Lassi Lilleberg

Generalized Schur functions and passive discrete-time realizations

ACTA WASAENSIA 456

Vaasan yliopisto
UNIVERSITY OF VAASA

ACADEMIC DISSERTATION

To be presented, with the permission of the Board of the School of Technology and Innovations of the University of Vaasa, for public examination on the 9th of April, 2021, at noon.

Reviewers

Professor Harald Woracek
Vienna University of Technology
Institute for Analysis and Scientific Computing
Wiedner Hauptstraße 8-10/101
1040 Vienna
AUSTRIA

Docent Mikael Kurula
Åbo Akademi
Faculty of Science and Engineering
Tuomiokirkontori 3
20500 Turku
FINLAND

Julkaisija	**Julkaisupäivämäärä**	
Vaasan yliopisto	Maaliskuu 2021	
Tekijä(t) Lassi Lilleberg	**Julkaisun tyyppi** Artikkeliväitöskirja	
ORCID tunniste	**Julkaisusarjan nimi, osan numero** Acta Wasaensia, 456	
Yhteystiedot Vaasan yliopisto Tekniikan ja innovaatiojohtamisen yksikkö Matematiikka PL 700 FI-65101 VAASA	**ISBN** 978-952-476-942-6 (painettu) 978-952-476-943-3 (verkkoaineisto) http://urn.fi/URN:ISBN:978-952-476-943-3 **ISSN** 0355-2667 (Acta Wasaensia 456, painettu) 2323-9123 (Acta Wasaensia 456, verkkoaineisto)	
	Sivumäärä 130	**Kieli** Englanti
Julkaisun nimike Yleistetyt Schur-funktiot ja passiiviset systeemit		

Tiivistelmä

Väitöskirjassa tarkastellaan passiivisia diskreettiaikaisia systeemejä Pontryaginin avaruuksissa. Kyseessä olevat systeemit voidaan samaistaa kontraktiivisten operaattorikolligaatioden kanssa. Niiden siirtofunktiot ovat yleistettyjä Schur-funktioita. D.Z. Arovin tulokset minimaalisista passiivisista systeemeistä Hilbertin avaruuksissa yleistetään indefiniitteihin avaruuksiin. Saatuja tuloksia sovelletaan M.G. Kreinin ja H. Langerin sekä L. de Brangesin teoriaan yleistetyistä Schur-funktioista, ydinavaruuksista ja koisometrisista kolligaatioista. Defektifunktiot määritellään yleistetyille Schur-funktioille käyttämällä optimaalisia ja minimaalisia realisaatioita. Defektifunktioiden ominaisuuksien ja yleistettyjen Schur-funktioiden realisaatioiden ominaisuuksien yhteyttä analysoidaan. Niiden yleistettyjen Schur-funktioiden, joiden defektifunktiot häviävät, realisaatiolla on vahvoja ominaisuuksia. Johdetut tulokset ovat yleistyksiä aiemmin tunnetuille unitaarisia rationaalifunktioita ja tavallisia sisäfunktioita koskeville tuloksille.

Tutkimusteemana ovat myös heikot ja unitaariset similariteetit yleistettyjen Schur-funktioiden realisaatioiden välillä. D.Z. Arovin ja M.A. Nudelmanin todistama kriteeri siitä, milloin kaikki minimaaliset ja passiiviset realisaatiot ovat unitaarisesti similaarisia, laajennetaan käsittämään myös yleistettyjen Schur-funktioiden luokka. Tutkimalla symmetrisen yleistetyn Schur-funktion optimaalisen ja minimaalisen realisaation sekä tämän duaalin välisen heikon similariteetin negatiivista indeksiä, saadaan kriteeri sille, milloin yleistetty symmetrinen Schur-funktio on myös yleistetty Nevanlinna-funktio.

Asiasanat

operaattorikolligaatio, passiivinen systeemi, kontraktiivinen kuvaus, siirtofunktio, yleistetty Schur-luokka, yleistetty Nevanlinna-luokka, sisäfunktio

Publisher Vaasan yliopisto	Date of publication March 2021	
Author(s) Lassi Lilleberg	**Type of publication** Doctoral thesis by publication	
ORCID identifier	**Name and number of series** Acta Wasaensia, 456	
Contact information University of Vaasa School of Technology and Innovations Mathematics P.O. Box 700 FI-65101 Vaasa Finland	**ISBN** 978-952-476-942-6 (print) 978-952-476-943-3 (online) http://urn.fi/URN:ISBN:978-952-476-943-3 **ISSN** 0355-2667 (Acta Wasaensia 456, print) 2323-9123 (Acta Wasaensia 456, online)	
	Number of pages 130	**Language** English

Title of publication

Generalized Schur functions and passive discrete-time realizations

Abstract

Passive discrete-time systems or contractive operator colligations in Pontryagin space setting are investigated. Transfer functions of such systems are generalized Schur functions, and hence these systems offer state space realizations for such functions. The approach here is to generalize the theory of minimal passive systems, pioneered by D.Z. Arov, from Hilbert space setting to indefinite setting, and then combine it with the theory of generalized Schur functions, reproducing kernels and co-isometric colligations; these are the subjects, which were pioneered by M. G. Krein and H. Langer and L. de Branges.

The concept of the defect function is expanded for generalized Schur functions by using optimal minimal realizations. Defect functions are used to analyze how behaviour of radial limit values on the unit circle affects the properties of certain realizations of generalized Schur functions. It is shown that generalized Schur functions with zero defect functions have stronger realizations than generic Schur functions. These results generalize the results obtained earlier in the Hilbert space setting for rational unitary functions and bi-inner functions.

Furthermore, weak and unitary similarity mappings between the state spaces of specific realizations of generalized Schur functions are investigated. A criterion involving a scattering suboperator obtained by D.Z. Arov and M.A. Nudelman, when all minimal passive realizations of the same Schur function are unitarily similar, is extended to the class of generalized Schur functions. For symmetric generalized Schur functions, it is shown that negative index of the weak similarity mapping between an optimal minimal realization and its dual can be used to decide when such a function is a generalized Nevanlinna function.

Keywords

operator colligation, passive system, contractive operator,
transfer function, generalized Schur class, generalized Nevanlinna class, inner function

ACKNOWLEDGEMENTS

First, I wish to thank my supervisor Professor Seppo Hassi for his invaluable advice and support during my doctoral studies. We have spent hours in discussion and he suggested the topic of this thesis. I also give my gratitude to members of the Department of Mathematics and Statistics of the University of Vaasa. It has been an honor to work with you. I have felt welcomed from the very first day I came here. Furthermore, I am thankful to the University of Vaasa for financial support and facilities that I have enjoyed. During the last months, I have worked under a grant received from the Vilho, Yrjö and Kalle Väisälä Foundation of the Finnish Academy of Science and Letters, and I express my gratitude for them.

I wish to thank my pre-examiners Professor Harald Woracek and Docent Mikael Kurula for reviewing my thesis and their feedbacks. I also wish to thank Dr Michael Dritschel for acting as my opponent.

My parents and sisters deserve praises for their lifelong support. Finally, my deepest gratitude goes to my family. My fiancée Lotta and our daughters Saara and Kiira have loved and supported me during all these years bringing joy to my life.

Laihia, February 2021
Lassi Lilleberg

CONTENTS

LIST OF PUBLICATIONS

The dissertation is based on the following three articles:

(I) Lilleberg, L. Passive Discrete-Time Systems with a Pontryagin State Space. *Complex Analysis and Operator Theory* **13**, 3767–3793 (2019).

(II) Lilleberg, L. Minimal Passive Realizations of Generalized Schur Functions in Pontryagin Spaces. *Complex Analysis and Operator Theory* **14**, 35 (2020).

(III) Lilleberg, L. Generalized Schur–Nevanlinna functions and their realizations. *Integral Equations Operator Theory* **92**, no. 5, 42 (2020).

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AUTHOR'S CONTRIBUTION

Publication I: "Passive Discrete-Time Systems with a Pontryagin State Space"

This is an independent work of the author.

Publication II: "Minimal Passive Realizations of Generalized Schur Functions in Pontryagin Spaces"

This is an independent work of the author.

Publication III: "Generalized Schur–Nevanlinna functions and their realizations"

This is an independent work of the author.

1 INTRODUCTION

The subjects of the study are discrete-time systems, or passive Pontryagin space operator colligations of the form

$$T_\Sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix},$$

and their transfer functions

$$\theta_\Sigma(z) = D + zC(I - zA)^{-1}B,$$

which are Pontryagin space operator valued generalized Schur functions. The motivation and the background of the subjects arise from the system theory. During the sixties, electrical engineer Rudolf Kálmán published several papers that later established the so-called state space representation of the dynamical system. Kálmán's work aroused the interest of theoretical mathematician, since they noticed that the mathematical techniques used in the papers made also possible to apply complex analytical methods effectively in operator theory. By late seventies, the basic theory of the realizations of analytic operator valued functions, operator models and the passive discrete-time systems and their connections to the class of ordinary Schur functions, i.e. Hilbert space operator valued functions holomorphic and bounded by one in the unit disc $\mathbb{D}$, were developed, for instance, by de Branges and Rovnyak (1966a, 1966b) , Sz.-Nagy and Foiaş (1970), Helton (1974), Brodskiĭ (1978), and Arov (1979b, 1979c) among others. The theory was initially developed and studied in the Hilbert space setting, and most of the achieved results were connected to theory of Hilbert space contractions and their dilations to unitary operators.

In the meantime, the study of indefinite inner product spaces advanced, and it was noticed soon that the theory could also be extended to investigate more general systems whose transfer functions go beyond the class of ordinary Schur functions. Such extensions allowed a reasonable treatment of some non-passive behavior, in the usual Hilbert space sense, for the underlying system. By relaxing the assumption of contractivity of the associated system operator in an appropriate way, one enters to linear spaces equipped with a natural inner product, which turned out to be generating a Kreĭn or a Pontryagin space structure on the associated realization space. It was shown that every Kreĭn space operator valued function analytic at the origin could be realized as a transfer function of a conservative discrete-time system in a Kreĭn space sense; see (Azizov & Iokhvidov, 1989, p. 269).

Due to natural difficulties of Kreĭn space operator theory, all the results from Hilbert space setting could not be generalized to the indefinite setting. However, when negative dimensions of the spaces were finite, i.e. Pontryagin spaces, the theory seemed to work very well. The case where the state space is a Pontryagin space while incoming and outgoing spaces are still Hilbert spaces, unitary systems were studied, for instance, by Dijksma, Langer, and de Snoo (1986a, 1986b), and passive systems by Saprikin (2001), Arov and Saprikin (2001), and Arov, Rovnyak, and Saprikin (2006). The case where all the spaces are Pontryagin spaces, theory of isometric, co-isometric and conservative systems is considered, for instance, by Dritschel and Rovnyak (1996), Alpay, Dijksma, Rovnyak, and de Snoo (1997) and Alpay, Azizov, Dijksma, and Rovnyak (2002) In the middle of nineties it was expected that most of the theory from standard Hilbert space systems could be extended in the Pontryagin space setting. However, after nineties, relatively few papers, related directly to the subject, were appearing to put the expected theory forward in a wider scale, while some further applications for the existing results have been coming out.

Still, the complete structures of Pontryagin space systems and transfer function realizations remain somewhat a mystery, and many problems are unsolved. In this thesis, the study of the realizations of the generalized Schur functions is continued. Through the thesis, one can find two main problems.

(1) The criterions for $\theta \in \mathrm{S}_\kappa(\mathcal{U},\mathcal{Y})$ which guarantee the existence of stronger realizations than the canonical realizations.

An arbitrary $\theta \in \mathrm{S}_\kappa(\mathcal{U},\mathcal{Y})$ has canonical realizations, see (Alpay et al., 1997, Chapter 2). The original reproducing kernel space model of an ordinary Schur function $\theta \in \mathrm{S}(\mathcal{U},\mathcal{Y})$, originated from the work of de Branges and Rovnyak (1966b), produces an observable co-isometric realization. The other model, which uses the theory of Hardy spaces of vector valued functions and which goes back to Sz.-Nagy and Foiaş (1970), produces a simple conservative realization of θ. However, in the special cases of finite dimensional conservative systems, the transfer functions are so called unitary rational functions, and it was noticed that in this case, the models above were essentially equal. Moreover, in this case models used by systems theorists, which were based on the minimality, were also equal to realizations described above.

Now question raises that when these abstract models are essentially same in general, or when they are equal to some larger classes of passive systems. One might conjecture from the facts stated above that solutions are classes of functions which are close is some sense to the rational unitary functions. This is in fact somehow true. The key is the unitarity, or almost unitarity in a certain sense; the functions whose defects functions are zeros. In Hilbert space setting, partial answers to problem were given in (Arov, 1979a) and (Y. M. Arlinskiĭ, Hassi, & de Snoo, 2007). In indefinite setting, the problem is briefly touched in (Alpay et al., 1997), but the concept of defect function is not defined. This is done in Article (I) for the class $\mathrm{S}_\kappa(\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, and some results from (Y. M. Arlinskiĭ et al., 2007) are generalized for the class $\mathrm{S}_\kappa(\mathcal{U},\mathcal{Y})$. In Article (II), definition of defect function is extended to case where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, and earlier results are further generalized and made sharper.

(2) Similarity mappings between realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ and the properties of similarity mappings.

Kálmán (1969) proved one the versions of the celebrated state space similarity theorem. He showed that in finite dimensional spaces, two minimal realizations of the same function are always similar. In infinite dimensional spaces, this is no longer true, as was shown by Helton (1974) and Arov (1979b); minimal realizations of the same function are only weakly similar in that case. However, for generalized Schur function θ, it may happen that all minimal passive realizations of θ are unitarily similar. For ordinary Schur functions, a criterion for this was obtained by Arov and Nudelman (2000, 2002). Article (II) provides a generalization of their results to the class $\mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index.

In general, for arbitrary $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, one needs more restrictive conditions to get a realization unique up to unitary change of state variable. For instance, every realization with the same properties what one of the canonical realizations has is unitarily similar with the canonical realization in question. Sometimes the other properties of those similarities can be used to analyze the properties of θ. In Article (III), it is shown that negative index of unitary or weak similarity mapping between the realization of symmetric $\theta \in \mathbf{S}_\kappa(\mathcal{U})$ and its dual, determines when θ is also a generalized Nevanlinna function with the specified negative index.

The rest of this overview is organized as follows. In Section 2, basic facts from the operator theory of indefinite inner product spaces needed in this thesis are recalled. Section 3 deals with analytic operator valued functions with associated reproducing kernels, concentrating to generalized Schur functions. In Section 4, operator colligations and they connections to the generalized Schur class are studied. Section 5 consists of summary of main results of articles.

2 FUNDAMENTALS ON LINEAR OPERATORS IN KREĬN AND PONTRYAGIN SPACES

For a general theory of indefinite inner product spaces and their operators, standard texts are the books of Bognár (1974), Azizov and Iokhvidov (1989) and Dritschel and Rovnyak (1996). In most of the parts, we follow the notation of (Dritschel & Rovnyak, 1996). A special attention is in contractive operators in Pontryagin spaces, since they provide a base for the subject of the thesis.

2.1 Geometry of indefinite inner product spaces

Throughout this thesis, an **inner product space** is a complex vector space $\mathcal{X}$ endowed with an indefinite inner product $\langle \cdot, \cdot \rangle_{\mathcal{X}}$. If the corresponding spaces are clear, subscripts in the notion of inner products will be left out. A vector $x \in \mathcal{X}$ is said to be positive (negative, neutral) if $E_{\mathcal{X}}(x) := \langle x, x \rangle > 0$ ($E_{\mathcal{X}}(x) < 0$, $E_{\mathcal{X}}(x) = 0$). Similarly, a vector subspace $\mathcal{X}_1$ of $\mathcal{X}$ is said to be positive, negative or neutral, if all non-zero vectors in $\mathcal{X}_1$ have the corresponding property. The anti-space $-\mathcal{X}$ of $\mathcal{X}$ is the space that coincide with $\mathcal{X}$ as a vector space and which has the inner product $\langle \cdot, \cdot \rangle_{-\mathcal{X}} = -\langle \cdot, \cdot \rangle_{\mathcal{X}}$. Vectors x and y from $\mathcal{X}$ are said to be orthogonal (with respect to the indefinite inner product of $\mathcal{X}$) if $\langle x, y \rangle = 0$, and it is denoted $x \perp y$. The orthogonality of subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$ and direct sum $\mathcal{X}_1 \oplus \mathcal{X}_2 = \binom{\mathcal{X}_1}{\mathcal{X}_2}$ are then defined as in the case of positive definite inner product spaces.

The space $\mathcal{X}$ is said to be a **Kreĭn space** if it can be represented as a direct sum of the form

$$\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-, \tag{2.1}$$

where $\mathcal{X}_+$ is a Hilbert space and $\mathcal{X}_-$ is an anti-space of some Hilbert space, that is, an anti-Hilbert space. For a Kreĭn space $\mathcal{X}$, the decomposition of the form (2.1) is not unique unless $\mathcal{X}_+$ or $\mathcal{X}_-$ is a zero space. However, if $\mathcal{X} = \mathcal{X}'_+ \oplus \mathcal{X}'_-$ is any other such a decomposition, it holds $\dim \mathcal{X}_+ = \dim \mathcal{X}'_+$ and $\dim \mathcal{X}_- = \dim \mathcal{X}'_-$. Any decomposition of the form (2.1) is called a **fundamental decomposition** of $\mathcal{X}$, and the positive and negative indices $\operatorname{ind}_+ \mathcal{X}$ and $\operatorname{ind}_- \mathcal{X}$ of $\mathcal{X}$ are defined by $\operatorname{ind}_+ \mathcal{X} = \dim \mathcal{X}_+$ and $\operatorname{ind}_- \mathcal{X} = \dim \mathcal{X}_-$. For a Kreĭn space $\mathcal{X}$ with the fixed fundamental decomposition (2.1), we define $|\mathcal{X}| = \mathcal{X}_+ \oplus (-\mathcal{X}_-)$. Then $|\mathcal{X}|$ is a Hilbert space which coincides with $\mathcal{X}$ as a vector space, and it induces a Hilbert space norm and a topology for the vector space $\mathcal{X}$. All the norms generated by different fundamental decompositions are equivalent, see (Dritschel & Rovnyak, 1996, Corollary on p. 5), and therefore they induce the same topology, which is called the **strong topology** of $\mathcal{X}$. In what follows, all notions related to the continuity, convergence and open sets in Kreĭn space $\mathcal{X}$ are understood to be with respect to the strong topology. All the spaces are assumed to be separable. A closed subspace $\mathcal{X}_1$ of a Kreĭn space $\mathcal{X}$ is called **regular** if it is also a Kreĭn space with the inherited inner product of $\mathcal{X}$. The subspace $\mathcal{X}_1$ is regular if and only its **orthogonal companion** $(\mathcal{X}_1)^{\perp} := \{x \in \mathcal{X} : \langle x, y\rangle = 0$ for all $y \in \mathcal{X}_1\}$ is regular. In that case, it holds $\mathcal{X} = \mathcal{X}_1 \oplus (\mathcal{X}_1)^{\perp}$. A regular subspace is Hilbert subspace if its negative index is zero and an anti-Hilbert subspace if its positive index is zero. A Kreĭn space $\mathcal{X}$ is called as a **Pontryagin space** if $\operatorname{ind}_- \mathcal{X}$ is finite, and these are the spaces we study most of the time in this thesis.

2.2 Bounded linear operators in Kreĭn and Pontryagin spaces

A linear operator $T : \mathcal{U} \to \mathcal{Y}$ where $\mathcal{U}$ and $\mathcal{Y}$ are Kreĭn spaces, is said to be bounded, that is, it belongs to the class $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ if and only if T belongs to $\mathcal{L}(|\mathcal{U}|, |\mathcal{Y}|)$. For a fixed fundamental decomposition $\mathcal{U} = \mathcal{U}_+ \oplus \mathcal{U}_-$, the operator $\mathcal{J}$ defined by

$$\mathcal{J}(u_+ + u_-) = u_+ - u_-, \qquad u_\pm \in \mathcal{U}_\pm, \tag{2.2}$$

is called a **fundamental symmetry**, and if $|\mathcal{U}|$ is an associated Hilbert space with the definite inner product $(\cdot,\cdot)_{|\mathcal{U}|}$, it holds

$$\begin{aligned} \langle u, y\rangle_{\mathcal{U}} &= (\mathcal{J}u, y)_{|\mathcal{U}|} = (u, \mathcal{J}y)_{|\mathcal{U}|} \\ (u, y)_{|\mathcal{U}|} &= \langle \mathcal{J}u, y\rangle_{\mathcal{U}} = \langle u, \mathcal{J}y\rangle_{\mathcal{U}}. \end{aligned} \tag{2.3}$$

By using the identities in (2.3) and basic results from Hilbert space operators, it follows that for an operator $T \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, there exists an unique operator $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{Y})$ such that $\langle Tu, y\rangle_{\mathcal{Y}} = \langle u, T^*y\rangle_{\mathcal{U}}$ for all $u \in \mathcal{U}$ and all $y \in \mathcal{Y}$, and T^* is called as the **adjoint** of T (with respect to the indefinite inner product). If T is considered as a Hilbert space operator $T \in \mathcal{L}(|\mathcal{U}|, |\mathcal{Y}|)$ and the Hilbert space adjoint of T is denoted as $T^{\times}$, it holds

$$T^* = \mathcal{J}_{\mathcal{U}} T^{\times} \mathcal{J}_{\mathcal{Y}}, \qquad T^{\times} = \mathcal{J}_{\mathcal{U}} T^* \mathcal{J}_{\mathcal{Y}}. \tag{2.4}$$

For any fundamental symmetry of a Kreĭn space $\mathcal{U}$, it holds

$$\mathcal{J} = \mathcal{J}^* = \mathcal{J}^{\times} = \mathcal{J}^{-1}.$$

Definition 2.1. The operator $T \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Kreĭn spaces, is called

(i) **contractive** if $\langle Tu, Tu\rangle_{\mathcal{Y}} \le \langle u, u\rangle_{\mathcal{U}}$ for every $u \in \mathcal{U}$;

(ii) **isometric** if $\langle Tu, Tu\rangle_{\mathcal{Y}} = \langle u, u\rangle_{\mathcal{U}}$ for every $u \in \mathcal{U}$;

(iii) **co-isometric** if T^* is isometric;

(iv) **unitary** if it is both isometric and co-isometric;

(v) **self-adjoint** if $\mathcal{U} = \mathcal{Y}$ and then $T = T^*$;

(vi) **positive** if it is self-adjoint and $\langle Tu, u \rangle \geq 0$ for every $u \in \mathcal{U}$;

(vi) a **projection** if it is self-adjoint and $T^2 = T$.

The image of projection is always a regular subspace, and every regular subspace arises uniquely in this way (Dritschel & Rovnyak, 1996, Theorem 1.3). The unique projection to the regular subspace $\mathcal{H} \subset \mathcal{U}$ is denoted as $P_{\mathcal{H}}$. The restriction of $T : \mathcal{U} \to \mathcal{Y}$ to the regular subspace $\mathcal{H} \subset \mathcal{U}$ is denoted as $T\restriction_{\mathcal{H}}$. If A and B are self-adjoint operators from the class $\mathcal{L}(\mathcal{U}) = \mathcal{L}(\mathcal{U},\mathcal{U})$, where $\mathcal{U}$ is Kreĭn space, then $A \leq B$ means that $B - A$ is a positive operator.

In the case where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, contractive operators and self-adjoint operators have similar spectral properties as the Hilbert space operators from the corresponding class. In the next proposition, we state two well-known results frequently used in the publications of this thesis. For the proofs, see for an instance, (Arov et al., 2006, Theorem 2.2) and (Azizov & Iokhvidov, 1989, Chapter 1, Corollary 3.15).

Proposition 2.2. *Let $\mathcal{U}$ and $\mathcal{Y}$ be Pontryagin spaces with the common negative index κ. Then:*

(i) if the operator $T \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is a contraction, the spectrum of T lies in the unit disc $\mathbb{D}$ with the exception of at most κ points;

(ii) if the operator $T \in \mathcal{L}(\mathcal{U})$ is self-adjoint, the spectrum of T lies in the real axis with the exception of at most 2κ points situated symmetrically with respect to the real axis.

The behavior of a self-adjoint operator $T \in \mathcal{L}(\mathcal{U})$, where $\mathcal{U}$ is a Kreĭn space, may be close to positive operator in a sense that the dimension of a subspace $\mathcal{K} \subset \mathcal{U}$ such that $\langle Tk, k \rangle_{\mathcal{U}} < 0$ holds for all $k \in \mathcal{K} \setminus \{0\}$, cannot be arbitrary large. To measure this precisely, we define the **negative index** $\operatorname{ind}_-(T)$, with respect to the inner product of $\mathcal{U}$, to be the supremum of all positive integers n such that there exists an invertible and nonpositive matrix of the form $\left(\langle Tu_j, u_i \rangle_{\mathcal{U}}\right)_{i,j=1}^n$, where $\{u_k\}_{k=1}^n \subset \mathcal{U}$. If such a matrix does not exists for any n, then $\operatorname{ind}_-(T)$ is defined to be zero. In that case, and in that case only, the operator T is said to be positive.

It is evident that the the operator $T \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Kreĭn spaces, is contractive if and only if $I_{\mathcal{U}} - T^*T$ is positive. If $\operatorname{ind}_-(I_{\mathcal{U}} - T^*T) = \kappa$ is finite, then the dimension of a subspace $\mathcal{K}$ such that

$$\langle Tk, Tk\rangle_{\mathcal{Y}} > \langle k, k\rangle_{\mathcal{U}}$$

for all $k \in \mathcal{K} \setminus \{0\}$, cannot exceed κ. In some sources, the operator T with this property is called quasi-contraction; see (Gheondea, 1993), since $\mathrm{ind}_{-}(I_{\mathcal{U}} - T^*T)$ measures how much T behaves like a contraction. Unlike in Hilbert spaces, if T is contractive, the adjoint T^* need not to be. However, if $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, then T is contractive if and only if T^* is (Dritschel & Rovnyak, 1996, Corollary 2.5).

2.3 Defect spaces and Julia operators

From the certain view of the general Kreĭn space operator theory, every bounded operator T is almost unitary; there exists a dilation of T such that the dilation is unitary; see (Azizov & Iokhvidov, 1989, Theorem 3.4 on p. 267) and (Dritschel & Rovnyak, 1996, Lecture 2). If $T \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Kreĭn spaces, a **dilation** of T is an operator $\widehat{T} \in \mathcal{L}(\widehat{\mathcal{U}}, \widehat{\mathcal{Y}})$, where $\widehat{\mathcal{U}}$ and $\widehat{\mathcal{Y}}$ are Kreĭn spaces such that $\mathcal{U}$ and $\mathcal{Y}$ are regular subspaces, respectively, of $\widehat{\mathcal{U}}$ and $\widehat{\mathcal{Y}}$, and it holds

$$T = P_{\mathcal{Y}} \widehat{T}\restriction_{\mathcal{U}}. \tag{2.5}$$

The dilation $\widehat{T}$ of T then has a block representation of the form

$$\widehat{T} = \begin{pmatrix} T & T_2 \\ T_3 & T_4 \end{pmatrix} : \begin{pmatrix} \mathcal{U} \\ \mathcal{U}^{\perp} \end{pmatrix} \to \begin{pmatrix} \mathcal{Y} \\ \mathcal{Y}^{\perp} \end{pmatrix}, \tag{2.6}$$

where the orthogonal companions $\mathcal{U}^{\perp}$ and $\mathcal{Y}^{\perp}$ are with respect to the spaces $\widehat{\mathcal{U}}$ and $\widehat{\mathcal{Y}}$.

A dilation $\widehat{T}$ of T of the form (2.6) is called **Julia operator**, or Julia dilation, if it is unitary, and T_2 and T_3^* have zero kernels. It is convenient to write this by using **defect operators**, since the notation then corresponds the celebrated dilation theory of Hilbert space contraction, which goes back to the work of Sz.-Nagy and Foiaş (1970). To this end, a defect operator of $T \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is a bounded operator $D_T : \mathfrak{D}_T \to \mathcal{U}$, where $\mathfrak{D}_T$ is a Kreĭn space, with the zero kernel such that it holds $I - T^*T = D_T D_T^*$. The space $\mathfrak{D}_T$ is called as the **defect space** of T. By using the defect operators, a Julia operator U_T of T is a unitary operator

$$U_T = \begin{pmatrix} T & D_{T^*} \\ D_T^* & -L^* \end{pmatrix} : \begin{pmatrix} \mathcal{U} \\ \mathfrak{D}_{T^*} \end{pmatrix} \to \begin{pmatrix} \mathcal{Y} \\ \mathfrak{D}_T \end{pmatrix}, \tag{2.7}$$

where D_T and D_{T^*} are defect operators, respectively, of T and T^*. It is known from

(Dritschel & Rovnyak, 1996, Theorem 2.3) that for every $T \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, there exists a Julia operator U_T of T, and for any defect operator $D_T : \mathfrak{D}_T \to \mathcal{U}$ of T, it holds $\mathrm{ind}\_\mathfrak{D}_T = \mathrm{ind}\_I - T^*T$. Moreover, if $\mathrm{ind}\_I - T^*T$ or $\mathrm{ind}\_I - TT^*$ is finite, then U_T is essentially unique in a sense that for any other Julia operator

$$U_T' = \begin{pmatrix} T & {D_{T^*}}' \\ {D_T}'^* & -L'^* \end{pmatrix} : \begin{pmatrix} \mathcal{X}_1 \\ {\mathfrak{D}_{T^*}}' \end{pmatrix} \to \begin{pmatrix} \mathcal{X}_2 \\ {\mathfrak{D}_T}' \end{pmatrix},$$

of T, there exists unitary operators $V_1 : \mathfrak{D}_{T^*} \to {\mathfrak{D}_{T^*}}'$ and $V_2 : \mathfrak{D}_T \to {\mathfrak{D}_T}'$ such that

$$D_{T^*} = {D_{T^*}}' V_1, \qquad D_T = {D_T}' V_2, \qquad V_1 L = L' V_2.$$

If $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index and $T : \mathcal{U} \to \mathcal{Y}$ is contractive, defect spaces of T and T^* are Hilbert spaces, and T can be dilated to a unitary operator without adding negative dimensions to the underlying spaces. Especially, if $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, one can always choose

$$\begin{aligned} &\mathfrak{D}_T = \overline{\mathrm{ran}}(I - T^*T)^{1/2}, && \mathfrak{D}_{T^*} = \overline{\mathrm{ran}}(I - TT^*)^{1/2}, \\ &D_T = (I - T^*T)^{1/2}, && D_{T^*} = (I - TT^*)^{1/2}, \qquad L = A, \end{aligned}$$

where $\overline{\mathrm{ran}}$ means the closure of the range.

3 OPERATOR VALUED ANALYTIC FUNCTIONS AND REPRODUCING KERNELS SPACES

Hilbert space operator valued functions holomorphic and bounded in the unit disc are studied in (Sz.-Nagy & Foiaş, 1970). Treatises of reproducing kernel Pontryagin spaces and operator valued analytic functions are given in (Dritschel & Rovnyak, 1996) and Alpay et al. (1997).

3.1 Reproducing kernels in Pontryagin spaces

Let $\Omega \subset \mathbb{C}$ be an open set, and let $K(w,z)$ be an $\mathcal{L}(\mathcal{Y})$-valued function on $\Omega \times \Omega$, where $\mathcal{Y}$ is a Kreĭn space. The function K is an **holomorphic Hermitian kernel** on $\Omega \times \Omega$ if

$$K(w,z) = K^*(w,z),$$

and if for every fixed w, it is analytic in z, and for every fixed z, it is analytic in $\bar{w}$. Here the notation $K^*(w,z)$ means $(K(w,z))^*$. In what follows, a holomorphic Hermitian kernel will be called as a kernel, since another kind of kernels are not considered.

A kernel K is **positive** on $\Omega \times \Omega$ if the matrix

$$\big(\langle K(w_j,w_i)y_j, y_i\rangle_{\mathcal{Y}}\big)_{i,j=1}^n \tag{3.1}$$

has no negative eigenvalues for any choice of $n \in \mathbb{N}$, $\{w_1,\dots,w_n\} \subset \Omega$ and $\{y_1,\dots,y_n\} \subset \mathcal{Y}$. It is a classical result, see Aronszajn (1950) for the scalar case $\mathbb{C} = \mathcal{Y}$, that then K generates an unique reproducing kernel Hilbert space $\mathcal{H}_K$ whose elements are $\mathcal{Y}$-valued functions holomorphic on Ω. The space $\mathcal{H}_K$ is the completion of a pre-Hilbert space

$$\mathcal{H}_0 = \operatorname{span}\{K(w,z)y : w \in \Omega, \quad y \in \mathcal{Y}\}, \tag{3.2}$$

endowed with an inner product

$$\left\langle \sum_{j=1}^n K(w_j,z)y_j, \sum_{i=1}^n K(w_i,z)y_i \right\rangle_{\mathcal{H}_o} = \left\langle \sum_{i,j=1}^n K(w_j,w_i)y_j, y_i \right\rangle_{\mathcal{Y}}, \tag{3.3}$$

where $K(w,z)y$ is treated as a function of z. Especially, $K(w,z)y \in \mathcal{H}_K$ for all $w \in \Omega$ and $y \in \mathcal{Y}$, and for any $h \in \mathcal{H}_K$, it holds

$$\langle h(z), K(w,z)y\rangle_{\mathcal{H}_K} = \langle h(w), y\rangle_{\mathcal{Y}}. \tag{3.4}$$

For scalar case $\mathcal{Y} = \mathbb{C}$, the function $K(w, z) \in \mathcal{H}_K$, and the identity (3.4) essentially reduces to

$$\langle h(z), K(w, z)\rangle_{\mathcal{H}_K} = h(w).$$

On the other hand, if $\mathcal{H}$ is a Hilbert space of $\mathcal{Y}$-valued holomorphic functions on Ω, the space $\mathcal{H}$ is generated by some kernel K if and only if an evaluation mapping $E_w : \mathcal{H} \to \mathcal{Y}$ defined by

$$E_w : f \mapsto f(w) \tag{3.5}$$

is bounded for every $w \in \Omega$. Moreover, the kernel K is unique. Indeed, most of extensively studied Hilbert spaces of holomorphic function are reproducing kernel spaces, and a construction of a Hilbert space of holomorphic function which is not a reproducing kernel space requires an effort, see (Alpay & Mills, 2003).

By passing Hilbert spaces to Pontryagin spaces and positive kernels to kernels with finite amount of negativity in a certain sense, the theory of reproducing kernel Hilbert spaces extends to the theory of reproducing kernel Pontryagin spaces, which were first studied by Schwartz (1964) and Sorjonen (1975). We say that the kernel $K(w, z)$ has κ **negative squares**, where $\kappa \in \mathbb{N}_0$, if the matrix of the form (3.1) has at most κ negative eigenvalues, counting multiplicities, for any choice of $n \in \mathbb{N}$, $\{w_1, \dots, w_n\} \subset \Omega$ and $\{y_1, \dots, y_n\} \subset \mathcal{Y}$, and there exist at least one such a matrix which has exactly κ negative eigenvalues, counting multiplicities. The case $k = 0$ corresponds the case of positive kernel. The kernel K with κ negative squares generates the unique reproducing kernel Pontryagin space $\mathcal{H}_K$ of $\mathcal{L}(\mathcal{Y})$-valued functions holomorphic on Ω, and $\text{ind}_{-}\mathcal{H}_K = \kappa$. Similarly as in the case of positive kernel, the Pontryagin space $\mathcal{H}_K$ is the completion of (3.2) endowed with the inner product (3.3); for the details, see (Alpay et al., 1997, Theorem 1.1.3). Moreover, if $\mathcal{H}$ is a Pontryagin space of holomorphic $\mathcal{L}(\mathcal{Y})$-valued functions on Ω with $\text{ind}_{-}\mathcal{H} = \kappa$, there exist an unique reproducing kernel $K(w, z)$ of $\mathcal{H}$ if and only if the point evaluations of the form (3.5) are bounded. If this happens, then K has κ negative squares and

$$K(w, z) = E_z E_w^*, \qquad z, w \in \Omega.$$

If the domain Ω of the holomorphic kernel $K(w, z)$ is a region, i.e. connected nonempty open set in the complex plane, the behaviour of K is locally similar in a sense that if Ω_0 is a subregion of Ω and K_0 is a restriction of K to Ω_0, then K_0 and K has the same number of negative squares (Alpay et al., 1997, Theorem 1.1.4).

The values of the kernel $K(w, z)$ above are, in general, Kreĭn space operators. In what follows, we mainly consider kernels with κ negative squares such that their values are Pontryagin space operators, and therefore, the functions in $\mathcal{H}_K$ take values in Pontryagin spaces.

3.2 Generalized Schur class functions

A classical Schur function is a complex valued function holomorphic and bounded by one in the unit disc $\mathbb{D}$. This is equivalent to that θ is holomorphic in $\mathbb{D}$ and the kernel

$$\frac{1-\theta(z)\overline{\theta(w)}}{1-z\bar{w}}, \qquad z,w\in\mathbb{D}, \tag{3.6}$$

is positive. The reproducing kernel Hilbert spaces generated by the kernels of the form (3.6) are known as **de Branges–Rovnyak spaces**, see (Fricain & Mashreghi, 2016a, 2016b). Especially, when θ is an inner function, that is, when θ is a Schur function and $\lim_{r\to 1^-}|\theta(re^{it})| = 1$ for a.e. on the unit circle $\mathbb{T}$, the spaces generated by the kernels (3.6) are called **model spaces**, and they are backward shift invariant closed subspaces $H^2 \ominus \theta H^2$ of the classical Hardy space H^2, see (Garcia, Mashreghi, & Ross, 2016).

A natural generalization is now to drop the requirement that the kernel (3.6) is positive, and require only that it has a finite number of negative squares. Moreover, the values of function θ are considered to be operators between Pontryagin spaces with the same negative index instead of scalars. Then, $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, belongs to **generalized Schur class** $\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, if it is holomorphic at the origin and the **Schur kernel**

$$K_\theta(w,z) = \frac{1-\theta(z)\theta^*(w)}{1-z\bar{w}}, \qquad z,w\in\mathbb{D}, \tag{3.7}$$

has κ negative squares. The class $\mathbf{S}_0(\mathcal{U},\mathcal{Y})$ is denoted by $\mathbf{S}(\mathcal{U},\mathcal{Y})$ and the class $\mathbf{S}_\kappa(\mathcal{U},\mathcal{U})$ by $\mathbf{S}_\kappa(\mathcal{U})$. The reproducing kernel Pontryagin space induced by the kernel (3.7) is denoted as $\mathcal{H}(\theta)$. An associated function $\theta^{\#}$ defined by $\theta^{\#}(z) = \theta^*(\bar{z})$ belongs to $\mathbf{S}_\kappa(\mathcal{Y},\mathcal{U})$ whenever $\theta\in\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$.

It is known, see (Alpay et al., 1997, Theorem 4.3.5), that every generalized Schur function has a meromorphic extension to the whole unit disc $\mathbb{D}$. Therefore, we consider the generalized Schur functions to be meromorphic functions on $\mathbb{D}$. A function θ belong to class $\mathbf{S}(\mathcal{U},\mathcal{Y})$ if and only if it is meromorphic on $\mathbb{D}$, holomorphic at the origin and has contractive values whenever defined on $\mathbb{D}$. When $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, $\theta\in\mathbf{S}(\mathcal{U},\mathcal{Y})$ is actually holomorphic on $\mathbb{D}$. Moreover, non-tangential strong boundary limit values $\theta(\zeta)$ exists and are contractive for a.e. $\zeta\in\mathbb{T}$; see (Sz.-Nagy & Foiaş, 1970, Chapter V). For the class $\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, we have the following alternative characterizations, which do not involve the kernel (3.7). For a proof, combine (Dijksma et al., 1986a, Proposition 7.11) and (Alpay et al., 1997, Theorem 4.2.1).

Proposition 3.1. *Let $\mathcal{U}$ and $\mathcal{Y}$ be Hilbert spaces, and let θ be an $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function holomorphic at the origin and meromorphic on $\mathbb{D}$. Then the following statements are equivalent:*

(i) $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$;

(ii) θ *has finite pole multiplicity* κ *and*

$$\lim_{r\to 1^-} \sup_{|z|=r} \|\theta(z)\| \le 1$$

holds;

(iii) θ *has factorizations of the form*

$$\theta(z) = \theta_r(z)B_r^{-1}(z) = B_l^{-1}(z)\theta_l(z), \tag{3.8}$$

where $\theta_r, \theta_l \in \mathbf{S}(\mathcal{U},\mathcal{Y})$, B_r *and* B_r *are Blaschke products of degree* κ *with values, respectively, in* $\mathcal{L}(\mathcal{U})$ *and* $\mathcal{L}(\mathcal{Y})$*, such that* $B_r(w)f = 0$ *and* $\theta_r(w)f = 0$ *for some* $w \in \mathbb{D}$ *only if* $f = 0$*, and* $B_l^*(w)g = 0$ *and* $\theta_l^*(w)g = 0$ *for some* $w \in \mathbb{D}$ *only if* $g = 0$.

In Proposition 3.1 above, an operator valued **Blaschke product of degree** κ is a finite product

$$B(z) = \prod_{k=1}^{\kappa}\left(I - P_k + \rho_k\frac{z-\alpha_k}{1-\bar{\alpha}_k z}P_k\right), \qquad |\rho_k| = 1, \qquad 0 < |\alpha_k| < 1, \tag{3.9}$$

of **simple Blaschke–Potapov factors** of the form

$$I - P + \rho\,\frac{z-\alpha}{1-\bar{\alpha}z}\,P, \qquad |\rho| = 1, \qquad 0 < |\alpha| < 1, \tag{3.10}$$

where $P \in \mathcal{L}(\mathcal{U})$ is an orthogonal projection from the Hilbert space $\mathcal{U}$ to an arbitrary one dimensional subspace. A straightforward calculations show that a Blaschke product is holomorphic on the closed unit disc $\overline{\mathbb{D}}$, it has unitary values everywhere on $\mathbb{T}$ and it is boundedly invertible whenever $z \in \overline{\mathbb{D}} \setminus \{\alpha\}$. The Blaschke product B of degree κ belongs to $\mathbf{S}(\mathcal{U})$, and $B^{-1} \in \mathbf{S}_\kappa(\mathcal{U})$. The space $\mathcal{H}(B)$ induced by the kernel (3.7) in the case $\theta = B$ is κ-dimensional Hilbert space, and $\mathcal{H}(B^{-1})$ is κ-dimensional anti-Hilbert space. The factorizations (3.8) of $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ are called, respectively, the **right and left Kreĭn–Langer factorizations** of θ. These factorizations are unique up to multiplication by unitary constant from left (right) for the left (right) Kreĭn–Langer factorization (Alpay et al., 1997, Theorem 4.2.4). It is easy to deduce that if (3.8) are Kreĭn–Langer factorizations of θ, then

$$\theta^{-\#} = \theta_r B_r^{-\#} = B_l^{-\#}\theta_l$$

are, respectively, left and right Kreĭn–Langer factorizations of $\theta^\#$. Moreover, it then holds

$$\mathcal{H}(\theta) = \mathcal{H}(\theta_r) \oplus \theta_r\mathcal{H}\left(B_r^{-1}\right), \qquad \mathcal{H}\left(\theta^\#\right) = \mathcal{H}\left(\theta_l^\#\right) \oplus \theta_l^\#\mathcal{H}\left(B_l^{-\#}\right),$$

where $\theta_r \mathcal{H}(B_r^{-1})$ and $\theta_l^{\#} \mathcal{H}\left(B_l^{-\#}\right)$ are the spaces induced by the kernels

$$\theta_r(z) K_{B_r^{-1}}(w,z) \theta_r(\bar{w}), \qquad \theta_l^{\#}(z) K_{B_l^{-\#}}(w,z) \theta_l^{\#}(\bar{w}).$$

Even for the class $\mathbf{S}(\mathcal{U},\mathcal{Y})$, the properties of the Pontryagin space operator valued generalized Schur functions are not so well understood than in the cases where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces. Some properties of the generalized Schur functions can be analyzed by using the **Potapov–Ginzburg transformation** $\theta \mapsto \theta_P$. Roughly saying, the Potapov–Ginzburg transformation connects $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ and the kernel K_θ to a function θ_P and the kernel K_{θ_P}, where θ_P, or $\theta_P(\varphi(z))$, where φ is an automorphism of the unit disc, belongs to $\mathbf{S}_\kappa(\mathcal{U}',\mathcal{Y}')$, where $\mathcal{U}'$ and $\mathcal{Y}'$ are Hilbert spaces; see details from Article (III) or (Alpay et al., 1997, Section 4.3). By using the Potapov–Ginzburg transformation, we obtain the following result published in Article (III); see also (Alpay & Dym, 1986, Theorem 6.8).

Theorem 3.2. *Let $\mathcal{U}$ and $\mathcal{Y}$ be Pontryagin spaces with the same negative index. If $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, then strong radial limit values $\lim_{r\to 1^-} \theta(r\zeta)$ exist for a.e. $\zeta \in \mathbb{T}$, and the limit values are contractive with respect to the indefinite inner products of $\mathcal{U}$ and $\mathcal{Y}$.*

To prove Theorem 3.2 in general case, one needs to know that it hold for the case when $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces. In fact, that case follows easily follows from Proposition 3.1 and Kreĭn–Langer factorizations. When $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces, there is no corresponding known result for the Kreĭn–Langer factorizations. The best known factorization result seems to be the following weak version of the Kreĭn–Langer factorizations. Represent

$$\mathcal{U} = \begin{pmatrix} \mathcal{U}_+ \\ \mathcal{U}_- \end{pmatrix}, \qquad \mathcal{Y} = \begin{pmatrix} \mathcal{Y}_+ \\ \mathcal{Y}_- \end{pmatrix},$$

for some fixed fundamental decompositions. With respect to this representation, one has a factorization of the form

$$\begin{pmatrix} b^{-1}(z) & 0 \\ 0 & I \end{pmatrix} \theta_0(z) \begin{pmatrix} I & 0 \\ 0 & b(z) \end{pmatrix}, \tag{3.11}$$

where b is a scalar finite Blaschke product and $\theta_0 \in \mathbf{S}(\mathcal{U},\mathcal{Y})$ (Alpay et al., 1997, Example 1 on p. 161). Unfortunately, the inverse Blaschke product factor in (3.11) does not necessarily cover all the poles of θ; the factor θ_0 can still has poles, even infinitely many. In the case where $\mathcal{U}$ and $\mathcal{Y}$ are anti-Hilbert spaces with same finite dimension, a function $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ has exactly κ zeros, counting multiplicities, but already in the case $\mathcal{U} = \mathcal{Y} = \mathbb{C} \oplus -\mathbb{C}$, a function $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ can have any countable number of zeros and poles; see Section 2 from Article (III).

3.3 Special subclasses of the generalized Schur functions

Theorem 3.2 shows that the radial limit values of the generalized Schur functions exists and are contractive a.e. on $\mathbb{T}$. As in the case of ordinary Schur function, the radial limit values can be isometric, co-isometric or unitary a.e., and we obtain the following generalization of an inner function.

Definition 3.3. A function $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, belongs to the class of

(i) generalized $\mathcal{J}$-inner functions $\mathbf{I}_\kappa(\mathcal{U},\mathcal{Y})$, if the radial limit values $\theta(\zeta)$ are isometric, with respect to the indefinite inner product, for a.e. $\zeta \in \mathbb{T}$;

(ii) generalized co-$\mathcal{J}$-inner functions $\mathbf{I}^*{}_\kappa(\mathcal{U},\mathcal{Y})$, if the radial limit values $\theta(\zeta)$ are co-isometric, with respect to the indefinite inner product, for a.e. $\zeta \in \mathbb{T}$;

(iii) generalized bi-$\mathcal{J}$-inner functions $\mathbf{U}_\kappa(\mathcal{U},\mathcal{Y})$, if the radial limit values $\theta(\zeta)$ are unitary, with respect to the indefinite inner product, for a.e. $\zeta \in \mathbb{T}$.

When $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, the letter $\mathcal{J}$ will be left out from the definition above, since $\mathcal{J}$ refers to the indefinite inner products. Indeed, if $\mathcal{J}_\mathcal{U}$ and $\mathcal{J}_\mathcal{Y}$ are fundamental symmetries of $\mathcal{U}$ and $\mathcal{Y}$ and $\times$ refers to adjoint with respect to associate Hilbert spaces, it holds $\theta^\times(\zeta)\mathcal{J}_\mathcal{Y}\theta(\zeta) = \mathcal{J}_\mathcal{U}$ a.e. on $\mathbb{T}$ for $\theta \in \mathbf{I}_\kappa(\mathcal{U},\mathcal{Y})$ and $\theta(\zeta)\mathcal{J}_\mathcal{U}\theta^\times(\zeta) = \mathcal{J}_\mathcal{Y}$ a.e. on $\mathbb{T}$ for $\theta \in \mathbf{I}^*{}_\kappa(\mathcal{U},\mathcal{Y})$. If $\theta \in \mathbf{U}_\kappa(\mathcal{U},\mathcal{Y})$, both relations above hold. In the matrix valued cases, that is, when $\mathcal{U}$ and $\mathcal{Y}$ are finite-dimensional, the class $\mathbf{U}_\kappa(\mathcal{U},\mathcal{Y})$ is extensively studied; see for an instance (Alpay & Dym, 1986) (Arov & Dym, 2008) or (Derkach & Dym, 2009). Nevertheless, the general case is not widely studied, and general definition of the generalized bi-$\mathcal{J}$-inner functions was recently introduced in Article (III).

It is possible that $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ does not belongs to the classes defined above, but it still is almost inner or co-inner in a sense that its **defect functions** are identically zeros. We describe here only the case where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces and a definition given in Article (I), since a definition for the general case treated in Article II requires a use of operator colligation and optimal minimal realizations; for details, see Article (II). By using (Sz.-Nagy & Foiaş, 1970, Theorem V.4.2) it can be deduced that for $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, there exist Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$, an **outer function** $\varphi_\theta \in \mathbf{S}(\mathcal{U},\mathcal{K})$ and **co-outer function** $\psi_\theta \in \mathbf{S}(\mathcal{H},\mathcal{Y})$ (for the definition of outer and co-outer functions, see (Sz.-Nagy & Foiaş, 1970, Chapter V)) such that

(i) $\varphi_\theta^*(\zeta)\varphi_\theta(\zeta) \le I_\mathcal{U} - \theta^*(\zeta)\theta(\zeta)$ a.e. on $\mathbb{T}$;

(ii) $\psi_\theta^(\zeta)\psi_\theta^*(\zeta) \leq I_\mathcal{Y} - \theta(\zeta)\theta^*(\zeta)$ a.e. on $\mathbb{T}$;

(iii) if $\widehat{\mathcal{K}}$ is a Hilbert space and $\widehat{\varphi} \in \mathbf{S}(\mathcal{U}, \widehat{\mathcal{K}})$ such that $\widehat{\varphi}^*(\zeta)\widehat{\varphi}(\zeta) \leq I_\mathcal{U} - \theta^*(\zeta)\theta(\zeta)$ a.e. on $\mathbb{T}$, then $\widehat{\varphi}^*(\zeta)\widehat{\varphi}(\zeta) \leq \varphi_\theta^*(\zeta)\varphi_\theta(\zeta)$ a.e. on $\mathbb{T}$;

(iv) if $\widehat{\mathcal{H}}$ is a Hilbert space and $\widehat{\psi} \in \mathbf{S}(\widehat{\mathcal{H}}, \mathcal{Y})$ such that $\widehat{\psi}(\zeta)\widehat{\psi}^*(\zeta) \leq I_\mathcal{Y} - \theta(\zeta)\theta^*(\zeta)$ a.e. on $\mathbb{T}$, then $\widehat{\psi}(\zeta)\widehat{\psi}^*(\zeta) \leq \psi_\theta(\zeta)\psi_\theta^*(\zeta)$ a.e. on $\mathbb{T}$.

