



Lebesgue type decompositions and Radon–Nikodym derivatives for pairs of bounded linear operators

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Abstract. For a pair of bounded linear Hilbert space operators A and B one considers the Lebesgue type decompositions of B with respect to A into an almost dominated part and a singular part, analogous to the Lebesgue decomposition for a pair of measures in which case one speaks of an absolutely continuous and a singular part. A complete parametrization of all Lebesgue type decompositions will be given, and the uniqueness of such decompositions will be characterized. In addition, it will be shown that the almost dominated part of B in a Lebesgue type decomposition has an abstract Radon–Nikodym derivative with respect to the operator A .

1. Introduction

Let \mathfrak{E} , \mathfrak{H} , and \mathfrak{K} be Hilbert spaces and $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ be bounded linear operators. In the present paper it will be shown that there are so-called *Lebesgue type decompositions* of the operator $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ relative to the operator $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ of the form

$$B = B_1 + B_2, \quad B_1, B_2 \in \mathbf{B}(\mathfrak{E}, \mathfrak{K}), \quad (1.1)$$

where $\text{ran } B_1 \perp \text{ran } B_2$, B_1 is almost dominated by A , and B_2 is singular with respect to A ; the terminology will be explained below. The collection of all Lebesgue type decompositions will be parametrized and a criterion for the uniqueness of such decompositions will be established. Furthermore, it will be shown that if B has the above Lebesgue type decomposition (1.1) with respect to A , then there exists a uniquely determined closed linear operator C from \mathfrak{H} to \mathfrak{K} satisfying a certain minimality condition, that, in general, is unbounded, such that

$$B_1 = CA; \quad (1.2)$$

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for details, see Theorem 7.4 below. This operator C will be called the *Radon–Nikodym derivative* of B_1 with respect to A . The above results are the abstract analogs of the usual Lebesgue decomposition of a finite measure into an absolutely continuous part and a singular part, and of the corresponding Radon–Nikodym derivative for the absolutely continuous part. There are, indeed, situations in measure theory where actually there is more than one Lebesgue type decomposition.

The above results can be interpreted as special cases of corresponding results for linear relations. For any linear relation T from a Hilbert space \mathfrak{H} to a Hilbert space \mathfrak{K} there exist the so-called Lebesgue type decompositions of the form

$$T = T_1 + T_2$$

such that $\text{ran } T_1 \perp \text{ran } T_2$, T_1 is a regular operator (an operator whose closure in $\mathfrak{H} \times \mathfrak{K}$ is an operator, i.e., a closable operator), and T_2 is a singular relation (a relation whose closure is the product of closed linear subspaces in \mathfrak{H} and \mathfrak{K} , respectively). A general treatment of Lebesgue type decompositions of a linear relation T was carried out in the recent paper [7]. For instance, in that paper an explicit parametrization of all Lebesgue type decompositions of T was established and the case where the Lebesgue type decomposition of T is unique has been characterized therein. In the setting of a pair of positive operators this kind of uniqueness result goes back to Ando [1]. The Lebesgue type decompositions for linear relations, in particular for linear operators, automatically give corresponding decompositions for a single semibounded sesquilinear form on a Hilbert space (see [31] and also [11]).

In the present paper Lebesgue type decompositions for linear relations which are simultaneously operator ranges are studied (see [4]); in what follows such linear relations are called shortly *operator range relations*. Operator range relations coincide with the linear relations of the form

$$L(A, B) = \{ \{Af, Bf\} : f \in \mathfrak{E} \}, \quad (1.3)$$

where $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$. The Lebesgue type decompositions of the relation $L(A, B)$ correspond to the Lebesgue type decompositions of the operator B with respect to the operator A in (1.1). In particular, B is almost dominated by A precisely if the corresponding relation $L(A, B)$ is regular, i.e., $L(A, B)$ is a closable operator; moreover, B is singular with respect to A precisely if $L(A, B)$ is singular. If $L(A, B)$ is regular, then its closure is the Radon–Nikodym derivative of B with respect to A mentioned in (1.2); the precise meaning of this statement will be explained later. The approach in the present paper makes it possible to specialize the results established in [7], including uniqueness results, to the setting of pairs of bounded linear operators. This leads, in particular, to the characterizations in Section 6, the

Radon–Nikodym derivatives in Section 7, and the decompositions in Section 8; see also [6, 8, 12].

The topic of the present paper was inspired by the work of Ando about Lebesgue type decompositions for pairs of bounded nonnegative operators [1], and the work of Simon about Lebesgue decompositions for nonnegative forms [31], see also [6] and [8], and Kosaki’s work on the Radon–Nikodym derivative in the setting of C^* -algebras [23]. The context of pairs of bounded linear operators which are not necessarily nonnegative is connected with the work of Mac Nerney, Kaufman, and others; see [18–22]. Moreover, there are strong connections with the work of Izumino [13–15] and of Izumino and Hirasawa [16]. They treated the case where $L(A, B)$ in (1.3) is a densely defined operator. The parametrization of the Lebesgue type decompositions and the notion of Radon–Nikodym seem to be new even in the case where $L(A, B)$ is densely defined. Moreover, at this point, it should be mentioned that, although the paper is inspired by [1], [31], and [23], the decompositions (1.1) here are concerned with pairs A and B , where B_1 is almost dominated by A and B_2 is singular with respect to A . However, the decompositions of Ando and Simon, and Kosaki’s Radon–Nikodym derivatives belong to a slightly different setting; this setting will be considered in further work.

Here is a brief description of the contents of the paper. The concept of operator ranges, including their natural topologies, will be reviewed in Section 2. Operator range relations and a special normalized class of them are treated in Section 3. As an application, a construction of the operator range representation of a closed relation is derived in Section 4. This construction resembles a measure-theoretic treatment of Radon–Nikodym derivatives for a pair of positive measures, and, as a bonus, leads to a natural introduction of Radon–Nikodym derivatives for pairs of bounded linear operators in Section 7. As a preparation for Lebesgue type decompositions for pairs of bounded operators, some characterizations of regular and singular operator range relations along the lines of [7] are given in Section 5. The corresponding classification for pairs of bounded linear operators can be found in Section 6. This involves the notions of domination and almost domination of a bounded operator with respect to another bounded operator, which correspond to the concept of absolute continuity of a positive measure with respect to another positive measure. Similarly, the notion of singularity of a pair of bounded operators is defined as an operator analog for the concept of singularity of a pair of positive measures. In Section 7 the abstract Radon–Nikodym derivative is introduced and investigated. The definition of Radon–Nikodym derivative given here involves an optimality property, see Lemma 7.1 and Theorem 7.4 and, in fact, this notion is uniquely determined in the general setting of operator range relations. Furthermore, it is shown that the Radon–Nikodym derivatives for operator

range relations admit similar properties known to hold for pairs of positive measures; see Theorem 7.7. Finally, all Lebesgue type decompositions for pairs of bounded linear operators are described in Section 8 with a uniqueness result analogous to that of Ando [1].

In [7] and in the present paper all Lebesgue type decompositions are orthogonal in the sense that the ranges of the components in the decomposition are orthogonal. Nonorthogonal Lebesgue type decompositions can be obtained by using decompositions of the Hilbert space via operator range spaces that are contractively included; see [10] and [11]. Further work will be concerned with the situation that the operators A and B are nonnegative or with the situation of a pair of nonnegative forms. Such cases can be found, for instance, in papers by T. Ando [1], H. Kosaki [23], and B. Simon [31], by Izumino and Hirasawa [13–16], and, more recently, by Z. Sebestyén, Zs. Tarsay, and T. Titkos [29, 30, 32–34], and in [6]. It will be made clear how these different situations fit in the context of linear relations. Moreover, further work will also be connected with the situation where at least one of the operators A and B is not bounded.

2. Linear relations admitting an operator range representation

In this section the special class of linear relations which, in addition, are operator ranges will be introduced. In what follows, such linear relations are briefly called *operator range relations*. This class extends the class of closed linear relations and the operator range relations have a number of useful properties. The introduction will be facilitated by a brief treatment of operator ranges in Hilbert spaces. The notions of operator ranges and operator range relations in the present sense go back to [4], [19], [24], [25], [26], [27]. A brief survey is given in this section. For the convenience of the reader, some proofs are included.

Definition 2.1. Let \mathfrak{X} be a Hilbert space with inner product (\cdot, \cdot) . A linear subspace \mathfrak{M} of \mathfrak{X} , together with an inner-product $(\cdot, \cdot)_+$ on \mathfrak{M} , is said to be an *operator range* in \mathfrak{X} if

- (a) \mathfrak{M} is a Hilbert space when equipped with $(\cdot, \cdot)_+$;
- (b) $\|u\|_+ \geq c\|u\|_{\mathfrak{X}}$, $u \in \mathfrak{M}$, for some $c > 0$.

In particular, a closed linear subspace is an operator range.

The terminology operator range is motivated by the following lemma. If $\mathfrak{M} \subset \mathfrak{X}$ is the range of a bounded operator, then there is a natural inner-product on \mathfrak{M} , that makes \mathfrak{M} an operator range.

Lemma 2.2. *Let \mathfrak{X} and \mathfrak{Y} be Hilbert spaces and let $Z \in \mathbf{B}(\mathfrak{Y}, \mathfrak{X})$. Then the*

linear space $\mathfrak{M} = \text{ran } Z$, equipped with the inner-product

$$(Zx, Zy)_+ = (x, y), \quad x, y \in \mathfrak{Y} \ominus \ker Z, \quad (2.1)$$

is an operator range.

Proof. Assume that Z is not the zero operator. Note that $\mathfrak{M} = \text{ran } Z$ with $(\cdot, \cdot)_+$ in (2.1) is indeed an inner-product space. To see that it is complete, let (Zx_n) be a Cauchy sequence in $(\mathfrak{M}, (\cdot, \cdot)_+)$ with $x_n \in \mathfrak{Y} \ominus \ker Z$. Then (x_n) is a Cauchy sequence in $\mathfrak{Y} \ominus \ker Z$. Thus, $x_n \rightarrow x$ for some $x \in \mathfrak{Y} \ominus \ker Z$ and $Zx_n \rightarrow Zx$, since $Z \in \mathbf{B}(\mathfrak{Y}, \mathfrak{X})$. This shows (a) in Definition 2.1. By definition $\|Zx\|_+ = \|x\|$, $x \in \mathfrak{Y} \ominus \ker Z$, and $\|Zx\| \leq \|Z\|\|x\|$, $x \in \mathfrak{Y}$, lead to the inequality

$$\|Zx\|_+ = \|x\| \geq \frac{1}{\|Z\|} \|Zx\|_{\mathfrak{X}}, \quad x \in \mathfrak{Y} \ominus \ker Z,$$

which gives (b) in Definition 2.1. ■

There is also a converse to Lemma 2.2.

Lemma 2.3. *Let $(\mathfrak{M}, (\cdot, \cdot)_+)$ be an operator range in \mathfrak{X} . Then there exists an operator $Z \in \mathbf{B}(\mathfrak{X})$ such that $\mathfrak{M} = \text{ran } Z$ and*

$$(Zx, Zy)_+ = (x, y), \quad x, y \in \mathfrak{X} \ominus \ker Z. \quad (2.2)$$

The operator Z may be chosen to be nonnegative.

Proof. Let $\iota : \mathfrak{M} \rightarrow \mathfrak{X}$ be the identification map, where each space has its own topology. Then it follows from (b) that $c\|\iota x\| \leq \|x\|_+$, so that ι is bounded. Its adjoint ι^\times from \mathfrak{X} to \mathfrak{M} is a bounded mapping and one has the polar decomposition (cf. [17, Section VI 2.7])

$$\iota^\times = Z|\iota^\times|,$$

where $Z : \mathfrak{X} \rightarrow \mathfrak{M}$ is the unique partial isometry with initial space $\overline{\text{ran } |\iota^\times|}$ and final space $\overline{\text{ran } \iota^\times}$. Since the last space is the orthogonal complement in \mathfrak{M} of $\ker \iota$, one sees that $\overline{\text{ran } \iota^\times} = \mathfrak{M}$. Consequently, $\text{ran } Z = \mathfrak{M}$, and (2.2) holds as Z is a partial isometry from \mathfrak{X} to \mathfrak{M} . Finally, it remains to observe that

$$c\|Zx\| \leq \|Zx\|_+ \leq \|x\|, \quad x \in \mathfrak{X},$$

thanks to (b) and the fact that Z is a partial isometry. Thus, $Z \in \mathbf{B}(\mathfrak{X})$.

