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# FACTORIZATION OF GENERALIZED NEVANLINNA FUNCTIONS AND THE INVARIANT SUBSPACE PROPERTY

HENDRIK LUIT WIETSMA

ABSTRACT. The well-known invariant subspace property of selfadjoint relations (multi-valued operators) in Pontryagin spaces is shown to be equivalent to the factorization property of (scalar) generalized Nevanlinna functions. This connection is established by a new realization for generalized Nevanlinna functions explicitly reflecting this connection. Combining this result with the new function-theoretic proof for the factorization property of generalized Nevanlinna functions contained in [20] immediately yields a new proof for the invariant subspace property of selfadjoint relations in Pontryagin spaces.

## 1. INTRODUCTION

A symmetric complex function  $f$  is an (ordinary) Nevanlinna function,  $f \in \mathfrak{N}$ , if  $\operatorname{Im}(f(z)) \cdot \operatorname{Im} z > 0$  for all  $z \in \mathbb{C} \setminus \mathbb{R}$ , see e.g. [9]. This class of functions was extended by M.G. Kreĭn and H. Langer, see [13], to the class of generalized Nevanlinna functions. A symmetric function  $f$  meromorphic on  $\mathbb{C} \setminus \mathbb{R}$  is a *generalized Nevanlinna function of index*  $\kappa \in \mathbb{N}_0$ ,  $f \in \mathfrak{N}_\kappa$ , if for arbitrary  $z_1, \dots, z_n$  contained in the intersection of  $\mathbb{C}_+$  with the domain of holomorphy of  $f$ , the Hermitian matrix  $(N_f(z_i, z_j))_{i,j=1,\dots,n}$  has at most  $\kappa$  negative eigenvalues, and there exists a choice  $z_1, \dots, z_n$  such that  $(N_f(z_i, z_j))_{i,j=1,\dots,n}$  has precisely  $\kappa$  negative eigenvalues. Here

$$(1.1) \quad N_f(z, w) := \frac{f(z) - \overline{f(w)}}{z - \overline{w}},$$

is the so-called Nevanlinna kernel of  $f$ . The class of generalized Nevanlinna functions with zero negative squares,  $\mathfrak{N}_0$ , coincides with the well-known class of (ordinary) Nevanlinna functions  $\mathfrak{N}$ . A fascinating property of generalized Nevanlinna functions is that they are characterized as being the product of a symmetric rational function and an ordinary Nevanlinna function.

**Theorem 1.1.** *A complex function  $f$  is a generalized Nevanlinna function of index  $\kappa$  if and only if there exists an ordinary Nevanlinna function  $f_0$  and a rational function  $r$  of degree  $\kappa$  such that  $f = r f_0 r^\#$ . Here  $r^\#(z) = \overline{r(\overline{z})}$ .*

*Moreover, if  $r f_0 r^\# = s g_0 s^\#$ , where  $f_0, g_0 \in \mathfrak{N}$  are not identically equal to zero, and  $r$  and  $s$  are rational functions, then there exists  $c \in \mathbb{C} \setminus \{0\}$  such that  $r = cs$  and  $g_0 = |c|^2 f_0$ .*

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Theorem 1.1 straight-forwardly follows from a factorization result for the class of so-called pseudo-Carathéodory established by P. Delsarte, Y. Genin and Y. Kamp in 1986, see [2]. As that paper had not obtained widespread attention, Theorem 1.1 was rediscovered some 12 years later by H. Langer. Not much later two papers, [4] and [6], appeared containing new proofs for it. Although these new proofs differ from the one in [2], all three proofs are function-theoretical. In particular, the two new proofs are a more or less direct consequence of the characterization of the index of a generalized Nevanlinna function in terms of (the multiplicities of) its so-called generalized poles of nonpositive type (GPNTs) contained in [16, Theorem 3.5], see also [17]. Here the GPNTs are points of exceptional growth of the generalized Nevanlinna function. However, this characterization of the index was established in [16] by operator-theoretical arguments. More specifically, by making use of an invariant subspace property of contractions in Pontryagin spaces.

In this paper Theorem 1.1 is derived directly from the following invariant subspace property without the use of the concept of GPNTs.

**Corollary 1.2.** *Let  $A$  be a selfadjoint relation with non-empty resolvent set in a Pontryagin space  $\{\Pi, [\cdot, \cdot]\}$  with nonzero negative index. Then there exists a  $\lambda \in \mathbb{C} \cup \{\infty\}$  for which  $\ker(A - \lambda)$  contains a non-trivial nonpositive vector.*

More precisely, combining Corollary 1.2 with the essentially algebraic result in Theorem 4.1 below provides a proof for the factorization in Theorem 1.1. Conversely, we also establish that Corollary 1.2 can be proven by combining Theorem 1.1 with Theorem 4.1. Thereby the main contribution of this article is established: The invariant subspace property in Corollary 1.2 and the factorization property in Theorem 1.1 are equivalent. This intimate relationship is expressed by the new realization result Theorem 4.1 which contains an explicit construction for a (selfadjoint operator) realization for  $rfr^\#$  given a (selfadjoint operator) realization for  $f$ , where  $f \in \mathfrak{R}_\kappa$  and  $r$  is a rational function of degree one.

Note that the following stronger invariant subspace property, which was first proven by L.S. Pontryagin for the case of operators (single-valued relations), see [19], can straight-forwardly be deduced from Corollary 1.2, see Section 6 below.

**Theorem 1.3.** *Let  $A$  be a selfadjoint relation with non-empty resolvent set in a Pontryagin space  $\{\Pi, [\cdot, \cdot]\}$  with negative index  $\kappa \in \mathbb{N}_0$ . Then there exist a  $\kappa$ -dimensional  $A$ -invariant nonpositive subspace  $\mathfrak{J}$  such that  $\operatorname{Im} \sigma(A \upharpoonright_{\mathfrak{J}}) \geq 0$ .*

When the proof of the invariant subspace property by means of the factorization property, presented in Section 6 below, is combined with the function-theoretic proof for the factorization property contained in [20] or in [2], one immediately obtains a new, essentially function-theoretic, proof for the invariant subspace property of selfadjoint relations in Pontryagin spaces.

Finally the organization of this paper is outlined. Section 2 and 3 contain short introductions to relations in Pontryagin spaces and (minimal) realizations for generalized Nevanlinna functions, respectively. Theorem 4.1 is proven in Section 4. That result is combined in Section 5 with Corollary 1.2 and basic results from Section 3 to obtain a simple proof for Theorem 1.1; that proof is inspired by the proof for a factorization of operator-valued generalized Nevanlinna functions contained in [18].

In the sixth and final section Theorem 1.1 is combined with Theorem 4.1 to provide a proof for Corollary 1.2, and it is shown how Theorem 1.3 can be straight-forwardly obtained from Corollary 1.2 and a basic result contained in Section 3.

## 2. RELATIONS IN PONTRYAGIN SPACES

A linear space  $\Pi$  together with a sesqui-linear form  $[\cdot, \cdot]$  defined on it, is a *Pontryagin space* if there exists an orthogonal decomposition  $\Pi^+ + \Pi^-$  of  $\Pi$  such that  $\{\Pi^+, [\cdot, \cdot]\}$  and  $\{\Pi^-, -[\cdot, \cdot]\}$  are Hilbert spaces, at least one of which is finite-dimensional. Here two subspaces  $\mathfrak{M}$  and  $\mathfrak{N}$  of a Pontryagin space  $\{\Pi, [\cdot, \cdot]\}$  are *orthogonal* if  $[f, g] = 0$  for all  $f \in \mathfrak{M}$  and  $g \in \mathfrak{N}$ . For our purposes it suffices to consider only Pontryagin spaces for which  $\Pi^-$  is finite-dimensional; its dimension (which is independent of the orthogonal decomposition  $\Pi^+ + \Pi^-$ ) is *the negative index* of  $\Pi$ . Recall that the dimension of every negative, nonpositive and neutral subspace of  $\{\Pi, [\cdot, \cdot]\}$  is less than or equal to this negative index.