The functions φ_θ and ψ_θ are called, respectively, the right and left defect functions of θ. Moreover, they are unique up to, respectively, a left constant unitary factor and a right constant unitary factor. When $\varphi \equiv 0$ or $\psi \equiv 0$, the function θ is not necessarily generalized inner or co-inner, but from the point of view of passive discrete-time system, they essentially have nearly all the same properties. Especially, the so-called canonical realization have stronger properties that is granted for an arbitrary $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, see Theorem 4.4 from Article (I) and Theorem 4.8 from Article (II). It is a worth of mentioning that when $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, definitions of right and left defect functions for generalized Schur functions given in Article II do not necessarily produce the same mathematical objects as definition given above unless special circumstances. Such a situation occurs when right or left defect function given by one of the two definitions is identically zero; in that case, and in that case only, a function given by other definition is also identically zero.

An $\mathcal{L}(\mathcal{U})$-valued function is called **symmetric**, or **real**, if $\theta(z) = \theta^\#(z)$ holds whenever defined. A symmetric $\mathcal{L}(\mathcal{U})$-valued function meromorphic on $\mathbb{C} \setminus \mathbb{R}$ belongs to the **generalized Nevanlinna class** $\mathbf{N}_\kappa(\mathcal{U})$ if the Nevanlinna kernel

$$N_\theta(w, z) = \frac{\theta(z) - \theta^*(w)}{z - \bar{w}}, \qquad w, z \in \rho(\theta), \tag{3.12}$$

has κ negative squares. These classes have been studied alongside with the Schur functions, mainly with scalar, matrix and Hilbert space operator valued cases; see for an instance (Hassi, de Snoo, & Woracek, 1998) and (Kreĭn & Langer, 1977). We do not give a systemic treatise of generalized Nevanlinna functions in this thesis. However, we mention an interesting subclass of the generalized Schur functions and the generalized Nevanlinna functions, which is the combined class $\mathbf{S}_{\kappa_1}(\mathcal{U}) \cap \mathbf{N}_{\kappa_2}(\mathcal{U})$, i.e., the functions that are both generalized Schur functions and generalized Nevanlinna functions for some indices. The class $\mathbf{S}(\mathcal{U}) \cap \mathbf{N}(\mathcal{U})$, where $\mathcal{U}$ is a Hilbert space, was first introduced and studied by Y. M. Arlinskiĭ, Hassi, and de Snoo (2009) and continued by Y. Arlinskiĭ and Hassi (2019). This class is connected to the **Stieltjes families**; see (Y. Arlinskiĭ & Hassi, 2020). The functions from the classes $\mathbf{S}_{\kappa_1}(\mathcal{U}) \cap \mathbf{N}_{\kappa_2}(\mathcal{U})$, where $\mathcal{U}$ is a Pontryagin space, and their operator colligation realizations are the main topic of Article (III).

4 OPERATOR COLLIGATIONS AND PASSIVE DISCRETE-TIME SYSTEMS

In a point of view of this thesis, the book (Alpay et al., 1997) provides a systematic treatise of isometric, co-isometric and unitary operator colligations. In Hilbert space setting or finite dimensional setting, more general type of discrete-time systems are studied in (Bart, Gohberg, Kaashoek, & Ran, 2008). We use a notation that combines features from system theory and from pure operator theory such that achieved new results can be easily compared with the theory developed earlier.

4.1 Operator colligations

Let $\mathcal{X}, \mathcal{U}$ and $\mathcal{Y}$ be Kreĭn spaces, and let $A \in \mathcal{L}(\mathcal{X})$, $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. Then, an **operator colligation** Σ is a quadruple $(T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, where $T_\Sigma \in \mathcal{L}(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y})$. The linear operator T_Σ is called as a **system operator**, and it can be presented in the following block form

$$T_\Sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix}. \tag{4.1}$$

If the **state space** $\mathcal{X}$ is a Pontryagin space with the negative index κ, the notation $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ is used. When needed, an operator colligation Σ will be written also as $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$. The operators A, B, C and D are called, respectively, the **main operator**, the **control operator**, the **observation operator** and the **feedthrough operator**. The space $\mathcal{U}$ is an **incoming space**, and $\mathcal{Y}$ is an **outgoing space**. A colligation $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ can also be seen as a linear discrete-time system of the form

$$\begin{cases} h_{k+1} &= Ah_k + B\xi_k \\ \sigma_k &= Ch_k + D\xi_k, \end{cases} \qquad k \geq 0, \tag{4.2}$$

or what is the same thing,

$$T_\Sigma \begin{pmatrix} h_k \\ \xi_k \end{pmatrix} = \begin{pmatrix} h_{k+1} \\ \sigma_k \end{pmatrix}, \qquad k \geq 0,$$

where $\{h_k\} \subset \mathcal{X}$, $\{\xi_k\} \subset \mathcal{U}$ and $\{\sigma_k\} \subset \mathcal{Y}$. Therefore, in what follows, a system refers to the colligation Σ and its operator expression of the form (4.1), although the actual linear system identification of the form (4.2) will be not used further in this thesis.

The transfer function of the system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, or in some sources, the characteristic function of the operator colligation, is an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ -valued analytic function defined by

$$\theta_\Sigma(z) := D + zC(I - zA)^{-1}B, \tag{4.3}$$

whenever $I - zA$ is invertible, so at least in some sufficiently small symmetric neighbourhood of the origin. A symmetric set $\Omega \subset \mathbb{C}$ now means that $\bar{z} \in \Omega$ whenever $z \in \Omega$.

4.2 Special classes of discrete-time systems

A problem where it is asked to represent an operator valued function θ analytic at the origin as a transfer of the system is called the **realization problem**, and any system Σ such that its transfer function θ_Σ coincides with θ in a neighbourhood of the origin, is called a **realization of** θ. It is often possible to obtain more information about θ by analyzing its realization and the operators in it. However, usually the realization is by no means unique, and often there is a need to obtain a realization with suitable properties, for instance, a realization admitting one or more properties defined below.

Definition 4.1. The system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is called

(a) **passive** if the system operator T_Σ of Σ is contractive;

(b) **isometric** if T_Σ is isometric;

(c) **co-isometric** if T_Σ is co-isometric

(d) **conservative** if T_Σ is unitary;

(e) **self-adjoint** if T_Σ is self-adjoint.

The following subspaces

$$\mathcal{X}^c := \overline{\text{span}}\,\{\text{ran}\, A^n B : n = 0, 1, \ldots\} \tag{4.4}$$
$$\mathcal{X}^o := \overline{\text{span}}\,\{\text{ran}\, A^{*n} C^* : n = 0, 1, \ldots\} \tag{4.5}$$
$$\mathcal{X}^s := \overline{\text{span}}\,\{\text{ran}\, A^n B, \text{ran}\, A^{*m} C^* : n, m = 0, 1, \ldots\}, \tag{4.6}$$

are called, respectively, controllable, observable and simple subspaces. The system Σ is said to be **controllable** (**observable**, **simple**) if $\mathcal{X}^c = \mathcal{X} (\mathcal{X}^o = \mathcal{X}, \mathcal{X}^s = \mathcal{X})$ and **minimal** if it is both controllable and observable.

When $\Omega \ni 0$ is some symmetric neighbourhood of the origin, then it can be deduced that

$$\begin{aligned}
\mathcal{X}^c &= \overline{\operatorname{span}}\,\{\operatorname{ran}\,(I - zA)^{-1}B : z \in \Omega\} \\
\mathcal{X}^o &= \overline{\operatorname{span}}\,\{\operatorname{ran}\,(I - zA^*)^{-1}C^* : z \in \Omega\} \\
\mathcal{X}^s &= \overline{\operatorname{span}}\,\{\operatorname{ran}\,(I - zA)^{-1}B, \operatorname{ran}\,(I - wA^*)^{-1}C^* : z, w \in \Omega\}.
\end{aligned}$$

Moreover, it holds

$$(\mathcal{X}^c)^\perp = \bigcap_{n=0}^{\infty} \ker\,(B^* A^{*n}) \tag{4.7}$$

$$(\mathcal{X}^o)^\perp = \bigcap_{n=0}^{\infty} \ker\,(B^* A^{*n}) \tag{4.8}$$

$$(\mathcal{X}^s)^\perp = (\mathcal{X}^c)^\perp \cap (\mathcal{X}^o)^\perp \tag{4.9}$$

If $\mathcal{U}$ and $\mathcal{Y}$ are Kreĭn spaces and θ is an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ -valued function holomorphic at the origin and no further requirements are made, there always exists a unitary realization Σ of θ; see (Azizov & Iokhvidov, 1989, Theorem 3.8, p. 269). However, in general, such a realization has a Kreĭn space as a state space, which makes a main operator A a Kreĭn space operator. From now on, we mainly study realizations of the generalized Schur functions described in Section 3. Moreover, we examine the properties that make a realization essentially unique in the following sense. Two systems $\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}; \kappa_1)$ and $\Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}; \kappa_2)$ are **unitarily similar** if $D_1 = D_2$ and there is a unitary operator $U : \mathcal{X}_1 \to \mathcal{X}_2$ such that

$$A_1 = U^{-1} A_2 U, \quad B_1 = U^{-1} B_2, \quad C_1 = C_2 U.$$

It is evident that this can happen only if $\dim \mathcal{X}_1 = \dim \mathcal{X}_2$ and $\kappa_1 = \kappa_2$. Unitarily similar systems differ only by a unitary change of state variable, and transfer functions of unitarily similar systems coincide. Moreover, unitary similarity preserves dynamical properties of the system and also the spectral properties of the main operator.

The systems Σ_1 and Σ_2 above are said to be **weakly similar** if $D_1 = D_2$ and there exists an injective closed densely defined possible unbounded linear operator $Z : \mathcal{X}_1 \to \mathcal{X}_2$ with the dense range such that

$$ZA_1 x = A_2 Z x, \quad C_1 x = C_2 Z x, \quad x \in \mathcal{D}(Z), \quad \text{and} \quad ZB_1 = B_2, \tag{4.10}$$

where $\mathcal{D}(Z)$ is the domain of Z. In general, as suggested by the name, this type of similarity does not promise very strong properties. Indeed, without any further information, nearly all that can be deduced is that two weakly similar systems have the same transfer function.

The following proposition collects some results of (Alpay et al., 1997, Chapter 2), (Saprikin, 2001, Theorem 2.3 and Proposition 3.3), Theorem 2.5 from Article (I) and Lemma 2.8 from Article (II).

Proposition 4.2. *Let $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index. Then, there exist realizations $\Sigma_k, k = 1,\dots,4$, of θ such that their state spaces are Pontryagin spaces with the negative index κ, and*

(i) *Σ_1 is simple conservative;*

(ii) *Σ_2 is controllable isometric;*

(iii) *Σ_3 is observable co-isometric;*

(iv) *Σ_4 is minimal passive.*

If the system Σ has some of the properties (i)–(iii)*, then $\theta_\Sigma \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, where κ is the negative index of the state space of Σ. When $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, this also happens when Σ has the property* (iv)*. Moreover, any two realizations of θ which both have the same property* (i), (ii) *or* (iii)*, are unitarily similar, and any two minimal passive realizations of θ are weakly similar.*

All the realizations in Proposition 4.2 are passive. In general, if a system $\Sigma = (T_\Sigma;\mathcal{X},\mathcal{U},\mathcal{Y};\kappa)$ is a passive realization of an $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function θ, then $\theta \in \mathbf{S}_{\kappa'}(\mathcal{U},\mathcal{Y})$, where $\kappa' \leq \kappa$. A realization Σ of θ is called **κ-admissible**, if the negative index of the state space of Σ is κ. For a passive κ-admissible realization $\Sigma = (A,B,C,D;\mathcal{X},\mathcal{U},\mathcal{Y};\kappa)$ of θ, subspaces (4.7)–(4.9) are Hilbert subspaces. In the case where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, this holds also in another direction. [1] To see this, consider a passive system $\Sigma = (A,B,C,D;\mathcal{X},\mathcal{U},\mathcal{Y};\kappa)$ such that (4.7)–(4.9) are Hilbert subspaces. Lemma 2.8 from Article (II) can be applied to obtain a system $\Sigma' = (A',B',C',D;\mathcal{X}',\mathcal{U},\mathcal{Y};\kappa)$ which is minimal passive and has the same transfer function as Σ. Since $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces and Σ' is minimal, its transfer function belongs to the class $\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ (Saprikin, 2001, Theorem 2.3), and therefore Σ' and Σ are κ-admissible.

By using the result derived above, an improved and precise version of Proposition 3.4 of Article III with a simple proof can be obtained.

Proposition 4.3. *If $\Sigma = (A,B,B^*,D;\mathcal{U};\kappa)$ is a passive self-adjoint system, its transfer function θ belongs to $\mathbf{S}_{\kappa_1}(\mathcal{U}) \cap \mathbf{N}_{\kappa_2}(\mathcal{U})$, where $\kappa_1 \leq \kappa_2$ and κ_2 is the dimension of a maximal negative subspace of*

$$\operatorname{span}\{\operatorname{ran}(I-zA)^{-1}B : z \in \Omega\} := \mathfrak{S},$$

[1] The mentioned result is improved and precise version of Lemma 3.5 of Article I, with a simple proof. The result was learned while considering questions raised by pre-examiner Mikael Kurula.

where Ω is a sufficiently small symmetric neighbourhood of the origin. Moreover, if $\mathcal{U}$ is a Hilbert space and $\kappa = \kappa_2$, then $\kappa_1 = \kappa_2$.

Proof. Only the claim stated in the last sentence will be proved, since the other claims are same as in Proposition 3.4 of Article III. To this end, since $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, it is enough to show that $(\mathcal{X}^o)^\perp$, $(\mathcal{X}^c)^\perp$ and $(\mathcal{X}^s)^\perp$ of $\Sigma = (A, B, B^*, D; \mathcal{U}; \kappa)$ are Hilbert spaces. Since Σ is self-adjoint, $(\mathcal{X}^o)^\perp$, $(\mathcal{X}^c)^\perp$ and $(\mathcal{X}^s)^\perp$ all coincide, and moreover, they coincide with $\mathfrak{S}^\perp$. Since the dimension of a maximal negative subspace of $\mathfrak{S}$ is $\kappa_2 = \kappa$, the space $\mathfrak{S}$ contains a maximal negative subspace of $\mathcal{X}$, and therefore $\mathfrak{S}^\perp = (\mathcal{X}^o)^\perp = (\mathcal{X}^c)^\perp = (\mathcal{X}^s)^\perp$ are Hilbert spaces, and the claim follows. □

Lemma 2.8 from Article (II) can also be applied for κ-admissible passive realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative indices. It is then possible to decompose Σ as Kálmán decomposition like manner, such that several useful new realizations of θ with desired properties can be obtained.

The defect functions of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, were described in Section 3. They are linked with realizations of the following definition.

Definition 4.4. Denote $E_\mathcal{X}(x) = \langle x, x \rangle_\mathcal{X}$ for a vector x in an inner product space $\mathcal{X}$. A κ-admissible passive realization $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ of a function $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, is called **optimal** if for any κ-admissible passive realization $\Sigma_0 = (A_0, B_0, C_0, D; \mathcal{X}_0, \mathcal{U}, \mathcal{Y}; \kappa)$ of θ it holds

$$E_\mathcal{X}\left(\sum_{n=0}^{N} A^n B u_n\right) \leq E_{\mathcal{X}_0}\left(\sum_{n=0}^{N} {A_0}^n B_0 u_n\right),$$

for any $N \in \mathbb{N}_0$ and $\{u_n\}_{n=0}^N \subset \mathcal{U}$. Moreover, an observable passive realization $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ is called ***-optimal** if for any observable κ-admissible passive realization $\Sigma_0 = (A_0, B_0, C_0, D; \mathcal{X}_0, \mathcal{U}, \mathcal{Y}; \kappa)$ of θ it holds

$$E_\mathcal{X}\left(\sum_{n=0}^{N} A^n B u_n\right) \geq E_{\mathcal{X}_0}\left(\sum_{n=0}^{N} {A_0}^n B_0 u_n\right),$$

for any $N \in \mathbb{N}_0$ and $\{u_n\}_{n=0}^N \subset \mathcal{U}$.

For $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, an optimal, or *-optimal, minimal κ-admissible passive realization always exists, and they are unique up to unitary similarity, see Theorem 3.8 from Article (II). By using optimal (*-optimal) minimal κ-admissible passive realization, one can give a new general definition of the right (left) defect function, which coincides essentially with the definition given earlier for defect functions of ordinary Schur functions.

We finish this section by mentioning that not only passive passive are interesting, although they were extensively studied. Especially, if $(T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{U}; \kappa)$ is self-adjoint, then the transfer function θ of Σ is a generalized Nevanlinna function from the class $N_{\kappa'}(\mathcal{U})$, where $\kappa' \leq \kappa$, and an equality $\kappa' = \kappa$ holds if and only if (4.7) is a Hilbert subspace of $\mathcal{X}$.

5 SUMMARY OF FINDINGS

The main results achieved in this thesis are summarized as follows.

I. Passive discrete-time systems with a Pontryagin state space

Generalized Schur functions from the class $\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, and corresponding passive discrete-time systems are investigated. Defect functions for generalized Schur functions are defined for the first time. After introducing a necessary machinery, a well-known result of D.Z. Arov and J.W. Helton about weak similarities between the minimal realizations of the same ordinary Schur function, is extended for the generalized Schur functions in Theorem 2.5.

In the later part of the article, realizations of generalized inner, generalized co-inner and generalized Schur functions with zero defect functions are studied. The main tools are cascade connections, or products, of the two passive systems, and Kreĭn–Langer factorizations $\theta = B_r^{-1}\theta_r = \theta_l B_l^{-1}$. A general criterion, when a product of two observable (controllable, simple) system is observable (controllable, simple), is derived in Lemma 3.3. By applying Lemma 3.3 and theory of canonical realizations of generalized Schur functions, a special condition when a product of two observable co-isometric (controllable isometric) systems is observable co-isometric (controllable isometric), is presented in Theorem 3.6 (3.7). The condition is geometrical and involves orthogonal decompositions of generalized de Branges–Rovnyak spaces.

It turns out in Theorem 3.9 that co-isometric observable (isometric controllable) realizations of a generalized Schur function θ always have product representations corresponding to the right(left) Kreĭn–Langer factorization of θ. This result is utilized in Theorem 4.2 to show that main operators of simple conservative realizations of generalized inner or co-inner functions have similar stability properties as in the case of realizations of ordinary inner and co-inner functions. The main result of this article is Theorem 4.4, where it is shown that when a right or a left defect function of generalized Schur function θ is identically zero, the canonical realizations of θ have strictly stronger properties than canonical realizations of arbitrary generalized Schur functions, giving a partial answer of the problem (I) stated in the Section 1.

II. Minimal Passive Realizations of Generalized Schur Functions in Pontryagin Spaces

Passive discrete-time systems such that all underlying spaces are Pontryagin spaces are studied. Transfer functions in that cases are generalized Schur functions from the class $S_\kappa(\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, and κ is not larger than the negative index of the state space. When κ coincide with the negative index of the state space, the realization is said to be κ-admissible. Such a notion is used the first time in this article, and it is proved in Lemma 2.8 that κ-admissible passive realization can be decomposed in suitable ways to obtain simple, observable, controllable or minimal restrictions of the original system. The concept of optimality and *-optimality, first used by D.Z. Arov in the Hilbert space setting, is extended to Pontryagin space operator valued generalized Schur functions. The existence of optimal minimal and *-optimal minimal realizations is proved in Theorem 3.5. Especially, it is proved that the first minimal restriction of simple conservative system is optimal, extending the results obtained by Arov, Kaashoek, and Pik (1997) and Saprikin (2001).

By using optimal and *-optimal systems, defect functions can be defined also for Pontryagin space operator valued Schur functions. This is done by reversing and modifying a process used by Arov (1979b) to obtain an optimal realization of an ordinary Schur functions. When defect functions are defined, some results from the first article are then further generalized and also made sharper in Theorem 4.8, giving a more rigorous answer of the problem (I) stated in the Section 1.

Under certain circumstances, it may happen that all minimal passive realizations of the same generalized Schur functions are unitarily similar, instead of being only weakly similar. For ordinary Schur functions, a criterion when this happen, is due to D.Z. Arov and M.A. Nudelman. Their criterion is generalized to the class of generalized Schur functions in this article in Theorem 4.10, giving some answers of the problem (II) stated in the Section 1. The approach used here is new; it relies just on the theory of passive systems, and this approach leads to a simpler proof.

III. Generalized Schur-–Nevanlinna functions and their realizations

Special subclasses of the Pontryagin space operator valued generalized Schur functions from the class $S_\kappa(\mathcal{U},\mathcal{Y})$ are studied. In the beginning of this article, structural properties and radial limit values of the Pontryagin space operator valued generalized Schur functions and generalized Nevanlinna functions are analyzed by using the Potapov–Ginzburg transformation. In the case where incoming and outgoing spaces are anti-Hilbert spaces, Corollary 2.3 and Remark 2.4 show that the behaviour of generalized Schur functions and generalized Nevanlinna functions is reciprocal to a case where incoming and outgoing spaces are Hilbert spaces. It is shown in Theorem 2.8 that strong radial limit values of the Pontryagin space operator valued generalized Schur functions are contractive, with respect to the indefinite inner product, almost everywhere on the unit circle. With this result, the notion of an inner function can be generalized to Pontryagin space operator valued setting.

Main results of this article concern operator colligation realizations of functions which are both generalized Schur and generalized Nevanlinna functions for some, not necessarily the same, indices. The transfer function of a self-adjoint system with Pontryagin state space is shown to be a generalized Nevanlinna function in Proposition 3.1. It can be then shown in Proposition 3.4 that the transfer function of a passive self-adjoint system with Pontryagin state space is both a generalized Nevanlinna function and a generalized Schur functions, such that the indices are not larger than the negative index of the state space. When $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, it is shown in Theorem 3.5 that a function $\theta \in \mathbf{S}_{\kappa_1}(\mathcal{U}) \cap \mathbf{N}_{\kappa_2}(\mathcal{U})$ always has a self-adjoint realization such that the state space is a Pontryagin space. Moreover, when a generalized Pontryagin space operator valued generalized Schur function is symmetric but not necessary a generalized Nevanlinna function, it can still be realized as a transfer function of self-adjoint system with a Kreĭn space as state space, as it is shown in Proposition 3.7. By using optimal and *-optimal minimal realizations, a criterion when a symmetric generalized Schur function is also a generalized Nevanlinna function is given in Theorem 3.10. The criterion involves weak similarity mappings between the optimal and *-optimal minimal realizations, thus giving some answers to the problem (II) stated in the Section 1.

In the last section, the concept of bi-inner dilation of a Schur functions is extended for the class $\mathbf{S}_{\kappa}(\mathcal{U}, \mathcal{Y})$. Is shown in Theorem 4.1 that a function $\theta \in \mathbf{S}_{\kappa}(\mathcal{U}, \mathcal{Y})$ which has a self-adjoint minimal κ-admissible realization, always has a bi-inner dilation, and this dilation can be chosen such that it is a generalized Nevanlinna function.

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Complex Analysis and Operator Theory (2019) 13:3767–3793
https://doi.org/10.1007/s11785-019-00930-1

Complex Analysis and Operator Theory

Passive Discrete-Time Systems with a Pontryagin State Space

Lassi Lilleberg[1]

Received: 28 January 2019 / Accepted: 20 May 2019 / Published online: 4 June 2019

Abstract

Passive discrete-time systems with Hilbert spaces as an incoming and outgoing space and a Pontryagin space as a state space are investigated. A geometric characterization when the index of the transfer function coincides with the negative index of the state space is given. In this case, an isometric (co-isometric) system has a product representation corresponding to the left (right) Kreĭn–Langer factorization of the transfer function. A new criterion, based on the inclusion of reproducing kernel spaces, when a product of two isometric (co-isometric) systems preserves controllability (observability), is obtained. The concept of the defect function is expanded for generalized Schur functions, and realizations of generalized Schur functions with zero defect functions are studied.

Keywords Operator colligation · Pontryagin space contraction · Passive discrete-time system · Transfer function · Generalized Schur class

Mathematics Subject Classification Primary 47A48 · 47A57 · 47B50; Secondary 93B05 · 93B07

1 Introduction

Let $\mathcal{U}$ and $\mathcal{Y}$ be separable Hilbert spaces. The **generalized Schur class** $\mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ consists of $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued functions $S(z)$ which are meromorphic in the unit disc $\mathbb{D}$ and holomorphic in a neighbourhood Ω of the origin such that the Schur kernel

Communicated by Bernd Kirstein.

This article is part of the topical collection "Linear Operators and Linear Systems" edited by Sanne ter Horst, Dmitry S. Kaliuzhnyi-Verbovetskyi and Izchak Lewkowicz.

Lassi Lilleberg
lassi.lilleberg@uva.fi

1 Department of Mathematics and Statistics, University of Vaasa, P.O. Box 700, 65101 Vaasa, Finland

$$K_S(w,z)=\frac{1-S(z)S^*(w)}{1-z\bar{w}}, \qquad w,z\in\Omega, \tag{1.1}$$

has κ negative squares ($\kappa=0,1,2,\dots$). This means that for any finite set of points $w_1,\dots,w_n$ in the domain of holomorphy $\rho(S)\subset\mathbb{D}$ of S and vectors $f_1,\dots,f_n\subset\mathcal{Y}$, the Hermitian matrix

$$\left(\langle K_S(w_j,w_i)f_j,f_i\rangle\right)_{i,j=1}^n \tag{1.2}$$

has at most κ negative eigenvalues, and there exists at least one such matrix that has exactly κ negative eigenvalues. It is known from the reproducing kernel theory [1,4,23,27,30] that the kernel (1.1) generates the reproducing kernel Pontryagin space $\mathcal{H}(S)$ with negative index κ. The spaces $\mathcal{H}(S)$ are called **generalized de Branges–Rovnyak spaces**, and the elements in $\mathcal{H}(S)$ are functions defined on $\rho(S)$ with values in $\mathcal{Y}$. The notation $S^*(z)$ means $(S(z))^*$, a function $S^\#(z)$ is defined to be $S^*(\bar{z})$ and $S^\#\in\mathbf{S}_\kappa(\mathcal{Y},\mathcal{U})$ whenever $S\in\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ [1, Theorem 2.5.2].

The class $\mathbf{S}_0(\mathcal{U},\mathcal{Y})$ is written as $\mathbf{S}(\mathcal{U},\mathcal{Y})$ and it coincides with the **Schur class**, that is, functions holomorphic and bounded by one in $\mathbb{D}$. The results first obtained by Kreĭn and Langer [26], see also [1, §4.2] and [21], show that $S\in\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ has **Kreĭn–Langer factorizations** of the form $S=S_rB_r^{-1}=B_l^{-1}S_l$, where $S_r,S_l\in\mathbf{S}_0(\mathcal{U},\mathcal{Y})$. The functions B_r^{-1} and B_l^{-1} are inverse Blaschke products, and they have unitary values everywhere on the unit circle $\mathbb{T}$. It follows from these factorizations that many properties of the functions in the Schur class $\mathbf{S}(\mathcal{U},\mathcal{Y})$ hold also for the generalized Schur functions.

The properties of the generalized Schur functions can be studied by using operator colligations and transfer function realizations. An **operator colligation** $\Sigma=(T_\Sigma;\mathcal{X},\mathcal{U},\mathcal{Y};\kappa)$ consists of a Pontryagin space $\mathcal{X}$ with the negative index κ (**state space**), Hilbert spaces $\mathcal{U}$ (**incoming space**), and $\mathcal{Y}$ (**outgoing space**) and a **system operator** $T_\Sigma\in\mathcal{L}(\mathcal{X}\oplus\mathcal{U},\mathcal{X}\oplus\mathcal{Y})$. The operator T_Σ can be written in the block form

$$T_\Sigma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}:\begin{pmatrix}\mathcal{X}\\ \mathcal{U}\end{pmatrix}\to\begin{pmatrix}\mathcal{X}\\ \mathcal{Y}\end{pmatrix}, \tag{1.3}$$

where $A\in\mathcal{L}(\mathcal{X})$ (**main operator**), $B\in\mathcal{L}(\mathcal{U},\mathcal{X})$ (**control operator**), $C\in\mathcal{L}(\mathcal{X},\mathcal{Y})$ (**observation operator**), and $D\in\mathcal{L}(\mathcal{U},\mathcal{Y})$ (**feedthrough operator**). Sometimes the colligation is written as $\Sigma=(A,B,C,D;\mathcal{X},\mathcal{U},\mathcal{Y};\kappa)$. It is possible to allow all spaces to be Pontryagin or even Kreĭn spaces, but colligations with only the state space $\mathcal{X}$ allowed to be a Pontryagin space will be considered in this paper. The colligation generated by (1.3) is also called a **system** since it can be seen as a **linear discrete-time system** of the form

$$\begin{cases}h_{k+1} &= Ah_k+B\xi_k,\\ \sigma_k &= Ch_k+D\xi_k,\end{cases}\qquad k\ge 0,$$

where $\{h_k\}\subset\mathcal{X}$, $\{\xi_k\}\subset\mathcal{U}$ and $\{\sigma_k\}\subset\mathcal{Y}$. In what follows, "system" always refers to (1.3), since other kind of systems are not considered.

When the system operator T_Σ in (1.3) is a contraction, the corresponding system is called **passive**. If T_Σ is isometric (co-isometric, unitary), then the corresponding system is called isometric (co-isometric, conservative). The **transfer function** of the system (1.3) is defined by

$$\theta_\Sigma(z) := D + zC(I - zA)^{-1}B, \tag{1.4}$$

whenever $I - zA$ is invertible. Especially, θ is defined and holomorphic in a neighbourhood of the origin. The values $\theta_\Sigma(z)$ are bounded operators from $\mathcal{U}$ to $\mathcal{Y}$. The **adjoint** or **dual system** is $\Sigma^* = (T_\Sigma^*; \mathcal{X}, \mathcal{Y}, \mathcal{U}; \kappa)$ and one has $\theta_{\Sigma^*}(z) = \theta_\Sigma{}^\#(z)$. Since contractions between Pontryagin spaces with the same negative indices are bi-contractions, Σ^* is passive whenever Σ is. If θ is an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function and $\theta_\Sigma(z) = \theta(z)$ in a neighbourhood of the origin, then the system Σ is called a **realization** of θ. A **realization problem** for the function $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ is to find a system Σ with a certain minimality property (controllable, observable, simple, minimal); for details, see Theorem 2.4, such that Σ is a realization of θ.

If $\kappa = 0$, the system reduces to the standard Hilbert space setting of the passive systems studied, for instance, by de Branges and Rovnyak [18,19], Ando [2], Sz.-Nagy and Foias [32], Helton [24], Brodskiĭ [20], Arov [5,6] and Arov et al. [7–10,13]. The theory has been extended to Pontryagin state space case by Dijksma et al. [21,22], Saprikin [28], Saprikin and Arov [12] and Saprikin et al. [11]. Especially, in [28], Arov's well-known results of minimal and optimal minimal systems are generalized to the Pontryagin state space settings. Part of those results are used in [11], where transfer functions, Kreĭn–Langer factorizations, and the corresponding product representation of system are studied and, moreover, the connection between bi-inner transfer functions and systems with bi-stable main operators are generalized to the Pontryagin state space settings. In this paper those results will be further expanded and improved.

The case when all the spaces are indefinite, the theory of isometric, co-isometric and conservative systems is considered, for instance, in [1], see also [23]. The indefinite reproducing kernel spaces were first studied by Schwartz in [29] and Sorjonen in [30].

The paper is organized as follows. In Sect. 2, basic notations and definitions about the indefinite spaces and their operators are given. Also, the left and right Kreĭn–Langer factorizations are formulated, and the boundary value properties of generalized Schur functions are introduced. After that, basic properties of linear discrete time systems, or operator colligations, especially in Pontryagin state space, are recalled without proofs. However, the extension of Arov's result about the weak similarity between two minimal passive realizations of the same transfer function, is given with a proof.

Section 3 deals mainly with the dilations, embeddings and products of two systems. The transfer function θ_Σ of the passive system $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ is a generalized Schur function with negative index no larger than the negative index of the state space $\mathcal{X}$, but the theory of passive systems will often be meaningful only if the indices are equal. A simple geometric criterion for these indices to coincides is given in Lemma 3.2. Main results in this section contain criteria when the product of two co-isometric (isometric) systems preserves observability (controllability). These results are obtained in Theorems 3.6 and 3.7. The criteria involve the reproducing kernel spaces induced by the generalized Schur functions. Moreover, Theorem 3.9 expands the results of [11] about the realizations of generalized Schur functions and their product representations corresponding to the Kreĭn–Langer factorizations. In the end of Sect. 3, it is obtained that if A is the main operator of $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ such that $\theta_\Sigma \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, then there exist unique fundamental decompositions $\mathcal{X} = \mathcal{X}_1^+ \oplus \mathcal{X}_1^- = \mathcal{X}_2^+ \oplus \mathcal{X}_2^-$ such that $A\mathcal{X}_1^+ \subset \mathcal{X}_1^+$ and $A\mathcal{X}_2^- \subset \mathcal{X}_2^-$, respectively; see Proposition 3.10.

Section 4 expands and generalizes the results of [6,11] about the realizations of bi-inner functions. It will be shown that the notions of stability and co-stability can be generalized to the Pontryagin state space settings in a similar manner as bi-stability is generalized in [11]. Moreover, the results of [3] about the realizations of ordinary Schur functions with zero defect functions will be generalized. This yields a class of generalized Schur functions with boundary value properties very close to those of inner functions in a certain sense.

2 Pontryagin Spaces, Kreĭn-Langer Factorizations and Linear Systems

Let $\mathcal{X}$ be a complex vector space with a Hermitian indefinite inner product $\langle\cdot,\cdot\rangle_\mathcal{X}$. The anti-space of $\mathcal{X}$ is the space $-\mathcal{X}$ that coinsides with $\mathcal{X}$ as a vector space but its inner product is $-\langle\cdot,\cdot\rangle_\mathcal{X}$. Notions of orthogonality and orthogonal direct sum are defined as in the case of Hilbert spaces, and $\mathcal{X} \oplus \mathcal{Y}$ is often denoted by $\begin{pmatrix}\mathcal{X} & \mathcal{Y}\end{pmatrix}^{\mathsf{T}}$. Space $\mathcal{X}$ is said to be a **Kreĭn space** if it admits a decomposition $\mathcal{X} = \mathcal{X}^+ \oplus \mathcal{X}^-$ where $(\mathcal{X}^\pm, \pm\langle\cdot,\cdot\rangle_\mathcal{X})$ are Hilbert spaces. Such a decomposition is called a **fundamental decomposition**. In general, it is not unique. However, a fundamental decomposition determines the Hilbert space $|\mathcal{X}| = \mathcal{X}^+ \oplus \left(-\mathcal{X}^-\right)$ with the strong topology which does not depend on the choice of the fundamental decomposition. The dimensions of $\mathcal{X}^+$ and $\mathcal{X}^-$, which are also independent of the choice of the fundamental decomposition, are called the **positive** and **negative indices** $\operatorname{ind}_\pm \mathcal{X} = \dim \mathcal{X}^\pm$ of $\mathcal{X}$. In what follows, all notions of continuity and convergence are understood to be with respect to the strong topology. All spaces are assumed to be separable. A linear manifold $\mathcal{N} \subset \mathcal{X}$ is a **regular subspace**, if it is itself a Kreĭn space with the inherited inner product of $\langle\cdot,\cdot\rangle_\mathcal{X}$. A **Hilbert subspace** is a regular subspace such that its negative index is zero, and a uniformly negative subspace is a regular subspace with positive index zero, i.e., an **anti-Hilbert space.** If $\mathcal{N} \subset \mathcal{X}$ is a regular subspace, then $\mathcal{X} = \mathcal{N} \oplus \mathcal{N}^\perp$, where $\perp$ refers to orthogonality w.r.t. indefinite inner product $\langle\cdot,\cdot\rangle_\mathcal{X}$. Observe that $\mathcal{N}$ is regular precisely when $\mathcal{N}^\perp$ is regular.

Denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of all continuous linear operators from the Kreĭn space $\mathcal{X}$ to the Kreĭn space $\mathcal{Y}$. Moreover, $\mathcal{L}(\mathcal{X})$ stands for $\mathcal{L}(\mathcal{X}, \mathcal{X})$. Domain of a linear operator T is denoted by $\mathcal{D}(T)$, kernel by $\ker T$ and $T\restriction_\mathcal{N}$ is a restriction of T to the linear manifold $\mathcal{N}$. The adjoint of $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is an operator $A^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that $\langle Ax, y\rangle_\mathcal{Y} = \langle x, A^*y\rangle_\mathcal{X}$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Classes of invertible, self-adjoint, isometric, co-isometric and unitary operators are defined as for Hilbert spaces, but with respect to the indefinite inner product. For self-adjoint operators $A, B \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, the inequality $A \leq B$ means that $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for all $x \in \mathcal{X}$. A self-adjoint operator $P \in \mathcal{L}(\mathcal{X})$ is an $\langle\cdot,\cdot\rangle$-**orthogonal projection** if $P^2 = P$. The unique orthogonal projection onto a regular subspace $\mathcal{N}$ of $\mathcal{X}$ exists and is denoted by $P_\mathcal{N}$. A **Pontryagin space** is a Kreĭn space $\mathcal{X}$ such that $\operatorname{ind}_- \mathcal{X} < \infty$. A linear operator $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a **contraction** if $\langle Ax, Ax\rangle \leq \langle x, x\rangle$ for all $x \in \mathcal{X}$. If $\mathcal{X}$ and

$\mathcal{Y}$ are Pontryagin spaces with the same negative index, then the adjoint of a contraction $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is still a contraction, i.e., A is a **bi-contraction**. The identity operator of the space $\mathcal{X}$ is denoted by $I_{\mathcal{X}}$ or just by I when the corresponding space is clear from the context. For further information about the indefinite spaces and their operators, we refer to [14,17,23].

For ordinary Schur class $\mathbf{S}(\mathcal{U}, \mathcal{Y})$, it is well known [32] that $S \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ has non-tangential strong limit values almost everywhere (a.e.) on the unit circle $\mathbb{T}$. It follows that $S \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ can be extended to $L^{\infty}(\mathcal{U}, \mathcal{Y})$ function, that is, the class of weakly measurable a.e. defined and essentially bounded $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued functions on $\mathbb{T}$. Moreover, $S(\zeta)$ is contractive a.e. on $\mathbb{T}$. If $S \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ has isometric (co-isometric, unitary) boundary values a.e. on $\mathbb{T}$, then S is said to be **inner** (**co-inner**, **bi-inner**).

If $\mathcal{U} = \mathcal{Y}$, then the notations $\mathbf{S}(\mathcal{U})$ and $\mathbf{S}_{\kappa}(\mathcal{U})$ are often used instead of $\mathbf{S}(\mathcal{U}, \mathcal{U})$ and $\mathbf{S}_{\kappa}(\mathcal{U}, \mathcal{U})$. Suppose that $P \in \mathcal{L}(\mathcal{U})$ is an orthogonal projection from the Hilbert space $\mathcal{U}$ to an arbitrary one dimensional subspace. Then a function defined by

$$b(z) = I - P + \rho \frac{z - \alpha}{1 - \bar{\alpha} z} P, \qquad |\rho| = 1, \qquad 0 < |\alpha| < 1, \tag{2.1}$$

is a **simple Blaschke-Potapov factor.** Easy calculations show that b is holomorphic in the closed unit disc $\overline{\mathbb{D}}$, it has unitary values everywhere on $\mathbb{T}$ and $b(z)$ is invertible whenever $z \in \overline{\mathbb{D}} \setminus \{\alpha\}$. In particular, $b \in \mathbf{S}_0(\mathcal{U})$ is bi-inner. A finite product

$$B(z) = \prod_{k=1}^{n} \left(I - P_k + \rho_k \frac{z - \alpha_k}{1 - \bar{\alpha}_k z} P_k \right), \qquad |\rho_k| = 1, \qquad 0 < |\alpha_k| < 1, \tag{2.2}$$

of simple Blaschke-Potapov factors is called **Blaschke product of degree** n, and it is also bi-inner and invertible on $\overline{\mathbb{D}} \setminus \{\alpha_1, \ldots, \alpha_n\}$. The following factorization theorem was first obtained by Kreĭn and Langer [26], see also [1, §4.2] and [21].

Theorem 2.1 *Suppose $S \in \mathbf{S}_{\kappa}(\mathcal{U}, \mathcal{Y})$. Then*

$$S(z) = S_r(z) B_r^{-1}(z) \tag{2.3}$$

where $S_r \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ and B_r is a Blaschke product of degree κ with values in $\mathcal{L}(\mathcal{U})$ such that $B_r(w) f = 0$ and $S_r(w) f = 0$ for some $w \in \mathbb{D}$ only if $f = 0$. Moreover,

$$S(z) = B_l^{-1}(z) S_l(z) \tag{2.4}$$

where $S_l \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ and B_l is a Blaschke product of degree κ with values in $\mathcal{L}(\mathcal{Y})$ such that $B_l^*(w) g = 0$ and $S_l^*(w) g = 0$ for some $w \in \mathbb{D}$ only if $g = 0$.

Conversely, any function of the form (2.3) or (2.4) belongs to $\mathbf{S}_{\kappa'}$ for some $\kappa' \leq \kappa$, and $\kappa' = \kappa$ exactly when the functions have no common zeros in sense as described above. Both factorizations are unique up to unitary constant factors.

The factorization (2.3) is called the **right Kreĭn-Langer factorization** and (2.4) is the **left Kreĭn-Langer factorization.** It follows that $S \in \mathbf{S}_{\kappa}(\mathcal{U}, \mathcal{Y})$ has κ poles (counting multiplicities) in $\mathbb{D}$, contractive strong limit values exist a.e. on $\mathbb{T}$ and S can also be extended to $L^\infty(\mathcal{U}, \mathcal{Y})$-function. Actually, these properties also characterize the generalized Schur functions. This result will be stated for reference purposes. For the proof of the sufficiency, see [21, Proposition 7.11].

Lemma 2.2 *Let S be an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function holomorphic at the origin. Then $S \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ if and only if S is meromorphic on $\mathbb{D}$ with finite pole multiplicity κ and*

$$\lim_{r\to 1} \sup_{|z|=r} \|S(z)\| \le 1$$

holds.

A function $S \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ and the factors S_r and S_l in (2.3) and (2.4) have simultaneously isometric (co-isometric, unitary) boundary values since the factors B_l^{-1} and B_r^{-1} have unitary values everywhere on $\mathbb{T}$.

The following result [32, Theorem V.4.2], which involves the notion of an **outer function** (for the definition, see [32]), will be utilized.

Theorem 2.3 *If $\mathcal{U}$ is a separable Hilbert space and $N \in L^\infty(\mathcal{U})$ such that $0 \le N(\zeta) \le I_\mathcal{U}$ a.e. on $\mathbb{T}$, then there exist a Hilbert space $\mathcal{K}$ and an outer function $\varphi \in \mathbf{S}(\mathcal{U}, \mathcal{K})$ such that*

(i) *$\varphi^*(\zeta)\varphi(\zeta) \le N^2(\zeta)$ a.e. on $\mathbb{T}$;*
(ii) *if $\widehat{\mathcal{K}}$ is a Hilbert space and $\widehat{\varphi} \in \mathbf{S}(\mathcal{U}, \widehat{\mathcal{K}})$ such that $\widehat{\varphi}^*(\zeta)\widehat{\varphi}(\zeta) \le N^2(\zeta)$ a.e. on $\mathbb{T}$, then $\widehat{\varphi}^*(\zeta)\widehat{\varphi}(\zeta) \le \varphi^*(\zeta)\varphi(\zeta)$ a.e. on $\mathbb{T}$.*

Moreover, φ is unique up to a left constant unitary factor.

For $S \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ with the Kreĭn–Langer factorizations $S = S_r B_r^{-1} = B_l^{-1} S_l$, define

$$N_S^2(\zeta) := I_\mathcal{U} - S^*(\zeta)S(\zeta), \qquad \text{a.e.} \quad \zeta \in \mathbb{T},$$
$$M_S^2(\zeta) := I_\mathcal{Y} - S(\zeta)S^*(\zeta), \qquad \text{a.e.} \quad \zeta \in \mathbb{T}.$$

Since Blaschke products are unitary on $\mathbb{T}$, it follows that

$$N_S^2(\zeta) = I_\mathcal{U} - S_l^*(\zeta)S_l(\zeta) = N_{S_l}^2(\zeta) \tag{2.5}$$
$$M_S^2(\zeta) = I_\mathcal{Y} - S_r(\zeta)S_r^*(\zeta) = M_{S_r}^2(\zeta). \tag{2.6}$$

Theorem 2.3 guarantees that there exists an outer function φ_S with properties introduced in Theorem 2.3 for N_S. An easy modification of Theorem 2.3 shows that there exists a Schur function ψ_S such that $\psi_S^\#$ is an outer function, $\psi_S(\zeta)\psi_S^*(\zeta) \le M_S^2(\zeta)$ a.e. $\zeta \in \mathbb{T}$ and $\psi_S(\zeta)\psi_S^*(\zeta) \le \widehat{\psi}(\zeta)\widehat{\psi}^*(\zeta)$ for every Schur function $\widehat{\psi}$ with a property $\widehat{\psi}_S(\zeta)\widehat{\psi}_S^*(\zeta) \le M_S^2(\zeta)$. Moreover,it follows from the identies (2.5) and (2.6) that

$$\varphi_S = \varphi_{S_l} \quad \text{and} \quad \psi_S = \psi_{S_r}. \tag{2.7}$$

The function φ_S is called the **right defect function** and ψ_S is the **left defect function**.

Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ be a passive system. The following subspaces

$$\mathcal{X}^c := \overline{\operatorname{span}}\{\operatorname{ran} A^n B : n = 0, 1, \ldots\}, \tag{2.8}$$
$$\mathcal{X}^o := \overline{\operatorname{span}}\{\operatorname{ran} A^{*n} C^* : n = 0, 1, \ldots\}, \tag{2.9}$$
$$\mathcal{X}^s := \overline{\operatorname{span}}\{\operatorname{ran} A^n B, \operatorname{ran} A^{*m} C^* : n, m = 0, 1, \ldots\}, \tag{2.10}$$

are called respectively controllable, observable and simple subspaces. The system Σ is said to be **controllable** (**observable**, **simple**) if $\mathcal{X}^c = \mathcal{X}$ ($\mathcal{X}^o = \mathcal{X}$, $\mathcal{X}^s = \mathcal{X}$) and **minimal** if it is both controllable and observable. When $\Omega \ni 0$ is some symmetric neighbourhood of the origin, that is, $\bar{z} \in \Omega$ whenever $z \in \Omega$, then also

$$\mathcal{X}^c = \overline{\operatorname{span}}\{\operatorname{ran} (I - zA)^{-1} B : z \in \Omega\}, \tag{2.11}$$
$$\mathcal{X}^o = \overline{\operatorname{span}}\{\operatorname{ran} (I - zA^*)^{-1} C^* : z \in \Omega\}, \tag{2.12}$$
$$\mathcal{X}^s = \overline{\operatorname{span}}\{\operatorname{ran} (I - zA)^{-1} B, \operatorname{ran} (I - wA^*)^{-1} C^* : z, w \in \Omega\}. \tag{2.13}$$

If the system operator T_Σ in (1.3) is a contraction, that is, Σ is passive, the operators

$$A : \mathcal{X} \to \mathcal{X}, \quad \begin{pmatrix} A \\ C \end{pmatrix} : \mathcal{X} \to \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix}, \quad \begin{pmatrix} A & B \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \mathcal{U} \end{pmatrix} \to \mathcal{X},$$

are also bi-contractions. Moreover, the operators B and C^* are contractions but not bi-contractions unless $\kappa = 0$.

The following realization theorem is known, and the parts (i)–(iii) can be found e.g. in [1, Chapter 2] and the part (iv) in [28, Theorem 2.3 and Proposition 3.3].

Theorem 2.4 *For $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ there exist realizations $\Sigma_k, k = 1, \ldots, 4$, of θ such that*

- (i) Σ_1 *is conservative and simple;*
- (ii) Σ_2 *is isometric and controllable;*
- (iii) Σ_3 *is co-isometric and observable;*
- (iv) Σ_4 *is passive and minimal.*

Conversely, if the system Σ has some of the properties (i)–(iv)*, then $\theta_\Sigma \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where κ is the negative index of the state space of Σ.*

It is also true that the transfer function of passive system is a generalized Schur function, but its index may be smaller than the negative index of the state space [28, Theorem 2.2]. For a conservative system Σ it is known from [1, Theorem 2.1.2 (3)] that the index of the transfer function θ_Σ of Σ co-insides with the negative index of the state space $\mathcal{X}$ of Σ if and only if the space $(\mathcal{X}^s)^\perp$ is a Hilbert subspace. This result holds also in more general settings when Σ is passive, as it will be proved in Lemma 3.2, after introducing some machinery.

Two realizations $\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}; \kappa)$ and $\Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}; \kappa)$ of the same function $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ are called **unitarily similar** if $D_1 = D_2$ and there exists a unitary operator $U : \mathcal{X}_1 \to \mathcal{X}_2$ such that

$$A_1 = U^{-1} A_2 U, \quad B_1 = U^{-1} B_2, \quad C_1 = C_2 U.$$

Moreover, the realizations Σ_1 and Σ_2 are said to be **weakly similar** if $D_1 = D_2$ and there exists an injective closed densely defined possibly unbounded linear operator $Z : \mathcal{X}_1 \to \mathcal{X}_2$ with the dense range such that

$$ZA_1 f = A_2 Z f, \quad C_1 f = C_2 Z f, \quad f \in \mathcal{D}(Z), \quad \text{and} \quad ZB_1 = B_2.$$

Unitary similarity preserves dynamical properties of the system and also the spectral properties of the main operator. If two realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ both have the same property (i), (ii) or (iii) of Theorem 2.4, then they are unitarily similar [1, Theorem 2.1.3]. In Hilbert state space case, results of Helton [24] and Arov [5] state that two minimal passive realizations of $\theta \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ are weakly similar. However, weak similarity preserves neither the dynamical properties of the system nor the spectral properties of its main operator. The following theorem shows that Helton's and Arov's statement holds also in Pontryagin state space settings. Proof is similar to the one given in the Hilbert space settings in [15, Theorem 3.2] and [16, Theorem 7.13].

Theorem 2.5 *Let $\Sigma_1 = (T_{\Sigma_1}; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}; \kappa)$ and $\Sigma_2 = (T_{\Sigma_2}; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}; \kappa)$ be two minimal passive realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$. Then they are weakly similar.*

Proof Decompose the system operators as in (1.3). In a sufficiently small neighbourhood of the origin, the functions θ_{Σ_1} and θ_{Σ_2} have the Neumann series which coincide. Hence $D_1 = D_2$ and $C_1 A_1^k B_1 = C_2 A_2^k B_2$ for any $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Since Σ_1 is controllable, vectors of the form $x = \sum_{k=0}^{N} A_1^k B_1 u_k$, $u_k \in \mathcal{U}$, are dense in $\mathcal{X}_1$. Define

$$Rx = \sum_{k=0}^{N} A_2^k B_2 u_k,$$

and let Z be the closure of the graph of R. Let $\{x_n\}_{n\in\mathbb{N}} \subset \operatorname{span}\{\operatorname{ran} A_1^k B_1 : k \in \mathbb{N}_0\} = \mathcal{D}(R)$ such that $x_n \to 0$ and $Rx_n \to y$ when $n \to \infty$. Since $C_1 A_1^k B_1 = C_2 A_2^k B_2$ for any $k \in \mathbb{N}_0$, also $C_1 A_1^k x_n = C_2 A_2^k R x_n$, and the continuity implies

$$C_2 A_2^k y = \lim_{n\to\infty} C_2 A_2^k R x_n = \lim_{n\to\infty} C_1 A_1^k x_n = 0.$$

Since Σ_2 is observable, it follows from (2.9) that

$$\bigcap_{k\in\mathbb{N}_0} \ker C_2 A_2^k = \{0\}, \tag{2.14}$$

and therefore $y = 0$. This implies that Z is a closed densely defined linear operator. Since Σ_2 is controllable, the range of Z is dense.

To prove the injectivity, let $x \in \mathcal{D}(Z)$ such that $Zx = 0$. Then there exists $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(R)$ such that $x_n \to x$ and $Rx_n \to 0$.

By the continuity,

$$C_1 A_1^k x = \lim_{n\to\infty} C_1 A_1^k x_n = \lim_{n\to\infty} C_2 A_2^k R x_n = 0$$

for any $k \in \mathbb{N}_0$. Since Σ_1 is observable, this implies that $x = 0$, and Z is injective.

