},$$
$$1-\big(4(0.24)(1-0.93)^{-4}\big)^{-1/4},$$
$$1-\big(4(0.24)(1-0.32)^{-4}\big)^{-1/4}\Big),$$
$$=(0.6363, 0.7474, 0.9293, 0.3130).$$
$$\hat{\gamma_4}=\Big(1-\big(4(0.31)(1-0.47)^{-4}\big)^{-1/4},$$
$$1-\big(4(0.31)(1-0.79)^{-4}\big)^{-1/4},$$
$$1-\big(4(0.31)(1-0.24)^{-4}\big)^{-1/4},$$
$$1-\big(4(0.33)(1-0.81)^{-4}\big)^{-1/4}\Big),$$
$$=(0.4977, 0.8010, 0.2798, 0.8199).$$

The score values of the above computed 4P^{F}Ns for $\lambda = -4$ are given as:

$$\mathcal{S}(\hat{\gamma}_1) = \frac{0.2723 + 0.4589 + 0.4122 + 0.6455}{4} = 0.4472,$$
$$\mathcal{S}(\hat{\gamma}_2) = \frac{0.4233 + 0.6636 + 0.6156 + 0.7958}{4} = 0.6245,$$
$$\mathcal{S}(\hat{\gamma}_3) = \frac{0.6363 + 0.7474 + 0.9293 + 0.3130}{4} = 0.6565,$$
$$\mathcal{S}(\hat{\gamma}_4) = \frac{0.4977 + 0.8010 + 0.2798 + 0.8199}{4} = 0.5996.$$

Since $\mathcal{S}(\hat{\gamma}_3) > \mathcal{S}(\hat{\gamma}_2) > \mathcal{S}(\hat{\gamma}_4) > \mathcal{S}(\hat{\gamma}_1)$,

$$\gamma_{\sigma(1)} = \hat{\gamma}_3 = (0.6363, 0.7474, 0.9293, 0.3130),$$
$$\gamma_{\sigma(2)} = \hat{\gamma}_2 = (0.4233, 0.6636, 0.6156, 0.7958),$$
$$\gamma_{\sigma(3)} = \hat{\gamma}_4 = (0.4977, 0.8010, 0.2798, 0.8199),$$
$$\gamma_{\sigma(4)} = \hat{\gamma}_1 = (0.2723, 0.4589, 0.4122, 0.6455).$$

Now using Equation (11), we have

$$\begin{aligned}
&mP^F SSHWA_{\nu,\theta}(\gamma_1, \ldots, \gamma_4)\\
&= \bigoplus_{\mathfrak{r}=1}^{4} (\nu_{\mathfrak{r}} \gamma_{\sigma(\mathfrak{r})}),\\
&= \Big(1 - \big(\sum_{\mathfrak{r}=1}^{4} \nu_{\mathfrak{r}}(1 - \mathfrak{p}_1 \circ \gamma_{\sigma(\mathfrak{r})})^{\lambda} - \sum_{\mathfrak{r}=1}^{4} \nu_{\mathfrak{r}} + 1\big)^{1/\lambda}, \ldots,\\
&\quad 1 - \big(\sum_{\mathfrak{r}=1}^{4} \nu_{\mathfrak{r}}(1 - \mathfrak{p}_4 \circ \gamma_{\sigma(\mathfrak{r})})^{\lambda} - \sum_{\mathfrak{r}=1}^{4} \nu_{\mathfrak{r}} + 1\big)^{1/\lambda}\Big),\\
&= \Big(1 - \big(0.31(1 - 0.6363)^{-4} + 0.24(1 - 0.4233)^{-4}\\
&\quad + \quad 0.33(1 - 0.4977)^{-4} + 0.12(1 - 0.2723)^{-4}\big)^{-1/4},\\
&\quad 1 - \big(0.31(1 - 0.7474)^{-4} + 0.24(1 - 0.6636)^{-4}\\
&\quad + \quad 0.33(1 - 0.8010)^{-4} + 0.12(1 - 0.4584)^{-4}\big)^{-1/4},\\
&\quad 1 - \big(0.31(1 - 0.9293)^{-4} + 0.24(1 - 0.6156)^{-4}\\
&\quad + \quad 0.33(1 - 0.2798)^{-4} + 0.12(1 - 0.4122)^{-4}\big)^{-1/4},\\
&\quad 1 - \big(0.31(1 - 0.3130)^{-4} + 0.24(1 - 0.7958)^{-4}\\
&\quad + \quad 0.33(1 - 0.8199)^{-4} + 0.12(1 - 0.6455)^{-4}\big)^{-1/4}\Big),\\
&= (0.5550, 0.7610, 0.9053, 0.7842).
\end{aligned}$$

Remark 2: The mP^{F}SSHA operators, as extension of both the mP^{F}SSWA and mP^{F}SSOWA operators satisfy the fundamental properties, including idempotency, boundedness, monotonicity, and commutativity.

B. M-POLAR FUZZY SCHWEIZER-SKLAR GEOMETRIC AGGREGATION OPERATORS

In this section, we present different types of Schweizer-Sklar geometric A$_g$Os for mP^{F}Ns, namely, mP^{F}SSWG, mP^{F}SSOWG, and mP^{F}SSHG operators.

Definition 8: For a collection of mP^{F}Ns $\gamma_{\mathfrak{r}} = (\mathfrak{p}_1 \circ \gamma_{\mathfrak{r}}, \ldots, \mathfrak{p}_m \circ \gamma_{\mathfrak{r}})$ where $\mathfrak{r} = 1, 2, \ldots, \mathfrak{n}$, a function $mP^F SSWG : \gamma^{\mathfrak{n}} \to \gamma$ is called an mP^{F}SSWG operator, which is given as:

$$mP^F SSWG_{\nu}(\gamma_1, \gamma_2, \ldots, \gamma_{\mathfrak{n}}) = \otimes_{\mathfrak{r}=1}^{\mathfrak{n}} (\gamma_{\mathfrak{r}})^{\nu_{\mathfrak{r}}}, \qquad (12)$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_{\mathfrak{n}})$ provides the weight of each mP^{F}N $\gamma_{\mathfrak{r}}, \forall\, \mathfrak{r} = 1, 2, \ldots, \mathfrak{n}$ with $\nu_{\mathfrak{r}} \in (0, 1]$, and $\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_{\mathfrak{r}} = 1$.

Now we provide a phenomenon to execute Schweizer-Sklar geometric operations on mP^{F}Ns by the following theorem and example.

Theorem 9: For a collection of mP^{F}Ns $\gamma_{\mathfrak{r}} = (\mathfrak{p}_1 \circ \gamma_{\mathfrak{r}}, \ldots, \mathfrak{p}_m \circ \gamma_{\mathfrak{r}})$ where $\mathfrak{r} = 1, 2, \ldots, \mathfrak{n}$, the aggregated value of these mP^{F}Ns is computed by the following formula:

$$\begin{aligned}
&mP^F SSWG_{\nu}(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{\mathfrak{n}})\\
&= \bigotimes_{\mathfrak{r}=1}^{\mathfrak{n}} (\gamma_{\mathfrak{r}})^{\nu_{\mathfrak{r}}},\\
&= \Big(\big(\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_{\mathfrak{r}}(\mathfrak{p}_1 \circ \gamma_{\mathfrak{r}})^{\lambda}\big)^{1/\lambda}, \ldots, \big(\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_{\mathfrak{r}}(\mathfrak{p}_m \circ \gamma_{\mathfrak{r}})^{\lambda}\big)^{1/\lambda}\Big).
\end{aligned} \qquad (13)$$

Proof: We utilize the induction technique to prove it.

Case 1: For $\mathfrak{n} = 1$, the Equation (13) becomes as:

$$\begin{aligned}
&mP^F SSWG_{\nu}(\gamma_1, \gamma_2, \ldots, \gamma_{\mathfrak{n}})\\
&= \gamma_1^{\nu_1}, \quad (\textit{since } \nu_1 = 1)\\
&= \Big(\big((\mathfrak{p}_1 \circ \gamma_1)^{\lambda}\big)^{1/\lambda}, \ldots, \big((\mathfrak{p}_m \circ \gamma_1)^{\lambda}\big)^{1/\lambda}\Big),\\
&= (\mathfrak{p}_1 \circ \gamma_1, \ldots, \mathfrak{p}_m \circ \gamma_1), \quad (\textit{using } \lambda = 1)\\
&= \gamma_1.
\end{aligned}$$

Thus, when $n = 1$, Equation (13) satisfied.

Case 2: Consider the Equation (13) holds for $\mathfrak{n} = \mathfrak{k}$ where $\mathfrak{k}$ is any arbitrarily natural integer, then

$$\begin{aligned}
&mP^F SSWG_{\nu}(\gamma_1, \gamma_2, \ldots, \gamma_{\mathfrak{k}})\\
&= \bigotimes_{\mathfrak{r}=1}^{\mathfrak{k}} (\gamma_{\mathfrak{r}})^{\nu_{\mathfrak{r}}},\\
&= \Big(\big(\sum_{\mathfrak{r}=1}^{\mathfrak{k}} \nu_{\mathfrak{r}}(\mathfrak{p}_1 \circ \gamma_{\mathfrak{r}})^{\lambda}\big)^{1/\lambda}, \ldots, \big(\sum_{\mathfrak{r}=1}^{\mathfrak{k}} \nu_{\mathfrak{r}}(p_m \circ \gamma_{\mathfrak{r}})^{\lambda}\big)^{1/\lambda}\Big).
\end{aligned}$$

Now for $\mathfrak{n} = \mathfrak{k} + 1$,

$$\begin{aligned}
&mP^F SSWG_{\nu}(\gamma_1, \gamma_2, \ldots, \gamma_{\mathfrak{k}}, \gamma_{\mathfrak{k}+1})\\
&= \bigotimes_{\mathfrak{r}=1}^{\mathfrak{k}} (\gamma_{\mathfrak{r}})^{\nu_{\mathfrak{r}}} \bigoplus (\gamma_{\mathfrak{k}+1})^{\nu_{\mathfrak{k}+1}},\\
&= \Big(\big(\sum_{\mathfrak{r}=1}^{\mathfrak{k}} \nu_{\mathfrak{r}}(\mathfrak{p}_1 \circ \gamma_{\mathfrak{r}})^{\lambda}\big)^{1/\lambda}, \ldots, \big(\sum_{\mathfrak{r}=1}^{\mathfrak{k}} \nu_j(\mathfrak{p}_m \circ \gamma_{\mathfrak{r}})^{\lambda}\big)^{1/\lambda}\Big)\\
&\quad \bigotimes \Big(\big(\nu_{\mathfrak{k}+1}(\mathfrak{p}_1 \circ \gamma_{\mathfrak{k}+1})^{\lambda}\big)^{1/\lambda}, \ldots, \big(\nu_{\mathfrak{k}+1}(\mathfrak{p}_m \circ \gamma_{\mathfrak{k}+1})^{\lambda}\big)^{1/\lambda}\Big),\\
&= \Big(\big(\sum_{j=1}^{\mathfrak{k}+1} \nu_{\mathfrak{p}}(\mathfrak{r}_1 \circ \gamma_{\mathfrak{r}})^{\lambda}\big)^{1/\lambda}, \ldots, \big(\sum_{\mathfrak{r}=1}^{\mathfrak{k}+1} \nu_j(\mathfrak{p}_m \circ \gamma_{\mathfrak{r}})^{\lambda}\big)^{1/\lambda}\Big).
\end{aligned}$$

Thus, Equation (13) is verified for $\mathfrak{n} = \mathfrak{k} + 1$. This implies that Equation (13) holds for all natural numbers $\mathfrak{n}$.