A mapping  $H$  from the Pontryagin space  $\{\Pi_1, [\cdot, \cdot]_1\}$  to the Pontryagin space  $\{\Pi_2, [\cdot, \cdot]_2\}$  is a *(linear) relation* (or a *(linear) multi-valued operator*) if  $H$  is defined on a (linear) subspace (called  $\text{dom } H$ ) of  $\Pi_1$ , maps each element  $x \in \text{dom } H$  to a subset  $Hx := H(x)$  of  $\Pi_2$  and is linear:

$$H(x + cy) = \{x' + cy' \in \Pi_2 : x' \in Hx, y' \in Hy\}, \quad x, y \in \Pi_1, \quad c \in \mathbb{C}.$$

In particular, (linear) relations from  $\Pi_1$  to  $\Pi_2$  can, and will, be identified with subspaces of  $\Pi_1 \times \Pi_2$  via their graphs. A relation  $H$  is an ordinary (single-valued) operator if the subspace  $\text{mul } H := \{f \in \Pi_2 : \{0, f\} \in H\}$ , *the multi-valued part* of  $H$ , is trivial. For any relation  $H$ , its adjoint, denoted as  $H^{[*]}$ , is defined as

$$H^{[*]} := \{\{f, f'\} \in \Pi_2 \times \Pi_1 : [f, g']_2 = [f', g]_1 \quad \forall \{g, g'\} \in H\}.$$

In particular, if  $H$  is a densely defined operator, then  $H^{[*]}$  is the operator such that

$$[f, Hg]_2 = [H^{[*]}f, g]_1, \quad \forall f \in \text{dom } H^{[*]}, \forall g \in \text{dom } H.$$

For any relation  $H$  in  $\{\Pi, [\cdot, \cdot]\}$ , i.e., a relation from  $\{\Pi, [\cdot, \cdot]\}$  to  $\{\Pi, [\cdot, \cdot]\}$ , its *resolvent* at  $z \in \mathbb{C}$ , defined as

$$(H - z)^{-1} = \{\{f' - zf, f\} \in \Pi \times \Pi : \{f, f'\} \in H\},$$

is a well-defined relation. Relations  $S$  and  $A$  in  $\{\Pi, [\cdot, \cdot]\}$  are called *symmetric* and *selfadjoint* if  $S \subseteq S^{[*]}$  and  $A = A^{[*]}$ , respectively. For a selfadjoint relation  $A$  the *resolvent set*  $\rho(A)$  is defined as usual:

$$\rho(A) := \{z \in \mathbb{C} : \text{dom } (A - z)^{-1} = \Pi\};$$

Thus defined  $\rho(A)$  is open and  $\sigma(A) := \mathbb{C} \setminus \rho(A)$ , the *spectrum* of  $A$ , is a closed subset of  $\mathbb{C}$ , see [8, Proposition 2.2]. The *point spectrum* of  $A$ ,  $\sigma_p(A)$ , is defined as

$$(2.1) \quad \sigma_p(A) := \{\alpha \in \mathbb{C} \cup \{\infty\} : \exists x \in \Pi \setminus \{0\} \text{ s.t. } \{x, \alpha x\} \in A\};$$

here the above should be interpreted to mean that  $\infty \in \sigma_p(A)$  if  $\text{mul}(A) \neq \{0\}$ . For a selfadjoint relation  $A$  the spectrum is symmetric with respect to the real line:

$$(2.2) \quad \sigma(A) = \overline{\sigma(A)}, \quad \sigma_p(A) = \overline{\sigma_p(A)}.$$

Moreover,  $\rho(A)$  contains all of  $\mathbb{C} \setminus \mathbb{R}$  except finitely many points if  $\rho(A) \neq \emptyset$ , see [8]. If  $A$  is a selfadjoint relation in  $\{\Pi, [\cdot, \cdot]\}$  with  $\rho(A) \neq \emptyset$ , then a subspace  $\mathfrak{L} \subseteq \Pi$

is called *A-invariant* if  $(A - z)^{-1}\mathfrak{L} \subseteq \mathfrak{L}$  for all  $z \in \rho(A)$ .

Finally, a (single-valued) operator  $U$  from (a Pontryagin space)  $\{\Pi_1, [\cdot, \cdot]_1\}$  to (a Pontryagin space)  $\{\Pi_2, [\cdot, \cdot]_2\}$  is a *standard unitary operator* if  $\text{dom } U = \Pi_1$ ,  $\text{ran } U = \Pi_2$  and

$$[f, g]_1 = [Uf, Ug]_2, \quad \forall f, g \in \text{dom } U.$$

### 3. MINIMAL REALIZATIONS FOR GENERALIZED NEVANLINNA FUNCTIONS

If  $A$  is a selfadjoint relation with non-empty resolvent set in a Pontryagin space  $\{\Pi, [\cdot, \cdot]\}$ , then  $f(z)$  defined for  $z \in \rho(A)$  and arbitrary, but fixed,  $z_0 \in \rho(A)$  by

$$(3.1) \quad f(z) := c + iz_0[\omega, \omega] + (z - \bar{z}_0) \left[ (I + (z - z_0)(A - z)^{-1}) \omega, \omega \right],$$

where  $c \in \mathbb{R}$  and  $\omega \in \Pi$ , is a generalized Nevanlinna function whose index is at most the negative index of  $\{\Pi, [\cdot, \cdot]\}$ , see [14, 15, 11]. The converse also holds: if  $f \in \mathfrak{N}_\kappa$ , then there exists a selfadjoint relation  $A$  with non-empty resolvent set in a Pontryagin space  $\{\Pi, [\cdot, \cdot]\}$ , whose negative index is (at least)  $\kappa$ , such that (3.1) holds for some  $\omega \in \Pi$  and  $c \in \mathbb{R}$ , see [14]; cf. [7, Section 2].

If (3.1) holds, then the pair  $\{A, \omega\}$  realizes  $f$ . In particular, in this terminology the *realizing space*  $\{\Pi, [\cdot, \cdot]\}$  and the selection of the arbitrarily fixed *realizing point*  $z_0 \in \rho(A)$  are suppressed. A pair  $\{A, \omega\}$  is said to realize  $f$  *minimally* if

$$(3.2) \quad \Pi = \text{c.l.s.} \{ (I + (z - z_0)(A - z)^{-1}) \omega : z \in \rho(A) \}.$$

If a realization  $\{A, \omega\}$  for  $f$  is minimal, then  $\rho(A)$  coincides with the domain of holomorphy of  $f$ , see [7, Theorem 1.1]. Minimal realizations for generalized Nevanlinna functions are unique up to transformation by standard unitary operators.