For $x \in \mathcal{D}(Z)$, there exists $\{x_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(R)$ such that $x_k \to x$ and $Rx_k \to Zx$. Then

$$A_1 x = \lim_{k\to\infty} A_1 x_k \tag{2.15}$$

$$A_2 Z x = \lim_{k\to\infty} A_2 R x_k = \lim_{k\to\infty} R A_1 x_k = \lim_{k\to\infty} Z A_1 x_k \tag{2.16}$$

$$C_1 x = \lim_{k\to\infty} C_1 x_k = \lim_{k\to\infty} C_2 R x_k = C_2 Z x \tag{2.17}$$

$$Z B_1 = R B_1 = B_2. \tag{2.18}$$

Since Z is closed, Eqs. (2.15) and (2.16) show that $A_1 x \in \mathcal{D}(Z)$ and $Z A_1 x = A_2 Z x$. Since (2.17) and (2.18) hold also, it has been shown that Z is a weak similarity. □

Remark 2.6 It should be noted that Theorem 2.5 holds also when all the spaces are Pontryagin, Kreĭn or, if one defines the observability criterion as $\cap_{n\in\mathbb{N}_0} \ker C A^n = \{0\}$, even Banach spaces. This result can also be derived from [31, p. 704].

3 Julia Operators, Dilations, Embeddings and Products of Systems

The system (1.3) can be expanded to a larger system either without changing the transfer function or without changing the main operator. Both of these can be done by using the **Julia operator**, see (3.1) below. For a proof of the next theorem and some further details about Julia operators, see [23, Lecture 2].

Theorem 3.1 *Suppose that $\mathcal{X}_1$ and $\mathcal{X}_2$ are Pontryagin spaces with the same negative index, and $A : \mathcal{X}_1 \to \mathcal{X}_2$ is a contraction. Then there exist Hilbert spaces $\mathfrak{D}_A$ and $\mathfrak{D}_{A^*}$, linear operators $D_A : \mathfrak{D}_A \to \mathcal{X}_1$, $D_{A^*} : \mathfrak{D}_{A^*} \to \mathcal{X}_2$ with zero kernels and a linear operator $L : \mathfrak{D}_A \to \mathfrak{D}_{A^*}$ such that*

$$U_A := \begin{pmatrix} A & D_{A^*} \\ D_A^* & -L^* \end{pmatrix} : \begin{pmatrix} \mathcal{X}_1 \\ \mathfrak{D}_{A^*} \end{pmatrix} \to \begin{pmatrix} \mathcal{X}_2 \\ \mathfrak{D}_A \end{pmatrix} \tag{3.1}$$

is unitary. Moreover, U_A is essentially unique.

A **dilation** of a system (1.3) is any system of the form $\widehat{\Sigma} = (\widehat{A}, \widehat{B}, \widehat{C}, D; \widehat{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; \kappa')$, where

$$\widehat{\mathcal{X}} = \mathcal{D} \oplus \mathcal{X} \oplus \mathcal{D}_*, \quad \widehat{A}\mathcal{D} \subset \mathcal{D}, \quad \widehat{A}^*\mathcal{D}_* \subset \mathcal{D}_*, \quad \widehat{C}\mathcal{D} = \{0\}, \quad \widehat{B}^*\mathcal{D}_* = \{0\}. \tag{3.2}$$

That is, the system operator $T_{\widehat{\Sigma}}$ of $\widehat{\Sigma}$ is of the form

$$\begin{aligned} T_{\widehat{\Sigma}} &= \begin{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix} & \begin{pmatrix} B_1 \\ B \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & C & C_1 \end{pmatrix} & D \end{pmatrix} : \begin{pmatrix} \begin{pmatrix} \mathcal{D} \\ \mathcal{X} \\ \mathcal{D}_* \end{pmatrix} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} \mathcal{D} \\ \mathcal{X} \\ \mathcal{D}_* \end{pmatrix} \\ \mathcal{Y} \end{pmatrix}, \\ \widehat{A} &= \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}, \qquad \widehat{B} = \begin{pmatrix} B_1 \\ B \\ 0 \end{pmatrix}, \qquad \widehat{C} = \begin{pmatrix} 0 & C & C_1 \end{pmatrix}. \end{aligned} \tag{3.3}$$

Then the system Σ is called a **restriction** of $\widehat{\Sigma}$, and it has an expression

$$\Sigma = (P_{\mathcal{X}}\widehat{A}\restriction_{\mathcal{X}}, P_{\mathcal{X}}\widehat{B}, \widehat{C}\restriction_{\mathcal{X}}, D; P_{\mathcal{X}}\widehat{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; \kappa). \tag{3.4}$$

Dilations and restrictions are denoted by

$$\widehat{\Sigma} = \mathrm{dil}_{\mathcal{X}\to\widehat{\mathcal{X}}}\Sigma, \quad \Sigma = \mathrm{res}_{\widehat{\mathcal{X}}\to\mathcal{X}}\widehat{\Sigma}, \tag{3.5}$$

mostly without subscripts when the corresponding state spaces are clear. A calculation show that the transfer functions of the original system and its dilation coincide.

The second way to expand the system (1.3) is called an **embedding**, which is any system determined by the system operator

$$\begin{aligned} T_{\widetilde{\Sigma}} &= \begin{pmatrix} A & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \widetilde{\mathcal{U}} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \widetilde{\mathcal{Y}} \end{pmatrix} \\ \iff & \begin{pmatrix} A & \begin{pmatrix} B & B_1 \end{pmatrix} \\ \begin{pmatrix} C \\ C_1 \end{pmatrix} & \begin{pmatrix} D & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \begin{pmatrix} \mathcal{U} \\ \mathcal{U}' \end{pmatrix} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \begin{pmatrix} \mathcal{Y} \\ \mathcal{Y}' \end{pmatrix} \end{pmatrix}, \end{aligned} \tag{3.6}$$

where $\mathcal{U}'$ and $\mathcal{Y}'$ are Hilbert spaces. The transfer function of the embedded system is

$$\begin{aligned} \theta_{\widetilde{\Sigma}}(z) &= \begin{pmatrix} D & D_{12} \\ D_{21} & D_{22} \end{pmatrix} + z \begin{pmatrix} C \\ C_1 \end{pmatrix} (I_{\mathcal{X}} - zA)^{-1} \begin{pmatrix} B & B_1 \end{pmatrix} \\ &= \begin{pmatrix} D + zC(I_{\mathcal{X}} - zA)^{-1}B & D_{12} + zC(I_{\mathcal{X}} - zA)^{-1}B_1 \\ D_{21} + zC_1(I_{\mathcal{X}} - zA)^{-1}B & D_{22} + zC_1(I_{\mathcal{X}} - zA)^{-1}B_1 \end{pmatrix} \\ &= \begin{pmatrix} \theta_{\Sigma}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{pmatrix}, \end{aligned} \tag{3.7}$$

where θ_{Σ} is the transfer function of the original system.

For a passive system there always exist a conservative dilation [28, Theorem 2.1] and a conservative embedding [11, p. 7]. Both of these can be constructed such that the system operator of the expanded system is the Julia operator of T_{Σ}. Such expanded systems are called **Julia dilation** and **Julia embedding**, respectively.

If the passive system (1.3) is simple (controllable, observable, minimal), then so is any conservative embedding (3.6) of it. This follows from the fact that $B\mathcal{U} \subset \widetilde{B}\widetilde{\mathcal{U}}$

and $C^*\mathcal{Y} \subset \widetilde{C}^*\widetilde{\mathcal{Y}}$. A detailed proof of simplicity can be found in [11, Theorem 4.3]. The same argument works also in the rest of the cases. However, it can happen that a simple passive system has no simple conservative dilation, even in the case when the original system is minimal, see the example on page 15 in [11].

Lemma 3.2 *Let θ_Σ be the transfer function of a passive system $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$. If $\theta_\Sigma \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, then the spaces $(\mathcal{X}^c)^\perp$, $(\mathcal{X}^o)^\perp$ and $(\mathcal{X}^s)^\perp$ are Hilbert subspaces of $\mathcal{X}$. Moreover, if one of the spaces $(\mathcal{X}^c)^\perp$, $(\mathcal{X}^s)^\perp$ and $(\mathcal{X}^s)^\perp$ is a Hilbert subspace, then so are the others and $\theta_\Sigma \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$.*

Proof If $\theta_\Sigma \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, it is proved in [28, Lemma 2.5] that $(\mathcal{X}^c)^\perp$ and $(\mathcal{X}^o)^\perp$ are Hilbert spaces. It easily follows from (2.8) and (2.9) that

$$(\mathcal{X}^s)^\perp = (\mathcal{X}^c)^\perp \cap (\mathcal{X}^o)^\perp, \tag{3.8}$$

so $(\mathcal{X}^s)^\perp$ is also a Hilbert space, and the first claim is proved.

Suppose next that $(\mathcal{X}^s)^\perp$ is a Hilbert space. Consider a conservative embedding $\widetilde{\Sigma}$ of Σ, and represent the system operator $T_{\widetilde{\Sigma}}$ as in (3.6). The first identity in (3.7) shows that the transfer function of any embedding of Σ has the same number of poles (counting multiplicities) as θ_Σ, and therefore it follows from Lemma 2.2 that the indices of θ_Σ and $\theta_{\widetilde{\Sigma}}$ coincides. Denote the simple subspace of the embedded system as $\widetilde{\mathcal{X}}^s$. Since $\mathcal{X}^s \subset \widetilde{\mathcal{X}}^s$, it holds $(\widetilde{\mathcal{X}}^s)^\perp \subset (\mathcal{X}^s)^\perp$, and therefore $(\widetilde{\mathcal{X}}^s)^\perp$ is also a Hilbert space. It follows from [1, Theorem 2.1.2 (3)] that the transfer function $\theta_{\widetilde{\Sigma}}$ of $\widetilde{\Sigma}$ belongs to $\mathbf{S}_\kappa(\widetilde{\mathcal{U}}, \widetilde{\mathcal{Y}})$, which implies now $\theta_\Sigma \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$. Then the first claim proved above implies that $(\mathcal{X}^c)^\perp$ and $(\mathcal{X}^o)^\perp$ are Hilbert subspaces.

If one assumes that $(\mathcal{X}^c)^\perp$ or $(\mathcal{X}^o)^\perp$ is a Hilbert space, the identity (3.8) shows that $(\mathcal{X}^s)^\perp$ is a Hilbert space as well. Then the argument above can be applied, and the second claim is proved. □

The **product** or **cascade connection** of two systems $\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}_1; \kappa_1)$ and $\Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}; \kappa_2)$ is a system $\Sigma_2 \circ \Sigma_1 = (T_{\Sigma_2 \circ \Sigma_1}; \mathcal{X}_1 \oplus \mathcal{X}_2, \mathcal{U}, \mathcal{Y}; \kappa_1 + \kappa_2)$ such that

$$T_{\Sigma_2\circ\Sigma_1} = \begin{pmatrix} \begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix} & \begin{pmatrix} B_1 \\ B_2 D_1 \end{pmatrix} \\ \begin{pmatrix} D_2 C_1 & C_2 \end{pmatrix} & D_2 D_1 \end{pmatrix} : \begin{pmatrix} \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{pmatrix} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{pmatrix} \\ \mathcal{Y} \end{pmatrix}. \tag{3.9}$$

Written in the form (1.3), one has $\mathcal{X} = \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{pmatrix}$ and

$$A = \begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 D_1 \end{pmatrix}, \qquad C = \begin{pmatrix} D_2 C_1 & C_2 \end{pmatrix}, \qquad D = D_2 D_1. \tag{3.10}$$

Note that $A_2 = A\restriction_{\mathcal{X}_2}$ and

$$\begin{pmatrix} A_1 & 0 & B_1 \\ B_2 C_1 & A_2 & B_2 D_1 \\ D_2 C_1 & C_2 & D_2 D_1 \end{pmatrix} = \begin{pmatrix} I_{\mathcal{X}_1} & 0 & 0 \\ 0 & A_2 & B_2 \\ 0 & C_2 & D_2 \end{pmatrix} \begin{pmatrix} A_1 & 0 & B_1 \\ 0 & I_{\mathcal{X}_2} & 0 \\ C_1 & 0 & D_1 \end{pmatrix} : \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ \mathcal{Y} \end{pmatrix}. \tag{3.11}$$

The product $\Sigma_2 \circ \Sigma_1$ is defined when the incoming space of Σ_2 is the outgoing space of Σ_1. Again, direct computations show that $\theta_{\Sigma_2 \circ \Sigma_1} = \theta_{\Sigma_2}\theta_{\Sigma_1}$ whenever both functions are defined. For the dual system one has $(\Sigma_2 \circ \Sigma_1)^* = \Sigma_1^* \circ \Sigma_2^*$. It follows from the identity (3.11) that the product $\Sigma_2 \circ \Sigma_1$ is conservative (isometric, co-isometric, passive) whenever Σ_1 and Σ_2 are. Also, if the product is isometric (co-isometric, conservative) and one factor of the product is conservative, then the other factor must be isometric (co-isometric, conservative).

The product of two systems preserves similarity properties introduced on page 7 in sense that if $\Sigma = \Sigma_2 \circ \Sigma_1$ and $\Sigma' = \Sigma_2' \circ \Sigma_1'$ such that Σ_1 is unitarily (weakly) similar with Σ_1' and Σ_2 is unitarily (weakly) similar with Σ_2', then easy calculations using (3.11) show that Σ and Σ' are unitarily (weakly) similar.

It is known (c.f. e.g. [1, Theorem 1.2.1]) that if $\Sigma_2 \circ \Sigma_1$ is controllable (observable, simple, minimal), then so are Σ_1 and Σ_2. The converse statement is not true. The following lemma gives necessary and sufficient conditions when the product is observable, controllable or simple. The simple case is handled in [11, Lemma 7.4].

Lemma 3.3 *Let* $\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}_1; \kappa_1)$, $\Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}; \kappa_2)$ *and* $\Sigma = \Sigma_2 \circ \Sigma_1$. *Let* $\Omega = \overline{\Omega}$ *be a symmetric neighbourhood of the origin such that the transfer function* $\theta_\Sigma = \theta_{\Sigma_2}\theta_{\Sigma_1}$ *of* Σ *is analytic in* Ω. *Consider the equations*

$$\theta_{\Sigma_2}(z)C_1(I - zA_1)^{-1}x_1 = -C_2(I - zA_2)^{-1}x_2, \quad \textit{for all } z \in \Omega; \tag{3.12}$$
$$\theta^{\#}_{\Sigma_1}(z)B_2^*(I - zA_2^*)^{-1}x_2 = -B_1^*(I - zA_1^*)^{-1}x_1, \quad \textit{for all } z \in \Omega, \tag{3.13}$$

where $x_1 \in \mathcal{X}_1$ *and* $x_2 \in \mathcal{X}_2$. *Then* Σ *is observable if and only if* (3.12) *has only the trivial solution, and* Σ *is controllable if and only if* (3.13) *has only the trivial solution. Moreover,* Σ *is simple if and only if the pair of equations consisting of* (3.12) *and* (3.13) *has only the trivial solution.*

Proof Write the system operator $T_{\Sigma_2 \circ \Sigma_1}$ in (3.9) in the form (1.3). It follows from (2.11)–(2.13) that

$$x \in (\mathcal{X}^o)^\perp \iff C(I - zA)^{-1}x = 0 \quad \text{for all } z \in \Omega; \tag{3.14}$$
$$x \in (\mathcal{X}^c)^\perp \iff B^*(I - zA^*)^{-1}x = 0 \quad \text{for all } z \in \Omega; \tag{3.15}$$
$$x \in (\mathcal{X}^s)^\perp \iff B^*(I - zA^*)^{-1}x = 0 \quad \text{and} \quad C(I - A)^{-1}x = 0 \quad \text{for all } z \in \Omega. \tag{3.16}$$

Decompose $x = x_1 \oplus x_2$, where $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$. With respect to the this decomposition, the definition of the main operator A from (3.10) yields

$$(I - zA)^{-1} = \begin{pmatrix} (I_{\mathcal{X}_1} - zA_1)^{-1} & 0 \\ z(I_{\mathcal{X}_2} - zA_2)^{-1}B_2C_1(I_{\mathcal{X}_1} - zA_1)^{-1} & (I_{\mathcal{X}_2} - zA_2)^{-1} \end{pmatrix}.$$

From this relation and (3.10), it follows that the right hand side of (3.14) is equivalent to

$$\begin{pmatrix} D_2C_1 & C_2 \end{pmatrix} \begin{pmatrix} (I_{\mathcal{X}_1} - zA_1)^{-1} & 0 \\ z(I_{\mathcal{X}_2} - zA_2)^{-1}B_2C_1(I_{\mathcal{X}_1} - zA_1)^{-1} & (I_{\mathcal{X}_2} - zA_2)^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

for all $z \in \Omega$. (3.17)

Similar calculations show that the right hand side of (3.15) is equivalent to

$$\begin{pmatrix} B_1^* & D_1^*B_2^* \end{pmatrix} \begin{pmatrix} (I_{\mathcal{X}_1} - zA_1^*)^{-1} & z(I_{\mathcal{X}_1} - zA_1^*)^{-1}C_1^*B_2^*(I_{\mathcal{X}_2} - zA_2^*)^{-1} \\ 0 & (I_{\mathcal{X}_2} - zA_2^*)^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

for all $z \in \Omega$. (3.18)

Expanding the identity (3.17) and using the definition of the transfer function

$$\theta_{\Sigma_2}(z) = D_2 + zC_2(I_{\mathcal{X}_2} - zA_2)^{-1}B_2$$

one gets that (3.17) is equivalent to

$$\begin{aligned} &\left(D_2 + C_2z(I_{\mathcal{X}_2} - zA_2)^{-1}B_2\right)C_1(I_{\mathcal{X}_1} - zA_1)^{-1}x_1 = -C_2(I_{\mathcal{X}_2} - zA_2)^{-1}x_2 \\ &\iff \quad \theta_{\Sigma_2}(z)C_1(I_{\mathcal{X}_1} - zA_1)^{-1}x_1 = -C_2(I_{\mathcal{X}_2} - zA_2)^{-1}x_2. \end{aligned}$$

That is, the identity (3.17) is equivalent to (3.12). Similar calculations and the identity

$$\theta^{\#}_{\Sigma_1}(z) = D_1^* + zB_1^*(I_{\mathcal{X}_1} - zA_1^*)^{-1}C_1^*$$

shows that the identity (3.18) is equivalent to (3.13). The results follow now by observing that if the system Σ is observable, controllable or simple, then, respectively, $(\mathcal{X}^o)^{\perp} = \{0\}$, $(\mathcal{X}^c)^{\perp} = \{0\}$ or $(\mathcal{X}^s)^{\perp} = \{0\}$. □

Part (iii) of the theorem below with an additional condition that all the realizations are conservative, is proved in [11, Theorem 7.3, 7.6]. Similar techniques will be used to expand this result as follows.

Theorem 3.4 *Let $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ and let $\theta = \theta_r B_r^{-1} = B_l^{-1}\theta_l$ be its Kreĭn–Langer factorizations. Suppose that*

$$\begin{aligned} \Sigma_{\theta_r} &= (T_{\Sigma_{\theta_r}}, \mathcal{X}_r^+, \mathcal{U}, \mathcal{Y}, 0), & \Sigma_{\theta_l} &= (T_{\Sigma_{\theta_l}}, \mathcal{X}_l^+, \mathcal{U}, \mathcal{Y}, 0), \\ \Sigma_{B_r^{-1}} &= (T_{\Sigma_{B_r^{-1}}}, \mathcal{X}_r^-, \mathcal{U}, \mathcal{U}, \kappa), & \Sigma_{B_l^{-1}} &= (T_{\Sigma_{B_l^{-1}}}, \mathcal{X}_l^-, \mathcal{Y}, \mathcal{Y}, \kappa), \end{aligned}$$

are the realizations of θ_r, θ_l, B_r^{-1} and B_l^{-1}, respectively. Then:

(i) *If Σ_{θ_r} and $\Sigma_{B_r^{-1}}$ are observable and passive, then so is $\Sigma_{\theta_r} \circ \Sigma_{B_r^{-1}}$;*
(ii) *If Σ_{θ_l} and $\Sigma_{B_l^{-1}}$ are controllable and passive, then so is $\Sigma_{B_l^{-1}} \circ \Sigma_{\theta_l}$;*
(iii) *If all the realizations described above are simple passive, then so are $\Sigma_{\theta_r} \circ \Sigma_{B_r^{-1}}$ and $\Sigma_{B_l^{-1}} \circ \Sigma_{\theta_l}$.*

Proof Suppose first that $\Sigma_{B_r^{-1}}$ is a simple passive system and Σ_{θ_r} is a passive system. The results from [11, Theorems 9.4 and 10.2] show that all the simple passive realizations of B_r^{-1} are conservative and minimal. Thus, the assumptions guarantees that $\Sigma_{B_r^{-1}}$ is conservative and minimal. Represent the system operators $T_{\Sigma_{B_r^{-1}}}$ and $T_{\Sigma_{\theta_r}}$ as

$$T_{\Sigma_{B_r^{-1}}} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} : \begin{pmatrix} \mathcal{X}_r^- \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \mathcal{X}_r^- \\ \mathcal{U} \end{pmatrix}, \qquad T_{\Sigma_{\theta_r}} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \begin{pmatrix} \mathcal{X}_r^+ \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \mathcal{X}_r^+ \\ \mathcal{Y} \end{pmatrix}. \tag{3.19}$$

Let $\Omega = \overline{\Omega}$ be a symmetric neighbourhood of the origin such that B_r^{-1} is analytic in Ω. Suppose that $x_1 \in \mathcal{X}_r^-$ and $x_2 \in \mathcal{X}_r^+$ satisfy

$$\theta_r(z)C_1(I - zA_1)^{-1}x_1 = -C_2(I - zA_2)^{-1}x_2, \quad \text{for all } z \in \Omega. \tag{3.20}$$

The space $\mathcal{X}_r^-$ is κ-dimensional anti-Hilbert space, and all the poles of B_r^{-1} are also poles of $C_1(I - zA_1)^{-1}x_1$. Since $\mathcal{X}_r^+$ is a Hilbert space, the operator A_2 is a Hilbert space contraction, and $(I - zA_2)^{-1}$ exists for all $z \in \mathbb{D}$. That is, the right hand side of (3.20) is holomorphic in $\mathbb{D}$, and then so is the left hand side also. Since θ_r and B_r have no common zeros in the sense of Theorem 2.1 and the zeros of B_r are the poles of B_r^{-1}, the factor $\theta_r(z)$ cannot cancel out the poles of $C_1(I - zA_1)^{-1}x_1$ (For a more detailed argument, see the proof of [11, Theorem 7.3]). That is, $\theta_r(z)C_1(I-zA_1)^{-1}x_1$ is holomorphic in $\mathbb{D}$ only if $C_1(I - zA_1)^{-1}x_1 \equiv 0$. Then also $C_2(I - zA_2)^{-1}x_2 \equiv 0$, and it follows from (2.12) that $x_1 \in (\mathcal{X}_r^{-o})^\perp$ and $x_2 \in (\mathcal{X}_r^{+o})^\perp$. Since the system $\Sigma_{B_r^{-1}}$ is minimal, $x_1 = 0$. If the system Σ_{θ_r} is observable, then $x_2 = 0$, and it follows from Lemma 3.3 that $\Sigma_{\theta_r} \circ \Sigma_{B_r^{-1}}$ is observable and passive, and part (i) is proven.

Next suppose that x_1 and x_2 satisfy (3.20) and

$$B_r^{-1\#}(z)B_2^*(I - zA_2^*)^{-1}x_2 = -B_1^*(I - zA_1^*)^{-1}x_1, \quad \text{for all } z \in \Omega. \tag{3.21}$$

The argument above gives $x_1 = 0$ and $x_2 \in (\mathcal{X}_r^{+o})^\perp$. Then,

$$B_r^{-1\#}(z)B_2^*(I - zA_2^*)^{-1}x_2 \equiv 0.$$

Since $B_r^{-1\#}(z)$ has just the trivial kernel for every $z \in \Omega$, also $B_2^*(I - zA_2^*)^{-1}x_2 \equiv 0$. The identity (2.11) implies now $x_2 \in (\mathcal{X}_r^{+c})^\perp$, and therefore

$$x_2 \in (\mathcal{X}_r^{+c})^\perp \cap (\mathcal{X}_r^{+o})^\perp = (\mathcal{X}_r^{+s})^\perp.$$

If the system Σ_{θ_r} is simple, then $x_2 = 0$, and it follows from Lemma 3.3 that $\Sigma_{\theta_r} \circ \Sigma_{B_r^{-1}}$ is simple and passive, and the first claim of the part (iii) is proven. The other claim in part (iii) and also part (ii) follow now by considering the dual systems. □

The product of the form $\Sigma_{B_l^{-1}} \circ \Sigma_{\theta_l}$ does not necessarily preserve observability as is shown in Example 3.8 below. A counter-example is constructed with the help of the following realization result. For the proof and more details, see [1, Theorem 2.2.1].

Lemma 3.5 *Let $S \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ and let $\mathcal{H}(S)$ be the Pontryagin space induced by the reproducing kernel (1.1). Then the system $\Sigma = (A, B, C, D, \mathcal{H}(S), \mathcal{U}, \mathcal{Y}; \kappa)$ where*

$$\begin{cases} A : h(z) \mapsto \dfrac{h(z) - h(0)}{z}, & B : u \mapsto \dfrac{S(z) - S(0)}{z} u, \\ C : h(z) \mapsto h(0), & D : u \mapsto S(0)u, \end{cases} \tag{3.22}$$

is co-isometric and observable realization of S. Moreover, $C(I - zA)^{-1}h = h(z)$ for $h \in \mathcal{H}(S)$.

The system Σ in Lemma 3.5 is called a **canonical co-isometric realization** of S.

If the systems Σ_1 and Σ_2 in Lemma 3.3 have additional properties, a criterion for observability that does not explicitly depend on a system operator can be obtained.

Theorem 3.6 *Let $\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}_1; \kappa_1)$ and $\Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}; \kappa_2)$ be co-isometric and observable realizations of the functions $S_1 \in \mathbf{S}(\mathcal{U}, \mathcal{Y}_1)$ and $S_2 \in \mathbf{S}(\mathcal{Y}_1, \mathcal{Y})$, respectively. Then $\Sigma = \Sigma_2 \circ \Sigma_1$ is co-isometric observable realization of $S = S_2 S_1$ if and only if the following two conditions hold:*

(i) $\mathcal{H}(S) = S_2\mathcal{H}(S_1) \oplus \mathcal{H}(S_2)$;
(ii) *The mapping $h_1 \mapsto S_2 h_1$ is an isometry from $\mathcal{H}(S_1)$ to $S_2\mathcal{H}(S_1)$.*

Proof Since all co-isometric observable realizations of S_1 and S_2 are unitarily similar, it can be assumed that Σ_1 and Σ_2 are realized as in Lemma 3.5. Let Ω be a neighbourhood of the origin such that S_1 and S_2 are analytic in Ω. By combining Lemma 3.5 and the condition (3.12) in Lemma 3.3, it follows that Σ is observable if and only if

$$S_2(z)h_1(z) = -h_2(z), \quad h_1 \in \mathcal{H}(S_1), \quad h_2 \in \mathcal{H}(S_2), \tag{3.23}$$

holds for every $z \in \Omega$ only when $h_1 \equiv 0$ and $h_2 \equiv 0$.

Assume the conditions (i) and (ii). Then $S_2(z)h_1(z) = -h_2(z)$ can hold only if $h_2 \equiv 0$. Since the mapping $h_1 \mapsto S_2 h_1$ is an isometry, it has only the trivial kernel. Therefore $h_1 \equiv 0$, and sufficiency is proven.

Conversely, assume that Σ is co-isometric and observable. The condition (3.23) shows that the mapping $h_1 \mapsto S_2 h_1$ has only the trivial kernel, and

$$S_2\mathcal{H}(S_1) \cap \mathcal{H}(S_2) = \{0\}. \tag{3.24}$$

It now follows from [1, Theorem 4.1.1] that $\mathcal{H}(S_1)$ and $S_2\mathcal{H}(S_1)$ are contained contractively in $\mathcal{H}(S)$, and $h_1 \mapsto S_2 h_1$ is a partial isometry. Since it has only the trivial kernel, it is an isometry, and (ii) holds. Since (3.24) holds and $\mathcal{H}(S_1)$ and $S_2\mathcal{H}(S_1)$ are contained contractively in $\mathcal{H}(S)$, a result from [1, Theorem 1.5.3] shows that $\mathcal{H}(S_1)$ and $S_2\mathcal{H}(S_1)$ are actually contained isometrically in $\mathcal{H}(S)$. Therefore $\mathcal{H}(S_1)^{\perp} = S_2\mathcal{H}(S_1)$ so the condition (i) holds and the necessity is proven. □

The dual version can be obtained by using the **canonical isometric** realizations from [1, Theorem 2.2.2] or taking adjoint systems in Theorem 3.6.

Theorem 3.7 *Let* $\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}_1; \kappa_1)$ *and* $\Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}; \kappa_2)$ *be isometric and controllable realizations of the functions* $S_1 \in \mathbf{S}(\mathcal{U}, \mathcal{Y}_1)$ *and* $S_2 \in \mathbf{S}(\mathcal{Y}_1, \mathcal{Y})$*, respectively. Then* $\Sigma = \Sigma_2 \circ \Sigma_1$ *is isometric and controllable realization of* $S = S_2 S_1$ *if and only if the following two conditions hold:*

(i) $\mathcal{H}(S^{\#}) = S_1^{\#}\mathcal{H}(S_2^{\#}) \oplus \mathcal{H}(S_1^{\#})$;
(ii) *The mapping* $h_2 \mapsto S_1^{\#}h_2$ *is an isometry from* $\mathcal{H}(S_2^{\#})$ *to* $S_1^{\#}\mathcal{H}(S_2^{\#})$.

In the Hilbert state space settings, a different criterion than in Theorems 3.6 and 3.7 was obtained in [25]. If Σ_1 and Σ_2 are simple conservative, a criterion for $\Sigma = \Sigma_2 \circ \Sigma_1$ to be simple conservative was obtained in the Hilbert state space case in [20] and generalized to the Pontryagin state space case in [11].

Here is the promised counter-example.

Example 3.8 Let $a \in H^{\infty}(\mathbb{D})$ such that $\|a\| \leq 1$ and let $b(z) = (z - \alpha)/(1 - z\bar{\alpha})$ where $\alpha \in \mathbb{D} \setminus \{0\}$. Define

$$S(z) := \frac{1}{\sqrt{2}} \begin{pmatrix} a(z) & \frac{1}{b(z)} \end{pmatrix}, \quad z \in \mathbb{D} \setminus \{\alpha\}. \tag{3.25}$$

Then $S \in \mathbf{S}_1(\mathbb{C}^2, \mathbb{C})$ and it has the left Kreĭn–Langer factorization

$$S(z) = b^{-1}(z) S_l(z) = b^{-1}(z) \begin{pmatrix} \frac{1}{\sqrt{2}} a(z) b(z) & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{3.26}$$

Consider the canonical co-isometric realizations $\Sigma_{b^{-1}}$ and Σ_{S_l} of b^{-1} and S_l, respectively. It follows from Theorem 3.6 that if $\Sigma_{b^{-1}} \circ \Sigma_{S_l}$ is observable, then $\mathcal{H}(S) = b^{-1}\mathcal{H}(S_l) \oplus \mathcal{H}(b^{-1})$. The argument in [1, p. 149] shows that this is false, so $\Sigma_{b^{-1}} \circ \Sigma_{S_l}$ is not observable. By considering the adjoint system one obtains a product of type $\Sigma_{S_r} \circ \Sigma_{B_r^{-1}}$ which is not controllable, while Σ_{S_r} and $\Sigma_{B_r^{-1}}$ are.

The function S in Example 3.8 is taken from [1, p. 149].

If the realization Σ of $\theta = \theta_r B_r^{-1} = B_l^{-1}\theta_l \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ has additional properties, it can be represented as the product of the form $\Sigma_{\theta_r} \circ \Sigma_{B_r^{-1}}$ or $\Sigma_{B_l^{-1}} \circ \Sigma_{\theta_l}$. The following theorem expands the results of [11, Theorem 7.2].

Theorem 3.9 *Let* $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ *and* $\theta = \theta_r B_r^{-1} = B_l^{-1}\theta_l$ *be its Kreĭn-Langer factorizations. Let* $\Sigma_k, k = 1, 2, 3,$ *be the realizations of* θ *which are respectively conservative, co-isometric and isometric such that the negative dimension of the state space in each realization is* κ*. Then:*

(i) *The realization* Σ_1 *can be represented as the products of the form*

$$\Sigma_1 = \Sigma_{\theta_r} \circ \Sigma_{B_r^{-1}} = \Sigma_{B_l^{-1}} \circ \Sigma_{\theta_l},$$

where $\Sigma_{\theta_r} = (T_{\Sigma_{\theta_r}}; \mathcal{X}_r^+, \mathcal{U}, \mathcal{Y}; 0)$ *and* $\Sigma_{\theta_l} = (T_{\Sigma_{\theta_l}}; \mathcal{X}_l^+, \mathcal{U}, \mathcal{Y}; 0)$ *are conservative realizations of the functions* θ_r *and* θ_l*, respectively, and* $\Sigma_{B_r^{-1}} = (T_{\Sigma_{B_r^{-1}}}; \mathcal{X}_r^-, \mathcal{U}, \mathcal{U}; \kappa)$ *and* $\Sigma_{B_l^{-1}} = (T_{\Sigma_{B_l^{-1}}}; \mathcal{X}_l^-, \mathcal{Y}, \mathcal{Y}; \kappa)$ *are conservative and minimal realizations of the functions* B_r^{-1} *and* B_l^{-1}*, respectively.*

(ii) *The realization* Σ_2 *can be represented as the product of the form*

$$\Sigma_2 = \Sigma_{\theta_r} \circ \Sigma_{B_r^{-1}},$$

where $\Sigma_{\theta_r} = (T_{\Sigma_{\theta_r}}; \mathcal{X}^+, \mathcal{U}, \mathcal{Y}; 0)$ *is a co-isometric realization of the function* θ_r *and* $\Sigma_{B_r^{-1}} = (T_{\Sigma_{B_r^{-1}}}; \mathcal{X}^-, \mathcal{U}, \mathcal{U}; \kappa)$ *is a conservative minimal realization of* B_r^{-1}.

(iii) *The realization* Σ_3 *can be represented as the product of the form*

$$\Sigma_3 = \Sigma_{B_l^{-1}} \circ \Sigma_{\theta_l},$$

where $\Sigma_{\theta_l} = (T_{\Sigma_{\theta_l}}; \mathcal{X}^+, \mathcal{U}, \mathcal{Y}; 0)$ *is an isometric realization of the function* θ_l *and* $\Sigma_{B_l^{-1}} = (T_{\Sigma_{B_l^{-1}}}; \mathcal{X}^-, \mathcal{Y}, \mathcal{Y}; \kappa)$ *is a conservative minimal realization of* B_l^{-1}.

Proof The theorem will be proved in two steps. In the first step, it is assumed that Σ_1 is simple, Σ_2 is observable and Σ_3 is controllable. In the second step, the general case will be proved by using the results from the first step.

Step 1 (i) This is stated essentially in [11, Theorem 7.2] but without proof. According to [21, Theorem 4.4], $\Sigma_1 = (T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ can be represented as the products of the form

$$\Sigma_1 = \Sigma_{r2} \circ \Sigma_{r1} = \Sigma_{l2} \circ \Sigma_{l1}$$

such that

$$\begin{aligned} \Sigma_{r1} &= (T_{\Sigma_{r1}}, \mathcal{X}_r^-, \mathcal{U}, \mathcal{U}, \kappa), \qquad \Sigma_{r2} = (T_{\Sigma_{r2}}, \mathcal{X}_r^+, \mathcal{U}, \mathcal{Y}, 0), \\ \Sigma_{l1} &= (T_{\Sigma_{l1}}, \mathcal{X}_l^+, \mathcal{U}, \mathcal{Y}, 0), \qquad \Sigma_{l2} = (T_{\Sigma_{l2}}, \mathcal{X}_l^-, \mathcal{Y}, \mathcal{Y}, \kappa), \end{aligned} \tag{3.27}$$

where $\mathcal{X}_r^-$ and $\mathcal{X}_l^-$ are κ-dimensional anti-Hilbert spaces. Subscripts refer "right" and "left", because it will be proved that the factorizations

$$\theta = \theta_{\Sigma_{r2}} \theta_{\Sigma_{r1}} = \theta_{\Sigma_{l2}} \theta_{\Sigma_{l1}}$$

of the transfer function θ of Σ_1 corresponding to the product representations above are actually Kreĭn-Langer factorizations. Since all the realizations in (3.27) are simple and conservative, it follows from Lemma 3.2 that $\theta_{\Sigma_{r2}}, \theta_{\Sigma_{l1}} \in \mathbf{S}(\mathcal{U}, \mathcal{Y}), \theta_{\Sigma_{r1}} \mathbf{S}_\kappa(\mathcal{U}), \theta_{\Sigma_{l2}} \in \mathbf{S}_\kappa(\mathcal{Y})$, and the spaces

$$\mathcal{X}_r^- \ominus \mathcal{X}_r^{-c}, \qquad \mathcal{X}_r^- \ominus \mathcal{X}_r^{-o}, \qquad \mathcal{X}_l^- \ominus \mathcal{X}_l^{-c}, \qquad \mathcal{X}_r^- \ominus \mathcal{X}_l^{-o}, \tag{3.28}$$

are Hilbert spaces. But since the state spaces $\mathcal{X}_r^-$ and $\mathcal{X}_l^-$ are anti-Hilbert spaces, all the spaces in (3.28) must be the zero spaces. Thus Σ_{r1} and Σ_{l2} are minimal. By using the unitary similarity introduced on page 7 it can be deduced now that all co-isometric observable realizations of θ_{r2} and θ_{l1} are conservative and minimal, and then it follows from [1, Theorem A3] that θ_{r2} and θ_{l1} are inverse Blaschke products, which gives the result.

(ii) It is known (cf. e.g. [1, Theorem 2.4.1]) that the co-isometric and observable realization $\Sigma_2 = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ of the function θ has a simple and conservative dilation $\widehat{\Sigma}_2 = (\widehat{A}, \widehat{B}, \widehat{C}, D; \widehat{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; \kappa)$ such that

$$T_{\widehat{\Sigma}_2} = \begin{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ 0 & A \end{pmatrix} & \begin{pmatrix} B_1 \\ B \end{pmatrix} \\ \begin{pmatrix} 0 & C \end{pmatrix} & D \end{pmatrix} : \begin{pmatrix} \begin{pmatrix} \mathcal{X}_0 \\ \mathcal{X} \end{pmatrix} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} \mathcal{X}_0 \\ \mathcal{X} \end{pmatrix} \\ \mathcal{Y} \end{pmatrix}, \tag{3.29}$$

where $\mathcal{X}_0$ is a Hilbert space. By [11, Theorem 7.7], there exist unique fundamental decompositions $\mathcal{X} = \mathcal{X}^+ \oplus \mathcal{X}^-$ and $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}^+ \oplus \widehat{\mathcal{X}}^-$ such that $A\mathcal{X}^+ \subset \mathcal{X}^+$ and $\widehat{A}\widehat{\mathcal{X}}^+ \subset \widehat{\mathcal{X}}^+$. Then $(\mathcal{X}_0 \oplus \mathcal{X}^+) \oplus \mathcal{X}^-$ is a fundamental decomposition of $\widehat{\mathcal{X}}$, and for $x_0 \in \mathcal{X}_0$ and $x_+ \in \mathcal{X}^+$

$$\widehat{A}(x_0 \oplus x_+) = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A \end{pmatrix} \begin{pmatrix} x_0 \\ x_+ \end{pmatrix} = \begin{pmatrix} A_{11}x_0 + A_{12}x_+ \\ Ax_+ \end{pmatrix} \in \begin{pmatrix} \mathcal{X}_0 \\ \mathcal{X}^+ \end{pmatrix}. \tag{3.30}$$

This yields $\widehat{\mathcal{X}}^+ = \mathcal{X}_0 \oplus \mathcal{X}^+$ and $\widehat{\mathcal{X}}^- = \mathcal{X}_2^-$. Part (i) shows that $\widehat{\Sigma}_2$ can be represented as $\widehat{\Sigma}_2 = \widehat{\Sigma}_{\theta_r} \circ \widehat{\Sigma}_{B_r^{-1}}$. The transfer functions of the components are θ_r and B_r^{-1}, respectively, and $\widehat{\Sigma}_{\theta_r}$ is simple and conservative and $\widehat{\Sigma}_{B_r^{-1}}$ is conservative and minimal. It follows from [11, Theorem 7.7] that the state spaces of $\widehat{\Sigma}_{\theta_r}$ and $\widehat{\Sigma}_{B_r^{-1}}$ are $\widehat{\mathcal{X}}^+$ and $\mathcal{X}^-$, respectively. Thus

$$\widehat{\Sigma}_{B_r^{-1}} = (A_1, B_1, C_1, D_1; \mathcal{X}^-, \mathcal{U}, \mathcal{U}; \kappa), \qquad \widehat{\Sigma}_{\theta_r} = (A_2, B_2, C_2, D_2; \widehat{\mathcal{X}}^+, \mathcal{U}, \mathcal{Y}; 0).$$

Now the representation $\widehat{\Sigma}_{\theta_r} \circ \widehat{\Sigma}_{B_r^{-1}}$, Eq. (3.11) and the representation (3.29) yield

$$T_{\widehat{\Sigma}_2} = \begin{pmatrix} I_{\mathcal{X}^-} & 0 & 0 \\ 0 & A_2 & B_2 \\ 0 & C_2 & D_2 \end{pmatrix} \begin{pmatrix} A_1 & 0 & B_1 \\ 0 & I_{\widehat{\mathcal{X}}^+} & 0 \\ C_1 & 0 & D_1 \end{pmatrix} : \begin{pmatrix} \mathcal{X}^- \\ \widehat{\mathcal{X}}^+ \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \mathcal{X}^- \\ \widehat{\mathcal{X}}^+ \\ \mathcal{Y} \end{pmatrix} \iff$$

$$T_{\widehat{\Sigma}_2} = \begin{pmatrix} I_{\mathcal{X}^-} & 0 & 0 & 0 \\ 0 & P_{\mathcal{X}_0} A_2 \restriction_{\mathcal{X}_0} & P_{\mathcal{X}_0} A_2 \restriction_{\mathcal{X}^+} & P_{\mathcal{X}_0} B_2 \\ 0 & P_{\mathcal{X}^+} A_2 \restriction_{\mathcal{X}_0} & P_{\mathcal{X}^+} A_2 \restriction_{\mathcal{X}^+} & P_{\mathcal{X}^+} B_2 \\ 0 & 0 & C_2 & D_2 \end{pmatrix} \begin{pmatrix} A_1 & 0 & 0 & B_1 \\ 0 & I_{\mathcal{X}^0} & 0 & 0 \\ 0 & 0 & I_{\mathcal{X}^+} & 0 \\ C_1 & 0 & 0 & D_1 \end{pmatrix} : \begin{pmatrix} \mathcal{X}_2^- \\ \mathcal{X}_0 \\ \mathcal{X}^+ \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \mathcal{X}_2^- \\ \mathcal{X}_0 \\ \mathcal{X}^+ \\ \mathcal{Y} \end{pmatrix}.$$

By using the representation above and (3.2)–(3.5), it follows that

$$\operatorname{res}_{\widehat{\mathcal{X}} \to \mathcal{X}} \widehat{\Sigma}_2 = \operatorname{res}_{\widehat{\mathcal{X}} \to \mathcal{X}} (\widehat{\Sigma}_{\theta_r} \circ \widehat{\Sigma}_{B_r^{-1}}) = \left(\operatorname{res}_{\widehat{\mathcal{X}}^+ \to \mathcal{X}^+} \widehat{\Sigma}_{\theta_r}\right) \circ \widehat{\Sigma}_{B_r^{-1}} = \Sigma_2.$$

Define $\widehat{\Sigma}_{B_r^{-1}} := \Sigma_{B_r^{-1}}$ and $\operatorname{res}_{\widehat{\mathcal{X}}^+ \to \mathcal{X}^+} \widehat{\Sigma}_{\theta_r} := \Sigma_{\theta_r}$. Since Σ_2 is co-isometric and observable and $\Sigma_{B_r^{-1}}$ is minimal and conservative, Σ_{θ_r} must be co-isometric and observable. That is, $\Sigma_2 = \Sigma_{\theta_r} \circ \Sigma_{B_r^{-1}}$ is the desired representation.

(iii) This can be done by using [1, Theorem 2.4.3] and then proceeding along the lines of the proof of (ii).

Step 2. (i) Denote $\Sigma_1 = (A, B, C, D; \mathcal{X}; \mathcal{U}, \mathcal{Y}; \kappa)$. Since the index of the transfer function θ coincides with the negative index of $\mathcal{X}$, Lemma 3.2 shows that $(\mathcal{X}^s)^\perp$ is a Hilbert space. It easily follows from (2.10) that $C(\mathcal{X}^s)^\perp = \{0\}$, $B^*(\mathcal{X}^s)^\perp = \{0\}$, $A\mathcal{X}^s \subset \mathcal{X}^s$ and $A(\mathcal{X}^s)^\perp \subset (\mathcal{X}^s)^\perp$. This implies that the system operator has the representation

$$T_{\Sigma_1} = \left(\begin{pmatrix} A_1 & 0 \\ 0 & A_0 \\ \end{pmatrix} \begin{pmatrix} 0 \\ B_0 \end{pmatrix} \atop \begin{pmatrix} 0 & C_0 \end{pmatrix} \quad D \right) : \left(\begin{pmatrix} (\mathcal{X}^s)^\perp \\ \mathcal{X}^s \end{pmatrix} \atop \mathcal{U} \right) \to \left(\begin{pmatrix} (\mathcal{X}^s)^\perp \\ \mathcal{X}^s \end{pmatrix} \atop \mathcal{Y} \right). \tag{3.31}$$

Easy calculations show that the restriction

$$\operatorname{res}_{\mathcal{X} \to \mathcal{X}^s} \Sigma_1 = (A_0, B_0, C_0, D; \mathcal{X}^s, \mathcal{U}, \mathcal{Y}; \kappa) := \Sigma_0$$

is conservative and simple. Step 1 (i) shows that $\Sigma_0 = \Sigma_{\theta_r} \circ \Sigma_{B_r^{-1}} = \Sigma_{B_l^{-1}} \circ \Sigma_{\theta_l}$, where

$$\Sigma_{\theta_r} = (T_{\Sigma_{\theta_r}}; \mathcal{X}_r^{s+}, \mathcal{U}, \mathcal{Y}; 0), \qquad \Sigma_{B_r^{-1}} = (T_{\Sigma_{B_r^{-1}}}; \mathcal{X}_r^{s-}, \mathcal{U}, \mathcal{U}; \kappa),$$
$$\Sigma_{\theta_l} = (T_{\Sigma_{\theta_l}}; \mathcal{X}_l^{s+}, \mathcal{U}, \mathcal{Y}; 0), \qquad \Sigma_{B_l^{-1}} = (T_{\Sigma_{B_l^{-1}}}; \mathcal{X}_l^{s-}, \mathcal{Y}, \mathcal{Y}; \kappa).$$

The spaces $\mathcal{X}_r^{s-}$ and $\mathcal{X}_l^{s-}$ are κ-dimensional anti-Hilbert spaces, Σ_{θ_r} and Σ_{θ_l} are conservative and simple and $\Sigma_{B_r^{-1}}$ and $\Sigma_{B_l^{-1}}$ are conservative and minimal. It can be now deduced that $\mathcal{X}$ has the fundamental decompositions $((\mathcal{X}^s)^\perp \oplus \mathcal{X}_r^{s+}) \oplus \mathcal{X}_r^{s-}$ and $((\mathcal{X}^s)^\perp \oplus \mathcal{X}_l^{s+}) \oplus \mathcal{X}_l^{s-}$. Moreover,

$$A((\mathcal{X}_r^s)^\perp \oplus \mathcal{X}_r^{s+}) \subset (\mathcal{X}_r^s)^\perp \oplus \mathcal{X}_r^{s+}, \quad A\mathcal{X}_l^{s-} \subset \mathcal{X}_l^{s-}.$$

Similar calculations as in the proof of Step 1 (ii) show that

$$\text{dil } \Sigma_0 = \left(\text{dil } \Sigma_{\theta_r}\right) \circ \Sigma_{B_r^{-1}} = \Sigma_{B_l^{-1}} \circ \left(\text{dil } \Sigma_{\theta_l}\right) = \Sigma_1.$$

Since Σ_1, $\Sigma_{B_l^{-1}}$ and $\Sigma_{B_r^{-1}}$ are conservative, dil Σ_{θ_r} and dil Σ_{θ_l} must be conservative. Moreover, the state spaces $(\mathcal{X}^s)^\perp \oplus \mathcal{X}_r^{s+}$ and $(\mathcal{X}^s)^\perp \oplus \mathcal{X}_l^{s+}$ of dil Σ_{θ_r} and dil Σ_{θ_l}, respectively, are Hilbert spaces. That is, $\Sigma_1 = \left(\text{dil } \Sigma_{\theta_r}\right) \circ \Sigma_{B_r^{-1}}$ and $\Sigma_1 = \Sigma_{B_l^{-1}} \circ \left(\text{dil } \Sigma_{\theta_l}\right)$ are the desired representations.

(ii) Denote $\Sigma_2 = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$. Lemma 3.2 show that $(\mathcal{X}^o)^\perp$ is a Hilbert space. From the identity (2.9) it follows easily that $A(\mathcal{X}^o)^\perp \subset (\mathcal{X}^o)^\perp$ and $C(\mathcal{X}^o)^\perp = \{0\}$. This implies that the system operator can be represented as

$$T_{\Sigma_2} = \begin{pmatrix} A_1 & A_2 & B_1 \\ 0 & A_0 & B_0 \\ 0 & C_0 & D \end{pmatrix} : \begin{pmatrix} (\mathcal{X}_2^o)^\perp \\ \mathcal{X}_2^o \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} (\mathcal{X}_2^o)^\perp \\ \mathcal{X}_2^o \\ \mathcal{Y} \end{pmatrix}. \tag{3.32}$$

Moreover, the restriction

$$\mathrm{res}_{\mathcal{X}_2 \to \mathcal{X}_2^o} \Sigma_2 = (A_0, B_0, C_0, D; \mathcal{X}_1^o, \mathcal{U}, \mathcal{Y}; \kappa) := \Sigma_0$$

is co-isometric and observable. Step 1 (ii) shows that Σ_0 has the representation $\Sigma_0 = \Sigma_{\theta_r} \circ \Sigma_{B_r^{-1}}$ such that the components

$$\Sigma_{\theta_r} = (T_{\Sigma_{\theta_r}}, \mathcal{X}^{o+}, \mathcal{U}, \mathcal{Y}, 0), \quad \Sigma_{B_r^{-1}} = (T_{\Sigma_{B_r^{-1}}}, \mathcal{X}^{o-}, \mathcal{U}, \mathcal{U}, \kappa)$$

have the properties introduced in Part 1 (ii). The final statement is obtained by proceeding as in the proof of (i).

(iii) The proof is similar to the proofs of (i) and (ii) and hence the details are omitted. □

Proposition 3.10 *Suppose that $A \in \mathcal{L}(\chi)$ is the main operator of a passive system $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ such that the index of the transfer function of Σ is κ. Then there exist unique fundamental decompositions $\mathcal{X} = \mathcal{X}_1^+ \oplus \mathcal{X}_1^- = \mathcal{X}_2^+ \oplus \mathcal{X}_2^-$ such that $A\mathcal{X}_1^+ \subset \mathcal{X}_1^+$ and $A\mathcal{X}_2^- \subset \mathcal{X}_2^-$, respectively.*

Proof Embed the system Σ in a conservative system $\widetilde{\Sigma} = (T_{\widetilde{\Sigma}}, \mathcal{X}, \widetilde{\mathcal{U}}, \widetilde{\mathcal{Y}}, \kappa)$ without changing the main operator and the state space. Now the first identity in (3.7) shows that the transfer function $\theta_{\widetilde{\Sigma}}$ of $\widetilde{\Sigma}$ has the same amount of poles (counting multiplicities) as the transfer function of the original system. Hence it follows from Lemma 2.2 that the index of $\theta_{\widetilde{\Sigma}}$ is κ. The representations in Theorem 3.9 (i) combined with the decomposition of the main operator A in (3.7) give the claimed fundamental decompositions. The decomposition $\mathcal{X}_1^+ \oplus \mathcal{X}_1^-$ corresponds to the one induced by the product representation $\widetilde{\Sigma} = \Sigma_{\widetilde{\theta}_r} \circ \Sigma_{\widetilde{B}_r^{-1}}$, where $\widetilde{\theta} = \widetilde{\theta}_r \widetilde{B}_r^{-1}$ is the right Kreĭn-Langer factorization of $\widetilde{\theta}$. Similarly, the decomposition $\mathcal{X}_2^+ \oplus \mathcal{X}_2^-$ corresponds to the one induced by the product representation $\widetilde{\Sigma} = \Sigma_{\widetilde{B}_l^{-1}} \circ \Sigma_{\widetilde{\theta}_l}$, where $\widetilde{\theta} = \widetilde{B}_l^{-1}\widetilde{\theta}_l$ is the left Kreĭn-Langer factorization of $\widetilde{\theta}$.