For the proof of the last statement, consider $Z \in \mathbf{B}(\mathfrak{X})$ and its polar decomposition $Z = |Z^*|C$, where $C \in \mathbf{B}(\mathfrak{X})$ is the unique partial isometry

with initial space $\overline{\text{ran}} Z^*$ and final space $\overline{\text{ran}} |Z^*|$. Observe that, by the Douglas Lemma (cf. [3]), $\text{ran } |Z^*| = \text{ran } Z$, so that $\text{ran } |Z^*| = \mathfrak{M}$ and, clearly,

$$(|Z^*|Cx, |Z^*|Cy)_+ = (x, y) = (Cx, Cy), \quad x, y \in (\ker Z)^\perp.$$

This implies that

$$(|Z^*|u, |Z^*|v)_+ = (u, v), \quad u, v \in (\ker |Z^*|)^\perp,$$

where $|Z^*| \in \mathbf{B}(\mathfrak{X})$ is nonnegative. ■

According to Lemma 2.2 and Lemma 2.3, operator ranges are parameterized by means of bounded operators $Z \in \mathbf{B}(\mathfrak{Y}, \mathfrak{X})$. The subspace $\ker Z \subset \mathfrak{Y}$ is called the *redundant part* of this parametrization, as it does not contribute to $\text{ran } Z$. The restriction Z_0 of Z to $\mathfrak{Y} \ominus \ker Z$ is called the *reduced part* or *reduction* of Z :

$$Z = (Z_0 ; O_{\ker Z}). \quad (2.3)$$

The topology of \mathfrak{M} is uniquely determined by the reduced part of the representing operator in a sense to be explained below.

Lemma 2.4. *Let \mathfrak{M} be an operator range in \mathfrak{X} . Assume that there exist Hilbert spaces \mathfrak{Y} and \mathfrak{Y}_1 and operators $Z \in \mathbf{B}(\mathfrak{Y}, \mathfrak{X})$ and $Z_1 \in \mathbf{B}(\mathfrak{Y}_1, \mathfrak{X})$, both of which satisfy the conditions in Lemma 2.2. Then there is a bounded and boundedly invertible operator $W \in \mathbf{B}(\mathfrak{Y}_1, \mathfrak{Y})$ such that*

$$Z_1 = ZW, \quad \ker W = \ker Z_1, \quad \text{ran } W = \overline{\text{ran}} Z^*. \quad (2.4)$$

Consequently, the topologies induced on \mathfrak{M} by Z and by Z_1 are equivalent.

Proof. If $\mathfrak{M} = \text{ran } Z_1$ with $Z_1 \in \mathbf{B}(\mathfrak{Y}_1, \mathfrak{X})$ with a Hilbert space \mathfrak{Y}_1 , then it is clear that the operators Z_1 and Z have the same range. Hence, by the Douglas Lemma, there is an operator $W \in \mathbf{B}(\mathfrak{Y}_1, \mathfrak{Y})$ such that (2.4) holds. Note that the operator W in Lemma 2.4 is a bounded bijective mapping from $(\ker Z_1)^\perp$ onto $(\ker Z)^\perp$, as follows from the closed graph theorem. ■

Lemma 2.5. *Let $\mathfrak{M} = \text{ran } Z$ be an operator range as in Lemma 2.1 and let Z_0 be the reduced part of Z . Then the following statements are equivalent:*

- (i) \mathfrak{M} is an operator range in \mathfrak{X} that is closed;
- (ii) $\text{ran } Z$ or, equivalently, $\text{ran } Z_0$ is closed;
- (iii) Z_0 is bounded and boundedly invertible.

In particular, operator ranges form a lattice; cf. [4].

The previous notion of operator range will now be extended to subspaces of product spaces; see for instance [25].

Definition 2.6. A linear relation T from \mathfrak{H} to \mathfrak{K} is said to be an *operator range relation* if its graph is an operator range in $\mathfrak{H} \times \mathfrak{K}$, thus $T = \text{ran } \Phi$ for some $\Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{H} \times \mathfrak{K})$, where \mathfrak{E} is some Hilbert space.

Lemma 2.7. Let T be a closed linear relation from the Hilbert space \mathfrak{H} to the Hilbert space \mathfrak{K} . Then T is an operator range relation.

Proof. Consider the orthogonal decomposition $\mathfrak{E} = \mathfrak{H} \times \mathfrak{K} = T \hat{\oplus} T^\perp$ and let $\Phi = P_T$ be the orthogonal projection from $\mathfrak{E} = \mathfrak{H} \times \mathfrak{K}$ to T , so that $T = \text{ran } \Phi$. ■

To characterize operator range relations the following notions and notations are needed. For a pair of operators $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ the linear relation $L(A, B)$ from \mathfrak{H} to \mathfrak{K} is defined by

$$L(A, B) = \{ \{Af, Bf\} : f \in \mathfrak{E} \}. \quad (2.5)$$

Let P_1 and P_2 be the orthogonal projections from $\mathfrak{H} \times \mathfrak{K}$ onto the component subspaces $\mathfrak{H} \times \{0\}$ and $\{0\} \times \mathfrak{K}$, respectively. The mappings $\iota_1\{\varphi, 0\} = \varphi$ and $\iota_2\{0, \psi\} = \psi$ identify $\mathfrak{H} \times \{0\}$ with \mathfrak{H} and $\{0\} \times \mathfrak{K}$ with \mathfrak{K} , respectively. In the following theorem the operator range relations are characterized; see [2, Theorem 1.10.1].

Theorem 2.8. Let \mathfrak{H} and \mathfrak{K} be Hilbert spaces and let T be a linear relation from \mathfrak{H} to \mathfrak{K} .

(a) If $T = \text{ran } \Phi$, where $\Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{H} \times \mathfrak{K})$, then $T = L(A, B)$ with

$$A = \iota_1 P_1 \Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{H}) \quad \text{and} \quad B = \iota_2 P_2 \Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{K}).$$

(b) If $T = L(A, B)$, where $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$, then $T = \text{ran } \Phi$ with

$$\Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{H} \times \mathfrak{K}) \quad \text{and} \quad \Phi f = \{Af, Bf\}, \quad f \in \mathfrak{E}.$$

Corollary 2.9. If T is an operator range relation from \mathfrak{H} to \mathfrak{K} , then $\text{dom } T$ and $\text{ker } T$ are operator ranges in \mathfrak{H} , while $\text{ran } T$ and $\text{mul } T$ are operator ranges in \mathfrak{K} .

The concept of an operator range relation extends in a certain way the notion of a closed linear relation. For instance, sums, and intersections, as well as Cartesian products of operator range relations are again operator range relations. In particular, operator range relations form a lattice; cf. [4]. Notice also that if the relation T is itself the range of some closed linear relation H from a Hilbert space \mathfrak{E} to the Hilbert space $\mathfrak{H} \times \mathfrak{K}$, then T is still an operator range relation.

3. Operator range relations and normalized pairs

Let \mathfrak{E} , \mathfrak{H} , and \mathfrak{K} be Hilbert spaces and consider the pair of operators $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$. It will be convenient to recall from [5, 28], see also [4], the following auxiliary column and row operators. The *column operator* $c(A, B)$ from \mathfrak{E} to $\mathfrak{H} \times \mathfrak{K}$ is defined by

$$c(A, B) = \begin{pmatrix} A \\ B \end{pmatrix}, \quad \text{i.e.,} \quad c(A, B)\varphi = \begin{pmatrix} A\varphi \\ B\varphi \end{pmatrix}, \quad \varphi \in \mathfrak{E}. \quad (3.1)$$

Then $c(A, B)$ belongs to $\mathbf{B}(\mathfrak{E}, \mathfrak{H} \times \mathfrak{K})$. Likewise, the *row operator* from $\mathfrak{H} \times \mathfrak{K}$ to \mathfrak{E} is defined by

$$r(A, B) = (A \quad B), \quad \text{i.e.,} \quad r(A, B) \begin{pmatrix} h \\ k \end{pmatrix} = Ah + Bk, \quad h \in \mathfrak{H}, \quad k \in \mathfrak{K}. \quad (3.2)$$

Then $r(A, B)$ belongs to $\mathbf{B}(\mathfrak{H} \times \mathfrak{K}, \mathfrak{E})$ and

$$\text{ran } r(A, B) = \text{ran } A + \text{ran } B. \quad (3.3)$$

Moreover, it is clear from (3.1) and (3.2) that

$$c(A, B)^* = r(A^*, B^*). \quad (3.4)$$

Therefore, it follows from (3.1) and (3.2) that

$$c(A, B)^* c(A, B) = A^* A + B^* B, \quad (3.5)$$

and that

$$c(A, B) c(A, B)^* = \begin{pmatrix} AA^* & AB^* \\ BA^* & BB^* \end{pmatrix}. \quad (3.6)$$

For a pair of operators $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ the linear relation $L(A, B)$ from \mathfrak{H} to \mathfrak{K} is defined by (2.5), so that $L(A, B)$ is an operator range relation as in Theorem 2.8, since in the present notation

$$L(A, B) = \text{ran } c(A, B). \quad (3.7)$$

The operator range relation $L(A, B)$ is sometimes called a *quotient*, as, indeed, in the sense of the product of linear relations one can write $L(A, B)$ as BA^{-1} ; cf. [18]. Note that the domain and the range of $L(A, B)$ in (2.5) are given by

$$\text{dom } L(A, B) = \text{ran } A, \quad \text{ran } L(A, B) = \text{ran } B,$$

while the kernel and the multivalued part of $L(A, B)$ are given by

$$\ker L(A, B) = A(\ker B), \quad \text{mul } L(A, B) = B(\ker A). \quad (3.8)$$

Recall that for a linear relation T from \mathfrak{H} to \mathfrak{K} the adjoint T^* is an automatically closed linear relation from \mathfrak{K} to \mathfrak{H} given by

$$T^* = JT^\perp = (JT)^\perp, \quad (3.9)$$

where J stands for the flip-flop operator $\{f, g\} \mapsto \{g, -f\}$; in other words

$$T^* = \{ \{h, k\} \in \mathfrak{K} \times \mathfrak{H} : (g, h) = (f, k) \text{ for all } \{f, g\} \in T \}. \quad (3.10)$$

In particular, $T^{**} = T^{\perp\perp}$.