**Proposition 3.1.** ([11, Theorem 3.2]) *Let  $A_i$  be a selfadjoint relation with  $\rho(A_i) \neq \emptyset$  in  $\{\Pi_i, [\cdot, \cdot]_i\}$  and let  $\omega_i \in \Pi_i$  be such that  $\{A_i, \omega_i\}$  realizes  $f_i \in \mathfrak{N}_\kappa$  minimally at the fixed point  $z_0$ , for  $i = 1, 2$ . Then  $f_1 = f_2 + c$  for some  $c \in \mathbb{R}$  if and only if there exists a standard unitary operator  $U$  from  $\{\Pi_1, [\cdot, \cdot]_1\}$  to  $\{\Pi_2, [\cdot, \cdot]_2\}$  such that  $A_2 = UA_1U^{-1}$  and  $\omega_2 = U\omega_1$ .*

The existence of a pair  $\{A, \omega\}$  minimally realizing an arbitrary  $f \in \mathfrak{N}_\kappa$  has been established in [14]; cf. [7, Section 2]. Since those minimal realizations are in a Pontryagin space with negative index  $\kappa$ , Proposition 3.1 yields that all minimal realizations for  $f \in \mathfrak{N}_\kappa$  are in Pontryagin spaces with negative index  $\kappa$ . Non-minimal realizations can be reduced to minimal ones, cf. e.g. [11, Section 2].

**Proposition 3.2.** *Let  $A$  be a selfadjoint relation with  $\rho(A) \neq \emptyset$  in the Pontryagin space  $\{\Pi, [\cdot, \cdot]\}$  and let  $\mathfrak{L}$  be a closed  $A$ -invariant subspace with isotropic part  $\mathfrak{L}_0: \mathfrak{L}_0 = \mathfrak{L} \cap \mathfrak{L}^{\perp}$ . Then the relation  $A_{\mathfrak{L}}$  defined via*

$$A_{\mathfrak{L}} := \{ \{f + [\mathfrak{L}_0], f' + [\mathfrak{L}_0]\} \in \mathfrak{L}/\mathfrak{L}_0 \times \mathfrak{L}/\mathfrak{L}_0 : \{f, f'\} \in A \},$$

*is a selfadjoint relation in the Pontryagin space  $\{\mathfrak{L}/\mathfrak{L}_0, [\cdot, \cdot]\}$ . If  $\omega \in \mathfrak{L}$ , then  $\{A, \omega\}$  and  $\{A_{\mathfrak{L}}, \omega + [\mathfrak{L}_0]\}$  realize the same generalized Nevanlinna function.*

Note that if  $\{\pi_+, \pi_-, \pi_0\}$  is the inertia index of  $\mathfrak{L}$ , see [1, Ch. 1: § 6], then the negative index of the Pontryagin space  $\{\mathfrak{L}/\mathfrak{L}_0, [\cdot, \cdot]\}$  is  $\pi_-$ , see [1, Ch. 1: 9.13].

*Proof.* The assumptions imply that the quotient space  $\{\mathfrak{L}/\mathfrak{L}_0, [\cdot, \cdot]\}$  is a Pontryagin space, see [1, Ch. 1: 9.13]. Since  $\mathfrak{L}_0$  is the isotropic part of  $\mathfrak{L}$  and  $A$  is a selfadjoint relation,  $A_{\mathfrak{L}}$  is a symmetric relation. To establish the selfadjointness of  $A_{\mathfrak{L}}$  it suffices by [8, Theorem 4.6] to show that

$$(3.3) \quad \rho(A) \subseteq \rho(A_{\mathfrak{L}}).$$

Let  $z \in \rho(A)$  be arbitrary, then  $\mathfrak{L} \subseteq \text{ran}(A - z)$ . Thus for every  $g \in \mathfrak{L}$  there exists  $\{f, f'\} \in A$ , such that  $g = f' - zf$ . Now the assumed  $A$ -invariance of  $\mathfrak{L}$  implies that  $f = (A - z)^{-1}g \in \mathfrak{L}$  and thus also  $f' \in \mathfrak{L}$ . Therefore,

$$\mathfrak{L} \subseteq \{f' - zf : \{f, f'\} \in A \cap (\mathfrak{L} \times \mathfrak{L})\}.$$

Consequently,  $\text{ran}(A_{\mathfrak{L}} - z) = \mathfrak{L}/\mathfrak{L}_0$  and this implies that  $z \in \rho(A_{\mathfrak{L}})$ . Since  $z \in \rho(A)$  was arbitrary, the above argument shows that (3.3) holds.

To prove the final assertion note first that  $\mathfrak{L}_0$  is  $A$ -invariant. This is a direct consequence of the fact that the assumed  $A$ -invariance of  $\mathfrak{L}$  together with selfadjointness of  $A$  implies that  $\mathfrak{L}^{[\perp]}$  is also  $A$ -invariant. Thus by definition of  $\mathfrak{L}$  and  $\mathfrak{L}_0$  we have that for every  $z \in \rho(A)$  and every  $h \in \mathfrak{L}_0$

$$[(I + (z - z_0)(A - z)^{-1})(\omega + h), (\omega + h)] = [(I + (z - z_0)(A - z)^{-1})\omega, \omega] = \frac{f(z) - f(\overline{z_0})}{z - \overline{z_0}}.$$

This shows that  $\{A, \omega\}$  and  $\{A_{\mathfrak{L}}, \omega\}$  realize the same generalized Nevanlinna function, see (3.1).  $\square$

**Corollary 3.3.** *Let  $\{A, \omega\}$  realize  $f \in \mathfrak{N}_{\kappa}$ , and define  $\mathfrak{L}$  and  $\mathfrak{L}_0$  to be*

$$\mathfrak{L} := \text{c.l.s.} \{(I + (z - z_0)(A - z)^{-1})\omega : z \in \rho(A)\} \quad \text{and} \quad \mathfrak{L}_0 := \mathfrak{L} \cap \mathfrak{L}^{[\perp]}.$$

*Then  $\mathfrak{L}$ ,  $\mathfrak{L}^{[\perp]}$  and  $\mathfrak{L}_0$  are  $A$ -invariant, and  $f$  is minimally realized by  $\{A_{\mathfrak{L}}, \omega + [\mathfrak{L}_0]\}$ , where  $A_{\mathfrak{L}}$  is as in Proposition 3.2.*

*Proof.* The subspace  $\mathfrak{L}$  contains  $\omega$  by definition and the resolvent identity shows that it is  $A$ -invariant. Since  $A$  is selfadjoint and  $\mathfrak{L}$  is  $A$ -invariant,  $\mathfrak{L}^{[\perp]}$  is also  $A$ -invariant and, hence, so is  $\mathfrak{L}_0$ . Therefore the statement follows from Proposition 3.2 and the definition of minimality, see (3.2).  $\square$

Let  $\mathfrak{L}$  be as in Corollary 3.3 for some realization  $\{A, \omega\}$ , then  $\omega \in \mathfrak{L} = (\mathfrak{L}^{[\perp]})^{[\perp]}$  and  $\mathfrak{L}^{[\perp]}$  is  $A$ -invariant. Conversely, if  $\mathfrak{M}$  is  $A$ -invariant and  $\omega \in \mathfrak{M}^{[\perp]}$ , then a direct calculation shows that  $\mathfrak{M} \subseteq \mathfrak{L}^{[\perp]}$ . Thus Corollary 3.3 has the following consequence.