To prove the uniqueness, the fact that A has no negative eigenvector with corresponding eigenvalue modulus one is needed. To this end, assume that $Ax = \lambda x$ for some $x \in \mathcal{X}$ and $\lambda \in \mathbb{T}$. Consider again a conservative embedding $\widetilde{\Sigma}$ of Σ, and represent $\widetilde{\Sigma}$ as in (3.6) . Then,

$$\begin{pmatrix} A & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda x \\ \widetilde{C} x \end{pmatrix}.$$

Since $\widetilde{\Sigma}$ is conservative, the system operator $T_{\widetilde{\Sigma}}$ of $\widetilde{\Sigma}$ is unitary. Therefore $\langle x, x \rangle_{\mathcal{X}} = \langle \lambda x, \lambda x \rangle_{\mathcal{X}} + \langle \widetilde{C}x, \widetilde{C}x \rangle_{\widetilde{\mathcal{Y}}}$, and since $\widetilde{\mathcal{Y}}$ is a Hilbert space and $|\lambda| = 1$, it must be $\widetilde{C}x = 0$. Then, $\widetilde{C}A^n x = \lambda^n \widetilde{C}x = 0$ for any $n \in \mathbb{N}_0$. That is, $x \in (\widetilde{\mathcal{X}}^o)^\perp$, where $\widetilde{\mathcal{X}}^o$

is the observable subspace of the system $\widetilde{\Sigma}$. Since the index of $\widetilde{\theta}$ is κ, the subspace $(\widetilde{\mathcal{X}}^o)^\perp$ is a Hilbert space by Lemma 3.2, and x must be non-negative.

Suppose now that $\mathcal{X}^+ \oplus \mathcal{X}^-$ is some other fundamental decomposition of $\mathcal{X}$ such that $A\mathcal{X}^- \subset \mathcal{X}^-$. It will be shown that $\mathcal{X}^- \subset \mathcal{X}_2^-$, since then $\mathcal{X}^- = \mathcal{X}_2^-$ because these subspaces have the same finite dimension, and thus $\mathcal{X}^+ \oplus \mathcal{X}^-$ is equal to $\mathcal{X}_2^+ \oplus \mathcal{X}_2^-$. It suffices to show that $\mathcal{X}_2^-$ contains all generalized eigenvectors of $A\restriction_{\mathcal{X}^-}$. Let x be a non-zero vector in $\mathcal{X}^-$ such that $(A - \lambda I)^n x = 0$ for some $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$. Since $\mathcal{X}_2^-$ is an anti-Hilbert space and $A\restriction_{\mathcal{X}^-}$ is a contraction, $|\lambda| \geq 1$. The fact proved above gives now $|\lambda| > 1$. Represent the vector x in the form $x = x^+ + x^-$, where $x^\pm \in \mathcal{X}_2^\pm$. Since $A\mathcal{X}_2^- \subset \mathcal{X}_2^-$, the operator A has a block representation

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{pmatrix} : \begin{pmatrix} \mathcal{X}_2^+ \\ \mathcal{X}_2^- \end{pmatrix} \to \begin{pmatrix} \mathcal{X}_2^+ \\ \mathcal{X}_2^- \end{pmatrix}.$$

Since A^* is also a contraction, A_{11}^* is a Hilbert space contraction, and therefore A_{11} must be a contraction. Now

$$(A - \lambda I)^n x = \begin{pmatrix} (A_{11} - \lambda I_{\mathcal{X}_2^+})^n & 0 \\ f(n) & (A_{22} - \lambda I_{\mathcal{X}_2^-})^n \end{pmatrix} \begin{pmatrix} x^+ \\ x^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $f(n)$ is an operator depending on n. This implies $(A_{11} - \lambda I_{\mathcal{X}_2^+})^n x^+ = 0$, but since A_{11} is a Hilbert space contraction and $|\lambda| > 1$, it must be $x^+ = 0$. Hence $x = x^- \in \mathcal{X}_2^-$, an the uniqueness of the decomposition $\mathcal{X} = \mathcal{X}_2^+ \oplus \mathcal{X}_2^-$ is proved. The uniqueness of the decomposition $\mathcal{X} = \mathcal{X}_1^+ \oplus \mathcal{X}_1^-$ can be proved by using the fact $A^*\mathcal{X}_1^- \subset \mathcal{X}_1^-$, and then proceeding as above. □

Proposition 3.10 is a generalization of [11, Theorem 7.7] in a sense that the condition that the system is simple can be relaxed. As proved, it suffices that the orthocomplement $(\mathcal{X}^s)^\perp$ of the simple subspace is a Hilbert space, see Lemma 3.2. The proof of Proposition 3.10 follows the lines of the proof of [11, Theorem 7.7].

The results of Theorem 3.9 (i) cannot be extended to isometric or co-isometric systems as the next example shows.

Example 3.11 Let S be as in Example 3.8 and let Σ be any co-isometric observable realization of S. Suppose that $\Sigma = \Sigma'_{b^{-1}} \circ \Sigma'_{S_l}$ for some co-isometric observable realizations of b^{-1} and S_l. Then the realizations $\Sigma'_{b^{-1}}$ and Σ'_{S_l} are unitarily similar, respectively, with the canonical co-isometric observable realizations $\Sigma_{b^{-1}}$ of b^{-1} and Σ_{S_l} of S_l. An easy calculation shows that $\Sigma'_{b^{-1}} \circ \Sigma'_{S_l}$ is unitarily similar with $\Sigma_{b^{-1}} \circ \Sigma_{S_l}$, which is a contradiction since $\Sigma_{b^{-1}} \circ \Sigma_{S_l}$ is not observable by Example 3.8. Thus Σ cannot be represented as a product of the form $\Sigma'_{b^{-1}} \circ \Sigma'_{S_l}$.

4 Stable Systems and Zero Defect Functions

A contraction $A \in \mathcal{L}(\mathcal{X})$, where $\mathcal{X}$ is a Hilbert space, belongs to the classes $C_{0\cdot}$ or $C_{\cdot 0}$ if, respectively, $\lim_{n\to\infty} A^n x = 0$ or $\lim_{n\to\infty} A^{*n} x = 0$ for every $x \in \mathcal{X}$.

The class C_{00} is defined to be $C_{0\cdot} \cap C_{\cdot 0}$. A system with a Hilbert state space is said to be **strongly stable** (**strongly co-stable**, **strongly bi-stable**) if the main operator of the system belongs to $C_{0\cdot}$ ($C_{\cdot 0}$, C_{00}). When the state space $\mathcal{X}$ is a Pontryagin space, stability cannot be defined verbatim, because for any contractive $A \in \mathcal{L}(\mathcal{X})$, the equality $\lim_{n\to\infty} A^n x = 0$ does not hold for any negative vector x. The stability property can therefore hold only in certain Hilbert subspaces. The following definition of stability generalizes and expands [11, Definition 9.1].

Definition 4.1 Let $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ be a passive system with the main operator A such that $\theta_\Sigma \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$. Let $\mathcal{X} = \mathcal{X}_1^+ \oplus \mathcal{X}_1^- = \mathcal{X}_2^+ \oplus \mathcal{X}_2^-$ be the unique fundamental decompositions of $\mathcal{X}$ introduced in Proposition 3.10 such that $A\mathcal{X}_1^+ \subset \mathcal{X}_1^+$ and $A\mathcal{X}_2^- \subset \mathcal{X}_2^-$. Then:

(i) Σ belongs to class $\mathbf{P}^\kappa_{0\cdot}$ if $A\restriction_{\mathcal{X}_1^+} \in C_{0\cdot}$;
(ii) Σ belongs to class $\mathbf{P}^\kappa_{\cdot 0}$ if $A^*\restriction_{\mathcal{X}_2^+} \in C_{0\cdot}$;
(iii) Σ belongs to class $\mathbf{P}^\kappa_{00}$ if $A\restriction_{\mathcal{X}_1^+} \in C_{00}$;
(iv) Σ belongs to class $\mathbf{C}^\kappa_{0\cdot}$ if Σ is simple conservative and $\Sigma \in \mathbf{P}^\kappa_{0\cdot}$;
(v) Σ belongs to class $\mathbf{C}^\kappa_{\cdot 0}$ if Σ is simple conservative and $\Sigma \in \mathbf{P}^\kappa_{\cdot 0}$;
(vi) Σ belongs to class $\mathbf{C}^\kappa_{00}$ if Σ is simple conservative and $\Sigma \in \mathbf{P}^\kappa_{00}$;
(vi) Σ belongs to class $\mathbf{I}^\kappa_{0\cdot}$ if Σ is controllable isometric and $\Sigma \in \mathbf{P}^\kappa_{0\cdot}$;
(vii) Σ belongs to class $\mathbf{I}^{*\kappa}_{\cdot 0}$ if Σ is observable co-isometric and $\Sigma \in \mathbf{P}^\kappa_{\cdot 0}$;

The classes $\mathbf{P}^\kappa_{00}$ and $\mathbf{C}^\kappa_{00}$ are defined in [11, Definition 9.1], as well as the class $\mathbf{P}^\kappa_{00}$ with the additional condition that Σ must be simple. It will be shown later that the realizations in the classes $\mathbf{C}^\kappa_{00}$, $\mathbf{I}^\kappa_{0\cdot}$ and $\mathbf{I}^{*\kappa}_{\cdot 0}$ are minimal, the realizations in $\mathbf{C}^\kappa_{0\cdot}$ are observable and the realizations in $\mathbf{C}^\kappa_{\cdot 0}$ are controllable.

Theorem 4.2 *A simple conservative system* $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ *belongs to*

(i) $\mathbf{C}^\kappa_{0\cdot}$ *if and only if* θ_Σ *has isometric boundary values a.e.;*
(ii) $\mathbf{C}^\kappa_{\cdot 0}$ *if and only if* θ_Σ *has co-isometric boundary values a.e.;*
(iii) $\mathbf{C}^\kappa_{00}$ *if and only if* θ_Σ *has unitary boundary values a.e.*

In the Hilbert state space case, i.e. $\kappa = 0$, the result is known and goes back essentially to [32]. For $\kappa > 0$, part (iii) is first proved in [11, Theorem 9.2].

Proof Since the results hold for $\kappa = 0$, it suffices to prove the them in case $\kappa > 0$. Consider the representations $\Sigma = \Sigma_{\theta_r} \circ \Sigma_{B_r^{-1}} = \Sigma_{B_l^{-1}} \circ \Sigma_{\theta_l}$ as in Theorem 3.9. Now the results follow by observing that the main operator of Σ_{θ_r} is $A\restriction_{\mathcal{X}_1^+}$ and the main operator of $\Sigma^*_{\theta_l}$ is $A^*\restriction_{\mathcal{X}_2^+}$, and then using the case $\kappa = 0$. □

In Sect. 2, the notions of defect functions were introduced. If the right or the left defect function of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ is identically equal to zero, the realizations of θ have some strong structural properties.

Lemma 4.3 *For a simple conservative system* $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ *with the transfer function* $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$*, the following statements hold:*

(i) Σ *is controllable if and only if* $\psi_\theta \equiv 0$*;*
(ii) Σ *is observable if and only if* $\varphi_\theta \equiv 0$*;*
(iii) Σ *is minimal if and only if* $\psi_\theta \equiv 0$ *and* $\varphi_\theta \equiv 0$.

Proof For the case $\kappa = 0$, see [3, Corollary 6.4] or [10, Theorem 1]. For $\kappa > 0$, consider the representations $\Sigma = \Sigma_{\theta_r} \circ \Sigma_{B_r^{-1}} = \Sigma_{B_l^{-1}} \circ \Sigma_{\theta_l}$ as in Theorem 3.9. If Σ is controllable, then so is Σ_{θ_l} and from case $\kappa = 0$ it follows that $\psi_{\theta_l} \equiv 0$. Now the identity (2.7) implies that $\psi_\theta \equiv 0$. Conversely, if $\psi_\theta \equiv 0$, the identity (2.7) shows that also $\psi_{\theta_l} \equiv 0$, and from the case $\kappa = 0$ it follows that Σ_{θ_l} is controllable. By Theorem 3.9 (i), $\Sigma_{B_l^{-1}}$ is minimal. Then it follows from Theorem 3.4 that $\Sigma = \Sigma_{B_l^{-1}} \circ \Sigma_{\theta_l}$ is controllable, and part (i) is proven. Proof of part (ii) is similar, and part (iii) follows by combining (i) and (ii). □

The following theorem in the Hilbert state space case was obtained in [3, Theorem 1.1]. The proof therein was based on the block parametrization of the system operator. The proof given here for the general case is based on the existence of minimal passive realizations. It also uses some techniques appearing in the proof of [6, Theorem 1] and, in addition, implements the product representations provided in Theorem 3.9.

Theorem 4.4 *Let* $\Sigma = (A, B, C, D, \mathcal{X}, \mathcal{U}, \mathcal{Y}, \kappa)$ *be a passive system with the transfer function* θ*. Then:*

(i) *If* Σ *is controllable and* $\varphi_\theta \equiv 0$*, then* Σ *is isometric and minimal. Moreover, if* θ *has isometric boundary values a.e., then* $\Sigma \in \mathbf{I}^{\kappa}_{0\cdot}$.
(ii) *If* Σ *is observable and* $\psi_\theta \equiv 0$*, then* Σ *is co-isometric and minimal. Moreover, if* θ *has co-isometric boundary values a.e., then* $\Sigma \in \mathbf{I}^{*\kappa}_{\cdot 0}$.
(iii) *If* Σ *is simple and* $\varphi_\theta \equiv 0$ *and* $\psi_\theta \equiv 0$*, then* Σ *is conservative and minimal. Moreover, if* θ *has unitary boundary values a.e., then* $\Sigma \in \mathbf{C}^{\kappa}_{00}$.

Proof (i) Denote the system operator of Σ by T, and consider the Julia embedding $\widetilde{\Sigma}$ of the system Σ. This means that the corresponding system operator is a unitary operator of the form

$$T_{\widetilde{\Sigma}} = \begin{pmatrix} A & \begin{pmatrix} B & D_{T^*_{,1}} \end{pmatrix} \\ \begin{pmatrix} C \\ D^*_{T_{,1}} \end{pmatrix} & \begin{pmatrix} D & D_{T^*_{,2}} \\ D^*_{T_{,2}} & -L^* \end{pmatrix} \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \begin{pmatrix} \mathcal{U} \\ \mathfrak{D}_{T^*} \end{pmatrix} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \begin{pmatrix} \mathcal{Y} \\ \mathfrak{D}_{T} \end{pmatrix} \end{pmatrix}, \tag{4.1}$$

where

$$D_{T^*} = \begin{pmatrix} D_{T^*_{,1}} \\ D_{T^*_{,2}} \end{pmatrix}, \quad D_T = \begin{pmatrix} D_{T_{,1}} \\ D_{T_{,2}} \end{pmatrix}, \quad D_{T^*} D^*_{T^*} = I_{\mathcal{X}} - TT^*, \quad D_T D^*_T = I_{\mathcal{X}} - T^*T,$$

such that D_T and D_{T^*} have zero kernels. The transfer function of the embedded system is

$$\theta_{\widetilde{\Sigma}}(z) = \begin{pmatrix} D + zC(I - zA)^{-1}B & D_{T^*_{,2}} + zC(I - zA)^{-1}D_{T^*_{,1}} \\ D^*_{T,2} + zD^*_{T,1}(I - zA)^{-1}B & -L^* + zD^*_{T,1}(I - zA)^{-1}D_{T^*_{,1}} \end{pmatrix}$$
$$= \begin{pmatrix} \theta(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{pmatrix}.$$

Notice that $\theta, \theta_{12}, \theta_{21}$ and θ_{22} all are generalized Schur functions. Because $I - \theta_{\widetilde{\Sigma}}(\zeta)\theta^*_{\widetilde{\Sigma}}(\zeta) \geq 0$ and $I - \theta^*_{\widetilde{\Sigma}}(\zeta)\theta_{\widetilde{\Sigma}}(\zeta) \geq 0$ a.e. on $\zeta \in \mathbb{T}$, one concludes that

$$I - \theta^*(\zeta)\theta(\zeta) \geq \theta^*_{21}(\zeta)\theta_{21}(\zeta); \tag{4.2}$$
$$I - \theta(\zeta)\theta^*(\zeta) \geq \theta_{12}(\zeta)\theta^*_{12}(\zeta). \tag{4.3}$$

Since $\varphi_\theta \equiv 0$, it follows from the identity (4.2) and Theorem 2.3 that $\theta_{21} \equiv 0$. Then $D^*_{T,2} = 0$ and $D^*_{T,1}(I - zA)^{-1}B = 0$ for every z in some neighbourhood of the origin. Since Σ is controllable, it follows from (2.11) that $D^*_{T,1} = 0$ and then $D_T = 0$, which means that Σ is isometric.

If Σ is chosen to be minimal passive, the previous argument shows that Σ is an isometric and minimal realization of θ. Since the controllable isometric realizations of θ are unitarily similar, they all are now also minimal. This proves the first statement in (i).

If θ has isometric boundary values a.e., then θ_l in the left Kreĭn-Langer factorization of θ is inner. Consider the product $\Sigma = \Sigma_{B^{-1}} \circ \Sigma_{\theta_l}$ as in the Theorem 3.9. Let $\mathcal{X}_1^+ \oplus \mathcal{X}_1^-$ and $\mathcal{X}_2^+ \oplus \mathcal{X}_2^-$ be the unique fundamental decompositions of $\mathcal{X}$ of, given by Proposition 3.10, such that $A\mathcal{X}_1^+ \subset \mathcal{X}_1^+$ and $A^*\mathcal{X}_2^+ \subset \mathcal{X}_2^+$. The case $\kappa = 0$ from [3, Theorem 1.1] shows that the main operator of Σ_{θ_l} belongs to $C_{0\cdot}$, and then the main operator of $\Sigma^*_{\theta_l}$, which is $A^*\restriction_{\mathcal{X}_2^+}$, belongs to $C_{\cdot 0}$. It suffices to show that this is equivalent to $A\restriction_{\mathcal{X}_1^+} \in C_{0\cdot}$. Consider a simple conservative embedding $\widetilde{\Sigma}$ of Σ. Represent $\widetilde{\Sigma}$ as in the products $\widetilde{\Sigma} = \Sigma_{\theta'_r} \circ \Sigma_{B_r^{-1\prime}} = \Sigma_{B_l^{-1\prime}} \circ \Sigma_{\theta'_l}$, see Theorem 3.9. In views of (3.11), the main operator $A^*\restriction_{\mathcal{X}_2^+}$ of $\Sigma^*_{\theta'_l}$ belongs to $C_{\cdot 0}$, and therefore the main operator of $\Sigma_{\theta'_l}$ belongs to $C_{0\cdot}$, see (3.9). It follows from Theorem 4.2 that θ'_l is inner. Then so is θ'_r, and again from the Theorem 4.2 it follows that the main operator $A\restriction_{\mathcal{X}_1^+}$ of the system $\Sigma_{\theta'_r}$ is in $C_{0\cdot}$. Then $\Sigma \in \mathbf{I}^\kappa_{0\cdot}$, and the second statement in (i) is proved.

(ii) If $\psi_\theta \equiv 0$, the identity (4.3) and Theorem 2.3 show that $\theta_{12} \equiv 0$, which means $D_{T^*_{,2}} = 0$ and $C(I - zA)^{-1}D_{T^*_{,1}} \equiv 0$. Since Σ is observable, one concludes as above that $D_{T^*} = 0$, which means that Σ is co-isometric. Similar arguments as above show that Σ is also minimal. Moreover, co-isometric boundary values of θ implies that $\Sigma \in \mathbf{I}^{*\kappa}_{\cdot 0}$.

(iii) If Σ is simple and $\varphi_\theta \equiv 0$ and $\psi_\theta \equiv 0$, arguments used in the proof of [11, Theorem 9.4] show that Σ is conservative. Minimality of Σ is obtained analogously as above. The last assertion is contained in Theorem 4.2. □

For the classes $\mathbf{I}^\kappa_{0\cdot}$ and $\mathbf{I}^{*\kappa}_{\cdot 0}$, conditions of Theorem 4.4 are also necessary.

Proposition 4.5 *An isometric controllable (co-isometric observable) system Σ belongs to $\mathbf{I}^\kappa_{0\cdot}$ ($\mathbf{I}^{*\kappa}_{\cdot 0}$) if and only if θ_Σ has isometric (co-isometric) boundary values a.e. on $\mathbb{T}$.*

Proof Only the proof of necessity needs to be given. For this, embed Σ to a conservative system $\widetilde{\Sigma}$ with the representation as in Theorem 3.9 and then apply Theorem 4.2. □

The existence of a co-isometric observable realization is guaranteed by Theorem 2.4. It is also possible that $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ has a co-isometric controllable realization that is neither observable nor conservative.

Example 4.6 Consider the function in Example 3.8 and choose a to be a scalar inner function. Easy calculations show that then S_l is co-inner and the right defect function φ_{S_l} of S_L is not identically zero. Theorem 4.4 shows that an observable passive realization Σ_{S_l} of S_l is co-isometric and minimal. The property $\varphi_{S_l} \neq 0$ and Lemma 4.3 show that Σ_{S_l} cannot be conservative. If $\Sigma_{b^{-1}}$ is a minimal conservative realization of b^{-1}, Theorem 3.4 shows that $\Sigma_{b^{-1}} \circ \Sigma_{S_l}$ is controllable while Example 3.8 shows that it is not observable. The product cannot be conservative either, and thus S has a co-isometric controllable realization.

If the defect functions of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ are zero functions, the results of Theorem 3.9 can be extended.

Proposition 4.7 *$\Sigma = (A, B, C, D, \mathcal{X}, \mathcal{U}, \mathcal{Y}, \kappa)$ be a passive system such that the transfer function θ of Σ belongs to $\mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$. Let $\theta = B_l^{-1}\theta_l = \theta_r B_r^{-1}$ be the Kreĭn–Langer factorizations of θ. Then the following statements hold:*

(i) *If $\varphi_\theta \equiv 0$, then Σ can be represented as in the product of the form*

$$\Sigma = \Sigma_{B_l^{-1}} \circ \Sigma_{\theta_l},$$

where $\Sigma_{B_l^{-1}}$ and Σ_{θ_l} and are minimal conservative realization of B_l^{-1} and passive realization of θ_l, respectively;

(ii) *If $\psi_\theta \equiv 0$, then Σ can be represented as in the product of the form*

$$\Sigma = \Sigma_{\theta_r} \circ \Sigma_{B_r^{-1}}$$

where $\Sigma_{B_r^{-1}}$ and Σ_{θ_r} are minimal conservative realization of B_r^{-1} and passive realization of θ_l, respectively;

(iii) *If $\varphi_\theta \equiv 0$ and $\psi_\theta \equiv 0$, then Σ can be represented as in the products of the form*

$$\Sigma = \Sigma_{B_l^{-1}} \circ \Sigma_{\theta_l} = \Sigma_{\theta_r} \circ \Sigma_{B_r^{-1}},$$

where $\Sigma_{B_l^{-1}}$ and $\Sigma_{B_r^{-1}}$ are minimal conservative realizations of B_l^{-1} and B_r^{-1}, respectively, and Σ_{θ_l} and Σ_{θ_r} are passive realizations of θ_l and θ_r, respectively.

Proof Only the proof of (ii) is provided, since the other assertions are obtained analogously. Suppose that $\psi_\theta \equiv 0$. Lemma 3.2 shows that the space $(\mathcal{X}^c)^\perp$ is a Hilbert space. It follows easily from the identity (2.8) that $A^*(\mathcal{X}^c)^\perp \subset (\mathcal{X}^c)^\perp$ and $B^*(\mathcal{X}^c)^\perp = \{0\}$. This implies that the system operator can be represented as

$$T_\Sigma = \begin{pmatrix} A_1 & 0 & 0 \\ A_2 & A_0 & B_0 \\ C_1 & C_0 & D \end{pmatrix} : \begin{pmatrix} (\mathcal{X}^c)^\perp \\ \mathcal{X}^c \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} (\mathcal{X}^c)^\perp \\ \mathcal{X}^c \\ \mathcal{Y} \end{pmatrix}. \tag{4.4}$$

Now easy calculations show that a restriction $\Sigma_0 = (A_0, B_0, C_0, D, \mathcal{X}^c, \mathcal{U}, \mathcal{Y}, \kappa)$ of Σ is controllable and passive, and then according to Theorem 4.4, Σ_0 is isometric and minimal. From Theorem 3.9 it follows that $\Sigma_0 = \Sigma_{B_l^{-1}} \circ \Sigma_{\theta_l}$ and the components have properties introduced in Theorem 3.9 (iii). The state space $\mathcal{X}^{c-}$ of $\Sigma_{B_l^{-1}}$ is invariant respect to A_0. Denote the state space of Σ_{θ_l} by $\mathcal{X}^{c+}$. Then $\left((\mathcal{X}^c)^\perp \oplus \mathcal{X}^{c+}\right) \oplus \mathcal{X}^{c-}$ is a fundamental decomposition of $\mathcal{X}$, and $A\mathcal{X}^{c-} \subset \mathcal{X}^{c-}$. Similar calculations as in the Step 1 (ii) of the proof of Theorem 3.9 show that

$$\Sigma = \operatorname{dil} \Sigma_0 = \operatorname{dil} \left(\Sigma_{B_l^{-1}} \circ \Sigma_{\theta_l}\right) = \Sigma_{B_l^{-1}} \circ \operatorname{dil} \Sigma_{\theta_l},$$

and this is the desired representation. □

Acknowledgements Open access funding provided by University of Vaasa (UVA). This paper will be a part of my forthcoming doctoral thesis. I wish to thank my supervisor Seppo Hassi for helpfull discussion while preparing this paper.

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Complex Analysis and Operator Theory (2020) 14:35
https://doi.org/10.1007/s11785-020-00993-5

Complex Analysis and Operator Theory

Minimal Passive Realizations of Generalized Schur Functions in Pontryagin Spaces

Lassi Lilleberg[1]

Received: 25 October 2019 / Accepted: 29 February 2020 / Published online: 9 March 2020

Abstract

Passive discrete-time systems in Pontryagin space setting are investigated. In this case the transfer functions of passive systems, or characteristic functions of contractive operator colligations, are generalized Schur functions. The existence of optimal and *-optimal minimal realizations for generalized Schur functions are proved. By using those realizations, a new definition, which covers the case of generalized Schur functions, is given for defects functions. A criterion due to D.Z. Arov and M.A. Nudelman, when all minimal passive realizations of the same Schur function are unitarily similar, is generalized to the class of generalized Schur functions. The approach used here is new; it relies completely on the theory of passive systems.

Keywords Operator colligation · Passive system · Transfer function · Defect functions · Generalized Schur class · Contractive operator

Mathematics Subject Classification Primary 47A48; Secondary 47A56 · 47B50 · 93B05 · 93B07 · 93B28

1 Introduction

An **operator colligation** $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ consists of separable Pontryagin spaces $\mathcal{X}$ (the **state space**), $\mathcal{U}$ (the **incoming space**), and $\mathcal{Y}$ (the **outgoing space**) and the **system operator** $T_\Sigma \in \mathcal{L}(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y})$, the space of bounded operators from $\mathcal{X} \oplus \mathcal{U}$ to $\mathcal{X} \oplus \mathcal{Y}$, where $\mathcal{X} \oplus \mathcal{U}$, or $\binom{\mathcal{X}}{\mathcal{U}}$, means the direct orthogonal sum with respect

Communicated by Bernd Kirstein.

Lassi Lilleberg
lassi.lilleberg@uva.fi

1 Department of Mathematics and Statistics, University of Vaasa, P.O. Box 700, 65101 Vaasa, Finland

to the indefinite inner product. The symbol κ is reserved for the finite negative index of the state space. The operator T_Σ has the block representation of the form

$$T_\Sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix}, \tag{1.1}$$

where $A \in \mathcal{L}(\mathcal{X})$ (the **main operator**), $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ (the **control operator**), $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ (the **observation operator**), and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ (the **feedthrough operator**). If needed, the colligation is written as $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$. It is always assumed in this paper that $\mathcal{U}$ and $\mathcal{Y}$ have the same negative index.

All notions of continuity and convergence are understood to be with respect to the strong topology, which is induced by any fundamental decomposition of the space in question.

The colligation (1.1) will be called as a **system** since it can be seen as a **linear discrete time system** of the form

$$\begin{cases} h_{k+1} = Ah_k + B\xi_k, \\ \sigma_k = Ch_k + D\xi_k, \end{cases} \qquad k \geq 0,$$

where $\{h_k\} \subset \mathcal{X}$, $\{\xi_k\} \subset \mathcal{U}$ and $\{\sigma_k\} \subset \mathcal{Y}$. In what follows, the "system" is identified with the operator expression appearing in (1.1). When the system operator T_Σ in (1.1) is contractive (isometric, co-isometric, unitary), with respect to the indefinite inner product, the corresponding system is called **passive** (isometric, co-isometric, conservative). In literature, conservative systems are also called unitary systems. The **transfer function** of the system (1.1) is defined by

$$\theta_\Sigma(z) := D + zC(I - zA)^{-1}B,$$

whenever $I - zA$ is invertible. Especially, θ_Σ is defined and holomorphic in a neighbourhood of the origin. The values $\theta_\Sigma(z)$ are bounded operators from $\mathcal{U}$ to $\mathcal{Y}$. Conversely, if θ is an operator valued function holomorphic in a neighbourhood of the origin, and transfer function of the system Σ coinsides with it, then Σ is a **realization** of θ. In some sources, transfer functions of the systems are also called characteristic functions of operator colligations.

The **adjoint** or **dual** of the system Σ is the system Σ^* such that its system operator is the indefinite adjoint T_Σ^* of T_Σ. That is, $\Sigma^* = (T_\Sigma^*; \mathcal{X}, \mathcal{Y}, \mathcal{U}; \kappa)$. In this paper, all the adjoints are with respect to the indefinite inner product. For an operator valued function φ, the notation $\varphi^*(z)$ is used instead of $(\varphi(z))^*$, and the function $\varphi^\#(z)$ is defined to be $\varphi^*(\bar{z})$. With this notation, for the transfer function θ_{Σ^*} of Σ^*, it clearly holds $\theta_{\Sigma^*}(z) = \theta_\Sigma{}^\#(z)$. Since contractions between Pontryagin spaces with the same negative index are bi-contractions (cf. eg. [24, Corollary 2.5]), Σ^* is passive whenever Σ is.

In the case where all the spaces are Hilbert spaces, the result that the transfer function of a passive system belongs to the Schur class has been established by Arov [4, Proposition 8]. In the case where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces and the state space $\mathcal{X}$

is a Pontryagin space, Saprikin showed in [30, Theorem 2.2] that the transfer function of the passive system (1.1) is a **generalized Schur function.** It will be proved later in Proposition 2.4 that this result holds also in the case when all the spaces are Pontryagin spaces. The **generalized Schur class** $\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, is the set of $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued functions $S(z)$ holomorphic in a neighbourhood Ω of the origin such that the Schur kernel

$$K_S(w,z) = \frac{1 - S(z)S^*(w)}{1 - z\bar{w}}, \qquad w, z \in \Omega, \tag{1.2}$$

has κ negative squares ($\kappa = 0, 1, 2, \ldots$). This means that for any finite set of points $w_1, \ldots, w_n$ in the domain of holomorphy $\rho(S)$ of S and set of vectors $\{f_1, \ldots, f_n\} \subset \mathcal{Y}$, the Hermitian matrix

$$\left(\left\langle K_S(w_j, w_i) f_j, f_i \right\rangle_{\mathcal{Y}}\right)_{i,j=1}^{n},$$

where $\langle \cdot, \cdot \rangle_{\mathcal{Y}}$ is the indefinite inner product of the space $\mathcal{Y}$, has no more than κ negative eigenvalues, and there exists at least one such matrix that has exactly κ negative eigenvalues. A function S belongs to $\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ if and only if $S_\kappa^\# \in \mathbf{S}(\mathcal{Y},\mathcal{U})$; see [1, Theorem 2.5.2]. The class $\mathbf{S}_0(\mathcal{U},\mathcal{Y})$ coinsides with the ordinary Schur class, and it is written as $\mathbf{S}(\mathcal{U},\mathcal{Y})$. The generalized Schur class was first studied by Kreĭn and Langer; see [26] for instance.

The direct connection between the transfer functions of passive systems of the form (1.1) and the generalized Schur functions allows to study the properties of generalized Schur functions by using passive systems, and vice versa. Therefore, a fundamental problem of the subject is, for a given $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, find a realization Σ of θ with the desired minimality or optimality properties (observable, controllable, simple, minimal, optimal, *-optimal); for details, see Theorems 2.6 and 3.5 and Lemma 2.8. The described problem is called a **realization problem.** In the standard Hilbert space setting, realizations problems, as well as other properties of passive systems, were studied, for instance, by Arov [4,5], Arov et al. [6–8], Ball and Cohen [13], de Branges and Rovnyak [20,21], Helton [25] and Nagy and Foias [29]. The case where the state space is a Pontryagin space while incoming and outgoing spaces are still Hilbert spaces, unitary systems were studied, for instance, by Dijksma et al. [22,23], and passive systems by Saprikin [30], Saprikin and Arov [10], Saprikin et al. [9] and by the author in [27]. The case where all the spaces are Pontryagin spaces, theory of isometric, co-isometric and conservative systems is considered, for instance, in [1,2,24].

Especially, Arov [5] proved the existence of so-called optimal minimal realizations of an ordinary Schur function; for definitions, see Sect. 3. The proof was based on the existence (right) **defect functions**. For an ordinary Schur function $S(\zeta)$, the (right) defect function φ of S is, roughly speaking, the maximal analytic minorant of $I - S^*(\zeta)S(\zeta)$. More precicely, this means that for almost everywhere (a.e.) ζ on the unit circle $\mathbb{T}$, it holds

$$\varphi^*(\zeta)\varphi(\zeta) \le I - S^*(\zeta)S(\zeta),$$

and for every other operator valued analytic function $\widehat{\varphi}$ with similar property, it holds

$$\widehat{\varphi}^*(\zeta)\widehat{\varphi}(\zeta) \leq \varphi^*(\zeta)\varphi(\zeta).$$

For the existence of defect functions, see [29, Theorem V.4.2], and for a detailed treatise, see [17–19]. Another names of defect functions are “spectral factors”, see [12]. Arov et al. [6] constructed (*-)optimal minimal passive systems in the Hilbert space setting without using defect functions. The construction can be done by taking an appropriate restriction of some system. In the indefinite setting, if one uses a suitable definition of optimality, a similar method as was used by Arov et al. still produces a (*-)optimal minimal passive system. In Pontryagin state space case, this was proved by Saprikin [30]. It will be shown in Theorem 3.5 that the same result still holds in the case where all the spaces are Pontryagin spaces.

The study of the class of generalized Schur functions $\mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ was continued in [9,10], in the case where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces and the state space is a Pontryagin space. Saprikin and Arov [10] used the right Kreĭn–Langer factorization of the form $S = S_r B_r^{-1}$ for $S \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, and proved that the existence of the optimal minimal realization of S is equivalent to the existence of the right defect function of S_r. However, they did not define the defect functions for the generalized Schur functions. This was done by the author in [27] by using the Kreĭn–Langer factorizations. With the definition given therein, the main results of [3] were generalized to the Pontryagin state space setting. The main subjects of [27] include some continuation of the study of products of systems and the stability properties of passive systems, subjects treated earlier by Saprikin et al. [9]. In the present paper, it will be shown that a concept of defect functions can be defined in the case where all the spaces are Pontryagin spaces. The key idea here is to use optimal minimal passive realizations and conservative embeddings. By using such a definition, it is shown that one can generalize and improve some of the main results from [3], using different proofs than those given in [3] or [27], see Theorem 4.8. Furthermore, in Theorem 4.10, the main results from [7,8] concerning the criterion when all the minimal realizations of a Schur function are unitarily similar, is generalized to the present indefinite setting. The proof will be carried out entirely by using the theory of passive systems, without applying Hardy space theory or the theory of Hankel operators as in the proof provided in [8].

The paper is organized as follows. In Sect. 2 basic facts of linear systems, Julia operators, dilations and embeddings are recalled. Moreover, Lemma 2.8 gives some usefull representations and restrictions of passive systems. That lemma will be used extensively later on in this paper.

In Sect. 3, the existence and basic properties of (*-)optimal minimal realizations are established. The main result of this section is Theorem 3.5.

The generalized defect functions are introduced in Sect. 4. In particularly, Theorem 4.10 in this section can be seen as the main result of the paper.

2 Linear Systems, Dilations and Embeddings

Let $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ be a linear system as in (1.1). The following subspaces

$$\mathcal{X}^c := \overline{\text{span}}\,\{\text{ran}\,A^n B : n = 0, 1, \ldots\} \tag{2.1}$$
$$\mathcal{X}^o := \overline{\text{span}}\,\{\text{ran}\,A^{*n} C^* : n = 0, 1, \ldots\} \tag{2.2}$$
$$\mathcal{X}^s := \overline{\text{span}}\,\{\text{ran}\,A^n B, \text{ran}\,A^{*m} C^* : n, m = 0, 1, \ldots\}, \tag{2.3}$$

are called, respectively, controllable, observable and simple subspaces. The system is said to be **controllable** (**observable**, **simple**) if $\mathcal{X}^c = \mathcal{X} (\mathcal{X}^o = \mathcal{X}, \mathcal{X}^s = \mathcal{X})$ and **minimal** if it is both controllable and observable.

When $\Omega \ni 0$ is some symmetric neighbourhood of the origin, that is, $\bar{z} \in \Omega$ whenever $z \in \Omega$, then also

$$\mathcal{X}^c = \overline{\text{span}}\,\{\text{ran}\,(I - zA)^{-1} B, z \in \Omega\} \tag{2.4}$$
$$\mathcal{X}^o = \overline{\text{span}}\,\{\text{ran}\,(I - zA^*)^{-1} C^*, z \in \Omega\} \tag{2.5}$$
$$\mathcal{X}^s = \overline{\text{span}}\,\{\text{ran}\,(I - zA)^{-1} B, \text{ran}\,(I - wA^*)^{-1} C^*, z, w \in \Omega\} \tag{2.6}$$

The system (1.1) can be expanded to a larger system without changing the transfer function. It can be done by using the so-called **defect operator** and **Julia operator**, see, respectively, (2.7) and (2.8) below. For a proof of the following theorem and more details about the defects operators and Julia operators, see [24]. The basic information about the indefinite inner product spaces and their operators can be recalled from [11,15,24].

Theorem 2.1 *Suppose that $\mathcal{X}_1$ and $\mathcal{X}_2$ are Pontryagin spaces with the same negative index, and let $A : \mathcal{X}_1 \to \mathcal{X}_2$ be a contraction. Then there exist Hilbert spaces $\mathfrak{D}_A$ and $\mathfrak{D}_{A^*}$, linear operators $D_A : \mathfrak{D}_A \to \mathcal{X}_1$, $D_{A^*} : \mathfrak{D}_{A^*} \to \mathcal{X}_2$ with zero kernels and a linear operator $L : \mathfrak{D}_A \to \mathfrak{D}_{A^*}$ such that it holds*

$$I - A^*A = D_A D_A^*, \qquad I - AA^* = D_{A^*} D_{A^*}^*, \tag{2.7}$$

and the operator

$$U_A := \begin{pmatrix} A & D_{A^*} \\ D_A^* & -L^* \end{pmatrix} : \begin{pmatrix} \mathcal{X}_1 \\ \mathfrak{D}_{A^*} \end{pmatrix} \to \begin{pmatrix} \mathcal{X}_2 \\ \mathfrak{D}_A \end{pmatrix} \tag{2.8}$$

is unitary. Moreover, D_A, D_{A^*} and U_A are unique up to unitary equivalence.

The notion of **dilation** of a discrete time-invariant system has been introduced by Arov [4]. A dilation of a system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ is any system of the form $\widehat{\Sigma} = (\widehat{A}, \widehat{B}, \widehat{C}, D; \widehat{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; \kappa)$, where

$$\widehat{\mathcal{X}} = \mathcal{D} \oplus \mathcal{X} \oplus \mathcal{D}_*, \quad \widehat{A}\mathcal{D} \subset \mathcal{D}, \quad \widehat{A}^*\mathcal{D}_* \subset \mathcal{D}_*, \quad \widehat{C}\mathcal{D} = \{0\}, \quad \widehat{B}^*\mathcal{D}_* = \{0\}. \tag{2.9}$$

The spaces $\mathcal{D}$ and $\mathcal{D}_*$ are required to be Hilbert spaces. The system operator $T_{\widehat{\Sigma}}$ of $\widehat{\Sigma}$ is of the form

$$T_{\widehat{\Sigma}} = \begin{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix} & \begin{pmatrix} B_1 \\ B \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & C & C_1 \end{pmatrix} & D \end{pmatrix} : \begin{pmatrix} \begin{pmatrix} \mathcal{D} \\ \mathcal{X} \\ \mathcal{D}_* \end{pmatrix} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} \mathcal{D} \\ \mathcal{X} \\ \mathcal{D}_* \end{pmatrix} \\ \mathcal{Y} \end{pmatrix}, \tag{2.10}$$
$$\widehat{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}, \qquad \widehat{B} = \begin{pmatrix} B_1 \\ B \\ 0 \end{pmatrix}, \qquad \widehat{C} = \begin{pmatrix} 0 & C & C_1 \end{pmatrix}.$$

The system Σ is called a **restriction** of $\widehat{\Sigma}$. Recall that subspace $\mathcal{N}$ of the Pontryagin space $\mathcal{H}$ is **regular** if it is itself a Pontryagin space with the inherited inner product of $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. The subspace $\mathcal{N}$ is regular precicely when $\mathcal{N}^{\perp}$ is regular, where $\perp$ refers to orthogonality with respect to the indefinite inner product of $\mathcal{H}$. Since $\mathcal{X}$ clearly is a regular subspace of $\widehat{\mathcal{X}}$, there exists the unique orthogonal projection $P_{\mathcal{X}}$ from $\widehat{\mathcal{X}}$ to $\mathcal{X}$. Let $\widehat{A}\restriction_{\mathcal{X}}$ be the restriction of $\widehat{A}$ to the subspace $\mathcal{X}$. Then, the system Σ can be represented as $\Sigma = (P_{\mathcal{X}}\widehat{A}\restriction_{\mathcal{X}}, P_{\mathcal{X}}\widehat{B}, \widehat{C}\restriction_{\mathcal{X}}, D; P_{\mathcal{X}}\widehat{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; \kappa)$. A calculation show that the transfer functions of the original system and its dilation coincide. Moreover, if Σ is passive, then is any retriction of it. The following proposition states that a passive system has a conservative dilation. For the Hilbert space case, this result is from [4], and for the Pontryagin state space case, see [30]. The similar proof as in [4] and [30] can be applied. For details, see the proof in [28, Proposition 2.3].

Proposition 2.2 *Let* $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ *be a passive system. Then there exists a conservative dilation* $\widehat{\Sigma} = (\widehat{A}, \widehat{B}, \widehat{C}, D; \widehat{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; \kappa)$ *of* Σ.

It is possible that $\mathcal{D} = \{0\}$ or $\mathcal{D}_* = \{0\}$ in (2.9). In those cases, the zero space and the corresponding row and column will be left out in (2.10). In particular, if the system Σ with the system operator T as in (1.1) is isometric (co-isometric), then $D_T = 0$ ($D_{T^*} = 0$).

There is also an another way to expand the system (1.1), and it is called an **embedding**. In this expansion, the state space and the main operator will not change. The embedding of the system (1.1) is any system determined by the system operator

$$T_{\widetilde{\Sigma}} = \begin{pmatrix} A & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \widetilde{\mathcal{U}} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \widetilde{\mathcal{Y}} \end{pmatrix} \iff \begin{pmatrix} A & \begin{pmatrix} B & B_1 \end{pmatrix} \\ \begin{pmatrix} C \\ C_1 \end{pmatrix} & \begin{pmatrix} D & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \begin{pmatrix} \mathcal{U} \\ \mathcal{U}' \end{pmatrix} \end{pmatrix}$$
$$\to \begin{pmatrix} \mathcal{X} \\ \begin{pmatrix} \mathcal{Y} \\ \mathcal{Y}' \end{pmatrix} \end{pmatrix},$$

where $\mathcal{U}'$ and $\mathcal{Y}'$ are Hilbert spaces. The transfer function of the embedded system is

$$\begin{aligned}\theta_{\widetilde{\Sigma}}(z) &= \begin{pmatrix} D + zC(I_{\mathcal{X}} - zA)^{-1}B & D_{12} + zC(I_{\mathcal{X}} - zA)^{-1}B_1 \\ D_{21} + zC_1(I_{\mathcal{X}} - zA)^{-1}B & D_{22} + zC_1(I_{\mathcal{X}} - zA)^{-1}B_1 \end{pmatrix} \\ &= \begin{pmatrix} \theta_{\Sigma}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{pmatrix},\end{aligned}$$

where θ_{Σ} is the transfer function of the original system. The embedded systems will be needed in Sect. 4.

It will be proved in Proposition 2.4 below that the transfer function of any passive system (1.1) is a generalized Schur function with index not larger than the negative index of the state space. For a special case where incoming and outcoming spaces are Hilbert spaces, this result is due to [30, Theorem 2.2]. The proof of the general case follows the lines of Saprikin's proof of the special case.

Lemma 2.3 *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ be a passive system with the transfer function θ. Denote the system operator of Σ as T. If*

$$D_T = \begin{pmatrix} D_{T,1} \\ D_{T,2} \end{pmatrix} : \mathfrak{D}_T \to \begin{pmatrix} \mathcal{X} \\ \mathcal{U} \end{pmatrix} \quad D_{T^*} = \begin{pmatrix} D_{T^*_{,1}} \\ D_{T^*_{,2}} \end{pmatrix} : \mathfrak{D}_{T^*} \to \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix},$$

are defect operators of T and T^*, respectively, then the identities

$$I_{\mathcal{Y}} - \theta(z)\theta^*(w) = (1 - z\bar{w})G(z)G^*(w) + \psi(z)\psi^*(w), \tag{2.11}$$
$$I_{\mathcal{U}} - \theta^*(w)\theta(z) = (1 - z\bar{w})F^*(w)F(z) + \varphi^*(w)\varphi(z), \tag{2.12}$$

with

$$\begin{aligned} G(z) &= C(I_{\mathcal{X}} - zA)^{-1}, & \psi(z) &= D_{T^*_{,2}} + zC(I_{\mathcal{X}} - zA)^{-1}D_{T^*_{,1}}, \\ F(z) &= (I_{\mathcal{X}} - zA)^{-1}B, & \varphi(z) &= D^*_{T,2} + zD^*_{T,1}(I_{\mathcal{X}} - zA)^{-1}B, \end{aligned} \tag{2.13}$$

hold for every z and w in a sufficiently small symmetric neighbourhood of the origin.

Proof By applying the results from [1, Theorem 1.2.4] and the identities in (2.7), the results follow by straightforward calculations. For details, see the proof in [28, Lemma 2.4]. □

Note that if Σ in Lemma 2.3 is isometric (co-isometric), then $D_T = 0$ ($D_{T^*} = 0$) and therefore $\varphi \equiv 0$ ($\psi \equiv 0$).

Proposition 2.4 *If $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ is a passive system, the transfer function θ of Σ belongs to $\mathbf{S}_{\kappa'}(\mathcal{U}, \mathcal{Y})$, where $\kappa' \leq \kappa$.*

Proof Denote the system operator of Σ as T. By Lemma 2.3, the kernel K_θ defined as in (1.2) has a representation

$$K_\theta(w, z) = G(z)G^*(w) + (1 - z\bar{w})^{-1}\psi(z)\psi^*(w), \tag{2.14}$$

where $G(z)$ and $\psi(z)$ are defined as in (2.13). Since the negative index of $\mathcal{X}$ is κ and the negative index of the Hilbert space $\mathfrak{D}_{T^*}$ is zero, it follows from [1, Lemma 1.1.1.], that for any finite set of points $w_1, \ldots, w_n$ in the domain of holomorphy of θ and the set of vectors $\{y_1, \ldots, y_n\} \subset \mathcal{Y}$, the Gram matrices

$$\left(\langle G^*(w_j)y_j, G^*(w_i)y_i\rangle_{\mathcal{X}}\right)_{i,j=1}^n, \qquad \left(\langle \psi^*(w_j)y_j, \psi^*(w_i)y_i\rangle_{\mathfrak{D}_{T^*}}\right)_{i,j=1}^n,$$

have, respectively, at most κ and zero negative eigenvalues.

The kernel $(1 - z\bar{w})^{-1}$ has no negative square, since it is the reproducing kernel of the classical Hardy space $H^2(\mathbb{D})$. The Schur product theorem shows that the kernel $(1 - z\bar{w})^{-1}\psi(z)\psi^*(w)$ has no negative square. Then it follows from [1, Theorem 1.5.5] that the kernel K_θ has at most κ negative square. That is, $\theta \in \mathbf{S}_{\kappa'}(\mathcal{U}, \mathcal{Y})$, where $\kappa' \leq \kappa$, and the proof is complete. □

Definition 2.5 A passive realization Σ of a generalized Schur function $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ is called κ-**admissible** if the negative index of the state space of Σ coinsides with the negative index κ of θ.

In what follows, this paper deals mostly with the κ-admissible realizations. It will turn out that the κ-admissible realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ are well behaved is some sense; they have many similar propeties than the standard passive Hilbert space systems.

The following realizations theorem is well known, see [1, Theorems 2.2.1, 2.2.2 and 2.3.1].

Theorem 2.6 *For a generalized Schur function $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ there exist realizations $\Sigma_k = (T_k; \mathcal{X}_k, \mathcal{U}, \mathcal{Y}; \kappa)$, $k = 1, 2, 3$, of θ such that*

(i) *Σ_1 is observable co-isometric;*
(ii) *Σ_2 is controllable isometric;*
(iii) *Σ_3 is simple conservative.*

Conversely, if the system Σ has some of the properties (i)–(iii)*, then $\theta_\Sigma \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where κ is the negative index of the state space of Σ.*

Recall that a **Hilbert subspace** of the Pontryagin space $\mathcal{X}$ is a regular subspace such that its negative index is zero. Conversely, **anti-Hilbert subspace** is a regular subspace such that its positive index is zero. When $\mathcal{U}$ and $\mathcal{Y}$ happens to be Hilbet spaces, the transfer function θ of the passive system $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ belongs to class $\mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ (with $\kappa = \text{ind}_- \mathcal{X}$) if and only if $(\mathcal{X}^s)^\perp$ is a Hilbert subspace [27, Lemma 3.2]. In the case when $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, the transfer function θ of the isometric (co-isometric, conservative) system $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ belongs to class $\mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ if and only if $(\mathcal{X}^c)^\perp$ $((\mathcal{X}^o)^\perp$,$(\mathcal{X}^s)^\perp)$ is a Hilbert subspace [1, Theorem 2.1.2]. For a passive system, one has the following result.