Example 4: Let $\gamma_1 = (0.47, 0.56, 0.62, 0.73)$, $\gamma_2 = (0.80, 0.91, 0.10, 0.27)$, $\gamma_3 = (0.55, 0.70, 0.97, 0.15)$ and $\gamma_4 = (0.73, 0.23, 0.84, 0.62)$ be 4P^{F}Ns with weights $\nu = (0.10, 0.50, 0.15, 0.25)$. Then, for $\lambda = -4$, we get

$$\begin{aligned}
&mP^F SSWG_\nu(\gamma_1, \gamma_2, \gamma_3, \gamma_4)\\
&= \bigotimes_{\mathfrak{r}=1}^{4} (\gamma_\mathfrak{r})^{\nu_\mathfrak{r}},\\
&= \Big(\big(\sum_{\mathfrak{r}=1}^{4} \nu_\mathfrak{r} (\mathfrak{p}_1 \circ \gamma_{\sigma(\mathfrak{r})})^\lambda \big)^{1/\lambda}, \ldots, \big(\sum_{\mathfrak{r}=1}^{4} \nu_\mathfrak{r} (\mathfrak{p}_4 \circ \gamma_{\sigma(\mathfrak{r})})^\lambda \big)^{1/\lambda} \Big),\\
&= \Big(\big(0.10(0.47)^{-4} + 0.50(0.80)^{-4} + 0.15(0.55)^{-4}\\
&\quad + 0.25(0.73)^{-4}\big)^{-1/4}, \big(0.10(0.56)^{-4}\\
&\quad + 0.50(0.91)^{-4} + 0.15(0.70)^{-4}\\
&\quad + 0.25(0.23)^{-4}\big)^{-1/4}, (0.10(0.62)^{-4}\\
&\quad + 0.50(0.10)^{-4} + 0.15(0.97)^{-4}\\
&\quad + 0.25(0.84)^{-4}\big)^{-1/4}, \big(0.10(0.73)^{-4}\\
&\quad + 0.50(0.27)^{-4} + 0.15(0.15)^{-4}\\
&\quad + 0.25(0.62)^{-4}\big)^{-1/4} \Big),\\
&= (0.6447, 0.3231, 0.1189, 0.2247).
\end{aligned}$$

Theorem 10 (Idempotent law): Suppose a collection of '$\mathfrak{n}$' mP^{F}Ns $\gamma_\mathfrak{r} = (\mathfrak{p}_1 \circ \gamma_\mathfrak{r}, \ldots, \mathfrak{p}_\mathfrak{m} \circ \gamma_\mathfrak{r})$, which are equal such that $\gamma_\mathfrak{r} = \gamma$, then

$$mP^F SSWG_\nu(\gamma_1, \gamma_2, \ldots, \gamma_\mathfrak{n}) = \gamma. \tag{14}$$

Proof: Since $\gamma_\mathfrak{r} = (\mathfrak{p}_1 \circ \gamma_\mathfrak{r}, \ldots, \mathfrak{p}_\mathfrak{m} \circ \gamma_\mathfrak{r})$, here $\mathfrak{r} = 1, 2, \ldots, \mathfrak{n}$, then using Equation (4),

$$\begin{aligned}
&mP^F SSWG_\nu(\gamma_1, \gamma_2, \ldots, \gamma_\mathfrak{n})\\
&= \bigotimes_{\mathfrak{r}=1}^{\mathfrak{n}} (\gamma_\mathfrak{r})^{\nu_\mathfrak{r}},\\
&= \Big(\big(\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_\mathfrak{r} (\mathfrak{p}_1 \circ \gamma_\mathfrak{r})^\lambda \big)^{1/\lambda}, \ldots, \big(\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_\mathfrak{r} (\mathfrak{p}_\mathfrak{m} \circ \gamma_\mathfrak{r})^\lambda \big)^{1/\lambda} \Big),\\
&= (\mathfrak{p}_1 \circ \gamma, \mathfrak{p}_2 \circ \gamma, \ldots, \mathfrak{p}_\mathfrak{m} \circ \gamma), \ \textit{for } \lambda = 1, \ \textit{and } \nu = 1,\\
&= \gamma.
\end{aligned}$$

Therefore, $mP^F SSWA_\nu(\gamma_1, \gamma_2, \ldots, \gamma_\mathfrak{n}) = \gamma$ satisfied if $\gamma_\mathfrak{r} = \gamma, \forall\, \mathfrak{r} = 1, \ldots, \mathfrak{n}$.

The following characteristics of the mP^{F}SSWG operators can be immediately shown by similar arguments of Theorem 10, thus, we omit them.

Theorem 11 (Bounded law): Suppose $\gamma_\mathfrak{r} = (\mathfrak{p}_1 \circ \gamma_\mathfrak{r}, \mathfrak{p}_2 \circ \gamma_\mathfrak{r}, \ldots, \mathfrak{p}_\mathfrak{m} \circ \gamma_\mathfrak{r})$ is a family of '$\mathfrak{n}$' mP^{F}Ns with $\gamma^- = \bigcap_{\mathfrak{r}=1}^{\mathfrak{n}} (\gamma_\mathfrak{r})$, and $\gamma^+ = \bigcup_{\mathfrak{r}=1}^{\mathfrak{n}} (\gamma_\mathfrak{r})$, then

$$\gamma^- \leq mP^F SSWG_\nu(\gamma_1, \gamma_2, \ldots, \gamma_\mathfrak{n}) \leq \gamma^+. \tag{15}$$

Theorem 12 (Monotonic law): For two families of mP^{F}Ns $\gamma_\mathfrak{r}$ and $\gamma'_\mathfrak{r}$ with $\mathfrak{r} = 1, 2, \ldots, \mathfrak{n}$, if $\gamma_\mathfrak{r} \leq \gamma'_\mathfrak{r}$, then

$$mP^F SSWG_\nu(\gamma_1, \gamma_2, \ldots, \gamma_\mathfrak{n}) \leq mP^F SSWG_\nu(\gamma'_1, \gamma'_2, \ldots, \gamma'_\mathfrak{n}). \tag{16}$$

Now we present the notion of mP^{F}SSOWG operator.

Definition 9: For a family of mP^{F}Ns $\gamma_\mathfrak{r} = (\mathfrak{p}_1 \circ \gamma_\mathfrak{r}, \ldots, \mathfrak{p}_\mathfrak{m} \circ \gamma_\mathfrak{r})$ with $\mathfrak{r} = 1, 2, \ldots, \mathfrak{n}$, an mP^{F}SSOWG operator is a function $mP^F SSOWG : \gamma^\mathfrak{n} \rightarrow \gamma$, which is given by

$$mP^F SSOWG_\nu(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_\mathfrak{n}) = \bigotimes_{\mathfrak{r}=1}^{\mathfrak{n}} (\gamma_{\sigma(\mathfrak{r})})^{\nu_\mathfrak{r}}, \tag{17}$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_\mathfrak{n})$ contains the weight of each $\gamma_\mathfrak{r}$, with $\nu_\mathfrak{r} \in (0, 1]$, and $\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_\mathfrak{r} = 1$, for all $\mathfrak{r} = 1, 2, \ldots, \mathfrak{n}$. Note that each $\sigma(\mathfrak{r})$ is a permutation, which satisfies $\gamma_{\sigma(\mathfrak{r}-1)} \geq \gamma_{\sigma(\mathfrak{r})}$.

Theorem 13: For a family of mP^{F}Ns $\gamma_\mathfrak{r} = (\mathfrak{p}_1 \circ \gamma_\mathfrak{r}, \ldots, \mathfrak{p}_\mathfrak{m} \circ \gamma_\mathfrak{r})$ where ($\mathfrak{r} = 1, 2, \ldots, \mathfrak{n}$), the aggregated value of these mP^{F}Ns is obtained by applying the suggested mP^{F}SSOWG operator as follows:

$$\begin{aligned}
&mP^F SSOWG_\nu(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_\mathfrak{n})\\
&= \bigotimes_{\mathfrak{r}=1}^{\mathfrak{n}} (\gamma_\mathfrak{r})^{\nu_\mathfrak{r}},\\
&= \Big(\big(\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_\mathfrak{r} (\mathfrak{p}_1 \circ \gamma_{\sigma(\mathfrak{r})})^\lambda \big)^{1/\lambda}, \ldots, \big(\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_\mathfrak{r} (\mathfrak{p}_\mathfrak{m} \circ \gamma_{\sigma(\mathfrak{r})})^\lambda \big)^{1/\lambda} \Big).
\end{aligned} \tag{18}$$

Example 5: Let $\gamma_1 = (0.23, 0.45, 0.72, 0.61)$, $\gamma_2 = (0.31, 0.61, 0.15, 0.25)$, $\gamma_3 = (0.83, 0.35, 0.55, 0.74)$ and $\gamma_4 = (0.43, 0.87, 0.50, 0.93)$ be 4P^{F}Ns having weights correspondingly from the vector $\nu = (0.30, 0.10, 0.15, 0.35)$. Then, for $\lambda = -4$, the score values of 4P^{F}Ns are computed as:

$$\begin{aligned}
\mathcal{S}(\gamma_1) &= \frac{0.23 + 0.45 + 0.72 + 0.64}{4} = 0.51,\\
\mathcal{S}(\gamma_2) &= \frac{0.31 + 0.61 + 0.15 + 0.25}{4} = 0.33,\\
\mathcal{S}(\gamma_3) &= \frac{0.83 + 0.35 + 0.55 + 0.74}{4} = 0.6175,\\
\mathcal{S}(\gamma_4) &= \frac{0.43 + 0.87 + 0.50 + 0.93}{4} = 0.6825.
\end{aligned}$$

Since $\mathcal{S}(\gamma_4) > \mathcal{S}(\gamma_3) > \mathcal{S}(\gamma_1) > \mathcal{S}(\gamma_2)$,

$$\begin{aligned}
\gamma_{\sigma(1)} &= \gamma_4 = (0.43, 0.87, 0.50, 0.93),\\
\gamma_{\sigma(2)} &= \gamma_3 = (0.83, 0.35, 0.55, 0.74),\\
\gamma_{\sigma(3)} &= \gamma_1 = (0.23, 0.45, 0.72, 0.64),\\
\gamma_{\sigma(4)} &= \gamma_2 = (0.31, 0.61, 0.15, 0.25).
\end{aligned}$$

Now using Equation (18), we obtain

$$mP^F SSOWG_\nu(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$$

$$= \bigotimes_{\mathfrak{r}=1}^{4} (\gamma_{\sigma(\mathfrak{r})})^{\nu_{\mathfrak{r}}},$$
$$= \Big(\big(\sum_{\mathfrak{r}=1}^{4} \nu_{\mathfrak{r}} (\mathfrak{p}_1 \circ \gamma_{\sigma(\mathfrak{r})})^{\lambda} \big)^{1/\lambda}, \ldots, \big(\sum_{\mathfrak{r}=1}^{4} \nu_{\mathfrak{r}} (\mathfrak{p}_m \circ \gamma_{\sigma(\mathfrak{r})})^{\lambda} \big)^{1/\lambda} \Big),$$
$$= \Big(\big(0.30(0.43)^{-4} + 0.10(0.83)^{-4} + 0.15(0.23)^{-4} + 0.35(0.31)^{-4}\big)^{-1/4}, \big(0.30(0.87)^{-4} + 0.10(0.35)^{-4} + 0.15(0.45)^{-4} + 0.35(0.61)^{-4}\big)^{-1/4}, \big(0.30(0.50)^{-4} + 0.10(0.55)^{-4} + 0.15(0.72)^{-4} + 0.35(0.15)^{-4}\big)^{-1/4}, \big(0.30(0.93)^{-4} + 0.10(0.74)^{-4} + 0.15(0.64)^{-4} + 0.35(0.25)^{-4}\big)^{-1/4} \Big),$$
$$= (0.3158, 0.5229, 0.1946, 0.3236).$$

Remark 3: Observe that $m\mathrm{P}^F$SSOWG operators fulfill some fundamental characteristics that consist of idempotency, monotonicity and boundedness, which are addressed in Theorems 10, 11, and 12 for $m\mathrm{P}^F$SSWG operators.

Theorem 14 (Commutative law) : Consider any two families of $m\mathrm{P}^F$Ns $\gamma_{\mathfrak{r}}$ and $\gamma'_{\mathfrak{r}}$ where $\mathfrak{r} = 1, 2, \ldots, \mathfrak{n}$, we get

$$mP^F SSOWG_{\nu}(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{\mathfrak{n}}) \tag{19}$$
$$= mP^F SSOWG_{\nu}(\gamma'_1, \gamma'_2, \gamma'_3, \ldots, \gamma'_{\mathfrak{n}}), \tag{20}$$

where each $\gamma'_{\mathfrak{r}}$ represents arbitrary permutation of $\gamma_{\mathfrak{r}}$.

Proof: Its proof is immediately followed by Theorem 9.

We are now ready to develop an innovative hybrid operator called $m\mathrm{P}^F$SSHG operator, which combines the features of both the $m\mathrm{P}^F$SSWG and $m\mathrm{P}^F$SSOWG operators.