**Corollary 3.4.** *Let  $\{A, \omega\}$  realize  $f \in \mathfrak{N}_{\kappa}$ . Then  $\{A, \omega\}$  realizes  $f$  minimally if and only if there exists no non-trivial  $A$ -invariant subspace  $\mathfrak{M}$  such that  $\omega \in \mathfrak{M}^{[\perp]}$ .*

If  $\rho(A) \neq \emptyset$ , then  $\rho(A)$  contains all of  $\mathbb{C} \setminus \mathbb{R}$  except at most finitely many points, see e.g. [8, Proposition 4.4]. Thus if  $f$  is not identically equal to zero, then the realizing point  $z_0$  in (3.1) can be taken from  $\{z \in \rho(A) : f(z) \neq 0\}$ ; such realizations are *regular*. If  $\{A, \omega\}$  is a regular realization for  $f \in \mathfrak{N}_{\kappa}$ , where  $f$  is not identically equal to zero, then  $\{A, \omega\}$  is a (minimal) realization for  $f$  if and only if  $\{\widehat{A}, \widehat{\omega}\}$  is a (minimal) realization for  $-f^{-1}$  ( $\in \mathfrak{N}_{\kappa}$ , cf. (1.1)), where  $\widehat{A}$  and  $\widehat{\omega}$  are defined via

$$(3.4) \quad \widehat{\omega} := -(f(z_0))^{-1}\omega \quad \text{and} \quad (\widehat{A} - z)^{-1} := (A - z)^{-1} - \gamma_z(f(z))^{-1}[\cdot, \gamma_{\overline{z}}],$$

see e.g. [18]. Here  $\gamma_z = (I + (z - z_0)(A - z)^{-1})\omega$ .

This section is concluded by presenting necessary criteria for a realization to be minimal. Proposition 3.5 below can be proven by making use of the characterization of the point spectrum of minimal realizations for a generalized Nevanlinna function in terms of its non-tangential growth; see e.g. [5, Lemmas 2.2 & 2.3]. Here an elementary proof is presented.

**Proposition 3.5.** *Let  $f \in \mathfrak{N}_\kappa$ , where  $f$  is not identically equal to zero, have the regular and minimal realization  $\{A, \omega\}$ . Then  $\sigma_p(A) \cap \sigma_p(\widehat{A}) = \emptyset$ .*

*Proof.* Assume that  $\beta \in \sigma_p(A) \cap \sigma_p(\widehat{A}) \cap \mathbb{C}$ ; the case  $\beta = \infty$  can be treated by similar arguments. Then there exist  $x_\beta, \widehat{x}_\beta \in \Pi \setminus \{0\}$  such that

$$(A - z_0)^{-1}x_\beta = (\beta - z_0)^{-1}x_\beta \quad \text{and} \quad (\widehat{A} - \overline{z_0})^{-1}\widehat{x}_\beta = (\overline{\beta} - \overline{z_0})^{-1}\widehat{x}_\beta;$$

see (2.1) and (2.2). Thus by (3.4) (with  $z = z_0$ )

$$-[x_\beta, \gamma_{\overline{z_0}}] \cdot [-(f(z_0))^{-1}\omega, \widehat{x}_\beta] = [(A - z_0)^{-1}x_\beta, \widehat{x}_\beta] - [x_\beta, (\widehat{A} - \overline{z_0})^{-1}\widehat{x}_\beta] = 0;$$

here  $\gamma_z = (I + (z - z_0)(A - z)^{-1})\omega$ . The above equation implies that either  $[x_\beta, \gamma_{\overline{z_0}}] = 0$  or that  $[\widehat{\omega}, \widehat{x}_\beta] = 0$ , see (3.4). In the former case the  $A$ -invariance of  $x_\beta$  implies that  $[x_\beta, \omega] = 0$ . Hence, the realization  $\{A, \omega\}$  is non-minimal by Corollary 3.4. In the latter case the realization  $\{\widehat{A}, \widehat{\omega}\}$  (for  $-f^{-1}$ ) is non-minimal by Corollary 3.4. From this the non-minimality of the realization  $\{A, \omega\}$  follows, see the remarks preceding (3.4).  $\square$

#### 4. CONNECTION BETWEEN THE FACTORIZATION AND INVARIANT SUBSPACE RESULTS

The key result used to prove the asserted connection between Theorem 1.1 and Corollary 1.2 is Theorem 4.1 below. Given a rational function  $r$  of degree one and a realization for  $f \in \mathfrak{N}_\kappa$ , Theorem 4.1 contains an explicit realization for  $rf r^\#$ . To increase the readability only the case that  $r$  does not have a pole or zero at infinity is presented; in the other cases a similar result holds, see Remark 4.2 below. For Theorem 4.1 (i) see also [3, Theorem 4.1]. Note that [12], based on an earlier version of this paper, contains a result similar to Theorem 4.1.

**Theorem 4.1.** *Let  $f \in \mathfrak{N}_\kappa$  not be identically equal to zero and let  $\alpha, \beta \in \mathbb{C}$  be such that  $\alpha \neq \beta$  and  $\alpha \neq \overline{\beta}$ . Moreover, let  $\{A, \omega\}$  realize  $f$ , where the realizing space  $\{\Pi, [\cdot, \cdot]\}$  has negative index  $\tilde{\kappa}$  ( $\tilde{\kappa} \geq \kappa$ ) and the realizing point  $z_0$  is contained in  $\rho(A) \setminus \{\beta, \overline{\beta}\}$ , and let  $r$  be defined as*

$$r(z) := \frac{z - \alpha}{z - \beta}.$$

*Then  $f_r := r f r^\#$  is realized by  $\{A_r, \omega_r\}$  where  $\omega_r := (d \quad \omega \quad \overline{\beta} - \alpha)^T$  and*

$$A_r := \{ \{ \{x_l, x_c, x_r\}, \{\beta x_l + [x_c, \omega] - e x_r, x'_c + \omega x_r, \overline{\beta} x_r\} \} \in \Pi_r \times \Pi_r : \\ x_l, x_r \in \mathbb{C}, \{x_c, x'_c\} \in A \};$$

*here  $d = -\frac{r(z_0)f(z_0)}{(z_0 - \beta)(\overline{z_0} - \beta)} + \frac{(2z_0 - \overline{\beta} - \alpha)[\omega, \omega]}{2(z_0 - \beta)(\overline{z_0} - \beta)}$  and  $e = \frac{f(z_0) + \overline{f(z_0)} + (\beta + \overline{\beta} - z_0 - \overline{z_0})[\omega, \omega]}{2(z_0 - \beta)(\overline{z_0} - \beta)}$ . Moreover, the realizing space  $\{\Pi_r, [\cdot, \cdot]_r\}$  corresponding to  $\{A_r, \omega_r\}$  is the Pontryagin space  $\{\Pi_r, [\cdot, \cdot]_r\}$  with negative index  $\tilde{\kappa} + 1$  defined via*

$$[g, h]_r := [g_c, h_c] + g_r \overline{h_l} + g_l \overline{h_r}, \quad g = \{g_l, g_c, g_r\}, h = \{h_l, h_c, h_r\} \in \Pi_r := \mathbb{C} \times \Pi \times \mathbb{C}.$$

Additionally, if  $\{A, \omega\}$  is a regular and minimal realization for  $f$ , then  $\tilde{\kappa} = \kappa$  and

- (i)  $\{A_r, \omega_r\}$  realizes  $f_r$  minimally if and only if  $\alpha \notin \sigma_p(A)$  and  $\beta \notin \sigma_p(\hat{A})$ ;
- (ii) if there exist  $x, \hat{x} \in \Pi \setminus \{0\}$  such that  $[x, x] \leq 0$ ,  $[\hat{x}, \hat{x}] \leq 0$ ,  $\{x, \bar{\alpha}x\} \in A$  and  $\{\hat{x}, \bar{\beta}\hat{x}\} \in \hat{A}$ , then  $f_r \in \mathfrak{N}_{\kappa-1}$ .

If the relation  $A$  in Theorem 4.1 is an operator, then  $A_r$  has with respect to the decomposition  $\mathbb{C} \times \Pi \times \mathbb{C}$  of  $\Pi_r$  the following block representation:

$$A_r = \begin{pmatrix} \beta & [\cdot, \omega] & -e \\ 0 & A & \omega \\ 0 & 0 & \bar{\beta} \end{pmatrix}.$$

*Proof.* The proof consists of five steps. In the first step all assertions except the last two are proven. In the second and third step some calculations are made to characterize the non-minimal part of the realization  $\{A_r, \omega_r\}$ . Those results are in the fourth and fifth step used to prove the assertions (i) and (ii), respectively.