Proposition 2.7 *For a passive realization $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, spaces $\mathcal{X}^c$, $\mathcal{X}^o$ and $\mathcal{X}^s$ are regular and their orthogonal complements are Hilbert subspaces.*

Proof Let Ω be a symmetric neighbourhood of the origin such that $(I - zA)^{-1}$ and $(I - zA^*)^{-1}$ exist for every $z \in \Omega$. Represent the kernel K_θ as in (2.14). Since K_θ has κ negative square, a similar argument used in the proof of 2.4 shows that the kernel $K_1(z, w) = G(z)G^*(w)$, where $G(z) = C(I - zA)^{-1}$, has κ negative square. It follows now from [1, Lemma 1.1.1'] that span{ran $(I - \overline{w}A^*)^{-1}C^*, \overline{w} \in \Omega\}$ contains a κ-dimensional maximal anti-Hilbert subspace $\mathcal{X}_\kappa$. Then, $\mathcal{X}_\kappa \oplus (\mathcal{X}_\kappa)^\perp = \mathcal{X}$ is a fundamental decomposition of $\mathcal{X}$. Especially, $(\mathcal{X}_\kappa)^\perp$ is a Hilbert subspace of $\mathcal{X}$. But

$$\left(\text{span}\{\text{ran}\,(I - \overline{w}A^*)^{-1}C^*, \overline{w} \in \Omega\}\right)^\perp = \left(\mathcal{X}^o\right)^\perp \subset (\mathcal{X}_\kappa)^\perp,$$

which implies that $(\mathcal{X}^o)^\perp$ is a Hilbert subspace, and therefore its orthocomplement $\mathcal{X}^o$ is regular.

By duality argument, the space $\mathcal{X}^c$ is a regular subspace and the space $(\mathcal{X}^c)^\perp$ is a Hilbert subspace. It easily follows from (2.1)–(2.3) that $(\mathcal{X}^s)^\perp = (\mathcal{X}^c)^\perp \cap (\mathcal{X}^o)^\perp$, and therefore $(\mathcal{X}^s)^\perp$ is also a Hilbert subspace and $\mathcal{X}^s$ is regular. □

It follows from the Proposition 2.7 above that the state space $\mathcal{X}$ of a κ-admissible realization Σ of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ can be decombosed to the controllable, observable and simple parts. Using this fact, the lemma below, which will be used extensively, can be proved.

Lemma 2.8 *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ be a passive system such that the spaces $(\mathcal{X}^o)^\perp$, $(\mathcal{X}^c)^\perp$ and $(\mathcal{X}^s)^\perp$ are Hilbert subspaces of $\mathcal{X}$. Then the system operator T of Σ has the following representations*

$$T = \begin{pmatrix} \begin{pmatrix} A_1 & A_2 \\ 0 & A_o \end{pmatrix} & \begin{pmatrix} B_1 \\ B_o \end{pmatrix} \\ \begin{pmatrix} 0 & C_o \end{pmatrix} & D \end{pmatrix} : \begin{pmatrix} \begin{pmatrix} (\mathcal{X}^o)^\perp \\ \mathcal{X}^o \end{pmatrix} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} (\mathcal{X}^o)^\perp \\ \mathcal{X}^o \end{pmatrix} \\ \mathcal{Y} \end{pmatrix} \tag{2.15}$$

$$T = \begin{pmatrix} \begin{pmatrix} A_3 & 0 \\ A_4 & A_c \end{pmatrix} & \begin{pmatrix} 0 \\ B_c \end{pmatrix} \\ \begin{pmatrix} C_1 & C_c \end{pmatrix} & D \end{pmatrix} : \begin{pmatrix} \begin{pmatrix} (\mathcal{X}^c)^\perp \\ \mathcal{X}^c \end{pmatrix} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} (\mathcal{X}^c)^\perp \\ \mathcal{X}^c \end{pmatrix} \\ \mathcal{Y} \end{pmatrix} \tag{2.16}$$

$$T = \begin{pmatrix} \begin{pmatrix} A_5 & 0 \\ 0 & A_s \end{pmatrix} & \begin{pmatrix} 0 \\ B_s \end{pmatrix} \\ \begin{pmatrix} 0 \; C_s \end{pmatrix} & D \end{pmatrix} : \begin{pmatrix} \begin{pmatrix} (\mathcal{X}^s)^\perp \\ \mathcal{X}^s \end{pmatrix} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} (\mathcal{X}^s)^\perp \\ \mathcal{X}^s \end{pmatrix} \\ \mathcal{Y} \end{pmatrix} \tag{2.17}$$

$$T = \begin{pmatrix} \begin{pmatrix} A'_{11} & A'_{12} & A'_{13} \\ 0 & A' & A'_{23} \\ 0 & 0 & A'_{33} \end{pmatrix} & \begin{pmatrix} B'_1 \\ B' \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & C' & C'_1 \end{pmatrix} & D \end{pmatrix} : \begin{pmatrix} \begin{pmatrix} (\mathcal{X}^o)^\perp \\ \overline{P_{\mathcal{X}^o}\mathcal{X}^c} \\ \mathcal{X}^o \cap (\mathcal{X}^c)^\perp \end{pmatrix} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} (\mathcal{X}^o)^\perp \\ \overline{P_{\mathcal{X}^o}\mathcal{X}^c} \\ \mathcal{X}^o \cap (\mathcal{X}^c)^\perp \end{pmatrix} \\ \mathcal{Y} \end{pmatrix} \tag{2.18}$$

$$T = \begin{pmatrix} \begin{pmatrix} A''_{11} & A''_{12} & A''_{13} \\ 0 & A'' & A''_{23} \\ 0 & 0 & A''_{33} \end{pmatrix} & \begin{pmatrix} B''_1 \\ B'' \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & C'' & C''_1 \end{pmatrix} & D \end{pmatrix} : \begin{pmatrix} \begin{pmatrix} \mathcal{X}^c \cap (\mathcal{X}^o)^\perp \\ \overline{P_{\mathcal{X}^c}\mathcal{X}^o} \\ (\mathcal{X}^c)^\perp \end{pmatrix} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} \mathcal{X}^c \cap (\mathcal{X}^o)^\perp \\ \overline{P_{\mathcal{X}^c}\mathcal{X}^o} \\ (\mathcal{X}^c)^\perp \end{pmatrix} \\ \mathcal{Y} \end{pmatrix} \tag{2.19}$$

The restrictions

$$\Sigma_o = (A_o, B_o, C_o, D; \mathcal{X}^o, \mathcal{U}, \mathcal{Y}; \kappa) \tag{2.20}$$
$$\Sigma_c = (A_c, B_c, C_c, D; \mathcal{X}^c, \mathcal{U}, \mathcal{Y}; \kappa) \tag{2.21}$$
$$\Sigma_s = (A_s, B_s, C_s, D; \mathcal{X}^s, \mathcal{U}, \mathcal{Y}; \kappa) \tag{2.22}$$
$$\Sigma' = (A', B', C', D; \overline{P_{\mathcal{X}^o}\mathcal{X}^c}, \mathcal{U}, \mathcal{Y}; \kappa) \tag{2.23}$$
$$\Sigma'' = (A'', B'', C'', D; \overline{P_{\mathcal{X}^c}\mathcal{X}^o}, \mathcal{U}, \mathcal{Y}; \kappa) \tag{2.24}$$

of Σ are passive, and Σ_o is observable, Σ_c is controllable, Σ_s is simple, and Σ' and Σ'' are minimal. For any $n \in \mathbb{N}_0$ and any z in a sufficiently small symmetric neighbourhood of the origin, it holds

$$A^n B = A_c^n B_c = A_s^n B_s, \tag{2.25}$$
$$(I - zA)^{-1} B = (I - zA_s)^{-1} B_s = (I - zA_c)^{-1} B_c, \tag{2.26}$$
$$A^{*n} C^* = A_o^{*n} C_o^* = A_s^{*n} C_s^*, \tag{2.27}$$
$$(I - zA^*)^{-1} C^* = (I - zA_s^*)^{-1} C_s^* = (I - zA_o^*)^{-1} C_o^*. \tag{2.28}$$

Moreover, if Σ is co-isometric (isometric), then so are Σ_o and Σ_s (Σ_c and Σ_s).

Proof Since $(\mathcal{X}^o)^\perp$, $(\mathcal{X}^c)^\perp$ and $(\mathcal{X}^s)^\perp$ are Hilbert spaces, the spaces $\mathcal{X}^o$, $\mathcal{X}^c$ and $\mathcal{X}^s$ are regular subspaces with the negative index κ. It follows from the identities (2.1)–(2.3) that

$$\begin{cases} (\mathcal{X}^o)^\perp, (\mathcal{X}^s)^\perp \text{ are } A\text{-invariant}, \\ (\mathcal{X}^c)^\perp, (\mathcal{X}^s)^\perp \text{ are } A^*\text{-invariant}, \\ \operatorname{ran} C^* \subset \mathcal{X}^o \subset \mathcal{X}^s, \\ \operatorname{ran} B \subset \mathcal{X}^c \subset \mathcal{X}^s, \end{cases}, \tag{2.29}$$

and the representations (2.15)–(2.17) follow. That is, Σ_o, Σ_c and Σ_s are restrictions of the passive system Σ, ans therefore they are passive.

Let T_{Σ_k} be the system operator of Σ_k where $k = o, c, s$, and let $\hat{x} \in \mathcal{X}^k \oplus \mathcal{U}$ and $\breve{x} \in \mathcal{X}^k \oplus \mathcal{Y}$. Calculation show that $T_{\Sigma_k}\hat{x} = T\hat{x}$, where $k = c, s$ and $T_{\Sigma_k}^*\breve{x} = T^*\breve{x}$ where $k = o, s$. It follows that if Σ is co-isometric (isometric), then so are Σ_o and Σ_s (Σ_c and Σ_s).

Suppose $x \in \mathcal{X}^o$ such that $C_o A_o^n x = 0$ for every $n = 0, 1, 2, \ldots$. Then

$$CA^n x = \begin{pmatrix} 0 & C_o \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ 0 & A_o \end{pmatrix}^n \begin{pmatrix} 0 \\ x \end{pmatrix} = C_o A_o^n x = 0,$$

and the identity (2.2) implies that $x \in \mathcal{X}^o \cap (\mathcal{X}^o)^\perp = \{0\}$. Thus $x = 0$, and it can be deduced that Σ_o is observable. Similar arguments show that Σ_c is controllable and Σ_s is simple, the details will be omitted.

Let $u \in \mathcal{U}$, and $n \in \mathbb{N}_0$. Then, by (2.16) and (2.17),

$$A^n Bu = \begin{pmatrix} A_3 & 0 \\ A_4 & A_c \end{pmatrix}^n \begin{pmatrix} 0 \\ B_c \end{pmatrix} = \begin{pmatrix} 0 \\ A_c^n B_c u \end{pmatrix} = A_c^n B_c u$$
$$A^n Bu = \begin{pmatrix} A_5 & 0 \\ 0 & A_s \end{pmatrix}^n \begin{pmatrix} 0 \\ B_s \end{pmatrix} = \begin{pmatrix} 0 \\ A_s^n B_s u \end{pmatrix} = A_s^n B_s u,$$

and (2.25) holds. By Neumann series, $(I - zA)^{-1}B = \sum_{n=0}^{\infty} z^n A^n B$ holds for all z in a sufficiently small symmetric neighbourhood of the origin, and (2.26) follows now from (2.25). The equalities (2.27) and (2.28) can be deduced similarly.

Since the orthocomplements $(\mathcal{X}^o)^{\perp}$ and $(\mathcal{X}^c)^{\perp}$ are Hilbert subspaces, it follows from [30, Lemma 3.1] that $\overline{P_{\mathcal{X}^o}\mathcal{X}^c}$ and $\overline{P_{\mathcal{X}^c}\mathcal{X}^o}$ are regular subspaces, and it holds

$$\mathcal{X}^o \cap (P_{\mathcal{X}^o}\mathcal{X}^c)^{\perp} = \mathcal{X}^o \cap (\mathcal{X}^c)^{\perp}, \qquad \mathcal{X}^c \cap (P_{\mathcal{X}^c}\mathcal{X}^o)^{\perp} = \mathcal{X}^c \cap (\mathcal{X}^o)^{\perp}.$$

Since $(\mathcal{X}^o)^{\perp} \subset (P_{\mathcal{X}^o}\mathcal{X}^c)^{\perp}$, $(\mathcal{X}^c)^{\perp} \subset (P_{\mathcal{X}^c}\mathcal{X}^o)^{\perp}$ and all the spaces are regular, simple calculations show that

$$(P_{\mathcal{X}^o}\mathcal{X}^c)^{\perp} = (\mathcal{X}^o)^{\perp} \oplus (\mathcal{X}^o \cap (P_{\mathcal{X}^o}\mathcal{X}^c)^{\perp}) \quad \text{and} \quad (P_{\mathcal{X}^c}\mathcal{X}^o)^{\perp}$$
$$= (\mathcal{X}^c)^{\perp} \oplus (\mathcal{X}^c \cap (P_{\mathcal{X}^c}\mathcal{X}^o)^{\perp}).$$

Therefore,

$$\mathcal{X} = P_{\mathcal{X}^o}\mathcal{X}^c \oplus (P_{\mathcal{X}^o}\mathcal{X}^c)^{\perp} = (\mathcal{X}^o)^{\perp} \oplus \overline{P_{\mathcal{X}^o}\mathcal{X}^c} \oplus (\mathcal{X}^o \cap (P_{\mathcal{X}^o}\mathcal{X}^c)^{\perp})$$
$$= (\mathcal{X}^o)^{\perp} \oplus \overline{P_{\mathcal{X}^o}\mathcal{X}^c} \oplus (\mathcal{X}^o \cap (\mathcal{X}^c)^{\perp}),$$

and similarly, $\mathcal{X} = (\mathcal{X}^c \cap (\mathcal{X}^o)^{\perp}) \oplus \overline{P_{\mathcal{X}^c}\mathcal{X}^o} \oplus (\mathcal{X}^c)^{\perp}$. Since $(\mathcal{X}^o \cap (\mathcal{X}^c)^{\perp}$ and $\mathcal{X}^c \cap (\mathcal{X}^o)^{\perp}$ are also Hilbert spaces, the spaces $\overline{P_{\mathcal{X}^o}\mathcal{X}^c}$ and $\overline{P_{\mathcal{X}^c}\mathcal{X}^o}$ are Pontryagin spaces with the negative index κ. By considering the properties in (2.29), the representations (2.18) and (2.19) follow now easily. That is, Σ' and Σ'' are restrictions of Σ, and therefore passive.

Denote $\mathcal{X}' := \overline{P_{\mathcal{X}^o}\mathcal{X}^c}$. Represent the system operator T of Σ as in (2.18). Then

$$P_{\mathcal{X}'}A^n B = P_{\mathcal{X}'} \begin{pmatrix} A'_{11} & A'_{12} & A'_{13} \\ 0 & A' & A'_{23} \\ 0 & 0 & A'_{33} \end{pmatrix}^n \begin{pmatrix} B'_1 \\ B' \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ A'^n B' \\ 0 \end{pmatrix} = A'^n B',$$

and similarly $A'^{*n}C'^{*} = P_{\mathcal{X}'}A^{*n}C^{*}$. Therefore,

$$\mathcal{X}'^c = \overline{\text{span}}\,\{\text{ran}\, A'^n B' : n = 0, 1, \ldots\} = \overline{\text{span}}\,\{\text{ran}\, P_{\mathcal{X}'}A^n B : n = 0, 1, \ldots\}$$
$$= \overline{P_{\mathcal{X}'}\overline{\text{span}}\,\{\text{ran}\, A^n B : n = 0, 1, \ldots\}} = \overline{P_{\mathcal{X}'}\mathcal{X}^c} = \overline{P_{\mathcal{X}'}P_{\mathcal{X}^o}\mathcal{X}^c} = \overline{P_{\mathcal{X}'}\mathcal{X}'} = \mathcal{X}',$$

and similarly $\mathcal{X}'^o = \overline{P_{\mathcal{X}'}\mathcal{X}^o} = \mathcal{X}'$, which implies that Σ' is minimal. A similar argument shows that Σ'' is minimal, and the proof is complete. □

Note that in particular, Lemma 2.8 implies the existence of a minimal passive realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$.

Definition 2.9 The restrictions Σ_o, Σ_c, Σ_s, Σ', and Σ'' in Lemma 2.8 are called, respectively, the observable, the controllable, the simple (or proper), the **first minimal** and the **second minimal** restrictions of Σ.

The first minimal and the second minimal restrictions will be considered later in Sects. 3 and 4.

Two realizations $\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}; \kappa_1)$ and $\Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}; \kappa_2)$ of the same function $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ are called **unitarily similar** if $D_1 = D_2$ and there exists a unitary operator $U : \mathcal{X}_1 \to \mathcal{X}_2$ such that

$$A_1 = U^{-1} A_2 U, \quad B_1 = U^{-1} B_2, \quad C_1 = C_2 U. \tag{2.30}$$

In that case, it easily follows that $\kappa_1 = \kappa_2$. Unitary similarity preserves dynamical properties of the system and also the spectral properties of the main operator. If two realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ both have the same property (i), (ii) or (iii) of Theorem 2.6, then they are unitarily similar [1, Theorem 2.1.3].

The realizations Σ_1 and Σ_2 above are said to be **weakly similar** if $D_1 = D_2$ and there exists an injective closed densely defined possible unbounded linear operator $Z : \mathcal{X}_1 \to \mathcal{X}_2$ with the dense range such that

$$Z A_1 x = A_2 Z x, \quad C_1 x = C_2 Z x, \quad x \in \mathcal{D}(Z), \quad \text{and} \quad Z B_1 = B_2, \tag{2.31}$$

where $\mathcal{D}(Z)$ is the domain of Z. In Hilbert state space case, a result of Helton [25] and Arov [4] states that two minimal passive realizations of $\theta \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ are weakly similar. However, weak similarity preserves neither dynamical properties of the system nor the spectral properties of its main operator.

Helton's and Arov's statement holds also in case where all the spaces are indefinite. This result is stated for reference purposes. Similar argument as Hilbert space case can be applied, definiteness of the inner product play no role. For a proof of special cases, see [14, Theorem 7.1.3], [31, p. 702] and [27, Theorem 2.5]. Note that the realizations are not assumed to be κ-admissible or passive.

Proposition 2.10 *Two minimal realizations of* $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ *are weakly similar.*

3 Optimal Minimal Systems

For κ-admissible realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, one can form the similar theory of optimal minimal passive systems as represented in the standard Hilbert space case in [6] and the Pontryagin state space case in [30]. Techniques, definitions and notations to be used here are similar to what appears in those papers.

Denote $E_{\mathcal{X}}(x) = \langle x, x \rangle_{\mathcal{X}}$ for a vector x in an inner product space $\mathcal{X}$. Following [6,10,30], a passive realization $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ is called **optimal** if for any passive realization $\Sigma' = (A', B', C', D'; \mathcal{X}', \mathcal{U}, \mathcal{Y}; \kappa)$ of θ, the inequality

$$E_{\mathcal{X}}\left(\sum_{k=0}^{n} A^k B u_k\right) \le E_{\mathcal{X}'}\left(\sum_{k=0}^{n} A'^k B' u_k\right), \qquad n \in \mathbb{N}_0, \quad u_k \in \mathcal{U}, \tag{3.1}$$

holds. On the other hand, the system Σ is called ***-optimal** if it is observable and

$$E_{\mathcal{X}}\left(\sum_{k=0}^{n} A^k B u_k\right) \ge E_{\mathcal{X}'}\left(\sum_{k=0}^{n} A'^k B' u_k\right), \qquad n \in \mathbb{N}_0, \quad u_k \in \mathcal{U}, \tag{3.2}$$

holds for every observable passive realization Σ' of θ. The requirement for observability must be included for avoiding trivialities, since otherwise every isometric realization of θ would be *-optimal; see Lemma 3.3 below and [6, Proposition 3.5 and example on page 144].

In the definition of optimality and *-optimality, the requirement that the considered realizations are κ-admissible is essential, as the example below shows.

Example 3.1 Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ and $\Sigma' = (A', B', C', D'; \mathcal{X}', \mathcal{U}, \mathcal{Y}; \kappa')$, where $\kappa < \kappa'$, be passive realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$. Suppose that (3.1) holds. By Lemma 2.8, if (3.1) holds for Σ, it holds also for the controllable restriction $\Sigma_c = (A_c, B_c, C_c, D'; \mathcal{X}^c, \mathcal{U}, \mathcal{Y}; \kappa)$ of Σ. For any vector x of the form $x = \sum_{n=0}^{M} A_c^n B_c u_n$ where $\{u_n\} \subset \mathcal{U}$ and $M \in \mathbb{N}_0$, define

$$Rx = \sum_{n=0}^{M} A'^n B' u_n.$$

It is easy to deduce that R is a linear relation. Moreover, since Σ_c is controllable by Lemma 2.8, R is densely defined. Since (3.1) holds, R is contractive. It follows now from [1, Theorem 1.4.2] that R can be extended to be everywhere defined contractive linear operator. Since $\mathrm{ind}_{-}\mathcal{X}^c = \kappa < \kappa' = \mathrm{ind}_{-}\mathcal{X}'$, it follows from [24, Theorem 2.4] that linear operator from $\mathcal{X}^c$ to $\mathcal{X}'$ cannot be contractive, and hence (3.1) cannot hold.

It will be shown in Theorem 3.5 below that an optimal (*-optimal) minimal realization exists, and it can be constructed by taking the first (second) minimal restriction, introduced in Definition 2.9, of simple conservative realizations. More lemmas will be needed before that.

Lemma 3.2 *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ is a passive realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, and let $\Sigma_s = (A_s, B_s, C_s, D; \mathcal{X}^s, \mathcal{U}, \mathcal{Y}; \kappa)$ be the restriction of Σ to the simple subspace. Then, the first (second) minimal restrictions of Σ and Σ_s coinside.*

Proof Only the proof of the statement concerning about the second minimal restrictions is provided, since the other case is similar. To make the notation less cumbersome, write $\mathcal{X}^s = \mathcal{X}_p$, where p refers to proper part. By Lemma 2.8, the equalities (2.25) and (2.27) hold, and it easily follows that it holds $\mathcal{X}^o = \mathcal{X}_p^o$, $\mathcal{X}^c = \mathcal{X}_p^c$, $(\mathcal{X}^o)^\perp = (\mathcal{X}^s)^\perp \oplus (\mathcal{X}_p^o)^\perp$ and $(\mathcal{X}^c)^\perp = (\mathcal{X}^s)^\perp \oplus (\mathcal{X}_p^c)^\perp$, where orthogonal complements $(\mathcal{X}_p^o)^\perp$ and $(\mathcal{X}_p^c)^\perp$ are taken with respect to the space $\mathcal{X}_p$. Therefore $P_{\mathcal{X}^c}\mathcal{X}^o = P_{\mathcal{X}_p^c}\mathcal{X}_p^o \subset \mathcal{X}^s = \mathcal{X}_p$, and consequently,

$$P_{P_{\mathcal{X}_p^c}\mathcal{X}_p^o} A_p\restriction_{P_{\mathcal{X}_p^c}\mathcal{X}_p^o} = P_{P_{\mathcal{X}^c}\mathcal{X}^o} A\restriction_{\mathcal{X}^s}\restriction_{P_{\mathcal{X}^c}\mathcal{X}^o} = P_{P_{\mathcal{X}^c}\mathcal{X}^o} A\restriction_{P_{\mathcal{X}^c}\mathcal{X}^o},$$
$$P_{P_{\mathcal{X}_p^c}\mathcal{X}_p^o} B_p = P_{P_{\mathcal{X}^c}\mathcal{X}^o} B, \qquad C_p\restriction_{P_{\mathcal{X}_p^c}\mathcal{X}_p^o} = C\restriction_{P_{\mathcal{X}^c}\mathcal{X}^o},$$

which shows that the second minimal restrictions of Σ and Σ_s co-inside. □

To prove the (*-)optimality of a system, the following lemma is helpful.

Lemma 3.3 *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}, \kappa)$, $\widehat{\Sigma} = (\widehat{A}, \widehat{B}, \widehat{C}, D; \widehat{\mathcal{X}}, \mathcal{U}, \mathcal{Y}, \kappa)$ and $\Sigma' = (A', B', C', D; \mathcal{X}', \mathcal{U}, \mathcal{Y}; \kappa)$ be realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ such that Σ is passive, $\widehat{\Sigma}$ is a passive dilation of Σ and Σ' is the first minimal restriction of $\widehat{\Sigma}$. Then*

$$E_{\mathcal{X}'}\left(\sum_{k=0}^{n} A'^k B' u_k\right) \le E_{\mathcal{X}}\left(\sum_{k=0}^{n} A^k B u_k\right), \qquad n \in \mathbb{N}_0, \quad u_k \in \mathcal{U}. \tag{3.3}$$

Moreover, for any isometric realization $\breve{\Sigma} = (\breve{A}_1, \breve{B}_1, \breve{C}_1, D; \breve{\mathcal{X}}, \mathcal{U}, \mathcal{Y}, \kappa)$ of θ, it holds

$$E_{\mathcal{X}}\left(\sum_{k=0}^{n} A^k B u_k\right) \le E_{\breve{\mathcal{X}}}\left(\sum_{k=0}^{n} \breve{A}^k \breve{B} u_k\right), \qquad n \in \mathbb{N}_0, \quad u_k \in \mathcal{U}. \tag{3.4}$$

Note that Proposition 2.2 guarantees the existence of a passive dilation $\widehat{\Sigma}$ of Σ with the properties described above.

Proof Since $\widehat{\Sigma}$ is a dilation of Σ, the system operator $T_{\widehat{\Sigma}}$ has a representation

$$T_{\widehat{\Sigma}} = \begin{pmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & D \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix} & \begin{pmatrix} B_1 \\ B \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & C_1 & C \end{pmatrix} & D \end{pmatrix} : \begin{pmatrix} \begin{pmatrix} \mathcal{D} \\ \mathcal{X} \\ \mathcal{D}_* \end{pmatrix} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} \mathcal{D} \\ \mathcal{X} \\ \mathcal{D}_* \end{pmatrix} \\ \mathcal{Y} \end{pmatrix}, \tag{3.5}$$

where $\mathcal{D}$ and $\mathcal{D}^*$ are Hilbert spaces. On the other hand, by Lemma 2.8, $\widehat{\Sigma}$ can also be represented as

$$T_{\widehat{\Sigma}} = \begin{pmatrix} \begin{pmatrix} A'_{11} & A'_{12} & A'_{13} \\ 0 & A' & A'_{23} \\ 0 & 0 & A'_{33} \end{pmatrix} & \begin{pmatrix} B'_1 \\ B' \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & C' & C'_1 \end{pmatrix} & D \end{pmatrix} : \begin{pmatrix} \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}' \\ \mathcal{X}_3 \end{pmatrix} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}' \\ \mathcal{X}_3 \end{pmatrix} \\ \mathcal{Y} \end{pmatrix},$$

where $\mathcal{X}_1 = (\widehat{\mathcal{X}}^o)^\perp$, $\mathcal{X}' = \overline{P_{\widehat{\mathcal{X}}^o}\widehat{\mathcal{X}}^c}$ and $\mathcal{X}_3 = \widehat{\mathcal{X}}^o \cap (\widehat{\mathcal{X}}^c)^\perp$. The spaces $\mathcal{X}_1$ and $\mathcal{X}_3$ are Hilbert spaces, and $\mathcal{X}'$ is a Pontryagin space with the negative index κ. Let $n \in \mathbb{N}_0$ and $\{u_k\}_{k=0}^n \subset \mathcal{U}$. Since $\mathcal{X}_3 \subset (\widehat{\mathcal{X}}^c)^\perp$, it holds

$$\begin{aligned}
&E_{\mathcal{X}'}\left(\sum_{k=0}^n A'^k B' u_k\right)\\
&\quad= E_{\widehat{\mathcal{X}}}\left(P_{\mathcal{X}'}\sum_{k=0}^n \widehat{A}^k \widehat{B} u_k\right)\\
&\quad= E_{\widehat{\mathcal{X}}}\left(\sum_{k=0}^n \widehat{A}^k \widehat{B} u_k\right) - E_{\widehat{\mathcal{X}}}\left(P_{\mathcal{X}_1}\sum_{k=0}^n \widehat{A}^k \widehat{B} u_k\right) - E_{\widehat{\mathcal{X}}}\left(P_{\mathcal{X}_3}\sum_{k=0}^n \widehat{A}^k \widehat{B} u_k\right)\\
&\quad= E_{\widehat{\mathcal{X}}}\left(\sum_{k=0}^n \widehat{A}^k \widehat{B} u_k\right) - E_{\widehat{\mathcal{X}}}\left(P_{\mathcal{X}_1}\sum_{k=0}^n \widehat{A}^k \widehat{B} u_k\right). \qquad (3.6)
\end{aligned}$$

With $\mathcal{D}$ and $\mathcal{D}^*$ as in (3.5), the identities in (2.9) hold. Therefore, it follows from the identities (2.1) and (2.2) that $\mathcal{D}_* \subset (\widehat{\mathcal{X}}^c)^\perp$ and $\mathcal{D} \subset (\widehat{\mathcal{X}}^o)^\perp = \mathcal{X}_1$. A similar calculation as above yields then

$$E_{\mathcal{X}}\left(\sum_{k=0}^n A^k B u_k\right) = E_{\widehat{\mathcal{X}}}\left(\sum_{k=0}^n \widehat{A}^k \widehat{B} u_k\right) - E_{\widehat{\mathcal{X}}}\left(P_{\mathcal{D}}\sum_{k=0}^n \widehat{A}^k \widehat{B} u_k\right). \qquad (3.7)$$

The inclusion $\mathcal{D} \subset \mathcal{X}_1$ and the fact that $\mathcal{D}$ and $\mathcal{X}_1$ are Hilbert spaces now implies the inequality $E_{\widehat{\mathcal{X}}}\left(P_{\mathcal{D}}\sum_{k=0}^n \widehat{A}^k \widehat{B} u_k\right) \le E_{\widehat{\mathcal{X}}}\left(P_{\mathcal{X}_1}\sum_{k=0}^n \widehat{A}^k \widehat{B} u_k\right)$. It follows now from the Eqs. (3.6) and (3.7) that $E_{\mathcal{X}'}\left(\sum_{k=0}^n A'^k B' u_k\right) \le E_{\mathcal{X}}\left(\sum_{k=0}^n A^k B u_k\right)$, and the inequality (3.3) is proved.

Assume that $\widehat{\Sigma}$ is isometric. Since $\mathcal{D}$ is a Hilbert space, it follows from (3.7) that

$$E_{\mathcal{X}}\left(\sum_{k=0}^n A^k B u_k\right) \le E_{\widehat{\mathcal{X}}}\left(\sum_{k=0}^n \widehat{A}^k \widehat{B} u_k\right). \qquad (3.8)$$

By Lemma 2.8, the controllable restriction $\widehat{\Sigma}_c = (\widehat{A}_c, \widehat{B}_c, \widehat{C}_c, D; \widehat{\mathcal{X}}^c, \mathcal{U}, \mathcal{Y}, \kappa)$ of $\widehat{\Sigma}$ is controllable isometric, and for every $n = 0, 1, 2, \ldots,$ it holds $\widehat{A}^n \widehat{B} = \widehat{A}_c^n \widehat{B}_c$. Therefore

$$E_{\widehat{\mathcal{X}}}\left(\sum_{k=0}^n \widehat{A}^k \widehat{B} u_k\right) = E_{\widehat{\mathcal{X}}^c}\left(\sum_{k=0}^n \widehat{A}_c^k \widehat{B}_c u_k\right). \qquad (3.9)$$

Similar argument show that if $\breve{\Sigma}^c = (\breve{A}_c, \breve{B}_c, \breve{C}_c, D; \breve{\mathcal{X}}^c, \mathcal{U}, \mathcal{Y}, \kappa)$ is the controllable restriction of the isometric system $\breve{\Sigma} = (\breve{A}, \breve{B}, \breve{C}, D; \breve{\mathcal{X}}, \mathcal{U}, \mathcal{Y}, \kappa)$, then $\breve{\Sigma}^c$ is controllable isometric and it holds

$$E_{\breve{\mathcal{X}}}\left(\sum_{k=0}^{n}\breve{A}^k\breve{B}u_k\right)=E_{\breve{\mathcal{X}}^c}\left(\sum_{k=0}^{n}\breve{A}_c^k\breve{B}_c u_k\right). \tag{3.10}$$

But $\widehat{\Sigma}^c$ and $\breve{\Sigma}^c$ are unitarily similar, and therefore

$$E_{\widehat{\mathcal{X}}^c}\left(\sum_{k=0}^{n}\widehat{A}_c^k\widehat{B}_c u_k\right)=E_{\breve{\mathcal{X}}^c}\left(\sum_{k=0}^{n}\breve{A}_c^k\breve{B}_c u_k\right). \tag{3.11}$$

By combining (3.8)–(3.11), the inequality (3.4) follows. □

Remark 3.4 It follows from the inequality (3.4) of Lemma 3.3 that if there exists an observable isometric realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, then it is *-optimal.

In the standard Hilbert space case, results of Arov [5] show that there exist optimal minimal realizations of a Schur function. The construction was based on the existence of the defect functions, see Sect. 4. Arov et. all provided new geometric proofs of these results in [6]. Saprikin used those new proofs and generalized Arov's results to Pontryagin state space case in [30]. It will be proved next that Arov's results holds in the case when all spaces are Pontryagin spaces. The geometric proofs in [6] can still be applied in the present setting with few appropriate changes.

Theorem 3.5 *Let $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index. Then:*

- (i) *The first minimal restriction of a simple conservative realization of θ is optimal minimal;*
- (ii) *The minimal passive system Σ^* is optimal if and only if the dual system Σ is *-optimal minimal;*
- (iii) *The second minimal restriction of a simple conservative realization of θ is *-optimal minimal;*
- (iv) *Optimal (*-optimal) minimal systems are unique up to unitary similarity, and every optimal (*-optimal) minimal realization of θ is the first minimal restriction (second minimal restriction) of some simple conservative realization of θ.*

Proof (i) Let $\Sigma'=(A',B',C',D;\mathcal{X}',\mathcal{U},\mathcal{Y};\kappa)$ be the first minimal restriction of a simple conservative realization $\widehat{\Sigma}'=(\widehat{A}',\widehat{B}',\widehat{C}',D;\widehat{\mathcal{X}}',\mathcal{U},\mathcal{Y};\kappa)$ of $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$. Let $\Sigma=(A,B,C,D;\mathcal{X},\mathcal{U},\mathcal{Y};\kappa)$ be the first minimal restriction of some conservative realization of θ such that its state space has negative index κ. To prove that Σ' is optimal, Lemma 3.3 shows that it is enough to prove

$$E_{\mathcal{X}'}\left(\sum_{k=0}^{n}A'^kB'u_k\right)\le E_{\mathcal{X}}\left(\sum_{k=0}^{n}A^kBu_k\right),\qquad n\in\mathbb{N}_0,\quad u_k\in\mathcal{U}. \tag{3.12}$$

By Lemma 3.2, it can be assumed that Σ is the first minimal restriction of some simple conservative realization $\widehat{\Sigma}=(\widehat{A},\widehat{B},\widehat{C},D;\widehat{\mathcal{X}},\mathcal{U},\mathcal{Y};\kappa)$ of θ. Since $\widehat{\Sigma}$ and $\widehat{\Sigma}'$ are both simple conservative, they are unitarily similar, so there exists a unitary operator $U : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}'$ such that $\widehat{A} = U^{-1}\widehat{A}'U$, $\widehat{B} = U^{-1}\widehat{B}'$ and $\widehat{C} = \widehat{C}'U$. Easy calculations shows that $\widehat{\mathcal{X}}'^{o} = U\widehat{\mathcal{X}}^{o}$, $\widehat{\mathcal{X}}'^{c} = U\widehat{\mathcal{X}}^{c}$, $(\widehat{\mathcal{X}}'^{o})^{\perp} = U(\widehat{\mathcal{X}}^{o})^{\perp}$, $(\widehat{\mathcal{X}}'^{c})^{\perp} = U(\widehat{\mathcal{X}}^{c})^{\perp}$ and $P_{\widehat{\mathcal{X}}'^{o}}\widehat{\mathcal{X}}'^{c} = UP_{\widehat{\mathcal{X}}^{o}}\widehat{\mathcal{X}}^{c}$. In particular,

$$P_{\mathcal{X}} = P_{P_{\widehat{\mathcal{X}}^{o}}\widehat{\mathcal{X}}^{c}} = U^{-1}P_{P_{\widehat{\mathcal{X}}'^{o}}\widehat{\mathcal{X}}'^{c}}U = U^{-1}P_{\mathcal{X}'}U,$$

which implies

$$A = P_{\mathcal{X}}\widehat{A}\restriction_{\mathcal{X}} = U^{-1}P_{\mathcal{X}'}\widehat{A}'U\restriction_{\mathcal{X}} = (U\restriction_{\mathcal{X}})^{-1}P_{\mathcal{X}'}\widehat{A}'\restriction_{\mathcal{X}'}U\restriction_{\mathcal{X}} = (U\restriction_{\mathcal{X}})^{-1}A'U\restriction_{\mathcal{X}}$$
$$B = (U\restriction_{\mathcal{X}})^{-1}B', \qquad C = C'U\restriction_{\mathcal{X}}.$$

It follows that Σ and Σ' are unitarily similar and the corresponding unitary operator is $U_0 = U\restriction_{\mathcal{X}}$. Then

$$E_{\mathcal{X}}\left(\sum_{k=0}^{n} A^k Bu_k\right) = E_{\mathcal{X}}\left(U_0^{-1}\sum_{k=0}^{n} A'^k B'u_k\right) = E_{\mathcal{X}'}\left(\sum_{k=0}^{n} A'^k B'u_k\right).$$

Therefore (3.12) holds, and Σ' is an optimal minimal system.

(ii) Let $\Sigma^* = (A^*, C^*, B^*, D^*; \mathcal{X}, \mathcal{Y}, \mathcal{U}; \kappa)$ be an optimal minimal passive realization of $\theta^{\#} \in \mathbf{S}_{\kappa}(\mathcal{Y}, \mathcal{U})$. Then $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ is a minimal passive realization of $\theta \in \mathbf{S}_{\kappa}(\mathcal{U}, \mathcal{Y})$. Consider an arbitrary observable passive realization $\Sigma' = (A', B', C', D; \mathcal{X}', \mathcal{U}, \mathcal{Y}; \kappa)$ of $\theta \in \mathbf{S}_{\kappa}(\mathcal{U}, \mathcal{Y})$. Then $\Sigma'^* = (A'^*, C'^*, B'^*, D^*; \mathcal{X}', \mathcal{Y}, \mathcal{U}; \kappa)$ is a controllable passive realization of $\theta^{\#}$. For a vector of the form $x' = \sum_{k=0}^{n} A'^{*k} C'^* y_k$, where $n \in \mathbb{N}_0$ and $y_k \in \mathcal{Y}$, define

$$Sx' = \sum_{k=0}^{n} (A^*)^k C^* y_k.$$

Since Σ'^* is controllable and Σ^* is optimal, the domain of S is dense, and it holds

$$E_{\mathcal{X}}(Sx) = E_{\mathcal{X}}\left(\sum_{k=0}^{n} (A^*)^k C^* y_k\right) \leq E_{\mathcal{X}'}\left(\sum_{k=0}^{n} A'^{*k} C'^* y_k\right) = E_{\mathcal{X}'}(x).$$

That is, S is a contractive linear relation with the dense domain. Then [1, Theorem 1.4.4] shows that the closure of S, which is still denoted as S, is contractive everywhere defined linear operator from $\mathcal{X}' \to \mathcal{X}$. Since $\mathcal{X}'$ and $\mathcal{X}$ are Pontryagin spaces with the same negative index, $S^* : \mathcal{X} \to \mathcal{X}'$, is contractive as well. The transfer functions of the Σ and Σ' coincide, and therefore $CA^m B = C'A'^k B'$ for every $m \in \mathbb{N}_0$. By definition, $S(A'^*)^m C'^* = (A^*)^m C^*$, or what is the same thing, $C'A'^m S^* = CA^m$, for every $m \in \mathbb{N}_0$. Then also

$$C'A'^{m+k}B' = CA^m A^k B = C'A'^m S^* A^k B \quad \text{for} \quad m, k \geq 0.$$

This implies $A'^k B' = S^* A^k B$ and moreover $S^* \left(\sum_{k=0}^{n} A^k B u_k\right) = \sum_{k=0}^{n} A'^k B' u_k$, since the system Σ' is observable. Therefore,

$$E_{\mathcal{X}'}\left(\sum_{k=0}^{n} A'^k B' u_k\right) = E_{\mathcal{X}'}\left(S^*\left(\sum_{k=0}^{n} A^k B u_k\right)\right) \leq E_{\mathcal{X}}\left(\sum_{k=0}^{n} A^k B u_k\right),$$

since S^* is contractive. This proves that Σ is *-optimal.
Suppose then that $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ is minimal passive *-optimal realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$. Then Σ^* is a minimal passive realization of $\theta^\# \in \mathbf{S}_\kappa(\mathcal{Y}, \mathcal{U})$. To prove the optimality of Σ^*, it suffices to consider all the minimal passive realizations of $\theta^\#$; see Lemma 3.3. Let $\Sigma'^* = (A'^*, C'^*, B'^*, D^*; \mathcal{X}', \mathcal{Y}, \mathcal{U}; \kappa)$ be a minimal passive realization of $\theta^\#$. Then Σ' is a minimal passive realization of θ. Since Σ is *-optimal, the inequality

$$E_{\mathcal{X}}\left(\sum_{k=0}^{n} A^k B u_k\right) \geq E_{\mathcal{X}'}\left(\sum_{k=0}^{n} A'^k B' u_k\right), \qquad n \in \mathbb{N}_0, \quad u_k \in \mathcal{U},$$

holds. Define $Kx = \sum_{k=0}^{n} A'^k B' u_k$ for $x = \sum_{k=0}^{n} A^k B u_k$. Using similar techniques as above, K can be extended to be a contractive operator from $\mathcal{X} \to \mathcal{X}'$ such that

$$K^*(A'^*)^k C'^* = (A^*)^k C^*.$$

Since K^* is contractive,

$$E_{\mathcal{X}}\left(\sum_{k=0}^{n} A^{*k} C^* y_k\right) = E_{\mathcal{X}}\left(K^* \sum_{k=0}^{n} A'^{*k} C'^* y_k\right) \leq E_{\mathcal{X}'}\left(\sum_{k=0}^{n} A'^{*k} C'^* y_k\right),$$

for $\{y_k\} \subset \mathcal{Y}$. This shows that Σ^* is optimal.

(iii) Let Σ be a simple conservative realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$. Then Σ^* is a simple conservative realization of $\theta^\#$, and the first minimal restriction $\Sigma^{*\prime}$ of Σ^* is optimal minimal by the part (i). By using the representations (2.18) and (2.19) from Lemma 2.8, it is easy to deduce that the dual system of $\Sigma^{*\prime}$ is the second minimal restriction Σ'' of Σ, and it follows from the part (ii) that Σ'' is *-optimal.

(iv) Only the proofs of the claims considering optimal minimal realizations will be given, since the claims considering *-optimal minimal realizations can be proved analogously. Let $\Sigma_j = (A_j, B_j, C_j, D; \mathcal{X}_j, \mathcal{U}, \mathcal{Y}; \kappa)$ for $j = 1, 2$, be optimal minimal realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$. In a sufficiently small neighbourhood of the origin, the transfer functions θ_{Σ_1} and θ_{Σ_2} of the systems Σ_1 and Σ_2 have the Neumann series and they coincide, so $C_1 A_1^k B_1 = C_2 A_2^k B_2$ for $k = 0, 1, 2, \ldots$ Define

$$Ux = \sum_{k=0}^{N} A_2^k B_2 u_k \tag{3.13}$$

for a vector x of the form $x = \sum_{k=0}^{N} A_1^k B_1 u_k$, where $\{u_k\} \in \mathcal{U}$. Since Σ_1 is controllable, such vectors are dense in $\mathcal{X}_1$. Because Σ_2 is controllable as well, vectors of the form Ux are dense in $\mathcal{X}_2$.

Since Σ_1 and Σ_2 both are optimal realizations, $E_{\mathcal{X}_1}(x) = E_{\mathcal{X}_1}(Ux)$, and therefore U is an isometric linear relation with the dense domain and the dense range. It follows now from [1, 1.4.2] that the closure of U is a unitary operator, which is still denoted as U. Then, trivially $B_1 = U^{-1}B_2$. For vector x in (3.13), it holds

$$UA_1x = U\sum_{k=0}^{N} A_1^{k+1}B_1u_k = \sum_{k=0}^{N} A_2^{k+1}B_2u_k = A_2Ux.$$

It follows that $UA_1x = A_2Ux$ holds in a dense set, and therefore by continuity, everywhere. Thus $A_1 = U^{-1}A_2U$. Moreover, for $k = 0, 1, 2, \ldots,$ one concludes $C_1A_1^kB_1 = C_2A_2^kB_2 = C_2UA_1^kB_1$. Since $\operatorname{span}_{k\in\mathbb{N}_0}A_1^kB_1$ is dense in $\mathcal{X}_1$, it must be $C_1 = C_2U$. It has been shown that the unitary operator U has all the properties of (2.30), and therefore Σ_1 and Σ_2 are unitarily similar.

Suppose then that $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ is an optimal minimal realization of θ. Let $\widehat{\Sigma}_0 = (\widehat{A}_0, \widehat{B}_0, \widehat{C}_0, D; \widehat{\mathcal{X}}_0, \mathcal{U}, \mathcal{Y}; \kappa)$ be some simple conservative realization of θ. Lemma 2.8 shows that the system operator of $\widehat{\Sigma}$ can be represented as

$$T_{\widehat{\Sigma}_0} = \begin{pmatrix} \begin{pmatrix} A'_{11} & A'_{12} & A'_{13} \\ 0 & A' & A'_{23} \\ 0 & 0 & A'_{33} \end{pmatrix} & \begin{pmatrix} B'_1 \\ B' \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & C' & C'_1 \end{pmatrix} & D \end{pmatrix} : \begin{pmatrix} \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}' \\ \mathcal{X}_2 \end{pmatrix} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}' \\ \mathcal{X}_2 \end{pmatrix} \\ \mathcal{Y} \end{pmatrix},$$

where $\mathcal{X}_1 = (\widehat{\mathcal{X}}^o)^\perp$, $\mathcal{X}' = \overline{P_{\widehat{\mathcal{X}}^o}\widehat{\mathcal{X}}^c}$ and $\mathcal{X}_2 = \widehat{\mathcal{X}}^o \cap (\widehat{\mathcal{X}}^c)^\perp$. Now $\Sigma' = (A', B', C', D; \mathcal{X}', \mathcal{U}, \mathcal{Y}; \kappa)$ is the first minimal restriction of $\widehat{\Sigma}$, and it follows from part (i) that Σ' is optimal minimal, and moreover, as proved above, unitarily similar with Σ. Therefore, there exists a unitary operator $U : \mathcal{X} \to \mathcal{X}'$ such that $A = U^{-1}A'U$, $B = U^{-1}B'$ and $C = C'U$. Define

$$T_{\widehat{\Sigma}} = \begin{pmatrix} \begin{pmatrix} A'_{11} & A'_{12}U & A'_{13} \\ 0 & A & U^{-1}A'_{23} \\ 0 & 0 & A'_{33} \end{pmatrix} & \begin{pmatrix} B'_1 \\ B \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & C & C'_1 \end{pmatrix} & D \end{pmatrix} : \begin{pmatrix} \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X} \\ \mathcal{X}_2 \end{pmatrix} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X} \\ \mathcal{X}_2 \end{pmatrix} \\ \mathcal{Y} \end{pmatrix},$$

and let $\widehat{\Sigma}$ be the system corresponding the system operator $T_{\widehat{\Sigma}}$. Easy calculations show that $\widehat{\Sigma}$ and $\widehat{\Sigma}_0$ are unitarily similar and

$$\widehat{U} = \begin{pmatrix} I & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & I \end{pmatrix} : \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X} \\ \mathcal{X}_2 \end{pmatrix} \to \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}' \\ \mathcal{X}_2 \end{pmatrix}$$

is the corresponding unitary operator. Therefore $\widehat{\Sigma}$ is a simple conservative system. Now $\widehat{U}$ maps $P_{\mathcal{X}^o}\mathcal{X}^c$ to $P_{\mathcal{X}'^o}\mathcal{X}'^c$, and $\widehat{U}\mathcal{X}' = U\mathcal{X}' = \mathcal{X}$. It follows that Σ is the first minimal restriction of $\widehat{\Sigma}$. □

4 Generalized Defect Functions

If $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, it is well known that $S \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ is holomorphic in the unit disk and it has non-tangential contractive strong limit values almost everywhere (a.e.) on the unit circle $\mathbb{T}$. Therefore, S can be extended to $L^\infty(\mathcal{U}, \mathcal{Y})$ function, that is, the class of weakly measurable a.e. defined and essentially bounded $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued functions on $\mathbb{T}$. Then it follows from [29, Theorem V.4.2] that there exist a Hilbert space $\mathcal{K}$ and an outer function $\varphi_S \in \mathbf{S}(\mathcal{U}, \mathcal{K})$ such that

$$\varphi_S^*(\zeta)\varphi_S(\zeta) \leq I - S^*(\zeta)S(\zeta) \tag{4.1}$$

a.e. on $\mathbb{T}$, and if a function $\widehat{\varphi} \in \mathbf{S}(\mathcal{U}, \widehat{\mathcal{K}})$, where $\widehat{\mathcal{K}}$ is a Hilbert space, has this same property, then

$$\widehat{\varphi}^*(\zeta)\widehat{\varphi}(\zeta) \leq \varphi_S^*(\zeta)\varphi_S(\zeta) \tag{4.2}$$

a.e. on $\mathbb{T}$. The function φ_S is called the **right defect function** of S. For the notions of the outer functions, *-outer functions, inner functions and *-inner functions, see [29, Chapter V]. From [29, Theorem V.4.2] it is also easy to deduce that there exists a Hilbert space $\mathcal{H}$ and a *-outer function $\psi_S \in \mathbf{S}(\mathcal{H}, \mathcal{Y})$ such that

$$\psi_S(\zeta)\psi_S^*(\zeta) \leq I - S(\zeta)S^*(\zeta) \tag{4.3}$$

a.e. $\zeta \in \mathbb{T}$ and if a Schur function $\widehat{\psi} \in \mathbf{S}(\widehat{\mathcal{H}}, \mathcal{Y})$ has this same property, then

$$\psi_S(\zeta)\psi_S^*(\zeta) \leq \widehat{\psi}(\zeta)\widehat{\psi}^*(\zeta). \tag{4.4}$$

The function ψ_S is called the **left defect function** of S. Both φ_S and ψ_S are unique up to a unitary constant.

The theory of the defect functions is considered, for instance, in [17–19]. Various connections of defect functions and passive realizations can be found in [3,7,8]. The definition of the defect functions was generalized for functions $S \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ in [27] by using the Kreĭn–Langer factorizations and the fact that all functions in $\mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ have also contractive strong limit values a.e. on $\mathbb{T}$. If $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces such that their negative index is not zero, the defect functions cannot be defined similarly as in the Hilbert space setting, since the boundary values of $S \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ may not be Hilbert space contractions. However, in the Hilbert state space case, Arov and Saprikin showed in [10] that for a function $S = S_r B_r^{-1} \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where $S_r B_r^{-1}$ is the right Kreĭn–Langer factorization of S, the existence of the optimal minimal realization of S is connected with the existence of the right defect function of S_r. In general, similar connections exist with certain functions constructed by embedded systems, and those function are called defect functions; this is the approach taken here.