Definition 10: For a collection of $m\mathrm{P}^F$Ns $\gamma_{\mathfrak{r}} = (\mathfrak{p}_1 \circ \gamma_{\mathfrak{r}}, \ldots, \mathfrak{p}_m \circ \gamma_{\mathfrak{r}})$, where $\mathfrak{r} = 1, 2, 3, \ldots, \mathfrak{n}$, an $m\mathrm{P}^F$SSHG operator is expressed as:

$$mP^F SSHG_{\nu,\theta}(\gamma_1, \gamma_2, \ldots, \gamma_{\mathfrak{n}}) = \bigotimes_{\mathfrak{r}=1}^{\mathfrak{n}} (\gamma_{\sigma(\mathfrak{r})})^{\nu_{\mathfrak{r}}},$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ indicates the weight of each $\gamma_{\mathfrak{r}}$, $\mathfrak{r} = 1, \ldots, \mathfrak{n}$, and $\gamma_{\mathfrak{r}} \in (0, 1]$ with $\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_{\mathfrak{r}} = 1$. Moreover, $\gamma_{\sigma(\mathfrak{r})}$ is the biggest $m\mathrm{P}^F$Ns at $\mathfrak{r}^{th}$ place, provided by $\gamma_{\sigma(\mathfrak{r})} = (\mathfrak{n}\theta_{\mathfrak{r}})\gamma_{\mathfrak{r}}$, $\mathfrak{r} = 1, 2, \ldots, \mathfrak{n}$, and $\theta = (\theta_1, \theta_2, \ldots, \theta_{\mathfrak{n}})$ is a weight-vector with $\theta_{\mathfrak{j}} \in (0, 1]$ that satisfy $\sum_{\mathfrak{j}=1}^{\mathfrak{n}} \theta_{\mathfrak{j}} = 1$.

Observe that in the case when $\nu = (\frac{1}{\mathfrak{n}}, \frac{1}{\mathfrak{n}}, \ldots, \frac{1}{\mathfrak{n}})$, the operator $m\mathrm{P}^F$SSHG transform into an $m\mathrm{P}^F$SSWG operator. However, the $m\mathrm{P}^F$SSHG operator generates $m\mathrm{P}^F$SSOWG operator, if $\theta = (\frac{1}{\mathfrak{n}}, \frac{1}{\mathfrak{n}}, \ldots, \frac{1}{\mathfrak{n}})$.

The following theorem can be easily shown by induction technique as used in Theorem 9.

Theorem 15: For a family of $m\mathrm{P}^F$Ns $\gamma_{\mathfrak{r}} = (\mathfrak{p}_1 \circ \gamma_{\mathfrak{r}}, \ldots, \mathfrak{p}_{\mathfrak{p}} \circ \gamma_{\mathfrak{r}})$ where $\mathfrak{r} = 1, 2, 3, \ldots, \mathfrak{n}$, the aggregated value of these $m\mathrm{P}^F$Ns using $m\mathrm{P}^F$SSHG operator is computed as:

$$mP^F SSHG\gamma_{\nu,\theta}(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{\mathfrak{n}}) = \otimes_{\mathfrak{r}=1}^{\mathfrak{n}} \Big(\gamma_{\sigma(\mathfrak{r})}\Big)^{\nu_{\mathfrak{r}}}, = \Big(\big(\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_{\mathfrak{r}} (\mathfrak{p}_1 \circ \gamma_{\sigma(\mathfrak{r})})^{\lambda} \big)^{1/\lambda}, \ldots, \big(\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \nu_{\mathfrak{r}} (\mathfrak{p}_m \circ \gamma_{\sigma(\mathfrak{r})})^{\lambda} \big)^{1/\lambda} \Big). \tag{21}$$

Example 6: Let $\gamma_1 = (0.23, 0.45, 0.72, 0.61)$, $\gamma_2 = (0.47, 0.56, 0.63, 0.23)$, $\gamma_3 = (0.26, 0.35, 0.55, 0, 24)$ and $\gamma_4 = (0.90, 0.43, 0.82, 0.24)$ be $4\mathrm{P}^F$Ns with $\nu = (0.40, 0.10, 0.25, 0.25)$, an associated vector of weights, and $\theta = (0.30, 0.10, 0.15, 0.35)$, a weight-vector from experts. Then, for $\lambda = -4$,

$$\gamma_1 = \Big(\big(\mathfrak{n}\theta_1(\mathfrak{p}_1 \circ \gamma_1)^{\lambda} - 1\big)^{1/\lambda}, \ldots, \big(\mathfrak{n}\theta_1(\mathfrak{p}_4 \circ \gamma_1)^{\lambda} - 1\big)^{1/\lambda} \Big),$$
$$= \Big(\big(4(0.30)(0.23)^{-4} - 1\big)^{-1/4}, \big(4(0.30)(0.45)^{-4} - 1\big)^{-1/4}, \big(4(0.30)(0.72)^{-4} - 1\big)^{-1/4}, \big(4(0.30)(0.61)^{-4} - 1\big)^{-1/4} \Big),$$
$$= (0.2199, 0.4337, 0.7329, 0.6010).$$

Similarly,

$$\gamma_2 = \Big(\big(4(0.10)(0.47)^{-4} - 1\big)^{-1/4}, \big(4(0.10)(0.56)^{-4} - 1\big)^{-1/4}, \big(4(0.10)(0.63)^{-4} - 1\big)^{-1/4}, \big(4(0.10)(0.23)^{-4} - 1\big)^{-1/4} \Big),$$
$$= (0.6105, 0.7556, 0.8978, 0.2897).$$
$$\gamma_3 = \Big(\big(4(0.15)(0.26)^{-4} - 1\big)^{-1/4}, \big(4(0.15)(0.35)^{-4} - 1\big)^{-1/4}, \big(4(0.15)(0.55)^{-4} - 1\big)^{-1/4}, \big(4(0.15)(0.24)^{-4} - 1\big)^{-1/4} \Big),$$
$$= (0.2960, 0.4002, 0.65113, 0.2731).$$
$$\gamma_4 = \Big(\big(4(0.35)(0.40)^{-4} - 1\big)^{-1/4}, \big(4(0.35)(0.43)^{-4} - 1\big)^{-1/4}, \big(4(0.35)(0.82)^{-4} - 1\big)^{-1/4}, \big(4(0.35)(0.24)^{-4} - 1\big)^{-1/4} \Big),$$
$$= (0.3694, 0.3978, 0.8310, 0.2208).$$

Then, the score values of these calculated $m\mathrm{P}^F$Ns for $\lambda = -4$ are given as:

$$\mathcal{S}(\gamma_1) = \frac{0.2199 + 0.4337 + 0.7329 + 0.6010}{4} = 0.4969,$$
$$\mathcal{S}(\gamma_2) = \frac{0.6105 + 0.7556 + 0.8978 + 0.2897}{4} = 0.6384,$$
$$\mathcal{S}(\gamma_3) = \frac{0.2960 + 0.4002 + 0.6513 + 0.2731}{4} = 0.4052,$$
$$\mathcal{S}(\gamma_4) = \frac{0.3694 + 0.3978 + 0.8310 + 0.2208}{4} = 0.45475.$$

Since $\mathcal{S}(\gamma_2) > \mathcal{S}(\gamma_1) > \mathcal{S}(\gamma_4) > \mathcal{S}(\gamma_3)$, so, their ranking order is provided as:

$$\gamma_{\sigma(1)} = \gamma_2 = (0.6105, 0.7556, 0.8978, 0.2897),$$

$$\gamma_{\sigma(2)} = \gamma_1 = (0.2199, 0.4337, 0.7329, 0.6010),$$
$$\gamma_{\sigma(3)} = \gamma_4 = (0.3694, 0.3978, 0.8310, 0.2208),$$
$$\gamma_{\sigma(4)} = \gamma_3 = (0.2960, 0.4002, 0.6513, 0.2731).$$

Now by Equation (21), we have

$$\begin{aligned}
&mP^F\,SSHG\gamma_{\nu,\theta}(\gamma_1,\gamma_2,\gamma_3,\gamma_4)\\
&= \otimes_{\mathfrak{r}=1}^{4}\left(\gamma_{\sigma(\mathfrak{r})}\right)^{\nu_{\mathfrak{r}}},\\
&= \Big(\big(\sum_{\mathfrak{r}=1}^{\mathfrak{n}}\nu_{\mathfrak{r}}(\mathfrak{p}_1\circ\gamma_{\sigma(\mathfrak{r})})^{\lambda}\big)^{1/\lambda},\ldots,\big(\sum_{\mathfrak{r}=1}^{\mathfrak{n}}\nu_{\mathfrak{r}}(\mathfrak{p}_{\mathfrak{m}}\circ\gamma_{\sigma(\mathfrak{r})})^{\lambda}\big)^{1/\lambda}\Big),\\
&= \Big(\big(0.40(0.6105)^{-4}+0.10(0.2199)^{-4}+0.25(0.3694)^{-4}\\
&\quad+0.25(0.2960)^{-4}\big)^{-1/4},\big(0.40(0.7556)^{-4}\\
&\quad+0.10(0.4337)^{-4}+0.25(0.3978)^{-4}\\
&\quad+0.25(0.4002)^{-4}\big)^{-1/4},\big(0.40(0.8978)^{-4}\\
&\quad+0.10(0.7329)^{-4}+0.25(0.8310)^{-4}\\
&\quad+0.25(0.6513)^{-4}\big)^{-1/4},\big(0.40(0.2897)^{-4}+0.10(0.6010)^{-4}\\
&\quad+0.25(0.2208)^{-4}+0.25(0.2731)^{-4}\big)^{-1/4}\Big),\\
&= (0.3232, 0.4528, 0.7679, 0.2634).
\end{aligned}$$

Remark 4: The $m\mathrm{P}^F$ SSHG operators, as a natural extension of both the $m\mathrm{P}^F$ SSWG and $m\mathrm{P}^F$ SSOWG operators satisfy certain basic properties of A_gOs, including idempotency, boundedness, monotonicity, and commutativity.

IV. APPLICATION TO MULTIPLE CRITERIA DECISION-MAKING

In the following section, we propose a MCDM strategy depending on our established $m\mathrm{P}^F$-Schweizer-Sklar-A_gOs to tackle different real-life MCDM challenges which comprise $m\mathrm{P}^F$ data. The terms employed for this objective are listed below.

Consider $\{\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{t}}\}$ as a universe and $\{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_{\mathfrak{n}}\}$ as a collection of parameters. Assume $\nu = (\nu_1, \nu_2, \ldots, \nu_{\mathfrak{n}})$ as a weight-vector satisfying $\sum_{\mathfrak{k}=1}^{\mathfrak{n}}\nu_{\mathfrak{k}} = 1$, $\nu_{\mathfrak{k}} \in (0, 1]$, $\forall\, \mathfrak{k} \in \{1, 2, \ldots, n\}$. Suppose an $m\mathrm{P}^F$ decision matrix $\tilde{\mathcal{M}} = (\mathfrak{d}_{\mathfrak{q}\mathfrak{k}})_{\mathfrak{t}\times\mathfrak{n}} = \left(\mathfrak{p}_1\circ\gamma_{\mathfrak{q}\mathfrak{k}}, \mathfrak{p}_2\circ\gamma_{\mathfrak{q}\mathfrak{k}}, \ldots, \mathfrak{p}_{\mathfrak{m}}\circ\gamma_{\mathfrak{q}\mathfrak{k}}\right)_{\mathfrak{t}\times\mathfrak{n}}$, which contains the membership degrees given by an expert or a group of expert. Concerning the above-mentioned terms, we develop the following algorithm to solve MCDM issues based on $m\mathrm{P}^F$ SSWA (or $m\mathrm{P}^F$ SSWG) operators.

Algorithm: Aggregation of MCDM issues using $m\mathrm{P}^F$ SSWA (or $m\mathrm{P}^F$ SSWG) operators

Input: i. $\{\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{t}}\}$, a universal set
ii. $\{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_{\mathfrak{n}}\}$, a collection of parameters
iii. $\nu = (\nu_1, \nu_2, \ldots, \nu_{\mathfrak{n}})$, a weight-vector.

Step I: Construct an $m\mathrm{P}^F$ decision matrix $\tilde{\mathcal{M}} = (\mathfrak{d}_{\mathfrak{q}\mathfrak{k}})_{\mathfrak{t}\times\mathfrak{n}} = \left(\mathfrak{p}_1\circ\gamma_{\mathfrak{q}\mathfrak{k}}, \mathfrak{p}_2\circ\gamma_{\mathfrak{q}\mathfrak{k}}, \ldots, \mathfrak{p}_{\mathfrak{m}}\circ\gamma_{\mathfrak{q}\mathfrak{k}}\right)_{\mathfrak{t}\times\mathfrak{n}}$ containing a finite set of '$\mathfrak{n}$' attributes and '$\mathfrak{t}$' alternatives.