Now apply the definition (3.9) to the linear relation $L(A, B)$ in (2.5). Then, by (3.7), the adjoint of $L(A, B)$ is given by $JL(A, B)^\perp = J(\text{ran } c(A, B))^\perp$, i.e.,

$$L(A, B)^* = J \ker c(A, B)^* = J \ker r(A^*, B^*).$$

In other words, the adjoint of $L(A, B)$ is given by

$$L(A, B)^* = \{ \{k, h\} \in \mathfrak{K} \times \mathfrak{H} : B^*k = A^*h \}, \quad (3.11)$$

cf. (3.10). At this point, it is useful to introduce the linear subspaces $\mathfrak{D}(A, B)$ and $\mathfrak{R}(A, B)$ of \mathfrak{K} and \mathfrak{H} by

$$\mathfrak{D}(A, B) = \{k \in \mathfrak{K} : B^*k \in \text{ran } A^*\}, \quad \mathfrak{R}(A, B) = \{h \in \mathfrak{H} : A^*h \in \text{ran } B^*\}, \quad (3.12)$$

respectively. In other words, $\mathfrak{D}(A, B)$ is the pre-image $(B^*)^{-1}(\text{ran } A^*)$ and $\mathfrak{R}(A, B)$ is the pre-image $(A^*)^{-1}(\text{ran } B^*)$. Note that

$$\mathfrak{D}(A, B) = \mathfrak{K} \Leftrightarrow \text{ran } B^* \subset \text{ran } A^*, \quad \mathfrak{R}(A, B) = \mathfrak{H} \Leftrightarrow \text{ran } A^* \subset \text{ran } B^*.$$

From the expression (3.11) it is seen that the domain and the range of $L(A, B)^*$ are given by the sets in (3.12):

$$\text{dom } L(A, B)^* = \mathfrak{D}(A, B), \quad \text{ran } L(A, B)^* = \mathfrak{R}(A, B). \quad (3.13)$$

Moreover, the kernel and multivalued part of $L(A, B)^*$ are given by

$$\ker L(A, B)^* = \ker B^*, \quad \text{mul } L(A, B)^* = \ker A^*, \quad (3.14)$$

respectively. Observe that (3.13) implies that

$$\text{mul } L(A, B)^{**} = \mathfrak{D}(A, B)^\perp, \quad \ker L(A, B)^{**} = \mathfrak{R}(A, B)^\perp. \quad (3.15)$$

It is helpful to state some general properties for the operators $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$, in which case $c(A, B) \in \mathbf{B}(\mathfrak{E}, \mathfrak{H} \times \mathfrak{K})$. By means of the Douglas Lemma, one sees, for instance, from (3.5) and (3.4), that

$$\begin{aligned} \text{ran } (A^*A + B^*B)^{\frac{1}{2}} &= \text{ran } (c(A, B)^*c(A, B))^{\frac{1}{2}} = \text{ran } c(A, B)^* \\ &= \text{ran } r(A^*, B^*) = \text{ran } A^* + \text{ran } B^*, \end{aligned} \quad (3.16)$$

cf. [4], and from (3.6) that

$$\operatorname{ran} \begin{pmatrix} AA^* & AB^* \\ BA^* & BB^* \end{pmatrix}^{\frac{1}{2}} = \operatorname{ran} (c(A, B)c(A, B)^*)^{\frac{1}{2}} = \operatorname{ran} c(A, B). \quad (3.17)$$

Moreover, it is clear that $\operatorname{ran} c(A, B)$ and $\operatorname{ran} c(A, B)^*$ are simultaneously closed, and therefore the following spaces

$$\operatorname{ran} c(A, B), \quad \operatorname{ran} c(A, B)^*c(A, B), \quad \operatorname{ran} c(A, B)^*, \quad \operatorname{ran} c(A, B)^*c(A, B), \quad (3.18)$$

are closed simultaneously; see, e.g., [2, Theorem 1.3.5]. The following characterization of closedness is a direct consequence of these considerations.

Lemma 3.1. *Let the linear relation $L(A, B)$ be given by (2.5). Then the following statements are equivalent:*

- (i) $L(A, B)$ is closed;
- (ii) $\operatorname{ran} (A^*A + B^*B)$ is closed in \mathfrak{E} ;
- (iii) $\operatorname{ran} A^* + \operatorname{ran} B^*$ is closed in \mathfrak{E} ;
- (iv) $\operatorname{ran} A^* + \operatorname{ran} B^* = \operatorname{ran} (A^*A + B^*B)$.

Proof. Thanks to the equivalences in (3.18), the assertions in (i), (ii), and (iii) follow from (3.7), (3.5), and (3.4) together with (3.3), respectively. For the equivalence with the remaining statement (iv), see [4, Theorem 2.1] and the corollaries following it, or [2, Lemma D.2]. ■

The next corollary shows that range space relations admit some specific properties which are well known in the case of closed operators; cf. the closed graph theorem. The following result goes back to Foias (see [4] and [25]); the present proof seems to be new.

Corollary 3.2. *Let T be a range space relation from \mathfrak{H} to \mathfrak{K} . Then the following implications hold:*

- (i) if $\operatorname{mul} T = \{0\}$, then $\operatorname{dom} T$ is closed implies that T is a bounded operator;
- (ii) if $\ker T = \{0\}$, then $\operatorname{ran} T$ is closed implies that T^{-1} is a bounded operator.

Proof. (i) By assumption one can write $T = L(A, B)$ with some operators A and B as in (2.5). Now the assumption $\operatorname{mul} T = \{0\}$ means that

$$\ker A \subset \ker B \quad \text{or, equivalently,} \quad \overline{\operatorname{ran} B^*} \subset \overline{\operatorname{ran} A^*}.$$

If $\operatorname{dom} T = \operatorname{ran} A$ is closed, then also $\operatorname{ran} A^*$ is closed. Hence, in fact, $\operatorname{ran} B^* \subset \operatorname{ran} A^*$ and

$$\operatorname{ran} A^* + \operatorname{ran} B^* = \operatorname{ran} A^* \quad \text{is closed.}$$

By Lemma 3.1 $L(A, B)$ is a closed operator and the statement follows from the closed graph theorem.

- (ii) This is obtained by applying (i) to the inverse T^{-1} . ■

Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$, $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$, and let the linear relation $L(A, B)$ be given by (2.5). In general, there will be some redundance in the representation (2.5); cf. (2.3). In fact, the redundant part is given by the closed linear subspace

$$\ker c(A, B) = \ker A \cap \ker B.$$

Observe that it follows from the identities

$$(\operatorname{ran}(A^*A + B^*B))^\perp = \ker(A^*A + B^*B) = \ker A \cap \ker B,$$

that the space \mathfrak{E} has the following orthogonal decomposition:

$$\mathfrak{E} = \overline{\operatorname{ran}}(A^*A + B^*B) \oplus (\ker A \cap \ker B). \quad (3.19)$$

Hence, one may reduce the representation in (2.5) by introducing the restrictions A_0 and B_0 of A and B to the closed linear subspace $\mathfrak{E}_0 = \overline{\operatorname{ran}}(A^*A + B^*B)$ of \mathfrak{E} , cf. (2.3), so that

$$A = r(A_0, O_{\ker A \cap \ker B}), \quad B = r(B_0, O_{\ker A \cap \ker B}), \quad (3.20)$$

with respect to the decomposition (3.19). It is clear that $\ker A_0 \cap \ker B_0 = \{0\}$, and thus

$$L(A, B) = \{ \{A_0 f, B_0 f\} : f \in \mathfrak{E}_0 \} \quad (3.21)$$

is a representation in terms of the reduced part $c(A_0, B_0)$ of $c(A, B)$. Recall that reduced representations as in (3.21) are uniquely defined up to everywhere defined operators which are bounded and boundedly invertible; cf. Lemma 2.4. Furthermore, via (3.6), one has from (3.20) that, with respect to the decomposition (3.19),

$$A^*A + B^*B = \begin{pmatrix} (A_0)^*A_0 + (B_0)^*B_0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.22)$$

It is clear that $\overline{\operatorname{ran}}((A_0)^*A_0 + (B_0)^*B_0) = \mathfrak{E}_0$; hence one obtains the following lemma.

Lemma 3.3. *Let the linear relation $L(A, B)$ be given by (2.5) and let A_0 and B_0 as in (3.20). Then the following statements are equivalent:*

- (i) $L(A, B)$ is closed;
- (ii) $\operatorname{ran}((A_0)^*A_0 + (B_0)^*B_0) = \mathfrak{E}_0$.

In the rest of this section attention is paid to the case where the linear relation $L(A, B)$ is closed. In this case the operators A and B representing $L(A, B)$ can be replaced by a normalized pair $A^*A + B^*B = I$, and certain formulas become simpler. In fact, it is instructive to first consider the case where $A^*A + B^*B$ is an orthogonal projection.

Lemma 3.4. *Assume that there exists an orthogonal projection Q in \mathfrak{E} such that*

$$A^*A + B^*B = Q, \quad (3.23)$$

in which case the redundant part is given by $\ker A \cap \ker B = \ker Q$. Then the relation $L(A, B)$ is closed and the orthogonal projection from $\mathfrak{H} \times \mathfrak{K}$ onto $L(A, B)$ is given by

$$P_{L(A,B)} = \begin{pmatrix} AA^* & AB^* \\ BA^* & BB^* \end{pmatrix}. \quad (3.24)$$

Moreover, the orthogonal projection from $\mathfrak{H} \times \mathfrak{K}$ onto $L(A, B)^$ is given by*

$$P_{L(A,B)^*} = \begin{pmatrix} I - BB^* & BA^* \\ AB^* & I - AA^* \end{pmatrix}. \quad (3.25)$$

Proof. Since $\text{ran } Q$ is closed, it follows from Lemma 3.1 that $L(A, B)$ is closed. Let $\Phi = c(A, B)$, so that $\Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{H} \times \mathfrak{K})$. Then (3.23) means that $\Phi^*\Phi = Q$ and $\text{ran } \Phi^* = \text{ran } Q$. Observe that

$$(\Phi\Phi^*)(\Phi\Phi^*) = \Phi(\Phi^*\Phi)\Phi^* = \Phi Q\Phi^* = \Phi\Phi^*,$$

so that $\Phi\Phi^*$ is an orthogonal projection, mapping onto $\text{ran } \Phi\Phi^* = \text{ran } \Phi$. Thus, the statement follows from (3.6); cf. [2, Appendix D, p. 703].

Note that condition (3.23) implies that $\ker A \cap \ker B = \ker \Phi = \ker Q$. Hence it follows that $\text{ran } r(A^*, B^*)$ is dense in \mathfrak{E}_0 and thus $\text{ran } r(A^*, B^*) = \mathfrak{E}_0 = \text{ran } Q$ by Lemma 3.1. Note that therefore also $\text{ran } r(B^*, -A^*) = \mathfrak{E}_0$, so that the selfadjoint mapping

$$\begin{pmatrix} B \\ -A \end{pmatrix} \begin{pmatrix} B \\ -A \end{pmatrix}^* = \begin{pmatrix} B \\ -A \end{pmatrix} (B^* \quad -A^*)$$

takes $\mathfrak{K} \times \mathfrak{H}$ onto $JL(A, B)$. Moreover, this mapping is, due to (3.23), idempotent. Thus, the orthogonal projection from $\mathfrak{K} \times \mathfrak{H}$ onto $JL(A, B)$ is given by

$$P_{JL(A,B)} = \begin{pmatrix} B \\ -A \end{pmatrix} \begin{pmatrix} B \\ -A \end{pmatrix}^* = \begin{pmatrix} BB^* & -BA^* \\ -AB^* & AA^* \end{pmatrix},$$

which is equivalent to (3.24).

Since $L(A, B)^* = (JL(A, B))^\perp$, the orthogonal projection onto $L(A, B)^*$ is given by

$$P_{L(A,B)^*} = I - P_{JL(A,B)} = \begin{pmatrix} I - BB^* & BA^* \\ AB^* & I - AA^* \end{pmatrix},$$

which gives (3.25). ■

Recall that the adjoint relation $L(A, B)^*$ has the representation (3.11). As a closed linear relation from \mathfrak{K} to \mathfrak{H} it is an operator range relation. Under condition (3.23) such a representation can be made explicit via Lemma 3.4.

Corollary 3.5. *Let the linear relation $L(A, B)$ be given by (2.5) and assume that condition (3.23) holds. Then the adjoint relation $L(A, B)^*$ is given by*

$$L(A, B)^* = \{ \{ (I - BB^*)\varphi + BA^*\psi, AB^*\varphi + (I - AA^*)\psi \} : \varphi \in \mathfrak{K}, \psi \in \mathfrak{H} \}.$$

If $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ satisfy (3.23) then the relation $L(A, B) = BA^{-1}$ is closed. In this case, its orthogonal operator part is given by

$$L(A, B)_s = \{ \{ A\varphi, (I - P)B\varphi \} : \varphi \in \mathfrak{E} \},$$

where P is the orthogonal projection onto $\text{mul } L(A, B)$; cf. [12, Section 3.2]. It can be rewritten in terms of the Moore–Penrose inverse $A^{(-1)}$ of A , which is an operator from $\text{ran } A \subset \mathfrak{H}$ to $(\ker A)^\perp \subset \mathfrak{E}$. In fact, it is defined as follows: for $\psi \in \text{ran } A$, there exists a unique $\varphi \in (\ker A)^\perp$ such that $A\varphi = \psi$, and $A^{(-1)}\psi := \varphi$; cf. [2]. Note that in the literature one often also extends $A^{(-1)}$ by $A^{(-1)}\psi = 0$ for $\psi \in (\text{ran } A)^\perp$.