Suppose that $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ is a passive realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$. Denote the system operator of Σ by T. Theorem 2.1 shows that T has a Julia operator of the form

$$\begin{pmatrix} T & D_{T^*} \\ D_T^* & -L^* \end{pmatrix} : \begin{pmatrix} \mathcal{X} \oplus \mathcal{U} \\ \mathfrak{D}_{T^*} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \oplus \mathcal{Y} \\ \mathfrak{D}_T \end{pmatrix}, \tag{4.5}$$

where $\mathfrak{D}_{T^*}$ and $\mathfrak{D}_T$ are Hilbert spaces, $D_{T^*} D_{T^*}^* = I - TT^*$ and $D_T D_T^* = I - T^*T$ such that D_T and D_{T^*} have zero kernels. Then, one can form the **Julia embedding** $\widetilde{\Sigma}$ of the system Σ; recall the embeddings from page 5. That is, the corresponding system operator $T_{\widetilde{\Sigma}}$ of the embedding $\widetilde{\Sigma}$ is a Julia operator of T, and it is of the form

$$T_{\widetilde{\Sigma}} = \begin{pmatrix} A & \begin{pmatrix} B & D_{T^*_{,1}} \end{pmatrix} \\ \begin{pmatrix} C \\ D^*_{T,1} \end{pmatrix} & \begin{pmatrix} D & D_{T^*_{,2}} \\ D^*_{T,2} & -L^* \end{pmatrix} \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \begin{pmatrix} \mathcal{U} \\ \mathfrak{D}_{T^*} \end{pmatrix} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \begin{pmatrix} \mathcal{Y} \\ \mathfrak{D}_T \end{pmatrix} \end{pmatrix}, \tag{4.6}$$

where $D_{T^*} = \begin{pmatrix} D_{T^*_{,1}} \\ D_{T^*_{,2}} \end{pmatrix}$ and $D_T = \begin{pmatrix} D_{T,1} \\ D_{T,2} \end{pmatrix}$. The transfer function of the Julia embedding is

$$\begin{aligned} \theta_{\widetilde{\Sigma}}(z) &= \begin{pmatrix} D + zC(I - zA)^{-1}B & D_{T^*_{,2}} + zC(I - zA)^{-1} D_{T^*_{,1}} \\ D^*_{T,2} + zD^*_{T,1}(I - zA)^{-1}B & -L^* + zD^*_{T,1}(I - zA)^{-1} D_{T^*_{,1}} \end{pmatrix} \\ &= \begin{pmatrix} \theta(z) & \psi(z) \\ \varphi(z) & \chi(z) \end{pmatrix}. \end{aligned} \tag{4.7}$$

Moreover, the identities (2.11) and (2.12) of Lemma 2.3 hold for the system Σ and its transfer function θ. If $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, similar arguments as used in the proof of Proposition 2.4 and in the proof of [27, Lemma 3.2] show that φ, ψ and χ are generalized Schur functions with the index not larger than κ.

Definition 4.1 Let $\mathcal{U}$ and $\mathcal{Y}$ be Pontryagin spaces with the same negative index. Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ be an optimal minimal passive realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, and let $\widetilde{\Sigma}$ be the Julia embedding of it, represented as in (4.6). Then the function φ in (4.7) is defined to be the right defect function φ_θ of θ.

Moreover, let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ be a *-optimal minimal passive realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, and let $\widetilde{\Sigma}$ be the Julia embedding of it, represented as in (4.6). Then the function ψ in (4.7) is defined to be the left defect function ψ_θ of θ.

Remark 4.2 Since optimal (*-optimal) minimal realizations are unitarily similar by Theorem 3.5, and Julia operators for contractive operator are essentially unique by Theorem 2.1, it can be deduced that the defect functions are essentially uniquely defined by $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$. The definition above is also slightly different from the one given in [27] for functions in the class $\mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces.

The right defect function of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ and the left defect function of $\theta^\#$ are closely related to each other.

Lemma 4.3 *For $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, it holds $\varphi_\theta^\# = \psi_{\theta^\#}$ and $\psi_\theta^\# = \varphi_{\theta^\#}$*

Proof Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ be an optimal (*-optimal) minimal realization of θ. Denote the system operator of Σ as T, and the Julia operator $T_{\widetilde{\Sigma}}$ of T as in (4.6). By Theorem 3.5, the system Σ^* is *-optimal (optimal) minimal, and a calculation shows that $T_{\widetilde{\Sigma}}^*$ is the Julia operator of T^*. Now the results follow means of (4.7). □

In the Hilbert space setting, $S \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ has factorizations of the form

$$S = S_i S_o = S_{*o} S_{*i},$$

where $S_i \in \mathbf{S}(\mathcal{Y}', \mathcal{Y})$ is inner, $S_o \in \mathbf{S}(\mathcal{U}, \mathcal{Y}')$ is outer, $S_{*o} \in \mathbf{S}(\mathcal{U}', \mathcal{Y})$ is *-outer, $S_{*i} \in \mathbf{S}(\mathcal{U}, \mathcal{U}')$ is *-inner, and $\mathcal{Y}'$ and $\mathcal{U}'$ are Hilbert spaces [29, p. 204]. The next proposition shows that for an ordinary Schur function $\theta \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$, the outer factor of φ_θ and the *-outer factor of ψ_θ defined above coincide essentially with the usual definition of defect functions.

Proposition 4.4 *Let $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces. Then*

$$\varphi_\theta^*(\zeta)\varphi_\theta(\zeta) \leq I - \theta^*(\zeta)\theta(\zeta)$$

a.e. on $\mathbb{T}$, and if a generalized Schur function $\widehat{\varphi} \in \mathbf{S}_{\kappa'}(\mathcal{U}, \widehat{\mathcal{K}})$, where $\widehat{\mathcal{K}}$ is a Hilbert space and κ' does not depend on κ, has this same property, then

$$\widehat{\varphi}^*(\zeta)\widehat{\varphi}(\zeta) \leq \varphi_\theta^*(\zeta)\varphi_\theta(\zeta),$$

a.e. on $\mathbb{T}$. If $\kappa = 0$, denote the inner and outer factors of φ_θ as φ_{θ_i} and φ_{θ_o}, respectively. Then, φ_{θ_i} is an isometric constant, and if φ' is an outer function with properties (4.1) and (4.2), then it holds $U\varphi_{\theta_o} = \varphi'$, where U is a unitary operator.

Moreover,

$$\psi_\theta(\zeta)\psi_\theta^*(\zeta) \leq I - \theta(\zeta)\theta^*(\zeta)$$

a.e. $\zeta \in \mathbb{T}$ and if a generalized Schur function $\widehat{\psi} \in \mathbf{S}_{\kappa'}(\widehat{\mathcal{H}}, \mathcal{Y})$, where $\widehat{\mathcal{K}}$ is a Hilbert space and κ' does not depend on κ, has this same property, then

$$\psi_\theta(\zeta)\psi_\theta^*(\zeta) \leq \widehat{\psi}(\zeta)\widehat{\psi}^*(\zeta)$$

a.e. $\zeta \in \mathbb{T}$. If $\kappa = 0$, denote the *-inner and *-outer factors of ψ_θ as $\psi_{\theta_{*i}}$ and $\psi_{\theta_{*o}}$, respectively. Then, $\psi_{\theta_{*i}}$ is a co-isometric constant, and if ψ' is a *-outer function with properties (4.3) and (4.4), then it holds $\psi_{\theta_{*o}}U' = \psi'$, where U' is a unitary operator.

Proof Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ be an optimal minimal realization of θ. Denote the system operator of Σ as T, the Julia operator $T_{\widetilde{\Sigma}}$ of T as in (4.6) and the function $\varphi = \varphi_\theta$ as in (4.7). Since $T_{\widetilde{\Sigma}}$ is unitary, the operator

$$T_{\Sigma'} = \begin{pmatrix} A & B \\ \begin{pmatrix} C \\ D^*_{T,1} \end{pmatrix} & \begin{pmatrix} D \\ D^*_{T,2} \end{pmatrix} \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \begin{pmatrix} \mathcal{Y} \\ \mathfrak{D}_T \end{pmatrix} \end{pmatrix}.$$

must be isometric, and therefore the system

$$\Sigma' = \left(A, B, \begin{pmatrix} C \\ D^*_{T,1} \end{pmatrix}, \begin{pmatrix} D \\ D^*_{T,2} \end{pmatrix}; \mathcal{X}, \mathcal{U}, \begin{pmatrix} \mathcal{Y} \\ \mathfrak{D}_T \end{pmatrix}; \kappa \right)$$

is an isometric realization of the function $\begin{pmatrix} \theta \\ \varphi_\theta \end{pmatrix}$. Since Σ' is an embedding of the minimal system Σ, the system Σ' is also minimal. It follows from Theorem 2.6 that $\begin{pmatrix} \theta \\ \varphi_\theta \end{pmatrix} \in \mathbf{S}_\kappa \left(\mathcal{U}, \mathcal{Y} \oplus \mathfrak{D}_T\right)$. Since contractive boundary values of generalized Schur functions exist for a.e. $\zeta \in \mathbb{T}$, it holds

$$\begin{pmatrix} \theta^*(\zeta) & \varphi^*_\theta(\zeta) \end{pmatrix} \begin{pmatrix} \theta(\zeta) \\ \varphi_\theta(\zeta) \end{pmatrix} \leq I \quad \Longleftrightarrow \quad \varphi^*_\theta(\zeta)\varphi_\theta(\zeta) \leq I - \theta^*(\zeta)\theta(\zeta)$$

for a.e. $\zeta \in \mathbb{T}$.

Suppose that a function $\widehat{\varphi} \in \mathbf{S}_{\kappa'}(\mathcal{U}, \widehat{\mathcal{K}})$, where $\widehat{\mathcal{K}}$ is a Hilbert space, has the property $\widehat{\varphi}^*(\zeta)\widehat{\varphi}(\zeta) \leq I - \theta^*(\zeta)\theta(\zeta)$ for a.e. $\zeta \in \mathbb{T}$. Since the function $\widehat{\varphi}$ has the left Kreĭn–Langer factorization of the form $\widehat{\varphi} = B_{\widehat{\varphi}}^{-1}\widehat{\varphi}_l$, where $\widehat{\varphi}_l$ is an ordinary Schur function, it holds $\widehat{\varphi}^*(\zeta)\widehat{\varphi}(\zeta) = \widehat{\varphi}^*_l(\zeta)\widehat{\varphi}_l(\zeta)$ for a.e. $\zeta \in \mathbb{T}$. Then the function

$$\breve{\theta} = \begin{pmatrix} \theta \\ \widehat{\varphi}_l \end{pmatrix}, \tag{4.8}$$

belongs to the Schur class $\mathbf{S}_\kappa \left(\mathcal{U}, \mathcal{Y} \oplus \widehat{\mathcal{K}}\right)$, and it has a controllable isometric realization $\breve{\Sigma}$ with the system operator

$$T_{\breve{\Sigma}} = \begin{pmatrix} A_1 & B_1 \\ \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} & \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} \end{pmatrix} : \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \mathcal{X}_1 \\ \begin{pmatrix} \mathcal{Y} \\ \widehat{\mathcal{K}} \end{pmatrix} \end{pmatrix}. \tag{4.9}$$

That is,

$$\begin{aligned} \breve{\theta}(z) = \begin{pmatrix} \theta(z) \\ \widehat{\varphi}_l(z) \end{pmatrix} &= \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} + z \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (I - zA_1)^{-1} B_1 \\ &= \begin{pmatrix} D_1 + zC_1(I - zA_1)^{-1}B_1 \\ D_2 + zC_2(I - zA_1)^{-1}B_1 \end{pmatrix}. \end{aligned}$$

It follows that

$$\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}; \kappa) \tag{4.10}$$

is a realization of θ, and since $\breve{\Sigma}$ is isometric and $\widehat{\mathcal{K}}$ is a Hilbert space, the system Σ_1 is passive. Since $T_{\breve{\Sigma}}$ is isometric, the defect operator $D_{T_{\breve{\Sigma}}}$ of $T_{\breve{\Sigma}}$ is zero, and it follows from Lemma 2.3 that

$$\begin{aligned} I - \breve{\theta}^*(z)\breve{\theta}(z) &= I - \theta^*(z)\theta(z) - \widehat{\varphi}_l^*(z)\widehat{\varphi}_l(z) \\ &= \left(1 - |z|^2\right) B_1^*(I - \bar{z}A_1^*)^{-1})(I - zA_1)^{-1} B_1 \end{aligned} \tag{4.11}$$

whenever the expressions are meaningful. By combining the identities (2.12) and (4.11) for optimal minimal realization Σ, one gets

$$\begin{aligned} &\left(1 - |z|^2\right) B_1^*(I - \bar{z}A_1^*)^{-1}(I - zA_1)^{-1} B_1 + \widehat{\varphi}_l^*(z)\widehat{\varphi}_l(z) \\ &\quad = \left(1 - |z|^2\right) B^*(I - \bar{z}A^*)^{-1}(I - zA)^{-1} B + {\varphi_\theta}^*(z)\varphi_\theta(z) \end{aligned} \tag{4.12}$$

for every z in a sufficiently small symmetric neighbourhood Ω of the origin. Since the system Σ is optimal, if follows by using Neumann series that

$$\begin{aligned} &\left\langle B^*(I - \bar{z}A^*)^{-1}(I - zA)^{-1}Bu, u\right\rangle \\ &= E_{\mathcal{X}}\left((I - zA)^{-1} Bu\right) = E_{\mathcal{X}}\left(\sum_{n=0}^{\infty} A^n Buz^n\right) \\ &\leq E_{\mathcal{X}_1}\left(\sum_{n=0}^{\infty} A_1^n B_1 uz^n\right) = \left\langle B_1^*(I - \bar{z}A_1^*)^{-1}(I - zA_1)^{-1}B_1u, u\right\rangle \end{aligned}$$

for every $z \in \Omega$ and for every $u \in \mathcal{U}$. Then it follows from (4.12) that $\widehat{\varphi}_l^*(z)\widehat{\varphi}_l(z) \leq {\varphi_\theta}^*(z)\varphi_\theta(z)$ for every $z \in \Omega$. By continuity,

$$\widehat{\varphi}_l^*(\zeta)\widehat{\varphi}_l(\zeta) = \widehat{\varphi}^*(\zeta)\widehat{\varphi}(\zeta) \leq {\varphi_\theta}^*(\zeta)\varphi_\theta(\zeta) \tag{4.13}$$

for a.e. $\zeta \in \mathbb{T}$.

Next suppose that $\kappa = 0$. By combining (4.2) and (4.13), it can be deduced that

$$\varphi^{'*}(\zeta)\varphi'(\zeta) = {\varphi_\theta}^*(\zeta)\varphi_\theta(\zeta) = {\varphi_{\theta_o}}^*(\zeta){\varphi_{\theta_i}}^*(\zeta)\varphi_{\theta_i}(\zeta)\varphi_{\theta_o}(\zeta) = {\varphi_{\theta_o}}^*(\zeta)\varphi_{\theta_o}(\zeta)$$

for a.e. $\zeta \in \mathbb{T}$. Then it follows from [29, Proposition V.4.1] that $\varphi' = U\varphi_{\theta_o}$, where U is a unitary operator. If one puts an outer function $\widehat{\varphi}_l = \varphi_{\theta_o} = U^{-1}\varphi'$ in (4.8) and constructs the operator $T_{\breve{\Sigma}}$ as in (4.9) , the construction of an optimal minimal system used in the proof of [5, Theorem 7] shows that the associated system Σ_1 in (4.10) is optimal. Since Σ is also optimal, for every $z \in \mathbb{D}$, it holds

$$B^*(I-\overline{z}A^*)^{-1}(I-zA)^{-1}B = B_1^*(I-\overline{z}A_1^*)^{-1}(I-zA_1)^{-1}B_1.$$

Then it follows from (4.12) that $\|\varphi_{\theta_i}(z)\varphi_{\theta_o}(z)u\| = \|\varphi_{\theta_o}(z)u\|$ for every $z \in \mathbb{D}$ and every $u \in \mathcal{U}$. The outer function $\varphi_{\theta_o}(z)$ has a dense range for every $z \in \mathbb{D}$ [29, Proposition V.2.4]. This implies that $\varphi_{\theta_i}(z)$ is an isometry for every $z \in \mathbb{D}$, and arguing as in the proof of [29, Proposition V.2.1] one deduces that φ_{θ_i} is an isometric constant. The claims involving φ_θ are proved.

The claims involving ψ_θ follow now directly by applying Lemma 4.3. □

Lemma 4.5 *Let $\Sigma_0 = (A_0, B_0, C_0, D; \mathcal{X}_0, \mathcal{U}, \mathcal{Y}; \kappa)$ and $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ be passive realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ such that Σ_0 is optimal. If for every z and w in a sufficiently small symmetric neighbourhood Ω of the origin the equality*

$$B^*(I-\overline{w}A^*)^{-1}(I-zA)^{-1}B = B_0^*(I-\overline{w}A_0^*)^{-1}(I-zA_0)^{-1}B_0 \tag{4.14}$$

holds, then Σ is optimal.

Proof It follows from Lemma 2.8 that the system operator T_Σ of Σ can be represented as in (2.16), the restriction $\Sigma_c = (A_c, B_c, C_c, D; \mathcal{X}^c, \mathcal{U}, \mathcal{Y}; \kappa)$ of Σ to the controllable subspace $\mathcal{X}^c$ is controllable passive, and (2.25) and (2.26) hold.

Define $Rx = \sum_{j=1}^M A_0^j B_0 u_j$ for the vectors of the form $x = \sum_{j=1}^M A_c^j B_c u_j$, where $M \in \mathbb{N}$ and $\{u_j\}_{j=1}^M \subset \mathcal{U}$. Since Σ_c is controllable, the domain of R is dense. Moreover, Σ_0 is optimal, and therefore $E_{\mathcal{X}_0}(Rx) \leq E_{\mathcal{X}^c}(x)$. That is, R is contractive, and it follows from [1, Theorem 1.4.2] that the closure of R is everywhere defined contractive linear operator. It is still denoted by R. Since

$$(I-zA_c)^{-1}B_c = \sum_{n=0}^\infty z^n A_c^n B_c, \qquad (I-zA_0)^{-1}B_0 = \sum_{n=0}^\infty z^n A_0^n B_0,$$

holds for every z in a sufficiently small symmetric neighbourhood Ω of the origin, it follows by continuity that $R\left((I-zA_c)^{-1}B_c u\right) = (I-zA_0)^{-1}B_0 u$ for every $z \in \Omega$ and $u \in \mathcal{U}$. Then

$$R\left(\sum_{j=1}^M (I-z_jA_c)^{-1}B_c u_j\right) = \sum_{j=1}^M (I_{\mathcal{X}_0} - z_jA_0)^{-1}B_0 u_j,$$

for all $M \in \mathbb{N}$, $\{z_j\}_{j=1}^M \subset \Omega$, and $\{u_j\}_{j=1}^M \subset \mathcal{U}$. Equalities (2.26) and (4.14) imply now

$$E_{\mathcal{X}^c}\left(\sum_{j=1}^M (I-z_jA_c)^{-1}B_c u_j\right) = \sum_{j=1}^M\sum_{k=1}^M \left\langle B_c^*(I-\overline{z_k}A_c^*)^{-1}(I-z_jA_c)^{-1}B_c u_j, u_k\right\rangle_{\mathcal{U}}$$
$$= \sum_{j=1}^M\sum_{k=1}^M \left\langle B_0^*(I-\overline{z_k}A_0^*)^{-1}(I-z_jA_0)^{-1}B_0 u_j, u_k\right\rangle_{\mathcal{U}}$$

$$= E_{\mathcal{X}_0}\left(\sum_{j=1}^{M}(I - z_j A_0)^{-1}B_0 u_j\right)$$
$$= E_{\mathcal{X}_0}\left(R\left(\sum_{j=1}^{M}(I - z_j A_c)^{-1}B_c u_j\right)\right).$$

This implies that R is isometric in span{ran $(I - zA_1)^{-1}B_1, z \in \Omega$}, which is a dense set, since Σ_1 is controllable. Since R is bounded, it is now isometric everywhere, and it follows that Σ_c is optimal. Then it follows from (2.25) that Σ is optimal, and the proof is complete. □

The main results of [3, Theorem 1.1] were generalized to the Pontryagin state space setting in [27, Theorem 4.4]. By using Definition 4.1, it can be shown that parts of this result, as well as [8, Theorem 1], hold also in the case when all the spaces are indefinite. Moreover, certain parts of [3, Theorem 1.1], [8, Theorem 1] and [27, Theorem 4.4] can be improved. Before stating these results, some lemmas are needed.

Lemma 4.6 *Let $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$. Then the following statements are equivalent:*

(i) all κ-admissible minimal passive realizations of θ are unitarily similar;
*(ii) there exists a minimal passive realization of θ such that it is both optimal and *-optimal;*
*(iii) all κ-admissible minimal passive realizations of θ are both optimal and *-optimal.*

Proof (i) ⇒ (iii). Suppose (i). Let the systems $\Sigma_1 = (A_1, B_1, C_1, D; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}; \kappa)$ and $\Sigma_2 = (A_2, B_2, C_2, D; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}; \kappa)$ be, respectively, minimal passive and optimal (*-optimal) minimal passive realizations of θ. Let U be the unitary operator from $\mathcal{X}_1$ to $\mathcal{X}_2$ with the properties described in (2.30). An easy calculation shows that

$$E_{\mathcal{X}_2}\left(\sum_{k=0}^{n} A_2^k B_2 u_k\right) = E_{\mathcal{X}_1}\left(U\sum_{k=0}^{n} A_1^k B_1 u_k\right) = E_{\mathcal{X}_1}\left(\sum_{k=0}^{n} A_1^k B_1 u_k\right)$$

for every $u \in \mathcal{U}$ and for every $n = 0, 1, 2, \ldots$ which implies that Σ_1 is actually optimal (*-optimal), and therefore (iii) holds.

(iii) ⇒ (ii). The claim (iii) trivially implies (ii).

(ii) ⇒ (i). Suppose (ii). Let the systems $\Sigma_1 = (A_1, B_1, C_1, D; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}; \kappa)$ and $\Sigma_2 = (A_2, B_2, C_2, D; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}; \kappa)$ be, respectively, optimal and *-optimal minimal passive realizations of θ. Let Z be the weak similarity mapping from $\mathcal{X}_1$ to $\mathcal{X}_2$ with the properties described in (2.31). It follows from (2.31) that all elements of the form $\sum_{k=0}^{n} A_1^k B_1 u_k$ belongs to the domain of Z, and $Z\left(\sum_{k=0}^{n} A_1^k B_1 u_k\right) = \sum_{k=0}^{n} A_2^k B_2 u_k$. Recall also here the construction of Z in the proof of [27, Theorem 2.5]. Since Σ_1 is both optimal and *-optimal,

$$E_{\mathcal{X}_2}\left(\sum_{k=0}^{n} A_2^k B_2 u_k\right) = E_{\mathcal{X}_2}\left(Z\sum_{k=0}^{n} A_1^k B_1 u_k\right) = E_{\mathcal{X}_1}\left(\sum_{k=0}^{n} A_1^k B_1 u_k\right).$$

Then it follows from [1, Theorem 1.4.2] that the operator Z has a unitary extension, and the properties in (2.30) follow by continuity. Therefore Σ_1 and Σ_2 are unitarily similar. Since unitary similarity clearly is a transitive property, (i) holds, and the proof is complete. □

Lemma 4.7 *If the system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ is an optimal passive realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, then $\mathcal{X}^c \subset \mathcal{X}^o$.*

Proof According to Proposition 2.7, the spaces $\mathcal{X}^o$ and $(\mathcal{X}^o)^\perp$ are regular subspaces and $(\mathcal{X}^o)^\perp$ is a Hilbert space. It follows from Lemma 2.8 that the system operator T of Σ can be represented as in (2.15), and the restriction $\Sigma_o = (A_o, B_o, C_o, D; \mathcal{X}^o, \mathcal{U}, \mathcal{Y}; \kappa)$ of Σ to the observable subspace $\mathcal{X}^o$ is observable passive realization of θ. For $n = 0, 1, 2, \ldots$, it holds

$$A^n = \begin{pmatrix} A_1^n & f(n) \\ 0 & A_0^n \end{pmatrix},$$

where $f(n)$ is an operator depending on n. Then for any $N \in \mathbb{N}_0$ and any $\{u_n\}_{n=0}^N \subset \mathcal{U}$, it holds

$$\sum_{n=0}^N A^n B u_n = \begin{pmatrix} \sum_{n=0}^N \left(A_1^n B_1 u_n + f(n) B_o u_n\right) \\ \sum_{n=0}^N A_o^n B_o u_n \end{pmatrix} = \begin{pmatrix} P_{(\mathcal{X}^o)^\perp}\left(\sum_{n=0}^N A^n B u_n\right) \\ P_{\mathcal{X}^o}\left(\sum_{n=0}^N A^n B u_n\right) \end{pmatrix}.$$

This implies

$$E_{\mathcal{X}}\left(\sum_{n=0}^N A^n B u_n\right) = E_{(\mathcal{X}^o)^\perp}\left(P_{(\mathcal{X}^o)^\perp}\left(\sum_{n=0}^N A^n B u_n\right)\right) + E_{\mathcal{X}^o}\left(\sum_{n=0}^N A_o^n B_o u_n\right).$$

But since Σ is optimal and $(\mathcal{X}^o)^\perp$ is a Hilbert space, one deduces $P_{(\mathcal{X}^o)^\perp}\left(\sum_{n=0}^N A^n B u_n\right) = 0$. That is, $\operatorname{span}\{A^n B : n = 0, 1, \ldots\} \subset \mathcal{X}^o$ and since $\mathcal{X}^o$ is closed, also $\overline{\operatorname{span}}\{A^n B : n \in \mathbb{N}_0\} = \mathcal{X}^c \subset \mathcal{X}^o$. □

The next Theorem contains promised extensions for some results of [3]. In particular, the fact that statements (I)(b), (II)(b) and (III)(b) implies the other statements, respectively, in parts (I), (II) and (III), is new also in the Hilbert space setting.

Theorem 4.8 *Let $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index.*

(I) The following statements are equivalent:

(a) $\varphi_\theta \equiv 0$;
(b) all κ-admissible controllable passive realizations of θ are minimal isometric;
(c) there exists an observable conservative realization of θ;
(d) all simple conservative realization of θ are observable;
(e) all observable co-isometric realizations of θ are conservative.

(II) The following statements are equivalent:

(a) $\psi_\theta \equiv 0$*;*
(b) all κ*-admissible observable passive realization of* θ *are minimal co-isometric;*
(c) there exists a controllable conservative realization of θ*;*
(d) all simple conservative realization of θ *are controllable;*
(e) all controllable isometric realizations of θ *are conservative.*

(III) The following statements are equivalent:

(a) $\varphi_\theta \equiv 0$ *and* $\psi_\theta \equiv 0$*;*
(b) all κ*-admissible simple passive realization of* θ *are minimal conservative;*
(d) there exists a minimal conservative realization of θ*.*

Proof (I) (a) $\Rightarrow$ (b). Suppose (a). Let the systems $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ and $\Sigma_0 = (A_0, B_0, C_0, D; \mathcal{X}_0, \mathcal{U}, \mathcal{Y}; \kappa)$ be, respectively, a controllable passive and an optimal minimal passive realizations of θ. Represent the Julia embeddings of Σ and Σ_0 as in (4.6). Then, (2.12) holds for Σ. Since $\varphi_\theta \equiv 0$, if follows from the definition of φ_θ that

$$I - \theta^*(w)\theta(z) = (1 - z\bar{w})B_0^*(I - \bar{w}A_0^*)^{-1}(I - zA_0)^{-1}B_0$$

holds for every z and w in a sufficiently small symmetric neighbourhood Ω of the origin. Since Σ_0 is optimal, by considering the Neuman series of $(I - zA_0)^{-1}B_0$ and $(I - zA_0)^{-1}B_0$, one deduces that

$$B_0^*(I - \bar{z}A_0^*)^{-1}(I - zA_0)^{-1}B_0 \leq B^*(I - \bar{z}A^*)^{-1}(I - zA)^{-1}B, \qquad z \in \Omega.$$

Then it holds $\varphi^*(z)\varphi(z) \leq= 0$ for every $z \in \Omega$. But since $\varphi(z)$ is an operator whose range belongs to the Hilbert space $\mathfrak{D}_T$, this implies $\varphi(z) = D_{T,2}^* + zD_{T,1}^*(I - zA)^{-1}B = 0$ for $z \in \Omega$. It follows that $D_{T,2}^* = 0$. Since Σ is controllable, span$\{(I - zA)^{-1}B; z \in \Omega\}$ is dense in $\mathcal{X}$ by the identity (2.4) and therefore also $D_{T,1}^* = 0$. Then $D_T = 0$, so T is isometric, and Σ is a controllable isometric system. In particular, if Σ is chosen to be minimal passive; for the existence, see Lemma 2.8, the previous argument shows that Σ is a minimal isometric realization of θ. Since all controllable isometric realizations of θ are unitarily similar, they are now also minimal, and (b) holds.

(b) $\Rightarrow$ (c). Suppose (b). Let $\Sigma' = (A', B', C', D; \mathcal{X}', \mathcal{U}, \mathcal{Y}; \kappa)$ be an optimal minimal passive realization of θ. The existence of Σ' follows from Theorem 3.5 (i). By assumption, Σ' is isometric. It follows from Theorem 3.5 (iv) that Σ' is the first minimal restriction of the simple conservative system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$. By Lemma 2.8, the system operator T_Σ of Σ can be represented as in (2.18), where now $\mathcal{X}' = \overline{P_{\mathcal{X}^o}\mathcal{X}^c}$.

$T_{\Sigma'}$ of Σ' is isometric and T_Σ is unitary, an easy calculation using the fact that the range space $(\mathcal{X}^o)^\perp$ is a Hilbert space shows that $B_1' = 0$ and $A_{12}' = 0$ in (2.18). But then for every $x \in (\mathcal{X}^o)^\perp$ and every $n = 0, 1, 2, \ldots$,

$$B^*A^{*n}x = \begin{pmatrix} 0 & B'^* & 0 \end{pmatrix} \begin{pmatrix} {A'_{11}}^* & 0 & 0 \\ 0 & {A'_0}^* & 0 \\ {A'_{13}}^* & {A'_{23}}^* & {A'_{33}}^* \end{pmatrix}^n \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = 0.$$

That is, $(\mathcal{X}^o)^\perp \subset (\mathcal{X}^c)^\perp$ and therefore $\mathcal{X}^c \subset \mathcal{X}^o$. Since Σ is simple, this implies now $\mathcal{X}^o = \mathcal{X}$. Then Σ is observable, and (c) holds.

(c) $\Rightarrow$ (a). Suppose (c). Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ be an observable conservative realization of θ. By Lemma 2.8, Σ can be represented as in (2.18). The first minimal restriction (2.23) of Σ is an optimal minimal realization of θ by Theorem 3.5 (i). But since Σ is observable, $\mathcal{X}^o = \mathcal{X}$ and $(\mathcal{X}^o)^\perp = \{0\}$. It follows that the reprentations (2.16) and (2.18) coinsides. That is, the first minimal restriction Σ' is just a restriction to the controllable subspace of Σ. By Lemma 2.8, Σ' is now isometric. Thus if one constructs a Julia operator of $T_{\Sigma'}$ as in (4.5), $D_{T_{\Sigma'}} = 0$, and then it follows from the definition of φ_θ and (4.7) that $\varphi_\theta \equiv 0$, and (a) holds.

The equivalences of the statements (c), (d) and (e) follow easily from the facts that all observable co-isometric realizations of θ are unitarily similar, all simple conservative realization of θ are unitarily similar and unitary similarity preserves the structural properties of the system and system operator. The part (I) is proven.

(II) The proof is analogous to the proof of the part (I), and the details are omitted.

(III) (a) $\Rightarrow$ (b). Suppose (a). By combining the parts (I) and (II), it follows that all controllable or observable passive realizations of θ are minimal conservative. Consider a simple passive realization $\Sigma = (A.B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ of θ. It follows from Lemma 2.8 that the contractive system operator T of Σ can be represented as in (2.15), where the restriction Σ_o in (2.20) is observable passive, and therefore now minimal conservative. Then the system operator T_{Σ_o} of Σ_o is unitary. Let $x \in \mathcal{X}^o$. Then, by contractivity of T and unitarity of T_{Σ_o}

$$E\left(\begin{pmatrix} A_1 & A_2 & B_1 \\ 0 & A_o & B_o \\ 0 & C_o & D \end{pmatrix}\begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}\right) = E\left(\begin{pmatrix} A_2x \\ A_ox \\ C_ox \end{pmatrix}\right) = E\left(A_2x\right) + E\left(\begin{pmatrix} A_0x \\ C_ox \end{pmatrix}\right)$$
$$= E\left(Tx\right) \leq E(x) = E(T_{\Sigma_o}x) = E\left(\begin{pmatrix} A_0x \\ C_ox \end{pmatrix}\right).$$

Since $A_2x \in (\mathcal{X}^o)^\perp$ and $(\mathcal{X}^o)^\perp$ is a Hilbert space, it follows that $A_2 = 0$. If one chooses $u \in \mathcal{U}$, a similar argument as above shows that $B_1 = 0$. Then for any $n \in \mathbb{N}$, it holds

$$A^nB = \begin{pmatrix} A_1 & 0 \\ 0 & A_o \end{pmatrix}^n \begin{pmatrix} 0 \\ B_o \end{pmatrix} = \begin{pmatrix} 0 \\ A_o^nB_o \end{pmatrix} \quad \text{and}$$
$$A^{*n}C^* = \begin{pmatrix} A_1^* & 0 \\ 0 & A_o^* \end{pmatrix}^n \begin{pmatrix} 0 \\ C_o^* \end{pmatrix} = \begin{pmatrix} 0 \\ A_o^{*n}C_o^* \end{pmatrix}.$$

This is only possible if $(\mathcal{X}^o)^\perp = 0$, since Σ is simple. But then the systems Σ_0 and Σ coincide, so the system Σ is minimal conservative, and (b) holds.

Now (b) trivially implies (c), and the fact that (c) implies (a) follows by combining the parts (I) and (II). The proof is complete. □

Remark 4.9 If $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, it follows from [27, Lemma 3.2] that simple passive realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ are κ-admissible. Therefore, in that case it is not necessary to assume the considered systems to be κ-admissible in Lemma 4.6 and Theorems 4.8 and 4.10, since the other assumptions already guarantee it. However, if $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, it is not known that are all simple passive, or even all minimal passive, realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ κ-admissible.

If $\varphi_\theta \equiv 0$ ($\psi_\theta \equiv 0$), then Theorem 4.8 shows that all κ-admissible minimal passive realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ are minimal isometric (co-isometric). In particular, they are controllable isometric (observable coisometric), and it follows from Theorem 2.6 that they are unitarily similar. This situation can occur also when the defect functions do not vanish identically. In what follows, the range of φ_θ and the domain of ψ_θ will be denoted, respectively, by $\mathfrak{D}_{\varphi_\theta}$ and $\mathfrak{D}_{\psi_\theta}$. In the Hilbert space setting, it is well known [18,19] that for a standard Schur function $\theta \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$, there exists a function $\chi_\theta \in L^\infty(\mathfrak{D}_{\psi_\theta}, \mathfrak{D}_{\varphi_\theta})$ such that the function

$$\Theta(\zeta) := \begin{pmatrix} \theta(\zeta) & \psi_\theta(\zeta) \\ \varphi_\theta(\zeta) & \chi_\theta(\zeta) \end{pmatrix} \tag{4.15}$$

has contractive values for a.e. $\zeta \in \mathbb{T}$. Under certain normalizing conditions for the functions φ_θ and ψ_θ, the function χ_θ is unique. In the Hilbert space setting, the important properties of the function $\chi_\theta(\zeta)$ established by Boiko and Dubovoj, were bublished without proof in the paper [16]. In general, χ_θ may has negative Fourier coefficients and therefore it is not a Schur function. In that case the function Θ in (4.15) is not a Schur function either. However, Arov and Nudelmann showed in [7,8] that Θ is a Schur function if and only if all minimal passive realizations of θ are unitarily similar. This result will be generalized to the indefinite setting in the following theorem. The proof uses optimal and *-optimal realizations as in [7,8], but it is more elementary.

Theorem 4.10 *Let $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, and let φ_θ and ψ_θ be defect functions of θ. Then all κ-admissible minimal passive realizations of θ are unitarily similar if and only if there exist an $\mathcal{L}(\mathfrak{D}_{\psi_\theta}, \mathfrak{D}_{\varphi_\theta})$-valued function χ_θ analytic in a neighbourhood of the origin such that*

$$\Theta = \begin{pmatrix} \theta & \psi_\theta \\ \varphi_\theta & \chi_\theta \end{pmatrix} \in \mathbf{S}_\kappa\left(\begin{pmatrix} \mathcal{U} \\ \mathfrak{D}_{\psi_\theta} \end{pmatrix}, \begin{pmatrix} \mathcal{Y} \\ \mathfrak{D}_{\varphi_\theta} \end{pmatrix}\right) \tag{4.16}$$

Proof Suppose that all κ-admissible minimal passive realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ are unitarily similar. Then it follows from Lemma 4.6 that every κ-admissible minimal passive realization is optimal and *-optimal. Take any κ-admissible minimal passive realization Σ of θ and consider its Julia embedding as in (4.6). Then the transfer function (4.7) of the Julia embedding belongs to the class $\mathbf{S}_\kappa\left(\mathcal{U} \oplus \mathfrak{D}_{T^*}, \mathcal{Y} \oplus \mathfrak{D}_T\right)$, and since Σ is both optimal and *-optimal, the upper right corner and lower left corner of (4.7) are defect functions of θ. Choose $\chi_\theta = \chi$ in (4.7), and the necessity is proven.

Suppose then that there exists an $\mathcal{L}(\mathfrak{D}_{\psi_\theta}, \mathfrak{D}_{\varphi_\theta})$-valued function χ_θ such that Θ in (4.16) belongs to the class $\mathbf{S}_\kappa\left(\mathcal{U} \oplus \mathfrak{D}_{\psi_\theta}, \mathcal{Y} \oplus \mathfrak{D}_{\varphi_\theta}\right)$. It suffices to show that there exists minimal passive realization Σ of θ such that it is both optimal and *-optimal; see Lemma 4.6. Let

$$\Sigma_\Theta = (A, \widetilde{B}, \widetilde{C}, \widetilde{D}; \mathcal{X}, \mathcal{U} \oplus \mathfrak{D}_{\psi_\theta}, \mathcal{Y} \oplus \mathfrak{D}_{\varphi_\theta}; \kappa)$$

be a simple conservative realization of $\Theta \in \mathbf{S}_\kappa\left(\mathcal{U} \oplus \mathfrak{D}_{\psi_\theta}, \mathcal{Y} \oplus \mathfrak{D}_{\varphi_\theta}\right)$. Then the system operator T_Θ of Σ_Θ can be represented as

$$T_\Theta = \begin{pmatrix} A & \begin{pmatrix} B & B_1 \end{pmatrix} \\ \begin{pmatrix} C \\ C_1 \end{pmatrix} & \begin{pmatrix} D & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \begin{pmatrix} \mathcal{U} \\ \mathfrak{D}_{\psi_\theta} \end{pmatrix} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \begin{pmatrix} \mathcal{Y} \\ \mathfrak{D}_{\varphi_\theta} \end{pmatrix} \end{pmatrix}.$$

In a sufficiently small symmetric neighbourhood Ω of the origin, it holds

$$\begin{aligned} \Theta(z) &= \begin{pmatrix} \theta(z) & \psi_\theta(z) \\ \varphi_\theta(z) & \chi_\theta(z) \end{pmatrix} \\ &= \begin{pmatrix} D + zC(I - zA)^{-1}B & D_{12} + zC(I - zA)^{-1}B_1 \\ D_{21} + zC_1 + (I - zA)^{-1}B & D_{22} + zC_1(I - zA)^{-1}B_1 \end{pmatrix}. \end{aligned}$$

The spaces $\mathfrak{D}_{\varphi_\theta}$ and $\mathfrak{D}_{\psi_\theta}$ are Hilbert spaces, and therefore it follows that the system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ is a passive realization of θ. Since Σ_Θ is conservative, Lemma 2.3 shows that

$$\begin{aligned} I - \Theta(z)\Theta^*(w) &= \begin{pmatrix} I_\mathcal{Y} - \theta(z)\theta^*(w) - \psi_\theta(z)\psi_\theta^*(w) & -\theta(z)\varphi_\theta^*(w) - \psi_\theta(z)\chi_\theta^*(w) \\ -\varphi_\theta(z)\theta^*(w) - \chi_\theta(z)\psi_\theta^*(w) & I_{\mathfrak{D}_{\varphi_\theta}} - \varphi_\theta(z)\varphi_\theta^*(w) - \chi_\theta(z)\chi_\theta^*(w) \end{pmatrix} \\ &= (1 - \bar{w}z)\widetilde{C}(I - zA)^{-1}(I - \bar{w}A^*)^{-1}\widetilde{C}^* \\ &= (1 - \bar{w}z)\begin{pmatrix} C(I - zA)^{-1}(I - \bar{w}A^*)^{-1}C^* & C(I - zA)^{-1}(I - \bar{w}A^*)^{-1}C_1^* \\ C_1(I - zA)^{-1}(I - \bar{w}A^*)^{-1}C^* & C_1(I - zA)^{-1}(I - \bar{w}A^*)^{-1}C_1^* \end{pmatrix} \\ I - \Theta^*(w)\Theta(z) &= \begin{pmatrix} I_\mathcal{U} - \theta^*(w)\theta(z) - \varphi_\theta^*(w)\varphi_\theta(z) & -\theta^*(w)\psi_\theta(z) - \varphi_\theta^*(w)\chi_\theta(z) \\ -\psi_\theta^*(w)\theta(z) - \chi_\theta^*(w)\varphi_\theta(z) & I_{\mathfrak{D}_{\psi_\theta}} - \psi_\theta^*(w)\psi_\theta(z) - \chi_\theta^*(w)\chi_\theta(z) \end{pmatrix} \\ &= (1 - \bar{w}z)\begin{pmatrix} B^*(I - \bar{w}A^*)^{-1}(I - zA)^{-1}B & B^*(I - \bar{w}A^*)^{-1}(I - zA)^{-1}B_1 \\ B_1^*(I - \bar{w}A^*)^{-1}(I - zA)^{-1}B & B_1^*(I - \bar{w}A^*)^{-1}(I - zA)^{-1}B_1 \end{pmatrix}. \end{aligned}$$

That is,

$$I_\mathcal{Y} - \theta(z)\theta^*(w) = (1 - \bar{w}z)C(I - zA)^{-1}(I - \bar{w}A^*)^{-1}C^* + \psi_\theta(z)\psi_\theta^*(w), \quad (4.17)$$

$$I_\mathcal{U} - \theta^*(w)\theta(z) = (1 - \bar{w}z)B^*(I - \bar{w}A^*)^{-1}(I - zA)^{-1}B + \varphi_\theta^*(w)\varphi_\theta(z). \quad (4.18)$$

An easy calculation and Lemma 4.3 show that the Eq. (4.17) is equivalent to

$$I_\mathcal{Y} - \theta^{\#*}(w)\theta^{\#}(z) = (1 - \bar{w}z)C(I - \bar{w}A)^{-1}(I - zA^*)^{-1}C^* + \varphi_{\theta^\#}{}^*(w)\varphi_{\theta^\#}(z).$$

Let $\Sigma' = (A', B', C', D; \mathcal{X}', \mathcal{U}, \mathcal{Y}; \kappa)$ and $\Sigma'' = (A'', B'', C'', D; \mathcal{X}'', \mathcal{U}, \mathcal{Y}; \kappa)$ be, respectively, an optimal minimal and a *-optimal minimal realizations of θ. It follows from Theorem 3.5 (ii) that Σ''^* is an optimal minimal realization of $\theta^\#$. Then, by the definition of φ_θ and $\varphi_{\theta^\#}$, it holds

$$I_{\mathcal{U}} - \theta^*(w)\theta(z) = (1 - \overline{w}z)B'^*(I - \bar{w}A'^*)^{-1}(I - zA')^{-1}B' + \varphi_\theta^*(w)\varphi_\theta(z)$$
$$I_{\mathcal{Y}} - \theta^{\#*}(w)\theta^\#(z) = (1 - \overline{w}z)C''(I - \bar{w}A'')^{-1}(I - zA''^*)^{-1}C''^* + \varphi_{\theta^\#}{}^*(w)\varphi_{\theta^\#}(z).$$

It follows that

$$B^*(I - \bar{w}A^*)^{-1}(I - zA)^{-1}B = B'^*(I - \bar{w}A'^*)^{-1}(I - zA')^{-1}B',$$
$$C(I - \bar{w}A)^{-1}(I - zA^*)^{-1}C^* = C''(I - \bar{w}A'')^{-1}(I - zA''^*)^{-1}C''^*.$$

By using Lemma 4.5, it can be deduced that Σ and Σ^* are optimal systems. Then it follows from Lemma 4.7 that $\mathcal{X}^c = \mathcal{X}^o$ and therefore $\mathcal{X}^s = \mathcal{X}^c = \mathcal{X}^o$. By Lemma 2.8, the restriction $\Sigma_s = (A_s, B_s, C_s, D; \mathcal{X}^s, \mathcal{U}, \mathcal{Y}; \kappa)$ of Σ to the simple subspace $\mathcal{X}^s$ is simple, and it holds $A^n B = A_s^n B_s$ and $A^{*n} C^* = A_s^{*n} C_s^*$ for every $n \in \mathbb{N}_0$. That is, Σ_s and Σ_s^* also are optimal systems. Moreover, they are minimal since $\mathcal{X}^s = \mathcal{X}^c = \mathcal{X}^o$. It follows now from Theorem 3.5 (ii) that Σ_s is also *-optimal, and the proof is complete. □

Acknowledgements Open access funding provided by University of Vaasa (UVA). I wish to thank Seppo Hassi for helpful discussions while preparing this paper.

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Integr. Equ. Oper. Theory (2020) 92:42
https://doi.org/10.1007/s00020-020-02600-w
Published online October 3, 2020

Integral Equations and Operator Theory

Generalized Schur–Nevanlinna functions and their realizations

Lassi Lilleberg

Abstract. Pontryagin space operator valued generalized Schur functions and generalized Nevanlinna functions are investigated by using discrete-time systems, or operator colligations, and state space realizations. It is shown that generalized Schur functions have strong radial limit values almost everywhere on the unit circle. These limit values are contractive with respect to the indefinite inner product, which allows one to generalize the notion of an inner function to Pontryagin space operator valued setting. Transfer functions of self-adjoint systems such that their state spaces are Pontryagin spaces, are generalized Nevanlinna functions, and symmetric generalized Schur functions can be realized as transfer functions of self-adjoint systems with Kreĭn spaces as state spaces. A criterion when a symmetric generalized Schur function is also a generalized Nevanlinna function is given. The criterion involves the negative index of the weak similarity mapping between an optimal minimal realization and its dual. In the special case corresponding to the generalization of an inner function, a concrete model for the weak similarity mapping can be obtained by using the canonical realizations.

Mathematics Subject Classification. Primary: 47A48; Secondary: 47A56, 47B50, 93B28.

Keywords. Operator colligation, Passive system, Self-adjoint system, Transfer function, Generalized Schur class, Generalized Nevanlinna class.

1. Introduction

Let $\mathcal{U}$ and $\mathcal{Y}$ be separable Pontryagin spaces with the same finite negative index, and let $\mathcal{L}(\mathcal{U},\mathcal{Y})$ be the class of bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$. An $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function θ belongs to **generalized Schur class** $\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, if it is holomorphic at the origin and the Schur kernel

$$K_\theta(w,z)=\frac{1-\theta(z)\theta^*(w)}{1-z\bar{w}},\qquad w,z\in\rho(\theta), \tag{1.1}$$

where $\theta^*(w)=(\theta(w))^*$, has κ negative squares. This means that for any finite sets of points $\{w_1,\ldots,w_n\}\subset\rho(\theta)$, where $\rho(\theta)$ is maximal domain of

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analyticity of θ, and vectors $\{f_1, \ldots, f_n\} \subset \mathcal{Y}$, the Hermitian matrix

$$\left(\langle K_\theta(w_j, w_i) f_j, f_i\rangle_{\mathcal{Y}}\right)_{i,j=1}^{n}, \tag{1.2}$$

where $\langle\cdot,\cdot\rangle_{\mathcal{Y}}$ is the inner product of $\mathcal{Y}$, has no more than κ negative eigenvalues, and there exists a matrix of the form (1.2) which has exactly κ negative eigenvalues. On the other hand, an $\mathcal{L}(\mathcal{U})$-valued function θ, where $\mathcal{U}$ is a Pontryagin space, belongs to **generalized Nevanlinna class** $\mathbf{N}_\kappa(\mathcal{U})$ if it is meromorphic on $\mathbb{C} \setminus \mathbb{R}$, **real**, or **symmetric**, in a sense that $\theta(z) = \theta^{\#}(z)$ for every $z \in \rho(\theta)$, where $\theta^{\#}(z)$ is defined to be $\theta^*(\bar{z})$, and the Nevanlinna kernel

$$N_\theta(w, z) = \frac{\theta(z) - \theta^*(w)}{z - \bar{w}}, \qquad w, z \in \rho(\theta), \tag{1.3}$$

has κ negative squares. If $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, the classes $\mathbf{S}_0(\mathcal{U}, \mathcal{Y})$ and $\mathbf{N}_0(\mathcal{U})$, which are denoted as $\mathbf{S}(\mathcal{U}, \mathcal{Y})$ and $\mathbf{N}(\mathcal{U})$, coincide with the ordinary Schur and Nevanlinna classes. That is, $\mathbf{S}(\mathcal{U}, \mathcal{Y})$ consists of $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued functions holomorphic and bounded by one in $\mathbb{D}$, and $\mathbf{N}(\mathcal{U})$ consists of $\mathcal{L}(\mathcal{U})$-valued functions holomorphic and symmetric in $\mathbb{C} \setminus \mathbb{R}$ such that their imaginary parts are nonnegative in the upper half plane. The classes of generalized Schur and Nevanlinna functions were first studied by Kreĭn and Langer in series of papers [26,27,27,29,30], first in the scalar case ($\mathcal{U} = \mathcal{Y} = \mathbb{C}$) and later in the operator valued case.

The study of the (generalized) Schur functions in infinite dimensional spaces naturally leads to contractive operators and passive **linear discrete-time systems**; or what is the same thing, contractive **operator colligations**. An operator colligation

$$\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \tag{1.4}$$

consists of a Kreĭn space $\mathcal{X}$ (the **state space**), Pontryagin spaces $\mathcal{U}$ (the **incoming space**) and $\mathcal{Y}$ (the **outgoing space**) with the same negative index, and the **system operator** $T_\Sigma \in \mathcal{L}(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y})$, where the direct orthogonal sum $\mathcal{X} \oplus \mathcal{U}$ or $\binom{\mathcal{X}}{\mathcal{U}}$ is with respect to the indefinite inner product. Here T_Σ is bounded and everywhere defined, and has the block representation of the form

$$T_\Sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix}, \tag{1.5}$$

where $A \in \mathcal{L}(\mathcal{X})$ is the **main operator**, and $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. The colligation will be usually called as a system, since it can be seen as a linear discrete-time system, and the system is identified with its operator expression (1.5). The system Σ is **passive** (isometric, co-isometric, conservative, self-adjoint), if the system operator T_Σ in (1.5) is contractive (isometric, co-isometric, unitary, self-adjoint) with respect to the indefinite inner product. The **transfer function** of the system (1.5), or characteristic function of the operator colligation, is defined by

$$\theta_\Sigma(z) := D + zC(I - zA)^{-1}B, \tag{1.6}$$

whenever $I - zA$ is invertible. Especially, θ_Σ is defined and holomorphic on a neighbourhood of the origin. The values $\theta_\Sigma(z)$ are bounded operators from $\mathcal{U}$ to $\mathcal{Y}$. Conversely, if θ is an operator valued function, and the transfer function of a system Σ coincides with it in a neighbourhood of the origin, then Σ is a (scattering) **realization** of θ, and a realization problem for the operator valued function θ analytic at the origin is to find system of the form (1.5) such that its transfer function coincides with θ.

For ordinary Schur functions, this connection was discovered and studied, for instance, by Arov [7,8], de Branges and Rovnyak [17,18], Brodskiĭ [19] and Sz.-Nagy and Foias [36]. The standard Hilbert space theory of ordinary Schur functions has a counterpart for the generalized Schur functions, and this will led to replacing the Hilbert state space, or all of the spaces, by Pontryagin, or in some cases, even by Kreĭn spaces. In the case where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, the generalized Schur class $\mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ and its connections to unitary colligations were studied, for instance, by Dijksma, Langer and de Snoo [22]. Arov's approach to use passive systems was utilized by Saprikin [34], Arov and Saprikin [13], Arov, Rovnyak and Saprikin [12] and by the author in [31] to study the class $\mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces. If $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, one encounters operator colligations, or systems, such that all the spaces are indefinite. Theory of canonical isometric, co-isometric and conservative systems in that case is considered, for instance, in [2,3,20,23], along with the other properties of the generalized Schur functions. Especially, symmetric generalized Schur functions, with a little bit more general definition than in this paper, were studied in [3]. The results about the unitary similarities between the canonical realizations obtained therein will be used.

Theory of passive systems and generalized Schur functions in the case where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, was studied by the author in [32]. On the other aspects, in the case where $\mathcal{U}$ and $\mathcal{Y}$ are finite dimensional, the class $\mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ is closely related to **generalized Potapov class** and generalized $\mathcal{J}$-inner functions; see for instance [1,6,8,21, 37].