Step II: Calculate preference values $\mathfrak{u}_{\mathfrak{q}}$, $(\mathfrak{q} = 1, 2, \ldots, \mathfrak{t})$ of objects in the $m\mathrm{P}^F$ decision-matrix $\tilde{\mathcal{M}}$ by the $m\mathrm{P}^F$ SSWA operator.

$$\begin{aligned}
\mathfrak{u}_{\mathfrak{q}} &= mP^F\,SSWA_{\nu}(\gamma_{\mathfrak{q}1}, \gamma_{\mathfrak{q}2}, \ldots, \gamma_{\mathfrak{q}\mathfrak{n}}) = \bigoplus_{\mathfrak{k}=1}^{\mathfrak{n}}(\nu_{\mathfrak{k}}\gamma_{\mathfrak{q}\mathfrak{k}}),\\
&= \Big(1-\big(\sum_{\mathfrak{k}=1}^{\mathfrak{n}}\nu_{\mathfrak{k}}(1-\mathfrak{p}_1\circ\gamma_{\mathfrak{q}\mathfrak{k}})^{\lambda}-\sum_{\mathfrak{k}=1}^{\mathfrak{n}}\nu_{\mathfrak{k}}+1\big)^{1/\lambda},\ldots,\\
&\quad 1-\big(\sum_{\mathfrak{k}=1}^{\mathfrak{n}}\nu_{\mathfrak{k}}(1-\mathfrak{p}_{\mathfrak{m}}\circ\gamma_{\mathfrak{q}\mathfrak{k}})^{\lambda}-\sum_{\mathfrak{k}=1}^{\mathfrak{n}}\nu_{\mathfrak{k}}+1\big)^{1/\lambda}\Big).
\end{aligned}$$

However, when we utilize $m\mathrm{P}^F$ SSWG operators, then

$$\begin{aligned}
&\mathfrak{u}_{\mathfrak{q}}\\
&= mP^F\,SSWG_{\nu}(\gamma_{\mathfrak{q}1}, \gamma_{\mathfrak{q}2}, \ldots, \gamma_{\mathfrak{q}\mathfrak{n}}) = \bigotimes_{\mathfrak{k}=1}^{\mathfrak{n}}(\gamma_{\mathfrak{q}\mathfrak{k}})^{\nu_{\mathfrak{k}}},\\
&= \Big(\big(\sum_{\mathfrak{k}=1}^{\mathfrak{n}}\nu_{\mathfrak{k}}(\mathfrak{p}_1\circ\gamma_{\mathfrak{q}\mathfrak{k}})^{\lambda}\big)^{1/\lambda},\\
&\qquad\ldots,\big(\sum_{\mathfrak{k}=1}^{\mathfrak{n}}\nu_{\mathfrak{k}}(\mathfrak{p}_{\mathfrak{m}}\circ\gamma_{\mathfrak{q}\mathfrak{k}})^{\lambda}\big)^{1/\lambda}\Big).
\end{aligned}$$

Step III: Compute the score values $\mathcal{S}(\mathfrak{u}_{\mathfrak{q}})$ where $\mathfrak{q} = 1, 2, \ldots, \mathfrak{t}$.

Step IV: Write all score values $\mathcal{S}(\mathfrak{u}_{\mathfrak{q}})$ in descending order. In the case if two or more values are equal then we can rank them using accuracy function.

Output: The object having maximum rank is decisive-object.

A. SELECTION OF THE BEST SITE FOR A WIND POWER STATION

Wind turbine stations, also known as wind farms, are very important to the global energy system. The fossil fuels, consisting of natural gas, petroleum, and lignite, have long been the main sources of energy because they are easy to find and have a high energy density. But burning fossil fuels releases green house gasses which harm people and environment by adding to climate change and air pollution. Furthermore, the extraction and use of fossil fuels have the potential to destroy ecosystems and cause environmental disasters. Economies that depend on fossil fuels are also more susceptible to change in prices.

For more cleaner and supportable energy future switching to renewable energy sources like hydro-power, solar electricity and wind energy is essential. Wind power plants are crucial because they provide electricity without harming the environment. One can depend on wind power for a very long time since it never run's out, unlike fossil fuels. Wind farms boost the economy and provide employment. Additionally, these power stations do not need a large amount of space and do not cause the disturbance to wildlife. Its a great way to acquire energy without harming the environment.

A nation's ability to progress cannot be determined just by its industrialization and energy output. The amount of power generated by a country is not a sufficient indicator of prosperity where the people are left destitute and homeless, and where the natural and cultural resources have been destroyed. The whole geographic framework of the nation, all of its living things, cultural, urban and social structures must be taken into account while evaluating development and advancement.

From this viewpoint, the move to renewable energy sources is absolutely necessary. In a wind power plant, wind turbine is a significant component, so, the process of wind turbine is described as: Wind turbines employ blades to harvest kinetic energy from the wind. Wind flows over the blades, creating lift and causing them to revolve. The blades are attached to a drive shaft that turns on an electric generator, which provides electricity (see Figure 1). As of 2020, hundreds of thousands of huge turbines in wind farms generates more than 650 gigawatts (GWs) of power, with 60 GWs installed per year. Wind turbines are becoming a growing major source of clean energy, and many countries employ them to minimize energy costs and reduce reliance on fossil fuels. In order to lower the cost of installation for the wind power plants and achieve optimum efficiency while in operation, site selection is crucial for that power plant. This opens the door to investigation of a site selection issue for a wind power station.

B. A CASE STUDY OF SINDH PROVINCE

This case study explores the establishment of a wind power station in the Sindh Province, Pakistan, highlighting the pressing need for renewable energy solutions in the region. Sindh, particularly the Gharo, Keti Bandar wind corridor, has emerged as a promising location for wind energy due to its consistent and high wind speeds, which often exceed 7 meter per second. These sites have become a significant hub for wind energy in Pakistan, with over 1,500 megawatt (MW) of installed capacity across different wind farms. The province suffers from significant energy shortages, primarily caused by reliance on fossil fuels, which hinders economic growth and exacerbates environmental issues. Transitioning to wind power not only addresses the energy crisis but also contributes to reducing greenhouse gas emissions. The climate in Sindh is characterized by high temperatures, ranging from 25 to 45 degree centigrade, with low annual rainfall averaging around 100-200 millimeter, creating stable conditions for wind energy generation without seasonal disruptions.

To identify the optimal site for a wind power station, four critical parameters were evaluated: cost of land, wind direction, wind speed, and locality. Data was collected from thirteen potential sites, (including Gharo, Keti Bandar, Thatta, Hyderabad (outskirts), Sukkur, Jacobabad, Shikarpur, Nawabshah, Larkana, Dadu, Mirpurkhas, Tando Allahyar, and Kotri) using field surveys, government records, and meteorological studies. The majority of operational wind power stations are concentrated in the Gharo-Keti Bandar wind corridor, which has emerged as a prominent hub for renewable energy development. Gharo and Keti Bandar host multiple wind farms, contributing significantly to the region's energy capacity, with a collective output exceeding 1,500 MW. Other locations such as Thatta have some wind projects in proximity, but their numbers are considerably lower than those in Gharo and Keti Bandar. Conversely, areas like Hyderabad (outskirts), Sukkur, Jacobabad, Shikarpur, Nawabshah, Larkana, Dadu, Mirpurkhas, Tando Allahyar, and Kotri currently lack substantial wind power installations, indicating limited activity in wind energy development outside the primary corridor. This concentration underscores the strategic importance of the Gharo-Keti Bandar region in Pakistan's efforts to diversify its energy sources and harness wind energy effectively. Each site was assigned numerical values based on qualitative assessments of these parameters, followed by a weighted scoring system to facilitate a comprehensive comparison. The results indicated that Gharo emerged as the most suitable site, scoring the highest due to its low land costs, optimal wind direction, and high wind speed. The project is expected to create local employment opportunities during both construction and operation phases while enhancing regional infrastructure and fostering economic development. Moreover, an Environmental Impact Assessment (EIA) will be crucial in evaluating the potential effects on local ecosystems, ensuring that mitigation measures are implemented to protect biodiversity. This systematic approach to site selection not only emphasizes the importance of renewable energy in addressing Sindh's energy needs but also serves as a model for future sustainable development initiatives in Pakistan.

In assessing potential sites for the wind power station in Sindh Province, four critical parameters were evaluated to ensure optimal site selection. Cost of Land was categorized into four levels: low, medium, high, and very high, allowing for a clear understanding of the economic feasibility of each site. Wind Direction was analyzed to determine prevailing wind patterns, which were classified as North, South, East, or West, ensuring that the chosen site could maximize energy generation based on favorable wind flows. Wind Speed was another key factor, classified into four categories: low, medium, high, and very high, reflecting the potential energy output that could be harnessed from each location. Lastly, Locality was examined to assess the proximity of each site to urban areas, residential communities, and major highways, as well as the accessibility for construction equipment, ensuring logistical efficiency and minimal disruption to local communities. These parameters collectively provide a comprehensive framework for evaluating the viability of each potential site for the wind power station. Suppose that the government of Pakistan is looking for a suitable site to build a new wind power plant to enhance their energy capacity according to the analysis(see Figure 2), and this critical task is assigned to a committee of experts. The possibilities in their

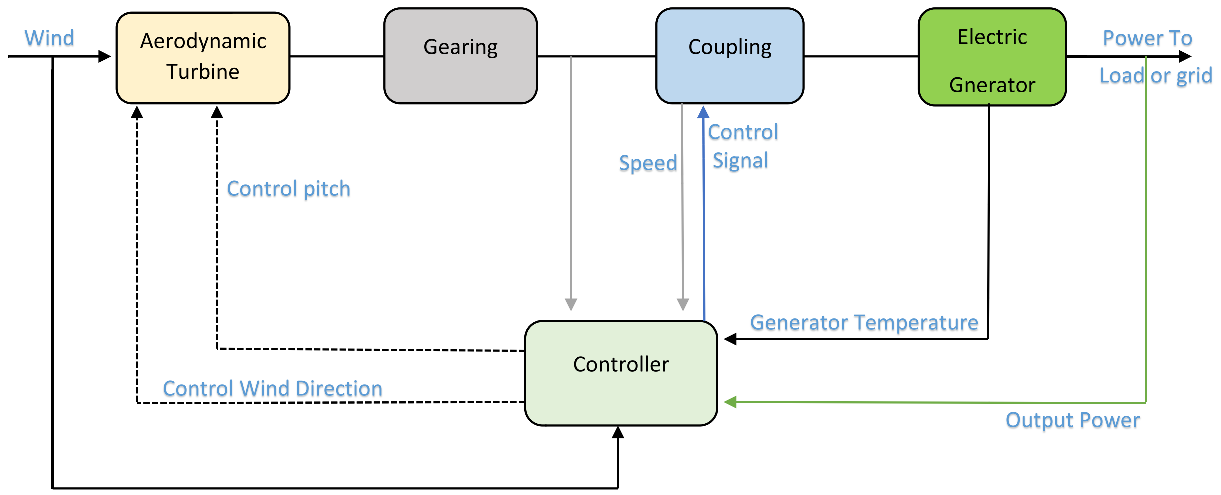

FIGURE 1. Turbine working process.

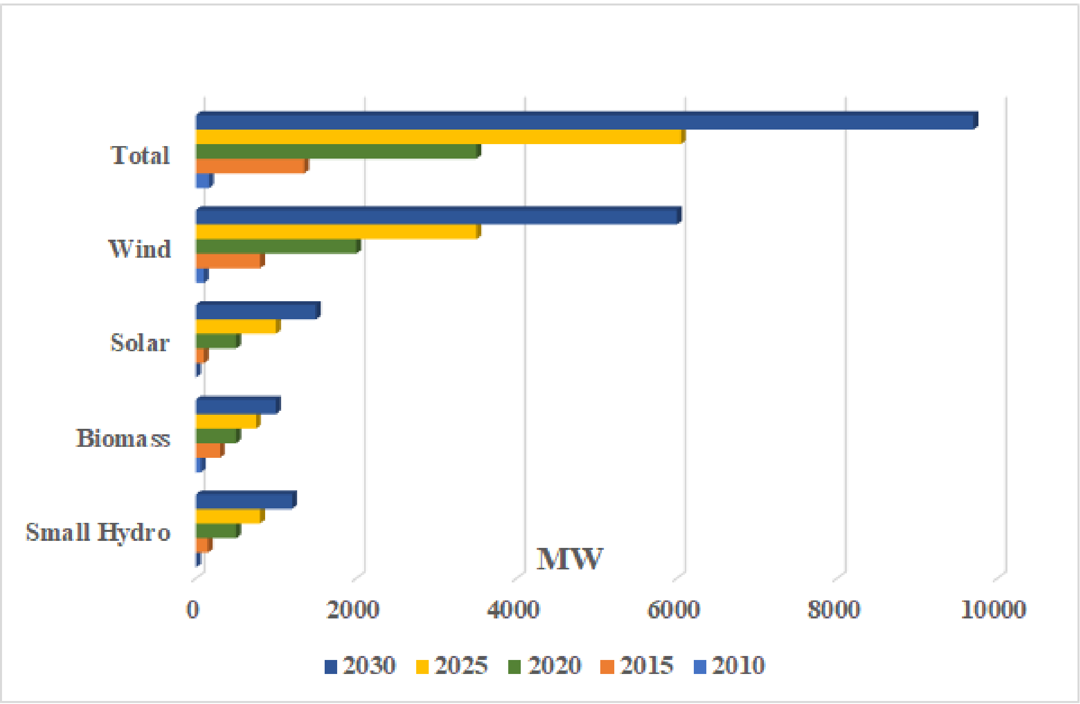

FIGURE 2. Pakistan government plan for utilization of renewable energy sources in power sector [52].

brain are Keti Bandar, Thatta, Gharo, Hyderabad (outskirts), Sukkur, Jacobabad, Shikarpur, Nawabshah, Larkana, Dadu, Mirpurkhas, Tando Allahyar, and Kotri, which are respectively denoted as: $\mathfrak{P}_1, \mathfrak{P}_2 \ldots, \mathfrak{P}_{13}$, and they further want the advice of a specialist on the options based on the following ideal parameters:

$\mathcal{X}_1$ represents the 'Cost of land',
$\mathcal{X}_2$ represents the 'Wind direction',
$\mathcal{X}_3$ represents the 'Wind speed',
$\mathcal{X}_4$ represents the 'Locality'.