Lemma 3.6. *Let the linear relation $L(A, B)$ be given by (2.5) and assume that condition (3.23) holds. Then the orthogonal operator part $L(A, B)_s$ is given by*

$$L(A, B)_s = \{ \{ A\varphi, B\varphi \} : \varphi \in (\ker A)^\perp \}, \quad (3.26)$$

and, consequently,

$$L(A, B)_s = BA^{(-1)}. \quad (3.27)$$

Proof. Consider the representation (2.5). Since $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$, the orthogonal decomposition

$$\mathfrak{E} = (\ker A)^\perp \oplus \ker A$$

leads to the alternative representation

$$L(A, B) = \{ \{ A\varphi, B\varphi + B\psi \} : \varphi \in (\ker A)^\perp, \psi \in \ker A \}, \quad (3.28)$$

cf. [7]. Recall from (3.8) that $\text{mul } L(A, B) = B(\ker A) = \{ B\psi : \psi \in \ker A \}$. Hence,

$$\begin{aligned} (B\varphi, B\psi) &= (B^*B\varphi, \psi) = ((Q - A^*A)\varphi, \psi) \\ &= (\varphi, \psi) - (A\varphi, A\psi) = 0, \quad \varphi \in (\ker A)^\perp \subset \text{ran } Q, \quad \psi \in \ker A, \end{aligned}$$

shows that the range decomposition in the representation (3.28) is orthogonal and that the orthogonal operator part of $L(A, B)$ has the representation (3.26). The representation (3.27) follows from the definition of $A^{(-1)}$. ■

The pair of bounded linear operators $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ in (2.5) is said to be *normalized* if

$$A^*A + B^*B = I. \quad (3.29)$$

In this case, the relation $L(A, B)$ is closed by Lemma 3.4 and the representation of $L(A, B)$ in (2.5) is reduced.

Lemma 3.7. *Let the linear relation $L(A, B)$ be given by (2.5) and assume that $L(A, B)$ is closed. Then $L(A, B)$ can be represented by a normalized pair.*

Proof. Since $L(A, B)$ is closed, it is known by Lemma 3.1 that $\text{ran}(A^*A + B^*B)$ is closed. Thus, for the reduced representation in (3.21) one now has

$$\text{ran}((A_0)^*A_0 + (B_0)^*B_0) = \mathfrak{E}_0$$

by Lemma 3.3. Replacing A_0 and B_0 by the equivalent pair

$$A'_0 = A_0((A_0)^*A_0 + (B_0)^*B_0)^{-\frac{1}{2}} \quad \text{and} \quad B'_0 = B_0((A_0)^*A_0 + (B_0)^*B_0)^{-\frac{1}{2}}$$

leads to $L(A, B) = L(A'_0, B'_0)$, where $A'_0 \in \mathbf{B}(\mathfrak{E}_0, \mathfrak{H})$ and $B'_0 \in \mathbf{B}(\mathfrak{E}_0, \mathfrak{K})$ form a normalized pair. ■

If $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ satisfy (3.23), then the statement in Corollary 3.5 is a direct consequence of Lemma 3.4. Under the stronger condition (3.29), a slightly different-looking result can be obtained by invoking the polar decomposition of the operator A , i.e.,

$$A = V_A(A^*A)^{\frac{1}{2}},$$

where V_A is the unique partial isometry from \mathfrak{E} to \mathfrak{H} with initial space $\overline{\text{ran}} A^*$ and final space $\overline{\text{ran}} A$. Recall that $I - V_A(V_A)^* = P_{\ker A^*}$. By means of the polar decomposition of A one sees that the normalization (3.29) leads to

$$\begin{aligned} I - AA^* &= I - V_A A^* A (V_A)^* \\ &= I - V_A (V_A)^* + V_A B^* B (V_A)^* = P_{\ker A^*} + V_A B^* B (V_A)^*. \end{aligned}$$

Moreover, by the well-known commutation relations

$$B(I - B^*B)^{\frac{1}{2}} = (I - BB^*)^{\frac{1}{2}}B, \quad (I - B^*B)^{\frac{1}{2}}B^* = B^*(I - BB^*)^{\frac{1}{2}},$$

one obtains from the normalization (3.29) that

$$AB^* = V_A(A^*A)^{\frac{1}{2}}B^* = V_A(I - B^*B)^{\frac{1}{2}}B^* = V_AB^*(I - BB^*)^{\frac{1}{2}}.$$

Therefore the orthogonal projection from $\mathfrak{H} \times \mathfrak{K}$ onto $L(A, B)^*$ in Lemma 3.4 is given by

$$\begin{aligned} P_{L(A, B)^*} &= \begin{pmatrix} I - BB^* & (I - BB^*)^{\frac{1}{2}} B(V_A)^* \\ V_A B^* (I - BB^*)^{\frac{1}{2}} & P_{\ker A^*} + V_A B^* B(V_A)^* \end{pmatrix} \\ &= \begin{pmatrix} (I - BB^*)^{\frac{1}{2}} \\ V_A B^* \end{pmatrix} \begin{pmatrix} (I - BB^*)^{\frac{1}{2}} \\ V_A B^* \end{pmatrix}^* + \begin{pmatrix} 0 & 0 \\ 0 & P_{\ker A^*} \end{pmatrix}. \end{aligned}$$

From this, the following result is now clear.

Corollary 3.8. *Let the linear relation $L(A, B)$ be given by (2.5) and assume that condition (3.29) holds. Then*

$$L(A, B)^* = \{ \{ (I - BB^*)^{\frac{1}{2}} k, V_A B^* k + P_{\ker A^*} h \} : h \in \mathfrak{H}, k \in \mathfrak{K} \},$$

where the range decomposition is orthogonal.

Note that the results in Corollary 3.5 and Corollary 3.8 are of practical importance in the description of boundary value problems (when one speaks of boundary values in parametrized form); cf. [2].

4. Construction of the closure of operator range relations

Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$. Then the linear relation $L(A, B)$ from \mathfrak{H} to \mathfrak{K} in (2.5) is an operator range relation. By Lemma 2.7 the closure $L(A, B)^{**}$ of $L(A, B)$ is an operator range relation and it will be shown how the closure $L(A, B)^{**}$ can be represented in terms of the original operators A and B .

First return to the treatment involving the pair of bounded operators A and B . The following construction is inspired by arguments appearing in measure theory. Due to the obvious inequalities

$$A^* A \leq A^* A + B^* B, \quad B^* B \leq A^* A + B^* B, \quad (4.1)$$

an application of the Douglas Lemma [3] shows that there exists a pair of contractions $C_A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $C_B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$, such that

$$A = C_A (A^* A + B^* B)^{\frac{1}{2}}, \quad B = C_B (A^* A + B^* B)^{\frac{1}{2}}, \quad (4.2)$$

or, equivalently,

$$A^* = (A^* A + B^* B)^{\frac{1}{2}} (C_A)^*, \quad B^* = (A^* A + B^* B)^{\frac{1}{2}} (C_B)^*. \quad (4.3)$$

The contractions $C_A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $C_B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ are uniquely determined by the conditions that C_A and C_B vanish on $(\text{ran } (A^* A + B^* B)^{\frac{1}{2}})^{\perp} = \ker A \cap \ker B$ or, equivalently,

$$\text{ran } (C_A)^* \subset \overline{\text{ran}} (A^* A + B^* B), \quad \text{ran } (C_B)^* \subset \overline{\text{ran}} (A^* A + B^* B), \quad (4.4)$$

and with these conditions one also has as a consequence

$$\ker (C_A)^* = \ker A^*, \quad \ker (C_B)^* = \ker B^*.$$

It follows from (4.2) that $L(A, B)$ can be written as

$$L(A, B) = \{ \{ C_A (A^* A + B^* B)^{\frac{1}{2}} f, C_B (A^* A + B^* B)^{\frac{1}{2}} f \} : f \in \mathfrak{E} \}. \quad (4.5)$$

This representation will be called the *canonical representation* of $L(A, B)$. Of course the pair C_A and C_B itself induces a linear relation $L(C_A, C_B)$ in $\mathfrak{H} \times \mathfrak{K}$, in other words,

$$L(C_A, C_B) = \{ \{ C_A f, C_B f \} : f \in \mathfrak{E} \}. \quad (4.6)$$

Analogously to (3.1) one may now introduce the column operator $c(C_A, C_B)$ from \mathfrak{E} to $\mathfrak{H} \times \mathfrak{K}$ by

$$c(C_A, C_B) = \begin{pmatrix} C_A \\ C_B \end{pmatrix}, \quad \text{i.e.,} \quad c(C_A, C_B)\varphi = \begin{pmatrix} C_A \varphi \\ C_B \varphi \end{pmatrix}, \quad \varphi \in \mathfrak{E}. \quad (4.7)$$

Then $c(C_A, C_B)$ belongs to $\mathbf{B}(\mathfrak{E}, \mathfrak{H} \times \mathfrak{K})$ and $L(C_A, C_B) = \text{ran } c(C_A, C_B)$; cf. (3.7). It follows from (4.5) that

$$L(A, B) = \text{ran } c(A, B) \subset \text{ran } c(C_A, C_B) = L(C_A, C_B). \quad (4.8)$$

Note that if $L(A, B)$ is closed, then $L(C_A, C_B) = L(A, B)$ by (4.5), (3.16), and Lemma 3.1.

The next lemma contains a first step in establishing the connection between the pairs A, B and C_A, C_B .

Lemma 4.1. *Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$, $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$, and let C_A, C_B be uniquely defined by (4.2) and (4.4). Then*

$$\ker C_A \cap \ker C_B = \ker A \cap \ker B, \quad (4.9)$$

and

$$[(C_A)^* C_A + (C_B)^* C_B]h = \begin{cases} h, & h \in \overline{\text{ran}}(A^* A + B^* B), \\ 0, & h \in \ker A \cap \ker B. \end{cases} \quad (4.10)$$

Consequently, $(C_A)^* C_A + (C_B)^* C_B$ is an orthogonal projection.

Proof. It follows from (4.2), (4.3), and (4.4) that

$$(A^* A + B^* B)^{\frac{1}{2}} [(C_A)^* C_A + (C_B)^* C_B] (A^* A + B^* B)^{\frac{1}{2}} = A^* A + B^* B.$$

Consequently, thanks to (4.4),

$$[(C_A)^* C_A + (C_B)^* C_B] (A^* A + B^* B)^{\frac{1}{2}} = (A^* A + B^* B)^{\frac{1}{2}},$$

which by the continuity of $(C_A)^*C_A + (C_B)^*C_B$ implies that

$$[(C_A)^*C_A + (C_B)^*C_B]h = h, \quad h \in \overline{\text{ran}}(A^*A + B^*B). \quad (4.11)$$

Hence the first part of (4.10) holds.

Keeping (3.19) in mind, it is clear that in the present context one also has

$$\mathfrak{E} = \overline{\text{ran}}((C_A)^*C_A + (C_B)^*C_B) \oplus (\ker C_A \cap \ker C_B).$$

Observe that it follows from (4.4) that

$$\overline{\text{ran}}((C_A)^*C_A + (C_B)^*C_B) \subset \overline{\text{ran}}(A^*A + B^*B),$$

or, equivalently,

$$\ker A \cap \ker B = \ker(A^*A + B^*B) \subset \ker C_A \cap \ker C_B.$$

Assume that these inclusions are strict. Then there exists a nontrivial element $h \in \overline{\text{ran}}(A^*A + B^*B)$ with $h \in \ker C_A \cap \ker C_B$, which contradicts (4.11). Thus,

$$\overline{\text{ran}}((C_A)^*C_A + (C_B)^*C_B) = \overline{\text{ran}}(A^*A + B^*B),$$

and (4.9) follows. Thus, also the second part of (4.10) holds. ■

Note that the pair C_A and C_B is normalized precisely when it is reduced or, equivalently, when the pair A and B is reduced.

The precise connection between the pairs A, B and C_A, C_B is now established in terms of the *polar decomposition* (cf. [17, Section VI 2.7]) of the column operator $c(A, B)$ in (3.1); cf. (4.3).