**Step 1:** By construction,  $\{\Pi_r, [\cdot, \cdot]_r\}$  is a Pontryagin space with negative index  $\tilde{\kappa} + 1$ . The selfadjointness of  $A_r$  is straight-forwardly established after noting that  $e \in \mathbb{R}$ . Clearly, for  $z \in \rho(A) \setminus \{\beta, \bar{\beta}\}$

$$(A_r - z)^{-1} = \begin{pmatrix} -\frac{1}{z-\beta} & \frac{[(A-z)^{-1}, \omega]}{z-\beta} & \frac{e + [(A-z)^{-1}, \omega]}{(z-\beta)(z-\bar{\beta})} \\ 0 & (A-z)^{-1} & \frac{(A-z)^{-1}\omega}{z-\bar{\beta}} \\ 0 & 0 & -\frac{1}{z-\bar{\beta}} \end{pmatrix}.$$

The proof of the first part of this statement is completed by showing that  $\{A_r, \omega_r\}$  realizes  $f_r = rfr^\#$ . Instead of proving this directly, an easier proof is obtained by a unitary transformation of the realization  $\{A_r, \omega_r\}$ . More specifically, define  $U := U_1 U_2$  in  $\{\Pi_r, [\cdot, \cdot]_r\}$  via

$$U_1 := \begin{pmatrix} 1 & -[\cdot, \omega] & -[\omega, \omega]/2 \\ 0 & I_\Pi & \omega \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad U_2 := \begin{pmatrix} \bar{z}_0 - \beta & 0 & 0 \\ 0 & I_\Pi & 0 \\ 0 & 0 & (z_0 - \bar{\beta})^{-1} \end{pmatrix}.$$

Then direct calculations show that  $U_1$  and  $U_2$  are standard unitary operators in  $\{\Pi_r, [\cdot, \cdot]_r\}$  and, hence, that  $U$  is a standard unitary operator in  $\{\Pi_r, [\cdot, \cdot]_r\}$ . Let

$$A_u := U A_r U^{-1} \quad \text{and} \quad \omega_u := U \omega_r.$$

Then  $\{A_u, \omega_u\}$  realizes the same function as  $\{A_r, \omega_r\}$ , see Proposition 3.1. Thus this part of the proof is completed by showing that  $\{A_u, \omega_u\}$  realizes  $f_r$ .

Let  $\gamma_z := (I + (z - z_0)(A - z)^{-1})\omega$ , then

$$(A_u - z)^{-1} = \begin{pmatrix} -\frac{1}{z-\beta} & -\frac{[\cdot, \gamma_z]}{z-\beta} & \frac{f(z)}{(z-\beta)(z-\bar{\beta})} \\ 0 & (A-z)^{-1} & \frac{-\gamma_z}{z-\bar{\beta}} \\ 0 & 0 & -\frac{1}{z-\bar{\beta}} \end{pmatrix}, \quad \omega_u = \begin{pmatrix} -\frac{r(z_0)f(z_0)}{z_0-\beta} \\ r(z_0)\omega \\ -\frac{\alpha-\bar{\beta}}{z_0-\bar{\beta}} \end{pmatrix}.$$

In the calculation to establish the above one has to make use of the identity  $f(z_0) - \overline{f(z_0)} = (z_0 - \bar{z}_0)[\omega, \omega]$ , see (3.1). A further straight-forward calculation shows that

$$\gamma_z^u := (I + (z - z_0)(A_u - z)^{-1})\omega_u = \begin{pmatrix} -\frac{r(z)f(z)}{z-\beta} & r(z)\gamma_z & -\frac{\alpha-\bar{\beta}}{z-\bar{\beta}} \end{pmatrix}^T.$$



Recall that the kernel  $N_g$  for  $g \in \mathfrak{N}_\kappa$  is defined in (1.1). In view of the identity

$$N_{f_r}(z, w) = r(z)N_f(z, w)\overline{r(w)} + f(z)r(z)\frac{\overline{r(\bar{z})} - \overline{r(w)}}{z - \bar{w}} + \overline{f(w)r(w)}\frac{r(z) - r(\bar{w})}{z - \bar{w}},$$

see e.g. [4, (3.14)], the identity  $N_{f_r}(z, z_0) = [\gamma_z^u, \omega_u]_r$  is now easily established. In other words,  $\{A_u, \omega_u\}$  realizes  $f_r$ .

**Step 2:** Henceforth the realization  $\{A, \omega\}$  of  $f$  is assumed to be minimal. In order to prove (i) and (ii), the orthogonal complement of

(4.1)

$$\mathfrak{M}_u := \text{span} \{(I + (z - z_0)(A_u - z)^{-1})\omega_u : z \in \rho(A_u)\} = \text{span} \{\gamma_z^u : z \in \rho(A_u)\}$$

should be determined; cf. Corollary 3.3. If  $\{0, x_c, 0\} \in \Pi_r$  is such that

$$0 = [\gamma_z^u, \{0, x_c, 0\}]_r = [r(z)\gamma_z, x_c] = r(z)[\gamma_z, x_c]$$

for all  $z \in \rho(A_u) = \rho(A) \setminus \{\beta, \bar{\beta}\}$ , then the assumed minimality of  $\{A, \omega\}$  yields  $x_c = 0$ . This shows that  $\mathfrak{M}_u^{[\perp]}$  is at most two-dimensional. Since  $\mathfrak{M}_u^{[\perp]}$  is  $A_u$ -invariant, see Corollary 3.3, it thus consists of (generalized) eigenvectors of  $A_u$ .

**Step 3:** Next the eigenvectors of  $A_u$  contained in  $\mathfrak{M}_u^{[\perp]}$  are determined; i.e., the eigenvectors  $x$  of  $A_u$  satisfying  $[x, \omega_u]_r = 0$ . A vector  $x = \{x_l, x_c, x_r\} \in \Pi_r \setminus \{0\}$  is an eigenvector of  $A_u$  satisfying  $[x, \omega_u]_r = 0$  if and only if there exists a  $\delta \in \mathbb{C} \cup \{\infty\}$  such that  $x$  satisfies for all  $z \in \rho(A_u) \setminus \{\delta\}$

$$(4.2) \quad \begin{aligned} \frac{x_l}{\beta - z} + \frac{[x_c, \gamma\bar{z}]}{\beta - z} + \frac{f(z)x_r}{(\beta - z)(\beta - z)} &= \frac{x_l}{\delta - z}, & (\bar{\beta} - z)^{-1}x_r &= \frac{x_r}{\delta - z}; \\ (A - z)^{-1}x_c + \frac{\gamma_z x_r}{\beta - z} &= \frac{x_c}{\delta - z}, & r(z_0) \left[ [x_c, \omega] + \frac{f(\bar{z}_0)x_r}{\beta - \bar{z}_0} \right] &= \frac{\beta - \bar{\alpha}}{\beta - \bar{z}_0}x_l. \end{aligned}$$

Here  $\frac{x_l}{\delta - z}$ ,  $\frac{x_r}{\delta - z}$  and  $\frac{x_c}{\delta - z}$  should be interpreted to be zero if  $\delta = \infty$ . The second equality in (4.2) shows that there are two cases to consider:  $\delta = \bar{\beta}$  or  $x_r = 0$ . In the latter case  $x_l \neq 0$  (see Step 2) and the remaining equalities in (4.2) reduce to

$$(4.3) \quad [x_c, \gamma\bar{z}] = \frac{\beta - \delta}{\delta - z}x_l, \quad (A - z)^{-1}x_c = \frac{x_c}{\delta - z}, \quad (\bar{\alpha} - \bar{z}_0)[x_c, \omega] = (\beta - \bar{\alpha})x_l.$$

Here the first and second equality should be interpreted to be  $[x_c, \gamma\bar{z}] = -x_l$  and  $(A - z)^{-1}x_c = 0$ , respectively, if  $\delta = \infty$ . The second equality in (4.3) shows that the first one is satisfied if and only if  $[x_c, \omega] = \frac{\beta - \delta}{\delta - \bar{z}_0}x_l$  ( $[x_c, \omega] = -x_l$  if  $\delta = \infty$ ). Thus the equalities in (4.3) are satisfied if and only if  $\delta = \bar{\alpha}$  and  $\{x_c, \bar{\alpha}x_c\} \in A$ ; i.e., if and only if  $\bar{\alpha} \in \sigma_p(A)$ .