On the other hand, the generalized Nevanlinna functions have been studied alongside with the Schur functions, mainly with scalar, matrix and Hilbert space operator valued cases. Instead of unitary and contractive operators, the study of the generalized Nevanlinna functions involves dissipative and self-adjoint linear operators and relations; see for instance [25,29].

The aim of this paper is to study connections of discrete-time systems, transfer functions and operator valued analytic functions which are both generalized Schur and generalized Nevanlinna functions for some indices, that is, which belongs to the class $\mathbf{S}_{\kappa_1}(\mathcal{U}) \cap \mathbf{N}_{\kappa_2}(\mathcal{U})$, where $\mathcal{U}$ is a Pontryagin space. Before involving the realization theory, the structural properties of the generalized Schur functions and generalized Nevanlinna functions are studied by using the Potapov–Ginzburg transformation. Especially, in the case where $\mathcal{U}$ and $\mathcal{Y}$ are finite dimensional anti-Hilbert spaces, the behaviour of the functions in the classes $\mathbf{S}_{\kappa_1}(\mathcal{U})$ and $\mathbf{N}_{\kappa_2}(\mathcal{U})$ is reciprocal to the Hilbert space case, see Corollary 2.3 and Proposition 2.6. Moreover, in Theorem 2.8, when $\mathcal{U}$

and $\mathcal{Y}$ are Pontryagin spaces with the same negative indices, it will be proved that for $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, the strong radial limit value $\theta(\zeta) := \lim_{r\to 1^-} \theta(r\zeta)$ where ζ belongs to the unit circle $\mathbb{T}$, exists almost everywhere (a.e.), and their values are contractive with respect to the underlying indefinite inner products. Theorem 2.8 also gives rise to a notion of a generalized $\mathcal{J}$-inner function in infinite dimensional spaces.

In realization theory, the study of the class $\mathbf{S}_{\kappa_1}(\mathcal{U}) \cap \mathbf{N}_{\kappa_2}(\mathcal{U})$, where $\mathcal{U}$ is a Pontryagin space, naturally leads to self-adjoint systems. For the ordinary Schur and Nevanlinna functions, these connections were studied by Arlinskiĭ, Hassi and de Snoo in [4] and by Arlinskiĭ and Hassi in [5]. One of their main results was that $\theta \in \mathbf{S}(\mathcal{U}) \cap \mathbf{N}(\mathcal{U})$, where $\mathcal{U}$ is a Hilbert space, if and only if θ has a minimal passive self-adjoint realization of the form (1.5) such that the state space is a Hilbert space [4, Theorem 5.4]. In the case $\theta \in \mathbf{S}_{\kappa_1}(\mathcal{U}) \cap \mathbf{N}_{\kappa_2}(\mathcal{U})$, one can obtain a similar realization which is self-adjoint, but not passive in the general case; see Theorem 3.5, Remark 3.6 and Proposition 3.7.

On the other hand, every $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ has a minimal passive realization Σ, and it can be chosen such that it is **optimal** or ***-optimal** [32, Theorem 3.5]; for the case where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, see also [34, Theorem 5.3]. For a symmetric $\theta \in \mathbf{S}_\kappa(\mathcal{U})$, these realizations have special properties. Namely, the dual system of the optimal minimal passive realization of θ is a *-optimal minimal passive realization of θ, and vice versa. One can form a weak similarity mapping Z between those systems such that Z is everywhere defined, contractive and self-adjoint. If θ has a meromorphic continuation to $\mathbb{C}\setminus\mathbb{R}$, then the negative index of the mapping Z with respect to the indefinite inner product in question determines the number of the negative squares of the Nevanlinna kernel (1.3); see Theorem 3.10. That is, the negative index of the Z, which roughly speaking tells that how much Z behaves like a positive operator with respect to the indefinite inner product in question, can be used to determine whether θ is also a generalized Nevanlinna function. If, in addition, the boundary values of θ on the unit disc $\mathbb{T}$ are unitary, then Z is also unitary and can be represented in an explicit form by using the canonical realizations from [2,3].

It is a classical problem to determine if an ordinary Schur function θ can represented as a corner of a bi-inner dilation of the form

$$\Theta = \begin{pmatrix} \theta & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix}; \tag{1.7}$$

see, for an instance, [8,14]. Arlinskiĭ and Hassi showed in [5] that every $\theta \in \mathbf{S}(\mathcal{U}) \cap \mathbf{N}(\mathcal{U})$, where $\mathcal{U}$ is a Hilbert space, has a bi-inner dilation, and moreover, a dilation (1.7) can be chosen such that it is an ordinary Nevanlinna function. In the last section of this paper, similar results will be obtained for the subclasses of $\mathbf{S}_{\kappa_1}(\mathcal{U}) \cap \mathbf{N}_{\kappa_2}(\mathcal{U})$, where $\mathcal{U}$ is a Pontryagin space. In particular, functions in $\mathbf{S}_{\kappa_1}(\mathcal{U}) \cap \mathbf{N}_{\kappa_2}(\mathcal{U})$ with the property that their minimal passive realizations are unitarily similar, always have a dilation with unitary boundary values almost everywhere on $\mathbb{T}$, and those functions in $\mathbf{S}_\kappa(\mathcal{U}) \cap \mathbf{N}_\kappa(\mathcal{U})$ which have a minimal passive self-adjoint realization, always have a dilation Θ with unitary boundary values almost everywhere on $\mathbb{T}$. Moreover, Θ can be chosen such that it is a generalized Nevanlinna function with the index κ; see Theorem 4.1.

2. Structural properties of the generalized Schur and generalized Nevanlinna functions

When $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, the full structure of the functions in $\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ and $\mathbf{N}_\kappa(\mathcal{U})$ is somewhat more complicated than in the better known Hilbert space case. For instance, when $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, **Kreĭn–Langer factorizations** shows that a function in $\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ has exactly κ poles, counting multiplicities; see Lemma 2.5. This does not hold anymore when the negative index of $\mathcal{U}$ and $\mathcal{Y}$ is not zero; a function $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ may has any countable number of poles, see Corollary 2.3 and Example 2.7 below. However, some properties of the function θ in $\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ or $\mathbf{N}_\kappa(\mathcal{U})$ can be analyzed by using a suitable transformation $\theta \mapsto \theta'$, where $\theta' \in \mathbf{S}_\kappa(\mathcal{U}',\mathcal{Y}')$ or $\mathbf{N}_\kappa(\mathcal{U}')$ for some Hilbert spaces $\mathcal{U}'$ and $\mathcal{Y}'$.

In what follows, all notions of continuity and convergence are understood to be with respect to the strong topology, which is induced by any fundamental decomposition of the space in question. Let θ be an $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function holomorphic on a set $\rho(\theta)$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index. Let $\mathcal{U} = \mathcal{U}_+ \oplus \mathcal{U}_-$ and $\mathcal{Y} = \mathcal{Y}_+ \oplus \mathcal{Y}_-$ be some fixed fundamental decompositions of $\mathcal{U}$ and $\mathcal{Y}$. Represent θ as

$$\theta(z) = \begin{pmatrix} \theta_{11}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{pmatrix} : \begin{pmatrix} \mathcal{U}_+ \\ \mathcal{U}_- \end{pmatrix} \to \begin{pmatrix} \mathcal{Y}_+ \\ \mathcal{Y}_- \end{pmatrix}, \tag{2.1}$$

and define $\mathcal{U}' = \mathcal{U}_+ \oplus |\mathcal{U}_-|$ and $\mathcal{Y}' = \mathcal{Y}_+ \oplus |\mathcal{Y}_-|$, where $|\mathcal{U}_-|$ and $|\mathcal{Y}_-|$ are antispaces of $\mathcal{U}_-$ and $\mathcal{Y}_-$. The antispace of an inner product space $\mathcal{H}$ is by definition the space that coincides with $\mathcal{H}$ as a vector space and is endowed with an inner product $-\langle \cdot,\cdot \rangle_{\mathcal{H}}$. Denote

$$\begin{cases} \sigma : \mathcal{U}_- \to |\mathcal{U}_-| & ,\tau : \mathcal{Y}_- \to |\mathcal{Y}_-|, \\ \sigma^* = -\sigma^{-1} & ,\tau^* = -\tau^{-1}, \end{cases} \tag{2.2}$$

for the identity mappings. The **Potapov–Ginzburg transformation**; see [2, Sect. 4.3] and [15, Sect. 5.§1], of θ is then defined to be an $\mathcal{L}(\mathcal{U}',\mathcal{Y}')$-valued function

$$\begin{aligned} \theta_P(z) &= \begin{pmatrix} \theta_{11}(z) - \theta_{12}(z)\theta_{22}^{-1}(z)\theta_{21}(z) & \theta_{12}(z)\theta_{22}^{-1}(z)\tau^{-1} \\ -\sigma\theta_{22}^{-1}(z)\theta_{21}(z) & \sigma\theta_{22}^{-1}(z)\tau^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \theta_{P11}(z) & \theta_{P12}(z) \\ \theta_{P21}(z) & \theta_{P22}(z) \end{pmatrix}, \end{aligned} \tag{2.3}$$

whose domain $\rho(\theta_P)$ consists of all the points $z \in \rho(\theta)$ such that $\theta_{22}(z)$ is invertible. A calculation shows that

$$\theta(z) = \begin{pmatrix} \theta_{P11}(z) - \theta_{P12}(z)\theta_{P22}^{-1}(z)\theta_{P21}(z) & \theta_{P12}(z)\theta_{P22}^{-1}(z)\sigma \\ -\tau^{-1}\theta_{P22}^{-1}(z)\theta_{P21}(z) & \tau^{-1}\theta_{P22}^{-1}(z)\sigma \end{pmatrix} \tag{2.4}$$

holds for every $z \in \rho(\theta_P)$. Note that the values of θ_{P22} are invertible whenever they exist. Define, respectively, $\mathcal{L}(\mathcal{Y}', \mathcal{Y})$ and $\mathcal{L}(\mathcal{U}', \mathcal{U})$-valued functions

$$\Phi(z) = \begin{pmatrix} I_{\mathcal{Y}_+} & -\theta_{12}(z)\sigma^{-1} \\ 0 & -\theta_{22}(z)\sigma^{-1} \end{pmatrix} \quad \text{and} \quad \Psi(z) = \begin{pmatrix} I_{\mathcal{U}_+} & -\theta_{21}^{\#}(z)\tau^{-1} \\ 0 & -\theta_{22}^{\#}(z)\tau^{-1} \end{pmatrix}. \tag{2.5}$$

Proposition 2.1. *Let $\mathcal{U}$ and $\mathcal{Y}$ be Pontryagin spaces with the same negative index $\pi \geq 1$, and let θ be an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function holomorphic on a set $\rho(\theta)$ and meromorphic on a set $\mathcal{D}$.*

(i) *If θ_P exists, it is meromorphic on $\mathcal{D}$, and if θ_P is meromorphic on a set $\mathcal{D}_P$, then so is θ.*
(ii) *The Potapov–Ginzburg transformation $\left(\theta^{\#}\right)_P$ of $\theta^{\#}$ is $(\theta_P)^{\#}$.*
(iii) *The identities*

$$I - \theta(z)\theta^*(w) = \Phi(z)\big(I - \theta_P(z)\theta_P^*(w)\big)\Phi^*(w) \tag{2.6}$$

$$I - \theta^{\#}(z)\theta^{\#*}(w) = \Psi(z)\left(I - \theta_P^{\#}(z)\theta_P^{\#*}(w)\right)\Psi^*(w) \tag{2.7}$$

$$\theta(z) - \theta(\bar{w}) = \Phi(z)\big(\theta_P(z) - \theta_P(\bar{w})\big)\Psi^*(w) \tag{2.8}$$

$$\theta^{\#}(z) - \theta^{\#}(\bar{w}) = \Psi(z)\left(\theta_P^{\#}(z) - \theta_P^{\#}(\bar{w})\right)\Phi^*(w) \tag{2.9}$$

hold whenever the corresponding functions are defined.
(iv) *If $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, then $\rho(\theta_P)$ is of the form $\rho(\theta) \setminus \Xi$, where Ξ contains at most κ points.*
(v) *If $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ and $\theta_{22}^{-1}(0)$ exists, then $\theta_P \in \mathbf{S}_\kappa(\mathcal{U}', \mathcal{Y}')$. If $\theta_P \in \mathbf{S}_\kappa(\mathcal{U}', \mathcal{Y}')$ then $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$.*
(vi) *If $\mathcal{U} = \mathcal{Y}$ and θ is a symmetric function such that θ_{22} is invertible for some $\alpha \in \mathbb{C} \setminus \mathbb{R}$, then $\theta \in \mathbf{N}_\kappa(\mathcal{U})$ if and only if $\theta_P \in \mathbf{N}_\kappa(\mathcal{U}')$.*
(vii) *If $\mathcal{U} = \mathcal{Y}$, $\theta = \theta^{\#}$ and $\theta_{22}^{-1}(0)$ exists, then $\theta \in \mathbf{S}_{\kappa_1}(\mathcal{U}) \cap \mathbf{N}_{\kappa_2}(\mathcal{U})$ if and only if $\theta_P \in \mathbf{S}_{\kappa_1}(\mathcal{U}') \cap \mathbf{N}_{\kappa_2}(\mathcal{U}')$.*

Proof. (i) Suppose θ_P exists, i.e. θ_{22} in decomposition (2.1) is invertible for some point $\alpha \in \rho(\theta)$. Since θ is meromorphic on $\mathcal{D}$, so are all the entries in (2.1). To prove that θ_P is meromorphic on $\mathcal{D}$, it is now sufficient to show that θ_{22}^{-1} is meromorphic on $\mathcal{D}$, since then all the entries in (2.3) are meromorphic. To this end, note that the values of θ_{22} are operators between the spaces with the same finite dimension. Therefore, $\theta_{22}(z)$ can be identified as a square matrix, and $\theta_{22}^{-1}(z)$ has a representation $\theta_{22}^{-1}(z) = \frac{\operatorname{cof}(\theta_{22}(z))}{\det(\theta_{22}(z))}$, where $\det(\theta_{22}(z))$ and $\operatorname{cof}(\theta_{22}(z))$ are, respectively, the determinant and the cofactor matrix of $\theta_{22}(z)$. The function $\det(\theta_{22})$ is not identically zero since $\theta_{22}(\alpha)$ is invertible. Since θ_{22} is meromorphic on $\mathcal{D}$, so are the functions $\det(\theta_{22}(z))$ and $\operatorname{cof}(\theta_{22}(z))$. It follows now that θ_{22}^{-1} exists and it is meromorphic on $\mathcal{D}$, and so is θ_P.

If θ_P is meromorphic on $\mathcal{D}_P$, by using the same argument as above, one can show that θ_{P22}^{-1} is meromorphic on $\mathcal{D}_P$, and then it follows from (2.4) that θ is meromorphic on $\mathcal{D}_P$.

For the proof of (ii), (iii) and (iv), see [2, Lemmas 4.3.1 and 4.3.2 and Theorem 4.3.3].

(v) From the part (iv) it follows that θ_P exists. By (2.6), it holds

$$K_\theta(w,z) = \Phi(z)K_{\theta_P}(w,z)\Phi^*(w) \tag{2.10}$$

for the Schur kernels K_θ and K_{θ_P} of the form (1.1), whenever the functions are defined. Let Ω be a region such that θ and θ_P both are holomorphic on Ω. Then, the values of θ_{22} are bijective in Ω, and it easily follows from this fact that $\Phi^*(w)$ is onto for every $w \in \Omega$. Then it follows from (2.10) that K_{θ_P} restricted to Ω has the same number of negative squares than K_θ restricted to Ω. Now an application of [2, Theorem 1.1.4] shows that unrestricted K_{θ_P} and K_θ have the same number of negative squares. Therefore, if $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ and $\theta_{22}^{-1}(0)$ exists, θ_P is holomorphic at the origin and K_{θ_P} has exactly κ negative squares, so $\theta_P \in \mathbf{S}_\kappa(\mathcal{U}',\mathcal{Y}')$. Conversely if $\theta_P \in \mathbf{S}_\kappa(\mathcal{U}',\mathcal{Y}')$, the function θ_P and then also θ are holomorphic at the origin, K_θ has exactly κ negative squares, so $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$.

(vi) It follows from the assumption $\mathcal{U} = \mathcal{Y}$ that $\mathcal{U}' = \mathcal{Y}'$, and the assumption that θ_{22} is invertible for some point guarantees that θ_P exists. Moreover, the function θ_P is also symmetric by part (ii). From these symmetry conditions it follows that $\sigma = \tau$ in (2.2) and $\Psi(z) = \Phi(z)$ in (2.5). By (2.8), it then holds

$$N_\theta(w,z) = \Psi(z)N_{\theta_P}(w,z)\Psi^*(w)$$

for the Nevanlinna kernels N_θ and N_{θ_P} of the form (1.3), whenever the functions are defined. Now the same argument as used in the proof of part (iii) shows that N_θ and N_{θ_P} have the same number of negative squares. Moreover, part (i) shows that if either θ or θ_P is meromorphic on $\mathbb{C} \setminus \mathbb{R}$, then so is the other. The claim now follows.

(vii) This follows straightforwardly from the parts (v) and (vi). □

Remark 2.2. The assumption that $\theta_{22}^{-1}(0)$ exists in parts (v) and (vii) of Proposition 2.1 is technical; it is needed because the generalized Schur function must be analytic at the origin. If $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ and $\theta_{22}(0)$ is not invertible, it follows from part (iv) that $\theta_{22}(\alpha)$ is invertible for some $\alpha \in \mathbb{D}$. The conclusions of the part (v) of Proposition 2.1 then hold if $\theta_P(z)$ is replaced by $\theta_P(\eta(z))$, where $\eta(z) = \frac{\alpha - z}{1-\bar{\alpha}z}$; see [2, Sect. 2.5 B]. The same is true in the part (vii) of Proposition 2.1, if $\alpha \in (-1,1)$, since then $\eta(\bar{z}) = \overline{\eta(z)}$ and $\theta_P(\eta(\overline{z})) = (\theta_P(\eta(z)))^*$. By part (iv), α can be chosen to be real.

In one dimensional cases, that is, when $\mathcal{U} = \mathcal{Y} = -\mathbb{C}$, where $-\mathbb{C}$ is the antispace of the complex numbers, the Potapov–Ginzburg transformation reduces to transformation of the form $\theta \mapsto \theta^{-1}$.

Corollary 2.3. *A function θ_1 such that $\theta_1(0) \neq 0$ belongs to $\mathbf{S}_{\kappa_1}(-\mathbb{C})$ if and only if $\theta_1^{-1} \in \mathbf{S}_{\kappa_1}(\mathbb{C})$, and a function θ_2 which is not identically zero belongs to $\mathbf{N}_{\kappa_2}(-\mathbb{C})$ if and only if $\theta_2^{-1} \in \mathbf{N}_{\kappa_2}(\mathbb{C})$. Moreover, a function θ such that $\theta(0) \neq 0$ belongs to $\mathbf{S}_{\kappa_1}(-\mathbb{C}) \cap \mathbf{N}_{\kappa_2}(-\mathbb{C})$ if and only if $\theta^{-1} \in \mathbf{S}_{\kappa_1}(\mathbb{C}) \cap \mathbf{N}_{\kappa_2}(\mathbb{C})$.*

Proof. The claims follow from parts (v)–(vii) of Proposition 2.1 by choosing $\mathcal{U} = \mathcal{Y} = -\mathbb{C}$, since then $\theta_P = \theta^{-1}$ and $\mathcal{U}' = \mathcal{Y}' = \mathbb{C}$. □

Remark 2.4. In Corollary 2.3, the roles of $-\mathbb{C}$ and $\mathbb{C}$ could be interchanged; it still holds, if one replaces $-\mathbb{C}$ by $\mathbb{C}$ and $\mathbb{C}$ by $-\mathbb{C}$. Moreover, Corollary 2.3 holds as stated, if one replaces the spaces $-\mathbb{C}$ and $\mathbb{C}$, respectively, by $-\mathbb{C}^n$ and $\mathbb{C}^n$, and changes the assumptions "θ_1 not identically zero" and "$\theta_2(0) \neq 0$", respectively, by "$\det(\theta_1)$ not identically zero" and "$\det(\theta_2(0)) \neq 0$". However, in that case, the roles of $-\mathbb{C}$ and $\mathbb{C}$ could not be interchanged, since if $n \geq 2$, there are matrix functions in $\mathbf{S}_\kappa(\mathbb{C}^n)$ such that their values are not invertible anywhere on $\mathbb{D}$.

When $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, the class $\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ has characterizations which do not involve the Schur kernel (1.1). For a proof of the following lemma, combine [22, Proposition 7.11] and [2, Theorem 4.2.1].

Lemma 2.5. *Let $\mathcal{U}$ and $\mathcal{Y}$ be Hilbert spaces, and let θ be an $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function holomorphic at the origin and meromorphic on $\mathbb{D}$. Then the following statements are equivalent:*

(i) *$\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$;*

(ii) *θ has finite pole multiplicity κ and*

$$\lim_{r\to 1^-} \sup_{|z|=r} \|\theta(z)\| \leq 1$$

holds;

(iii) *θ has factorizations of the form*

$$\theta(z) = \theta_r(z)B_r^{-1}(z) = B_l^{-1}(z)\theta_l(z),$$

where $\theta_r, \theta_l \in \mathbf{S}(\mathcal{U},\mathcal{Y})$, B_r and B_r are Blaschke products of degree κ with values, respectively, in $\mathcal{L}(\mathcal{U})$ and $\mathcal{L}(\mathcal{Y})$, such that $B_r(w)f = 0$ and $\theta_r(w)f = 0$ for some $w \in \mathbb{D}$ only if $f = 0$, and $B_l^*(w)g = 0$ and $\theta_l^*(w)g = 0$ for some $w \in \mathbb{D}$ only if $g = 0$.

When $\mathcal{U}$ and $\mathcal{Y}$ are finite dimensional anti-Hilbert spaces with the same negative index, i.e. $\mathcal{U} = \mathcal{Y} = -\mathbb{C}^n$, the results of Lemma 2.5 have counterparts; in particular, the analog for Lemma 2.5(ii) will be stated and proved in proposition below.

For a meromorphic function θ such that the values of θ are operators between the spaces with the same finite dimension, z is called a zero of θ if it is a pole of θ^{-1}. If $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, the lower bound of an operator $T : \mathcal{X} \to \mathcal{Y}$ is a value $L \geq 0$ satisfying $\|Tx\|_\mathcal{Y} \geq L$ for all $x \in \mathcal{X}$ such that $\|x\|_\mathcal{X} = 1$. The operator T is called bounded below if a non-zero lower bound exists, and the best possible choice of all the lower bounds, i.e. the greatest one, is denoted as $\gamma(T)$.

Proposition 2.6. *An $n \times n$-matrix valued function θ meromorphic on $\mathbb{D}$ and holomorphic at the origin belongs to $\mathbf{S}_\kappa(-\mathbb{C}^n)$ if and only if θ has exactly κ zeros in $\mathbb{D}$, counting multiplicities, and*

$$\lim_{r\to 1^-} \inf_{|z|=r} \gamma(\theta(z)) \geq 1. \tag{2.11}$$

where $\gamma(\theta(z))$ is taken with respect to the usual norm of $\mathcal{L}(\mathbb{C}^n)$.

Proof. The values of θ can be considered as the operators in $\mathcal{L}(-\mathbb{C}^n)$. Then, the Potapov–Ginzburg transformation θ_P of θ is θ^{-1}. Suppose $\theta \in \mathbf{S}_\kappa(-\mathbb{C}^n)$. Then, by Proposition 2.1, θ^{-1} is meromorphic on $\mathbb{D}$. It can be assumed that θ^{-1} exists at the origin, since if not, one only has to consider $\theta^{-1}(\eta(z))$ as in Remark 2.2. Then $\theta^{-1} \in \mathbf{S}_\kappa(\mathbb{C}^n)$ by Proposition 2.1. It follows from Lemma 2.5 that θ^{-1} has exactly κ poles in $\mathbb{D}$, counting multiplicities, and it holds

$$\lim_{r\to 1^-} \sup_{|z|=r} \|\theta^{-1}(z)\| \leq 1. \tag{2.12}$$

It follows now that θ has exactly κ zeros, counting multiplicities, in $\mathbb{D}$, and (2.11) holds.

Assume then that θ has κ zeros in $\mathbb{D}$ and (2.11) holds. It can be again assumed that $z = 0$ is not a zero of θ. Then θ^{-1} is meromorphic on $\mathbb{D}$ and holomorphic at the origin, it has κ poles and (2.12) holds. It follows from Lemma 2.5 that $\theta^{-1} \in \mathbf{S}_\kappa(\mathbb{C}^n)$, and then by Proposition 2.1 that $\theta \in \mathbf{S}_\kappa(-\mathbb{C}^n)$. □

Lemma 2.5 and Proposition 2.6 show that when $\mathcal{U}$ and $\mathcal{Y}$ are definite, that is, Hilbert spaces or anti-Hilbert spaces, functions in the class $\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ can have only finite number of poles or zeros, respectively. This does not hold in general, when the spaces $\mathcal{U}$ and $\mathcal{Y}$ are indefinite. In that case, it is possible that $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ has infinite number of zeros and poles, as Example 2.7 below shows. However, a function $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ still has some properties similar to (2.11) or (2.12). Indeed, the radial limit values of $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ exists a.e. on $\mathbb{T}$, and they are contractive with respect to the indefinite inner product of $\mathcal{U}$ and $\mathcal{Y}$; see Theorem 2.8 below.

Example 2.7. Let b_1 and b_2 be scalar infinite Blaschke products such that $b_2(0) \neq 0$. Consider an $\mathcal{L}(\mathbb{C} \oplus -\mathbb{C})$-valued function $\theta(z) = \begin{pmatrix} b_1(z) & 0 \\ 0 & b_2^{-1}(z) \end{pmatrix}$. A calculation shows that the Potapov–Ginzburg transformation θ_P of the function θ is the $\mathcal{L}\left(\mathbb{C}^2\right)$-valued function $\theta_P(z) = \begin{pmatrix} b_1(z) & 0 \\ 0 & b_2(z) \end{pmatrix}$. It easily follows from Lemma 2.5 that $\theta_P \in \mathbf{S}(\mathbb{C}^2)$, and then by Proposition 2.1 that $\theta \in \mathbf{S}(\mathbb{C} \oplus -\mathbb{C})$. Moreover, θ has infinite number of zeros and poles.

Theorem 2.8. *Let $\mathcal{U}$ and $\mathcal{Y}$ be Pontryagin spaces with the same negative index.*

(i) *If $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, then strong radial limit values $\lim_{r\to 1^-} \theta(r\zeta)$ exist for a.e. $\zeta \in \mathbb{T}$, and the limit values are contractive with respect to the indefinite inner products of $\mathcal{U}$ and $\mathcal{Y}$.*

(ii) *If $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, strong radial limit values of the function θ are isometric (co-isometric) a.e. on $\mathbb{T}$ if and only if strong radial limit values of θ_P are isometric (co-isometric) a.e. on $\mathbb{T}$.*

Proof. (i)The Hilbert space case is known. For ordinary Schur functions, the result is classical, see [36, Chapter V]. If $\kappa > 0$ and $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, θ has Kreĭn–Langer factorizations of the form (2.5). Since inverse Blaschke products are rational functions with unitary values everywhere on $\mathbb{T}$, the result now follows from the case $\kappa = 0$.

Assume then that the negative index of $\mathcal{U}$ and $\mathcal{Y}$ is not zero. By Proposition 2.1, the Potapov–Ginzburg transform θ_P of θ exists. It can be assumed that $\theta_{22}(0)$ invertible; if not, one only need to consider $\theta_P(\eta(z))$, where η is as in Remark 2.2. By Proposition 2.1, $\theta_P \in \mathbf{S}_\kappa(\mathcal{U}', \mathcal{Y}')$, and since $\mathcal{U}'$ and $\mathcal{Y}'$ are Hilbert spaces, θ_P is meromorphic on $\mathbb{D}$, has strong contractive radial limit values almost everywhere on $\mathbb{T}$, and the same holds for the entries $\theta_{P_{11}}$, $\theta_{P_{12}}$, $\theta_{P_{21}}$ and $\theta_{P_{22}}$ in (2.3). By Lemma 2.5, θ_P has exactly κ poles in $\mathbb{D}$, counting multiplicities, and therefore $\theta_{P_{22}}$ has no more than κ poles in $\mathbb{D}$. It now follows again from Lemma 2.5 that $\theta_{P_{22}} \in \mathbf{S}_{\kappa'}(|\mathcal{U}_-|, |\mathcal{Y}_-|)$, where $\kappa' \leq \kappa$. Then, $\theta_{P_{22}}$ has the Kreĭn–Langer factorization of the form

$$\theta_{P_{22}} = B^{-1}\theta_0, \tag{2.13}$$

where B^{-1} is an inverse Blaschke product and $\theta_0 \in \mathbf{S}_0(|\mathcal{U}_-|, |\mathcal{Y}_-|)$. The values of $\theta_{P_{22}}$ are operators between the spaces with same finite dimension, and they can be identified with square matrices. Moreover, the values of $\theta_{P_{22}}$ are by construction and Lemma 2.1 invertible at least on $\rho(\theta) \setminus \Xi$, where Ξ contains at most κ points. Since the values of B^{-1} are invertible whenever they exists, it follows that the values of θ_0 are invertible on $\rho(\theta) \setminus \Xi$. In particular, the function $\det(\theta_0)$, is not identically zero. These facts combined with (2.13) show that $\theta_{P_{22}}^{-1}(z)$ has a representation

$$\theta_{P_{22}}^{-1}(z) = \frac{\operatorname{cof}(\theta_{P_{22}}(z))}{\det(\theta_{P_{22}}(z))} = \frac{\operatorname{cof}(\theta_{P_{22}}(z))}{\det(B^{-1}(z))\det(\theta_0(z))}, \tag{2.14}$$

where cof means the cofactor matrix. The function $\theta_{P_{22}}$ is meromorphic on $\mathbb{D}$ and has strong contractive radial limit values a.e. on $\mathbb{T}$, so clearly $\operatorname{cof}(\theta_{P_{22}})$ is meromorhic in $\mathbb{D}$ and has strong radial limit values a.e. on $\mathbb{T}$. Since the values of Blaschke product B^{-1} are unitary everywhere on the unit cirle, $|\det(B^{-1}(\zeta))| = 1$ for every $\zeta \in \mathbb{T}$. The values of θ_0 are contractive everywhere on $\mathbb{D}$, and therefore $\det(\theta_0)$ is bounded holomorphic function in $\mathbb{D}$. This implies that radial limit values of $\det(\theta_0)$ exist, and since $\det(\theta_0)$ is not identically zero, the radial limit values also differ from zero a.e. on $\mathbb{T}$. It now follows from (2.14) that $\theta_{P_{22}}^{-1}$ is meromorphic on $\mathbb{D}$ and has radial limit values a.e. on $\mathbb{T}$. It has been proved that all the entries in the representation of θ in (2.4) are meromorphic in $\mathbb{D}$ and have strong radial limit values a.e. on $\mathbb{T}$, so the same holds for θ. The fact that the radial limit values of θ are contractive with the respect to the inner products of $\mathcal{U}$ and $\mathcal{Y}$ follows now easily from the identity (2.6) in Proposition 2.1, since the radial limit values of θ_P are contractive.

(ii) Consider the identities (2.6) and (2.7) from Proposition 2.1. The claims follow from these identities if one proves that the strong radial limit values of Φ and Ψ exist and are onto a.e. on $\mathbb{T}$. It follows from the part (i) that all the entries in the definition of Φ in (2.5) have strong radial limit values a.e. on $\mathbb{T}$, so the same holds for Φ. Since $\theta_{22}^{-1} = \sigma^{-1}\theta_{P22}\tau$ and the strong radial limit values of θ_{P22} exist a.e. on $\mathbb{T}$, the strong radial limit values of θ_{22}^{-1} also exist a.e. on $\mathbb{T}$. Especially, the strong radial limit values of θ_{22} are invertible a.e. on $\mathbb{T}$. An easy calculation then shows that the strong radial limit values of Φ are onto a.e. on $\mathbb{T}$. Similar argument shows that the same holds for Ψ, and the claims follow. □

In the special case where $\mathcal{U} = \mathcal{Y}$ and $\mathcal{U}$ is finite dimensional, Theorem 2.8 above could be derived from [1, Theorem 6.8]. A function $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, is called inner (co-inner, bi-inner), if the radial limit values of θ are isometric (co-isometric, unitary) a.e. on $\mathbb{T}$. By using a similar notion as in [1,6,8,21], a function $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, is called a **generalized $\mathcal{J}$-inner (co-$\mathcal{J}$-inner, bi-$\mathcal{J}$-inner) function**, if the radial limit values of θ are isometric (co-isometric, unitary) a.e. on $\mathbb{T}$, with respect to the inner product of $\mathcal{U}$ and $\mathcal{Y}$. Following [12]; see also [31, Sect. 4], the class $\mathbf{U}_\kappa(\mathcal{U}, \mathcal{Y})$ is defined to be the subclass of the generalized bi-$\mathcal{J}$-inner functions in $\mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$. The class $\mathbf{U}_\kappa(\mathcal{U},\mathcal{U})$ is written as $\mathbf{U}_\kappa(\mathcal{U})$. For a symmetric function, it is evident that if the radial values are isometric or co-isometric a.e., they are also unitary.

3. Linear systems, self-adjoint realizations and similarity mappings in state spaces

If needed, the colligation, or the system, of the form (1.4) will be written as $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$. Often in this paper, $\mathcal{U} = \mathcal{Y}$ and it will be then written $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U})$. In what follows, unless otherwise stated, the state space $\mathcal{X}$ and the spaces $\mathcal{U}$ and $\mathcal{Y}$ are assumed to be Pontryagin spaces, which will be indicated by the notation $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ where κ is reserved for the negative index of $\mathcal{X}$. Note that the common negative index of $\mathcal{U}$ and $\mathcal{Y}$ is not assumed to be related to κ. The **adjoint** or **dual** of the system Σ is the system Σ^* such that its system operator is the indefinite adjoint T_Σ^* of T_Σ. That is, $\Sigma^* = (T_\Sigma^*; \mathcal{X}, \mathcal{Y}, \mathcal{U})$. In this paper, all the adjoints are with respect to the indefinite inner products in question. The identity $\theta_{\Sigma^*}(z) = \theta_\Sigma{}^\#(z)$ holds for the transfer function θ_{Σ^*} of the dual system Σ^*.

The following subspaces

$$\mathcal{X}^c := \overline{\text{span}}\,\{\text{ran}\, A^n B : n = 0, 1, \ldots\} \tag{3.1}$$
$$\mathcal{X}^o := \overline{\text{span}}\,\{\text{ran}\, A^{*n} C^* : n = 0, 1, \ldots\} \tag{3.2}$$
$$\mathcal{X}^s := \overline{\text{span}}\,\{\text{ran}\, A^n B, \text{ran}\, A^{*m} C^* : n, m = 0, 1, \ldots\}, \tag{3.3}$$

are called, respectively, controllable, observable and simple subspaces. The system is said to be **controllable** (**observable**, **simple**) if $\mathcal{X}^c = \mathcal{X} (\mathcal{X}^o = \mathcal{X}, \mathcal{X}^s = \mathcal{X})$ and **minimal** if it is both controllable and observable. When $\Omega \ni 0$ is some symmetric neighbourhood of the origin, that is, $\bar{z} \in \Omega$ whenever $z \in \Omega$, then also

$$\mathcal{X}^c = \overline{\text{span}}\,\{\text{ran}\,(I - zA)^{-1} B : z \in \Omega\} \tag{3.4}$$
$$\mathcal{X}^o = \overline{\text{span}}\,\{\text{ran}\,(I - zA^*)^{-1} C^* : z \in \Omega\} \tag{3.5}$$
$$\mathcal{X}^s = \overline{\text{span}}\,\{\text{ran}\,(I - zA)^{-1} B, \text{ran}\,(I - wA^*)^{-1} C^* : z, w \in \Omega\} \tag{3.6}$$

In the case where all the spaces are Hilbert spaces, it is well known; see for instance [8, Proposition 8], that the transfer function of the passive system is an ordinary Schur function. In general case where $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces such that $\mathcal{U}$ and $\mathcal{Y}$ have the same negative index, the transfer function of the passive system $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U}; \kappa)$ is a generalized Schur function, with the index not larger that the negative index of the state space [32, Proposition 2.4]. Conversely, every $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ has a realization of the form (1.5), and the realization can be chosen such that it is controllable isometric (observable co-isometric, simple conservative, minimal passive) [2, Chapter 2], [32, Lemma 2.8]. Any two controllable isometric (observable co-isometric, simple conservative) realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ are **unitarily similar** [2, Theorem 2.1.3]. Two given realizations $\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}; \kappa_1)$ and $\Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}; \kappa_2)$ of the same $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function θ analytic at the origin are called unitarily similar if $D_1 = D_2$ and there exists a unitary operator $U : \mathcal{X}_1 \to \mathcal{X}_2$ such that

$$A_1 = U^{-1} A_2 U, \quad B_1 = U^{-1} B_2, \quad C_1 = C_2 U. \tag{3.7}$$

Unitary similarity preserves dynamical properties of the system and also the spectral properties of the main operator. Moreover, it easily follows that if the realizations are unitarily similar, their state spaces have the same negative index.

The realizations Σ_1 and Σ_2 above are said to be **weakly similar** if $D_1 = D_2$ and there exists an injective closed densely defined possible unbounded linear operator $Z : \mathcal{X}_1 \to \mathcal{X}_2$ with the dense range such that

$$Z A_1 x = A_2 Z x, \quad C_1 x = C_2 Z x, \quad x \in \mathcal{D}(Z), \quad \text{and} \quad Z B_1 = B_2, \tag{3.8}$$

where $\mathcal{D}(Z)$ is the domain of Z. It is known that two minimal realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, or more generally, any $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function holomorphic at the origin, are weakly similar; see [32, Proposition 2.2], [31, Theorem 2.5] and [35, p. 702].

For a generalized Nevanlinna function $\theta \in \mathbf{N}_\kappa(\mathcal{U})$ in the special case where $\mathcal{U}$ is a Hilbert space, the realization of θ usually means a representation of the form

$$\theta(z) = \theta(z_0)^* + (z - \overline{z_0}) \Gamma^* (I + (z - z_0)(H - z)^{-1}) \Gamma, \tag{3.9}$$

such that $\mathcal{X}$ is a Pontryagin space, $\Gamma \in \mathcal{L}(\mathcal{U}, \mathcal{X})$, H is a self-adjoint linear relation in $\mathcal{X}$ and z_0 is some fixed point in $\rho(H) \cap \mathbb{C}^+$, where $\rho(H)$ is the field of regularity of H. In fact, θ is a generalized Nevanlinna function if and only if it has a representation of the form (3.9) [25,29]. The realization can be chosen such that the negative index of $\mathcal{X}$ coincides with the index κ of $\theta \in \mathbf{N}_\kappa(\mathcal{U})$, and it holds

$$\mathcal{X} = \overline{\operatorname{span}} \left\{ (I + (z - z_0)(H - z)^{-1}) \Gamma u : \quad z \in \rho(H), \quad u \in \mathcal{U} \right\}.$$

In that case, the realization is unique up to unitary equivalence.

In general, a function $\theta \in \mathbf{N}_\kappa(\mathcal{U})$ is not necessary holomorphic at the origin, and therefore it cannot be realized in the form (1.6). However, a self-adjoint system with a Pontryagin state space always induces some generalized Nevanlinna function.

Proposition 3.1. *Let $\Sigma = (A, B, B^*, D; \mathcal{X}, \mathcal{U}; \kappa)$ be a self-adjoint system. Then the transfer function θ of Σ belongs to the generalized Nevanlinna class $\mathbf{N}_{\kappa'}$ $(\mathcal{U})$, where κ' is the dimension of a maximal negative subspace of*

$$\operatorname{span}\{\operatorname{ran}(I - zA)^{-1}B : z \in \Omega\}, \tag{3.10}$$

where Ω is some sufficiently small symmetric neighbourhood of the origin.

Proof. Since Σ is self-adjoint, A and D must be self-adjoint operators, $\mathcal{U} = \mathcal{Y}$, $\theta(z) = \theta^{\#}(z)$, and $B^* = C$. Then the spaces (3.1)–(3.3) coincide. It follows from [15, Corollary 3.15, pp. 106] that the non-real spectrum of A consists of not more than 2κ (counting multiplicities) eigenvalues situated symmetrically with respect to the real axis. Since $(I - zA)^{-1}$ exists whenever $1/z$ is in the resolvent set $\rho(A)$ of A, it follows that $\theta(z) = D + zB^*(I - zA)^{-1}B$ is meromorphic on $\mathbb{C} \setminus \mathbb{R}$ with at most 2κ non-real poles. By using the resolvent identity; cf. also [2, Theorem 1.2.4], and the fact that the system operator is self-adjoint, one deduces that the Nevanlinna kernel of θ can be represented as

$$N_\theta(w, z) = \frac{\theta(z) - \theta^*(w)}{z - \bar{w}} = B^*(I - zA^*)^{-1}(I - \bar{w}A)^{-1}B. \tag{3.11}$$

Therefore, it follows from [2, Lemma 1.1.1'] that the number of negative eigenvalues of the Gram matrix of the form

$$\left(\langle N_\theta(w_j, w_i)f_j, f_i\rangle_{\mathcal{U}}\right)_{i,j=1}^{n} = \left(\left\langle (I - \bar{w}_jA)^{-1}Bf_j, (I - \bar{w}_iA)^{-1}Bf_i\right\rangle_{\mathcal{X}}\right)_{i,j=1}^{n},$$

where $f_i \in \mathcal{U}$ and $w_i \in \mathbb{C} \setminus \mathbb{R}$, $i = 1, \ldots, n$, coincides with the dimension of a maximal negative subspace of the span of $\{(I - \bar{w}_iA)^{-1}Bf_i\}_{i=1}^{n}$. It now follows that the Nevanlinna kernel N_θ has κ' negative squares, where κ is the dimension of a maximal negative subspace of (3.10), and the proof is complete. □

By using the fact that the transfer function of the passive system (1.5) is a generalized Schur function with the index not larger than the negative index of the state space of Σ, it follows from Proposition 3.1 that the transfer function of a passive self-adjoint system is both a generalized Schur function and a generalized Nevanlinna function. Moreover, if $\mathcal{U}$ is a Hilbert space, the negative indices coincide. Some further machinery from the Kreĭn space operator theory will be needed to prove this.

Let $\mathcal{X}$ be a Kreĭn space. The negative index $\operatorname{ind}_-(H)$, with respect to the inner product of $\mathcal{X}$, of the bounded self-adjoint operator $H \in \mathcal{L}(\mathcal{X})$ is defined to be the supremum of all positive integers n such that there exists an invertible and nonpositive matrix of the form $\left(\langle Hx_j, x_i\rangle_{\mathcal{X}}\right)_{i,j=1}^{n}$, where $\{x_k\}_{k=1}^{n} \subset \mathcal{X}$. If such a matrix does not exists for any n, then $\operatorname{ind}_-(H)$ is defined to be zero. In that case, the operator H is nonnegative with respect to the inner product of $\mathcal{X}$.. In general, the negative index of the self adjoint operator measures how much the operator behaves like a positive operator. For an arbitrary $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, the operator T^*T is a bounded self adjoint operator in $\mathcal{L}(\mathcal{X})$, and it is easy to deduce that T is contractive if and only if $\operatorname{ind}_-(I - T^*T) = 0$.

It well known; cf. [15, Theorem 3.4 on p. 267.], [23, Lecture 2], that every bounded linear operator between Kreĭn spaces can be dilated to unitary operator. In this paper, the following version of that result, which can be derived from [23, Theorems 2.3 and 2.4], is needed.

Theorem 3.2. *Suppose that* $A \in \mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$ *where* $\mathcal{X}_1$ *and* $\mathcal{X}_2$ *are Pontryagin spaces with the same negative index. Then there exist Kreĭn spaces* $\mathfrak{D}_A$ *and* $\mathfrak{D}_{A^*}$ *with*

$$\operatorname{ind}_-(I - A^*A) = \operatorname{ind}_-(\mathfrak{D}_A) = \operatorname{ind}_-(\mathfrak{D}_{A^*}) = \operatorname{ind}_-(I - AA^*),$$

and linear operators $D_A \in \mathcal{L}(\mathfrak{D}_A, \mathcal{X}_1)$ *and* $D_{A^*} \in \mathcal{L}(\mathfrak{D}_{A^*}, \mathcal{X}_2)$ *with zero kernels and a linear operator* $L \in \mathcal{L}(\mathfrak{D}_A, \mathfrak{D}_{A^*})$ *such that it holds*

$$I - A^*A = D_A D_A^*, \qquad I - AA^* = D_{A^*} D_{A^*}^*.$$

Furthermore, the operator

$$U_A := \begin{pmatrix} A & D_{A^*} \\ D_A^* & -L^* \end{pmatrix} : \begin{pmatrix} \mathcal{X}_1 \\ \mathfrak{D}_{A^*} \end{pmatrix} \to \begin{pmatrix} \mathcal{X}_2 \\ \mathfrak{D}_A \end{pmatrix} \tag{3.12}$$

is unitary. Moreover, if $\operatorname{ind}_-(I - A^*A) = \operatorname{ind}_-(I - AA^*)$ *is finite, then* U_A *is essentially unique.*

The operator U_A in Theorem 3.2 is called as a **Julia operator** of A, the operators D_A and D_{A^*} are called, respectively, **defect operators** of A and A^*, and the spaces $\mathfrak{D}_A$ and $\mathfrak{D}_{A^*}$ are called, respectively, **defect spaces** of A and A^*. In general, any bounded operator V with the zero kernel is called as a defect operator of A if it holds $I - A^*A = VV^*$. Julia operator of A is essentially unique, if for any other Julia operator

$$U_A' = \begin{pmatrix} A & {D_{A^*}}' \\ {D_A}'^* & -L'^* \end{pmatrix} : \begin{pmatrix} \mathcal{X}_1 \\ {\mathfrak{D}_{A^*}}' \end{pmatrix} \to \begin{pmatrix} \mathcal{X}_2 \\ {\mathfrak{D}_A}' \end{pmatrix},$$

of A, there exists unitary operators $H_1 : \mathfrak{D}_{A^*} \to {\mathfrak{D}_{A^*}}'$ and $H_2 : \mathfrak{D}_A \to {\mathfrak{D}_A}'$ such that

$$D_{A^*} = {D_{A^*}}' H_1, \qquad D_A = {D_A}' H_2, \qquad H_1 L = L' H_2.$$

If θ is the transfer function of the system (1.5), the Schur kernel of the form (1.1) can be represented as a sum of two kernels. This can be done by using the defect operators of the system operator and its adjoint. A special case, where the system is passive, i.e. the system operator is contractive, is proved in [32, Lemma 2.4]; see also the proof of [34, Theorem 2.2]. The proofs given therein can be applied word by word to get the next result, since the existence of defect operator is guaranteed by Theorem 3.2. Therefore, the proof will not be repeated here.

Lemma 3.3. *Let* $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ *be a system with the transfer function* θ*. Denote the system operator of* Σ *as* T*. If*

$$D_T = \begin{pmatrix} D_{T,1} \\ D_{T,2} \end{pmatrix} : \mathcal{D}_T \to \begin{pmatrix} \mathcal{X} \\ \mathcal{U} \end{pmatrix} \quad D_{T^*} = \begin{pmatrix} D_{T^*_{,1}} \\ D_{T^*_{,2}} \end{pmatrix} : \mathcal{D}_{T^*} \to \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix}, \tag{3.13}$$

are defect operators of T and T^*, respectively, then the identities

$$I_{\mathcal{Y}} - \theta(z)\theta^*(w) = (1 - z\bar{w})G(z)G^*(w) + \psi(z)\psi^*(w), \tag{3.14}$$

$$I_{\mathcal{U}} - \theta^*(w)\theta(z) = (1 - z\bar{w})F^*(w)F(z) + \varphi^*(w)\varphi(z), \tag{3.15}$$

with

$$G(z) = C(I_{\mathcal{X}} - zA)^{-1}, \qquad \psi(z) = D_{T^*_{,2}} + zC(I_{\mathcal{X}} - zA)^{-1}D_{T^*_{,1}}, \tag{3.16}$$

$$F(z) = (I_{\mathcal{X}} - zA)^{-1}B, \qquad \varphi(z) = D^*_{T,2} + zD^*_{T,1}(I_{\mathcal{X}} - zA)^{-1}B, \tag{3.17}$$

hold for every z and w in a sufficiently small symmetric neighbourhood of the origin.

The system (1.5) can be expanded to a larger system such that the state space and the main operator will not change. This expansion is called an **embedding**. The embedding of the system (1.5) is any system determined by the system operator

$$T_{\widetilde{\Sigma}} = \begin{pmatrix} A & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} = \begin{pmatrix} A & \begin{pmatrix} B & B_1 \end{pmatrix} \\ \begin{pmatrix} C \\ C_1 \end{pmatrix} & \begin{pmatrix} D & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \begin{pmatrix} \mathcal{U} \\ \mathcal{U}' \end{pmatrix} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \begin{pmatrix} \mathcal{Y} \\ \mathcal{Y}' \end{pmatrix} \end{pmatrix},$$

where $\mathcal{U}'$ and $\mathcal{Y}'$ are Hilbert spaces. The transfer function of the embedded system is

$$\theta_{\widetilde{\Sigma}}(z) = \begin{pmatrix} D + zC(I_{\mathcal{X}} - zA)^{-1}B & D_{12} + zC(I_{\mathcal{X}} - zA)^{-1}B_1 \\ D_{21} + zC_1(I_{\mathcal{X}} - zA)^{-1}B & D_{22} + zC_1(I_{\mathcal{X}} - zA)^{-1}B_1 \end{pmatrix} = \begin{pmatrix} \theta_\Sigma(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{pmatrix},$$

where θ_Σ is the transfer function of the original system.

Proposition 3.4. *If $\Sigma = (A, B, B^*, D; \mathcal{U}; \kappa)$ is a passive self-adjoint system, its transfer function θ belongs to $\mathbf{S}_{\kappa_1}(\mathcal{U}) \cap \mathbf{N}_{\kappa_2}(\mathcal{U})$, where $\kappa_1 \le \kappa_2$ and κ_2 is the dimension of a maximal negative subspace of*

$$\operatorname{span}\{\operatorname{ran}(I - zA)^{-1}B : z \in \Omega\},$$

where Ω is a sufficiently small symmetric neighbourhood of the origin . Moreover, if $\mathcal{U}$ is a Hilbert space, then $\kappa_1 = \kappa_2$.

Proof. It follows from Proposition 3.1 that $\theta \in \mathbf{N}_{\kappa_2}(\mathcal{U})$. Moreover, since Σ is passive, θ is also a generalized Schur function with the negative index κ_1, which is not larger than the negative index κ of the state space $\mathcal{X}$. By using Lemma 3.3, the equation (3.14) and a result from [2, Theorem 1.5.5], it follows that $\kappa_1 \le \kappa_1' + \kappa_2'$, where κ_1' and κ_2' are the negative indices of the kernels $(1 - z\bar{w})^{-1}(\psi(z)\psi^*(w))$ and $G(z)G^*(w)$ in (3.14), respectively. Since Σ is self-adjoint system, $A = A^*$ and $C = B^*$. Then the same argument as in the proof of Proposition 3.1 shows that $\kappa_2' = \kappa_2$. Since Σ is passive, the system operator of T and its adjoint T^* are contractive. Therefore, $\psi^*(w)$ is an operator in a Hilbert space $\mathfrak{D}_{T^*}$, and it follows that the kernel $(1 - z\bar{w})^{-1}(\psi(z)\psi^*(w))$ has no negative square; for details, see the proof of [32, Proposition 2.4]. Therefore $\kappa_1 \le \kappa_2$.