Lemma 4.2. *Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$, $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$, and let C_A, C_B be uniquely defined by (4.2) and (4.4). Let $c(A, B)$ and $c(C_A, C_B)$ be as defined in (3.1) and (4.7), respectively. Then the identity*

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} C_A \\ C_B \end{pmatrix} (A^*A + B^*B)^{\frac{1}{2}}$$

*is the polar decomposition of the column operator $c(A, B)$, where the column operator $c(C_A, C_B)$ is the unique partial isometry whose initial and final spaces are given by $\overline{\text{ran}}(A^*A + B^*B)$ and $\overline{\text{ran}} c(A, B)$, respectively.*

Proof. By Lemma 4.1, $(C_A)^*C_A + (C_B)^*C_B$ is the orthogonal projection from \mathfrak{E} onto

$$\text{ran}((C_A)^*C_A + (C_B)^*C_B) = \overline{\text{ran}}(A^*A + B^*B).$$

In particular, $\text{ran}[(C_A)^*C_A + (C_B)^*C_B]$ is closed and, equivalently, $L(C_A, C_B)$ is closed; cf. Lemma 3.1. Then the column operator $c(C_A, C_B)$ is a partial

isometry with initial space $\overline{\text{ran}}(A^*A + B^*B)$ and final space $\text{ran } c(C_A, C_B)$ (cf. [2, Appendix D]). Since $\text{ran}(A^*A + B^*B)^{\frac{1}{2}}$ is dense in $\overline{\text{ran}}(A^*A + B^*B)$, also its image under $c(C_A, C_B)$ is dense in $\text{ran } c(C_A, C_B)$. By construction, $c(C_A, C_B)$ maps $\text{ran}(A^*A + B^*B)^{\frac{1}{2}}$ onto $\text{ran } c(A, B)$, cf. (4.5), and thus the final space of $c(C_A, C_B)$ is equal to $\overline{\text{ran}} c(A, B)$. \blacksquare

Recall that the reduction of $L(A, B)$ in (3.20) is with respect to the orthogonal decomposition (3.19) of the space \mathfrak{E} :

$$\mathfrak{E} = \mathfrak{E}_0 \oplus (\ker A \cap \ker B) \quad \text{with} \quad \mathfrak{E}_0 = \overline{\text{ran}}(A^*A + B^*B)^{\frac{1}{2}}.$$

Thus, A and B are row operators:

$$A = r(A_0, O_{\ker A \cap \ker B}) \quad \text{and} \quad B = r(B_0, O_{\ker A \cap \ker B}), \quad (4.12)$$

such that $A_0 \in \mathbf{B}(\mathfrak{E}_0, \mathfrak{H})$ and $B_0 \in \mathbf{B}(\mathfrak{E}_0, \mathfrak{K})$. As in (4.2), there exists a pair of operators $C_{A_0} \in \mathbf{B}(\mathfrak{E}_0, \mathfrak{H})$ and $C_{B_0} \in \mathbf{B}(\mathfrak{E}_0, \mathfrak{K})$, such that

$$A_0 = C_{A_0}((A_0)^*A_0 + (B_0)^*B_0)^{\frac{1}{2}}, \quad B_0 = C_{B_0}((A_0)^*A_0 + (B_0)^*B_0)^{\frac{1}{2}}, \quad (4.13)$$

and these operators are unique. It will be shown that, in fact, the operators C_{A_0} and C_{B_0} are the reductions of the pair C_A and C_B in (4.2).

Lemma 4.3. *Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$, $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$, let C_A, C_B be uniquely defined by (4.2) and (4.4), and let A_0 and B_0 be the restrictions of A and B as in (3.20). Then the operators C_{A_0} and C_{B_0} in (4.13) are the restrictions of C_A and C_B . In other words,*

$$L(C_A, C_B) = \{ \{C_{A_0}f, C_{B_0}f\} : f \in \mathfrak{E}_0 \}$$

is a reduced representation of $L(C_A, C_B)$.

Proof. With the restrictions A_0 and B_0 , it follows from (3.22) and (4.2) that

$$\begin{aligned} A &= C_A(A^*A + B^*B)^{\frac{1}{2}} = C_A \begin{pmatrix} ((A_0)^*A_0 + (B_0)^*B_0)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix}, \\ B &= C_B(A^*A + B^*B)^{\frac{1}{2}} = C_B \begin{pmatrix} ((A_0)^*A_0 + (B_0)^*B_0)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (4.14)$$

It has been shown in (4.9) that $\mathfrak{E}_0 = \ker A \cap \ker B = \ker C_A \cap \ker C_B$. Now let $\tilde{C}_A \in \mathbf{B}(\mathfrak{E}_0, \mathfrak{H})$ and $\tilde{C}_B \in \mathbf{B}(\mathfrak{E}_0, \mathfrak{K})$ be the restrictions of C_A and C_B , so that $C_A = (\tilde{C}_A, 0)$ and $C_B = (\tilde{C}_B, 0)$. Then with $A = r(A_0, 0)$ and $B = r(B_0, 0)$, it follows from (4.14), that

$$A_0 = \tilde{C}_A((A_0)^*A_0 + (B_0)^*B_0)^{\frac{1}{2}}, \quad B_0 = \tilde{C}_B((A_0)^*A_0 + (B_0)^*B_0)^{\frac{1}{2}}.$$

A comparison with (4.13) gives that $\tilde{C}_A = C_{A_0}$ and $\tilde{C}_B = C_{B_0}$. \blacksquare

The description of $L(A, B)^{**}$ and its orthogonal operator part follows easily from Lemma 4.2 and Lemma 3.6.

Theorem 4.4. *Let the linear relation $L(A, B)$ be given by (2.5) and let C_A and C_B be the uniquely defined operators satisfying (4.2) and (4.4). Then the linear relation $L(C_A, C_B)$ is closed and*

$$L(A, B)^{**} = L(C_A, C_B) = \{ \{C_A\varphi, C_B\varphi\} : \varphi \in \mathfrak{E} \} = C_B(C_A)^{-1}. \quad (4.15)$$

Consequently, the orthogonal operator part of $L(A, B)^{**}$ is given by

$$(L(A, B)^{**})_s = \{ \{C_A\varphi, C_B\varphi\} : \varphi \in (\ker C_A)^\perp \} = C_B(C_A)^{(-1)}. \quad (4.16)$$

Proof. As shown in Lemma 4.2 one has $\operatorname{ran} c(C_A, C_B) = \overline{\operatorname{ran}} c(A, B)$. Hence, it follows from (4.8), after taking closures, that

$$L(A, B)^{**} = \overline{\operatorname{ran}} c(A, B) = \operatorname{ran} c(C_A, C_B) = L(C_A, C_B),$$

and it is clear that $L(C_A, C_B) = C_B(C_A)^{-1}$. Thus, (4.15) has been shown.

Next observe that by Lemma 4.1 the condition (3.23) is satisfied. Hence (4.16) follows from Lemma 3.6. \blacksquare

Finally, the representation of the adjoint relation $L(C_A, C_B)^*$ will be considered. Recall that the identity (3.11) gives the following representation:

$$L(C_A, C_B)^* = \{ \{k, h\} \in \mathfrak{K} \times \mathfrak{H} : (C_B)^*k = (C_A)^*h \}.$$

An application of Lemma 3.4 and Lemma 4.1 gives the following result; cf. Corollary 3.5.

Corollary 4.5. *Let the linear relation $L(A, B)$ be given by (2.5) and let C_A and C_B be the uniquely defined operators satisfying (4.2) and (4.4). Then the adjoint relation $L(C_A, C_B)^*$ is given by*

$$\{ \{ (I - C_B(C_B)^*)\varphi + C_B(C_A)^*\psi, C_A(C_B)^*\varphi + (I - C_A(C_A)^*)\psi \} : \varphi \in \mathfrak{K}, \psi \in \mathfrak{H} \}.$$

In order to rewrite the result in Corollary 4.5, consider the polar decomposition of the contraction $C_A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$:

$$C_A = W_A((C_A)^*C_A)^{\frac{1}{2}},$$

where W_A is the unique partial isometry from $\mathfrak{H}_{A,B}$ to \mathfrak{H} with initial space and final space given by

$$(\ker W_A)^\perp = \overline{\operatorname{ran}} ((C_A)^*C_A)^{\frac{1}{2}} \quad \text{and} \quad \operatorname{ran} W_A = \overline{\operatorname{ran}} C_A,$$

respectively. Under the assumption that the pair A and B is reduced it follows that the pair C_A and C_B is normalized, see Lemma 4.1. Hence the argument preceding Corollary 3.8 now gives the following result.

Corollary 4.6. *Let the linear relation $L(A, B)$ be given by (2.5) and let C_A and C_B be the uniquely defined operators satisfying (4.2) and (4.4). Assume that the pair A and B is reduced. Then the adjoint relation $L(C_A, C_B)^*$ is given by*

$$L(C_A, C_B)^* = \{ \{ (I - C_B(C_B)^*)^{\frac{1}{2}}h, W_A(C_B)^*h + k \} : h \in \mathfrak{K}, k \in \ker (C_A)^* \},$$

where the range decomposition is orthogonal.

5. Regular and singular operator range relations

Let \mathfrak{E} , \mathfrak{H} , and \mathfrak{K} be Hilbert spaces and consider the pair of bounded linear operators $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$. Let the operator range relation $L(A, B)$ be given by (2.5). Recall that $L(A, B)$ is regular if $L(A, B)$ is (the graph of) a closable operator and $L(A, B)$ is singular if its closure is the Cartesian product of closed linear subspaces. Criteria will be given for the regularity and singularity of $L(A, B)$ in terms of A and B , and in terms of C_A and C_B in (4.2).

First the regularity and the singularity of $L(A, B)$ will be expressed in terms of the pair A and B ; see [7] for some further equivalent statements which hold for general linear relations. The regularity of $L(A, B)$ is described by the following lemma.

Lemma 5.1. *Let the linear relation $L(A, B)$ and $\mathfrak{D}(A, B)$ be given by (2.5) and (3.12), respectively. Then the following statements are equivalent:*

- (i) $L(A, B)$ is regular;
- (ii) the set $\mathfrak{D}(A, B)$ is dense in \mathfrak{K} .

Moreover, the following statements are equivalent:

- (iii) $L(A, B)$ is a bounded operator;
- (iv) $\mathfrak{D}(A, B) = \mathfrak{K}$, i.e., $\text{ran } B^* \subset \text{ran } A^*$.

Finally, the following statements are equivalent:

- (v) $L(A, B) \in \mathbf{B}(\mathfrak{H}, \mathfrak{K})$;
- (vi) $\text{ran } B^* \subset \text{ran } A^*$ and $\text{ran } A = \mathfrak{H}$.

Proof. (i) \Leftrightarrow (ii) Recall that $L(A, B)$ is regular if and only if $\text{mul } L(A, B)^{**} = \{0\}$. It follows from (3.15) that this is equivalent to $\mathfrak{D}(A, B)$ being dense in \mathfrak{K} ; see (3.13).

(iii) \Leftrightarrow (iv) The relation $L(A, B)$ is a bounded operator precisely if there exists $c \geq 0$ such that

$$\|Bf\| \leq c\|Af\|, \quad f \in \mathfrak{E}, \quad (5.1)$$

or, equivalently, if $B^*B \leq c^2 A^*A$. By the Douglas Lemma [3] this is equivalent to $\text{ran } B^* \subset \text{ran } A^*$. This last inclusion is the same as saying $\text{dom } L(A, B)^* = \mathfrak{D}(A, B) = \mathfrak{K}$; see (3.12) and (3.13).

(v) \Leftrightarrow (vi) This is now clear. ■

Likewise, the singularity of $L(A, B)$ can be expressed in various equivalent useful ways.

Lemma 5.2. *Let the linear relation $L(A, B)$ be given by (2.5). Then the following statements are equivalent:*

- (i) $L(A, B)$ is singular;
- (ii) $L(A, B)^{**} = \overline{\text{ran } A} \times \overline{\text{ran } B}$;
- (iii) $\mathfrak{D}(A, B) \subset \ker B^*$;
- (iv) $\mathfrak{R}(A, B) \subset \ker A^*$;
- (v) $\text{ran } A^* \cap \text{ran } B^* = \{0\}$;
- (vi) $L(A, B)^* = \ker B^* \times \ker A^*$.

Proof. This is a straightforward application of [7, Proposition 2.8], together with (3.13) and (3.14). It suffices to prove (iii) \Leftrightarrow (v).

(iii) \Rightarrow (v) Let $\ell \in \text{ran } A^* \cap \text{ran } B^*$. Then $\ell \in \text{ran } A^*$ and $\ell = B^*h$ for some $h \in \mathfrak{H}$. Thus, $h \in \mathfrak{D}(A, B)$ and by assumption one sees that $h \in \ker B^*$. Thus, it follows that $\ell = 0$, and hence (v) holds.