On the other hand, if  $\delta = \bar{\beta}$ , then (4.2) reduces to

$$(4.4) \quad \begin{aligned} [x_c, \gamma\bar{z}] + \frac{f(z)x_r}{\beta - z} &= \frac{\beta - \bar{\beta}}{\beta - z}x_l, & (A - z)^{-1}x_c + \frac{\gamma_z x_r}{\beta - z} &= \frac{x_c}{\beta - z}; \\ [x_c, \omega] + \frac{f(\bar{z}_0)x_r}{\beta - \bar{z}_0} &= \frac{\beta - \bar{\alpha}}{\bar{\alpha} - \bar{z}_0}x_l. \end{aligned}$$

Taking  $z$  to be  $\bar{z}_0$  in the first equality and comparing with the third equation yields, in view of the assumption  $\alpha \neq \beta$ , that  $x_l = 0$ . In particular,  $x_r \neq 0$  (see Step 2)

and the third condition is satisfied when the first one is. Thus the equalities in (4.4) are equivalent to

$$(4.5) \quad -[x_c, \gamma_{\bar{z}}] = f(z) \frac{x_r}{\bar{\beta} - z}, \quad (A - z)^{-1} x_c + \frac{\gamma_z x_r}{\bar{\beta} - z} = \frac{x_c}{\bar{\beta} - z}.$$

Recall that  $f(z)$  is nonzero for  $z \in \rho(\widehat{A})$ , cf. (3.4). Combining the preceding two equalities yields the following constraint on  $x_c$ :

$$(\bar{\beta} - z)^{-1} x_c = (A - z)^{-1} x_c + \gamma_{z_0} (-f(z))^{-1} [x_c, \gamma_{\bar{z}}] = (\widehat{A} - z)^{-1} x_c$$

for all  $z \in (\rho(A) \cap \rho(\widehat{A})) \setminus \{\beta, \bar{\beta}\}$ . Since  $\rho(A)$  and  $\rho(\widehat{A})$  correspond to the domain of holomorphy of  $f$  and  $f^{-1}$ , respectively, see [7, Theorem 1.1],  $(\rho(A) \cap \rho(\widehat{A})) \setminus \{\beta, \bar{\beta}\}$  is dense in  $\mathbb{C} \setminus \mathbb{R}$ . Consequently, (4.5) has a solution if and only if  $\{x_c, \bar{\beta} x_c\} \in \widehat{A}$ ; i.e., if and only if  $\bar{\beta} \in \sigma_p(\widehat{A})$ .

**Step 4:** The argument in Step 2 showed that  $\mathfrak{M}_u^{[\perp]}$ , see (4.1), consists of (generalized) eigenvectors of  $A_u$ . Hence the calculations in Step 3 showed that  $\mathfrak{M}_u^{[\perp]} = \{0\}$  if and only if  $\alpha \notin \sigma_p(A)$  and  $\beta \notin \sigma_p(\widehat{A})$ ; cf. (2.2). This proves (i), because by Corollary 3.3 the realization  $\{A_u, \omega_u\}$  (and, hence, also the unitary equivalent realization  $\{A_r, \omega_r\}$ ) for  $f_r$  is minimal if and only if  $\mathfrak{M}_u^{[\perp]} = \{0\}$ .

**Step 5:** If the assumptions in (ii) hold, then the calculations in Step 3 show that

$$x_e = \{(\bar{\alpha} - \bar{z}_0)(\beta - \bar{\alpha})^{-1} [x, \omega], x, 0\} \quad \text{and} \quad \widehat{x}_e = \{0, \widehat{x}, (\bar{\beta} - \bar{z}_0)[\widehat{x}, \omega]\}$$

are nonpositive vectors contained in  $\mathfrak{M}_u^{[\perp]}$ , see (4.1). In fact, the calculations in Step 3 showed that  $\{x_e, \bar{\alpha} x_e\}, \{\widehat{x}_e, \bar{\beta} \widehat{x}_e\} \in A_u$ , i.e.,  $x_e$  and  $\widehat{x}_e$  are eigenvectors of  $A_u$  corresponding to the eigenvalues  $\bar{\alpha}$  and  $\bar{\beta}$ . Since  $\alpha \neq \beta$  and  $\alpha \neq \bar{\beta}$  by assumption, the eigenvectors  $x_e$  and  $\widehat{x}_e$  are orthogonal. Therefore  $\mathfrak{M}_u^{[\perp]}$ , being at most two-dimensional, see Step 2, is under the assumptions in (ii) equal to the two-dimensional nonpositive subspace  $\text{span}\{x_e, \widehat{x}_e\}$ . Thus the index of  $f_r$  is equal to  $\kappa + 1 - 2 = \kappa - 1$  by Corollary 3.3 (see also Proposition 3.2).  $\square$

**Remark 4.2.** Theorem 4.1 also holds if  $\alpha = \infty$  or  $\beta = \infty$ . If  $\alpha = \infty$ , i.e. if  $r(z) = (z - \bar{\beta})^{-1}$ , then  $f_r = r f r^\#$  is realized by  $\{A_r, \omega_r\}$ , where  $A_r$  is as in Theorem 4.1 and

$$\omega_r = \begin{pmatrix} -\frac{r(z_0)f(z_0)}{(z_0 - \bar{\beta})(\bar{z}_0 - \bar{\beta})} + \frac{[\omega, \omega]}{2(\bar{z}_0 - \bar{\beta})(z_0 - \bar{\beta})} \\ 0 \\ 1 \end{pmatrix}.$$

The condition  $\{x, \bar{\alpha} x\} \in A$  that occurs in (ii) should in this case be interpreted to be  $\{0, x\} \in A$ .

The case  $\beta = \infty$ , i.e.  $r(z) = z - \alpha$ , can be obtained from the case  $\alpha = \infty$  by means of (3.4). In this case the condition  $\{\widehat{x}, \bar{\beta} \widehat{x}\} \in \widehat{A}$  that occurs in (ii) should be interpreted to be  $\{0, \widehat{x}\} \in \widehat{A}$  and  $f_r = r f r^\#$  is realized by  $\{A_r, \omega_r\}$  where

$$(A_r - z)^{-1} = \begin{pmatrix} 0 & [\cdot, \gamma_{\bar{z}}] & f(z) \\ 0 & (A - z)^{-1} & \gamma_z \\ 0 & 0 & 0 \end{pmatrix}, \quad \omega_r = \begin{pmatrix} -(z_0 - \bar{\beta})f(z_0) \\ (z_0 - \bar{\beta})\omega \\ 1 \end{pmatrix}.$$

## 5. PROOF OF THE PRODUCT REPRESENTATION

*Proof of Theorem 1.1.* The proof consists of three steps: first the existence of the factorization is established, thereafter its uniqueness and, finally, its sufficiency.