For a detailed evaluation, four sub-characteristics of each parameter are studied in the following, which provide $4P^F$Ns.

- 'Cost of land' contains low, medium, high and very high.
- 'Wind direction' contains North, South, East and West.
- 'Wind speed' contains low, high, medium and very high.
- 'Locality' contains near to urban areas, near to residential communities, near to highway and accessibility for construction equipment.

Consequently, the $4P^F$ matrix is shown in tabular format (see Table 3).

According to committee, the specialist gives a suitable weight to each parameter as follows:

$$\nu_1 = 0.30, \quad \nu_2 = 0.10, \quad \nu_3 = 0.35, \quad \nu_4 = 0.25.$$

Clearly, $\Sigma_{\mathfrak{k}=1}^{4} \nu_{\mathfrak{k}} = 1$. For choosing the most suitable location for a wind power plant, we employ two operators ($m\mathrm{P}^F$SSWA and $m\mathrm{P}^F$SSWG) as below:

(1) Employing the $m\mathrm{P}^F$SSWA operator, we obtain the outcomes $\mathfrak{u}_\mathfrak{q}$ for each site $\mathfrak{P}_\mathfrak{q}$ with $\lambda = -4$ where $\mathfrak{q} = 1, 2, \ldots, 13$.

$$\begin{aligned}
\mathfrak{u}_1 &= (0.5218, 0.5871, 0.5488, 0.7144),\\
\mathfrak{u}_2 &= (0.7326, 0.6962, 0.5416, 0.9308),\\
\mathfrak{u}_3 &= (0.7055, 0.9326, 0.8411, 0.9010),\\
\mathfrak{u}_4 &= (0.6544, 0.7983, 0.9315, 0.6108),\\
\mathfrak{u}_5 &= (0.6062, 0.7748, 0.9350, 0.9291),\\
\mathfrak{u}_6 &= (0.8650, 0.7853, 0.9112, 0.5635),\\
\mathfrak{u}_7 &= (0.8830, 0.9610, 0.7608, 0.6790),\\
\mathfrak{u}_8 &= (0.7604, 0.8071, 0.9325, 0.5537),\\
\mathfrak{u}_9 &= (0.8622, 0.5493, 0.7713, 0.6173),\\
\mathfrak{u}_{10} &= (0.7923, 0.5617, 0.7402, 0.4373),\\
\mathfrak{u}_{11} &= (0.7355, 0.8154, 0.9644, 0.9740),\\
\mathfrak{u}_{12} &= (0.8961, 0.6902, 0.9576, 0.8109),\\
\mathfrak{u}_{13} &= (0.6508, 0.5333, 0.9610, 0.7716).
\end{aligned}$$

(2) Determine the score values $\mathcal{S}(\mathfrak{u}_\mathfrak{q})$ for each object $\mathfrak{P}_\mathfrak{q}$, $\mathfrak{q} = 1, 2, \ldots, 13$.

$$\begin{aligned}
&\mathcal{S}(\mathfrak{u}_1) = 0.5930, \quad \mathcal{S}(\mathfrak{u}_2) = 0.7253, \quad \mathcal{S}(\mathfrak{u}_3) = 0.8451,\\
&\mathcal{S}(\mathfrak{u}_4) = 0.7488, \quad \mathcal{S}(\mathfrak{u}_5) = 0.8113, \quad \mathcal{S}(\mathfrak{u}_6) = 0.7813,\\
&\mathcal{S}(\mathfrak{u}_7) = 0.8210, \quad \mathcal{S}(\mathfrak{u}_8) = 0.7634, \quad \mathcal{S}(\mathfrak{u}_9) = 0.7000,\\
&\mathcal{S}(\mathfrak{u}_{10}) = 0.6329, \quad \mathcal{S}(\mathfrak{u}_{11}) = 0.8723, \quad \mathcal{S}(\mathfrak{u}_{12}) = 0.8387,\\
&\mathcal{S}(\mathfrak{u}_{13}) = 0.7299.
\end{aligned}$$

(3) Arrange the scores $\mathcal{S}(\mathfrak{u}_\mathfrak{q})$, where $\mathfrak{q} = 1, 2, \ldots, 13$ in descending order calculated in the previous step as below: $\mathfrak{P}_{11} > \mathfrak{P}_3 > \mathfrak{P}_{12} > \mathfrak{P}_7 > \mathfrak{P}_5 > \mathfrak{P}_6 > \mathfrak{P}_8 > \mathfrak{P}_4 > \mathfrak{P}_{13} > \mathfrak{P}_2 > \mathfrak{P}_9 > \mathfrak{P}_{10} > \mathfrak{P}_1 > \mathfrak{P}_{10}$.

(4) Hence, $\mathfrak{P}_{11}$ (Mirpurkhas) has an optimum score value, making it the ideal location for a wind power station.

The step-wise implementation of the developed MCDM method is provided in the following Figure 3.

Similarly, we apply the same Algorithm using $m\mathrm{P}^F$SSWG operator on the the explored Application.

(1) Using the $m\mathrm{P}^F$SSWG operator with $\lambda = -4$, we get the outcomes $\mathfrak{u}_\mathfrak{q}$ for each site $\mathfrak{P}_\mathfrak{q}$ where $\mathfrak{q} = 1, 2, \ldots, 13$.

$$\begin{aligned}
\mathfrak{u}_1 &= (0.3941, 0.2584, 0.4164, 0.4937),\\
\mathfrak{u}_2 &= (0.2584, 0.6169, 0.1410, 0.6443),\\
\mathfrak{u}_3 &= (0.1413, 0.7016, 0.5096, 0.3273),\\
\mathfrak{u}_4 &= (0.1349, 0.4065, 0.5949, 0.2803),\\
\mathfrak{u}_5 &= (0.4227, 0.3359, 0.1300, 0.5988),\\
\mathfrak{u}_6 &= (0.2584, 0.5620, 0.4290, 0.1764),
\end{aligned}$$

TABLE 3. $4\mathrm{P}^F$ decision matrix.

$\tilde{\mathcal{M}}$	$\mathcal{X}_1$	$\mathcal{X}_2$
$\mathfrak{P}_1$	(0.57,0.67,0.47,0.78)	(0.40,0.56,0,26,0.57)
$\mathfrak{P}_2$	(0.80,0.57,0.60,0.73)	(0.58,0.70,0.25,0.40)
$\mathfrak{P}_3$	(0.54,0.60,0.55,0.29)	(0.40,0.56,0.91,0.54)
$\mathfrak{P}_4$	(0.10,0.85,0.90,0.55)	(0.57,0.52,0.35,0.70)
$\mathfrak{P}_5$	(0.56,0.80,0.75,0.60)	(0.40,0.85,0.10,0.96)
$\mathfrak{P}_6$	(0.90,0.45,0.77,0.35)	(0.45,0.80,0.95,0.10)
$\mathfrak{P}_7$	(0.31,0.12,0.74,0.76)	(0.91,0.22,0.47,0.39)
$\mathfrak{P}_8$	(0.27,0.81,0.95,0.32)	(0.36,0.91,0.16,0.47)
$\mathfrak{P}_9$	(0.33,0.18,0.83,0.25)	(0.92,0.22,0.63,0.40)
$\mathfrak{P}_{10}$	(0.41,0.33,0.19,0.49)	(0.15,0.27,0.34,0.60)
$\mathfrak{P}_{11}$	(0.75,0.86,0.47,0.28)	(0.81,0.76,0.98,0.21)
$\mathfrak{P}_{12}$	(0.67,0.77,0.35,0.83)	(0.57,0.15,0.67,0.40)
$\mathfrak{P}_{13}$	(0.28,0.42,0.74,0.83)	(0.79,0.47,0.29,0.63)
	$\mathcal{X}_3$	$\mathcal{X}_4$
$\mathfrak{P}_1$	(0.55,0.20,0.57,0.40)	(0.30,0.54,0.60,0.65)
$\mathfrak{P}_2$	(0.20,0.75,0.57,0.90)	(0.50,0.56,0.10,0.95)
$\mathfrak{P}_3$	(0.77,0.92,0.68,0.30)	(0.10,0.95,0.40,0.93)
$\mathfrak{P}_4$	(0.40,0.55,0.89,0.65)	(0.75,0.30,0.95,0.20)
$\mathfrak{P}_5$	(0.35,0.40,0.95,0.50)	(0.70,0.25,0.10,0.85)
$\mathfrak{P}_6$	(0.20,0.65,0.40,0.59)	(0.56,0.84,0.36,0.63)
$\mathfrak{P}_7$	(0.43,0.97,0.10,0.30)	(0.91,0.79,0.82,0.37)
$\mathfrak{P}_8$	(0.40,0.25,0.57,0.30)	(0.83,0.76,0.82,0.67)
$\mathfrak{P}_9$	(0.82,0.37,0.18,0.70)	(0.25,0.67,0.10,0.38)
$\mathfrak{P}_{10}$	(0.84,0.65,0.14,0.10)	(0.23,0.40,0.75,0.31)
$\mathfrak{P}_{11}$	(0.45,0.66,0.10,0.98)	(0.73,0.68,0.30,0.25)
$\mathfrak{P}_{12}$	(0.92,0.14,0.76,0.83)	(0.25,0.16,0.97,0.44)
$\mathfrak{P}_{13}$	(0.60,0.45,0.97,0.39)	(0.30,0.63,0.40,0.19)

$$\begin{aligned}
\mathfrak{u}_7 &= (0.3900, 0.1609, 0.1300, 0.3567),\\
\mathfrak{u}_8 &= (0.3380, 0.3236, 0.2826, 0.3397),\\
\mathfrak{u}_9 &= (0.3245, 0.2315, 0.1370, 0.3210),\\
\mathfrak{u}_{10} &= (0.2406, 0.3658, 0.1816, 0.1297),\\
\mathfrak{u}_{11} &= (0.5543, 0.7133, 0.1297, 0.2792),\\
\mathfrak{u}_{12} &= (0.3496, 0.1609, 0.4627, 0.5379),\\
\mathfrak{u}_{13} &= (0.3317, 0.4627, 0.4459, 0.2632).
\end{aligned}$$

(2) We compute the score values $\mathcal{S}(\mathfrak{u}_\mathfrak{q})$ for each site $\mathfrak{P}_\mathfrak{q}$ where $\mathfrak{q} = 1, 2, \ldots, 13$.

$$\begin{aligned}
&\mathcal{S}(\mathfrak{u}_1) = 0.3970, \quad \mathcal{S}(\mathfrak{u}_2) = 0.4145, \quad \mathcal{S}(\mathfrak{u}_3) = 0.4180,\\
&\mathcal{S}(\mathfrak{u}_4) = 0.3542, \quad \mathcal{S}(\mathfrak{u}_5) = 0.3718, \quad \mathcal{S}(\mathfrak{u}_6) = 0.3565,\\
&\mathcal{S}(\mathfrak{u}_7) = 0.2594, \quad \mathcal{S}(\mathfrak{u}_8) = 0.3210, \quad \mathcal{S}(\mathfrak{u}_9) = 0.2535,\\
&\mathcal{S}(\mathfrak{u}_{10}) = 0.2294, \quad \mathcal{S}(\mathfrak{u}_{11}) = 0.4191, \quad \mathcal{S}(\mathfrak{u}_{12}) = 0.3778,\\
&\mathcal{S}(\mathfrak{u}_{13}) = 0.3759.
\end{aligned}$$

(3) Next, we arrange the calculated scores $\mathcal{S}(\mathfrak{u}_\mathfrak{q})$ in descending order where $\mathfrak{q} = 1, 2, \ldots, 13$.
$\mathfrak{P}_3 > \mathfrak{P}_{11} > \mathfrak{P}_2 > \mathfrak{P}_1 > \mathfrak{P}_{12} > \mathfrak{P}_{13} > \mathfrak{P}_6 > \mathfrak{P}_4 > \mathfrak{P}_5 > \mathfrak{P}_8 > \mathfrak{P}_7 > \mathfrak{P}_9 > \mathfrak{P}_{10}$.