(v) \Rightarrow (iii) This implication is trivial. ■

Next the regularity and the singularity of $L(A, B)$ will be expressed in terms of the pair C_A and C_B in (4.2). Recall from Theorem 4.4 that $L(A, B)^{**} = L(C_A, C_B)$. Thus, the following characterization of the regularity of $L(A, B)$ is clear.

Lemma 5.3. *Let the linear relation $L(A, B)$ be given by (2.5) and let C_A and C_B be the uniquely defined operators satisfying (4.2) and (4.4). Then the following statements are equivalent:*

- (i) $L(A, B)$ is regular;
- (ii) $L(C_A, C_B)$ is an operator;
- (iii) $\ker C_A \subset \ker C_B$.

Likewise, the singularity of $L(A, B)$ can be expressed as follows.

Corollary 5.4. *Let the linear relation $L(A, B)$ be given by (2.5) and let C_A and C_B be the uniquely defined operators satisfying (4.2) and (4.4). Then the following statements are equivalent:*

- (i) $L(A, B)$ is singular;
- (ii) $\ker C_A + \ker C_B = \mathfrak{E}$.

Proof. Due to $L(A, B)^{**} = L(C_A, C_B)$ (see Theorem 4.4), it follows from Lemma 5.2 that $L(A, B)$ is singular if and only if $L(C_A, C_B) = \text{ran } C_A \times \text{ran } C_B$, or, equivalently, precisely when

$$\text{ran } c(C_A, C_B) = \text{ran } C_A \times \text{ran } C_B. \quad (5.2)$$

(i) \Rightarrow (ii) Assume that $L(A, B)$ is singular. Then, thanks to (5.2), for every $e \in \mathfrak{E}$ there exist $e_1, e_2 \in \mathfrak{E}$ such that

$$c(C_A, C_B)e = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad c(C_A, C_B)e_1 = \begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \quad c(C_A, C_B)e_2 = \begin{pmatrix} 0 \\ h_2 \end{pmatrix}.$$

Then, clearly, $e_1 \in \ker C_B$, $e_2 \in \ker C_A$, and

$$e_0 = e - e_1 - e_2 \in \ker c(C_A, C_B) = \ker C_A \cap \ker C_B.$$

This shows that $e \in \ker C_A + \ker C_B$. Therefore, $\mathfrak{E} = \ker C_A + \ker C_B$.

(ii) \Rightarrow (i) Assume that $\mathfrak{E} = \ker C_A + \ker C_B$. Let $a = a_1 + a_2$ and $b = b_1 + b_2$ with $a_1, b_1 \in \ker C_B$ and $a_2, b_2 \in \ker C_A$. Then the equality

$$\begin{pmatrix} C_A a \\ C_B b \end{pmatrix} = \begin{pmatrix} C_A a_1 \\ C_B b_2 \end{pmatrix} = \begin{pmatrix} C_A(a_1 + b_2) \\ C_B(a_1 + b_2) \end{pmatrix},$$

shows that $\text{ran } C_A \times \text{ran } C_B \subset \text{ran } c(C_A, C_B)$. Hence, (5.2) is satisfied. \blacksquare

6. Classification of pairs of bounded linear operators

Let \mathfrak{E} , \mathfrak{H} , and \mathfrak{K} be Hilbert spaces and let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ be a pair of bounded linear operators. The characterizations of the operator range relation $L(A, B)$ from (2.5) will now be augmented by further characterizations in terms of A and B that are influenced by similar observations in measure theory.

Definition 6.1. Let \mathfrak{E} , \mathfrak{H} , and \mathfrak{K} be Hilbert spaces and let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$. Then the operator B is said to be *dominated* by A , denoted by $B \prec A$, if there exists some $c > 0$ such that

$$\|Bf\| \leq c\|Af\| \quad \text{for all } f \in \mathfrak{E}.$$

By the Douglas Lemma this definition is equivalent to the factorization $B = CA$ where $C \in \mathbf{B}(\mathfrak{H}, \mathfrak{K})$; the operator C is uniquely determined when $\text{ran } C^* \subset \overline{\text{ran } A}$, in which case $\ker C^* = \ker B^*$. Note that in the present context Definition 6.1 agrees with the definition of domination in [9]. For the following, it is useful to recall from the Douglas Lemma that B is dominated by A if and only if $\text{ran } B^* \subset \text{ran } A^*$, i.e., $\mathfrak{D}(A, B) = \mathfrak{K}$.

The following simple result is immediate from Lemma 5.1.

Lemma 6.2. Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ and let the relation $L(A, B)$ be defined by (2.5). Then the following statements are equivalent:

- (i) B is dominated by A ;

- (ii) $\mathfrak{D}(A, B) = \mathfrak{K}$;
- (iii) $L(A, B)$ is a bounded operator.

The notion of domination in Definition 6.1 is now extended.

Definition 6.3. Let \mathfrak{E} , \mathfrak{H} , and \mathfrak{K} be Hilbert spaces and let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$. Then the operator B is said to be *almost dominated* by A if there exists a sequence of bounded operators $B_n \in \mathbf{B}(\mathfrak{E}, \mathfrak{K}_n)$, where \mathfrak{K}_n are Hilbert spaces, and a sequence $c_n \geq 0$, such that for all $f \in \mathfrak{E}$

- (a) $\|B_n f\| \leq c_n \|A f\|$;
- (b) $\|B_n f\| \leq \|B_{n+1} f\|$;
- (c) $\|B_n f\| \nearrow \|B f\|$.

It is clear that if $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ is dominated by $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$, then B is automatically almost dominated by A by taking $B_n = B$ and $c_n = c$.

The analog of Lemma 6.2 for the almost dominated case is contained in the following theorem.

Theorem 6.4. Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ and let the relation $L(A, B)$ be defined by (2.5). Then the following statements are equivalent:

- (i) B is almost dominated by A ;
- (ii) $\mathfrak{D}(A, B)$ is dense in \mathfrak{K} ;
- (iii) $L(A, B)$ is regular.

Proof. (i) \Rightarrow (iii) Assume that B is almost dominated by A . Then there exists a sequence of operators $B_n \in \mathbf{B}(\mathfrak{E}, \mathfrak{K}_n)$ as in Definition 6.3. Note that if $f \in \ker A$ then $B_n f = 0$ due to (a), and hence $\|B f\| = \sup \|B_n f\| = 0$ due to (c). One concludes that $\ker A \subset \ker B$, so that $L(A, B)$ is an operator, see (3.8). Define the sequence of linear relations T_n from $\overline{\text{ran}} A$ to \mathfrak{K}_n by

$$T_n = \text{clos} \{ \{A f, B_n f\} : f \in \mathfrak{E} \}.$$

Due to (a) it follows that each T_n is a closed bounded operator from $\overline{\text{ran}} A$ to \mathfrak{K}_n . Furthermore, by (b) one sees that for $m \leq n$

$$\|T_m A f\| = \|B_m f\| \leq \|B_n f\| = \|T_n A f\|, \quad f \in \mathfrak{E},$$

which implies that

$$\|T_m h\| \leq \|T_n h\|, \quad h \in \overline{\text{ran}} A. \quad (6.1)$$

Moreover, if $h \in \text{dom } L(A, B)$ so that $h = A f$ for some $f \in \mathfrak{E}$, then it follows from (c) that

$$\|T_n h\| = \|T_n A f\| = \|B_n f\| \nearrow \|B f\| = \|L(A, B) h\|. \quad (6.2)$$

Hence the sequence $T_n \in \mathbf{B}(\overline{\text{ran}} A, \mathfrak{K}_n)$ satisfies (6.1) and (6.2). Thus, [7, Theorem 8.8] implies that the operator $L(A, B)$ is closable.

(iii) \Rightarrow (i) Assume that $L(A, B)$ is regular, so that $L(A, B)$ is a closable operator from \mathfrak{H} to \mathfrak{K} . Then by [7, Theorem 8.9] there exists a sequence $T_n \in \mathbf{B}(\overline{\text{ran}} A, \mathfrak{H})$ of bounded operators with the property (6.1) such that

$$\|T_n h\| \nearrow \|L(A, B)h\|, \quad h \in \text{dom } L(A, B). \quad (6.3)$$

Define the operators $B_n = T_n A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$. It will be shown that the conditions of Definition 6.3 are satisfied with $\mathfrak{K}_n = \mathfrak{H}$. First note that

$$\|B_n f\| \leq \|T_n\| \|A f\|, \quad f \in \mathfrak{E},$$

so that (a) is satisfied. Secondly, observe that for all $m \leq n$ it follows from (6.1) that

$$\|B_m f\| = \|T_m A f\| \leq \|T_n A f\| = \|B_n f\|, \quad f \in \mathfrak{E},$$

so that (b) is satisfied. Finally note that (6.3) implies

$$\|B_n f\| = \|T_n A f\| \nearrow \|L(A, B)A f\| = \|B f\|, \quad f \in \mathfrak{E},$$

so that (c) is satisfied. Thus, B is almost dominated by A .

(ii) \Leftrightarrow (iii) See Lemma 5.1. ■

The following definition finds its inspiration in a similar notion which is current in measure theory.

Definition 6.5. Let \mathfrak{E} , \mathfrak{H} , and \mathfrak{K} be Hilbert spaces and let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$. Then the operator B is said to be *singular* with respect to A or, equivalently, the operator A is said to be *singular* with respect to B if for every $D \in \mathbf{B}(\mathfrak{E})$

$$D \prec A \quad \text{and} \quad D \prec B \quad \Rightarrow \quad D = 0.$$

Note that an equivalent statement is that $\text{ran } D^* \subset \text{ran } A^*$ and $\text{ran } D^* \subset \text{ran } B^*$ imply that $D = 0$. It is straightforward to characterize the property “ B is singular with respect to A ” in terms of the operators A and B .

Theorem 6.6. Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ and let the relation $L(A, B)$ be defined by (2.5). Then the following statements are equivalent:

- (i) B is singular with respect to A ;
- (ii) $\text{ran } A^* \cap \text{ran } B^* = \{0\}$;
- (iii) $L(A, B)$ is singular.

Proof. (i) \Rightarrow (ii) Assume that B is singular with respect to A . To prove (ii), suppose that $\text{ran } A^* \cap \text{ran } B^* \neq \{0\}$. Then there exists a proper orthogonal projection D in \mathfrak{E} with $\text{ran } D \subset \text{ran } A^* \cap \text{ran } B^*$, or

$$\text{ran } D \subset \text{ran } A^* \quad \text{and} \quad \text{ran } D \subset \text{ran } B^*.$$

Since D is an orthogonal projection, it is selfadjoint and one concludes that $D \prec A$ and $D \prec B$. Hence $D = 0$. This contradiction implies that $\text{ran } A^* \cap \text{ran } B^* = \{0\}$.

(ii) \Rightarrow (i) Assume that $\text{ran } A^* \cap \text{ran } B^* = \{0\}$. To prove (i), suppose that $D \in \mathbf{B}(\mathfrak{E})$ satisfies $D \prec A$ and $D \prec B$ or, equivalently, $\text{ran } D^* \subset \text{ran } A^*$ and $\text{ran } D^* \subset \text{ran } B^*$. This leads to $\text{ran } D^* \subset \text{ran } A^* \cap \text{ran } B^*$. Hence $D^* = 0$ and thus $D = 0$. Therefore, B is singular with respect to A .

(ii) \Leftrightarrow (iii) See Lemma 5.2. ■

In the situation of Definition 6.5 one sometimes says that A and B are mutually singular.

7. Almost domination and the Radon–Nikodym derivative

Let \mathfrak{E} , \mathfrak{H} , and \mathfrak{K} be Hilbert spaces and let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$. Let $L(A, B)$ be the corresponding operator range relation defined in (2.5). If B is dominated or almost dominated by A , then there is a factorization of B with respect to A , which gives the notion of the Radon–Nikodym derivative in the abstract setting of the operator range relation $L(A, B)$.