**Existence of the factorization:** Let  $\{A, \omega\}$  be a minimal and regular realization for  $f \in \mathfrak{N}_\kappa$  with  $\kappa \neq 0$  (for  $\kappa = 0$  Theorem 1.1 trivially holds), see [14]; cf. [7, Section 2]. Moreover, let  $\hat{A}$  and  $\hat{\omega}$  be as in (3.4). Then by Corollary 1.2 there exist non-trivial nonpositive eigenvectors  $y_\alpha$  of  $A$  and  $\hat{y}_\beta$  of  $\hat{A}$  corresponding to some eigenvalues  $\alpha \in \mathbb{C} \cup \{\infty\}$  and  $\beta \in \mathbb{C} \cup \{\infty\}$  of  $A$  and  $\hat{A}$ , respectively. Since  $\{A, \omega\}$  is a minimal and regular realization for  $f$ , Proposition 3.5 together with (2.2) implies that  $\alpha \neq \beta$  and  $\beta \neq \bar{\alpha}$ . Let

$$r(z) := \frac{z - \bar{\alpha}}{z - \beta}, \quad \text{if } \alpha, \beta \in \mathbb{C}, \quad r(z) := \frac{1}{z - \beta}, \quad \text{if } \alpha = \infty, \quad r(z) = z - \bar{\alpha}, \quad \text{if } \beta = \infty.$$

Then  $f_r = r f r^\# \in \mathfrak{N}_{\kappa-1}$  by Theorem 4.1; see also Remark 4.2. Repeating the above procedure shows that  $f \in \mathfrak{N}_\kappa$  has the factorization in Theorem 1.1.

**Uniqueness of the factorization:** Recall that if  $h_0 \in \mathfrak{N}$ , then

$$(5.1) \quad \lim_{z \xrightarrow{\beta} \beta \in \mathbb{R}} (\beta - z) h_0(z) \in [0, \infty), \quad \lim_{z \xrightarrow{\infty}} \frac{h_0(z)}{z} \in [0, \infty),$$

see e.g. [4, (3.3) and (3.5)]. Here the notation  $\lim_{z \xrightarrow{x} x \in \mathbb{R} \cup \{\infty\}}$  is used to denote the *non-tangential limit* to  $x \in \mathbb{R} \cup \{\infty\}$  from the upper half-plane. If  $h_0 \in \mathfrak{N}$  is not identically equal to zero, then also  $-h_0^{-1} \in \mathfrak{N}$ . Hence for such functions (5.1) yields

$$(5.2) \quad \lim_{z \xrightarrow{\beta}} (\beta - z)^{-1} h_0(z) < 0 \quad \text{or} \quad \lim_{z \xrightarrow{\beta}} |(\beta - z)^{-1} h_0(z)| = \infty;$$

here  $(\beta - z)^{-1} h_0(z)$  should be interpreted to be  $z h_0(z)$  if  $\beta = \infty$ .

Now let  $r f_0 r^\# = s g_0 s^\#$ , where  $f_0, g_0 \in \mathfrak{N}$  and  $r$  and  $s$  are rational functions. Then  $f_0 = (s/r) g_0 (s/r)^\#$ . Clearly, the rational function  $s/r$  has no poles (or zeros) in  $\mathbb{C} \setminus \mathbb{R}$ , because  $f_0$  and  $g_0$  are holomorphic on  $\mathbb{C} \setminus \mathbb{R}$  being ordinary Nevanlinna functions. Assume that  $\beta \in \mathbb{R}$  is a pole of  $s/r$  ( $\beta = \infty$  can be treated similarly) of multiplicity  $n \in \mathbb{N}$ ; i.e., that there exists a rational function  $t$  not having  $\beta$  as a zero or pole such that  $s(z)/r(z) = t(z)/(z - \beta)^n$ . In that case

$$\lim_{z \xrightarrow{\beta}} (\beta - z) f_0(z) = t(\beta) t^\#(\beta) \lim_{z \xrightarrow{\beta}} (\beta - z)^{1-2n} g_0(z).$$

By (5.1) the left-hand side is nonnegative, while by (5.2), taking into account that  $t(\beta) t^\#(\beta) > 0$ , the right-hand side can not be nonnegative. This contradiction shows that the rational function  $s/r$  has no poles in  $\mathbb{C} \cup \{\infty\}$  and, hence, is constant.

**Sufficiency of the factorization:** Let  $f = r f_0 r^\#$ , where  $f_0 \in \mathfrak{N}$  is not identically equal to zero and  $r$  is a rational function of degree  $\kappa$ . Since  $f_0 \in \mathfrak{N}$ , it can be realized by  $\{A_0, \omega_0\}$  where the realizing space is a Hilbert space, see e.g. [10]. Thus Theorem 4.1 (applied  $\kappa$  times) yields a realization  $\{A_f, \omega_f\}$  of  $f$  where the negative index of the realizing space is  $\kappa$ . Thus  $f$  is a generalized Nevanlinna function whose index  $\kappa_f$  is smaller than or equal to  $\kappa$ , cf. Corollary 3.3. Hence, by the proven part of Theorem 1.1,  $f = s g_0 s^\#$ , where  $g_0 \in \mathfrak{N}$  and  $s$  is a rational function of degree

$\kappa_f \leq \kappa$ . Finally, the proven uniqueness of the factorization yields that  $\kappa_f$  (the degree of  $s$ ) is equal to  $\kappa$  (the degree of  $r$ ).  $\square$

## 6. PROOF OF THE INVARIANT SUBSPACE THEOREM

Here the invariant subspace result Corollary 1.2 is proven by means of Theorems 1.1 and 4.1. We start, however, by showing that the general invariant subspace result Theorem 1.3 can be obtained from Corollary 1.2 by means of Proposition 3.2.

*Proof of Theorem 1.3.* The proof proceeds by induction on the negative index  $\kappa$  of the Pontryagin space starting with the case  $\kappa = 1$ . Then there exists by Corollary 1.2 a nonpositive eigenvector  $x_\lambda \neq 0$  of  $A$  for an eigenvalue  $\lambda$ . If  $\text{Im } \lambda < 0$ , then by (2.2) there exists an eigenvector  $x_{\bar{\lambda}} \neq 0$  for the eigenvalue  $\bar{\lambda}$ . Since eigenvectors for non-real eigenvalues are easily seen to be neutral, Theorem 1.3 holds in this case.