(4) Thus, an extreme score value of $\mathfrak{P}_3$ (Gharo) making it the ideal location for a wind power station.

V. COMPARISON ANALYSIS AND DISCUSSION

This section provides qualitative as well as quantitative comparisons and disadvantages of the developed $m\mathrm{P}^F$ Schweizer-Sklar $\mathrm{A_gOs}$ with existing $\mathrm{A_gOs}$, including $m\mathrm{P}^F$ Yager $\mathrm{A_gOs}$, $m\mathrm{P}^F$ Aczel-Alsina $\mathrm{A_gOs}$ and $m\mathrm{P}^F$ Dombi $\mathrm{A_gOs}$.

BY mPFSSWA aggregation operator.

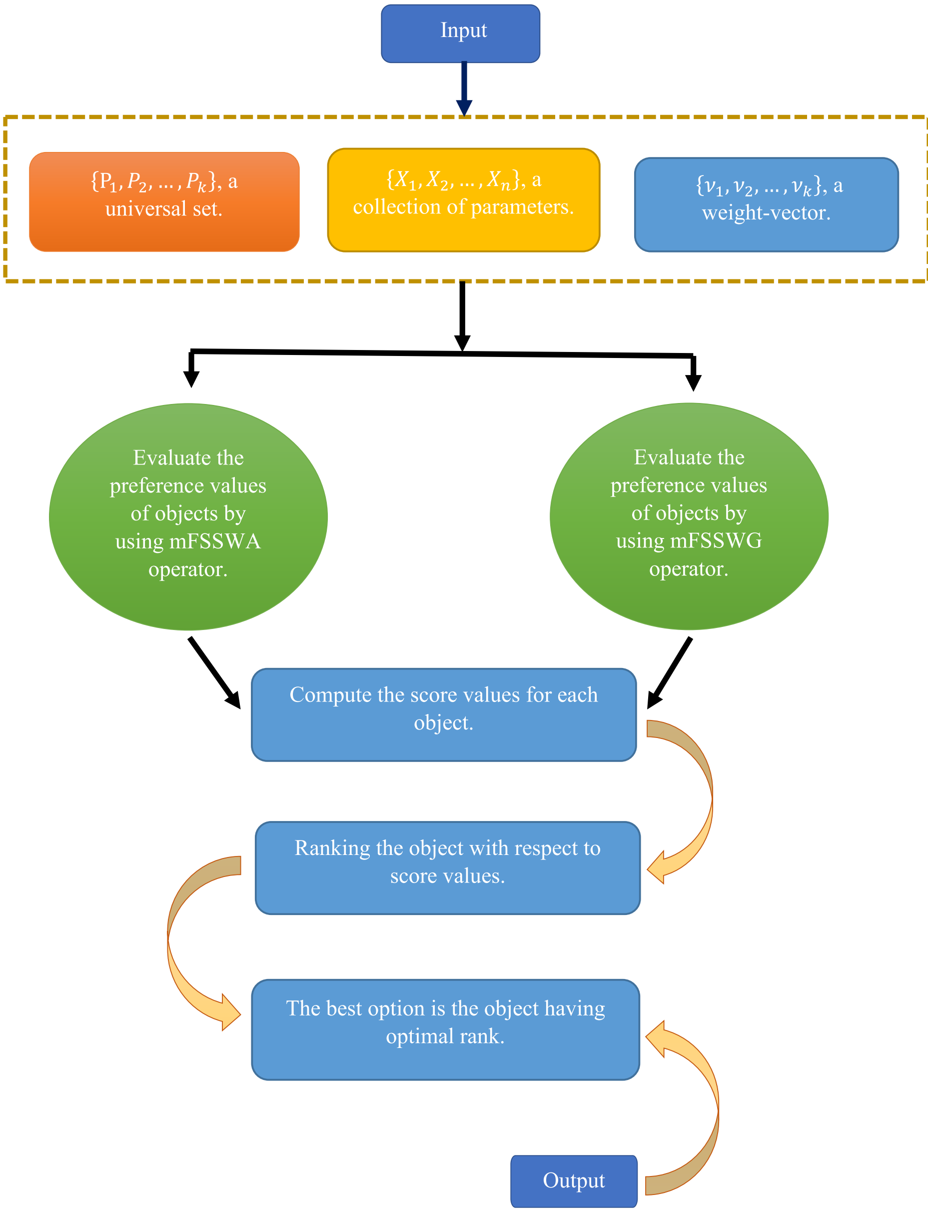

FIGURE 3. Flowchart diagram.

A. COMPARISON

Since the A_gOs make it easier to collect and understand huge data-set by combining the meanings of several connected pieces of data into a single piece. To tackle $m\mathrm{P}^F$ information, several $AgOs$ have been proposed by integrating the concepts of $m\mathrm{P}^F$ sets with different t-Ns and t-CoNs-based operations, such as $m\mathrm{P}^F$ Yager $AgOs$ [30], $m\mathrm{P}^F$ Aczel-Alsina $AgOs$ [39], and $m\mathrm{P}^F$ Dombi $AgOs$ [34]. However, Schweizer-Sklar $AgOs$ are introduced in this work due to several reasons or advantages over existing $AgOs$. First, the fusion of Schweizer-Sklar operations with $m\mathrm{P}^F$ sets has not been addressed to date. Second, the range of parameters involved in $m\mathrm{P}^F$ Schweizer-Sklar operators is broader in comparison to other $AgOs$, such as those based on Hamacher, Dombi, Yager, Frank, and Aczel-Alsina operations. The application 'choosing an appropriate site for a wind power station' is considered for the comparison purpose. Clearly, the results of $m\mathrm{P}^F$ Yager A_gOs, $m\mathrm{P}^F$ Aczel-Alsina A_gOs and $m\mathrm{P}^F$ Dombi A_gOs are compared with the results of the proposed $m\mathrm{P}^F$ Schweizer-Sklar weighted averaging ($m\mathrm{P}^F$SSWA) A_gOs and $m\mathrm{P}^F$ Schweizer-Sklar weighted geometric ($m\mathrm{P}^F$SSWG) A_gOs. Tables 4-5 and Figure 4

TABLE 4. Comparison of mP^F Schweizer-Sklar A_gOs with mP^F Yagar A_gOs [30], mP^F Aczel-Alsina A_gOs [39], mP^F Dombi A_gOs [34], mP^F AHP [40], mP^F TOPSIS [33], [41] and mP^F ELECTRE-I [33] methods.

Scores\ A_gOs	$m\mathrm{P^F}$SSWA	$m\mathrm{P^F}$SSWG	$m\mathrm{P^F}$YWA	$m\mathrm{P^F}$YWG	$m\mathrm{P^F}$-AHP	$m\mathrm{P^F}$ TOPSIS
$\mathcal{S}(\mathfrak{u}_1)$	0.5930	0.3970	0.3454	0.5877	0.2417	0.2048
$\mathcal{S}(\mathfrak{u}_2)$	0.7253	0.4145	0.3701	0.7206	0.7320	0.6311
$\mathcal{S}(\mathfrak{u}_3)$	0.8451	0.4180	0.3888	0.8412	0.6403	0.6325
$\mathcal{S}(\mathfrak{u}_4)$	0.7488	0.3542	0.2941	0.7438	0.3911	0.5713
$\mathcal{S}(\mathfrak{u}_5)$	0.8113	0.3718	0.3400	0.8071	0.4921	0.5849
$\mathcal{S}(\mathfrak{u}_6)$	0.7813	0.3565	0.3209	0.7776	0.4579	0.6100
$\mathcal{S}(\mathfrak{u}_7)$	0.8210	0.2594	0.2364	0.8163	0.3353	0.4870
$\mathcal{S}(\mathfrak{u}_8)$	0.7634	0.3210	0.2835	0.7666	0.3550	0.4966
$\mathcal{S}(\mathfrak{u}_9)$	0.7000	0.2535	0.2271	0.6905	0.3168	0.4823
$\mathcal{S}(\mathfrak{u}_{10})$	0.6329	0.2294	0.2017	0.5994	0.2074	0.2137
$\mathcal{S}(\mathfrak{u}_{11})$	0.8723	0.4191	0.3961	0.8705	0.8622	0.6474
$\mathcal{S}(\mathfrak{u}_{12})$	0.8387	0.3778	0.3347	0.8347	0.2813	0.3914
$\mathcal{S}(\mathfrak{u}_{13})$	0.7299	0.3759	0.3416	0.7220	0.3634	0.5429

Scores\ A_gOs	$m\mathrm{P^F}$AAWA	$m\mathrm{P^F}$AAWG	$m\mathrm{P^F}$DWA	$m\mathrm{P^F}$DWG	$m\mathrm{P^F}$ ELECTRE-I
$\mathcal{S}(\mathfrak{u}_1)$	0.3777	0.6001	0.3646	0.1392	0.3026
$\mathcal{S}(\mathfrak{u}_2)$	0.3971	0.7310	0.3805	0.2638	0.6952
$\mathcal{S}(\mathfrak{u}_3)$	0.4064	0.8488	0.3934	0.1477	0.7045
$\mathcal{S}(\mathfrak{u}_4)$	0.3318	0.7555	0.3141	0.2384	0.6419
$\mathcal{S}(\mathfrak{u}_5)$	0.3592	0.8159	0.3502	0.1796	0.6720
$\mathcal{S}(\mathfrak{u}_6)$	0.3458	0.7895	0.3356	0.2106	0.6743
$\mathcal{S}(\mathfrak{u}_7)$	0.2539	0.8257	0.2489	0.1709	0.5520
$\mathcal{S}(\mathfrak{u}_8)$	0.3111	0.7813	0.3027	0.2115	0.5647
$\mathcal{S}(\mathfrak{u}_9)$	0.2478	0.7112	0.2425	0.2789	0.4398
$\mathcal{S}(\mathfrak{u}_{10})$	0.2224	0.6249	0.2179	0.3614	0.3964
$\mathcal{S}(\mathfrak{u}_{11})$	0.4106	0.8742	0.4273	0.4103	0.7231
$\mathcal{S}(\mathfrak{u}_{12})$	0.3615	0.8426	0.3476	0.1541	0.4106
$\mathcal{S}(\mathfrak{u}_{13})$	0.3655	0.7375	0.3562	0.2542	0.5963

TABLE 5. Comparison between the ranking results of the proposed mP^F Schweizer-Sklar A_gOs with mP^F Yagar A_gOs [30] and mP^F Aczel-Alsina A_gOs [39], mP^F Dombi A_gOs [34], mP^F TOPSIS [33], [41] and mP^F ELECTRE-I [33] methods.