Assume then that $\mathcal{U}$ is a Hilbert space. Denote the system operator of Σ as T. Theorem 3.2 guarantees the existence of the defect operator D_T of

T with the properties described therein. Since $T = T^*$ is contractive, the domain $\mathcal{D}_T$ of D_T is a Hilbert space. Moreover, (3.12) shows that it holds

$$\begin{pmatrix} T & D_T \end{pmatrix} \begin{pmatrix} T & D_T \end{pmatrix}^* = TT^* + D_T D_T^* = T^2 + D_T D_T^* = I.$$

That is, the operator $\begin{pmatrix} T & D_T \end{pmatrix}$ is co-isometric. Therefore the system $\widetilde{\Sigma}$ with the system operator

$$T_{\widetilde{\Sigma}} = \begin{pmatrix} A & \begin{pmatrix} B & D_{T,1} \end{pmatrix} \\ B^* & \begin{pmatrix} D & D_{T,2} \end{pmatrix} \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \begin{pmatrix} \mathcal{U} \\ \mathcal{D}_T \end{pmatrix} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \mathcal{U} \end{pmatrix}$$

is a co-isometric embedding of Σ. The transfer function of $\widetilde{\Sigma}$ is given by

$$\widetilde{\theta} = \begin{pmatrix} D & D_{T,2} \end{pmatrix} + zB^*(I - zA^*)^{-1} \begin{pmatrix} B & D_{T,1} \end{pmatrix} = \begin{pmatrix} \theta & \psi \end{pmatrix} \tag{3.18}$$

where ψ is defined as in (3.16). Since the system Σ is self-adjoint, the identity (3.11) holds. By applying (3.14) from Lemma 3.3, it follows that

$$K_{\widetilde{\theta}}(w,z) = B^*(I - zA^*)^{-1}(I - \bar{w}A)^{-1}B = N_\theta(w,z).$$

Since N_θ has κ_2 negative squares, so has $K_{\widetilde{\theta}}$, and therefore $\widetilde{\theta}$ is a generalized Schur function with the index κ_2. The first identity in (3.18) shows that the total number of poles, counting multiplicities, of $\widetilde{\theta}$ and θ are equal. It then follows from Lemma 2.5 that $\widetilde{\theta}$ and θ have the same index, and the proof is complete. □

Theorem 3.5. *If $\mathcal{U}$ is a Hilbert space, then $\theta \in \mathbf{S}_{\kappa_1}(\mathcal{U}) \cap \mathbf{N}_{\kappa_2}(\mathcal{U})$ has a minimal self-adjoint realization $\Sigma = (A, B, B^*, D; \mathcal{X}, \mathcal{U}; \kappa_2)$.*

Proof. Define $\breve{\theta}(z) = -\theta(1/z) + \theta(0)$. Then $\breve{\theta}$ clearly is meromorphic on $\mathbb{C} \setminus \mathbb{R}$, analytic at the infinity with $\breve{\theta}(\infty) = 0$ and it holds $\breve{\theta}(z) = \breve{\theta}^{\#}(z)$. Moreover, for any finite set of points $\{w_1, \ldots, w_n\} \subset \rho(\breve{\theta})$, and vectors $\{f_1, \ldots, f_n\} \subset \mathcal{U}$, it holds

$$\begin{aligned}
\left(\langle N_{\breve{\theta}}(w_j, w_i) f_j, f_i \rangle_{\mathcal{U}}\right)_{i,j=1}^n &= \left(\left\langle \frac{\theta^*\left(\frac{1}{w_j}\right) - \theta\left(\frac{1}{w_i}\right)}{w_i - \overline{w_j}} f_j, f_i \right\rangle_{\mathcal{U}} \right)_{i,j=1}^n \\
&= \left(\left\langle \frac{\theta\left(\frac{1}{w_i}\right) - \theta^*\left(\frac{1}{w_j}\right)}{\frac{1}{w_i} - \frac{1}{\overline{w_j}}} w_i f_j, w_j f_i \right\rangle_{\mathcal{U}} \right)_{i,j=1}^n \\
&= \left(\left\langle N_\theta\left(\frac{1}{w_j}, \frac{1}{w_i}\right) w_i f_j, w_j f_i \right\rangle_{\mathcal{U}} \right)_{i,j=1}^n \\
&= \left(\langle N_\theta\left(w_j', w_i'\right) f_j', f_i' \rangle_{\mathcal{U}} \right)_{i,j=1}^n .
\end{aligned}$$

The identity above yields that N_θ and $N_{\breve{\theta}}$ have the same number of negative squares, and therefore $\breve{\theta} \in \mathbf{N}_{\kappa_1}(\mathcal{U})$. Since θ is holomorphic at the origin, it has the Neumann series of the form $\theta(z) = \sum_{n=0}^{\infty} \theta_n z^n$, for every $z \in \Omega$, where

Ω is a sufficiently small symmetric neighbourhood of the origin. Therefore $\lim_{z\to 0} z^{-1}(\theta(z)-\theta(0)) = \theta_1$, and also

$$\lim_{z\to\infty} z\check{\theta}(z) = -\lim_{z\to\infty} z\left(\theta\left(\frac{1}{z}\right)-\theta(0)\right) = -\lim_{z'\to 0} z'^{-1}\left(\theta\left(z'\right)-\theta(0)\right) = -\theta_1.$$

This implies $y\lim_{y\to\infty}\left|\langle\check{\theta}(iy)f,f\rangle_{\mathcal{U}}\right| < \infty$ for every $f\in\mathcal{U}$. Therefore, $\check{\theta}$ has the realization of the form (3.9) which reduces to $\check{\theta}(z) = \widehat{B}^*(\widehat{A}-z)^{-1}\widehat{B}$, where $\widehat{B}\in\mathcal{L}(\mathcal{U},\mathcal{X})$ and $\widehat{A}$ is a self-adjoint operator in a Pontryagin space $\widehat{\mathcal{X}}$ with the negative index κ_2 [28], [33, pp. 348–349]. But then θ can be realized as

$$\theta(z) = D - \widehat{B}^*\left(\widehat{A}-\frac{1}{z}\right)^{-1}\widehat{B} = D + z\widehat{B}^*\left(I-z\widehat{A}\right)^{-1}\widehat{B},$$

where $D=\theta(0)=\theta^*(0)$. That is, $\widehat{\Sigma} = (\widehat{A},\widehat{B},\widehat{B}^*,D;\widehat{\mathcal{X}},\mathcal{U};\kappa_2)$ is a self-adjoint realization of θ. It follows from Proposition 3.1 that the dimension of the maximal negative subspace of $\operatorname{span}\{\operatorname{ran}(I-z\widehat{A})^{-1}\widehat{B} : z\in\Omega\} := \mathfrak{S}$, where Ω is some sufficiently small symmetric neighbourhood of the origin, is κ_2, the negative index of $\widehat{\mathcal{X}}$. Then, the closure $\widehat{\mathcal{X}}^c$ of $\mathfrak{S}$ must be a regular subspace of $\widehat{\mathcal{X}}$. Therefore, $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}^c \oplus \left(\widehat{\mathcal{X}}^c\right)^{\perp}$, and $\left(\widehat{\mathcal{X}}^c\right)^{\perp}$ is a Hilbert subspace of $\widehat{\mathcal{X}}$. Since Σ is self-adjoint system, the spaces $\widehat{\mathcal{X}}^c$, $\widehat{\mathcal{X}}^o$ and $\widehat{\mathcal{X}}^s$ coincide. These facts and (3.1) imply that the system operator $\widehat{T}$ of $\widehat{\Sigma}$ can be represented as

$$\widehat{T} = \begin{pmatrix}\begin{pmatrix}A_1 & 0\\ 0 & A\end{pmatrix} & \begin{pmatrix}0\\ B\end{pmatrix}\\ \begin{pmatrix}0 & B^*\end{pmatrix} & D\end{pmatrix} : \begin{pmatrix}\begin{pmatrix}(\widehat{\mathcal{X}}^c)^{\perp}\\ \widehat{\mathcal{X}}^c\end{pmatrix}\\ \mathcal{U}\end{pmatrix} \to \begin{pmatrix}\begin{pmatrix}(\widehat{\mathcal{X}}^c)^{\perp}\\ \widehat{\mathcal{X}}^c\end{pmatrix}\\ \mathcal{U}\end{pmatrix}. \tag{3.19}$$

Define $\mathcal{X} = \widehat{\mathcal{X}}^c$. A calculation shows that $\widehat{A}^n\widehat{B} = A^nB$ for every $n\in\mathbb{N}_0 = \{0,1,2,\ldots\}$ and therefore $\overline{\operatorname{span}}\{\operatorname{ran} A^nB : n=0,1,\ldots\} = \mathcal{X}$. That is, $\Sigma = (A,B,C,D;\mathcal{X},\mathcal{U};\kappa_1)$ is a self-adjoint minimal realization of θ. □

Remark 3.6. The realization Σ in Theorem 3.5 is not shown to be passive. In the case where $\mathcal{U}$ is a Hilbert space and $\theta\in\mathbf{S}(\mathcal{U})\cap\mathbf{N}(\mathcal{U})$, that is, when θ is an ordinary Schur and Nevanlinna function, it is known from [4, Theorem 5.1] that there exists a minimal self-adjoint *passive* realization Σ of θ. In general, if $\theta\in\mathbf{S}_{\kappa_1}(\mathcal{U})\cap\mathbf{N}_{\kappa_2}(\mathcal{U})$, where $\mathcal{U}$ is Pontryagin space, it follows from Proposition 3.4 that a self-adjoint minimal realization $\Sigma = (T_\Sigma;\mathcal{X},\mathcal{U};\kappa)$ of θ can be passive only if $\kappa_1\le\kappa_2=\kappa$, and in the case where $\mathcal{U}$ is a Hilbert space, only if $\kappa_1=\kappa_2=\kappa$.

The conditions of Theorem 3.5 that $\mathcal{U}$ is a Hilbert space there and $\theta\in\mathbf{N}_{\kappa_2}(\mathcal{U})$ can be relaxed slightly; with a cost of weakened conclusions.

Proposition 3.7. *Let $\mathcal{U}$ be a Pontryagin space. Then a symmetric function $\theta\in\mathbf{S}_\kappa(\mathcal{U})$ has a self-adjoint realization $\Sigma=(A,B,C,D;\mathcal{X},\mathcal{U})$ where $\mathcal{X}$ is a Kreĭn space.*

Proof. Let $\Sigma_1 = (A,B,C,D;\mathcal{X}_1,\mathcal{U};\kappa)$ be a simple conservative realization of θ. Since θ is symmetric in sense $\theta(z)=\theta^{\#}(z)$, it holds $D=D^*$ and the dual system $\Sigma_1^* = (A^*,C^*,B^*,D;\mathcal{X}_1,\mathcal{U};\kappa)$ of Σ_1 is also a simple conservative realization of θ. Therefore Σ_1 and Σ_1^* are unitarily similar, that is, there exists a unitary mapping $J : \mathcal{X}_1 \to \mathcal{X}_1$ such that, see (3.7),

$$A = J^* A^* J, \qquad JB = C^*, \qquad C = B^* J, \qquad J^{-1} = J^*. \tag{3.20}$$

The letter J is used, because the operator J is now also self-adjoint in $\mathcal{X}_1$. Indeed, let $N \in \mathbb{N}_0$. Easy calculations show that it holds

$$J\left(\sum_{n=0}^{N} A^n B\right) = \sum_{n=0}^{N} {A^*}^n C^* = \sum_{n=0}^{N} {A^*}^n J^* B = J^*\left(\sum_{n=0}^{N} A^n B\right), \tag{3.21}$$

and similarly

$$J\left(\sum_{n=0}^{N} {A^*}^n C^*\right) = J^*\left(\sum_{n=0}^{N} {A^*}^n C^*\right). \tag{3.22}$$

Since Σ_1 is simple, it follows from (3.21) and (3.22) that J and J^* coincide on a dense lineal of $\mathcal{X}_1$, and then by continuity, everywhere. That is, J is unitary and self-adjoint. Now introduce the inner product space $\mathcal{X}$, which coincides with $\mathcal{X}_1$ as a vector space but which is endowed with the inner product $\langle x, y\rangle_{\mathcal{X}} = \langle Jx, y\rangle_{\mathcal{X}_1}$. Then $\mathcal{X}$ is a Kreĭn space. Moreover, it holds

$$\langle Ax, y\rangle_{\mathcal{X}} = \langle JAx, y\rangle_{\mathcal{X}_1} = \langle x, A^*Jy\rangle_{\mathcal{X}_1} = \langle x, JAy\rangle_{\mathcal{X}_1} = \langle x, Ay\rangle_{\mathcal{X}},$$
$$\langle Bu, x\rangle_{\mathcal{X}} = \langle JBu, x\rangle_{\mathcal{X}_1} = \langle u, B^*Jx\rangle_{\mathcal{U}} = \langle u, B^*Jx\rangle_{\mathcal{U}} = \langle u, Cx\rangle_{\mathcal{U}}.$$

This implies that A is self-adjoint in the Kreĭn space $\mathcal{X}$, and the adjoint of $B : \mathcal{U} \to \mathcal{X}$ is C viewed as operator from $\mathcal{X}$ to $\mathcal{U}$. Then, A, B, C and their adjoints all are everywhere defined, and therefore bounded also with respect to the topology induced by $\mathcal{X}$ [16, Chapter VI 2]. Define $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U})$. Then, Σ is a self-adjoint realization of θ, and the proof is complete. □

In Proposition 3.7, the self-adjoint realization $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U})$ with the Kreĭn space $\mathcal{X}$ was constructed from a simple conservative realization $\Sigma_1 = (T_{\Sigma_1}; \mathcal{X}_1, \mathcal{U}; \kappa)$. If $\mathcal{X}$ is a Pontryagin space with the negative index κ'', it follows from Proposition 3.1 that the transfer function $\theta \in \mathbf{S}_\kappa(\mathcal{U})$ belongs also in the class $\mathbf{N}_{\kappa'}(\mathcal{U})$, where $\kappa' \le \kappa''$. One might conjecture that this happens for every $\theta \in \mathbf{S}_\kappa(\mathcal{U}) \cap \mathbf{N}_{\kappa'}(\mathcal{U})$. However, Theorem 3.8 below shows that this is not true; it happens only when $\theta \in \mathbf{U}_\kappa(\mathcal{U}) \cap \mathbf{N}_{\kappa'}(\mathcal{U})$. That is, the values of θ must also be unitary for all but finitely many points on the unit circle $\mathbb{T}$; see the page 11.

Theorem 3.8. *Let $\theta \in \mathbf{S}_{\kappa_1}(\mathcal{U})$ be symmetric in a sense $\theta(z) = \theta^{\#}(z)$, and let $\Sigma_1 = (A, B, C, D; \mathcal{X}_1, \mathcal{U}; \kappa_1)$ be a simple conservative realization of θ. Construct the Kreĭn space $\mathcal{X}$ and the self-adjoint realization $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U})$ by using the method given in the proof of Proposition 3.7. Then $\theta \in \mathbf{U}_{\kappa_1}(\mathcal{U}) \cap \mathbf{N}_{\kappa_2}(\mathcal{U})$ if and only if $\mathcal{X}$ is a Pontryagin space with the negative index κ_2.*

Proof. Let $*$ and $[*]$ refer, respectively, to the adjoint with respect to the inner product of $\mathcal{X}_1$ and $\mathcal{X}$.

$\Leftarrow$: Suppose that $\mathcal{X}$ is a Pontryagin space with the negative index κ_2. Let J be the unitary similarity mapping used in Proposition 3.7 with the properties (3.20). Then, since

$$\langle A^*x, y\rangle_{\mathcal{X}} = \langle JA^*x, y\rangle_{\mathcal{X}_1} = \langle x, AJy\rangle_{\mathcal{X}_1} = \langle x, JA^*y\rangle_{\mathcal{X}_1} = \langle x, A^*y\rangle_{\mathcal{X}},$$

A and A^* are both self-adjoint operators with respect to the inner product of the Pontryagin space $\mathcal{X}$, and it follows from [15, Corollary 3.15, pp. 106] that the non-real spectra of A and A^* consist only of finitely many points. Then, $(I-\zeta A)^{-1}$ and $(I-\overline{\zeta}A^*)^{-1}$ exist for all but finitely many $\zeta \in \mathbb{T}$. Since Σ_1 is conservative, the system operator T of Σ is unitary, and therefore the defect spaces of T and T^* in (3.13), are zero spaces. By using (3.15) from Lemma 3.3, it can be now deduced that

$$I - \theta^*(\zeta)\theta(\zeta) = (1-\zeta\overline{\zeta})B^*(I-\overline{\zeta}A^*)^{-1}(I-\zeta A)^{-1}B = 0$$

for all but finitely many $\zeta \in \mathbb{T}$, which shows that $\theta = \theta^{\#} \in \mathbf{U}_{\kappa_1}(\mathcal{U})$. Choose some fundamental decomposition of $\mathcal{U}$, and consider the Potapov–Ginzburg transformation θ_P as in (2.3). It can be assumed that θ_P is holomorphic at the origin, since if not, one only has to consider $\theta_P(\eta(z))$ as in Remark 2.2. Then by Proposition 2.1 and Theorem 2.8, θ_P is symmetric and the radial limit values of θ_P are unitary a.e. on $\mathbb{T}$. It follows then from [12, Theorems 9.4 and 10.2]; see also [31, Theorem 4.4], that all simple passive realizations of θ_P are conservative and minimal. Then it follows from [2, Theorems 2.1.3 and 4.3.3] that all simple conservative realizations of θ are minimal. Therefore Σ_1 is minimal, which implies that Σ is also minimal, since the norms of spaces $\mathcal{X}_1$ and $\mathcal{X}$ are equivalent. Therefore, Σ is a minimal self-adjoint system with a Pontryagin state space, and it follows from Proposition 3.1 that θ is a generalized Nevanlinna function whose negative index coincides with the index of the maximal negative subspace of the space of the form (3.10). But since Σ is minimal, the space (3.10) is dense in $\mathcal{X}$, and by [16, Theorem 1.4 on p. 185], it contains a maximal uniformly negative subspace of $\mathcal{X}$. It follows that $\theta \in \mathbf{N}_{\kappa_2}(\mathcal{U})$.

$\Rightarrow$: Let $\theta \in \mathbf{U}_{\kappa_1}(\mathcal{U}) \cap \mathbf{N}_{\kappa_2}(\mathcal{U})$. By using the Potapov–Ginzburg transformation similarly as above, it can be deduced that Σ_1 and Σ are both minimal. By using a similar argument as in the proof Proposition 3.1, one deduces that the Nevanlinna kernel of θ can be represented as

$$\begin{aligned} N_\theta(w,z) &= C(I-zA)^{-1}(I-\bar{w}A)^{-1}B \\ &= B^{[*]}(I-zA^{[*]})^{-1}(I-\bar{w}A)^{-1}B. \end{aligned} \tag{3.23}$$

Then the matrix of the form

$$\left(\langle N_\theta(w_j,w_i)f_j, f_i\rangle_{\mathcal{U}}\right)_{i,j=1}^n = \left(\langle (I-\bar{w}_jA)^{-1}Bf_j, (I-\bar{w}_iA)^{-1}Bf_i\rangle_{\mathcal{X}}\right)_{i,j=1}^n,$$

such that $f_i \in \mathcal{U}$, $w_i \in \Omega$, $i = 1, \ldots, n$, where Ω is a sufficiently small symmetric neighbourhood of the origin, is a Gram matrix. Since $\theta \in \mathbf{N}_{\kappa_2}(\mathcal{U})$, the kernel N_θ has κ_2 negative squares. These facts combined with [2, Lemma 1.1.1'] imply that there exists a finite sequence $\{(I-\overline{w_i}A)^{-1}Bf_i\}_{i=1}^n \subset \mathcal{X}$ of vectors such that the linear span of $\{(I-\overline{w_i}A)^{-1}Bf_i\}_{i=1}^n$ contains a κ_2-dimensional anti-Hilbert subspace of $\mathcal{X}$. Therefore, ind$_-\mathcal{X}$ is at least κ_2. Suppose that ind$_-\mathcal{X} > \kappa_2$. Then there exists a finite sequence $\{x_i\}_{i=1}^{\kappa'} \subset \mathcal{X}$ of linearly independent negative vectors such that $\kappa' > \kappa_2$. Since Σ is minimal, span$\{$ran $(I-$

$zA)^{-1}B : z \in \Omega\} := \mathfrak{S}$, where Ω is a sufficiently small symmetric neighbourhood of the origin, is dense in $\mathcal{X}$. Then, each x_i can be approximated as closely as desired by a vector x'_i from $\mathfrak{S}$. By choosing a good enough approximation x'_i for each x_i, one obtains κ' negative linearly independent vectors $x'_1, \ldots, x'_{\kappa'} \in \mathfrak{S}$. That is, $\mathfrak{S}$ contains k'-dimensional anti-Hilbert subspace, and by using a similar argument as in the proof of Proposition 3.1, it can be shown that the kernel N_θ has at least $\kappa' > \kappa_2$ negative squares. This contradicts the assumption that $\theta \in \mathbf{N}_{\kappa_2}(\mathcal{U})$. Therefore, $\operatorname{ind}_- \mathcal{X}$ must be κ_2, and the proof is complete. □

A simple conservative realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ is unique up to unitary similarity, and therefore the results of Theorem 3.8 do not depend on the choice of Σ and J. The concrete models for J can be obtained by using the canonical realizations of θ. If $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, for an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function θ holomorphic on a neighbourhood Ω of the origin, the kernel (1.1) has κ negative squares if and only if the related $\mathcal{L}(\mathcal{Y} \oplus \mathcal{U})$-valued kernel

$$D_\theta(w, z) = \begin{pmatrix} K_\theta(w, z) & \dfrac{\theta(z) - \theta(\overline{w})}{z - \overline{w}} \\ \dfrac{\theta^\#(z) - \theta^\#(\overline{w})}{z - \overline{w}} & K_{\theta^\#}(w, z) \end{pmatrix}, \qquad w, z \in \Omega, \tag{3.24}$$

has κ negative squares; see [2, Theorem 2.5.2]. The Pontryagin space generated by the kernel (3.24), that is, the completion of the space

$$\operatorname{span}\left\{ D_\theta(w, z)\binom{y}{u} : w \in \Omega, \quad y \in \mathcal{Y}, \quad u \in \mathcal{U} \right\}, \tag{3.25}$$

where $D_\theta(w, z)\binom{y}{u}$ is treated as a function of z, is denoted by $\mathcal{D}(\theta)$. The function space $\mathcal{D}(\theta)$ is continuously contained in $\mathcal{H}(\theta) \oplus \mathcal{H}(\theta^\#)$, where $\mathcal{H}(\theta)$ is the Pontryagin space generated by the kernel (1.1). The spaces $\mathcal{H}(\theta)$ and $\mathcal{D}(\theta)$ can be chosen as state spaces of, respectively, an observable co-isometric realization and a simple conservative realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$. For $h \in \mathcal{H}(\theta)$, $\binom{h}{k} \in \mathcal{D}(\theta)$ and $u \in \mathcal{U}$, define respectively

$$\begin{cases} A_1 : h(z) \mapsto \dfrac{h(z) - h(0)}{z}, & B_1 : u \mapsto \dfrac{\theta(z) - \theta(0)}{z} u, \\ C_1 : h(z) \mapsto h(0), & D : u \mapsto \theta(0)u, \end{cases} \tag{3.26}$$

and

$$\begin{cases} A_2 : \begin{pmatrix} h(z) \\ k(z) \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{h(z) - h(0)}{z} \\ zk(z) - \theta^\#(z)h(0) \end{pmatrix}, & B_2 : u \mapsto \begin{pmatrix} \dfrac{\theta(z) - \theta(0)}{z} u \\ \left(I_{\mathcal{U}} - \theta^\#(z)\theta^{\#*}(0)\right) u \end{pmatrix}, \\ C_2 : \begin{pmatrix} h \\ k \end{pmatrix} \mapsto h(0), & D : u \mapsto \theta(0)u. \end{cases} \tag{3.27}$$

Then $\Sigma_1 = (A_1, B_1, C_1, D, \mathcal{H}(\theta), \mathcal{U}, \mathcal{Y}, \kappa)$ and $\Sigma_2 = (A_2, B_2, C_2, D, \mathcal{D}(\theta), \mathcal{U}, \mathcal{Y}, \kappa)$ are, respectively, an observable co-isometric realization of θ, and a simple conservative realization of θ; for the proof, see [2, Theorems 2.2.1 and 2.3.1].

These systems are called, respectively, the **canonical co-isometric realization** and the **canonical unitary** (or **conservative**) **realization** of θ. Any observable co-isometric realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ is unitarily similar with the system Σ_1, and any simple conservative realization is unitarily similar with Σ_2.

Suppose next the simple conservative realization of the symmetric function $\theta \in \mathbf{S}_{\kappa_1}(\mathcal{U})$ in Theorem 3.8 is chosen to be the canonical unitary realization. Then it can be derived from [3, Theorem 3.6] that the self-adjoint unitary similarity J is of the form $J = \widehat{J}\restriction_{\mathcal{D}(\theta)}$, where

$$\widehat{J} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H}(\theta) \\ \mathcal{H}(\theta) \end{pmatrix} \to \begin{pmatrix} \mathcal{H}(\theta) \\ \mathcal{H}(\theta) \end{pmatrix}. \tag{3.28}$$

In addition, if also $\theta \in \mathbf{U}_{\kappa_1}(\mathcal{U})$, it has been shown in the proof of Theorem 3.8 that all co-isometric observable or isometric controllable realizations of θ are minimal conservative. Therefore it can be assumed that Σ_1 in Theorem 3.8 is the canonical co-isometric realization. In that case, it can be derived from [3, Corollary 3.7] that J is the closure of a linear relation Λ defined by

$$\Lambda\left(\sum_i K_\theta(w_i,z)f_i\right) = \sum_i N_\theta(w_i,z)f_i,$$

where $K_\theta(w_i,z)f_i \in \mathcal{H}(\theta)$ and $N_\theta(w_i,z)f_i \in \mathcal{H}(\theta)$ are treated as a function of z.

Proposition 3.9. *Let $\mathcal{U}$ be Pontryagin space and let θ be a symmetric $\mathcal{L}(\mathcal{U})$-valued function holomorphic at the origin and meromorphic on $\mathbb{D}\cup\mathbb{C}\setminus\mathbb{R}$. Then the following statements are equivalent:*

- (i) *θ has a minimal conservative self-adjoint realization Σ such that the state space of Σ is a Hilbert space;*
- (ii) *$\theta \in \mathbf{U}(\mathcal{U})\cap\mathbf{N}(\mathcal{U})$;*
- (iii) *$K_\theta(w,z) = N_\theta(w,z)$ and the kernels are nonnegative.*

Proof. (i) $\Rightarrow$ (ii). Denote $\Sigma = (A,B,B^*,D;\mathcal{X},\mathcal{U};0)$. Since Σ is a minimal conservative self-adjoint realization of θ such that $\mathcal{X}$ is a Hilbert space, it follows from Proposition 3.4 that $\theta \in \mathbf{S}(\mathcal{U})\cap\mathbf{N}(\mathcal{U})$. Moreover, the main operator A is self-adjoint operator in the Hilbert space $\mathcal{X}$, and a similar argument as used in the proof of Theorem 3.8 can be used to show that the values of θ are unitary for every $\zeta \in \mathbb{D}\setminus\{-1,1\}$. Therefore $\theta \in \mathbf{U}(\mathcal{U})\cap\mathbf{N}(\mathcal{U})$.

(ii) $\Rightarrow$ (iii). By definition of $\mathbf{U}(\mathcal{U})$ and $\mathbf{N}(\mathcal{U})$, the kernels K_θ and N_θ are nonnegative. If $\mathcal{U}$ is a Hilbert space, the other claim now follows by combining [4, Theorem 5.1] and [5, Proposition 3.1]. Therefore assume $\operatorname{ind}_-\mathcal{U} > 0$. Fix some fundamental decomposition of $\mathcal{U}$, and consider the Potapov–Ginzburg transformation θ_P of θ, defined by (2.3). Since θ is symmetric, the functions Φ and Ψ defined by (2.5) coincide. It can be assumed that $\theta_{22}^{-1}(0)$ exists; otherwise, consider $\theta_P(\eta(z))$ as in Remark 2.2. Then $\theta_P \in \mathbf{U}(\mathcal{U}')\cap\mathbf{N}(\mathcal{U}')$ by Proposition 2.1(vii), and since $\mathcal{U}'$ is a Hilbert space, it now holds $K_{\theta_P}(w,z) = N_{\theta_P}(w,z)$. It follows then from (2.6) and (2.8) that also $K_\theta(w,z) = N_\theta(w,z)$.

(iii) $\Rightarrow$ (i) Since $K_\theta(w,z)$ is nonnegative, $\theta \in \mathbf{S}(\mathcal{U})$, and the canonical unitary realization Σ_2 defined by the operators in (3.27) is simple conservative and the state space is a Hilbert space. Since $\theta = \theta^{\#}$ and $K_\theta(w,z) = N_\theta(w,z)$, the kernel $D_\theta(w,z)$ in (3.24) reduces to

$$\begin{pmatrix} K_\theta(w,z) & K_\theta(w,z) \\ K_\theta(w,z) & K_\theta(w,z) \end{pmatrix}.$$

Therefore, all the functions in the space (3.25) are of the form

$$\begin{pmatrix} h_0(z) \\ h_0(z) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} \alpha_j K_\theta(w_j,z)u_j \\ \sum_{j=1}^{n} \alpha_j K_\theta(w_j,z)u_j \end{pmatrix},$$

such that $\alpha_j \in \mathbb{C}$, $u_j \in \mathcal{U}$ and $w_j \in \Omega$, where Ω is the domain of holomorphy of θ. It follows that all the $\mathcal{L}(\mathcal{U} \oplus \mathcal{U})$-valued functions in the completion $\mathcal{D}(\theta)$ of (3.25) are of the form $\left(\begin{smallmatrix} h(z) \\ h(z) \end{smallmatrix}\right)$. Then, the self-adjoint unitary similarity mapping $J = \widehat{J}\restriction_{\mathcal{D}(\theta)}$ between Σ_2 and Σ_2^*, where $\widehat{J}$ where is defined by (3.28), is identity. That is, Σ_2 is self-adjoint, and since it is simple, it is now minimal, and the proof is complete. $\square$

In Theorem 3.8, the condition that the space $\mathcal{X}$ induced by the mapping J is a Pontryagin space with the negative index κ is equivalent to $\operatorname{ind}_- J = \kappa$, where $\operatorname{ind}_- J$ is with respect to the state space $\mathcal{X}_1$. By considering minimal passive realizations instead of simple conservative realizations, one can obtain a similar type of characterization when $\theta \in \mathbf{S}_{\kappa_1}(\mathcal{U}) \cap \mathbf{N}_{\kappa_2}(\mathcal{U})$.

Denote $E_{\mathcal{X}}(x) = \langle x, x\rangle_{\mathcal{X}}$ for the vector x in an inner product space $\mathcal{X}$. For $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, the realization Σ of θ is called κ-**admissible**, if the negative index of the state space of Σ is κ. A κ-admissible passive realization $\Sigma = (A,B,C,D;\mathcal{X},\mathcal{U},\mathcal{Y};\kappa)$ of $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ is called **optimal** if for any κ-admissible passive realization $\Sigma_0 = (A_0,B_0,C_0,D;\mathcal{X}_0,\mathcal{U},\mathcal{Y};\kappa)$ of θ it holds

$$E_{\mathcal{X}}\left(\sum_{n=0}^{N} A^n B u_n\right) \le E_{\mathcal{X}_0}\left(\sum_{n=0}^{N} {A_0}^n B_0 u_n\right),$$

for any $N \in \mathbb{N}_0$ and $\{u_n\}_{n=0}^N \subset \mathcal{U}$. Moreover, an observable passive realization $\Sigma = (A,B,C,D;\mathcal{X},\mathcal{U},\mathcal{Y};\kappa)$ of $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ is called *-**optimal** if for any observable κ-admissible passive realization $\Sigma_0 = (A_0,B_0,C_0,D;\mathcal{X}_0,\mathcal{U},\mathcal{Y};\kappa)$ of θ it holds

$$E_{\mathcal{X}}\left(\sum_{n=0}^{N} A^n B u_n\right) \ge E_{\mathcal{X}_0}\left(\sum_{n=0}^{N} {A_0}^n B_0 u_n\right),$$

for any $N \in \mathbb{N}_0$ and $\{u_n\}_{n=0}^N \subset \mathcal{U}$. The requirement of the observability in the definition of *-optimality is essential to avoid trivialities, see [9, Proposition 3.5 and example on page 144]. Moreover, the requirement that the considered realizations are κ-admissible is also essential, see [32, Example 3.1].

Let $\theta \in \mathbf{S}_{\kappa_1}(\mathcal{U})$ be symmetric. It follows from [34, Theorem 5.3] and [32, Theorem 3.5] that there exists a *-optimal minimal passive realization $\Sigma = (A,B,C,D;\mathcal{X},\mathcal{U};\kappa)$ of θ. Since $\theta = \theta^{\#}$, it follows from [34, Theorem 5.2] and [32, Theorem 3.5] that the dual system $\Sigma^* = (A^*,C^*,B^*,D;\mathcal{X},\mathcal{U};\kappa)$ of

Σ is optimal minimal passive. Define

$$Z\left(\sum_{n=0}^{N} A^n B u_n\right) = \sum_{n=0}^{N} A^{*n} C^* u_n$$

for the vectors of the form $\sum_{n=0}^{N} A^n B u_n$. Since Σ and Σ^* are minimal and Σ^* is optimal, the linear relation Z is densely defined, contractive, and it has a dense range in $\mathcal{X}$. It follows from [2, Theorem 1.4.2] that the closure of Z, which is still denoted as Z, is an everywhere defined bounded contractive linear operator in $\mathcal{X}$. By proceeding as in the proof of [31, Theorem 2.5], one deduces that Z is injective, it has a dense range, and it holds

$$ZA = A^*Z, \quad C = B^*Z \quad \text{and} \quad ZB = C^*. \tag{3.29}$$

That is, Z is an everywhere defined weak similarity. Moreover, it holds

$$Z^*\left(\sum_{n=0}^{N} A^n B u_n\right) = \sum_{n=0}^{N} A^{*n} Z^* B u_n = \sum_{n=0}^{N} A^{*n} C^* u_n = Z\left(\sum_{n=0}^{M} A^n B u_n\right).$$

Since Σ is minimal, it follows now that $Z : \mathcal{X} \to \mathcal{X}$ is self-adjoint. That is, Z is bounded injective self-adjoint operator. Moreover, an optimal (*-optimal) minimal passive realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ is unique up to unitary similarity [32, Theorem 3.5]. Therefore, the mapping Z is unique up to unitary equivalence, and the properties of Z in Theorem 3.10 below do not depend of the choice of a *-optimal minimal passive realization of θ.

Theorem 3.10. *Let $\theta \in \mathbf{S}_{\kappa_1}(\mathcal{U})$ be symmetric and let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}; \kappa_1)$ be a *-optimal minimal passive realization of θ. Then $\theta \in \mathbf{S}_{\kappa_1}(\mathcal{U}) \cap \mathbf{N}_{\kappa_2}(\mathcal{U})$ if and only if $\operatorname{ind}_- Z = \kappa_2$, where Z is the self-adjoint contraction in $\mathcal{X}$ with the properties (3.29).*

Proof. Since Z is self-adjoint, bounded and injective with a dense range, it has a Bognár–Krámli factorization of the form $Z = V_Z V_Z^*$, where $V_Z : \mathcal{D}_Z \to \mathcal{X}$, is bounded, $\mathcal{D}_Z$ is a Kreĭn space with $\operatorname{ind}_- \mathcal{D}_Z = \operatorname{ind}_- Z$, and V_Z and V_Z^* have zero kernels and dense ranges; see [23, Theorem 2.1]. By using the realization Σ, the properties (3.29) of the self-adjoint mapping Z imply that $Z(I - zA)^{-1} = (I - zA^*)^{-1}Z$ whenever $(I - zA)^{-1}$ and $(I - zA^*)$ exist. Hence the Nevanlinna kernel of θ can be represented as

$$\begin{aligned} N_\theta(w, z) &= C(I - zA)^{-1}(I - \bar{w}A)^{-1}B = B^*Z(I - zA)^{-1}(I - \bar{w}A)^{-1}B \\ &= B^*(I - zA^*)^{-1}Z(I - \bar{w}A)^{-1}B = B^*(I - zA^*)^{-1}V_Z V_Z^*(I - \bar{w}A)^{-1}B; \end{aligned}$$

cf. (3.23) on page 20. This yields the identity

$$\begin{aligned} \left(\langle N_\theta(w_j, w_i) f_j, f_i \rangle_{\mathcal{U}}\right)_{i,j=1}^{n} &= \left(\langle Z(I - \bar{w}_j A)^{-1} B f_j, (I - \bar{w}_i A)^{-1} B f_i \rangle_{\mathcal{X}}\right)_{i,j=1}^{n} \\ &= \left(\langle V_Z^*(I - \bar{w}_j A)^{-1} B f_j, V_Z^*(I - \bar{w}_i A)^{-1} B f_i \rangle_{\mathcal{D}_Z}\right)_{i,j=1}^{n} \end{aligned} \tag{3.30}$$

where $n \in \mathbb{N}$, $\{f_j\}_{j=1}^{n} \subset \mathcal{U}$, and $\{w_j\}_{j=1}^{n} \subset \Omega$ for some sufficiently small symmetric neighbourhood Ω of the origin, for the kernel N_θ. Moreover, since Σ is minimal, the space $\operatorname{span}\{\operatorname{ran}(I - zA)^{-1}B : z \in \Omega\} := \mathfrak{S}$ is dense in $\mathcal{X}$. Let $y \in \mathcal{D}_Z$ such that $\langle y, V_Z^* x \rangle_{\mathcal{D}_Z} = 0$ for all $x \in \mathfrak{S}$. Then, $\langle y, V_Z^* x \rangle_{\mathcal{D}_Z} =$

$\langle V_Z y, x\rangle_{\mathcal{D}_Z} = 0$, which implies $V_Z y = 0$ and then $y = 0$, since $\mathfrak{S}$ is a dense set and V_Z has only the trivial kernel. That is, $V_Z^*\mathfrak{S}$ is a dense set in $\mathcal{D}_Z$.

$\Leftarrow$: Suppose that $\mathrm{ind}_- Z = \kappa_2$. Then $\mathrm{ind}_- \mathcal{D}_Z = \kappa_2$, and it follows from the identity (3.30) that N_θ has at most κ_2 negative squares. Since $V_Z^*\mathfrak{S}$ is a dense set in $\mathcal{D}_Z$, [2, Lemma 1.1.1] shows that there exists a finite sequence $\{V_Z^*(I-\overline{w_i'}A)^{-1}Bf_i'\}_{i=1}^n \subset \mathcal{D}_Z$ of vectors such that the linear span of $\{V_Z^*(I-\overline{w_i'}A)^{-1}Bf_i'\}_{i=1}^n$ contains a κ_2-dimensional anti-Hilbert subspace of $\mathcal{D}_Z$. Then by the identity (3.30) the kernel N_θ has at least κ_2 negative squares. It has been showed that N_θ has exactly κ_2 negative squares, and therefore $\theta \in \mathbf{N}_{\kappa_2}(\mathcal{U})$.

$\Rightarrow$: Assume $\theta \in \mathbf{N}_{\kappa_2}(\mathcal{U})$. Then the identity (3.30) shows that $\mathrm{ind}_- Z$ is at least κ_2. Since $V_Z^*\mathfrak{S}$ is a dense set, a similar argument as in the proof of Theorem 3.8 can be used to conclude that $\mathrm{ind}_- Z = \kappa_2$. □

4. Dilations and subclasses of generalized Schur–Nevanlinna functions

A dilation of the function $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ is any function Θ holomorphic at the origin, which is of the form

$$\Theta(z) = \begin{pmatrix} \theta(z) & \theta_2(z) \\ \theta_3(z) & \theta_4(z) \end{pmatrix} \tag{4.1}$$

and has values in $\mathcal{L}(\mathcal{U}\oplus\mathcal{U}', \mathcal{Y}\oplus\mathcal{Y}')$, where $\mathcal{U}'$ and $\mathcal{Y}'$ are Hilbert spaces. In the case where $\mathcal{U}'$ or $\mathcal{Y}'$ is a zero space, the corresponding row or column in (4.1) will be left out. This definition is a straightforward generalization of the definition of a dilation of $\theta \in \mathbf{S}(\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, as represented in [14]. A function $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ has a **generalized bi-$\mathcal{J}$-inner dilation**, if there exists a dilation Θ of θ such that $\Theta \in \mathbf{U}_\kappa(\mathcal{U}\oplus\mathcal{U}', \mathcal{Y}\oplus\mathcal{Y}')$. The case when $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces and $\kappa = 0$ corresponds to the ordinary bi-inner dilation. It is known from [14] that $\theta \in \mathbf{S}(\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, has a bi-inner dilation if and only if there exists $\varphi \in \mathbf{S}(\mathcal{U},\mathcal{Y}')$ and $\psi \in \mathbf{S}(\mathcal{Y},\mathcal{U}')$, where such that $I_\mathcal{U} - \theta^*(\zeta)\theta(z) = \varphi^*(\zeta)\varphi(\zeta)$ and $I_\mathcal{Y} - \theta(\zeta)\theta^*(\zeta) = \psi(\zeta)\psi^*(\zeta)$ for a.e. $\zeta \in \mathbb{T}$. Moreover, every $\theta \in \mathbf{S}(\mathcal{U}) \cap \mathbf{N}(\mathcal{U})$ has a bi-inner dilation Θ, which can be chosen such that $\Theta \in \mathbf{U}(\mathcal{U}\oplus\mathcal{U}') \cap \mathbf{N}(\mathcal{U}\oplus\mathcal{U}')$ [5, pp. 4)].

As mentioned on page 12, any two minimal passive realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ are only weakly similar in general. However, for some generalized Schur functions, any two minimal passive κ-admissible realizations are unitary similar. This happens, for an example, if the boundary values of $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ are (co-)isometric a.e. on $\mathbb{T}$, or when the generalized right or left defect function is identically zero; see [32, pp. 25]. Let $\mathbf{S}_\kappa^U(\mathcal{U},\mathcal{Y})$ be the class that consist of those functions $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ with the property that any two minimal passive κ-admissible realizations of θ are unitarily similar. A criterion for θ to be in $\mathbf{S}^U(\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, were obtained by Arov and Nudelman in [10,11]. This result was generalized for the class $\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative indices, by the author in [32, pp. 25].

Define $\mathcal{RS}_\kappa(\mathcal{U})$ to be the class of those functions in $\mathbf{S}_\kappa(\mathcal{U})\cap\mathbf{N}_\kappa(\mathcal{U})$ which have κ-admissible passive self-adjoint realizations. If $\mathcal{U}$ is a Hilbert space and $\kappa=0$, the class $\mathcal{RS}_0(\mathcal{U})$ coincides with $\mathbf{S}(\mathcal{U})\cap\mathbf{N}(\mathcal{U})$ [5].

Theorem 4.1. *Every $\theta\in\mathbf{S}^U_{\kappa_1}(\mathcal{U})\cap\mathbf{N}_{\kappa_2}(\mathcal{U})$, where $\mathcal{U}$ is a Pontryagin space, has a generalized bi-$\mathcal{J}$-inner dilation $\Theta\in\mathbf{U}_{\kappa_1}(\mathcal{U}\oplus\mathcal{U}')$, and every $\theta\in\mathcal{RS}_\kappa(\mathcal{U})$ has a generalized bi-$\mathcal{J}$-inner dilation $\Theta\in\mathbf{U}_\kappa(\mathcal{U}\oplus\mathcal{U}')\cap\mathbf{N}_\kappa(\mathcal{U}\oplus\mathcal{U}')$.*

Proof. Suppose $\theta\in\mathbf{S}^U_{\kappa_1}(\mathcal{U})\cap\mathbf{N}_{\kappa_2}(\mathcal{U})$. Consider a *-optimal minimal realization $\Sigma=(A,B,C,D;\mathcal{X},\mathcal{U};\kappa_1)$ of θ and the self-adjoint contraction Z with the properties (3.29). Since $\theta\in\mathbf{S}^U_{\kappa_1}$ and Σ and Σ^* are both minimal passive κ_1-admissible realizations of θ, the system Σ is unitarily similar with Σ^*. It follows then easily that Z is actually unitary in $\mathcal{X}$, and therefore $Z=Z^*=Z^{-1}$ i.e. Then, since Z in boundedly invertible, the space $\mathcal{X}_Z$, which coincide with $\mathcal{X}$ as a vector space but which is endowed with the inner product $\langle x,y\rangle_{\mathcal{X}_Z}=\langle Zx,y\rangle_{\mathcal{X}}$, is a Kreĭn space [15, 6.13 on page 40]. The same arguments as used in the proof of Proposition 3.7 shows that A and A^* are self-adjoint with respect to the inner product of $\mathcal{X}_Z$, and $\Sigma_Z:=(A,B,C,D;\mathcal{X}_Z,\mathcal{U})$ is a self-adjoint realization of θ. Since Σ is minimal and $\theta\in\mathbf{N}_{\kappa_2}(\mathcal{U})$, similar arguments as in the proof of Theorem 3.8 can be used to conlude that Σ_Z is minimal and $\mathcal{X}_Z$ is a Pontryagin space with the negative index κ_2. Then, the non-real spectra of A and A^* consist only of finitely many points, i,e., $(I-\zeta A)^{-1}$ and $(I-\zeta A^*)^{-1}$ are defined for all but finitely many $\zeta\in\mathbb{T}$.

Denote the system operator of Σ as T. By Theorem 3.2, the system operator T has a Julia operator of the form

$$\begin{pmatrix} T & D_{T^*}\\ D_T^* & -L^*\end{pmatrix}:\begin{pmatrix}\mathcal{X}\oplus\mathcal{U}\\ \mathfrak{D}_{T^*}\end{pmatrix}\to\begin{pmatrix}\mathcal{X}\oplus\mathcal{U}\\ \mathfrak{D}_T\end{pmatrix},\tag{4.2}$$

where $\mathfrak{D}_{T^*}$ and $\mathfrak{D}_T$ are Hilbert spaces, $D_{T^*}D_{T^*}^*=I-TT^*$ and $D_TD_T^*=I-T^*T$ such that D_T and D_{T^*} have zero kernels. Then, one can form the **Julia embedding** $\widetilde{\Sigma}$ of the system Σ; recall the embeddings introduced after Lemma 3.3. That is, the corresponding system operator $T_{\widetilde{\Sigma}}$ of the embedding $\widetilde{\Sigma}$ is a Julia operator of T, and it is of the form

$$T_{\widetilde{\Sigma}}=\begin{pmatrix} A & \begin{pmatrix} B & D_{T^*_{,1}}\end{pmatrix}\\ \begin{pmatrix} C\\ D^*_{T,1}\end{pmatrix} & \begin{pmatrix} D & D_{T^*_{,2}}\\ D^*_{T,2} & -L^*\end{pmatrix}\end{pmatrix}:\begin{pmatrix}\mathcal{X}\\ \begin{pmatrix}\mathcal{U}\\ \mathfrak{D}_{T^*}\end{pmatrix}\end{pmatrix}\to\begin{pmatrix}\mathcal{X}\\ \begin{pmatrix}\mathcal{U}\\ \mathfrak{D}_T\end{pmatrix}\end{pmatrix},$$

where $D_{T^*}=\begin{pmatrix}D_{T^*_{,1}}\\ D_{T^*_{,2}}\end{pmatrix}$ and $D_T=\begin{pmatrix}D_{T,1}\\ D_{T,2}\end{pmatrix}$. The system $\widetilde{\Sigma}$ is conservative, and since Σ is minimal, the inclusions $\operatorname{ran}B\subset\operatorname{ran}\begin{pmatrix}B & D_{T^*_{,1}}\end{pmatrix}$ and $\operatorname{ran}C^*\subset\operatorname{ran}\begin{pmatrix}C^* & D_{T,1}\end{pmatrix}$ imply that $\widetilde{\Sigma}$ is also minimal. The transfer function of the Julia embedding is

$$\Theta(z)=\begin{pmatrix} D+zC(I-zA)^{-1}B & D_{T^*_{,2}}+zC(I-zA)^{-1}D_{T^*_{,1}}\\ D^*_{T,2}+zD^*_{T,1}(I-zA)^{-1}B & -L^*+zD^*_{T,1}(I-zA)^{-1}D_{T^*_{,1}}\end{pmatrix}=\begin{pmatrix}\theta(z) & \psi(z)\\ \varphi(z) & \chi(z)\end{pmatrix}.$$

The funtion Θ is a dilation of θ, and since it is the transfer function of the minimal conservative system, it is a generalized Schur function with the index κ_1 [2, Theorem 2.1.2]. Since $\widetilde{\Sigma}$ is conservative, it follows from Lemma 3.3 that

$$I - \Theta(z)\Theta^*(w) = (1 - z\overline{w}) \begin{pmatrix} C \\ D^*_{T_{,1}} \end{pmatrix} (I - zA)^{-1}(I - \overline{w}A^*)^{-1} \begin{pmatrix} C^* & D_{T_{,1}} \end{pmatrix}$$
$$I - \Theta^*(w)\Theta(z) = (1 - z\overline{w}) \begin{pmatrix} B^* \\ D^*_{T^*_{,1}} \end{pmatrix} (I - \overline{w}A^*)^{-1}(I - zA)^{-1} \begin{pmatrix} B & D_{T^*_{,1}} \end{pmatrix},$$

and therefore that $I - \Theta(\zeta)\Theta^*(\zeta) = 0$ and $I - \Theta^*(\zeta)\Theta(\zeta) = 0$ for all but finitely many $\zeta \in \mathbb{T}$. That is, the radial limit values of Θ are unitary for all but finitely many $\zeta \in \mathbb{T}$, and therefore Θ is a unitary dilation of θ.

Assume then that $\theta \in \mathcal{RS}_\kappa(\mathcal{U})$, and let $\widehat{\Sigma} = (\widehat{A}, \widehat{B}, \widehat{C}, D; \widehat{\mathcal{X}}, \mathcal{U}; \kappa)$ be a κ-admissible passive self-adjoint realization of θ. Since $\widehat{\Sigma}$ is a κ-admissible and passive, the space $\widehat{\mathcal{X}}^s$ is a regular subspace with the negative index κ, and $(\widehat{\mathcal{X}}^s)^\perp$ is a Hilbert space [32, Proposition 2.7]. This implies that the system operator $T_{\widehat{\Sigma}}$ can be represented as in (3.19). It then easily follows that the restriction $\Sigma = (A, B, C, D; \widehat{\mathcal{X}}^s, \mathcal{U}; \kappa)$ of $\widehat{\Sigma}$ to the simple subspace $\mathcal{X}^s$ is a minimal passive self-adjoint κ-admissible realization of θ. Denote the system operator of Σ as T. By Theorem 3.2 there exists a Julia operator U_T of T of the form (4.2) where $\mathfrak{D}_{T^*}$ and $\mathfrak{D}_T$ are Hilbert spaces. Since T is self-adjoint, it follows from [24, Theorem 5 and pp. 88] that U_T can be chosen such that $\mathfrak{D}_{T^*} = \mathfrak{D}_T := \mathcal{U}'$ and $U_T \in \mathcal{L}(\mathcal{X} \oplus \mathcal{U} \oplus \mathcal{U}')$ is self-adjoint; cf. (3.13). Now construct a Julia embedding $\widetilde{\Sigma}$ of Σ similarly as above, by using U_T, and denote the transfer function of $\widetilde{\Sigma}$ as Θ. Then Θ is a dilation of θ and $\widetilde{\Sigma}$ is minimal self-adjoint conservative. Therefore $\Theta \in \mathbf{S}_\kappa \in (\mathcal{U} \oplus \mathcal{U}')$, and by Proposition 3.1 also $\Theta \in \mathbf{N}_\kappa(\mathcal{U} \oplus \mathcal{U}')$. Since $A = A^*$ is a self-adjoint operator in a Pontryagin space with the negative index κ, a similar argument as above shows that the values of Θ are unitary for all but finitely many $\zeta \in \mathbb{T}$. Therefore also $\Theta \in \mathbf{U}_\kappa(\mathcal{U} \oplus \mathcal{U}')$, and the proof is complete. □

Acknowledgements

I wish to thank the Vilho, Yrjö and Kalle Väisälä Foundation of the Finnish Academy of Science and Letters for a received grant. I also wish to thank Seppo Hassi for helpful discussions while preparing this paper.

Funding Open access funding provided by University of Vaasa (UVA).

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

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Lassi Lilleberg ( )
Department of Mathematics and Statistics
University of Vaasa
P.O. Box 70065101 Vaasa
Finland
e-mail: lassi.lilleberg@uva.fi

Received: May 28, 2020.
Revised: September 10, 2020.