Suppose that Theorem 1.3 holds for  $\kappa = 1, \dots, k-1$ ,  $k \in \mathbb{N}$ , and let  $A$  be a selfadjoint relation in a Pontryagin space with negative index  $k$ . Then the argument from the case  $\kappa = 1$  shows that there exists a nonpositive eigenvector  $x_\lambda \neq 0$  of  $A$  for an eigenvalue  $\lambda$  with  $\text{Im } \lambda \geq 0$ . Now apply Proposition 3.2 with  $\mathfrak{L} := (\text{span } \{x_\lambda\})^{\perp\perp}$ , then  $A_{\mathfrak{L}}$  as in Proposition 3.2 is a selfadjoint relation in a Pontryagin space with negative index  $k-1$ . Hence, by the induction hypothesis there exists a  $(k-1)$ -dimensional nonpositive subspace  $\mathfrak{T}_{\mathfrak{L}}$  which is  $A_{\mathfrak{L}}$ -invariant and satisfies  $\text{Im } \sigma(A_{\mathfrak{L}} \upharpoonright_{\mathfrak{T}_{\mathfrak{L}}}) \geq 0$ . Hence  $\mathfrak{T} := \mathfrak{T}_{\mathfrak{L}} + \text{span } \{x_\lambda\}$  is a  $k$ -dimensional nonpositive  $A$ -invariant subspace such that  $\text{Im } \sigma(A \upharpoonright_{\mathfrak{T}}) \geq 0$ .  $\square$

*Proof of Corollary 1.2.* Let  $A$  be a selfadjoint relation with  $\rho(A) \neq \emptyset$  in a Pontryagin space  $\{\Pi, [\cdot, \cdot]\}$  with nonzero negative index  $\kappa$ . For arbitrary  $\omega \in \Pi$ , let  $f$  be the generalized Nevanlinna function realized by  $\{A, \omega\}$  at an arbitrary point  $z_0 \in \rho(A)$  via (3.1). By means of  $\omega$  and  $A$  define  $\mathfrak{L}$  and  $\mathfrak{L}_0$  to be

$$\mathfrak{L} := \text{c.l.s. } \{(I + (z - z_0)(A - z)^{-1})\omega : z \in \rho(A)\} \quad \text{and} \quad \mathfrak{L}_0 := \mathfrak{L} \cap \mathfrak{L}^{\perp\perp}.$$

Since the subspace  $\mathfrak{L}_0$  is by definition neutral,  $A$ -invariant (see Corollary 3.3) and finite-dimensional (being neutral in a Pontryagin space), it consists of (generalized) eigenvectors of  $A$ . Thus the statement holds if  $\mathfrak{L}_0 \neq \{0\}$ .

Next consider the case that  $\mathfrak{L}_0 = \{0\}$ . I.e., the case that the realization  $\{A, \omega\}$  for  $f$  is minimal, cf. Corollary 3.3. Then  $f \in \mathfrak{N}_\kappa$ . Since  $\kappa > 0$  by assumption, there exists by Theorem 1.1 a rational function  $r$  of degree one and  $g \in \mathfrak{N}_{\kappa-1}$  such that  $f = rgr^\#$ . The zero and pole of  $r$  will be denoted by  $\alpha$  and  $\bar{\beta}$ , respectively; note that  $\alpha \neq \beta$  and  $\alpha \neq \bar{\beta}$ . Let  $\{A_g, \omega_g\}$  be any minimal realization for  $g$  whose realization point is also  $z_0$ , see e.g. [10, Theorem 4.2]; in particular the corresponding realizing space has negative index  $\kappa-1$ . Note that the selection of  $z_0$  as the realization point of  $\{A_g, \omega_g\}$  is possible, because, with  $\mathcal{D}(f)$  and  $\mathcal{D}(g)$  denoting the sets of holomorphy of  $f$  and  $g$  respectively, one has that

$$\rho(A) = \mathcal{D}(f) \subseteq \mathcal{D}(g) = \rho(A_g);$$

here the equalities on the above line are a consequence of the assumed minimality of the realizations, see [7, Theorem 1.1]. Now Theorem 4.1 yields a realization

$\{A_f, \omega_f\}$  for  $f$  where

$$(6.1) \quad (A_f - z)^{-1} = \begin{pmatrix} \frac{1}{\beta - z} & * & * \\ 0 & (A_g - z)^{-1} & * \\ 0 & 0 & \frac{1}{\bar{\beta} - z} \end{pmatrix};$$

here  $(\beta - z)^{-1}$  and  $(\bar{\beta} - z)^{-1}$  should be interpreted to be 0 if  $\beta = \infty$ , see Remark 4.2. The above expression shows that  $x_\beta := \{1, 0, 0\} \in \ker(A_f - \beta)$ ; here  $\ker(A_f - \beta)$  should be interpreted to be  $\text{mul } A_f$  if  $\beta = \infty$ . Moreover,  $x_\beta$  is also neutral in the associated realizing space, see Theorem 4.1.

If  $\{A_f, \omega_f\}$  is a minimal realization for  $f$ , then by Proposition 3.1 there exists a standard unitary operator  $U$  such that  $A = UA_fU^{-1}$ . Hence  $Ux_\beta$  is a non-trivial nonpositive eigenvector corresponding to the eigenvalue  $\beta$  for  $A$ . Thus the statement holds in this case.

The proof is completed by considering the case that  $\{A_f, \omega_f\}$  is not a minimal realization for  $f$ . In that case the non-minimal part of that realization has to be considered. I.e., the orthogonal complement of following set has to be investigated:

$$\mathfrak{M} := \text{c.l.s.} \{ (I + (z - z_0)(A_f - z)^{-1}) \omega : z \in \rho(A_f) \};$$

cf. Corollary 3.3. The negative index of the realizing space corresponding to the constructed realization  $\{A_f, \omega_f\}$  is  $(\kappa - 1) + 1 = \kappa$ , see Theorem 4.1. Since  $f \in \mathfrak{N}_\kappa$ , the negative index of the realizing space corresponding to any minimal realization is  $\kappa$ , see the discussion following Proposition 3.1. Therefore  $\mathfrak{M}^{[\perp]}$  does not contain non-trivial nonpositive vectors by Corollary 3.3 and Proposition 3.2. Hence, Steps 2 and 3 of the proof of Theorem 4.1 show that it consists at most of two positive eigenvectors corresponding to the eigenvalues  $\bar{\alpha}$  and  $\bar{\beta}$ . In particular,

$$(6.2) \quad \mathfrak{M} \cap \mathfrak{M}^{[\perp]} = \{0\}.$$

The proof is completed by showing that in all cases  $\mathfrak{M}$  contains a non-trivial nonpositive eigenvector corresponding to the eigenvalue  $\beta$ . For if that is true Corollary 3.3 implies, in light of (6.2), that the minimal realization for  $f$  obtained by reducing the realization  $\{A_f, \omega_f\}$  contains a non-trivial nonpositive eigenvector for  $\beta$ . Thus the proof would then be completed by another application of Proposition 3.1.

If  $\mathfrak{M}^{[\perp]}$  does not contain a (positive) eigenvector corresponding to the eigenvalue  $\bar{\beta}$ , then  $x_\beta$ , as defined following (6.1), is obvious contained in  $\mathfrak{M} = (\mathfrak{M}^{[\perp]})^{[\perp]}$ , because  $\mathfrak{M}^{[\perp]}$  contains in that case at most an eigenvector for  $\bar{\alpha} (\neq \beta)$ . In view of the fact that  $\mathfrak{M}^{[\perp]}$  contains only positive eigenvectors and that eigenvectors for non-real eigenvalues are neutral, the preceding reasoning also holds if  $\beta \in \mathbb{C} \setminus \mathbb{R}$ . It remains to consider the case that  $\beta \in \mathbb{R} \cup \{\infty\}$  and that  $\mathfrak{M}^{[\perp]}$  contains a positive vector  $x_\beta^+ \in \ker(A_f - \beta)$ , then

$$x_\beta - \frac{[x_\beta, x_\beta^+]}{[x_\beta^+, x_\beta^+]} x_\beta^+$$

is nonpositive and contained in  $\ker(A_f - \beta)$ . The above vector is an element of  $(\mathfrak{M}^{[\perp]})^{[\perp]} = \mathfrak{M}$ , because it is by construction orthogonal to  $x_\beta^+$  and being an eigenvector for  $\beta$  it is orthogonal to any eigenvector for  $\alpha (\neq \bar{\beta})$  possibly contained in  $\mathfrak{M}^{[\perp]}$ .  $\square$

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