A_gO	Ranking order	Choice
$m\mathrm{P^F}$SSWA	$\mathfrak{P}_{11} > \mathfrak{P}_3 > \mathfrak{P}_{12} > \mathfrak{P}_7 > \mathfrak{P}_5 > \mathfrak{P}_6 > \mathfrak{P}_8 > \mathfrak{P}_4 > \mathfrak{P}_{13} > \mathfrak{P}_2 > \mathfrak{P}_9 > \mathfrak{P}_{10} > \mathfrak{P}_1$	$\mathfrak{P}_{11}$
$m\mathrm{P^F}$SSWG	$\mathfrak{P}_1 1 > \mathfrak{P}_3 > \mathfrak{P}_2 > \mathfrak{P}_1 > \mathfrak{P}_{12} > \mathfrak{P}_{13} > \mathfrak{P}_6 > \mathfrak{P}_4 > \mathfrak{P}_5 > \mathfrak{P}_8 > \mathfrak{P}_7 > \mathfrak{P}_9 > \mathfrak{P}_{10}$	$\mathfrak{P}_{11}$
$m\mathrm{P^F}$YWA	$\mathfrak{P}_{11} > \mathfrak{P}_3 > \mathfrak{P}_2 > \mathfrak{P}_1 > \mathfrak{P}_{13} > \mathfrak{P}_5 > \mathfrak{P}_{12} > \mathfrak{P}_6 > \mathfrak{P}_4 > \mathfrak{P}_8 > \mathfrak{P}_7 > \mathfrak{P}_9 > \mathfrak{P}_{10}$	$\mathfrak{P}_{11}$
$m\mathrm{P^F}$YWG	$\mathfrak{P}_{11} > \mathfrak{P}_3 > \mathfrak{P}_{12} > \mathfrak{P}_7 > \mathfrak{P}_5 > \mathfrak{P}_6 > \mathfrak{P}_8 > \mathfrak{P}_4 > \mathfrak{P}_2 > \mathfrak{P}_{13} > \mathfrak{P}_9 > \mathfrak{P}_{10} > \mathfrak{P}_1$	$\mathfrak{P}_{11}$
$m\mathrm{P^F}$AAWA	$\mathfrak{P}_{11} > \mathfrak{P}_3 > \mathfrak{P}_2 > \mathfrak{P}_1 > \mathfrak{P}_{13} > \mathfrak{P}_{12} > \mathfrak{P}_5 > \mathfrak{P}_6 > \mathfrak{P}_4 > \mathfrak{P}_8 > \mathfrak{P}_7 > \mathfrak{P}_9 > \mathfrak{P}_{10}$	$\mathfrak{P}_{11}$
$m\mathrm{P^F}$AAWG	$\mathfrak{P}_{11} > \mathfrak{P}_3 > \mathfrak{P}_{12} > \mathfrak{P}_7 > \mathfrak{P}_5 > \mathfrak{P}_6 > \mathfrak{P}_8 > \mathfrak{P}_4 > \mathfrak{P}_{13} > \mathfrak{P}_2 > \mathfrak{P}_9 > \mathfrak{P}_{10} > \mathfrak{P}_1$	$\mathfrak{P}_{11}$
$m\mathrm{P^F}$DWA	$\mathfrak{P}_{11} > \mathfrak{P}_3 > \mathfrak{P}_2 > \mathfrak{P}_1 > \mathfrak{P}_{13} > \mathfrak{P}_5 > \mathfrak{P}_{12} > \mathfrak{P}_6 > \mathfrak{P}_4 > \mathfrak{P}_8 > \mathfrak{P}_7 > \mathfrak{P}_9 > \mathfrak{P}_{10}$	$\mathfrak{P}_{11}$
$m\mathrm{P^F}$DWG	$\mathfrak{P}_{11} > \mathfrak{P}_{10} > \mathfrak{P}_9 > \mathfrak{P}_2 > \mathfrak{P}_{13} > \mathfrak{P}_4 > \mathfrak{P}_8 > \mathfrak{P}_6 > \mathfrak{P}_5 > \mathfrak{P}_7 > \mathfrak{P}_{12} > \mathfrak{P}_3 > \mathfrak{P}_1$	$\mathfrak{P}_{11}$
$m\mathrm{P^F}$-AHP	$\mathfrak{P}_{11} > \mathfrak{P}_2 > \mathfrak{P}_3 > \mathfrak{P}_5 > \mathfrak{P}_6 > \mathfrak{P}_4 > \mathfrak{P}_{13} > \mathfrak{P}_8 > \mathfrak{P}_7 > \mathfrak{P}_9 > \mathfrak{P}_{12} > \mathfrak{P}_1 > \mathfrak{P}_{10}$	$\mathfrak{P}_{11}$
$m\mathrm{P^F}$ TOPSIS	$\mathfrak{P}_{11} > \mathfrak{P}_3 > \mathfrak{P}_2 > \mathfrak{P}_6 > \mathfrak{P}_5 > \mathfrak{P}_4 > \mathfrak{P}_{13} > \mathfrak{P}_8 > \mathfrak{P}_7 > \mathfrak{P}_9 > \mathfrak{P}_{12} > \mathfrak{P}_{10} > \mathfrak{P}_1$	$\mathfrak{P}_{11}$
$m\mathrm{P^F}$ ELECTRE-I	$\mathfrak{P}_{11} > \mathfrak{P}_3 > \mathfrak{P}_2 > \mathfrak{P}_6 > \mathfrak{P}_5 > \mathfrak{P}_4 > \mathfrak{P}_{13} > \mathfrak{P}_8 > \mathfrak{P}_7 > \mathfrak{P}_9 > \mathfrak{P}_{12} > \mathfrak{P}_{10} > \mathfrak{P}_1$	$\mathfrak{P}_{11}$

TABLE 6. The behavior of different parametric values (λ) using mP^F SSWA operators on the explored application.

λ	$\mathcal{S}(\hat{\mathfrak{u}_1})$	$\mathcal{S}(\hat{\mathfrak{u}_2})$	$\mathcal{S}(\hat{\mathfrak{u}_3})$	$\mathcal{S}(\hat{\mathfrak{u}_4})$	$\mathcal{S}(\hat{\mathfrak{u}_5})$	$\mathcal{S}(\hat{\mathfrak{u}_6})$	$\mathcal{S}(\hat{\mathfrak{u}_7})$	$\mathcal{S}(\hat{\mathfrak{u}_8})$	$\mathcal{S}(\hat{\mathfrak{u}_9})$	$\mathcal{S}(\hat{\mathfrak{u}_{10}})$	$\mathcal{S}(\hat{\mathfrak{u}_{11}})$	$\mathcal{S}(\hat{\mathfrak{u}_{12}})$	$\mathcal{S}(\hat{\mathfrak{u}_{13}})$
-1	0.5537	0.6711	0.7513	0.6787	0.7337	0.6936	0.7423	0.6826	0.5905	0.5544	0.8136	0.7597	0.6541
-2	0.5700	0.6964	0.8015	0.7112	0.7752	0.7425	0.7849	0.7265	0.6435	0.5917	0.8514	0.8040	0.6888
-3	0.5830	0.7136	0.8293	0.7337	0.7978	0.7674	0.8075	0.7548	0.6775	0.6161	0.8647	0.8264	0.7123
-4	0.5930	0.7253	0.8450	0.7487	0.8113	0.7813	0.8210	0.7735	0.7000	0.6329	0.8723	0.8387	0.7292
-5	0.6009	0.7335	0.8547	0.7591	0.8203	0.7900	0.8296	0.7863	0.7155	0.6452	0.8776	0.8460	0.7415
-6	0.6071	0.7394	0.8611	0.7665	0.8267	0.7960	0.8355	0.7953	0.7264	0.6546	0.8814	0.8508	0.7509
-7	0.6120	0.7439	0.8655	0.7721	0.8316	0.8003	0.8397	0.8020	0.7344	0.6621	0.8843	0.8541	0.7580
-8	0.6160	0.7474	0.8688	0.7763	0.8355	0.8036	0.8430	0.8070	0.7405	0.6681	0.8867	0.8566	0.7637
-9	0.6194	0.7501	0.8714	0.7798	0.8386	0.8062	0.8455	0.8109	0.7451	0.6730	0.8886	0.8585	0.7682
-10	0.6222	0.7523	0.8734	0.7825	0.8411	0.8083	0.8475	0.8140	0.7488	0.6772	0.8903	0.8599	0.7719

provide the comparative analysis between the score values and ranking results of mP^F Schweizer-Sklar A_gOs with mP^F Yagar weighted (mP^FYW) averaging and geometric A_gOs, mP^F Aczel-Alsina weighted (mP^FAAW) averaging and geometric A_gOs, mP^F Dombi weighted (mP^FDW) averaging and geometric A_gOs, mP^F AHP, mP^F TOPSIS, and

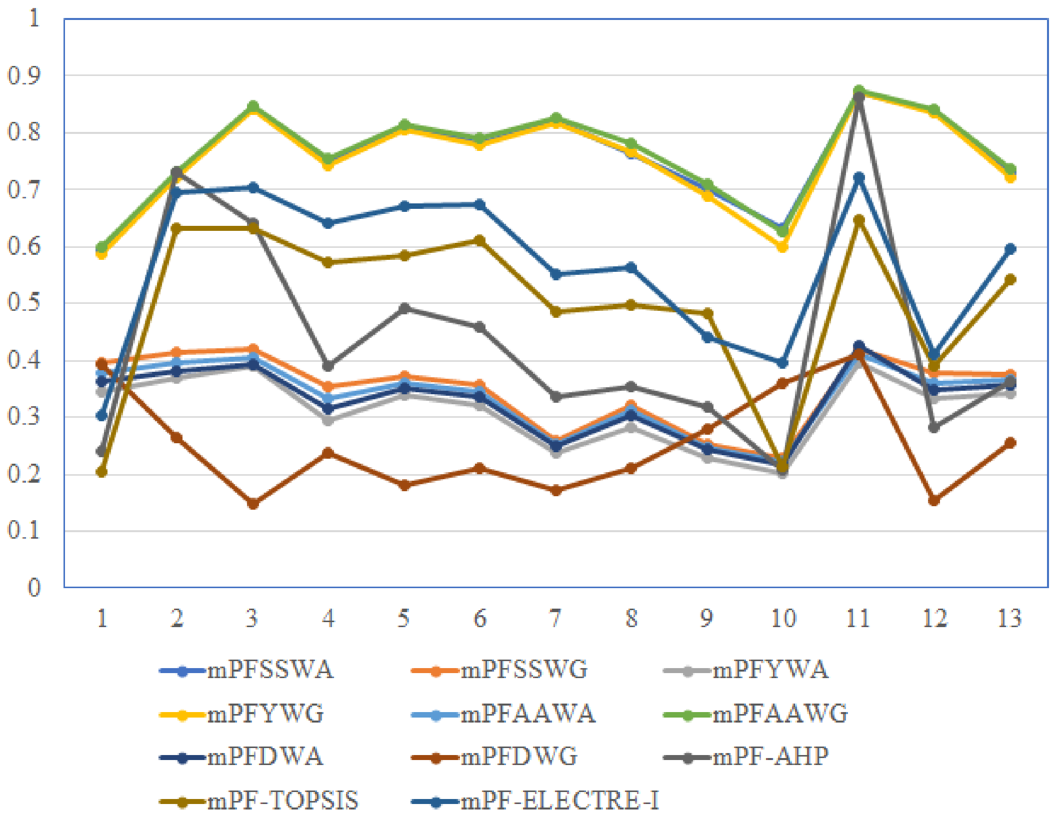

FIGURE 4. Comparison among the initiated $m\mathrm{P}^F$ Schweizer-Sklar A_gOs and existing $m\mathrm{P}^F$ Yagar A_gOs [30], $m\mathrm{P}^F$ Aczel-Alsina A_gOs [39], $m\mathrm{P}^F$ Dombi A_gOs [34], $m\mathrm{P}^F$ TOPSIS [33], [41] and $m\mathrm{P}^F$ ELECTRE-I [33] methods.

TABLE 7. The behavior of different parametric values (λ) using $m\mathrm{P}^F$ SSWA operators on the proposed application.