Observe the following straightforward remarks. Let $L(A, B)$ be an operator range relation as in (2.5) and recall that $L(A, B)$ is equal to the quotient BA^{-1} . In the case that $L(A, B)$ is an operator, one may write

$$Bf = L(A, B)Af \quad \text{for all } f \in \mathfrak{E}. \quad (7.1)$$

If also $L(A, B)^{**}$ is an operator, then it follows from

$$L(A, B) = \{\{Af, Bf\} : f \in \mathfrak{E}\} \subset L(A, B)^{**}$$

that one may write

$$Bf = L(A, B)^{**}Af \quad \text{for all } f \in \mathfrak{E}. \quad (7.2)$$

Note that in this identity only the reduced part of the pair A and B plays a role.

First the case of domination will be considered.

Lemma 7.1. *Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$. Then the following statements are equivalent:*

- (i) B is dominated by A ;
- (ii) $B = CA$ holds for some bounded linear operator C from \mathfrak{H} to \mathfrak{K} ;
- (iii) $B = CA$ holds for some closed bounded linear operator C from \mathfrak{H} to \mathfrak{K} .

*If one of these conditions is satisfied, then $L(A, B)$ is a bounded linear operator and $B = L(A, B)A$. Moreover, if $B = CA$ holds for some bounded linear operator C from \mathfrak{H} to \mathfrak{K} , then $L(A, B) \subset C$; and if $B = CA$ holds for some closed bounded linear operator C from \mathfrak{H} to \mathfrak{K} , then $L(A, B)^{**} \subset C$.*

Proof. (i) \Rightarrow (ii) It follows from Lemma 6.2 that $L(A, B)$ is a bounded linear operator. Hence it follows from (7.1) that (ii) holds.

(ii) \Rightarrow (iii) To see this replace C in $B = CA$ by its closure C^{**} .

(iii) \Rightarrow (i) This is clear.

These equivalent statements imply that $L(A, B)$ is a bounded linear operator and it follows from (2.5) that $B = L(A, B)A$. Therefore, if $B = CA$ holds for some bounded linear operator C from \mathfrak{H} to \mathfrak{K} , then $L(A, B) \subset C$ and, if in addition C is closed, it follows that $L(A, B)^{**} \subset C$. \blacksquare

Notice that the Radon–Nikodym derivative in the following definition satisfies the minimality property expressed in Lemma 7.1.

Definition 7.2. Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ and assume that B is dominated by A . Then the *Radon–Nikodym derivative* $R(A, B)$ of B with respect to A is the bounded closed operator $L(A, B)^{**}$ from \mathfrak{H} to \mathfrak{K} .

For an illustration of such a Radon–Nikodym derivative, return to the inequalities $A^*A \leq A^*A + B^*B$ and $B^*B \leq A^*A + B^*B$. These inequalities imply that there are $C_A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $C_B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ such that the identities in (4.2) hold, and they are unique when (4.4) is assumed. It is clear from Definition 6.1 that A and B are dominated by the operator $(A^*A + B^*B)^{\frac{1}{2}}$, so that each of the following relations from \mathfrak{E} to \mathfrak{H} and from \mathfrak{E} to \mathfrak{K} , respectively,

$$L((A^*A + B^*B)^{\frac{1}{2}}, A) \quad \text{and} \quad L((A^*A + B^*B)^{\frac{1}{2}}, B)$$

is not only regular, but (the graph of) a bounded operator. Recall that A_0 and B_0 are the reduction of the pair A and B ; see (4.12) and Lemma 4.3.

Lemma 7.3. Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$, and let C_A and C_B be the uniquely defined operators satisfying (4.2) and (4.4). Then the Radon–Nikodym derivatives of A and B with respect to $(A^*A + B^*B)^{\frac{1}{2}}$ are given by

$$R((A^*A + B^*B)^{\frac{1}{2}}, A) = C_{A_0}, \quad R((A^*A + B^*B)^{\frac{1}{2}}, B) = C_{B_0}, \quad (7.3)$$

Proof. Since the operators C_A and C_B satisfy the identities (4.2), it follows from Lemma 7.1 that

$$R((A^*A + B^*B)^{\frac{1}{2}}, A)^{**} \subset C_A, \quad R((A^*A + B^*B)^{\frac{1}{2}}, B)^{**} \subset C_B.$$

Since all these operators are closed and bounded,

$\text{dom } R((A^*A + B^*B)^{\frac{1}{2}}, A)^{**} = \text{dom } R((A^*A + B^*B)^{\frac{1}{2}}, B)^{**} = \overline{\text{ran}}(A^*A + B^*B)$
and the identities (7.3) follow. \blacksquare

Next, the notion of almost domination in the general case will be taken up again.

Theorem 7.4. *Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$. Then the following statements are equivalent:*

- (i) B is almost dominated by A ;
- (ii) $B = CA$ holds for some closable linear operator C from \mathfrak{H} to \mathfrak{K} ;
- (iii) $B = CA$ holds for some closed linear operator C from \mathfrak{H} to \mathfrak{K} .

*If one of these conditions is satisfied, then $L(A, B)$ is a closable linear operator such that $B = L(A, B)A$. Moreover, if $B = CA$ holds for some closable linear operator C from \mathfrak{H} to \mathfrak{K} , then $L(A, B) \subset C$ and if $B = CA$ holds for some closed linear operator C from \mathfrak{H} to \mathfrak{K} , then $L(A, B)^{**} \subset C$.*

Proof. (i) \Rightarrow (ii) Assume that B is almost dominated by A . By Theorem 6.4 this implies that $L(A, B)$ is a closable operator. In particular, it follows from (7.2) that (ii) holds.

(ii) \Rightarrow (iii) As in the previous lemma this is seen by replacing C in $B = CA$ by its closure.

(iii) \Rightarrow (i) Assume that $B = CA$ with a closed operator C . It follows from $B = CA$ that $A^*C^* \subset B^*$, in other words

$$\{k, k'\} \in C^* \Rightarrow \{k, A^*k'\} \in B^* \Rightarrow B^*k = A^*k'.$$

Hence $\text{dom } C^* \subset \mathfrak{D}(A, B)$. Since C is a closed operator, it follows that $\text{dom } C^*$ is dense and thus that $\mathfrak{D}(A, B)$ is dense. By Theorem 6.4 this means that B is almost dominated by A .

It remains to prove the last statements. Notice that if C satisfies (ii), then $\text{ran } A \subset \text{dom } C$ and

$$L(A, B) = \{\{Ah, Bh\} : h \in \mathfrak{E}\} = \{\{Ah, CAh\} : h \in \mathfrak{E}\} \subset C.$$

If, in addition, C is closed, then one sees that $L(A, B)^{**} \subset C$. ■

Similarly to what happens in the dominated case, the Radon–Nikodym derivative in the following definition satisfies the minimality property expressed in Theorem 7.4.

Definition 7.5. Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ and assume that B is almost dominated by A . Then the *Radon–Nikodym derivative* $R(A, B)$ of B with respect to A is the closed operator $L(A, B)^{**}$ from \mathfrak{H} to \mathfrak{K} .

Corollary 7.6. *Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ and assume that B is almost dominated by A . Then the Radon–Nikodym derivative $R(A, B)$ of B with respect to A is bounded if and only if B is dominated by A .*

Now recall that $L(A, B)^{**} = L(C_A, C_B)$ and, moreover, that $L(C_A, C_B)$ is an operator precisely when $\ker C_A \subset \ker C_B$. The Radon–Nikodym derivative $R(A, B)$ can be expressed in terms of the Radon–Nikodym derivatives in Lemma 7.3.

Theorem 7.7. *Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ and assume that B is dominated or almost dominated by A . Moreover, let C_A and C_B be as defined in (4.2) and (4.4) and let C_{A_0} and C_{B_0} be the Radon–Nikodym derivatives in (7.3). Then the Radon–Nikodym derivative $R(A, B)$ of B with respect to A is given by the quotient*

$$R(A, B) = C_{B_0}(C_{A_0})^{-1}, \quad (7.4)$$

where $\ker C_{A_0} \subset \ker C_{B_0}$.

Proof. The Radon–Nikodym derivative $R(A, B)$ is given by the closed operator $L(C_A, C_B)$. Now consider the reduction A_0 and B_0 of the pair A and B . By Lemma 4.3 one has $R(A, B) = (L(A, B))^{**} = L(C_A, C_B) = L(C_{A_0}, C_{B_0})$ with $\ker C_{A_0} \subset \ker C_{B_0}$. The identity (7.4) follows by rewriting the above result as a quotient of the operators C_{A_0} and C_{B_0} . \blacksquare

It will be helpful to compare the results in Lemma 7.3 and Theorem 7.7, together with Theorem 4.4, with the following construction known from measure theory. Let (μ, ν) be a pair of finite positive measures. Then μ and ν are absolutely continuous with respect to the sum measure $\rho = \mu + \nu$: $\mu \ll \rho$, $\nu \ll \rho$. This gives rise to the existence of the corresponding Radon–Nikodym derivatives $f = \frac{d\mu}{d\rho}$ and $g = \frac{d\nu}{d\rho}$ and in this case

$$f + g = 1 \quad \rho\text{-a.e.} \quad (7.5)$$

If, in addition, $\nu \ll \mu$ with the Radon–Nikodym derivative $h = \frac{d\nu}{d\mu}$, then $\nu \ll \mu \ll \rho$ implies that

$$g = \frac{d\nu}{d\rho} = \frac{d\nu}{d\mu} \frac{d\mu}{d\rho} = hf \quad \rho\text{-a.e.}$$

Since $\rho = \mu + \nu$ and $\nu \ll \mu$, one has also $\rho \ll \mu$. Consequently, one has $f > 0$ ρ -a.e. ($\Leftrightarrow \mu$ -a.e.) and thus, in fact, the Radon–Nikodym derivative h is given by

$$\frac{d\nu}{d\mu} = \frac{g}{f} \quad \mu\text{-a.e.} \quad (7.6)$$

Remark 7.8. The concept of Radon–Nikodym derivative given in Definitions 7.2 and 7.5 is applicable and uniquely determined for general operator range relations, which are regular (i.e., closable operators). Indeed, by Theorem 2.8 the operator T has the representation $T = L(A, B)$ and if $T = \text{ran } Z$ for some other operator $Z \in \mathbf{B}(\mathfrak{Y}, \mathfrak{H} \times \mathfrak{K})$, then Lemma 2.4 shows that there exists a bounded and boundedly invertible operator $W \in \mathbf{B}(\mathfrak{E}, \mathfrak{Y})$ such that $c(A, B) = ZW$. Then

$$L(A, B) = \text{ran } c(A, B) = \text{ran } ZW = \text{ran } Z$$

and taking closures leads to $\overline{\text{ran}} Z = R(A, B)$.

8. Lebesgue type decompositions for pairs of bounded linear operators

Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ be bounded linear operators. In this section it will be shown that there exist Lebesgue type decompositions $B = B_1 + B_2$ such that B_1 is almost dominated by A and B_2 is singular with respect to A . The main idea is to go back to the corresponding operator range relation $L(A, B)$ and to use the Lebesgue type decompositions of $L(A, B)$; cf. [7].

Definition 8.1. Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ be bounded linear operators and let P be the orthogonal projection onto $\mathfrak{D}(A, B)^\perp$. The regular part B_{reg} and the singular part B_{sing} are defined by

$$B_{\text{reg}} = (I - P)B, \quad B_{\text{sing}} = PB. \quad (8.1)$$

The corresponding decomposition

$$B = B_{\text{reg}} + B_{\text{sing}} \quad (8.2)$$

is called the Lebesgue decomposition of B with respect to A .

Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ be as in Definition 8.1. Let $L(A, B)$ from \mathfrak{H} to \mathfrak{K} be defined as in (2.5) and recall that

$$\text{dom } L(A, B)^* = \mathfrak{D}(A, B) \quad \text{and} \quad \text{mul } L(A, B)^{**} = \mathfrak{D}(A, B)^\perp;$$

cf. (3.15). Then the relation $L(A, B)$ has the Lebesgue decomposition

$$L(A, B) = L(A, B)_{\text{reg}} + L(A, B)_{\text{sing}}, \quad (8.3)$$

where the regular and singular components are given by

$$L(A, B)_{\text{reg}} = (I - P)L(A, B), \quad L(A, B)_{\text{sing}} = PL(A, B); \quad (8.4)$$

here P stands for the orthogonal projection from \mathfrak{K} onto $\text{mul } L(A, B)^{**} = \mathfrak{D}(A, B)^\perp$; cf. [7]. Via the decomposition (8.3) one may now obtain the Lebesgue decomposition of B with respect to A as in Definition 8.1.