λ	$\mathcal{S}(\hat{u_1})$	$\mathcal{S}(\hat{u_2})$	$\mathcal{S}(\hat{u_3})$	$\mathcal{S}(\hat{u_4})$	$\mathcal{S}(\hat{u_5})$	$\mathcal{S}(\hat{u_6})$	$\mathcal{S}(\hat{u_7})$	$\mathcal{S}(\hat{u_8})$	$\mathcal{S}(\hat{u_9})$	$\mathcal{S}(\hat{u_{10}})$	$\mathcal{S}(\hat{u_{11}})$	$\mathcal{S}(\hat{u_{12}})$	$\mathcal{S}(\hat{u_{13}})$
-1	0.8252	0.5591	0.5389	0.3578	0.4105	0.4774	0.3924	0.5254	0.7574	0.9165	0.2449	0.3724	0.6184
-2	0.4805	0.3301	0.2061	0.3112	0.2392	0.2751	0.2263	0.2885	0.3937	0.4689	0.1530	0.2057	0.3429
-3	0.4295	0.2921	0.1717	0.2704	0.2050	0.2364	0.2463	0.2762	0.3299	0.4250	0.1359	0.1748	0.2941
-4	0.4107	0.2762	0.1551	0.2522	0.1894	0.2198	0.1793	0.2237	0.3017	0.3720	0.1278	0.1615	0.2726
-5	0.4003	0.2670	0.1453	0.2411	0.1799	0.2103	0.1705	0.2107	0.2850	0.3565	0.1225	0.1540	0.2590
-6	0.3934	0.2607	0.1389	0.2335	0.1733	0.2041	0.1645	0.2016	0.2737	0.3460	0.1186	0.1492	0.2493
-7	0.3881	0.2561	0.1345	0.2279	0.1684	0.1997	0.1603	0.1949	0.2656	0.3381	0.1157	0.1459	0.2420
-8	0.3840	0.2526	0.1312	0.2237	0.1645	0.1964	0.1570	0.1899	0.2595	0.3320	0.1133	0.1434	0.2363
-9	0.3806	0.2499	0.1286	0.2202	0.1614	0.1938	0.1545	0.1860	0.2549	0.3270	0.1114	0.1415	0.2318
-10	0.3778	0.2477	0.1266	0.2175	0.1589	0.1917	0.1525	0.1829	0.2512	0.3229	0.1097	0.1401	0.2281

$m\mathrm{P}^F$ ELECTRE-I methods. It is evident from Tables 4 and 5 that the optimal object obtained by applying the proposed $m\mathrm{P}^F$SSWA and preexisting $m\mathrm{P}^F$YW averaging, $m\mathrm{P}^F$YW geometric, $m\mathrm{P}^F$AAW averaging, $m\mathrm{P}^F$AAW geometric, and $m\mathrm{P}^F$DW averaging operators is same, i.e., $\mathfrak{P}_{11}$. Since, the range of parameter (λ) in the Schweizer-Sklar t-N and t-C_oN is $-\infty$ to $+\infty$ that is greater than other existing operations, including Hamacher, Dombi, Yager, Frank and Aczel-Alsina t-Ns and t-C_oNs. However, the parametric range of Hamacher, Dombi, Yager, Frank and Aczel-Alsina t-Ns and t-C_oNs is from 0 to ∞, which is clearly lesser than the parametric range of Schweizer-Sklar t-N and t-C_oN. That is why, by taking the values of λ from -1 to -10, the initiated $m\mathrm{P}^F$SSWA and $m\mathrm{P}^F$SSWG operators are implemented on the studied application in previous section, and the obtained results are provided in Tables 6 and 7, respectively. From Figures 5 and 6, it is clear that the ranking of alternatives under consideration is stable, which validate the flexibility of parameter regarding a particular scenario. Additionally, for more understanding about the advantages of the initiated AgOs, their certain characteristics are provided in comparison with some existing AgOs, which are displayed by Table 8. All in all, our initiated $m\mathrm{P}^F$ Schweizer-Sklar A_gOs are more flexible and adaptable to handle $m\mathrm{P}^F$ information as compared to the existing operators, including $m\mathrm{P}^F$YW averaging and $m\mathrm{P}^F$YW geometric, $m\mathrm{P}^F$AAW

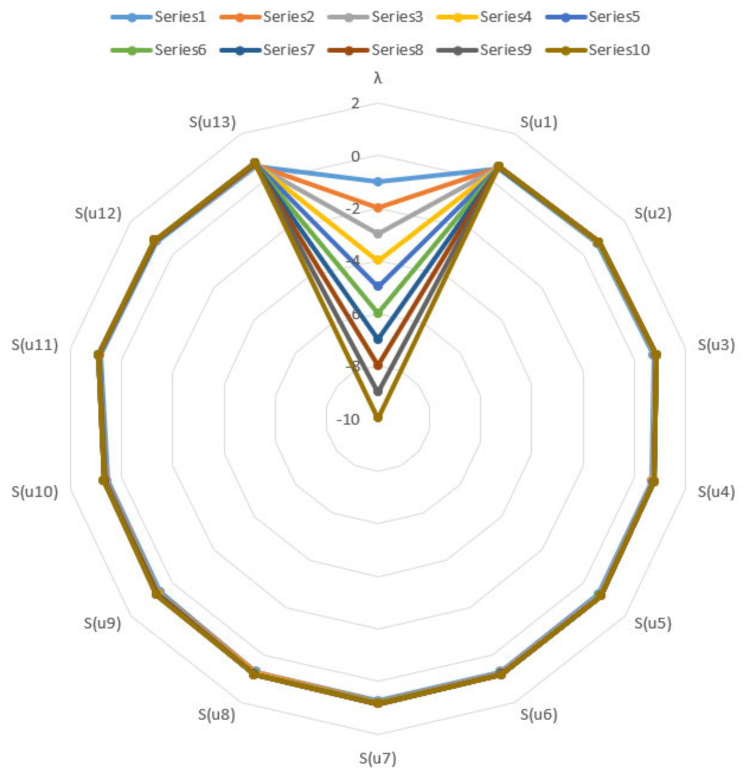

FIGURE 5. Comparison between different parametric values of λ on the presented application using $m\mathrm{P}^{F}$ SSWA operators.

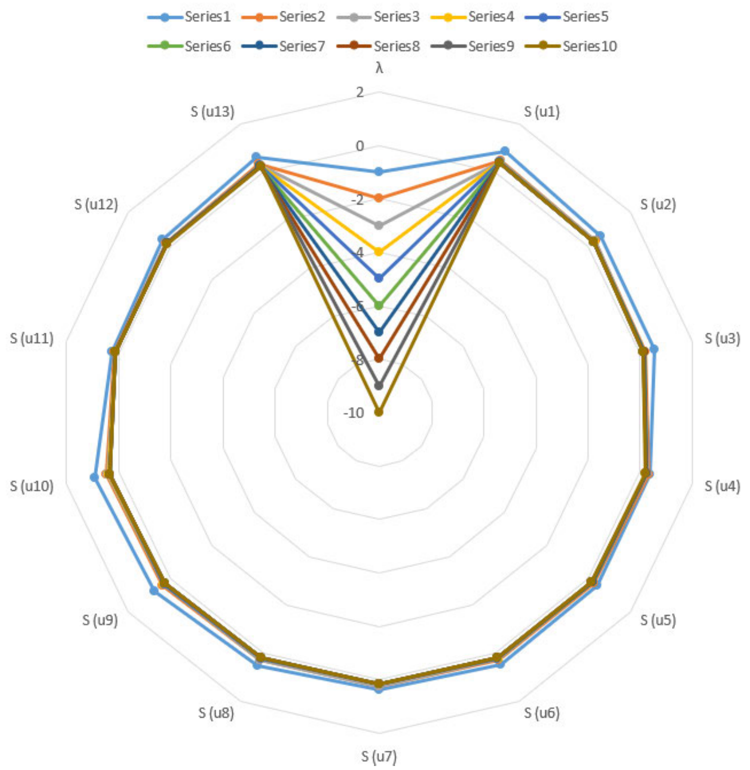

FIGURE 6. Comparison between different parametric values of λ on the presented application using $m\mathrm{P}^{F}$ SSWG operators.

TABLE 8. Comparison between different AgOs.

Aggregation operators	Whether permits to tackle sub-features of attributes	Whether allows parametric values from the interval $[-\infty, \infty]$	Whether flexible for decision-makers' to evaluate multipolar characteristics of attributes	Whether it assigns criteria weights
q-Rung orthopair fuzzy Archimedean A_gOs [42]	No	No	No	Yes
Intuitionistic linguistic A_gOs [43]	No	No	No	Yes
Power partitioned neutral A_gOs based on T-spherical FSs [44]	No	No	No	Yes
$m\mathrm{P}^{\mathrm{F}}$ Yager A_gOs [30]	Yes	No	Yes	Yes
$m\mathrm{P}^{\mathrm{F}}$ Aczél–Alsina A_gOs [39]	Yes	No	Yes	Yes
Spherical fuzzy neutrality A_gOs [45]	No	No	No	Yes
Dombi exponential A_gOs based on neutrosophic cubic hesitant FSs [46]	No	No	No	Yes
Pythagorean fuzzy interaction A_gOs [53]	No	No	No	No
q-rung orthopair hesitant fuzzy rough Einstein A_gOs [47]	No	No	No	Yes
Spherical fuzzy Yager A_gOs [51]	No	No	No	Yes
$m\mathrm{P}^{\mathrm{F}}$ Dombi A_gOs [34]	Yes	No	Yes	Yes
$m\mathrm{P}^{\mathrm{F}}$ TOPSIS [33]	Yes	No	Yes	No
$m\mathrm{P}^{\mathrm{F}}$ ELECTRE-I [33]	Yes	No	Yes	No
Proposed $m\mathrm{P}^{\mathrm{F}}$ Schweizer-Sklar A_gOs	Yes	Yes	Yes	Yes

averaging and $m\mathrm{P}^F$ AAW geometric, $m\mathrm{P}^F$ DW averaging and $m\mathrm{P}^F$ DW geometric A_gOs, $m\mathrm{P}^F$ AHP, $m\mathrm{P}^F$ TOPSIS, and $m\mathrm{P}^F$ ELECTRE-I methods.

B. LIMITATIONS

Despite the benefits of the suggested MCDM methods based on Schweize-Sklar A_gOs, they have certain drawbacks. One of the major drawback is the lengthy computation, which becomes more difficult when complex information is involved. Due to this fact, the process becomes very challenging when dealing with large data-sets. The mathematical software like MATLAB can be used to make calculation process smarter and valuable. One more problem is that different A_gOs may produce different outcomes (as seen in the above comparison subsection that the proposed $m\mathrm{P}^F$ SSWG operator and the existing geometric operators yields different optimal objects). As a result, actual outcomes may differ from predictions.

VI. CONCLUSION AND FUTURE DIRECTIONS

Traditional MCDM approaches are unable to tackle complicated multipolar challenges precisely. Consequently, the fusion of multipolar information with A_gOs is currently gaining significant attention as a mathematical topic for unifying diverse inputs into a single useful output, as polarized data and many characteristics are common in a wide range of real-life issues. We fixed these problems by combining $m\mathrm{P}^F$ concept with Schweize-Sklar t-N and t-C_oN operations, and have initiated certain $m\mathrm{P}^F$ set-based averaging and geometric operators, namely, the $m\mathrm{P}^F$ SSWA operator, $m\mathrm{P}^F$ SSOWA operator, $m\mathrm{P}^F$ SSHA operator, $m\mathrm{P}^F$ DWG operator, $m\mathrm{P}^F$ SSOWG operator and $m\mathrm{P}^F$ SSHG operator. To better understand these new concepts, we have explained them with corresponding numerical examples. Furthermore, we have investigated certain fundamental properties of these presented A_gOs, such as idempotency, boundedness, monotonicity, and commutativity. In order to validate the authenticity of proposed A_gOs, we have studied a real-life problem, that is, a case study of Sindh province in Pakistan for choosing an appropriate site to build a wind power station. Finally, we have compared the developed $m\mathrm{P}^F$ Schweizer-Sklar A_gOs with some preexisting operators such as $m\mathrm{P}^F$ Aczel-Alsina, $m\mathrm{P}^F$ Yagar, $m\mathrm{P}^F$ Dombi A_gOs, $m\mathrm{P}^F$ AHP, $m\mathrm{P}^F$ TOPSIS, and $m\mathrm{P}^F$ ELECTRE-I methods.

A. FUTURE DIRECTIONS

The proposed $m\mathrm{P}^F$ Schweizer-Sklar AgOs are valuable mathematical tools for industry stakeholders, particularly for MCDM in various domains like project evaluation, supplier selection, and risk assessment, where the significance of attributes such as quality, cost, and delivery timeline varies from one stakeholder to another. The main feature of the proposed operators is their ability to aggregate estimations regarding multiple attributes/criteria from different experts/stakeholders, allowing for more representative and nuanced decision-making processes. Moreover, the developed AgOs are useful in scenarios such as environmental impact assessments, where stakeholders need to estimate effective outputs from both negative and positive viewpoints to select the best area for investment, particularly in site selection problems. Since site selection problems involve complex information, a thorough investigation of each criterion for every available site is essential to achieve an optimal result. These characteristics make the proposed operators superior to existing Schweizer-Sklar A_gOs based on other uncertainty theories, such as fuzzy and IF sets, particularly in site selection scenarios. Due to the broader applicability of multipolar fuzzy set theory, this work can be expanded as follows:

(1) m-polar fuzzy prioritized Schweizer-Sklar aggregation operators,
(2) m-polar fuzzy soft Schweizer-Sklar aggregation operators,
(3) Picture m-polar fuzzy Schweizer-Sklar aggregation operators,
(4) m-polar fuzzy rough Schweizer-Sklar aggregation operators,

AUTHOR CONTRIBUTIONS

Ghous Ali.: Conceptualization, Methodology, writing–original draft, and Writing–review and editing; Muhammad Anwar: Methodology, Analysis, and Proof reading; Bander Almutairi: Methodology and Funding acquisition; Muhammad Faheem: Writing–review and editing; and Sabeeha Kanwal: Writing–original draft and Writing–review and editing.

DATA AVAILABILITY

The paper includes the information used to verify the study's findings.

DECLARATIONS

Conflicts of Interest: The authors declare that they have no conflicts of interest regarding the publication of the paper.
Ethical approval: Not applicable.

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