Theorem 8.2. (Lebesgue decomposition) *Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ be bounded linear operators. Then B_{reg} is almost dominated by A , B_{sing} is singular with respect to A , and B has the Lebesgue decomposition (8.2) with respect to A . The regular part B_{reg} can be written as*

$$B_{\text{reg}} = R(A, B_{\text{reg}})A, \quad (8.5)$$

where $R(A, B_{\text{reg}})$ is the Radon–Nikodym derivative of B_{reg} with respect to A :

$$R(A, B_{\text{reg}}) = L(A, B_{\text{reg}})^{**}. \quad (8.6)$$

In fact, if $L(A, B)$ is given by (8.3) and (8.4), then

$$L(A, B)_{\text{reg}} = L(A, B_{\text{reg}}), \quad L(A, B)_{\text{sing}} = L(A, B_{\text{sing}}). \quad (8.7)$$

Proof. The decomposition (8.2) follows from (8.1). It follows from (8.4) that $L(A, B)_{\text{reg}}$ and $L(A, B)_{\text{sing}}$ have the representations

$$\begin{aligned} L(A, B)_{\text{reg}} &= \{ \{Af, (I - P)Bf\} : f \in \mathfrak{E} \} = \{ \{Af, B_{\text{reg}}f\} : f \in \mathfrak{E} \}, \\ L(A, B)_{\text{sing}} &= \{ \{Af, PBf\} : f \in \mathfrak{E} \} = \{ \{Af, B_{\text{sing}}f\} : f \in \mathfrak{E} \}, \end{aligned}$$

which give (8.7). Since the relation $L(A, B)_{\text{reg}}$ is regular, it follows from Theorem 6.4 that B_{reg} is almost dominated by A . Likewise, since the relation $L(A, B)_{\text{sing}}$ is singular, it follows from Theorem 6.6 that B_{sing} is singular with respect to A .

The statements about the Radon–Nikodym derivative in (8.5) and (8.6) follow from Theorem 7.4. ■

The Lebesgue decomposition in (8.2) is an example of a so-called Lebesgue type decomposition.

Definition 8.3. Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ be bounded operators. The operator B is said to have a Lebesgue type decomposition with respect to A , if $B = B_1 + B_2$ where $B_1, B_2 \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ have the following properties:

- (a) $\text{ran } B_1 \perp \text{ran } B_2$;
- (b) B_1 is almost dominated by A ;
- (c) B_2 is singular with respect to A .

Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ as in Definition 8.3. Let the linear relation $L(A, B)$ from \mathfrak{H} to \mathfrak{K} be defined by (2.5). According to [7] the Lebesgue type decompositions of $L(A, B)$ are in one-to-one correspondence with the closed linear subspaces $\mathfrak{L} \subset \mathfrak{K}$ such that

$$\mathfrak{L} \subset \overline{\text{dom } L(A, B)^*} \setminus \text{dom } L(A, B)^*, \quad (8.8)$$

which satisfy the condition

$$\text{clos}(\mathfrak{L}^\perp \cap \mathfrak{D}(A, B)) = \mathfrak{L}^\perp \cap \text{clos } \mathfrak{D}(A, B). \quad (8.9)$$

Define the closed linear subspace \mathfrak{M} by

$$\mathfrak{M} = \mathfrak{D}(A, B)^\perp \oplus \mathfrak{L}, \quad (8.10)$$

and let $P_{\mathfrak{M}}$ be the orthogonal projection from \mathfrak{K} onto \mathfrak{M} . Then the corresponding Lebesgue type decomposition of $L(A, B)$ is given by

$$L(A, B) = L(A, B)_1 + L(A, B)_2, \quad (8.11)$$

where the regular and singular components are given by

$$L(A, B)_1 = (I - P_{\mathfrak{M}})L(A, B), \quad L(A, B)_2 = P_{\mathfrak{M}}L(A, B), \quad (8.12)$$

respectively. Via the decomposition (8.11) one may now obtain the Lebesgue type decompositions of B with respect to A as in Definition 8.3.

Theorem 8.4. *Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ be bounded linear operators. Then the Lebesgue type decompositions of B with respect to A are in one-to-one correspondence with the closed linear subspaces $\mathfrak{L} \subset \mathfrak{K}$ in (8.8) which satisfy the condition (8.9). In particular, the corresponding Lebesgue type decomposition is given by*

$$B = B_1 + B_2, \quad B_1 = (I - P_{\mathfrak{M}})B, \quad B_2 = P_{\mathfrak{M}}B,$$

where \mathfrak{M} is as in (8.10) and $P_{\mathfrak{M}}$ is the orthogonal projection from \mathfrak{K} onto \mathfrak{M} , while B_1 is almost dominated by A and B_2 is singular with respect to A . The regular part B_1 can be written as

$$B_1 = R(A, B_1)A,$$

where $R(A, B_1)$ is the Radon–Nikodym derivative of B_1 with respect to A :

$$R(A, B_1) = L(A, B_1)^{**}.$$

In fact, $L(A, B)$ is given by (8.11) and (8.12) precisely, when

$$L(A, B)_1 = L(A, B_1), \quad L(A, B)_2 = L(A, B_2).$$

Proof. Consider the linear relation $L(A, B)$ from \mathfrak{H} to \mathfrak{K} defined by (2.5).

First it is shown that every Lebesgue type decomposition of the linear relation $L(A, B)$ (in the sense of [7]) generates a Lebesgue type decomposition of B with respect to A as in Definition 8.3. To see this let $\mathfrak{L} \subset \overline{\text{dom } L(A, B)^*} \setminus \text{dom } L(A, B)$ be a linear subspace which satisfies (8.9) and let \mathfrak{M} be as defined in (8.10) with the corresponding orthogonal projection $P_{\mathfrak{M}}$ onto \mathfrak{M} . According to [7, Theorem 5.4] the formula $L(A, B) = (I - P_{\mathfrak{M}})L(A, B) + P_{\mathfrak{M}}L(A, B)$ determines a Lebesgue type decomposition of $L(A, B)$, where $(I - P_{\mathfrak{M}})L(A, B)$ is the regular part and $P_{\mathfrak{M}}L(A, B)$ is the singular part generated uniquely by the subspace \mathfrak{L} . From the representation of the regular part $(I - P_{\mathfrak{M}})L(A, B) = \{ \{Af, (I - P_{\mathfrak{M}})Bf\} : f \in \mathfrak{E} \}$ and Theorem 6.4 it follows that $B_1 = (I - P_{\mathfrak{M}})B$ is almost dominated by A . Likewise from the representation of the singular part $P_{\mathfrak{M}}L(A, B) = \{ \{Af, P_{\mathfrak{M}}Bf\} : f \in \mathfrak{E} \}$ and Theorem 6.6 it follows that $B_2 = P_{\mathfrak{M}}B$ is singular with respect to A . Hence, the identity $B = B_1 + B_2$ is a Lebesgue type decomposition of B with respect to A in the sense of Definition 8.3.

Conversely, assume that $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$, $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$, and that B has a Lebesgue type decomposition as in Definition 8.3. Then it is clear that the corresponding relations satisfy

$$L(A, B) = L(A, B_1) + L(A, B_2), \tag{8.13}$$

where, due to Theorem 6.4 and Theorem 6.6, the relation $L(A, B_1)$ is regular and the relation $L(A, B_2)$ is singular. Hence (8.13) is a Lebesgue type decomposition

for $L(A, B)$. Again by [7, Theorem 5.4] there exists a linear subspace \mathfrak{L} , such that (8.8) and (8.9) are satisfied, and a subspace \mathfrak{M} given by (8.10) such that

$$\begin{aligned} L(A, B_1) &= (I - P_{\mathfrak{M}})L(A, B) = L(A, (I - P_{\mathfrak{M}})B), \\ L(A, B_2) &= P_{\mathfrak{M}}L(A, B) = L(A, P_{\mathfrak{M}}B). \end{aligned} \quad (8.14)$$

Thanks to the first identities in (8.14), for every $h \in \mathfrak{E}$ there exists $f \in \mathfrak{E}$, such that $Ah = Af$ and $(I - P_{\mathfrak{M}})Bh = B_1f$. Thus, $f = h + \varphi$ for some $\varphi \in \ker A$. Since $L(A, B_1)$ is regular, it is an operator and hence $\text{mul } L(A, B_1) = B_1(\ker A) = 0$; cf. (3.8). This shows that $B_1\varphi = 0$, and thus $(I - P_{\mathfrak{M}})Bh = B_1h$. Therefore, $(I - P_{\mathfrak{M}})B = B_1$ and, consequently, $P_{\mathfrak{M}}B = B_2$. This proves the one-to-one correspondence between the Lebesgue type decompositions of $B = B_1 + B_2$ in Definition 8.3 and of $L(A, B)$ in (8.11) and (8.12). The one-to-one correspondence between the closed subspaces \mathfrak{L} satisfying the conditions (8.8) and (8.9) is obtained from [7, Theorem 5.4].

The statement about the Radon–Nikodym derivative of B_1 with respect to A follows from Theorem 6.4. ■

For any Lebesgue type decomposition of B with respect to A , the part B_1 , which is almost dominated by A , is, in fact, dominated by the regular part B_{reg} .

Corollary 8.5. *Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$. If $B = B_1 + B_2$ is a Lebesgue type decomposition of B , then $\|B_1h\| \leq \|B_{\text{reg}}h\|$, $h \in \mathfrak{E}$.*

Proof. Let $B = B_1 + B_2$ be a Lebesgue type decomposition of B with respect to A . Then as in the proof of Theorem 8.4 one finds that

$$B_1 = (I - P_{\mathfrak{M}})B = (I - P_{\mathfrak{M}})(I - P)B = (I - P_{\mathfrak{M}})B_{\text{reg}},$$

where it was used that $\text{ran } P \subset \text{ran } P_{\mathfrak{M}}$; cf. (8.10). ■

The uniqueness of Lebesgue type decompositions of B with respect to A in Definition 8.3 can be characterized as follows.

Corollary 8.6. *Let $A \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $B \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$. Then the following statements are equivalent:*

- (i) B admits a unique Lebesgue type decomposition with respect to A ;
- (ii) $L(A, B)$ admits a unique Lebesgue type decomposition;
- (iii) $\mathfrak{D}(A, B)$ is closed;
- (iv) B_{reg} is dominated by A ;
- (v) the Radon–Nikodym derivative $R(A, B_{\text{reg}})$ is a bounded operator.

In this case, all Lebesgue type decompositions of B with respect to A coincide with the Lebesgue decomposition of $B = B_{\text{reg}} + B_{\text{sing}}$ in Theorem 8.2.

Proof. (i) \Leftrightarrow (ii) The Lebesgue type decompositions of $B = B_1 + B_2$ with respect to A correspond to the Lebesgue type decompositions of $L(A, B)$ via Theorem 8.4, see (8.11), (8.12). Hence B has a unique Lebesgue type decomposition with respect to A if and only if $L(A, B)$ has a unique Lebesgue type decomposition.

(ii) \Leftrightarrow (iii) By [7, Theorem 6.1] $L(A, B)$ has a unique Lebesgue type decomposition if and only if $\text{dom } L(A, B)^*$ is closed, i.e., $\mathfrak{D}(A, B)$ is closed; cf. (3.13).

(ii) \Leftrightarrow (iv) Again by [7, Theorem 6.1] $L(A, B)$ has a unique Lebesgue type decomposition if and only if $L(A, B)_{\text{reg}} = L(A, B_{\text{reg}})$ is bounded; cf. Theorem 8.2. Now $L(A, B_{\text{reg}})$ is bounded if and only if B_{reg} is dominated by A ; cf Lemma 6.2.

(iv) \Leftrightarrow (v) This follows from Corollary 7.6.

The last statement is clear from (8.8), since if $\mathfrak{D}(A, B) = \text{dom } L(A, B)^*$ is closed, then $\mathfrak{L} = \{0\}$ and \mathfrak{M} in (8.10) coincides with $\mathfrak{D}(A, B)^\perp$. \blacksquare

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